Lecture 20 Notes

market $\begin{cases} \text{bank interest rate } r : C_t = C_0 e^{rt} \\ \text{stock} : GBM(\alpha, \sigma) \end{cases}$

$dS_t = \alpha S_t dt + \sigma S_t dW_t$

$S_t = S_0 \exp((\alpha - \frac{\sigma^2}{2})t + \sigma W_t)$

$\frac{d}{dt} C_t = r C_t$

market → bank : $C_t = C_0 e^{rt}$ → $dC_t = r C_t dt$

market → $dS_t = \alpha S_t dt + \sigma S_t dW_t$

$S_t = S_0 \exp[(\alpha - \frac{\sigma^2}{2})t + \sigma W_t]$

this is all we need...

consider security that pays $g(S_T) = V_T$ at time T

Thm 1 : assume security is replicable and X_t = wealth of replicating portfolio at time $t = f(t, S_t)$

then f solves the black scholes PDE

assume replicable and wealth measured by $f(t, S_t)$

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what does this mean?

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f satisfies BS PDE

1. $\partial_t f + rx \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = rf$ for $x > 0, t < T$
2. $f(T, x) = g(x)$ (terminal wealth = payoff) $f(T,x) = g(x)$
3. $f(t, 0) = e^{-r\tau} g(0)$ where $\tau = T - t$
4. BV at ∞ $\quad \partial_t f + rx \partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f - rf = 0$

(note wealth of rep portfolio at time $T = X_T = f(T, S_T) = g(S_T)$ ⟹ replicates payoff

if $S_t = 0$ then $S_T = 0$ ⟹ security pays $g(0)$ at time T. repl by putting $e^{-r(T-t)} g(0)$ \$ in bank at time t

stock hits 0, then always 0

discount back

Thm 2. if f solves the BSPDE, then the security is replicable, X_t = wealth of replicating portfolio = AFP = $f(t, S_t)$

next thm : if f satisfies, then the security is repl. and $f(t, S_t)$ is the AFP

claim : $f(t,x) = e^{-r\tau} \int g(e^{(r - \frac{\sigma^2}{2})y + \sigma\sqrt{\tau} y}) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$ with $\tau = T - t$

solves the BSPDE

plug in $dS_t = \alpha S_t dt + \sigma S_t dW_t$

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derive 1 $\quad$ start w/ self financing condition : $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt$

X_t is wealth of self financing portfolio at t then : $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt$

Step 1 : know $X_t = f(t, S_t)$ = wealth of self financing portfolio

⟹ $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt = \Delta_t (\alpha S_t dt + \sigma S_t dW_t) + r(X_t - \Delta_t S_t) dt$

$$dX_t = [(\alpha - r)\Delta_t S_t + rX_t]\,dt + \sigma \Delta_t S_t\, dW_t \quad (*)$$ (*) via self finance + definitions

Step 2: Itô $df(t, S_t)$ next, we Ito $df(t, S_t)$ which will give another expression for dX_t

$$dX_t = df(t, S_t) = \partial_t f\, dt + \partial_x f\, dS_t + \tfrac{1}{2}\partial_x^2 f\, d[S,S]_t$$
$$= \partial_t f\, dt + \partial_x f(\alpha S_t\, dt + \sigma S_t\, dW_t) + \tfrac{1}{2}\partial_x^2 f \sigma^2 S_t^2\, dt$$

$$dX_t = (\partial_t f + \alpha \partial_x f S_t + \tfrac{1}{2}\sigma^2 S_t^2 \partial_x^2 f)\,dt + \sigma \partial_x f S_t\, dW_t \quad (**)$$

Step 3: equate dW_t terms in * and ** then we can equate the dt and dW_t terms

$$\Rightarrow \sigma \Delta_t S_t = \sigma \partial_x f S_t \Rightarrow \Delta_t = \partial_x f(t, S_t)$$ delta hedging rule

Step 4: equate dt terms in * and **

$$(\alpha - r)\Delta_t S_t + rX_t = \partial_t f + \alpha \partial_x f S_t + \tfrac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$$

$$\Rightarrow (\cancel{\alpha} - r)\partial_x f S_t + rf = \partial_t f + \cancel{\alpha \partial_x f S_t} + \tfrac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$$

$rf = \partial_t f + r\partial_x f S_t + \tfrac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$. replace S_t with x $\Rightarrow$ formula 1

then we arrive at BS PDE

prove theorem 2:

1. choose $\Delta_t = \partial_x f(t, S_t)$
2. choose $X_0 = f(0, S_0)$

will show that $X_t = f(t, S_t)\ \forall t \in (0,T)$

how to approach thm 2?

1. choose $\Delta_t = \partial_x f(t, S_t)$
2. choose $X_0 = f(0, S_0)$

WTS $X_t = f(t, S_t)\ \forall t \in (0,T)$

$$\Rightarrow X_T = \lim_{t \to T} X_t = \lim_{t \to T} f(t, S_t) = \underbrace{f(T, S_T)}_{= g(T)}$$

(limits imply $X_T = f(T, S_T) = g(T)$)

$\Rightarrow$ wealth at time T = payoff of security $\Rightarrow$ security is replicable and the wealth of the replicating portfolio = AFP = X_t