Final God's Key

I. First Half

We assume that all times are multiples of 6 months and semiannual compounding is employed.

SDP = securities with deterministic payments. payments are non-negative with at least one payment being strictly positive

- discount factors in terms of spot rates and forward rates

$$\frac{1}{d(T)} = \left(1+\frac{\hat{r}(t)}{2}\right)^{2T} = \left(1+\frac{f(0.5)}{2}\right)\left(1+\frac{f(1)}{2}\right)\left(1+\frac{f(1.5)}{2}\right)\cdots\left(1+\frac{f(t)}{2}\right)$$

- to replicate a forward loan, you buy one and sell another (replicate obligations)
- a coupon bond = an annuity + a ZCB
- if $f(T+0.5) > q$ then keeping the face and coupon rate fixed and increasing the maturity from T to T+0.5 will decrease the price of a coupon bond
- big coupon bond = small coupon bond + annuity
- if the spot rate curve is flat at level y, then all securities have YTM = y
- a bond trades at par if and only if $q = y$. a bond trades above par iff $q > y$. a bond trades below par iff $q < y$:

→ to prove, observe that $\frac{\lambda}{1-\lambda} = \frac{2}{y}$ so that $P = F\left(\lambda^{2T} + \frac{q}{y}\frac{\lambda(1-\lambda^{2T})}{1-\lambda}\right)$ becomes $P = F\left[\lambda^{2T} + \frac{q}{y}(1-\lambda^{2T})\right]$. now observe, keeping $y > 0$ fixed we can think of the RHS as a linear function in the variable $x = \frac{q}{y}$. when $x = 1$, this function takes the value F; for $x > 1$ the value is greater than F and for $x < 1$ the value is less than F. □

- for SDPs the YTM is between the smallest and largest spot rates corresponding to payment times
- if you have a long position on two SDPs, the YTM of the portfolio is between the YTMs of the components
- if the spot rate curve is upward sloping then $y_a(t) \le y_{pe}(t) \le \hat{r}(t)$
- $\hat{r}(t) \ge \hat{r}(t-0.5)$ if and only if $f(t) \ge \hat{r}(t)$

→ observe that $f_c(t) = \hat{r}_c(t) + t\hat{r}_c'(t)$ ⇒ directly implies the desired result

- $\hat{r}(t) \geq \hat{r}(t-0.5)$ if and only if $f(t) \geq \hat{r}(t-5)$
- if the spot rate curve is upward sloping then so is the annuity yield curve and the par coupon yield curve
- if the forward rate curve is upward sloping on $[0,t]$ then so is the spot rate curve

→ shown by observing that $y_i = \left(1 + \frac{\hat{r}(i/2)}{2}\right)$ is the geometric mean of $x_i = \left(1 + \frac{f(i/2)}{2}\right)$
this is because we deduce forward rates from spot rates : $y_i = (x_1 \cdot x_2 \cdots x_i)^{\frac{1}{i}}$

- the par coupon yield for maturity T satisfies

$$F = Fd(T) + F\frac{y_{pc}(T)}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right)$$

- assume the spot rate curve is upward sloping on $[0,T]$. for a coupon bond with maturity T, lowering the coupon rate will increase the YTM ; increasing the coupon rate will lower the YTM
- DV01s add : a portfolio's DV01 consists of the sum of its components' DV01s
- the duration of a portfolio is a weighted average of the durations of the pieces
- the convexity of a portfolio is a weighted average of the convexities of the pieces
- for a ZCB with maturity T we have $D_{mac} = T$ and $C \approx T^2$
- for a random payment that is guaranteed to be positive, the DV01 need not be positive
- the price of a floating rate just before a coupon reset equals face value
- at initiation, a plain vanilla receiver swap can be replicated by going long a par coupon bond and short a floating rate bond

→ $X_0 = F\left(d(T) + \frac{q}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right)\right) - F = 0 \Rightarrow q_{swap} = \dfrac{2(1-d(T))}{\sum_{i=1}^{2T} d\left(\frac{i}{2}\right)}$

- the swap rate for a plain vanilla swap is equal to the par coupon rate
- an inverse floater is a bond that pays coupons whose values move in the opposite direction of some benchmark rate : pays coupons $\frac{F}{2}(q^* - r_{t-0.5,t})$ for $t = 0.5, 1, \cdots, T$ and with some fixed $q^* > 0$

- the time-0 of a payment of $\frac{F}{2} r_{t-0.5,t}$ at time t is $F[d(t-.5)-d(t)] = \frac{F}{2} f(t) d(t)$

⟶ use replication. at time 0, we purchase a ZCB with face value F and maturity $t-0.5$, and also short a ZCB with face value F and maturity t. at time $t-0.5$ we invest F until time t at the rate $r_{t-0.5,t}$ and then at time T, we pay F to close out the short bond position:

$$x_0 = F(d(t-0.5)-d(t)) = Fd(t)\left[\frac{d(t-0.5)}{d(t)} - 1\right] = Fd(t)\frac{f(t)}{2}$$

where $f(t) = 2\left[\frac{d(t-0.5)}{d(t)} - 1\right]$ in the last equality

so this must be equal to the time-0 price to receive $\frac{F}{2} r_{t-0.5,t}$ at time t

- holding all other parameters fixed, increasing the coupon will increase the DV01, but decrease the duration and decrease the convexity
- holding all other parameters fixed, increasing the yield will decrease the DV01, duration and convexity
- if $q \geq y$, then holding other parameters fixed, DV01 and duration will increase if the maturity is increased
- if $0 < q < y$, it could theoretically happen that a bond with longer maturity has smaller duration, but this will not occur under typical market conditions
- holding F and y fixed, the DV01 for a ZCB is maximized for $T \approx \frac{1}{y}$

II. Second Half

- perturbed yield curve: $y_{new}(t) = y(t) + (y_1 - y_1^*)Y_1(t) + (y_2 - y_2^*)Y_2(t) + \cdots + (y_4 - y_4^*)Y_4(t)$
- $P = f(y_1, y_2, y_3, y_4) \Rightarrow \Delta P = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4$
- multifactor hedge key idea: match each of the four key rate DV01s of the security with the corresponding DV01 of the hedging portfolio
- reference yield + perturbations $\rightarrow$ new yield curve $\rightarrow$ new discount factors $\rightarrow$ new price
- hedging with PCA: sold security with $P = f(z_1, z_2, z_3)$ and want to hedge sale with portfolio $P^{(h)} = g(z_1, z_2, z_3)$. choose the portfolio such that $\frac{\partial f}{\partial z_i}(0,0,0) = \frac{\partial g}{\partial z_i}(0,0,0)$
- discount process: $$D_m = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{m-1})} \quad \text{with } D_0 = 1 \quad [D_n(1+R_{n-1}) = D_{n-1}]$$

$\hookrightarrow$ note that the discount process is predictable, ie. D_m is $\mathcal{F}_{m-1}$-measurable

- Jensen's Inequality: let $\phi: \mathbb{R} \mapsto \mathbb{R}$ be convex, then $\tilde{\mathbb{E}}_n[\phi(Y)] \geq \phi(\tilde{\mathbb{E}}_n[Y_n])$
- One Step Ahead: if X is time $n+1$ measurable, then
$\tilde{\mathbb{E}}_n[X](\omega_1, \cdots, \omega_n) = \tilde{p}X(\omega_1, \cdots, \omega_n, H) + \tilde{q}X(\omega_1, \cdots, \omega_n, T)$
- discount factors: $d(n) = \tilde{\mathbb{E}}[D_n]$
- risk neutral pricing: for a security that makes a single payment $V_m(\omega_1, \cdots, \omega_m)$
$V_n = \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m V_m] \Rightarrow V_0 = \tilde{\mathbb{E}}[D_m V_m]$

$\hookrightarrow$ once the first n tosses are known we will be at a certain location in the tree - we ignore all of the paths that are now impossible and compute the expectation going forward from that location

- ZCBs: $B_{n,m}$ is the price at time n of a ZCB with maturity m and face \$1
$B_{n,m} = \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m]$; $B_{0,m} = d(m) = \tilde{\mathbb{E}}[D_m]$; $B_{n,n} = 1$; $B_{n,n+1} = \frac{1}{1+R_n}$
- Coupon Bonds: $C^q_{n,m} = B_{n,m} + q\sum_{i=n+1}^{m} B_{n,i}$ (price after coupon paid)
- Annuities: $A_{n,m} = \sum_{i=n+1}^{m} B_{n,i}$
- General security w/ adapted cash flow: time-0 of a security paying C_n at each $n = 1, \cdots, m$
$\tilde{\mathbb{E}}[\sum_{n=1}^{m} D_n C_n] = \sum_{n=1}^{m}\tilde{\mathbb{E}}[D_n C_n]$
- Forward Rates: $F_{n,m}$ interest rate agreed upon at time n for loan initiated at time m and

settled at time m+1. replicate:

$B_{n,m} - B_{n,m+1}(1+F_{n,m}) = 0 \implies F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1$

- float note: pays R_{n-1} at each of times $n = 1, \cdots, m$ and $F = \$1$ at maturity

first price floating payment

$\longrightarrow \tilde{E}[D_n R_{n-1}] = B_{0,n-1} - B_{0,n}$ (similar to $F\frac{r_{t-0.5,t}}{2} \rightarrow F(d(t-0.5) - d(t))$)

now the float note:

$\tilde{E}[D_m + \sum_{n=1}^{m} D_n R_{n-1}] = B_{0,m} + \sum_{n=1}^{m}(B_{0,n-1} - B_{0,n}) = B_{0,0} = 1$

- m-period interest rate swap with fixed rate K is a contract that pays $K - R_{n-1}$ at each of the times $n = 1, \cdots, m$. the m period swap rate $SR_{0,m}$ is the value of K that makes the time 0 price of this contract equal to 0
- m-period interest rate cap with fixed rate K is a contract that pays $(R_{n-1} - K)^+$ at each of the times $n = 1, \cdots, m$
- m-period interest rate floor with fixed rate K is a contract that pays $(K - R_{n-1})^+$

time-0 prices:

$\text{Swap}^K_{0,m} = \tilde{E}[\sum_{i=1}^{m} D_i(K - R_{i-1})] = C^K_{0,m} - 1$

$\text{Cap}^K_{0,m} = \tilde{E}[\sum_{i=1}^{m} D_i(R_{i-1} - K)^+]$

$\text{Floor}^K_{0,m} = \tilde{E}[\sum_{i=1}^{m} D_i(K - R_{i-1})^+]$

note: $\text{Swap}^K_{0,m} + \text{Cap}^K_{0,m} = \text{Floor}^K_{0,m}$

- Caplets/Floorlets: a single payment of $(R_{K-1} - K)^+$ at time K is called an interest rate caplet. a single payment of $(K - R_{K-1})^+$ at time K is called an interest rate floorlet
- More swaps:

$\text{Swap}^K_{0,m} = (K\sum_{n=1}^{m} B_{0,n}) + B_{0,m} - 1$

$\longrightarrow$ setting this equal to zero and solving for K: $SR_{0,m} = \frac{1 - B_{0,m}}{\sum_{i=1}^{m} B_{0,i}}$

- Discounted price: with no intermediate payments, $(D_n P_n)$ is a martingale

$\tilde{E}_n[D_m P_m] = D_n P_n$

- Backward induction: $V_n = \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}]$

where we have $\tilde{E}_n[V_{n+1}](\omega) = \tilde{p}V_{n+1}(\omega, H) + \tilde{q}V_{n+1}(\omega, T)$

- **Securities with multiple payments**: security that pays $A_n(\omega_1,\dots,\omega_n)$ at each time n. let V_n be the time-n price of the security after the payment A_n has been made:
$V_m = 0$; $V_n = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[A_{n+1}+V_{n+1}]$
- **Simplified Backward Induction**: security making single payment V_m at time m where we have
$V_m = v_m(\#H_m(\omega)) \Longrightarrow v_n(k) = \frac{1}{1+r_n(k)}[\tilde{p}v_{n+1}(k+1) + \tilde{q}v_{n+1}(k)]$
⟶ for securities making multiple payments A_n with $A_n = a_n(\#H_n(\omega))$ or $a_n(\#H_{n-1}(\omega))$
then $v_n(k) = \frac{a_{n+1}(k)}{1+r_n(k)} + \frac{1}{1+r_n(k)}[\tilde{p}v_{n+1}(k+1) + \tilde{q}v_{n+1}(k)]$
- **American Options**: $V_m = G_m$, $V_n = \max\{G_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]\}$
- **Bermudan Options**: have a set of possible exercise dates
if n is not an exercise date: $V_n = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]$
if n is an exercise date: $V_n = \max\{G_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]\}$
- **Callable Bond**: a bond is callable with call dates $\mathcal{E}$ and call prices $(F_n)_{n\in\mathcal{E}}$ provided that at each time $n\in\mathcal{E}$, the issuer has the right to pay the bond holder $F_q + F_n$ and is relieved of the obligation to make any future payments ($p^{callable} = p^{bond} - p^{call}$)
- **Putable Bond**: the bond is putable if the holder can sell the bond back to the issuer at the price F_n after the coupon has been paid (ie. the holder receives the amount $F_q + F_n$ at time n and no further payments are made) ($p^{putable} = p^{bond} + p^{put}$)
⟶ Pricing algorithm:
if $n\notin\mathcal{E}$: $V_n^c = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^c + F_q]$; $V_n^p = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^p + F_q]$
if $n\in\mathcal{E}$: $V_n^c = \min\{F_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^c + F_q]\}$; $V_n^p = \max\{F_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^p + F_q]\}$
- **Early exercise**: if interest rates are positive, and $G_n = \phi(P_n)$, $n=1,\dots,m$ for some nonnegative convex function, and (D_mP_m) is a martingale, then it is never beneficial to exercise V prior to m
⟶ similarly for concave functions, it is beneficial to exercise V prior to m
- **Forward Contract**: agreement between two parties concerning the sale of an asset at a future date m. the party taking the long position agrees to buy the asset at price K at time m. define

$For_{n,m}$ to be the value of K that makes the price of both positions 0 at time n

$\hookrightarrow For_{n,m} = \dfrac{\tilde{\mathbb{E}}_n[D_m P_m]}{\tilde{\mathbb{E}}_n[D_m]}$; if no intermediate pymts $\Rightarrow For_{n,m} = \dfrac{D_n P_n}{D_n B_{n,m}} = \dfrac{P_n}{B_{n,m}}$

- Futures : an m-futures process is an adapted process $(Fut_{n,m})$ such that

i. $Fut_{m,m} = P_m$ ii. $\tilde{\mathbb{E}}_n[\sum_{i=n}^{m-1} D_{i+1}(Fut_{i+1,m} - Fut_{i,m})] = 0$

$\hookrightarrow$ an investor with a long position receives $Fut_{k+1,m} - Fut_{k,m}$ on $k+1$ for each $k = 0, \cdots, m-1$

$\Rightarrow Fut_{n,m} = \tilde{\mathbb{E}}_n[P_m]$. note that $\mathbb{E}[Fut_{n,m}] = \mathbb{E}\tilde{\mathbb{E}}_n P_m = \mathbb{E}[P_m] = Fut_{0,m}$ (mg!)

- Forwards vs. Futures (under $\tilde{\mathbb{P}}$) :

(a) if D_m and P_m are uncorrelated then $For_{0,m} = Fut_{0,m}$

(b) if D_m and P_m are positively correlated then $For_{0,m} > Fut_{0,m}$

(c) if D_m and P_m are negatively correlated then $Fut_{0,m} > For_{0,m}$

- Fixed rate mortgage with level payments : individual borrows an amount P and agrees to pay the bank the same amount A every month for the next T years

$\hookrightarrow \lambda = \dfrac{1}{1 + y/12}$; $P = A \sum_{i=1}^{12T} \lambda^i = A \dfrac{\lambda}{1-\lambda}(1 - \lambda^{12T})$

- denote B_n as the outstanding principal balance after the nth payment has been made. the interest component of the $n+1^{st}$ payment is : $B_n \dfrac{y}{12}$

the principal component of this payment is $A - B_n \dfrac{y}{12}$

and thus the outstanding principal balance satisfies the following : $B_{n+1} = B_n\left(1 + \dfrac{y}{12}\right) - A$

- Prepayments : the monthly prepayment rate, $SMM = 1 - (1 - CPR)^{\frac{1}{12}}$ where CPR is the annual prepayment rate

$\hookrightarrow$ apply this rate to decrease the outstanding principal, and reducing the total monthly payment

- Receiver Swap : with fixed rate K initiated at n and having maturity date m is an agreement to receive the fixed amount K in exchange for a variable amount $R_n, R_{n+1}, \cdots$. we have

$Swap_{n,m}^K = \dfrac{1}{D_n} \tilde{\mathbb{E}}_n[\sum_{i=n+1}^{m} D_i (K - R_{i-1})]$

$\hookrightarrow Swap_{n,m}^K = K A_{n,m} - (1 - B_{n,m})$; $SR_{n,m} = \dfrac{1 - B_{n,m}}{A_{n,m}}$

- A receiver swaption is an option to enter a receiver swap at a future date. If the option is exercised at n, then the exchange of payments starts at time $n+1$; the value at exercise is $(Swap^{K}_{n,m})^{+}$

- A payer swap is a swap in which the holder receives floating R_{i-1} and pays fixed K. a payer swaption is an option to enter a payer swap
- the time 0 price of a European receiver swaption with exercise date and a termination date m for the swap is the same as the time 0 price of the security V that makes a single payment of $V_n = (K - SR_{n,m})^{+} A_{n,m}$
- A swap is said to be callable if the payer of fixed has the right to cancel

$\Rightarrow$ callable swap = receiver swap − receiver swaption

- A swap is putable if the receiver of fixed has the right to cancel

$\Rightarrow$ putable swap = receiver swap + payer swaption

- Construct Spot Rate Curve : suppose we have N securities we want to use : $V^{(i)}$ for $i = 1, \cdots, N$, where T_i is the largest time V^i makes a payment and $T^1 < T^2 < \cdots < T^N$

the time 0 price of security V^i is given by $P^i = \sum_{j=1}^{2T_N} F^i_j d(\frac{j}{2})$, $i = 1, \cdots, N$

$\longrightarrow$ where F^i_j is the payment made by the ith security at time $j/2$

- Piecewise constant forwards : assume there are constants α_i , $i = 1, \cdots, N$ so that $f_c(t) = \alpha_i$, $T_{i-1} < t < T_i$

now all of the discount factors $d(t)$, $0 \le t \le T_N$ can be expressed in terms of α_i. we solve for the α_i and use these to compute the discount factors

- ex.

consider interest rate swaps that exchange payments twice a year, suppose $q^{swap}(1) = 0.02$ and $q^{swap}(2) = 0.04$. this tells us the following (can infer about par coupon bonds)

(S1) $100 = d(0.5) + 101 d(1)$

(S2) $100 = 2d(0.5) + 2d(1) + 2d(1.5) + 102 d(2)$

we write

$$f_c(t) = \begin{cases} \alpha_1 , & 0 < t < 1 \\ \alpha_2 , & 1 < t < 2 \end{cases} \longrightarrow \text{(\# of equations)}$$

recall that $d(t) = \exp(-t\hat{r}_c(t)) \implies d(t) = \exp(-\int_0^t f_c(s)\,ds)$

this brings us to:

$$d(t) = \begin{cases} \exp(-\alpha_1 t) & , 0 \leq t \leq 1 \\ \exp(-\alpha_1 - \alpha_2(t-1)) & , 1 < t \leq 2 \end{cases}$$

we can the relate this back to (S1) and (S2) as follows:

$$100 = e^{-0.5\alpha_1} + 100e^{-\alpha_1}$$

$$100 = 2e^{-0.5\alpha_1} + 2e^{-\alpha_1} + 2e^{-\alpha_1 - 0.5\alpha_2} + 102e^{-\alpha_1 - \alpha_2}$$

solve and arrive at: $\alpha_1 = 0.0199066$ and $\alpha_2 = 0.060112$

$\longrightarrow$ then plug back in and find the $d(t)$'s and use these to find $\hat{r}(t)$'s