Lecture 1

① class website (no canvas)

② midterm 1 : 2/22 , midterm 2 : 4/5 , final : TBD

③ homework : every Wed (1:59 pm)

→ lowest 2 homeworks dropped

brownian motion ≈ continuous analog to discrete random walk

continuous time :

simplest model for stock price : $dS_t = \alpha S_t dt$ (differential equation) → $S_t = S_0 e^{\alpha t}$

↓

$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

$dS_t = \alpha S_t dt + \sigma S_t dW_t$ ✓

brownian motion (continuous time random walk)

need Itô calculus for this

brownian motion : not differentiable anywhere

add randomness via BM

goals :

construct brownian motion (W_t)

→ "make sense" of such $dS_t = \alpha S_t dt + \sigma S_t dW_t$. "surprise" : chain rule has a correction → Itô formula

make sense of this stock model : need Itô calculus

has to treat dW_t

① central limit theorem

ξ_i are iid RV's . $\mathbb{E}\xi_i = 0$ (mean 0) , $\mathbb{E}\xi_i^2 = 1$ (variance 1)

let $S_n = \sum_{k=1}^{n} \xi_k$ → S_n is a sum of iid RVs where each has $\mathbb{E}=0$ and $Var = 1$

sum of iid RVs

$\frac{1}{n} S_n \to 0$

independent, so variances add

behavior of S_n as $n \to \infty$

LLN : $\lim_{n\to\infty} \frac{S_n}{n} = 0$

$\lim_{n\to\infty} \frac{1}{n} S_n = 0$ (law of large numbers)

→ "empirical mean"

note $\mathbb{E}S_n^2$ = sum of all ξ_i individual variances $= n$

$(\mathbb{E}[(\frac{1}{n}S_n)^2])$

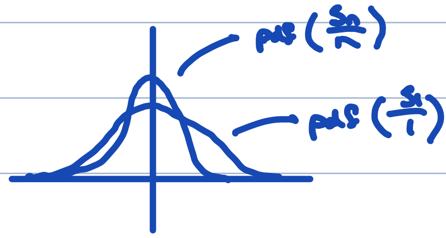
"pf :" compute the variance : $Var(\frac{1}{n}S_n) = \mathbb{E}(\frac{1}{n}S_n)^2 = \frac{1}{n^2}\mathbb{E}S_n^2$ [by iid] $= \frac{1}{n^2}(n) = \frac{1}{n}$

show the variance → 0.

→ $n = \frac{1}{n} \to \infty$

$\xrightarrow{n\to\infty} 0$ ie. $Var\left(\frac{S_n}{n}\right) \xrightarrow{n\to\infty} 0$

shown



clusters closer to 0

what about $\frac{1}{\sqrt{n}} S_n$?

CLT : consider $\frac{S_n}{\sqrt{n}}$ instead

CLT now considers $\frac{S_n}{\sqrt{n}}$ instead

note the variance of $\frac{S_n}{\sqrt{n}}$: $Var\left(\frac{S_n}{\sqrt{n}}\right) = \frac{1}{n} Var(S_n) = 1$

$X \sim N(0,1)$ iff

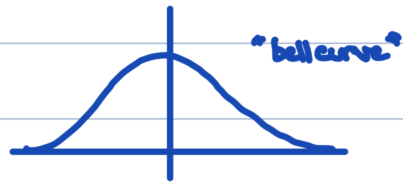
PDF is : $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

converges to $N(0,1)$

$\frac{1}{\sqrt{n}} S_n \xrightarrow{n\to\infty} N(0,1)$ (standard normal)

show this converges to a standard normal

std normal : $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



mean std. dev

$N(0,1)$

we say $X \sim N(0,1)$ if PDF of X is $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, ✓

$X \sim N(\mu, \sigma^2)$ if PDF is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$; $\mathbb{E}X = \mu$, $Var(X) = \sigma^2$

$X \sim N(0,1)$ iff $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

CLT : cdf of $\left(\frac{1}{\sqrt{n}} S_n\right) \xrightarrow[\text{pointwise}]{n\to\infty}$ cdf $(N(0,1))$

$X \sim N(\mu, \sigma^2)$ iff

show that the distributions converge pointwise

$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

characteristic functions ~ moment generating fns.

mean μ, var σ^2

characteristic function :

let X be a RV. the characteristic fn of X, denoted by $\varphi_\lambda(x)$ is defined by

$\varphi_\lambda(x) = \mathbb{E}e^{i\lambda X}$ where $i = \sqrt{-1}$

characteristic function. $\varphi_X(\lambda) = \mathbb{E}e^{i\lambda X}$

moment generating function :

mgf. $M_X(\lambda) = \mathbb{E}e^{\lambda X}$

to prove distribution, show that CF's converge.

defined by

$M_\lambda(x) = \mathbb{E}e^{\lambda X}$

annoying because this might not be finite

mgf worse than cf because the expectation may not be defined. with cf, always lies on unit circle

remark : MGF of RV's need not be finite $\forall \lambda$, but CF is defined

$\underline{e^{i\lambda}} \to |e^{i\theta}| = 1$

always

Q : $X \sim N(0,1)$

$\varphi_\lambda(x) =$

$M_X(\lambda) = \mathbb{E}e^{\lambda X}$ $\varphi_X(\lambda) = \mathbb{E}e^{i\lambda X}$

$M_\lambda(x) = \mathbb{E}e^{\lambda X} = e^{\frac{\lambda^2}{2}}$

CF always defined, on unit circle

Lecture 2 Notes

last time: CLT $\frac{S_n}{\sqrt{n}} \to N(0,1)$ $[\mathbb{E}\xi_i = 0, \mathbb{E}\xi_i^2 = 1, \xi_i \text{ iid}, S_n = \sum_i^n \xi_i]$

CLT: $\frac{S_n}{\sqrt{n}} \to N(0,1)$ where $S_n = \sum \xi_i$ with $\mathbb{E}\xi_i = 0$, $\mathbb{E}\xi_i^2 = 1$

① CF + MGF

recall X an RV

CF/MGF unique to distribution

the characteristic function of X is defined by $\varphi_\lambda(X) = \mathbb{E}e^{i\lambda X}$

(for a given RV, $\lambda \to \varphi_\lambda(X)$ is a complex valued function)

MGF/CF and moments:

$\mathbb{E}X^n = M_X^{(n)}(0)$

the moment generating function of X is $M_\lambda(X) = \mathbb{E}e^{\lambda X}$

$\mathbb{E}X = M_0'(X) = \frac{\partial}{\partial\lambda} M_\lambda(X)\big|_{\lambda=0}$. recall n^{th} moment is $\mathbb{E}X^n$. we have $\mathbb{E}X^n = M_X^{(n)}(0)$

check:

$M_\lambda(X) = \mathbb{E}e^{\lambda X}$: $\partial_\lambda M_\lambda(X) = \partial_\lambda \mathbb{E}e^{\lambda X} = \mathbb{E}\partial_\lambda e^{\lambda X} = \mathbb{E}Xe^{\lambda X}$

when $\lambda = 0$, get $\partial_\lambda M_\lambda(X)\big|_{\lambda=0} = \mathbb{E}Xe^0 = \mathbb{E}X$ evaluated at $\lambda = 0$

iterate this: get $\mathbb{E}X^n = \partial_\lambda^n M_\lambda(X)\big|_{\lambda=0}$

need to account for the i

for CF: $\mathbb{E}X^n = (-i)^n \varphi_X^{(n)}(0)$

in terms of the CF:

$\mathbb{E}X = (-i)\partial_\lambda \varphi_\lambda(X)\big|_{\lambda=0}$ $\to$ note: $\partial_\lambda^n \varphi_\lambda(X) = \mathbb{E}\partial_\lambda^n(e^{i\lambda X}) = \mathbb{E}(iX)^n e^{i\lambda X}$

$\Rightarrow \partial_\lambda^n \varphi_\lambda(X)\big|_{\lambda=0} = \partial_\lambda^n \varphi_0(X) = \mathbb{E}(iX)^n = (i)^n \mathbb{E}X^n$

$\Rightarrow \mathbb{E}X^n = (-i)^n \partial_\lambda \varphi_\lambda(X)\big|_{\lambda=0}$ similar property for CF: $\mathbb{E}X^n = (-i)^n \varphi_X^{(n)}(0)$

lazy stat: $\mathbb{E}f(X) = \int_{-\infty}^{\infty} f(x)p(x)\,dx$

thus we have

ex. compute MGF and CF of $N(0,1)$

$\mathbb{E}X^n = (-i)^n \varphi_X^{(n)}(0) = M_X^{(n)}(0)$: utilize this property

① MGF: $X \sim N(0,1)$

directly compute w/ lazy stat

$M_\lambda(X) = \mathbb{E}e^{\lambda X} = \int_{-\infty}^{\infty} e^{\lambda x} e^{-x^2/2} \frac{1}{\sqrt{2\pi}}\,dx$

$= e^{\frac{\lambda^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\lambda x + \lambda)^2} \frac{dx}{\sqrt{2\pi}} = e^{\lambda^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\lambda)^2} \frac{dx}{\sqrt{2\pi}}$

change vars of int

$= e^{\lambda^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{=1} = e^{\lambda^2/2}$

covers entire distribution

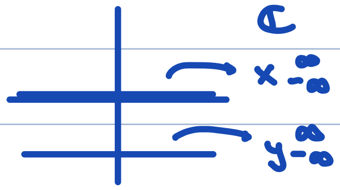
to compute MGF: use law of unconcious statistician, and defn of pdf normalized

$\Rightarrow \mathbb{E}e^{\lambda X} = M_\lambda(X) = e^{\lambda^2/2}$ when $X \sim N(0,1)$

Standard normal:
$M_X(\lambda) = e^{\lambda^2/2}$ ✓, $\varphi_X(\lambda) = e^{-\frac{\lambda^2}{2}}$ ✓

$\varphi_\lambda(X) = e^{-\lambda^2/2}$

compute similarly: $\mathbb{E}e^{i\lambda X} = \int_{-\infty}^{\infty} e^{i\lambda x - x^2/2} \frac{dx}{\sqrt{2\pi}} = e^{-\lambda^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-i\lambda)^2} \frac{dx}{\sqrt{2\pi}}$



$\underbrace{y = x - i\lambda}_{\text{reasonable}}$ shifting domain of integration by factor of i

can treat the i has usual constant when integrating

can still integrate this normally

complex analysis implies $\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-i\lambda)^2} \frac{dx}{\sqrt{2\pi}} = 1$

$\Rightarrow X \sim N(0,1)$, $\varphi_\lambda(X) = \mathbb{E}e^{i\lambda X} = e^{-\lambda^2/2}$

theorem: a sequence of RVs X_n converges to X in distribution (ie. CDF of $X_n \to$ CDF of X at points of continuity) $\Leftrightarrow$ for every bounded continuous function f:
$\lim_{n\to\infty} \mathbb{E}f(X_n) = \mathbb{E}f(X)$

p_n = PDF of X_n, p = PDF of X

$p_n \xrightarrow{n\to\infty} p$ $\quad \mathbb{E}f(X_n) = \int_{-\infty}^{\infty} f(x)p_n(x)dx$; $\mathbb{E}f(X) = \int_{-\infty}^{\infty} f(x)p(x)dx$

$\Leftrightarrow$

$\varphi_\lambda(X_n) \xrightarrow{n\to\infty} \varphi_\lambda(X) \; \forall \lambda \in \mathbb{R}$

$X \sim N(0,1)$:

$M_X(\lambda) = e^{\lambda^2/2}$, $\varphi_X(\lambda) = e^{-\lambda^2/2}$

Lecture 3 Notes

few properties: following are equivalent

(1) X and Y have the same distribution (PDF) (2) X and Y have the same CDF (3) X and Y have same CF (4) X and Y have same MGF

$\varphi_X(\lambda) = \mathbb{E}e^{i\lambda X}$ (proper notation)

CLT: $\frac{S_n}{\sqrt{n}} \longrightarrow N(0,1)$, $S_n = \sum_1^n \xi_k$, ξ_k iid with $E\xi_k = 0$, $E\xi_k = 1$

convergence in distribution

easiest thing to use: converge in distribution $\Longleftrightarrow$ converge in CFs/MGFs

last time:

$X_n \longrightarrow X$ in distribution $\Longleftrightarrow \varphi_{X_n}(\lambda) \xrightarrow{n\to\infty} \varphi_X(\lambda)$ $\forall \lambda \in \mathbb{R}$

$X \sim N(0,1)$ then $\varphi_X(\lambda) = e^{-\lambda^2/2}$

$X \perp\!\!\!\perp Y \Rightarrow \varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$

if $X \perp\!\!\!\perp Y \Rightarrow \varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$

proof. X, Y are two independent RVs. then $\varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$

$\varphi_{X+Y}(\lambda) = \mathbb{E}e^{i\lambda(X+Y)} = \mathbb{E}e^{i\lambda X}e^{i\lambda Y}$ (by ind.) $= \mathbb{E}e^{i\lambda X}\mathbb{E}e^{i\lambda Y} = \varphi_X(\lambda)\varphi_Y(\lambda)$

by independence ($EXY = EXEY$)

theorem: if $\varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$ $\forall \lambda \in \mathbb{R}$, then X and Y are independent

proof: if $\alpha \in \mathbb{R}$ and X is an RV, then $\varphi_{\alpha X}(\lambda) = \varphi_X(\alpha\lambda)$

$\varphi_{\alpha X} = \mathbb{E}e^{i\lambda(\alpha X)} = \mathbb{E}e^{i(\alpha\lambda)X} = \varphi_X(\alpha\lambda)$ also note $\varphi_{\alpha X}(\lambda) = \varphi_X(\alpha\lambda)$

can easily prove these with properties of the formula $\mathbb{E}e^{i\lambda X}$

uses properties of exponents
derive yourself if needed

proof of CLT:

① ξ_i are iid, $E\xi_i = 0$, $E\xi_i^2 = 1$ ② $S_n = \sum_{k=1}^n \xi_k$

NTS $\frac{S_n}{\sqrt{n}} \longrightarrow N(0,1) \Longleftrightarrow CF\left(\frac{S_n}{\sqrt{n}}\right)$ is $e^{-\lambda^2/2}$

ie. NTS $\forall \lambda \in \mathbb{R}$, $\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = e^{-\lambda^2/2}$ want to examine dist. of $\frac{S_n}{\sqrt{n}}$

① $\varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \varphi_{S_n}\left(\frac{\lambda}{\sqrt{n}}\right) = \varphi_{\xi_1}\left(\frac{\lambda}{\sqrt{n}}\right)\varphi_{\xi_2}\left(\frac{\lambda}{\sqrt{n}}\right)\cdots\varphi_{\xi_n}\left(\frac{\lambda}{\sqrt{n}}\right)$ ⊛

let ξ have the same distribution as $\xi_1 \cdots$

then $\varphi_{\xi_i}(\lambda) = \varphi_\xi(\lambda)$ $\forall \lambda \in \mathbb{R}$

$\varphi_{S_n/\sqrt{n}}(\lambda) = \varphi_{\underset{\text{sum of iids}}{\underline{S_n}}}\left(\frac{\lambda}{\sqrt{n}}\right)$

$= \left[\varphi_{\xi_i}\left(\frac{\lambda}{\sqrt{n}}\right)\right]^n$

$* \Rightarrow \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \varphi_\xi\left(\frac{\lambda}{\sqrt{n}}\right)^n$

we know the moments $\quad f(x) \approx f(0) + f'(0)(x) + \frac{1}{2}f''(0)(x)^2$ $\quad$ we approximate the CF centered at $\frac{\lambda}{\sqrt{n}}$

② note: $\varphi_\xi(0) = \mathbb{E}e^{i(0)\xi} = 1$

$\varphi'_\xi(0) = i\mathbb{E}\xi' = 0$ $\qquad [\varphi_\xi(\frac{\lambda}{\sqrt{n}})]^n = [\varphi_\xi(0) + \varphi'_\xi(0)(\frac{\lambda}{\sqrt{n}}) + \frac{1}{2}\varphi''_\xi(0)(\frac{\lambda^2}{n})]$

$\varphi''_\xi(0) = i^2\mathbb{E}\xi^2 = -1$

now we approximate w/ taylors

$$\text{so } \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = [\varphi_\xi(\tfrac{\lambda}{\sqrt{n}})]^n = [\varphi_\xi(0) + \tfrac{\lambda}{\sqrt{n}}\varphi'_\xi(0) + \tfrac{\lambda^2}{2n}\varphi''_\xi(0) + \text{small}]^n,$$

$$= (1 + 0 - \tfrac{\lambda^2}{2n} + O(\tfrac{1}{n^{3/2}}))^n \xrightarrow{n\to\infty} e^{-\lambda^2/2} \qquad \square$$

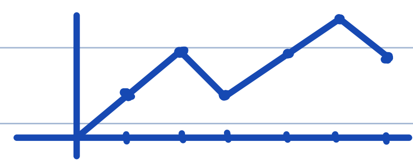
recall: simplest stock model, $dS_t = \alpha S_t dt + \sigma S_t dW_t$ $\quad$ what is this W?

explain "W_t" which is "Brownian Motion" ≈ 'Random' motion

Nobert Wiener → "Wiener Process" $\quad$ brownian motion ≈ continuous time RW

W_t → "continuous time random walk" $\quad$ +1 if heads, −1 if tails

DT random walk: 1 coin flip per second



recall standard random walk

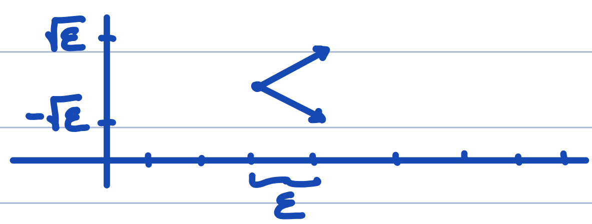
flip every 1/2 second $\quad$ if heads → +1/2, if tails → −1/2

let $\varepsilon > 0$, every ε seconds flip a coin: if H → $+\alpha$, if T → $-\alpha$ $\quad$ flip coin every ε secs, up/down by $\sqrt{\varepsilon}$

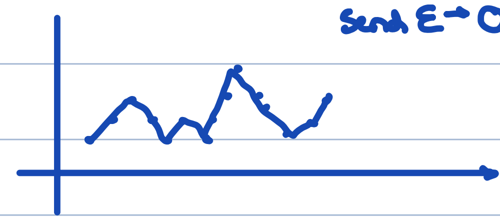
compute variance after time 1 = Variance after $\frac{1}{\varepsilon}$ steps of random walk $= \frac{1}{\varepsilon}\mathrm{Var}(\text{each step})$

$= \frac{1}{\varepsilon}(\frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^2) = \frac{\alpha^2}{\varepsilon} \longrightarrow \alpha = \sqrt{\varepsilon}$

should choose $\alpha = \sqrt{\varepsilon}$



connect dots, send $\varepsilon \to 0$



the limit is called Brownian motion

what is BM? think DT random walk: → rescale

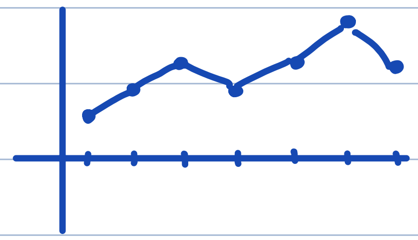
flip coin every ε seconds, up/down by $\sqrt{\varepsilon}$

Lecture 4 Notes

kept just DT random walk, flipping every ε seconds, step sizes of $\sqrt{\varepsilon}$

recall : simple stock model $dS_t = \alpha S_t dt + \sigma S_t dW_t$

W_t : Brownian motion , intuition $\longrightarrow$ continuous time random walk



SRW (DT) : $S_n = \sum_{k=1}^{n} \varepsilon_k$, ε_k iid w/ $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$

$S_t = S_n + (t-n)\varepsilon_{n+1}$ $\quad t \in [n, n+1]$

need step size $\sqrt{\varepsilon}$ to ensure variance remains $\approx t$

flip every ε seconds , steps of size $\sqrt{\varepsilon}$, send $\varepsilon \to 0$: results in continuous time RW i.e. Brownian motion

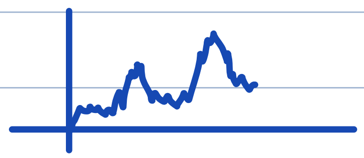
let $S_t^{\varepsilon} = \sqrt{\varepsilon}\, S_{t/\varepsilon}$ (random walk , flip every ε seconds w/ step size $\sqrt{\varepsilon}$)

let $W_t = \lim_{\varepsilon \to 0} S_t^{\varepsilon} = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, S_{t/\varepsilon}$ $\qquad$ $S_t^{\varepsilon} = \sqrt{\varepsilon}\, S_{t/\varepsilon}$: RW, flip every ε secs , step size of $\sqrt{\varepsilon}$

(can claim : ① this limit exists ② the function $t \longmapsto W_t$ is a continuous function of t)

now BM is just as $\varepsilon \to 0$. $W_t = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, S_{t/\varepsilon}$

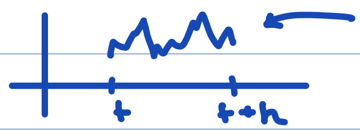
then W_t : $W_t = \lim_{\varepsilon \to 0} S_t^{\varepsilon}$

Q :  is $t \longmapsto W_t$ differentiable ?

absolutely not differentiable

$\frac{d}{dt} S_t^{1} = \pm 1$ when it exists , $\frac{d}{dt} S_t^{\varepsilon} = \sqrt{\varepsilon}\, S_{t/\varepsilon} = \sqrt{\varepsilon} \cdot \frac{1}{\varepsilon}(\pm 1)$ where it exists $= \pm \frac{1}{\sqrt{\varepsilon}}$

no hope of the $\frac{d}{dt} S_t^{\varepsilon}$ existing as $\varepsilon \to 0$



in between t and $t+h$ there are infinitely many zigzags

no hope of this function being differentiable

theorem :

with probability 1 , $\frac{dW_t}{dt}$ does not exist for every $t \geq 0$: always continuous , never differentiable

BM : always continuous , never differentiable

W is always cts , but never diff

note : ① distribution of W_t ? dist $(W_t) \approx$ dist $(S_t^{\varepsilon}) \approx$ dist $(\sqrt{\varepsilon}\, S_{t/\varepsilon})$

$\approx$ dist $\left(\sqrt{\varepsilon} \sum_{k=1}^{t/\varepsilon} \varepsilon_k\right) \approx$ dist $\left(\sqrt{t}\, \frac{1}{\sqrt{t/\varepsilon}} \sum_{k=1}^{t/\varepsilon} \varepsilon_k\right)$

$\xrightarrow{\varepsilon \to 0}$ $N(0,t)$ $\qquad$ use CLT to show dist of brownian mtn $\qquad$ by CLT , $\varepsilon \to 0$ $\longrightarrow$ $N(0,1)$

expect distribution of W_t : dist$(W_t) = N(0,t)$

② dist of $W_t - W_s$ $\Longrightarrow$ dist $(W_t - W_s) \approx$ dist $\left(\sqrt{\varepsilon} \sum_{s/\varepsilon}^{t/\varepsilon} \varepsilon_k\right) \approx$ dist $\left(\sqrt{t-s}\, \frac{1}{\sqrt{t-s/\varepsilon}} \sum_{s/\varepsilon}^{t/\varepsilon} \varepsilon_k\right)$

$\xrightarrow{\varepsilon \to 0}$ $N(0, t-s)$

$W_t \approx$ dist $(\sqrt{\varepsilon}\, S_{t/\varepsilon})$

③ moreover, $W_t - W_s$ is independent of W_s — $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s \rightarrow$ includes W_s

reason: $W_t - W_s \approx \sqrt{\varepsilon}\sum_{s/\varepsilon}^{t/\varepsilon} \xi_k$, $W_s \approx \sqrt{\varepsilon}\sum_{1}^{s/\varepsilon} \xi_k$

coin tosses after s/ε — coin tosses before s/ε $\rightarrow$ which are independent

"better definition of BM"

def: W is a standard brownian motion if ① the trajectories are continuous (ie. the function $t \longmapsto W_t$ is always continuous) ② $W_t - W_s \sim N(0, t-s)$ ③ $W_t - W_s$ is independent of $\mathcal{F}_s$ (where $\mathcal{F}$ = "all information up to time s) ie. $W_t - W_s$ is independent of W_r $\forall r \leq s$

process W: ① $W_t - W_s \sim N(0, t-s)$

discrete probability

$\Omega \rightarrow$ sample space $= \{(\omega_1, \cdots, \omega_N) \mid \omega_i$ is outcome of each coin toss$\}$

PMF: $p(\omega) \in [0,1]$, $p(\omega)$ = prob of some coin toss appearing

$A \subseteq \Omega$, $p(A) = \sum_{\omega \in A} p(\omega)$

note because $W_t - W_s \sim N(0, t-s)$ and $W_0 = 0$

take $t=t, s=0 \rightarrow W_t - 0 \sim N(0, t-0) \Rightarrow W_t \sim N(0,t)$ — this will be our working definition for W

(same can be said, $W_s \sim N(0,s)$)

Standard BM, W_t:

1. $W_t - W_s \sim N(0, t-s)$ $\rightarrow$ take $W_0 = 0$, $W_t - W_0 = W_t \sim N(0,t)$
2. $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

W_t is a standard BM if :

1. the fn $t \longmapsto W_t$ is continuous (as a fn of t)

($W_0 = 0$)

2. $W_t - W_s \sim N(0, t-s)$

3. $W_t - W_s$ is independent of $\mathcal{F}_s$ (intuition : $W_t - W_s$ is independent of $W_r \ \forall r \leq s$)

W: ① $W_t - W_s \sim N(0, t-s)$ (indep steps)
② $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

✓

$W_t - W_s$ is independent of all $W_r \ \forall r \leq s$

Sample space :

$\Omega = \{\omega \mid \omega = (\omega_1, \cdots, \omega_N)\}$ ω_i = outcome of ith toss

$p(\omega)$ = probability of $\omega_1, \cdots, \omega_N$ being tossed

$A \subseteq \Omega$, $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$

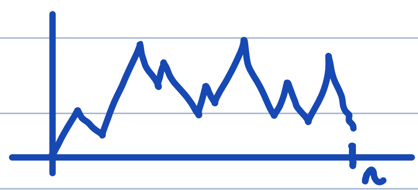
what is sample space in CTS ?

$\Omega = \{$all trajectories of BM$\}$

→ all continuous functions

"trajectory ≈ function"

continuous time :



let $S_n = \sum_1^n \xi_i$

can "redo" the DT :

in continuous time :

$\Omega = \{$trajectories of the SRW S_n up to time n

$\Omega = \{$all of the trajectories of BM$\}$

↓

$\Omega = C([0, \infty))$

"the set of all continuous functions"

continuous time : sample space = "all trajectories of BM" ✓

$= \mathring{C}[0,\infty) = \{$all continuous fn's on $[0,\infty)\}$ (starts at 0)

$= \{\omega \mid \omega : [0,\infty) \mapsto \mathbb{R}$ is a cts fn and $\omega(0) = 0\}$ ← infinite dimension

"events" : DT - event is simply a subset of Ω

→ this is an uncountably infinite sample space

⇓

no probability mass function

"$\mathbb{P}(A)$" if A is an event

horrible sample space.

→ satisfies certain properties

→ can we restrict to set of measurable functions ?

1. $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

2. $A \cap B = \emptyset \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

2*. if $A_1, A_2, \cdots$ are pairwise disjoint $\Rightarrow \mathbb{P}(\bigcup_1^\infty A_i) = \sum_1^\infty \mathbb{P}(A_i)$

want basic properties to hold

① impossible to define $\mathbb{P}(A)$ for every $A \subseteq \Omega$ and still be mathematically consistent

② restrict to measurable sets

impossible to define $\mathbb{P}(A)$ for EVERY $A \in \Omega$

$\mathcal{P}(\Omega)$ = powerset of Ω restrict to mb sets

"cannot assign a prob to uncountably infinite # of cts fns and still retain nice mathematical properties"

$\mathcal{G} \subseteq \mathcal{P}(\Omega)$ = "collection of measurable (good) sets"

restrict to all "measurable" sets : $\mathcal{P}(\Omega)$ = set of all subsets of Ω

mathematically possible to define $\to$ $\mathcal{G} \subseteq \mathcal{P}(\Omega)$

$\mathcal{P}(A)$ $\forall A \in \mathcal{G}$ and be consistent with

$\mapsto$ take some subset of Ω where all events behave "measurably"

① and ②nd

pick these such that $\mathbb{P}(A)$ $\forall A \in \mathcal{G}$ follow props of $\mathbb{P}(A)$

$\mathcal{G}$ is a σ-alg : any event in $\mathcal{G}$ is an event that you can make sense of the prob of

$W \to$ a BM , fix $t > 0$

$\mathbb{P}$ is a prob measure on $(\Omega, \mathcal{G})$

$\{W_t > 5\} = \{\omega \in \Omega \mid W_t(\omega) > 5\} \in \mathcal{G}$

$\hookrightarrow$ $\mathbb{P}(N(0,t) > 5) = \int_5^\infty e^{-y^2/2t} \frac{dy}{\sqrt{2\pi t}}$

$\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{G})$

1. $\mathbb{P} : \mathcal{G} \mapsto [0,1]$, $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$
2. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint
3. if $A_n \in \mathcal{G}$, $\mathbb{P}(\bigcup_1^\infty A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$

expected values :

what is $\mathbb{E}X$?

$X \to$ RV : what is $\mathbb{P}(X > 0) = \mathbb{P}(\{\omega \mid X(\omega) > 0\})$

discrete time :

RV is any fn $X : \Omega \mapsto \mathbb{R}$

continuous time :

RV is a measurable fn $X : \Omega \mapsto \mathbb{R}$

$\to$ RV is a measurable fn

means that for any $\alpha \in \mathbb{R}$, granted $\{X > \alpha\} \in \mathcal{G}$, $\forall \alpha, \beta$ $\{\alpha \leq X \leq \beta\} \in \mathcal{G}$

what is $\mathbb{E}X$

if range $X = \{x_1, \cdots, x_n\}$

$\{X = x_i\} \in \mathcal{G}$, $\mathbb{P}(\{X = x_i\})$ is defined

lecture 6 notes

last time:

$\Omega \to$ sample space. $C[0,\infty) = \{w \mid w: [0,\infty) \mapsto \mathbb{R}$ is cts and $w(0) = 0\}$

$\mathcal{G} \subseteq \mathcal{P}(\Omega) = \{$all "measurable" sets$\}$

$A \in \mathcal{G}$ called an event, $\mathbb{P}(A)$ is defined, $A, B \in \mathcal{G}$, $A \cup B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$, $A^c \in \mathcal{G}$, etc.

$$\{X > 0\} = \{w \in \Omega \mid X(w) > 0\}$$

expected values

the RV must be $\mathcal{G}$ mb

$X: \Omega \mapsto \mathbb{R}$ is a RV if $X: \Omega \mapsto \mathbb{R}$ is a $\mathcal{G}$-measurable function

ie. $\{X > \alpha\} \in \mathcal{G}\ \forall \alpha \in \mathbb{R}$, $\{X = \alpha\} \in \mathcal{G}$, $\{X < \alpha\}$, $\{X \in (\alpha, \beta)\}$, etc.

X is a RV.

if range(X) is finite, then say range$(X) = \{x_1, \cdots, x_n\}$ then $\mathbb{E}X = \sum x_i \mathbb{P}(X = x_i)$

if not $\to$ no formula

if the range is finite then we have

integration over measurable sets

$\mathbb{E}X =$ "Lebesgue integral"

$$\mathbb{E}X = \sum_{\text{range}(x)} x_i\, \mathbb{P}(X = x_i)$$

$\to$ otherwise no formula

$\to$ $\mathbb{E}X$ is a lebesgue integral: it is the integration of functions over measurable sets:

$\mathbb{E}X$ satisfies same properties that you're used to

if p_X is the pdf of X (ie. $\forall A \in \mathbb{R}$, $\mathbb{P}(X \in A) = \int_A p_X(x)\,dx$

$$\mathbb{E}X = \int_\Omega X(w)\, d\mathbb{P}(w)$$

lazy stat: $\mathbb{E}f(X) = \int_{-\infty}^{\infty} f(x) p(x)\,dx$

$f: \mathbb{R} \mapsto \mathbb{R}$ is some fn, $\mathbb{E}f(X) = \int_{-\infty}^{\infty} f(x) p_X(x)\,dx$

note: $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$

layer cake: $\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq t)\,dt$

$$\mathbb{1}_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

2. (layercake) if $X \geq 0$ is an RV.

then $\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq t)\,dt$

properties:

1. linearity
2. positivity: if $X \geq 0$ then $\mathbb{E}X \geq 0$. if $X \geq 0$ and $\mathbb{E}X = 0$, then $X = 0$ almost surely
3. unconscious stat: $\mathbb{E}f(X) = \int_{-\infty}^{\infty} f(x) p_X(x)\,dx$

filtrations:

"adapted" trading strategies

layercake:

choose position on an asset without looking into the future

$X \geq 0$, then $\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq t)\,dt$

adapted process is a sequence of $\mathcal{F}_i$ mb RVs for $i \leq n$

adapted : sequence of $\mathcal{F}_i$ measurable RVs for $i \leq n$

discrete time : $\mathcal{F}_n$ = "all information obtained by observing the first n coin tosses"

do not want to be able to look into the future

X is adapted if X_n is $\mathcal{F}_n$ measurable

RVs are $\mathcal{F}$ measurable

Stoch processes are adapted

continuous time :

work with trajectories

coin tosses lose meaning : work with trajectory of BM instead

$\mathcal{F}_t$ = {all events that can be observed using the trajectory of W up to time t}

$\mathcal{F}_t$ = {all events observed by observing BM trajectory}

deduced events from observing the trajectory of BM

is $s \leq t$, $\{W_s > 0\} \in \mathcal{F}_t$

up to time t, you're okay

in DT: $\mathcal{F}_0 = \{\emptyset, \Omega\}$

in CT: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \mathbb{P}(A) \in \{0,1\}\}$

(set of all mb sets where probability is nicely defined)

discrete time : $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$

cts : $\forall s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$ → in cts : $\forall s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$

X adapted iff X_t is $\mathcal{F}_t$-mb $\forall t \geq 0$

adaptedness : a process X is adapted if $\forall t \geq 0$, X_t is $\mathcal{F}_t$-mb

adapted : X_t is $\mathcal{F}_t$-mb $\forall t \geq 0$

conditional expectation :

X → some RV, $\mathbb{E}_t X$ = conditional expectation of X given $\mathcal{F}_t$

$= \mathbb{E}[X \mid \mathcal{F}_t]$ "conditioned on the information up until time t"

trajectory of BM up until t

= best approximation of X by an $\mathcal{F}_t$-measurable RV

ie. $\mathbb{E}_t X$ is a $\mathcal{F}_t$-mb RV such that

$\left(\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \right)$

for every $A \in \mathcal{F}_t$:

$\mathbb{E}(\mathbb{1}_A X) = \mathbb{E}[\mathbb{1}_A \mathbb{E}_t X]$, $\frac{1}{\mathbb{P}(A)} \mathbb{E}(\mathbb{1}_A X)$ = average of X restricted on event A

Lecture 7 Notes

$E_t X$ is the unique $\mathcal{F}_t$-mb RV such that:

conditional expectation

$$\mathbb{E}[\mathbb{1}_A X] = \mathbb{E}[\mathbb{1}_A E_t X]$$

$\mathcal{F}_t \rightarrow$ all events that can be observed by using the trajectory of W for times $\leq t$

$E_t X = \mathbb{E}[X \mid \mathcal{F}_t]$: best approximation of X by an $\mathcal{F}_t$-mb RV

$E_t X$: best approximation of X from a $\mathcal{F}_t$-mb RV

definition: $E_t X$ is the unique $\mathcal{F}_t$-mb RV such that $\forall A \in \mathcal{F}_t$,

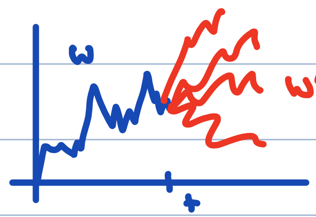
$$\frac{1}{P(A)} \mathbb{E}[\mathbb{1}_A X] = \frac{1}{P(A)} \mathbb{E}[\mathbb{1}_A E_t X] \implies \mathbb{E}[\mathbb{1}_A X] = \mathbb{E}[\mathbb{1}_A E_t X]$$

$E_t X$ is the unique $\mathcal{F}_t$-mb RV st.

$\mathbb{E}[\mathbb{1}_A X] = \mathbb{E}[\mathbb{1}_A E_t X]$

$\forall A \in \mathcal{F}_t$

discrete time: formula for $E_n X(\omega) = \dfrac{\mathbb{E}(\mathbb{1}_{\Pi_t(\omega)} X)}{\mathbb{P}(\Pi_t(\omega))}$



$\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega(s) = \omega'(s) \ \forall s \leq t\}$

doesn't work in continuous time because $P(\Pi_t(\omega)) = 0$

let Z be the cond expectation of X on information up until t:

1. Z is $\mathcal{F}_t$-mb
2. $\mathbb{E}[\mathbb{1}_A Z] = \mathbb{E}[\mathbb{1}_A X] \ \forall A \in \mathcal{F}_t$

we call this $Z = E_t[X]$

properties of conditional expectation:

1. $\mathbb{E}(E_t X) = \mathbb{E}(X)$

proof. choose $A = \Omega$ in (*), LHS $= \mathbb{E}X$, RHS $= \mathbb{E}(E_t X)$

2. If X, Y are 2 RVs and $\alpha \in \mathbb{R}$ (not random)

$E_t(X + \alpha Y) = E_t X + \alpha E_t$

Z be the cond exp of X at t:

1. Z is $\mathcal{F}_t$-mb
2. $\mathbb{E}[\mathbb{1}_A X] = \mathbb{E}[\mathbb{1}_A Z] \ \forall A \in \mathcal{F}_t$

check

$E_t(X+Y) = E_t X + E_t Y$

$E_t(X+Y) \rightarrow \mathbb{E}(\mathbb{1}_A (X+Y)) = \mathbb{E}(\mathbb{1}_A E_t(X+Y))$

note

$\mathbb{E}(\mathbb{1}_A (E_t X + E_t Y))$

$= \mathbb{E}(\mathbb{1}_A E_t X) + \mathbb{E}(\mathbb{1}_A E_t Y)$ utilize the defn in these proofs

by *, $= \mathbb{E}(\mathbb{1}_A X) + \mathbb{E}(\mathbb{1}_A Y) = \mathbb{E}(\mathbb{1}_A (X+Y))$

$\Longrightarrow \mathbb{E}(\mathbb{1}_A(\mathbb{E}_t X + \mathbb{E}_t Y)) = \mathbb{E}(\mathbb{1}_A(X+Y)) = \mathbb{E}(\mathbb{1}_A \mathbb{E}_t(X+Y))$

3. tower property,

if $s \leq t$ then $\mathbb{E}_s \mathbb{E}_t X = \mathbb{E}_s X$

tower property: for any $s \leq t$, $\mathbb{E}_s \mathbb{E}_t X = \mathbb{E}_s$

4. if $X \in \mathcal{F}_t$ then $\mathbb{E}_t X = X$

more generally, if $X \in \mathcal{F}_t$ and Y is anything then $\mathbb{E}_t[XY] = X\mathbb{E}_t[Y]$

"take out what is known"

5. if X is independent of $\mathcal{F}_t$ then $\mathbb{E}_t X = \mathbb{E}X$

$\longrightarrow$ treat $\mathcal{F}_t$ mb RVs as constants

Lecture 8 notes

$\mathbb{E}_t X$: $\forall A \in \mathcal{F}_t$, $\mathbb{E}(\mathbb{1}_A \mathbb{E}_t X) = \mathbb{E}(\mathbb{1}_A X)$

Q. $\mathbb{E}(\mathbb{1}_A X) = \mathbb{E}(\mathbb{1}_A Y) \Rightarrow X = Y$ **

reason: ** $\Rightarrow \mathbb{E}(\mathbb{1}_A (X-Y)) = 0 \quad \forall A \in \mathcal{G}$ ✓

because for $X \geq 0$, $\mathbb{E}[X] = 0 \Rightarrow X = 0$ a.s.

$\mathcal{G}$ is full space

choose $A = \{X > Y\} \Rightarrow \mathbb{E}(\underbrace{\mathbb{1}_{\{X \geq Y\}}(X-Y)}_{\geq 0}) = 0$

$\Leftrightarrow P(X > Y) = 0$

similarly, $A = \{X < Y\}$, $\mathbb{E}[\underbrace{\mathbb{1}_{\{X<Y\}}(X-Y)}_{\leq 0}] = 0 \Rightarrow \mathbb{P}(X<Y) = 0$

$\Rightarrow P(X \neq Y) = 0$

Q. know $\mathbb{E}(\mathbb{1}_A \mathbb{E}_t X) = \mathbb{E}(\mathbb{1}_A X) \; \forall A \in \mathcal{F}_t$ → not $\mathcal{G}$, but $\mathcal{F}_t$

$\Rightarrow X = \mathbb{E}_t X$? no because we don't have equality for every $A \in \mathcal{G}$

bottom line: need $\mathcal{F}_t$-mb!!

note. if $Y + Z$ are both $\mathcal{F}_t$-mb, $\mathbb{E}(\mathbb{1}_A Y) = \mathbb{E}(\mathbb{1}_A Z) \Rightarrow Y = Z$ a.s.

only can

for equality to hold, need $\mathcal{F}_t$ measurability

last time: $X \rightarrow$ independent of $\mathcal{F}_t \Rightarrow \mathbb{E}_t X = \mathbb{E} X$

if $X \perp\!\!\!\perp \mathcal{F}_t$, $\mathbb{E}_t X = \mathbb{E} X$
(conditioning on any info $\mathcal{F}_t$ does not give us any info)

(DT justification - use the formula → independence

CT: if $A \in \mathcal{F}_t$, $\mathbb{E}[\mathbb{1}_A X] = \mathbb{E}(\mathbb{1}_A)\mathbb{E}(X)$

$\mathbb{E}[\mathbb{1}_A \mathbb{E}_t X] \overset{*}{=} \mathbb{E}(\mathbb{1}_A X) = \mathbb{E}\mathbb{1}_A \, \mathbb{E} X$ → independence

$\Rightarrow \mathbb{E}(\mathbb{1}_A \mathbb{E}_t X) = \mathbb{E}(\mathbb{1}_A \underbrace{\mathbb{E}X}_{\text{a constant so } \mathcal{F}_t\text{-mb}}) \; \forall A \in \mathcal{F}_t \Rightarrow \mathbb{E}_t X = \mathbb{E} X$ a.s.

ex. 1

compute $\mathbb{E}(W_s W_t)$ with $s \leq t$

two tricks: get to $W_t - W_s + W_s$ or use props of W as a mg

ind

sol 1: $\mathbb{E}(W_s(W_t - W_s + W_s)) = \mathbb{E} W_s (W_t - W_s) + \mathbb{E} W_s^2 = (\mathbb{E} W_s)(\mathbb{E}(W_t - W_s)) + s = s$

sol 2: $\mathbb{E}(W_s W_t) = \mathbb{E}\mathbb{E}_s(W_s W_t) = \mathbb{E} W_s \underbrace{\mathbb{E}_s W_t}_{\text{martingale!}} = \mathbb{E} W_s \mathbb{E}_s (W_t - W_s + W_s)$

$= \mathbb{E}(W_s(\;0\; + W_s)) = s$

reasons:

$\mathbb{E}_s W_s = W_s$ because $W_s \in \mathcal{F}_s$, $\mathbb{E}_s(W_s - W_t) = \mathbb{E}(W_s - W_t)$ because $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

normal $\mathbb{E}$

mg. iff $\mathbb{E}_s[M_t] = M_s \; \forall s \leq t$

M must be adapted

classic trick: rewrite W_t as $(W_t - W_s) + W_s$

we say a process M is a (continuous) martingale if

1. M is adapted (ie. M_t is $\mathcal{F}_t$-mb)
2. ~~$\mathbb{E}_n[M_{n+1}] = M_n$ (discrete time)~~ $\longrightarrow$ $\forall s \le t$, $\mathbb{E}_s M_t = M_s$

mg: must be adapted and $\forall s \le t$, $\mathbb{E}_s M_t = M_s$

proposition: BM is a mg

proof. $\mathbb{E}_s W_t = \mathbb{E}_s(W_t - W_s + W_s) = W_s$

Q. W_t^2 a mg? no, $\mathbb{E}W_t^2$ is not a const

better answer: $\mathbb{E}_s W_t^2 = \mathbb{E}_s(W_t + W_s - W_s)^2 = \underbrace{\mathbb{E}_s(W_t - W_s)^2}_{\mathbb{E}(W_t - W_s)^2 = t-s} + \underbrace{2\mathbb{E}_s W_s(W_t - W_s)}_{2W_s \mathbb{E}_s W_t - W_s \\ 2W_s \mathbb{E} W_t - W_s = 0} + \underbrace{\mathbb{E}_s W_s^2}_{W_s^2}$

$\mathbb{E}_s W_t^2 = t - s + W_s^2 \implies W_t^2$ is not mg

note $\implies \mathbb{E}_s(W_t^2 - t) = W_s^2 - s$: this is a martingale

$\implies W_t^2 - t$ is a MG

$(W_t^2 - t)$ is a mg $\longrightarrow$ $(W_t^2 - [W,W]_t)$ is a mg

$W_t^2 - t$ a mg,
$W_t^2 - [W,W]_t$ a mg
$M_t^2 - [M,M]_t$ is a mg

independence lemma

assume X is $\mathcal{F}_t$-mb, Y is independent of $\mathcal{F}_t$, and $f(x,y)$ is some function

$\mathbb{E}_t[f(X,Y)]$ = "average of $f(x,y)$ leaving X alone

$= g(X)$ where $g(x) = \mathbb{E}[f(x,Y)]$

independence lemma:

let X be $\mathcal{F}$mb and $Y \perp\!\!\!\perp \mathcal{F}_t$

then $\mathbb{E}[f(X,Y) \mid \mathcal{F}] = g(X)$ where $g(x) = \mathbb{E}[f(x,Y)]$

$\downarrow$

to remember: basically treat the $\mathcal{F}$mb var X

as the constant we plug into $g(x)$

independence lemma:

two RVs X, Y. let X be $\mathcal{F}$ mb and $Y \perp\!\!\!\perp \mathcal{F}$

$\mathbb{E}[f(X,Y) \mid \mathcal{F}] = g(X)$ where $g(x) = \mathbb{E}[f(x,Y)]$

hold X constant

traditional expectation across Y

W_t a mg:

WTS $\mathbb{E}_s[W_t] = W_s$

$$\mathbb{E}_s[W_t] = \mathbb{E}_s[(W_t - W_s) + W_s]$$

$$= \mathbb{E}_s[\underbrace{(W_t - W_s)}]+ W_s$$

$\mathcal{F}_s$ mb

$W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

$W_t - W_s \sim N(0, t-s)$

$$= \mathbb{E}[(W_t - W_s)] + W_s$$

$$= 0 + W_s$$

QED

Lecture 9 Notes

independence lemma :

$X \to \mathcal{F}_t$-mb , $Y \to \mathcal{F}_t$ independent

leave X alone , average Y

$f(x,y)$ some function : $\mathbb{E}_t f(X,Y) = g(X)$ where $g(x) = \mathbb{E} f(x,Y)$

if we have $\mathcal{F} = \sigma(X)$

then X is $\mathcal{F}$ mb always

ex.

$\mathcal{F} = \sigma(X)$ allows us to treat X as $\sigma(X)$ mb

$X, Y \to$ independent standard normals

$X \sim N(0,1)$, $Y \sim N(0,1)$ with $X \perp\!\!\!\perp Y$. compute $\mathbb{E} e^{iXY} \longrightarrow \mathbb{E}[\mathbb{E}(e^{iXY} \mid \mathcal{F})]$

utilize tower rule

try $\mathcal{F} = \sigma(X)$ = all events that can be observed using only X

$X \to \mathcal{F}$ mb , $Y \to \mathcal{F}$ independent : $\mathbb{E}(e^{iXY} \mid \mathcal{F}) = g(X)$ where $g(x) = \mathbb{E} e^{ixY}$

$= e^{-x^2/2} \Rightarrow \mathbb{E}(e^{iXY} \mid \mathcal{F}) = e^{-\frac{X^2}{2}} \to \mathbb{E}[\mathbb{E}(e^{iXY} \mid \mathcal{F})] = \mathbb{E} e^{-\frac{X^2}{2}}$ use unconscious stat

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \left(\text{know } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}} \frac{dt}{\sqrt{2\pi t}} = 1 \right) = \sqrt{1/2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(1/2)}} \frac{dx}{\sqrt{2\pi \, 1/2}}$$

$$= \frac{1}{\sqrt{2}}$$

✓

Stochastic integration

recall simple stock model , make sense by integrating

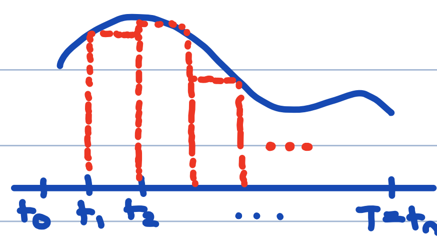
motivation : $dS_t = \alpha S_t + \underline{\sigma S_t \, dW_t}$ Add this BM term to make more realistic stock model

$\hookrightarrow$ make sense by :

stochastic integral (calculus rules don't work)

$$S_T - S_0 = \int_0^T dS_t = \underbrace{\int_0^T \alpha S_t \, dt}_{\text{r-integral / we know this}} + \overbrace{\int_0^T \sigma S_t \, dW_t}$$

$\hookrightarrow$ how to evaluate ?



1. riemann integrals : $\int_0^T f(t)\,dt = \lim_{\|P\| \to 0} \sum f(t_i)(t_{i+1} - t_i)$

$\lim_{\|P\| \to 0} \sum f(t_i)(t_{i+1} - t_i)$

any partition of time

$P = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ (partition of $(0, T]$)

$\|P\|$ = mesh size of P = $\max_i (t_{i+1} - t_i)$

mesh size : max distance between partition intervals

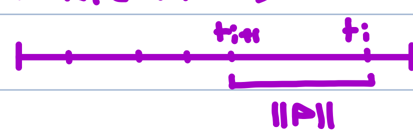
what is standard riemann integral ?

$\int_0^T f(t)\,dt = \lim_{\|P\| \to 0} \sum \underbrace{f(t_i)}_{\sim f(t)} \underbrace{(t_{i+1} - t_i)}_{\sim dt}$,

given some process S ($S_t \to$ price of some asset at time t)

$\Delta_t \to$ # of shares of stock held at time t

$\|P\| = \max_i \{t_{i+1} - t_i\}$



$\|P\| = \max_i (t_{i+1} - t_i)$

Say we only trade at times $t_0, t_1, \ldots, t_n = T$. net change in wealth between time 0 and T

$\underline{\Delta_{t_0}(S_{t_1} - S_{t_0})} + \Delta_{t_1}(S_{t_2} - S_{t_1}) + \cdots + \Delta_{t_{n-1}}(S_{t_n} - S_{t_{n-1}})$

net change: sum of all differences in asset price * the # of shares held

net change in wealth $= \underbrace{\sum_{i=0}^{n-1} \Delta_{t_i}(S_{t_{i+1}} - S_{t_i})}_{\text{sort of intuition}}$

if S was differentiable, then net change in wealth $= \sum \Delta_{t_i} \frac{(S_{t_{i+1}} - S_{t_i})}{t_{i+1} - t_i} t_{i+1} - t_i$

expect net CIW $\xrightarrow{\|P\| \to 0} \int_0^T \Delta_t \underline{\frac{dS}{dt}} dt = \int_0^T \Delta_t \, dS_t$

only if S is differentiable

if S was differentiable, could make sense with a riemann integral

called the riemann-stieltjes method

$\underline{\sum \Delta_{t_i}(S_{t_{i+1}} - S_{t_i})}$

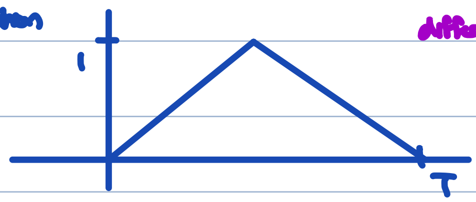
when S is not differentiable, need to use an "Ito Integral"

in our case, asset price is not differentiable

theorem: if process S has finite first variation, then standard integral exists

finite first variation $\downarrow$ riemann integral exists

first variation



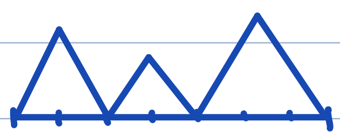
differentiable → finite first variation

elevator cost = vertical distance travelled = 2

vert dist travelled

$V_{[0,T]}(X)$: first variation of X on interval $[0,T]$

defn: given a process X, define $V_{[0,T]}(X)$ = first variation of X on $[0,T]$ ✓



$= \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \underline{|X_{t_{i+1}} - X_{t_i}|}$

absolute value of vertical distance travelled. sum this up to get full vertical distance travelled

1. if X is differentiable, then $\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}| = \sum \frac{|X_{t_{i+1}} - X_{t_i}|}{t_{i+1} - t_i} t_{i+1} - t_i$

$\xrightarrow{\|P\| \to 0} \int_0^T \left|\frac{dX}{dt}\right| dt$

if monotonically increasing, becomes a telescoping series

first variation:

1. always true that $V_{[0,T]}(X) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|$
2. if X is differentiable, then $V_{[0,T]}(X) = \int_0^T \left|\frac{dX}{dt}\right| dt$

use these!

↳ note if X is differentiable $\Rightarrow V_{[0,T]}(X) < \infty$ (finite first variation) first variation :

$$V_{[0,T]}(X) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|$$

$\longrightarrow$ if monotonically increasing, this becomes $X_T - X_0$

for differentiable functions, $V_{[0,T]}(X) = \int_0^T \left|\frac{dX}{dt}\right| dt$

if X is differentiable $\Longrightarrow V_{[0,T]}(X) < \infty$

Lecture 10 Notes

goal: make sense of $\int_0^T D_t \, dS_t$ (Dt represents position on asset)

riemann stieltjes integral: $\lim_{\|P\| \to 0} \sum D_{t_i}(S_{t_{i+1}} - S_{t_i})$

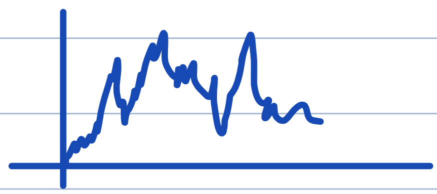
if S has finite first variation, then $\lim_{\|P\| \to 0} \sum D_{t_i}(S_{t_{i+1}} - S_{t_i}) \overset{\text{exists}}{=} \int_0^T D_t \, dS_t$

finite first variation implies riemann integral exists

note: S is differentiable $\rightarrow$ S has finite first variation and $V_{[0,T]} S = \int_0^T \left| \frac{dS}{dt} \right| dt$

recall $V_{[0,T]} S = \lim_{\|P\| \to 0} \sum |S_{t_{i+1}} - S_{t_i}|$

traditional defn



S differentiable: $V_{[0,T]} S = \int_0^T \left| \frac{dS}{dt} \right| dt$

Riemann integral: $\int_0^T D_t \, dS_t = \lim_{\|P\| \to 0} \sum D_{t_i}(S_{t_{i+1}} - S_{t_i})$

(S has finite first)

thm. $V_{[0,T]} W = \infty$ a.s. showing that BM has infinite first var

proof of $\mathbb{E}(V_{(0,T]} W) = +\infty$

0 ⊢—⊢—⊢—⊢—⊢ T $\rightarrow$ uniform partition into subintervals of size $\frac{T}{n}$

$V_{[0,T]} W = \infty$ a.s.

$t_i = \frac{iT}{n}$, $i \in \{0, 1, \dots, n\}$

$\rightarrow$ normalize so we know this

compute $\mathbb{E} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \sum_{i=0}^{n-1} \mathbb{E} |W_{t_{i+1}} - W_{t_i}|$

note $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i) = N(0, \frac{T}{n})$

now we know distribution

$\Rightarrow \mathbb{E} |W_{t_{i+1}} - W_{t_i}| = \mathbb{E} \frac{|W_{t_{i+1}} - W_{t_i}|}{\sqrt{t_{i+1} - t_i}} \sqrt{t_{i+1}} = \sqrt{t_{i+1} - t_i} \, \mathbb{E}|N(0,1)|$, $\underbrace{\lim_{n \to \infty}}_{\text{take lim}} \mathbb{E} \sum_{i=1}^{n-1} |W_{t_{i+1}} - W_{t_i}|$

$= \lim_{n \to \infty} \sum_{i=0}^{n-1} \underbrace{\sqrt{\frac{T}{n}}}_{\text{because partition is uniform}} \mathbb{E}|N(0,1)| = \lim_{n \to \infty} \sqrt{nT} \, \mathbb{E}|N(0,1)| \to \infty$

important trick:

$|W_{t_{i+1}} - W_{t_i}| = \frac{|W_{t_{i+1}} - W_{t_i}|}{\sqrt{t_{i+1} - t_i}} \sqrt{t_{i+1} - t_i}$

$\sigma^2 = t_{i+1} - t_i$

normalize RV by $X^* = \frac{(X - \mu)}{\sigma}$

quadratic variation

let X be any process. define

$[X, X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$

quadratic variation simply squares the differences

remark: if X is continuous w/ finite first variation ($V_{[0,T]} X < \infty$)

then $[X,X]_T = 0$ → proved in HW. if $V_{[0,T]}X < \infty$, then $[X,X]_T = 0$

prop. W → std BM

$[W,W]_T = T$ a.s. → quadratic variation of W is $[W,W]_T = T$

proof. will show $\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = T$ in "L^2"

quadratic variation:

$$[X,X]_T = \lim_{\|P\|\to 0} \sum (X_{t_{i+1}} - X_{t_i})^2$$

0 ... T → $\frac{T}{n}$ (uniform partition)

if Z_n is a sequence of RVs

to show that $Z_n \to \mu$

enough to show $\mathbb{E}Z_n \to \mu$ & $Var(Z_n) \to 0$

for a stochastic process:

to show that $Z \to \mu$ (some value)

it is enough to show

1. $\mathbb{E}Z \to \mu$ AND 2. $Var(Z) \to 0$

will show

1. $\lim_{n\to\infty} \mathbb{E} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = T$ ✓
2. $\lim_{n\to\infty} Var\left(\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2\right) = 0$ ✓

} can apply that here

1.

simply the Var of $W_{t_{i+1}} - W_{t_i}$

$$\mathbb{E}\sum (W_{t_{i+1}} - W_{t_i})^2 = \sum \mathbb{E}(W_{t_{i+1}} - W_{t_i})^2 = \sum (t_{i+1} - t_i) = T$$

telescopes to T

by defn

2.

by independence

$$Var\left(\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2\right) = \sum Var(W_{t_{i+1}} - W_{t_i})^2$$

compute

* variance of a sequence of independent RVs is the sum of their individual variances

$$Var(W_{t_{i+1}} - W_{t_i})^2 = \mathbb{E}(W_{t_{i+1}} - W_{t_i})^4 - (\mathbb{E}(W_{t_{i+1}} - W_{t_i})^2)^2$$

$$= (t_{i+1} - t_i)^2 \left(\mathbb{E}\left(\frac{W_{t_{i+1}} - W_{t_i}}{\sqrt{t_{i+1} - t_i}}\right)^4 - \left(\mathbb{E}\left(\frac{W_{t_{i+1}} - W_{t_i}}{\sqrt{t_{i+1} - t_i}}\right)^2\right)^2 \right)$$

recall

$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$

normalize the RVs

$$= (t_{i+1} - t_i)^2 \left[\mathbb{E}N(0,1)^4 - 1\right]$$

$$Var\left(\sum (W_{t_{i+1}} - W_{t_i})^2\right) = \sum (t_{i+1} - t_i)^2 (\mathbb{E}N(0,1)^4 - 1)$$

$$= \sum \left(\frac{T}{n}\right)^2 (\mathbb{E}N(0,1)^4 - 1) = \frac{T^2}{n} \mathbb{E}(N(0,1)^4 - 1) \xrightarrow{n\to\infty} 0$$

trick

$$|W_{t_{i+1}} - W_{t_i}| = \sqrt{t_{i+1}-t_i}\,\frac{|W_{t_{i+1}} - W_{t_i}|}{\sqrt{t_{i+1}-t_i}}$$

$\downarrow$ $\sigma^2 = t_{i+1} - t_i$

$\downarrow$ $\sigma^2 = 1$. normalized by $X^* = \dfrac{(X-\mu)}{\sigma}$

$[W,W]_T = T$

if $V_{[0,T]} X < \infty$, then $[X,X]_T = 0$. pf.

$$[X,X]_T = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(X_{t_{i+1}} - X_{t_i})$$

$$\leq \lim_{\|P\|\to 0} \max_i (|X_{t_{i+1}} - X_{t_i}|) \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|$$

as $\|P\| \to 0$, $t_{i+1} - t_i \to 0$, thus bc. cts, $|X_{t_{i+1}} - X_{t_i}| \to 0$

further $\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}| \to V_{[0,T]}(X)$

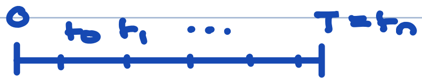
$$\leq (0)(\underbrace{V_{[0,T]}(X)}_{<\infty}) \leq 0$$

lecture 11 notes

prop. $[W,W]_T = T$ almost surely

computed $\lim_{N\to\infty} \sum (W_{t_{i+1}} - W_{t_i})^2$

$= T$ (in L^2)



showed $W_t^2 - t$ is a mg $\rightarrow$ $W_t^2 - [W,W]_t$ is a mg

proposition. $W_t^2 - [W,W]_t$ is a martingale

proof:

know from before, $W_t^2 - t$ is a martingale. (computed $\mathbb{E}_s(W_t^2 - t) = W_s^2 - s$

know $[W,W]_t = t$

$\Rightarrow W_t^2 - [W,W]_t = W_t^2 - t$ which is a martingale

if M is a cts mg: $M_t^2 - [M,M]_t$ is also a cts mg

let M be a continuous martingale

if M is a continuous mg: $\mathbb{E}M_t^2 < \infty \Leftrightarrow \mathbb{E}[M,M]_t < \infty$

(1) $\mathbb{E}M_t^2 < \infty$ iff $\mathbb{E}[M,M]_t < \infty$

(2) in this case, $M_t^2 - [M,M]_t$ is a continuous mg *

(3) conversely, if $M_t^2 - A_t$ is a mg for any continuous, increasing process A such that $A_0 = 0$, then we must have $A_t = [M,M]_t$

if $M_t^2 - A_t$ is a mg for any cts increasing process A with $A_0 = 0$, then $A_t = [M,M]_t$

say $M_t^2 - A_t$ is a mg. the process A is cts, increasing, and $A_0 = 0$.

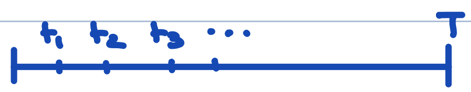
$\mapsto$ both of these conditions $\Rightarrow A_t = [M,M]_t$

if $M_t^2 - A_t$ is a cts mg

$\downarrow$

must have $A_t = [M,M]_t$

intuition why $M^2 - [M,M]$ is a mg



NTS. $\mathbb{E}_s(M_t^2 - [M,M]_t) = M_s^2 - [M,M]_s$ $\rightarrow$ showing mg condition for $M_t^2 - [M,M]_t$

simpler check:

$\mathbb{E}_{t_{n-1}}[M_{t_n}^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2]$ (note $\lim_{\|P\|\to 0} \sum (M_{t_{i+1}} - M_{t_i})^2 = [M,M]_T$)

have $s = t_{n-1}$

show the "discrete version" to imply the cts case

$$\mathbb{E}_{t_{n-1}}\left[M_{t_n}^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}}-M_{t_i})^2\right] = \mathbb{E}_{t_{n-1}} M_{t_n}^2 - \underbrace{\mathbb{E}_{t_{n-1}}(M_{t_n}-M_{t_{n-1}})^2}_{\text{compute}} - \underbrace{\mathbb{E}_{t_{n-1}}\left(\sum_{i=0}^{n-2}(M_{t_{i+1}}-M_{t_i})^2\right)}_{\mathcal{F}_{t_{n-1}}\text{-mb}}$$

$$\mathbb{E}_{t_{n-1}}(M_{t_n}-M_{t_{n-1}})^2 = \mathbb{E}_{t_{n-1}}(M_{t_n}^2 + M_{t_{n-1}}^2 - 2M_{t_n}M_{t_{n-1}})$$

$$= \mathbb{E}_{t_{n-1}} M_{t_n}^2 + M_{t_{n-1}}^2 - 2M_{t_{n-1}}\mathbb{E}_{t_{n-1}} M_{t_n}$$

$$= \mathbb{E}_{t_{n-1}} M_{t_n}^2 + M_{t_{n-1}}^2 - 2M_{t_{n-1}}^2 \longrightarrow M \text{ is mg} : = M_{t_{n-1}}$$

$$= \mathbb{E}_{t_{n-1}} M_{t_n}^2 - M_{t_{n-1}} \checkmark$$

using cond. expec + mg props

$$\mathbb{E}_{t_{n-1}}\left[M_{t_n}^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}}-M_{t_i})^2\right] = \mathbb{E}_{t_{n-1}} M_{t_n}^2 - \mathbb{E}_{t_{n-1}} M_{t_n}^2 + M_{t_{n-1}} - \sum_{i=0}^{n-2}(M_{t_{i+1}}-M_{t_i})^2$$

$$\mathbb{E}_{t_{n-1}}\left[M_{t_n}^2 - \sum_{i=0}^{n-1}(M_{t_{i+1}}-M_{t_i})^2\right] = M_{t_{n-1}} - \sum_{i=0}^{n-2}(M_{t_{i+1}}-M_{t_i})^2$$

shows a "discrete" version of $M^2 - [M,M]$ is a mg! $\Longrightarrow$ shows "DT"ish version. full proof would need to $\lim \|P\| \to 0$

if X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$

$\delta t \to$ small

if X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t})$

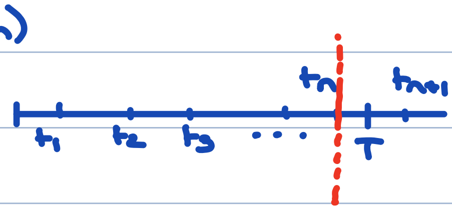
$\|P\| = \max_i\{t_{i+1}-t_i\}$

Ito integrals :

Ito sum : $I_P(T) = \sum_{i=0}^{n-1} D_{t_i}\Delta_i W + D_{t_n}(W_T - W_{t_n})$

- $D_t = D(t)$ some adopted process ✓
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times (trade times)
- $\|P\| = \max_i\{t_{i+1}-t_i\}$ and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$
- $I_p(T) = \sum_{i=0}^{n-1} D_{t_i}\Delta_i W + D_{t_n}(W_T - W_{t_n})$ ($I_p(T)$ = Itô sum)

$$\hookrightarrow I_p(T) = \underbrace{\sum_{i=0}^{n-1} D_{t_i}(W_{t_{i+1}} - W_{t_i})}_{\text{wealth up to } t_n} + D_{t_n}(W_T - W_{t_n})$$



the Ito integral of D with respect to Brownian motion is

$I_T = \int_0^T \Delta_t dW_t = \lim_{\|P\| \to 0} I_P(T)$, then we have $I_T = \underbrace{\lim_{\|P\| \to 0} I_P(T)}_{\text{lim of the Itô sum}}$

difference between Ito and Riemann:

Riemann integral: $\lim_{\|P\| \to 0} \sum \Delta_{t_i}(W_{t_{i+1}} - W_{t_i})$ may not exist because $V_{[0,T]} W = \infty$

Ito integral: $\lim_{\|P\| \to 0} \sum \Delta_{t_i}(W_{t_{i+1}} - W_{t_i})$ exists w.p. 1 $= \int_0^T \Delta_t dW_t$

for riemann integrals: can consider $\lim_{\|P\| \to 0} \sum \Delta_{t_i}(W_{t_{i+1}} - W_{t_i})$ or $\lim_{\|P\| \to 0} \sum \Delta_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})$ or $\lim_{\|P\| \to 0} \sum \Delta_{\frac{t_i + t_{i+1}}{2}}(W_{t_{i+1}} - W_{t_i})$ etc... any $\xi_i \in [t_i, t_{i+1}]$

for ito, NEED $\xi_i = t_i$

riemann integrals: $\lim_{\|P\| \to 0} \sum \Delta_{\xi_i} \Delta_i W$ exists for any $\xi_i \in [t_i, t_{i+1}]$

for ito, must have $\xi_i = t_i$ for the limit to exist

(1) riemann: $\lim_{\|P\| \to 0} \sum \Delta_{\xi_i} \Delta_i W$ exists for any $\xi_i \in [t_i, t_{i+1}]$

(2) ito: NEED $\xi_i = t_i$ for the limit to exist

riemann: $\int_0^T (\ ) \underline{dt}$

ito: $\int_0^T (\ ) \underline{dW_t}$

notation ↓ what are we differentiating wrt

thm: if $\mathbb{E} \underbrace{\int_0^T \Delta_t^2 dt}_{\text{riemann } (dt)} < \infty$ as then

then

(1) $I_T = \lim_{\|P\| \to 0} I_P(T)$ exists w.p. 1, and $\mathbb{E} I(T)^2 < \infty$

(2) the process I_T is a mg. $\mathbb{E}_s I_t = \mathbb{E}_s \int_0^t \Delta_r dW_r = \int_0^s \Delta_r dW_r = I_s$

(3) $[I, I]_T = \int_0^T \Delta_t^2 dt$ a.s.

consider $\mathbb{E} \int_0^T \Delta_t^2 dt$.

if $\mathbb{E} \int_0^T \Delta_t^2 dt$ exists, then:

1. $I_T = \lim_{\|P\| \to 0} I_P(T)$ exists w.p. 1, and $\mathbb{E} I(T)^2 < \infty$
2. I_T is a martingale: $\mathbb{E}_s I_t = \mathbb{E}_s \int_0^t \Delta_r dW_r = \int_0^s \Delta_r dW_r = I_s$
3. $[I, I]_T = \int_0^T \Delta_t^2 dt$ a.s.

$\int_0^T D_t\,dt = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} D_{\varepsilon_i}(t_{i+1} - t_i)$: $\varepsilon_i \in [t_i, t_{i+1}]$

$\int_0^T D_t\,dW_t = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} D_{\varepsilon_i}(t_{i+1} - t_i)$: $\varepsilon_i = t_i$ this MUST be the case

for any adapted process D_t

if $\mathbb{E} \int_0^T D_t^2\,dt < \infty$ (finiteness condition)

1. the Ito integral exists

2. $I_T = \int_0^T D_t\,dW_t$ is a continuous martingale

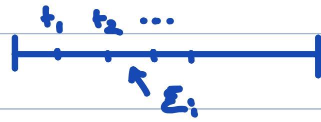
$\longrightarrow$ $\mathbb{E}_s I_t = I_s$. thus $\mathbb{E}_0 \int_0^T D_t\,dW_t = 0$

3. $[I,I]_T = \underbrace{\int_0^T D_t^2\,dt}_{\text{square the process, change } dW_t \text{ to } dt}$

Lecture 12 Notes

Ito integral. $I_T = \int_0^T D_r dW_r = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} D_{t_i}(t_{i+1} - t_i)$ f(t) dt

Ito integrals:



Riemann Stieltjes integral: $\int_0^T D_r dS_r = \lim_{\|P\|\to 0} \sum D_{\varepsilon_i}(S_{t_{i+1}} - S_{t_i})$

ε_i is any point between t_i and t_{i+1} — any point $\varepsilon_i \in [t_i, t_{i+1}]$. exists for differentiable processes

exists as long as S has finite first variation

Ito integral: $\int_0^T D_r dW_r = \lim_{\|P\|\to 0} \sum D_{t_i}(W_{t_{i+1}} - W_{t_i})$ (*) Ito integral must have D be adapted

1. required to have D_{t_i}
2. required to have D to be adapted

with these conditions, the (*) exists for almost every $\omega \in \Omega$ (wp 1)

thm. assume $\mathbb{E}\int_0^T D_r^2 dr < \infty$ and D is an adapted process. then

1. $I_T = \int_0^T D_r dW_r$ (Itô integral of D wrt W) exists wp 1 AND is a continuous martingale

ie. $\mathbb{E}_s I_T = I_s \quad \forall s \le T$ — Ito integral of D wrt W is a continuous mart

ie. $\mathbb{E}_s \int_0^t D_r dW_r = \int_0^s D_r dW_r$

and

2. quadratic variation of $\int_0^T D_s dW_s = \underbrace{\int_0^T D_s^2 ds}_{\text{riemann}}$

ie. if $I_t = \int_0^t D_r dW_r$

then $[I, I]_t = \int_0^t D_r^2 dr$

itô integral * $I_t = \int_0^t D_r dW_r$

intuition for why Ito integral is mg: $\mathbb{E}_s \int_0^t D_r dW_r = \int_0^s D_r dW_r$

take a partition $P = \{0 = t_0 < t_1 < \cdots < t_n\}$. for simplicity $t = t_n$ and $s = t_m$ for some $m \le n$

compute and check that $\mathbb{E}_{t_m}\left[\sum_{k=0}^{n-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})\right] = \sum_{k=0}^{m-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})$ (**)

taking $\lim_{\|P\|\to 0}$ of (**) $\Rightarrow \mathbb{E}_s I_t = I_s \Rightarrow$ Ito integral is mg

proof of (**)

LHS $= \sum_{k=0}^{n-1} E_{t_m} D_{t_k}(W_{t_{k+1}} - W_{t_k}) = E_{t_m}$ when showing Ito integral a mg, split the summations

$= \sum_{k=0}^{m-1} E_{t_m} D_{t_k}(W_{t_{k+1}} - W_{t_k}) + \sum_{k=m}^{n-1} E_{t_m} D_{t_k}(W_{t_{k+1}} - W_{t_k})$

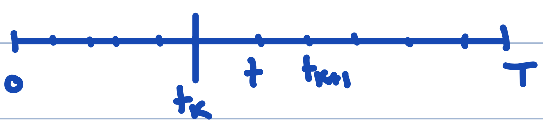
$= \sum_{k=0}^{m-1} D_{t_k}(W_{t_{k+1}} - W_{t_k}) + \sum_{k=m}^{n-1} E_{t_m} D_{t_k} E_{t_k}(W_{t_{k+1}} - W_{t_k})$ ind.

$= \sum_{k=0}^{m-1} D_{t_k}(W_{t_{k+1}} - W_{t_k}) + \sum_{k=m}^{n-1} E_{t_m} D_{t_k} \underbrace{E(W_{t_{k+1}} - W_{t_k})}_{=0}$

$= \sum_{k=0}^{m-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})$

Ito integral → split summation

intuition for $[I,I]_t = \int_0^t D_s^2\,ds$



pick $t \in [0,T]$ find k s.t. $t \in (t_k, t_{k+1})$

$P = \{0 = t_0 < t_1 < \cdots < t_n = T\}$

let $I_P(t) = \sum_{i=0}^{k-1} D_{t_i}(W_{t_{i+1}} - W_{t_i}) + D_{t_k}(W_t - W_{t_k})$

note.

$I_T = \lim_{\|P\| \to 0} I_P(t)$

will compute and check $[I_P, I_P]_t = \sum_{i=0}^{k-1} D_{t_i}^2 (t_{i+1} - t_i) + D_{t_k}^2 (t - t_k)$ (***)

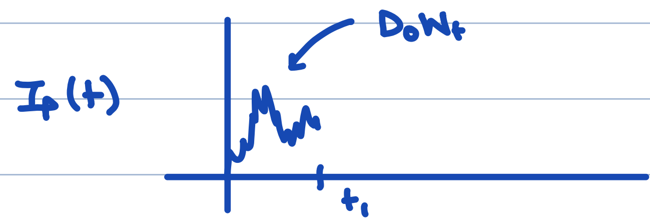
note that (***) $\Rightarrow [I,I]_t = \int_0^t D_s^2\,ds$

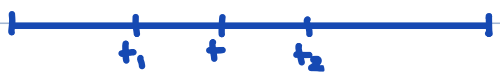
check (***)

suppose $0 \le t < t_1$

$I_P(t) = D_0(W_t - W_0) = D_0 W_t$

$\Rightarrow [I_P, I_P] = D_0^2 t$

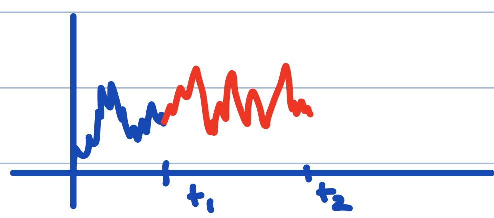




say $t_1 \le t < t_2$

$I_P(t) = D_{t_0}(W_{t_1} - W_{t_0}) + D_{t_1}(W_t - W_{t_1})$

$[I_P, I_P]_t = D_0^2(t_1 - t_0) + D_{t_1}^2 (t - t_1)$



$\Rightarrow$ (***) after repeating

Lecture 13 Notes

Itô integral :

D adapted process, W brownian motion

$I_T = \int_0^T D_t\, dW_t = \underline{\text{Itô integral}} = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})$

Itô integral

Thm : if $\mathbb{E}\int_0^T D_t^2\, dt < \infty$ (and D is adapted of course)

1. Itô integral is a mg : $\mathbb{E}_s \int_0^t D_r\, dW_r = \int_0^s D_r\, dW_r$
2. QV of Itô integral is given by : $[I,I]_T = \int_0^t D_s^2\, ds$

$QV\left(\int_0^t D_s\, dW_s\right) = \int_0^t D_s^2\, ds$

Ito isometry . $\mathbb{E}\left(\int_0^T D_t\, dW_t\right)^2$

$= \mathbb{E}\int_0^T D_t^2\, dt$

cor : Itô isometry

note $\mathbb{E}\int_0^t D_s\, dW_s = mg = \mathbb{E}\int_0^0 D_s\, dW_s = 0$

$\mathbb{E}\left(\int_0^t D_s\, dW_s\right)^2$?

Ito isometry $\longrightarrow$ easy application of $M_t^2 - [M,M]_t$ being a martingale

recall : if M is a mg, then $M^2 - [M,M]$ is a martingale

$\Rightarrow \mathbb{E}(M_t^2 - [M,M]_t) = M_0^2 - [M,M]_0 = M_0^2$

$\longrightarrow \mathbb{E}M_t^2 = M_0^2 + \mathbb{E}[M,M]_t$

so, $\mathbb{E}\left(\int_0^t D_s\, dW_s\right)^2 = 0 + \mathbb{E}\int_0^t D_s^2\, ds$ ✓

alternate proof of Itô isometry

Will show $\mathbb{E}\left(\sum_{k=0}^{n-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})\right)^2 = \mathbb{E}\sum_{k=0}^{n-1} D_{t_k}^2 (t_{k+1} - t_k)$ (*)

$\lim_{\|P\|\to 0}$ of (*) : LHS $\longrightarrow \mathbb{E}\left(\int_0^t D_s\, dW_s\right)^2$, RHS $\longrightarrow \mathbb{E}\int_0^t D_s^2\, ds$

proof of (*)

$\mathbb{E}\left(\sum_{k=0}^{n-1} D_{t_k}(W_{t_{k+1}} - W_{t_k})\right)^2 = \mathbb{E}\sum_{k=0}^{n-1} D_{t_k}^2 (W_{t_{k+1}} - W_{t_k})^2$ ①

alternate proof using summations

$+ 2\mathbb{E}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} D_{t_i}(W_{t_{i+1}} - W_{t_i}) D_{t_j}(W_{t_{j+1}} - W_{t_j})$ ②

1. $\mathbb{E}\sum_{k=0}^{n-1} D_{t_k}^2 (W_{t_{k+1}}-W_{t_k})^2$

$\hookrightarrow \mathbb{E} D_{t_k}^2 (W_{t_{k+1}}-W_{t_k})^2 = \mathbb{E}\,\mathbb{E}_{t_k} D_{t_k}^2 (W_{t_{k+1}}-W_{t_k})^2 = \mathbb{E} D_{t_k}^2\, \mathbb{E}_{t_k} (W_{t_{k+1}}-W_{t_k})^2$

$= \mathbb{E} D_{t_k}^2\, \mathbb{E}(W_{t_{k+1}}-W_{t_k})^2 = \mathbb{E} D_{t_k}^2 (t_{k+1}-t_k)$

you check : $\mathbb{E}$ ② $= 0$

$\Rightarrow$ (*) QED

properties of Ito integrals : similar to riemann

prop.

1. $\int_0^T (X_s+Y_s)\,dW_s = \int_0^T X_s\,dW_s + \int_0^T Y_s\,dW_s$
2. $\int_0^t X_r\,dW_r = \int_0^s X_r\,dW_r + \int_s^t X_r\,dW_r$
3. $\int_0^T \alpha X_s\,dW_s = \alpha \int_0^T X_s\,dW_s$ f. $\alpha \in \mathbb{R}$

riemann integral : $\frac{d}{dt}\int_0^t F_s\,ds = F_t$ Ito integral certainly not differentiable

is $\int_0^t X_s\,dW_s$ diff ? NO : wp 1, not diff anywhere

what is $\int_0^t dW_s = W_t - W_0 = W_t$

$\lim_{\|P\|\to 0} \sum \underbrace{D_{t_k}}_{1}(W_{t_{k+1}}-W_{t_k}) = \lim_{\|P\|\to 0} W_T - W_0$

Q: $F \geq 0 \Rightarrow \int_0^t F_s\,ds \geq 0$

ito. $X \geq 0 \Rightarrow \int_0^t X_s\,dW_s \geq 0$? NO. eg $X=1$, $\int_0^t dW_s = W_t$ not ≥ 0 always

compute ito integral :

$\int_0^t W_s\,dW_s$. need Itô formula

lecture 14 notes

itô's formula: $\int_0^T W_s \, dW_s$??

how would you compute $\int_0^t W_r \, dW_r$

now we know mg can always be expressed in form $dM_t = b_t dt + \sigma_t dW_t$

semimartingales = martingale + process of finite first var

$X_t = \underbrace{B_t}_{\text{bounded var}} + \underbrace{M_t}_{\text{mg}} + X_0$ want $B_0 = M_0 = 0$ and B, M adapted and continuous

semimartingale: marty + process of finite first var

$M_t = \int_0^t \sigma_s \, dW_s$ (itô integral) the ito integral is the mg

$B_t = \int_0^t b_s \, ds$ (riemann integral) riemann integral is the process with finite first variance

$B_t = \int_0^t b_s ds \Rightarrow \frac{d}{dt} B_t = \frac{d}{dt} \int_0^t b_s ds = b_t$

notation: $B_t - B_0 = \int_0^t b_s ds \Longleftrightarrow dB_t = b_t dt$

ito decomp. take process and split into ito component + riemann component

$X_t - X_0 = B_t + M_t$

$dX_t = B_t dt + M_t dW_t$

$M_t = M_t - M_0 = \int_0^t \sigma_s dW_s$. notation $dM_t = \sigma_t dW_t$

$X_t - X_0 = \int_0^t dX_s$ notation: $M_t - M_0 = \int_0^t \sigma_s dW_s$

$\longrightarrow dM_t = \sigma_t dW_t$

notation. $\underbrace{X_t - X_0}_{\int_0^t dX_s} = \int_0^t b_s ds + \int_0^t \sigma_s dW_s$

shorthand: $dX_t = b_t dt + \sigma_s dW_s \Longleftrightarrow X_t - X_0 = \int_0^t b_s ds + \int_0^t \sigma_s dW_s$

notation shorthand: $dX_t = b_t dt + \sigma_s dW_s \Longleftrightarrow X_t - X_0 = \int_0^t b_s ds + \int_0^t \sigma_s dW_s$

suppose $f(t,x)$ is diff and X_t is diff (fn of t)

compute $\frac{d}{dt} f(t, X_t) \overset{\text{chain}}{=} \partial_t f(t,X_t) \frac{d}{dt} t + \partial_x f(t, X_t) \frac{dX_t}{dt}$

$df(t,X_t) = \partial_t f(t,X_t) dt + \partial_x f(t,X_t) dX_t$ (chain rule. holds if f, X are diff)

itos formula: some $C^{1,2}$ function $f(t,x)$

itô's formula:

1. $f = f(t,x)$ is $C^{1,2}$ ($\partial_t f, \partial_x f, \partial_x^2 f$ exist and are continuous)

2. X is a semimg : $dX_t = b_t dt + \sigma_t dW_t$ $\left(X_t - X_0 = \int_0^t b_r dr + \int_0^t \sigma_r dW_r\right)$

then X is a semimg $dX_t = b_t dt + \sigma_t dW_t$

$$df(t,X_t) = \underbrace{\partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t}_{\text{chain rule}} + \underbrace{\frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t}_{\text{Ito correction}}$$

integral version :

$$X_T - X_0 = \int_0^T b_t dt + \int_0^T \sigma_t dW_t$$

$$f(T,X_T) - f(0,X_0) = \int_0^T \partial_t f(t,X_t)dt + \int_0^T \partial_x f(t,X_t)dX_t + \int_0^T \frac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t$$

note :

$$\int_0^T \partial_t f(t,X_t)dt = \int_0^T \partial_x f(t,X_t) b_t dt + \int_0^T \partial_x f(t,X_t)\sigma_t dW_t$$

rule to compute

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

QV = 0 ; $\longrightarrow$ QV $= \int_0^t \sigma_s^2 ds$

$$[X,X]_t = \int_0^t \sigma_s^2 ds$$

$$d[X,X]_t = \sigma_t^2 dt$$

finite first var. contributes nothing to QV

$dX_t = \underbrace{b_t dt} + \sigma_t dW_t$

$d[X,X]_t = \sigma_t^2 dt$

square it, remain

compute QV of $X_t = W_t^2$

idea : write $d(W_t^2) = b_t dt + \sigma_t dW_t$

$d[W^2,W^2]_t = \sigma_t^2 dt$

$X_T - X_0 = \int_0^T b_t dt + \int_0^T \sigma_t dW_t$

$[X,X]_T = \int_0^T \sigma_t^2 dt$

$\longrightarrow d[X,X]_T = \sigma_T^2 dt$

use Itô to write $d(W_t^2) = (\quad)dt + (\quad)dW_t$

choose $f(t,x) = x^2$, $X_t = W_t$

$W_t^2 = f(t,W_t) \Rightarrow d(W_t^2) = df(t,W_t) = $ Ito

$\partial_t f = 0$

$\partial_x f = 2x$ $\Longrightarrow$ $dW_t^2 = \partial_t f dt + \partial_x f dX_t + \frac{1}{2}\partial_x^2 f d[X,X]_t$

$\partial_x^2 f = 2$ $\qquad = 0dt + 2X_t dX_t + d[X,X]_t$

$d[X,X]_t = dt$ $\qquad dW_t^2 = dt + 2W_t dW_t$

$\Longrightarrow d[W^2, W^2]_t = 4W_t^2 dt$

$[W^2, W^2]_T = \int_0^T 4W_t^2 dt$

$dX_t = b_t dt + \sigma_t dW_t$

$X_T = X_0 + \int_0^T b_t dt + \int_0^T \sigma_t dW_t$

standard 1D Ito Formula

Lecture 15 Notes

$$X_t = X_0 + \underbrace{B_t}_{BV} + \underbrace{M_t}_{mg}$$

$$X_t = X_0 + \underbrace{B_t}_{\text{bounded var : riemann}} + \overset{\text{mg : ito}}{\overline{M_t}} \Longrightarrow X_t = X_0 + \underbrace{\int_0^t b_r\,dr}_{\text{riemann}} + \underbrace{\int_0^t \sigma_r\,dW_r}_{\text{ito}}$$

$$X_t = X_0 + \underbrace{\int_0^t b_s\,ds}_{\text{riemann}} + \underbrace{\int_0^t \sigma_s\,dW_s}_{\text{itô}}$$

notation: $dX_t = b_t\,dt + \sigma_t\,dW_t$

differential notation:

$$dX_t = b_t\,dt + \sigma_t\,dW_t$$

f must be $C^{1,2}$

rule #1. **Itô formula**: must have $f(t,x) \in C^{1,2}$: need ∂_t, ∂_x, and ∂_x^2 to exist

$$df(t,X_t) = \underbrace{\partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dx}_{\text{traditional chain rule}} + \underbrace{\tfrac{1}{2}\partial_x^2 f(t,X_t)\,d[X,X]_t}_{\text{ito correction}}$$

rule #2. quadratic variation

must have $f \in C^{1,2}$ to apply ito formula

if $dX_t = b_t\,dt + \sigma_t\,dW_t$, then $d[X,X]_t = \underbrace{\sigma_t^2\,dt}_{\text{square it, riemann}}$

$$df(t,x) = \underbrace{\partial_t f\,dt + \partial_x f\,dX_t}_{\text{chain rule}} + \underbrace{\tfrac{1}{2}\partial_x^2 f\,d[X,X]_t}_{\text{ito correction}}$$

$dX_t = b_t\,dt + \sigma_t\,dW_t$ (plug in for dX_t)

$$\partial_x f(t,X_t)\,dX_t = \partial_x f(t,X_t)\,dt + \partial_x f(t,X_t)\,\sigma_t\,dW_t$$

if: $dX_t = b_t\,dt + \sigma_t\,dW_t$

$\left(X_t = X_0 + \int_0^t b_r\,dr + \int_0^t \sigma_r\,dW_r\right)$

then: $d[X,X]_t = \sigma_t^2\,dt$

$\left([X,X]_t = \int_0^t \sigma_r^2\,dr\right)$

examples.

compute $\mathbb{E}_s W_t^3$ $\longrightarrow$ ito then use properties

option 1. $\mathbb{E}_s W_t^3 = \mathbb{E}_s(W_t - W_s + W_s)^3$ expand ...

option 2. itô formula

compute $dW_t^3 \longrightarrow X_t = W_t,\ f = x^3$

choosing $X_t = W_t$, $f(t,x) = x^3$

$[W,W]_t = t$,

because $X_t = W_t$.

$[W,W]_t = t$

$\partial_t f = 0$, $\partial_x f = 3x^2$, $\partial_x^2 = 6x$, $d[X,X]_t = t$

$$df(t,W_t) = \partial_t f\,dt + \partial_x f\,dX_t + \tfrac{1}{2}\partial_x^2 f\,d[X,X]_t$$

$$= 0\,dt + 3X_t^2\,dX_t + \tfrac{1}{2}6X_t\,dt$$

$$d(W_t^3) = 3W_t^2\,dW_t + 3W_t\,dt$$

now integrate both sides → can then integrate, making the bounds whatever we want

$W_t^3 - W_0^3 = \int_0^t 3W_r^2 dW_r + \int_0^t 3W_r dr$

$\mathbb{E}_s W_t^3 = \mathbb{E}_s \int_0^t 3W_r^2 dW_r + \mathbb{E}_s \int_0^t 3W_r dr$ and solve... but what if we did

instead, integrate from s to t: this will make the (mg) ito integral disappear

$W_t^3 - W_s^3 = \int_s^t 3W_r^2 dW_r + \int_s^t 3W_r dr$

$\mathbb{E}_s W_t^3 = \mathbb{E}_s W_s^3 + \mathbb{E}_s \int_s^t 3W_r^2 dW_r + \mathbb{E}_s \int_s^t 3W_r dr$

$= W_s^3 + \underbrace{\int_s^s 3W_r^2 dW_r}_{0} + \int_s^t 3 \underbrace{\mathbb{E}_s W_r}_{W_s} dr$ pull in the $\mathbb{E}_s$

$\mathbb{E}_s W_t^3 = W_s^3 + \int_s^t 3W_s dr$ → a constant wrt the integral

$+ 3W_s \int_s^t dr = \underline{W_s^3 + 3W_s(t-s)}$

ex2. let $M_t = \int_0^t W_s dW_s$. let $\mathcal{E}_t = \exp(M_t - \int_0^t b_s ds)$

find adapted process b_s such that $\mathcal{E}_t$ is a mg → such that a martingale: ito it, and want the riemann integrals to be 0. (ito integrals are martingales)

strategy: compute $d\mathcal{E}_t = (\text{something})dt + (\text{some else})dW_t$

if could compute $d\mathcal{E}_t$, need ↑ to be $= 0$, because then $\mathcal{E}_t$ is a mg

① compute $d\mathcal{E}$ using itô

strategy: finding adapted process such that something is a martingale

↳ ito, and set the riemann integral terms to be zero

↓ need the riemann terms to go to zero

$X_t = M_t = \int_0^t W_s dW_s$

$d[X,X]_t = W_t^2 dt$ (rule from prev lectures)

$f(t,x) = \exp(x - \int_0^t b_s ds)$ ← square it, make riemann

$\partial_t f = -\exp(x - \int_0^t b_s ds) b_t$

$\partial_x f = \exp(x - \int_0^t b_s ds)$

$\partial_x^2 f = \exp(x - \int_0^t b_s ds)$

ito isometry:

$\mathbb{E}(\int_0^T D_t dW_t)^2 = \mathbb{E}\int_0^T D_t^2 dt$

can be useful when computing expected values of squares

after compute, plug in

$d\mathcal{E}_t = df(t, X_t)$

$= \partial_t f dt + \partial_x f dX_t + \frac{1}{2}\partial_x^2 d[X,X]_t$

$= -b_t \mathcal{E}_t dt + \mathcal{E}_t dX_t + \frac{1}{2}\mathcal{E}_t W_t^2 dt$

$= \underbrace{(\frac{1}{2}W_t^2 - b_t)\mathcal{E}_t dt}_{\text{need to be } 0} + \mathcal{E}_t W_t dW_t \qquad (\mathcal{E}_t dX_t = \mathcal{E}_t W_t dW_t)$

$b_t = \frac{1}{2}W_t^2$ by setting remainder to 0, we get our sufficient condition

$\Rightarrow d\mathcal{E}_t = 0 + \mathcal{E}_t W_t dW_t$

$\Rightarrow \mathcal{E}_t = \int_0^t \mathcal{E}_s W_s dW_s$ which is a martingale

ex3.

find $g(t)$ such that $X_t = g(t)\sin(W_t)$ is a mg

solution. Itô

$dX_t = g'(t)\sin(W_t)dt + g(t)\cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt$

equate the dt terms to be 0

Ito formula:

$$df(t,X_t) = \underbrace{\partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t}_{\text{standard chain rule}} + \overbrace{\frac{1}{2}\partial_x^2 f(t,X_t)d[X,X]_t}^{\text{Ito correction}}$$

lecture 16 notes

intuition behind Ito formula

2. $dW_t = b_t dt + \underline{\sigma_t dW_t}$, $d[W,W]_t = \sigma_t^2 dt$

$X_t = X_0 + B_t + M_t$ → $\sigma_t^2 dt$

M_t → $\int_0^t \sigma_s dW_s$

$X_t = X_0 + B_t + M_t$

M_t: martingale

B_t: finite first variation

recall

$dX_t = b_t dt + \sigma_t dW_t$

Ito decomposition

$df(t,X_t) = \partial_t f dt + \partial_x f dX_t + \frac{1}{2}\partial_x^2 f d[X,X]_t$

compute $\underline{\int_0^t W_s dW_s}$ → chose this function such that $\partial_x = X$.

$\Rightarrow$ gives us $\underline{W_t dW_t}$ when applying Ito

choose $X_t = W_t$, $f(t,x) = \frac{1}{2}x^2$

$\partial_t = 0$, $\partial_x = X$, $\partial_x^2 = 1$, $d[X,X]_t = dt$

$d\frac{1}{2}W_t^2 = 0 + W_t dW_t + \frac{1}{2}\cdot 1 dt$

$\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt$

$\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}t$

$\underline{\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)}$ then can compute

recall the results :

1. W_t is a martingale
2. $W_t^2 - t$ is a martingale
3. $W_t^2 - [W,W]_t$ is a martingale
4. For any mg M_t, $M_t^2 - [M,M]_t$ is a martingale

Levy's. M is cts mg with $M_0 = 0$, $d[M,M]_t = dt \Rightarrow M$ is standard BM

thm (Levy's):

If M is a continuous martingale with $M_0 = 0$ and $d[M,M]_t = dt$

then M is a Brownian motion

ie. $M_t - M_s \sim N(0, t-s)$ and $M \perp\!\!\!\perp \mathcal{F}_s$

fact : Levy's Theorem :

if 1. M is a continuous martingale

2. $M_0 = 0$ and 3. $[M,M]_t = t$

then M is a standard BM

$M_t = \int_0^t \mathrm{sign}(W_s)\, dW_s$

then M is a BM. M is a cts mg, $M_0 = 0$, $d[M,M]_t = dt$

$\mathrm{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

proof of Levy's → use characteristic functions

proof.

observation 1 : if M is a mg and D is anything adapted, then $\int_0^t D_s dM_s$ is a martingale.

check 1. look at $E_n \sum_0^{n-1} D_{t_i}(M_{t_{i+1}} - M_{t_i}) = \sum_0^{n-1} D_{t_i}(M_{t_{i+1}} - M_{t_i})$

if M is mg and D is adapted, $\int_0^t D_s dM_s$ is a mg check 2. if $dM_t = \sigma_t \, dW_t$, $\int_0^t D_s dM_s = \underbrace{\int_0^t D_s \sigma_s dW_s}_{mg}$

FACT: if M is a mg and D is an adapted process, then $\int_0^t D_s dM_s$ is a mg

compute $\mathbb{E} e^{i\lambda M_t}$ (want to get $e^{-\frac{1}{2}\lambda^2 t}$)

let $\varphi_t = \mathbb{E} e^{i\lambda M_t}$

goal: compute $\mathbb{E} e^{i\lambda M_t}$ using itô.

apply ito: $X_t = M_t$, $f(t,x) = e^{i\lambda x}$

$$d(e^{i\lambda M_t}) = 0 dt + i\lambda e^{i\lambda M_t} dM_t - \frac{1}{2}\lambda^2 e^{i\lambda M_t} \underbrace{d[M,M]_t}_{dt} \quad (*)$$

by assumption!

$$\Rightarrow e^{i\lambda M_t} - 1 = i\lambda \int_0^t e^{i\lambda M_s} dM_s - \frac{1}{2}\lambda^2 \int e^{i\lambda M_t} dt \quad : \text{take } \mathbb{E}$$

$$\Rightarrow \varphi_t(t) - 1 = 0 - \frac{1}{2}\lambda^2 \int_0^t \underbrace{\mathbb{E} e^{i\lambda M_s}}_{\varphi(s)} ds$$

note that the dM_s integral is a martingale. thus it goes to zero!

$$\varphi_t(t) - 1 = -\frac{1}{2}\lambda^2 \int_0^t \varphi(s) ds$$

$$\Rightarrow \varphi'(t) = -\frac{\lambda^2}{2}\varphi(t) \rightarrow \varphi(t) = \varphi(0) e^{-\frac{\lambda^2}{2}t} = e^{-\frac{\lambda^2}{2}t}$$

$$\Rightarrow M_t \sim N(0,t)$$

differential equation $\rightarrow$ solves to $\varphi(t) = e^{\frac{-\lambda^2}{2}t}$

to show $M_t - M_s \sim N(0, t-s)$, integrate (*) from s to t and take $\mathbb{E}$

to show independence, integrate (*) from s to t and take conditional expectation

to show independent increments, change bounds of integration

$$e^{i\lambda M_t} - e^{i\lambda M_s} = i\lambda \int_s^t e^{i\lambda M_r} dr - \frac{1}{2}\lambda^2 \int_s^t e^{i\lambda M_r} dr \quad (**)$$

let $\varphi(t) = \mathbb{E}_s e^{i\lambda(M_t - M_s)}$

divide everything by $e^{i\lambda M_s}$

$$** \Longrightarrow \varphi(t) - 1 = i\lambda \mathbb{E}_s\left(e^{-i\lambda M_s} \int_s^t e^{i\lambda M_r} dM_r\right) - \frac{\lambda^2}{2} \mathbb{E}_s\left(e^{-i\lambda M_s} \int_s^t e^{i\lambda M_r} dr\right)$$

1. $i\lambda \mathbb{E}_s\left(e^{-i\lambda M_s} \int_s^t e^{i\lambda M_r} dM_r\right) = i\lambda e^{-i\lambda M_s} \mathbb{E}_s \int_s^t e^{i\lambda M_r} dM_r = 0$

martingale. goes to 0

2. $\mathbb{E}_s\left(e^{-i\lambda M_s} \int_s^t e^{i\lambda M_r} dr\right)$

$$= e^{-i\lambda M_s} \int_s^t \mathbb{E}_s e^{i\lambda M_r} dr = \int_s^t \mathbb{E}_s\left(e^{i\lambda(M_r - M_s)}\right) dr$$

$$= \int_s^t \varphi(r) dr \quad \checkmark \; \varphi(r)$$

$$\Rightarrow \varphi(t) - 1 = -\frac{\lambda^2}{2} \int_s^t \varphi(r) dr \rightarrow \varphi(t) = \mathbb{E}_s e^{i\lambda(M_t - M_s)} = e^{\frac{-\lambda^2(t-s)}{2}} \Longrightarrow M_t - M_s \perp\!\!\!\perp \mathcal{F}_s$$

once again left w/ a differential equation

$$\varphi(t)-1 = \frac{-\lambda^2}{2}\int_s^t \varphi(r)\,dr$$

$$\varphi'(t) = \frac{-\lambda^2}{2}\varphi(t)$$

$$\rightarrow \varphi(t) = \varphi(0)e^{-\frac{\lambda^2}{2}t}$$

$$\downarrow$$

$$\varphi(0) = \mathbb{E}_s e^{-i\lambda M_s}$$

$$= e^{-i\lambda M_s}$$

Lecture 17 notes

computed $\mathbb{E}_s e^{i\lambda(M_t - M_s)} = e^{-\lambda^2/2 (t-s)}$ $(*)$

FACT.
$X \perp\!\!\!\perp Y \iff$ joint CF of X,Y = prod of CF
$\mathbb{E}e^{i(\lambda x + \mu y)} = \mathbb{E}e^{i\lambda x}\mathbb{E}e^{i\lambda Y}$

$M_t - M_s$ is $\perp\!\!\!\perp$ of $\mathcal{F}_s$

$M_t - M_s \sim N(0, t-s)$ ✓

$(*) \Rightarrow \mathbb{E}e^{i\lambda(M_t - M_s)} = e^{-\lambda^2/2 (t-s)}$ = CF of $\underline{N(0,t-s)}$

let X be any $\mathcal{F}_s$ mb var : show $M_t - M_s \perp\!\!\!\perp X$

why does this show independence

say X is $\underline{\mathcal{F}_s}$-mb. aim : to show $M_t - M_s \perp\!\!\!\perp X$

recall :

want to show $M_t - M_s \perp\!\!\!\perp X$ where X is $\mathcal{F}_s$-mb

$X \perp\!\!\!\perp Y \iff$ joint CF of X,Y = prod of CFs

$\hookrightarrow \mathbb{E}e^{i(\lambda X + \mu Y)} = \mathbb{E}e^{i\lambda X}\mathbb{E}e^{i\mu Y}$

FACT :
$X \perp\!\!\!\perp Y \iff$ joint CF of X,Y = prod of CFs
$\hookrightarrow \mathbb{E}e^{i(\lambda X + \mu Y)} = \mathbb{E}e^{i\lambda X}\mathbb{E}e^{i\mu Y}$

in our case ,

$\mathbb{E}e^{i\lambda X + \mu(M_t - M_s)} = \mathbb{E}e^{i\lambda X}e^{\mu(M_t - M_s)}$

$= \mathbb{E}\mathbb{E}_s e^{i\lambda X}e^{i\mu(M_t - M_s)}$ tower rule

$= \underbrace{\mathbb{E}e^{i\lambda X}}\underbrace{\mathbb{E}_s e^{i\mu(M_t - M_s)}} = \underbrace{\mathbb{E}e^{i\lambda X}}_{\text{CF of } X}\underbrace{e^{-\mu^2/2 (t-s)}}_{\text{CF of } M_t - M_s} \Rightarrow M_t - M_s \perp\!\!\!\perp$ of X $\square$

both computed during proof of Levy

product of the two

intuition behind Itô

$df(t, X_t) = \partial_t f dt + \partial_x f dX_t + \frac{1}{2}\partial_x^2 f \, d[X,X]_t$

special case : $X_t = W_t$ and $f(t,x) = f(x)$

itô : $df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$ $(**)$

will show $(**)$ this is what we want to show

$f(W_t) - f(0) = \int_0^T f(W_t)dW_t + \frac{1}{2}\int_0^T f''(W_t)dt$

ito intuition $\longrightarrow$ taylor expand

partition of $[0,T)$ into $\{0 = t_0 < \cdots < t_n = T\}$ ✓

$f(W_T) - f(W_0) = \sum (f(W_{t_{i+1}}) - f(W_{t_i}))$ start with taylor approx

$f(x+h)-f(x) = f'(x)(h) + \frac{1}{2}f''(x)(h)^2 + O(h^3)$

reminder: $f(x+h)-f(x) = f'(x)(h) + \frac{1}{2}f''(x)(h)^2 + O(h^3)$

$\Rightarrow f(W_{t_{i+1}})-f(W_{t_i}) = \underline{f'(W_{t_i})(W_{t_{i+1}}-W_{t_i})} + \frac{1}{2}\underline{f''(W_{t_i})(W_{t_{i+1}}-W_{t_i})^2}$

telescopes to   directly plug into taylor

$f(W_T)-f(0) = \underbrace{\sum f'(W_{t_i})(W_{t_{i+1}}-W_{t_i})}_{\mathrm{I}} + \frac{1}{2}\underbrace{\sum f''(W_{t_i})(W_{t_{i+1}}-W_{t_i})^2}_{\mathrm{II}} + \text{small}$

Ito integral

1. $\lim_{\|P\|\to 0} \mathrm{I} = \int_0^T f'(W_t)\,dW_t$

2. $\lim_{\|P\|\to 0} \mathrm{II} = \lim_{\|P\|\to 0} \sum f''(W_{t_i})(t_{i+1}-t_i) + \lim_{\|P\|\to 0} \overbrace{\sum f''(W_{t_i})\left((W_{t_{i+1}}-W_{t_i})^2-(t_{i+1}-t_i)\right)}^{\mathrm{III}}$

$= \int_0^T f''(W_t)\,dt + \lim_{\|P\|\to 0} \mathrm{III}$

NTS: $\lim_{\|P\|\to 0} \mathrm{III} = 0$ ($\Rightarrow$ done)

proof.

estimate the variance and compute mean

1. $\mathbb{E} \sum f''(W_{t_i})\left((W_{t_{i+1}}-W_{t_i})^2-(t_{i+1}-t_i)\right) = 0$

2. variance

$f'' \le C$ (bounded).

$\mathrm{Var}\left(\sum C\left((W_{t_{i+1}}-W_{t_i})^2-(t_{i+1}-t_i)\right)\right)$

$= C^2\,\mathrm{Var}\left(\sum (W_{t_{i+1}}-W_{t_i})^2\right) = C^2 \sum \mathrm{Var}\left((W_{t_{i+1}}-W_{t_i})^2\right)$

$= C^2 \sum \mathrm{Var}\left(\left(\frac{W_{t_{i+1}}-W_{t_i}}{\sqrt{t_{i+1}-t_i}}\right)^2 (t_{i+1}-t_i)\right)$

$= C^2 \sum \mathrm{Var}\left(N(0,1)^2\right)(t_{i+1}-t_i)^2$

$= C^2\,\mathrm{Var}\left(N(0,1)^2\right) \sum (t_{i+1}-t_i)^2$

$\le C^2\,\mathrm{Var}\left(N(0,1)^2\right) \underbrace{\max_i (t_{i+1}-t_i)}_{\|P\|} \underbrace{\sum (t_{i+1}-t_i)}_{T}$

$\xrightarrow{\|P\| \to 0} 0$

Lecture 18 Notes

Lemma: $dX_t = b_t\,dt + \sigma_t\,dW_t$, and $dX_t = c_t\,dt + \gamma_t\,dW_t$

then $b = c$ and $\sigma = \gamma$

alternatively, $X_t = X_0 + B_t + M_t$ and $X_t = Y_0 + C_t + N_t$

$B_0 = C_0 = 0$, $M_0 = N_0 = 0$, $B, C \rightarrow$ finite first var, $N, M \rightarrow$ martingales

then:

$X_0 = Y_0$, $B_t = C_t$, and $M_t = N_t$

uniqueness of Ito decomposition

if $X_t = X_0 + B_t + M_t$, $X_t = Y_0 + C_t + N_t$

with B, C having finite first variation, $B_0 = C_0 = 0$

and M, N martingales, $M_0 = N_0 = 0$

$\Downarrow$

then $X_0 = Y_0$, $M = N$, and $B = C$

proof: put $t = 0 \Rightarrow X_0 = Y_0$ ✓

$\Rightarrow B_t + M_t = C_t + N_t \iff B_t - C_t = N_t - M_t$

let $Z_t = N_t - M_t$ (must be a martingale) (difference in martingales)

set $Z_t = N_t - M_t = \underbrace{B_t - C_t}_{\text{finite first var}}$

know $Z_t = B_t - C_t$ (must have finite first variation)

$\Rightarrow [Z, Z]_t = 0$ ✓ (finite first var $\Rightarrow$ quadratic var $= 0$)

know $Z_t^2 - [Z,Z]_t = 0 \Rightarrow Z_t^2$ is a martingale $\Rightarrow \mathbb{E}[Z_t^2] = \mathbb{E}[Z_0^2] = 0 \Rightarrow Z_t = 0$ a.s.

$\Rightarrow B = C$ and $M = N$

$\mathbb{E}[Z_t^2]$ where Z_t^2 is nonnegative

$\Rightarrow \mathbb{E}[Z_t^2] = 0$ iff $Z_t = 0$

continuous time markets

① cash: $\partial_t C_t = r C_t$ where r = interest rate

continuously compounded

DT:

fact:

$\mathbb{E}[X^2] = 0 \iff X = 0$

bank $\longrightarrow$ interest rate $= \rho$ after time T

\$ $C_0 \longrightarrow (1+\rho T)C_0$ after time T

(time 0)

simple interest r. Δt small

$C_{n\Delta t}$ is the cash in the bank at time $n\Delta t$

withdraw at time $n\Delta t$ and reinvest: $C_{(n+1)\Delta t} = (1 + r\Delta t)C_{n\Delta t}$

set $t = n\Delta t$, send $\Delta t \to 0$:

$\partial_t C = rC$ and $C_t = C_0 e^{rt}$

bank 2: pays interest r over time periods Δt: \$ $C_{k\Delta t} \longrightarrow \underbrace{(1 + r\Delta t)C_{k\Delta t}}_{\text{at time } k+1}$

(cash at time $k\Delta t$)

$C_t = C_0 e^{rt} \rightarrow \dfrac{\partial C_t}{\partial t} = rC_t$: $dC_t = rC_t\,dt$

start with: \$ C_0 in bank 2 at time 0, after time T: have $(1+r\Delta t)^{\frac{T}{\Delta t}} C_0$

$(1+r\Delta t)^{\frac{T}{\Delta t}} = (1+\rho T)$ (from no arbitrage)

send $\Delta t \to 0$

$\lim_{\Delta t \to 0} (1+r\Delta t)^{\frac{T}{\Delta t}} = e^{rT}$ → cash continuously compounded

$dC_t = rC_t\,dt$

$dD_t = -rD_t\,dt$

continuous time: r → continuously compounded interest rate

C_0 in bank → $C_0 e^{rt}$ at time t

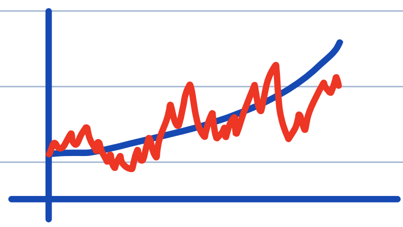
so

equation for cash

C_t = cash in bank at t → $C_t = C_0 e^{rt} \Rightarrow \partial_t C_t = rC_0 e^{rt} = rC_t$: $dC_t = rC_t\,dt$

$dC_t = rC_t\,dt$

now for stock price:



S_t = stock price at time t

$dS_t = \alpha S_t\,dt +$ "noise"

↓ $\sigma\,dW_t$ need to model stock with "noise"

Key formulas:

$dC_t = rC_t\,dt$

$dD_t = -rD_t\,dt$

try #1: $dS_t = \alpha S_t\,dt + \sigma\,dW_t \Longrightarrow$ bad because S_t is not always > 0

try #2: $dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$ → variance of noise should be proportional to S_t

with this def, S_t stays $> 0 \;\forall t$

def: S is a geometric Brownian motion (GBM) with mean return rate α and volatility σ if:

$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$ (α, σ are constants)

geometric brownian motion:

$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$

proposition: mean return, volatility

If S is a GBM(α, σ) then $S_t = S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t}$

$S_t = S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t}$

proof: compute $d(\ln S_t)$ by Itô

$d(\ln S_t) = 0\,dt + \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} d[S,S]_t$

to prove:
compute $d(\ln S_t)$ with Itô

$= \alpha\,dt + \sigma\,dW_t - \frac{1}{2S_t^2}\sigma^2 S_t^2\,dt$

$= \alpha\,dt + \sigma\,dW_t - \frac{1}{2}\sigma^2\,dt$

$$d(\ln S_t) = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \Rightarrow \ln S_t - \ln S_0 = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$d(\ln S_t) = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} d[S,S]_t$$

$\hookrightarrow$ $dS_t = \alpha S_t dt + \underbrace{\sigma S_t}\, dW_t$

$\hookrightarrow$ $d[S,S]_t = \sigma^2 S_t^2 dt$

$$d(\ln S_t) = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} \sigma^2 S_t^2 dt$$

$$d(\ln S_t) = \alpha dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$d(\ln S_t) = \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$$\ln S_t - \ln S_0 = \left(\alpha - \frac{1}{2}\sigma^2\right)\int_0^t dt + \sigma \int_0^t dW_t$$

$$\ln S_t - \ln S_0 = \left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

$$S_t = S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]$$