Lecture 19 Notes

continuous time market

$r \to$ interest rate (continuously compounded)

$C_t = C_0 e^{rt}$, simple diffeq

cash in bank: $C_t = C_0 e^{rt} \iff dC_t = rC_t\,dt$

$\frac{d}{dt} C_t = rC_t \implies \underline{\underline{dC_t = rC_t\,dt}}$

discount factor: $D_t = e^{-rt}$ (D_t \$ in bank at time 0 $\to$ \$1 at time t)

$C_t = C_0 e^{rt}$

$dD_t = -rD_t\,dt$

discount factor: $D_t = e^{-rt}$

if you have $D_t = e^{-rt}$ dollars in bank at $t=0$

$\mapsto D_t e^{rt} = e^{-rt}e^{rt} = 1$ dollar at time t

stock price: geometric brownian motion

GBM(α, σ)

$\alpha \to$ mean return, $\sigma \to$ volatility

$S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t$

market: 1 stock $\to$ GBM(α,σ) ; bank/MM $\to$ interest rate r

1. assets are liquid   liquid assets
2. borrowing rate = lending rate = r borrowing rate = lending rate
3. no transaction costs   no transaction costs

consider a security that <u>matures at time T</u> and pays $V_T = g(S_T)$ ($g \to$ some nonrandom fn)

examples:

$V_T = g(S_T)$ payoff is a deterministic fn of the terminal stock price

european call: $\underline{g(x) = (x-K)^+}$ (strike K), european put: $g(x) = (K-x)^+$

ex. call, put, etc.

BS formula:

AFP of security at time t

$= V_t = f(t, S_t)$ and $f(t,x) = e^{-r\tau}\int_{-\infty}^{\infty} g\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\tau y}\right)\frac{e^{-y^2/2\tau}}{\sqrt{2\pi\tau}}\,dy$

black scholes pde: gives the AFP of security V which pays $g(x)$ at T

thm 1: if the security is <u>replicable</u> and the wealth of the replicating portfolio is of the form:

$f(t,S_t)$, then f satisfies the Black Scholes PDE:

1. $\partial_t f + rx\,\partial_x f + \frac{\sigma^2}{2}\partial_x^2 f = rf$ for $x>0,\ t<T$
2. $f(T,x) = g(x)$ (terminal wealth = payoff)
3. $f(t,0) = e^{-r\tau}g(0)$ where $\tau = T-t$
4. BV at ∞

If the security is repl. and the wealth of the repl portfolio is $f(t,S_t)$. then f satisfies the BS PDE

boundary values: $f(T,x) = g(x)$, $f(t,0) = e^{-r(T-t)}g(0)$

$$\partial_t f + rx\,\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0$$

Theorem 2: suppose the BS PDE ①,②,③ have a $C^{1,2}$ solution $f(t,x)$, then the security is replicable and wealth of the replicating portfolio is $V_t = f(t,S_t)$

TODO: pf of thm 1, pf of thm 2, check that ④ satisfies BS PDE (skip)

replication: need a self financing portfolio that has the same payoff as the security

self financing:

replication: self financing, same payoff as security

portfolio that only uses tradeable assets and has no external cash flow

$\longrightarrow$ X_t is the wealth of a self financing portfolio if $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$

portfolio $\begin{cases} \to \Delta_t \text{ shares of stock} \\ \to \text{rest in cash} \end{cases}$

self financing condition:

$X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$

$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$

proof of thm 1: assume X_t = wealth of rep port $= f(t,S_t)$

$X \to$ self financing $\Longrightarrow dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$ ($dS_t = \alpha S_t dt + \sigma S_t dW_t$)

assume replicable, with wealth $f(t,S_t) = X_t$

$dX_t = (\Delta_t \alpha S_t - r\Delta_t S_t + rX_t)dt + \Delta_t \sigma S_t dW_t$

$= ((\alpha - r)\Delta_t S_t + rX_t)dt + r\Delta_t S_t dW_t$

$dX_t = ((\alpha - r)\Delta_t S_t + rX_t)dt + r\Delta_t S_t dW_t$

condition on wealth

always use GBM(α,σ):

$dS_t = \alpha S_t dt + \sigma S_t dW_t$

self financing condition:

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$$

Lecture 20 Notes

market $\begin{cases} \text{bank interest rate } r : C_t = C_0 e^{rt} \\ \text{stock} : GBM(\alpha, \sigma) \end{cases}$

$\frac{d}{dt} C_t = r C_t$

$dC_t = r C_t \, dt$

market: bank : $C_t = C_0 e^{rt}$

market: $dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$

$dS_t = \alpha S_t dt + \sigma S_t dW_t$

$S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

$S_t = S_0 \exp\left[\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$

this is all we need...

consider security that pays $g(S_T) = V_T$ at time T

Thm 1 : assume security is replicable and X_t = wealth of replicating portfolio at time $t = f(t, S_t)$

then f solves the black scholes PDE

assume replicable and wealth measured by $f(t, S_t)$

↓

what does this mean?

↓

f satisfies BS PDE

1. $\partial_t f + rx \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = rf$ for $x > 0, t < T$
2. $f(T, x) = g(x)$ (terminal wealth = payoff) $f(T,x) = g(x)$
3. $f(t, 0) = e^{-r\tau} g(0)$ where $\tau = T - t$
4. BV at ∞ $\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2} \partial_x^2 f - rf = 0$

(note wealth of rep portfolio at time $T = X_T = f(T, S_T) = g(S_T)$ ⟹ replicates payoff

if $\underline{S_t = 0 \text{ then } S_T = 0}$ ⟹ security pays $g(0)$ at time T. repl by putting $e^{-r(T-t)} g(0)$ \$ in bank at time t

stock hits 0, then always 0

discount back

Thm 2. if f solves the BSPDE, then the security is replicable, X_t = wealth of replicating portfolio = AFP = $f(t, S_t)$

next thm: if f satisfies, then the security is repl. and $f(t, S_t)$ is the AFP

claim: $f(t, x) = e^{-r\tau} \int g\left(e^{\left(r - \frac{\sigma^2}{2}\right)y + \sigma\sqrt{\tau} y}\right) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$ with $\tau = T - t$

solves the BSPDE

plug in $dS_t = \alpha S_t dt + \sigma S_t dW_t$

derive 1 start w/ self financing condition : $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt$

X_t is wealth of self financing portfolio at t then : $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt$

Step 1 : know $X_t = f(t, S_t)$ = wealth of self financing portfolio

⟹ $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t) dt = \Delta_t (\alpha S_t dt + \sigma S_t dW_t) + r(X_t - \Delta_t S_t) dt$

$dX_t = [(\alpha - r)\Delta_t S_t + rX_t]\,dt + \sigma \Delta_t S_t\, dW_t$ (*) (*) via self finance + definitions

Step 2: Itô $df(t, S_t)$ next, we Ito $df(t,S_t)$ which will give another expression for dX_t

$dX_t = df(t, S_t) = \partial_t f\,dt + \partial_x f\,dS_t + \frac{1}{2}\partial_x^2 f\,d[S,S]_t$

$= \partial_t f\,dt + \partial_x f(\alpha S_t\,dt + \sigma S_t\,dW_t) + \frac{1}{2}\partial_x^2 f \sigma^2 S_t^2\,dt$

$dX_t = (\partial_t f + \alpha \partial_x f S_t + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 f)\,dt + \sigma \partial_x f S_t\,dW_t$ (**)

Step 3: equate dW_t terms in * and ** then we can equate the dt and dW_t terms

$\Rightarrow \sigma \Delta_t S_t = \sigma \partial_x f S_t \Rightarrow \Delta_t = \partial_x f(t, S_t)$ delta hedging rule

Step 4: equate dt terms in * and **

$(\alpha - r)\Delta_t S_t + rX_t = \partial_t f + \alpha \partial_x f S_t + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$

$\Rightarrow (\cancel{\alpha} - r)\partial_x f S_t + rf = \partial_t f + \cancel{\alpha \partial_x f} S_t + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$

$rf = \partial_t f + r\partial_x f S_t + \frac{1}{2}\sigma^2 S_t^2 \partial_x^2 f$. replace S_t with x $\Rightarrow$ formula ①

then we arrive at BS PDE

prove theorem 2:

1. choose $\Delta_t = \partial_x f(t, S_t)$
2. choose $X_0 = f(0, S_0)$

will show that $X_t = f(t, S_t)\ \forall t \in (0,T)$

$\Rightarrow X_T = \lim_{t\to T} X_t = \lim_{t \to T} f(t, S_t) = \underbrace{f(T, S_T)}_{= g(T)}$

(limits imply $X_T = f(T, S_T) = g(T)$)

$\Rightarrow$ wealth at time T = payoff of security $\Rightarrow$ security is replicable and the wealth of the replicating portfolio = AFP = X_t

how to approach thm 2?

1. choose $\Delta_t = \partial_x f(t, S_t)$
2. choose $X_0 = f(0, S_0)$

WTS $X_t = f(t, S_t)\ \forall t \in (0,T)$

Lecture 22 Notes

$$N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

$$N(-x) = 1 - N(x)$$

AFP at time $t = f(t, S_t)$

$$f(t,x) = e^{-r\tau} \int_0^\infty g\left(xe^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}, \quad \tau = T-t$$

"black scholes formula" $\longrightarrow$ explicit BS PDE solution:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\left(x\exp\left((r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y\right)\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

can use this to price securities: simply use whatever payoff $g(x)$ you want

use * to price a european call with strike K and maturity T

payoff: $(S_T - K)^+$

choose $\underline{g(x) = (x-K)^+}$

$$c(t,x) = f(t,x) = e^{-r\tau} \int_0^\infty \left(xe^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y} - K\right)^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$= \int_{y=(\ )}^{\infty} \cdots$$

change bounds, suffer

explicit solution, plug in payoff

simplified final answer

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)$$

arrive at result for call option

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln(x/K) + (r \pm \frac{\sigma^2}{2})\tau\right), \quad \tau = T-t$$

$$N(x) = \text{CDF of } N(0,1) = \mathbb{P}(N(0,1) \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

FACTS:

1. $N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$
2. $1 - N(d) = N(-d)$
3. $-\ln(a/b) = \ln(b/a)$

AFP of a eur put strike K and mat T

$p(t,x) =$ rather than recompute, use put call parity

put call parity: buy 1 call and short 1 put

payoff time T: $(S_T - K)^+ - (K - S_T)^+ = S_T - K$ = payoff of F.C.

replicate FC: buy 1 share, borrow Ke^{-rT} at time 0

$\Rightarrow$ AFP of FC at time t is $S_t - Ke^{-rT}e^{rt} = S_t - Ke^{-r(T-t)}$ = AFP 1 call − AFP 1 put

$= c(t, S_t) - p(t, S_t)$

$$\Rightarrow p(t,x) = c(t,x) - x + Ke^{-r\tau}$$

buy 1 call, short 1 put

$T: (S_T - K)^+ - (K - S_T)^+ = S_T - K$

buy 1 share, borrow Ke^{-rT}

$S_t - Ke^{-r(T-t)} = c(t, S_t) - p(t, S_t)$

$$\longrightarrow p(t,x) = c(t,x) - x + Ke^{-r\tau}$$

"greeks"

put call parity: $c(t,x) - p(t,x) = x - Ke^{-r(T-t)}$

delta : $\partial_x c$, gamma : $\partial_x^2 c$, theta : $\partial_t c$

can compute the partial derivatives, ie. "the greeks"

$$\partial_x c(t,x) = N(d_+) + \underbrace{x N'(d_+)\partial_x d_+ - K e^{-r\tau} N'(d_-)\partial_x d_-}_{\text{goes to } 0}$$

$$= N(d_+)$$

compute partials

$$\partial_x^2 c(t,x) = N'(d_+)\partial_x d_+ = \frac{e^{-d_+^2/2}}{\sqrt{2\pi}}\left(\frac{1}{\sigma\sqrt{\tau}}\frac{1}{x}\right)$$

$$\partial_t c(t,x) = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)$$

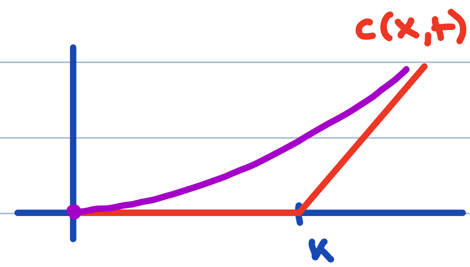
used in trading practice

1. $c(t,x)$ is increasing as a fn of x

$(\partial_x c = N(d_+) > 0)$

2. $c(t,x)$ is convex as a fn of x (since $\partial_x^2 c > 0$)

3. $c(t,x)$ is decreasing as a fn of t (since $\partial_t c < 0$



following observations:

(1) c is an increasing function of x

(2) c is convex as a function of x

(3) c is decreasing as a function of t

call strike K, mat T

X_t = wealth of rep portfolio at t = $c(t,S_t)$

know Δ_t = # shares = $\partial_x c(t,S_t)$ (delta hedge)

$= N(d_+(t,S_t))$

cash balance : $X_t - \Delta_t S_t = c(t,S_t) - N(d_+(t,S_t))S_t$

$= -Ke^{-r\tau}N(d_-(t,S_t)) < 0$

hedge a call, always borrow from bank

note : to hedge a short call position, you will always borrow from bank

Lecture 23 Notes

$4ab = (a+b)^2 - (a-b)^2$

1. multidimensional Itô $\rightarrow$ Girsanov $\rightarrow$ RNM $\rightarrow$ more general pricing

Joint QV.

$[X,Y]_T = \lim_{\|P\| \to 0} \sum (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$

X, Y two processes

$t_0 = 0 \quad t_1 \quad t_2 \quad \ldots \quad T = t_n$

reminder : $[X,X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)^2$

notation, $\Delta_i X = X_{t_{i+1}} - X_{t_i}$

we knew $[X,X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$

joint QV

now we can define joint quadratic variation

let $\Delta_i X = X_{t_{i+1}} - X_{t_i}$

$[X,Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)(\Delta_i Y)$

intuition, increment for stoch processes : $\Delta_i X \approx O(\sqrt{t_{i+1} - t_i})$, $\Delta_i Y \approx O(\sqrt{t_{i+1} - t_i})$

$[X,Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$

note : $4ab = (a+b)^2 - (a-b)^2$

note that $4ab = (a+b)^2 - (a-b)^2$

fact : $4ab = (a+b)^2 - (a-b)^2$

$\Rightarrow (\Delta_i X)(\Delta_i Y) = \frac{1}{4}(\Delta_i (X+Y)^2 - \Delta_i (X-Y)^2)$

$= \frac{1}{4}((X_{t_{i+1}} + Y_{t_{i+1}} - (X_{t_i} + Y_{t_i}))^2 - ((X_{t_{i+1}} - Y_{t_{i+1}}) - (X_{t_i} - Y_{t_i}))^2)$

$[X,Y]_T = \lim_{\|P\| \to 0} \sum (\Delta_i X)(\Delta_i Y)$

$= \lim_{\|P\| \to 0} \sum \frac{1}{4} (\Delta_i (X+Y))^2 - (\Delta_i (X-Y))^2$

$= \frac{1}{4}[[X+Y, X+Y]_T - [X-Y, X-Y]_T]$

$(\Delta_i X)(\Delta_i Y)$

$= \frac{1}{4}[(\Delta_i X + \Delta_i Y)^2 - (\Delta_i X - \Delta_i Y)^2)$

$= \frac{1}{4}[(\Delta_i (X+Y))^2 - (\Delta_i (X-Y))^2)$

$[X,Y]_T = \frac{1}{4}([X+Y, X+Y]_T - [X-Y, X-Y]_T)$

fact :

$[X,Y]_T = \frac{1}{4}([X+Y, X+Y]_T - [X-Y, X-Y]_T)$

$\frac{d}{dt}(XY) = X\frac{dY}{dt} + Y\frac{dX}{dt}$

for differentiable processes : $d(X,Y) = XdY + YdX$

$[X+Y]_T =$

$\frac{1}{4}([X+Y, X+Y]_T - [X-Y, X-Y]_T)$

proposition :

If X, Y are stochastic processes, then

$d(XY) = XdY + YdX + d[X,Y]$

Stochastic product rule :

$d(XY) = XdY + YdX + \underline{[X,Y]_T}$

need this for stochastic !

proof : proof by Itô formula

$XY = \frac{1}{4}[(X+Y)^2 - (X-Y)^2]$

$d(\underbrace{(X+Y)}_{\text{this is our "x"}}{}^2) \stackrel{\text{itô}}{=} 2(X+Y)d(X+Y) + d[X+Y, X+Y]_t$

$d((X+Y)^2) = 2(X+Y)dX + 2(X+Y)dY + d[X+Y, X+Y]$
$d((X-Y)^2) = 2(X-Y)dX + 2(X-Y)dY + d[X-Y, X-Y]$

distribute the dt

combine these two terms

$d(XY) = \frac{1}{4} d((X+Y)^2 - (X-Y)^2)$

$= \frac{1}{4}(4YdX + 4XdY + \underbrace{d[X+Y, X+Y] - d[X-Y, X-Y]}_{\text{we know this } = [X,Y]_T})$

$= \underbrace{XdY + YdX + d[X,Y]}_{\text{product rule proved}}$

proposition: $X \to$ continuous stochastic process

$B \to$ continuous process w/ finite first variation, then $[X,B]_T = 0$

pf. $P = \{0 = t_0 < t_1 \cdots < t_n = T\}$

fact: if X is a continuous stochastic process and B has finite first variation: then $[X,B]_T = 0$

NTS: $\lim_{\|P\| \to 0} \sum (\Delta_i X)(\Delta_i B) = 0$

$\left| \sum_{i=1}^{n-1} (\Delta_i X)(\Delta_i B) \right| \leq \sum_{i=1}^{n-1} |\Delta_i X| |\Delta_i B|$

$\leq \underbrace{\max_i |\Delta_i X|}_{\xrightarrow{\|P\| \to 0} 0} \underbrace{\sum_{i=1}^{n-1} |\Delta_i B|}_{\xrightarrow{\|P\| \to 0} V_{[0,T]} B < \infty}$

"bounded variation" ie. finite first variation

$\Rightarrow [X,B]_T = 0$

multi D Itô: say X and Y are differentiable functions of t. say $f(t,x,y)$ is also diff

$\frac{d}{dt} f(t, X_t, Y_t) = \partial_t f \cdot \frac{dt}{dt} + \partial_x f \frac{dX_t}{dt} + \partial_y f \frac{dY_t}{dt}$

multi D ito: want to able to Itô functions of more than 1 variable, ie. $f(t, X_t, Y_t)$

assume: $X, Y \to$ 2 stoch processes

$f(t,x,y)$ is $C^{1,2}$ (1 deriv in t, 2 deriv in x and y: $\partial_x^2 f, \partial_y^2 f, \partial_x \partial_y f, \partial_x f, \partial_y f$)

must be $C^{1,2}$: 1 deriv in t, 2 derivs in x and y

thm: (Itô formula)

$df(t, X_t, Y_t) = \partial_t f dt + \partial_x f dX_t + \partial_y f dY_t + \frac{1}{2}[\partial_x^2 f d[X,X]_t + \partial_y^2 f d[Y,Y]_t + 2\partial_{xy} f d[X,Y]_t]$

Itô:

$df(t, X_t, Y_t) = \partial_t f dt + \partial_x f dX_t + \partial_y f dY_t + \frac{1}{2}[\partial_x^2 f d[X,X]_t + \partial_y^2 f d[Y,Y]_t + 2\partial_x \partial_y f d[X,Y]_t]$

fact: if X is a continuous stochastic process and B has finite first variation:

then $[X,B]_T = 0$

proof: by defn,

$$[X,B]_T = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(B_{t_{i+1}} - B_{t_i})$$

$$\leq \lim_{\|P\|\to 0} \max_i (X_{t_{i+1}} - X_{t_i}) \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})$$

$$\leq \underbrace{\lim_{\|P\|\to 0} \max_i (X_{t_{i+1}} - X_{t_i})}_{\longrightarrow 0 \text{ by continuity + definition}} \underbrace{\lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})}_{\longrightarrow V_{[0,T]}(B) < \infty \text{ by assumption}}$$

$$\leq (0)(\text{constant}) = 0$$

$[X,B]_T = 0$

Lecture 24 Notes

Ito formula. $f \in C^{1,2}$, $f = f(t,x,y)$

X, Y – 2 Ito processes

2d Itô formula

$$df(t,X_t,Y_t) = \partial_t f\,dt + \partial_x f\,dX_t + \partial_y f\,dY_t + \frac{1}{2}\left[\partial_x^2 f\,d[X,X]_t + \partial_y^2 f\,d[Y,Y]_t + 2\partial_x\partial_y f\,d[X,Y]_t\right]$$

intuition behind Itô   we had the following intuition for 1D Itô:

– 1D Ito: $f(X_T) - f(X_0) = \sum f(X_{t_{i+1}}) - f(X_{t_i})$

$$= \sum f'(X_{t_i})\Delta_i X + \frac{1}{2}\sum f''(X_{t_i})(\Delta_i X)^2 + \text{small}$$

as $\|P\| \to 0$

$$f(X_T) - f(X_0) = \int_0^T f'(X_t)\,dX_t + \lim_{\|P\|\to 0}\frac{1}{2}\sum f''(X_{t_i})\underbrace{\Delta_i[X,X]}_{(\Delta_i X)^2}$$

$$+ \frac{1}{2}\int_0^T f''(X_t)\,\underline{d[X,X]_t}$$

$[X,X]_{t_{j+1}} \approx \sum_{i=0}^{j} (\Delta_i X)^2$

$[X,X]_{t_j} \approx \sum_{i=0}^{j-1} (\Delta_i X)^2$

$d[X,X] = (\Delta_i X)^2$

$f(X_T) - f(X_0) = \underbrace{\sum f(X_{t_{i+1}}) - f(X_{t_i})}_{\text{telescopes}}$

the change between these two is just one $(\Delta_i X)^2 \Longrightarrow d[X,X]$

approximated by

$$= \sum\left[f'(X_{t_i})(\Delta_i X) + \frac{1}{2}f''(X_{t_{i+1}})(\Delta_i X)^2\right]$$

suppose X and Y are two processes

$f = f(x,y)$ now for intuition behind 2D

1. $f(X_T,Y_T) - f(X_0,Y_0) = \underline{\sum f(X_{t_{i+1}},Y_{t_{i+1}}) - f(X_{t_i},Y_{t_i})}$ telescopes to $f(X_T,Y_T) - f(X_0,Y_0)$

2D Taylor

$$f(x+h,y+k) \approx f(x,y) + \underbrace{h\,\partial_x f(x,y) + k\,\partial_y f(x,y)}_{\text{first partials}} + \underbrace{\frac{1}{2}\left[\partial_x^2 f(x,y)(h)^2 + \partial_y^2 f(x,y)(k)^2 + \partial_x\partial_y f(x,y)(hk)\right]}_{\text{second partials (second order)}}$$

instead, now use 2D taylor approx.

use $\underline{x = X_{t_i}}$, $\underline{y = Y_{t_i}}$, $\underline{h = \Delta_i X}$, $\underline{k = \Delta_i Y}$

$f(X_{t_{i+1}},Y_{t_{i+1}}) - f(X_{t_i},Y_{t_i}) =$ just plug in with these values

$$\partial_x f\,\Delta_i X + \partial_y f\,\Delta_i Y + \frac{1}{2}\left[\partial_x^2 f(\Delta_i X)^2 + \partial_y^2 f(\Delta_i Y)^2 + 2\partial_x\partial_y f(\Delta_i X)(\Delta_i Y)\right]$$

note $\sum \partial_x f(\Delta_i X) \xrightarrow{\|P\|\to 0} \int_0^T \partial_x f(X_t,Y_t)\,dX_t$ ✓

$\sum \partial_y f(\Delta_i Y) \xrightarrow{\|P\|\to 0} \int_0^T \partial_y f(X_t,Y_t)\,dY_t$ ✓

$\Delta_i X \to dX_t$

$\Delta_i Y \to dY_t$

$\sum \partial_x^2 f(\Delta_i X)^2 \xrightarrow{\|P\|\to 0} \int_0^T \partial_x^2 f(X_t,Y_t)\,d[X,X]$

$\sum \partial_y^2 f(\Delta_i Y)^2 \xrightarrow{\|P\|\to 0} \int_0^T \partial_y^2 f(X_t,Y_t)\,d[Y,Y]$

$(\Delta_i X)^2 \to d[X,X]$

$(\Delta_i Y)^2 \to d[Y,Y]$

$\sum \partial_x\partial_y f(\Delta_i X)(\Delta_i Y) \xrightarrow{\|P\|\to 0} \int_0^T \partial_x\partial_y f\,d[X,Y]$ $(\Delta_i X)(\Delta_i Y) \to d[X,Y]$

(incremental change in $d[X,Y]$)

properties of joint QV.

1. X, Y, Z are 3 stochastic processes

property:

$[X,Y+Z]_T = [X,Y]_T + [X,Z]_T$

$[X,Y+Z]_T = [X,Y]_T + [X,Z]_T$

$$[X,Y+Z]_T = \lim_{\|P\|\to 0} \sum \Delta_i X\,(\Delta_i(Y+Z))$$

directly prove with defining equation

$\Delta_i(Y+Z) = \Delta_i Y + \Delta_i Z$

$$= \lim_{\|P\|\to 0} \sum \Delta_i X(\Delta_i Y + \Delta_i Z)$$

$$= \lim_{\|P\|\to 0} \sum \Delta_i X \Delta_i Y + \lim_{\|P\|\to 0} \sum \Delta_i X \Delta_i Z$$

prop: say $X_T = X_0 + B_T + M_T$

$Y_T = Y_0 + C_T + N_T$

we have: $X_T = X_0 + B_T + M_T$

$Y_T = Y_0 + C_T + N_T$

B, C finite first variation. $\underline{B_0 = C_0 = M_0 = N_0 = 0}$, $M, \underline{N \text{ are mgs}}$

all start at 0

M and N are mgs

$[X,X]_T = [M,M]$

$[X,Y]_T = ?\ [M,N]$

we proved already that the B_T with finite first variation contributes nothing to the quadratic var:

ie. $[X,X] = [M,M]$

if $dX_t = b\,dt + dM_t$

$dY_t = c\,dt + dN_t$

then $d[X,Y]_t = d[M,N]_t$

pf. $[X,Y] = [B+M, C+N]$

$= [B,C]_t + [B,N]_t + [M,C]_t + [M,N]_t$

last time: $[B,C] = [B,N] = [M,C] = 0$

WTS. $[X,Y] = [M,N]$

$[X,Y] = [B+M, C+N]$

$$= \underbrace{[B,C] + [B,N] + [M,C]}_{\substack{\text{all of these contain a term with} \\ \text{finite first variation.}}} + [M,N]$$

$\rightarrow\ = 0$

$[X,Y] = [M,N]$ this leaves only the $[M,N]$

Lecture 25 Notes

notation: $\Delta_i X = X_{t_{i+1}} - X_{t_i}$, $\Delta_i Y = Y_{t_{i+1}} - Y_{t_i}$

$[X,Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)(\Delta_i Y)$

note the following:

$$\lim_{\|P\| \to 0} \sum |t_{i+1} - t_i|^\alpha = \begin{cases} \infty & \text{for } \alpha < 1 \\ T & \text{for } \alpha = 1 \\ 0 & \text{for } \alpha > 1 \end{cases}$$

stoch process $\rightarrow \Delta_i X \approx$ size $\sqrt{(t_{i+1} - t_i)}$

differentiable fn $\rightarrow \Delta_i B \approx$ size $(t_{i+1} - t_i)$

note: $\underline{\lim_{\|P\| \to 0} \sum |t_{i+1} - t_i|^\alpha} = \begin{cases} \infty & \alpha < 1 \\ T & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$

think about the arguments for each of these cases

computing joint QV:

rule 1: ignore first variation terms $\rightarrow$ already proved this. finite first-variation terms contribute nothing to joint quadratic var

$\left.\begin{array}{l} dX_t = b\,dt + dM_t \\ dY_t = c\,dt + dN_t \end{array}\right\}$ then $d[X,Y]_t = d[M,N]_t$

(recall: $d[X,X]_t = \sigma_t^2 d[M,M]_t$)

rule 2: if $X_t = \int_0^t \sigma_s dM_s$, $Y_t = \int_0^t \tau_s dN_s$

joint QV, $d[X,Y]_t = \sigma_t \tau_t d[M,N]_t$

FACT:
for $X_t = \int_0^t \sigma_s dM_s$
$\hookrightarrow d[X,X]_t = \sigma_t^2 d[M,M]_t$

rule 3:

If M and N are continuous martingales, and M is independent of N (M and N are independent), then $\underline{d[M,N]_t} = 0$

$= 0$

FACT:
if $X_t = \int_0^t \sigma_s dM_s$ and $Y_t = \int_0^t \tau_s dN_s$
$\hookrightarrow d[X,Y]_t = \sigma_t \tau_t d[M,N]_t$

intuition for rule 2:

$X_T - X_0 = \int_0^T \sigma_s dM_s \approx \underline{\sum \sigma_{t_i} \Delta_i M}$

$Y_T - Y_0 = \int_0^T \tau_s dN_s \approx \underline{\sum \tau_{t_i} \Delta_i N}$

$[X,Y]_T = \lim_{\|P\| \to 0} \sum (\Delta_i X)(\Delta_i Y) \approx \sum (\Delta_i X)(\Delta_i Y)$

note: $\left.\begin{array}{l} \Delta_i X \approx \sigma_{t_i} (\Delta_i M) \\ \Delta_i Y \approx \tau_{t_i} (\Delta_i N) \end{array}\right\}$ by the above justification

FACT:
if M and N are continuous, independent martingales $\rightarrow$ then $d[M,N]_t = 0$

$\Rightarrow [X,Y]_t = \sum \sigma_{t_i} \tau_{t_i} (\Delta_i M)(\Delta_i N)$ $d[M,N]$

$[M,N]_t \approx \sum (\Delta_i M)(\Delta_i N)$ incremental change

$d[M,N]_t \approx \Delta_i [M,N] = (\Delta_i M)(\Delta_i N)$

$[X,Y] = \int_0^T \sigma_s \tau_s d[M,N]$

$[X,Y]_T \approx \sum \sigma_{t_i} \tau_{t_i} \Delta_i [M,N] \xrightarrow{\|P\| \to 0} \underline{\int_0^T \sigma_s \tau_s d[M,N]}$

justification for rule 3: independent cts martingales: $[M,N]_T = 0$

M and N are continuous independent Mgs. Show $[M,N]_T = 0$, ie. $\lim_{\|P\|\to 0} \underline{\underline{\sum (\Delta_i M)(\Delta_i N) = 0}}$

will show $\lim_{\|P\|\to 0} \mathbb{E}\left(\sum (\Delta_i M)(\Delta_i N)\right)^2 = 0$ → this would imply ↗ this almost-surely

note $\mathbb{E}\left(\sum_i (\Delta_i M)(\Delta_i N)\right)^2 = \mathbb{E}\sum_i (\Delta_i M)^2 (\Delta N)^2$ I

$+ 2\mathbb{E}\sum_{j=0}^{n} \sum_{i=0}^{j-1} (\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N)$ II

I. $\left|\sum (\Delta_i M)^2 (\Delta_i N)^2\right| \leq \underbrace{\max (\Delta_i M)^2}_{\to 0 \text{ by cts.}} \underbrace{\sum (\Delta_i N)^2}_{\text{finite, } [N,N]_T < \infty}$ $\left(\underline{\|P\|\to 0}\right)$

can pull out either and use continuity + bounded quadratic var

II.

$\mathbb{E}\sum_{j=0}^{n} \sum_{i=0}^{j-1} (\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N)$

independence $= \sum_{j=0}^{n} \sum_{i=0}^{j-1} \mathbb{E}(\Delta_i M \Delta_j N)\,\mathbb{E}(\Delta_i N \Delta_j N)$ → use independence to distribute the expectation

tower $= \sum_{j=0}^{n} \sum_{i=0}^{j-1} \mathbb{E}\mathbb{E}_{t_j}(\Delta_i M)(\Delta_j M)\,\mathbb{E}\mathbb{E}_{t_j}(\Delta_i N \Delta_j N)$

then apply tower

(since $i < j$, $\Delta_i M$ is $\mathcal{F}_{t_j}$ mb)

$= \underbrace{\sum_{j=0}^{n-1} \sum_{i=0}^{j-1} \mathbb{E}\left[\Delta_i M\, \mathbb{E}_{t_j}(\Delta_j M)\right] \mathbb{E}\left[\Delta_i N\, \mathbb{E}_{t_j}(\Delta_j N)\right]}_{=0}$ $\left(\mathbb{E}_{t_j} \Delta_j M = \mathbb{E}_{t_j}(M_{t_{j+1}} - M_{t_j}) = 0\right)$

all equal to 0

because martingale!

summation fact:

$\left[\sum_{i=0}^{n} (\Delta_i M)(\Delta_i N)\right]^2$

$= \underbrace{\sum_{i=0}^{n} (\Delta_i M)^2 (\Delta_i N)^2}_{\text{same terms}} + 2\sum_{j=0}^{n} \underbrace{\sum_{i=0}^{j-1} (\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N)}_{\text{cross terms}}$

computing joint QV:

If $X_t = \int_0^t \sigma_s \, dM_s$ and $Y_t = \int_0^t \tau_s \, dN_s$

$\Rightarrow d[X,Y]_t = \sigma_t \tau_t \, d[M,N]_t$

further, recall

$d[X,X]_t = \sigma_t^2 \, d[M,M]_t$

lastly,

if M and N are continuous independent mgs

then $d[M,N] = 0$

Lecture 26 Notes

$dX_t = b_t dt + \sigma_t dM_t$

rules.

$dY_t = c_t dt + \tau_t dN_t$

$dX_t = b_t dt + \sigma_t dM_t$, $dY_t = c_t dt + \tau_t dN_t$

$\Rightarrow d[X,Y]_t = \sigma_t \tau_t \, d[M,N]_t$

$\Rightarrow d[X,Y]_t = \sigma_t \tau_t \, d[M,N]_t$

also : 1. if M and N are continuous independent martingales then $d[M,N] = 0$

2. if $M = N = W$ (BM), then $d[M,N]_t = dt$

if M, N are cts + indep.

$d[M,N] = 0$

↓

MUST BE INDEPENDENT

if not, this does not hold

if $d[M,N]_t = 0$, must $M \perp\!\!\!\perp N$? NO!

ex. $M=N=W$

$d[M,N]_t = dt$

ex. $M_t = \int_0^t \mathbb{1}\{W_s > 0\} \, dW_s$, $N_t = \int_0^t \mathbb{1}\{W_s \le 0\} \, dW_s$

$d[M,N]_t = \mathbb{1}\{W_t > 0\} \mathbb{1}\{W_t \le 0\} \, dt = 0$ (uses traditional rule)

note $M_t + N_t = \int_0^t 1 \, dW_s = W_t$ (use this to show, M and N are not independent)

important note. $[M,N] = 0$ does NOT imply that M and N are independent

notation in higher dimensions

$v \in \mathbb{R}^d \rightarrow v = (v_1, \cdots, v_d)$ or $v = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$ d-dimensional vector : $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$

for us :

vectors superscript denotes coordinate, subscript denotes time

scalars → superscript denotes powers

superscript = coordinate

subscript = time

$X \rightarrow$ "$\mathbb{R}^d$ valued process"

$X_t = (X_t^1, X_t^2, \cdots, X_t^d)$ each X_t^i is a $\mathcal{F}_t$ meas RV

X_t^d

Ito formula in vector notation

$X_t = (X_t^1, X_t^2, \cdots, X_t^d)$

1. $X \rightarrow \mathbb{R}^d$ valued process

↘

each coordinate is a stochastic process

2. $f \in C^{1,2}$: $f(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$

Ito :

$\partial_i f = \frac{\partial f}{\partial x_i}$

$$df(t, X_t) = \partial_t f \, dt + \sum_{i=1}^d \partial_i f \, dX_t^i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \partial_i \partial_j f \, d[X^i, X^j]_t$$

def: we say W is a standard d-dimensional brownian motion if

1. each W^i is a standard 1d BM
2. $W^1, W^2, \cdots, W^d$ are all independent

W is a d-dimensional BM if:

1. each W^i is a standard 1D BM
2. $W^1, W^2, \cdots, W^d$ are all independent

What does this independence imply?

note:

$$d[W^i, W^j]_t = \begin{cases} dt & i=j \\ 0dt & i\neq j \end{cases}$$

FACT: $d[W^i,W^j]_t = \begin{cases} dt, & i=j \\ 0, & i\neq j \end{cases}$

ex. let $W \to$ 2 dim std. BM

$f: \mathbb{R}^2 \longmapsto \mathbb{R}$ defined by $f(x) = \ln(1+|x|)$, $x \in \mathbb{R}^2$ $\quad |x| = \sqrt{x_1^2+x_2^2}$

length, magnitude

$df(W_t) = d\ln(1+|W_t|)$

1. $\partial_t f = 0$, 2. $\partial_1 f = \frac{1}{1+|x|}\partial_1\sqrt{x_1^2+x_2^2} = \frac{1}{1+|x|}\frac{2x_1}{2\sqrt{x_1^2+x_2^2}} = \frac{1}{1+|x|}\frac{x_1}{|x|}$

3. $\partial_2 f = \frac{1}{1+|x|}\frac{x_2}{|x|}$

4. $\partial_i\partial_j f = \cdots \quad \partial_1^2 f + \partial_2^2 f = 0$ (you check)

$$df(t,W_t) = \partial_t f + \sum \partial_i f\, dW_t^i + \frac{1}{2}\sum_i\sum_j \partial_i\partial_j d[W^i,W^j]$$

plug into Itô

$$= 0 + \sum \partial_i f\, dW_t^i + \frac{1}{2}\sum_{i=1}^{2}\partial_i^2 f\, dt$$

only case where $i=j$

$$= \frac{W_t^1}{(1+|W_t|)|W_t|}dW_t^1 + \frac{W_t^2}{(1+|W_t|)|W_t|}dW_t^2 + 0$$

$M_t = \ln(1+|W_t|)$

only Ito integrals. look like martingale?

is M a martingale? guess yes?

NOPE.

multi-dim Itô formula:

- X is a d-dimensional Itô process: $X_t = (X_t^1, \cdots, X_t^d)$
- $f = f(t,x)$ is a function defined for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$
- f is $C^{1,2}$:
  - $\to$ f is once differentiable in t
  - $\to$ f is twice differentiable in each coordinate x_i
  - $\to$ all the above partial derivatives are continuous

then we have:

$d(f(t, X_t)) =$

$$\underbrace{\partial_t f(t, X_t)\, dt}_{\text{time}} + \underbrace{\sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i}_{\text{each coord}} + \frac{1}{2} \underbrace{\sum_{(i,j)}}_{\sum_{i=1}^{d} \sum_{j=1}^{d}} \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

Lecture 27 Notes

$W \to$ 2D brownian motion $\qquad X_t = W_t + (1,0)$ some const basis vector

$X_t = W_t + (1,0)$, $f(x) = \ln|x|$, $x = (x_1, x_2) \in \mathbb{R}^2$, $|x| = \sqrt{x_1^2 + x_2^2}$

compute $d\ln|X_t|$ by Itô : use multi Itô

$$df(X_t) = \partial_t f\,dt + \partial_1 f\,dX_t^1 + \partial_2 f\,dX_t^2 + \frac{1}{2}\sum_{i,j} \partial_i\partial_j f\, d[X^i, X^j]_t$$

$$d[X^i, X^j] = \begin{cases} dt & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$df(X_t) = \partial_1 f\,dW_t^1 + \partial_2 f\,dW_t^2 + \frac{1}{2}\sum_{i=1}^{2} \partial_i^2 f\,dt$$

$f(x) = \ln|x|$

$$\partial_1 f = \frac{1}{|x|}\partial_1\sqrt{x_1^2+x_2^2} = \frac{1}{|x|}\frac{2x_1}{2\sqrt{x_1^2+x_2^2}} = \frac{x_1}{|x|^2}$$

$$\partial_2 f = \frac{x_2}{|x|^2}$$

can check $\partial_1^2 f + \partial_2^2 f = 0$

$$\Rightarrow \quad \underline{d\ln|X_t| = \frac{X_t^1}{|X_t|^2}dW_t^1 + \frac{X_t^2}{|X_t|^2}dW_t^2 + 0}$$

the resulting computation formula

cannot forget about Ito finiteness cond.

Q: ?? $\Rightarrow \ln|X_t|$ is a mg?

note: $\ln|X_T| - \underbrace{\ln|X_0|}_{\ln(1)=0} = \int_0^t \underbrace{\frac{X_s^1\,dW_s^1}{|X_s|^2}}_{*} + \underbrace{\frac{X_s^2\,dW_s^2}{|X_s|^2}}_{**}$

explore expectation using lazy statistician

note: $\mathbb{E}\ln|X_t|$ (compute using lazy statistician)

$$\mathbb{E}\ln(|W_t + (1,0)|) = \int_{\mathbb{R}^2} \ln(|x + (1,0)|)\, e^{-\frac{1}{2}|x|^2}\frac{dx_1dx_2}{2\pi}$$

magnitude

double integral over $\mathbb{R}$

standard joint normal

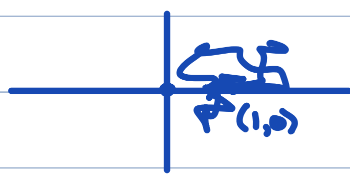
$$Z \sim N\left((0,0), \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$\phi_Z(z_1, z_2) = \frac{1}{\sqrt{2\pi}}e^{-x_1^2/2}\frac{1}{\sqrt{2\pi}}e^{-x_2^2/2} = \frac{1}{2\pi}e^{-\frac{1}{2}|x|^2}$$

$$\mathbb{E}\ln(|\sqrt{t}\underbrace{\tfrac{W_t}{\sqrt{t}}}_{\text{normalize}} + (1,0)|) = \int_{\mathbb{R}^2} \ln(\underbrace{|\sqrt{t}x + (1,0)|}_{\text{now lazy statistician with the normalized}})\, e^{-\frac{1}{2}|x|^2}\frac{dx_1dx_2}{2\pi}$$

$=$ NOT constant in t $\Rightarrow$ not a martingale

note $f(x) = \ln|x| \in \mathbb{R}^2$ $\underline{\text{is not diff at } 0}$



discontinuous at (0,0)

however the process hits 0 with prob 0 $\Rightarrow$ hence applying Ito is legal

recall :

$M_t = \int_0^t \sigma_s dW_s$ is a Martingale if σ_s is adapted and $\mathbb{E}\int_0^t \sigma_s^2 ds < \infty$

this finiteness condition for * and ** $\Rightarrow$ $\ln|x|$ is not a mg

for an Ito integral to be a martingale, must not forget about the finiteness condition :

Levy's Criterion

$\mathbb{E}\int_0^t \sigma_s^2 ds < \infty$

1D : if M is a cts mg with $M_0 = 0$ and $d[M,M]_t = dt$ $\Rightarrow$ M is a BM

d-dim : if M is a d-dim process

1. $M_0 = 0$
2. M^i is a cts mg for every $i \in \{1, \cdots, d\}$
3. $d[M^i, M^j] = \begin{cases} dt & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\Rightarrow$ M is a d-dim BM

multidimensional joint normal :

$Z \sim N\left((0,0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\phi(z_1, z_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} = \frac{1}{2\pi} e^{-\frac{|x|^2}{2}}$

Lecture 28 Notes

Levy's Criterion (d-dimensions)

Say M is a d-dim process such that

1. $M_0 = 0$ ✓

2. M is a continuous mg ✓

ie. each coordinate M^i is a mg ✓

3. $d[M^i, M^j] = \begin{cases} 1dt & i=j \\ 0dt & i \neq j \end{cases}$ ✓

$\Rightarrow$ M is a d-dim BM

(ie. $M^1, M^2, \cdots, M^d$ are all independent and each M^i is a standard 1dim BM)

$\Rightarrow M_t - M_s \sim N\left(0, \begin{pmatrix} t-s & & 0 \\ & \ddots & \\ 0 & & t-s \end{pmatrix}\right)$

d-dim normal

Levy's Criterion:

M is a d-dim stochastic process such that

1. $M_0 = 0$

2. M is a continuous mg (each M^i is a mg)

3. $d[M^i, M^j] = \begin{cases} dt, & i=j \\ 0, & i \neq j \end{cases}$

$\Rightarrow$ then M is a d-dim BM

ie. $M^1, \cdots, M^d$ are independent standard 1D BMs

this also implies

$\Rightarrow M_t - M_s \sim N\left(0, \begin{pmatrix} t-s & & 0 \\ & \ddots & \\ 0 & & t-s \end{pmatrix}\right)$

(d dimensional normal)

$$dM^1 = \frac{W^1 dW^1}{|W|^2} + \frac{W^2 dW^2}{|W|^2}$$

$$dM^2 = \frac{W^1 dW^1}{|W|^2} - \frac{W^2 dW^2}{|W|^2}$$

compute: $d[M^1, M^1] = \frac{(W_t^1)^2}{|W_t|^2} dt + \frac{(W_t^2)^2}{|W_t|^2} dt + 0 = 1dt$

$d[M^2, M^2] = \frac{(W_t^1)^2}{|W_t|^2} dt + \frac{(W_t^2)^2}{|W_t|^2} dt + 0 = 1dt$

$d[M^1, M^2] = \frac{(W_t^1)^2}{|W_t|^2} dt - \frac{(W_t^2)^2}{|W_t|^2} dt + 0 \neq$ cannot apply Levy ☹

doesn't satisfy $d[M^1, M^2] = 0$

$\Downarrow$

would need this to apply Levy

proof of Levy:

Will show $CF(M_t) = CF\left(N\left(0, \begin{pmatrix} t-s & & 0 \\ & \ddots & \\ 0 & & t-s \end{pmatrix}\right)\right)$

$\overbrace{\varphi(\lambda)}^{f(t)} = \mathbb{E} e^{i\lambda \cdot M_t}, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$

similar to 1D proof.

use characteristic functions

find a differential equation for f and solve $\qquad \varphi_{M_t}(\lambda) = \mathbb{E} e^{i\lambda \cdot M_t}$ where $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$

Ito $e^{i\lambda \cdot M_t}$: want to multidim Ito $e^{i\lambda \cdot M_t}$

let $g(x) = e^{i\lambda \cdot x}$ $\quad (x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d)$

$\partial_t g = 0$, $\partial_j g = i\lambda_j e^{i\lambda \cdot x}$

$\partial_j \partial_k = -\lambda_j \lambda_k e^{i\lambda \cdot x}$ $\quad$ partial derivs with respect to coord.

$$d(e^{i\lambda \cdot M_t}) = dg(t, M_t)$$

$$= \partial_t g \, dt + \sum \partial_j g \, dM_t^j + \frac{1}{2} \sum_{jk} \partial_j \partial_k g \, d[M^j, M^k]$$

$$= 0 + i \sum_j \lambda_j e^{i\lambda \cdot M_t} dM_t^j + \frac{1}{2} \sum_{j=1}^d \partial_j^2 g \, dt$$

$$= i \sum_j \lambda_j e^{i\lambda \cdot M_t} dM_t^j - \frac{1}{2} \sum_{j=1}^d \lambda_j^2 e^{i\lambda \cdot M_t} dt$$

$$e^{i\lambda \cdot M_t} - \underbrace{e^{i\lambda \cdot M_0}}_{1} = i \sum \lambda_j \int_0^t e^{i\lambda \cdot M_s} dM_s^j - \frac{1}{2} \sum_{j=1}^d \lambda_j^2 \int_0^t e^{i\lambda \cdot M_s} ds$$

$$\mathbb{E} e^{i\lambda \cdot M_t} = i \sum \lambda_j \mathbb{E} \int_0^t e^{i\lambda \cdot M_s} dM_s^j - \frac{1}{2} \sum_{j=1}^d \lambda_j^2 \mathbb{E} \int_0^t e^{i\lambda \cdot M_s} ds$$

$$f(t) - 1 = 0 - \frac{1}{2} |\lambda|^2 \int_0^t f(s) \, ds$$

$$f'(t) = -\frac{1}{2} |\lambda|^2 f(t) \Rightarrow f(t) = f(0) e^{-\frac{|\lambda|^2}{2} t}$$

solve diffeq

$$\Rightarrow f(t) = \mathbb{E} e^{i\lambda \cdot M_t} = e^{-\frac{|\lambda|^2}{2} t} = CF(N(0, \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix}))$$

quick check

1D : $N(0,t) \rightarrow CF = e^{-\lambda^2/2 \, t}$ $\qquad$ CF of d-dim normal

dD : $N(0, \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix}) \rightarrow CF = e^{-|\lambda|^2/2 \, t}$ $\qquad$ $CF = e^{-|\lambda|^2/2 \, t}$

if $Z \sim N(0, \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix})$, $Z = (Z^1, \cdots, Z^d)$

then $Z^1, \cdots, Z^d$ are all independent and $Z^j \sim N(0,t)$

compute $\varphi_\lambda(Z) = \mathbb{E} e^{i\lambda \cdot Z}$

$= \mathbb{E} e^{i \sum \lambda_j Z^j}$ ✓

$= \mathbb{E}\left(\prod_{j=1}^{d} e^{i\lambda_j Z^j}\right) = \prod_{j=1}^{d} \mathbb{E} e^{i\lambda_j Z^j} = \prod_{j=1}^{d} e^{-\frac{\lambda_j^2}{2}t} = e^{-\frac{|\lambda|^2}{2}t}$

product of individual CFs

Lecture 29 Notes

$dM_t^1 = \frac{W_t^1 dW_t^1}{|W_t|} + \frac{W_t^2 dW_t^2}{|W_t|}$; $dM_t^2 = \frac{W_t^2 dW_t^1}{|W_t|} - \frac{W_t^1 dW_t^2}{|W_t|}$; $M_0 = 0$

looks like M_1 and M_2 depend on each other, but will show that M_1 and M_2 are independent

we can show M^1 and M^2 are independent by showing M is a standard BM

proof. apply Levy : 1. $M_0 = 0$, 2. M is a continuous mg

compute QV:

$d[M^1, M^1]_t = \frac{(W_t^1)^2}{|W_t|^2} dt + \frac{(W_t^2)^2}{|W_t|^2} dt + 0 = 1dt$

$d[M^2, M^2]_t = \frac{(W_t^2)^2}{|W_t|^2} dt + \frac{(W_t^1)^2}{|W_t|^2} dt + 0 = 1dt$

and

$d[M^1, M^2] = \frac{W_t^1 W_t^2}{|W_t|^2} dt + \frac{W_t^2(-W_t^1)}{|W_t|^2} dt + 0 = 0dt$

3. this implies $d[M^i, M^j] = \begin{cases} dt, & i=j \\ 0, & i \neq j \end{cases}$. thus we can apply 2d Levy

$\Rightarrow$ M is a 2D brownian motion $\Rightarrow$ M^1 and M^2 are independent!

risk neutral measures : we now have d-stocks. want to construct risk neutral measures

market
- bank $\rightarrow$ interest rate R_t (R_t is any adapted process)
- d stocks : S_t^i = spot price of ith stock at time t

price of ith stock

cash : C_0 \$ in bank at time 0

at time t, $C_t = C_0 e^{\int_0^t R_s ds}$ ($\because dC_t = R_t C_t dt \Rightarrow \frac{dC_t}{C_t} = R_t dt$

$\Rightarrow d(\ln C_t) = R_t dt \Rightarrow \ln C_t - \ln C_0 = \int_0^t R_s ds \Rightarrow C_t = C_0 \exp(\int_0^t R_s ds))$

discount factor : $D_t = \exp(-\int_0^t R_s ds)$

no constant interest rate :

now $C_t = C_0 \exp(\underbrace{\int_0^t R_s ds}_{\text{accumulation of interest rates}})$

note, $dD_t = -R_t D_t dt$

we have : $dC_t = R_t C_t dt$, $dD_t = -R_t D_t dt$

def : a risk neutral measure $\tilde{\mathbb{P}}$ is a measure such that

1. $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ (*)

what is a risk neutral measure : measure $\tilde{\mathbb{P}}$ such that

1. $\tilde{\mathbb{P}}$ is "equivalent" to $\mathbb{P}$

2. discounted stock is a $\tilde{\mathbb{P}}$-mg

2. discounted stock price is a $\tilde{\mathbb{P}}$-martingale

note. already have a measure $\mathbb{P}$. $\forall A \in \Omega$, $\mathbb{P}(A) \in [0,1]$ is prob of event A occurring

$\longrightarrow$ define $\mathbb{E}$, cond exp, mg, etc.

$\tilde{\mathbb{P}} \longrightarrow$ different measure on the same probability space

$\tilde{\mathbb{E}}$ = expectation under $\tilde{\mathbb{P}}$; M is a $\tilde{\mathbb{P}}$-mg $\Longleftrightarrow$ $\tilde{\mathbb{E}}_s M_t = M_s$; etc.

ex.

let Z be a random var. (need $\mathbb{E}Z = 1$ and $Z \geq 0$)

define $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{P}}(A) = \int_A Z \, dP = \mathbb{E}(\mathbb{1}_A Z)$

note if A + B are disjoint, $\tilde{\mathbb{P}}(A \cup B) = \tilde{\mathbb{P}}(A) + \tilde{\mathbb{P}}(B)$

check.

$\tilde{\mathbb{P}}(A \cup B) = \mathbb{E}(\mathbb{1}_{A \cup B} Z) = \mathbb{E}((\mathbb{1}_A + \mathbb{1}_B) Z) = \mathbb{E}(\mathbb{1}_A Z) + \mathbb{E}(\mathbb{1}_B Z) =$

$\tilde{\mathbb{P}}(A) + \tilde{\mathbb{P}}(B)$

need $\tilde{\mathbb{P}}(\Omega) = 1 \Longleftrightarrow \mathbb{E}(\mathbb{1}_\Omega Z) = 1 \Longleftrightarrow \mathbb{E}Z = 1$

Lecture 30 Notes

risk neutral measures:

market → bank → interest rate R_t.
market → d-stocks. S^i_t = price of ith stock at time t

$C_t = C_0 e^{\int_0^t R_s ds}$, ie. $dC_t = R_t C_t dt$. $D_t = e^{-\int_0^t R_s ds}$, ie. $dD_t = -R_t D_t dt$

$C_t = C_0 e^{\int_0^t R_s ds}$, $dC_t = R_t C_t dt$, $D_t = e^{-\int_0^t R_s ds}$, $dD_t = -R_t D_t dt$

def. we say $\tilde{\mathbb{P}}$ is a risk neutral measure if:

1. $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ → $\tilde{\mathbb{P}}$ must be equivalent to $\mathbb{P}$
2. the discounted wealth of each S^i is a $\tilde{\mathbb{P}}$-mg → DS^i must be a mg $\forall i$

~~equivalent~~ measures.

lebesgue integral notation

eg: $\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbb{1}_A Z) = \int_A Z \, d\mathbb{P}$

what is a measure? $\tilde{\mathbb{P}}(A) = \mathbb{E}[\mathbb{1}_A Z] = \int_A Z \, d\mathbb{P}$

need ① $\mathbb{E}Z = 1$ ($\because$ want $\tilde{\mathbb{P}}(\Omega) = 1$)

② $Z \geq 0$ ($\because$ want $\tilde{\mathbb{P}}(A) \in [0,1]$)

need these conditions: 1. $\mathbb{E}Z = 1$, 2. $Z > 0$ } sufficient

def. we say $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ if for every event A, $\mathbb{P}(A) = 0 \Longleftrightarrow \tilde{\mathbb{P}}(A) = 0$

note. say $\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbb{1}_A Z)$, $\underline{\underline{\mathbb{E}Z = 1}}$, $\underline{\underline{Z \geq 0}}$

$\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ iff:

$\mathbb{P}(A) = 0 \Longleftrightarrow \tilde{\mathbb{P}}(A) = 0$ $\forall A$

say $\mathbb{P}(A) = 0$

$\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbb{1}_A Z) = 0$

$\mathbb{1}_A = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$, $\mathbb{1}_A Z = \begin{cases} Z & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \Rightarrow \mathbb{1}_A Z = 0$ a.s.

other direction

want $\tilde{\mathbb{P}}(A) = 0 \Rightarrow \mathbb{P}(A) = 0$, ie. $\mathbb{E}(\mathbb{1}_A Z) = 0 \Rightarrow \mathbb{P}(A) = 0$

(bad ex. $B \subseteq \Omega$ some event, $\mathbb{P}(B) = \frac{1}{2}$. try $Z = 2\mathbb{1}_B$)

$\tilde{\mathbb{P}}(B^c) = \mathbb{E}(\mathbb{1}_{B^c} Z) = \mathbb{E}(\mathbb{1}_{B^c} \mathbb{1}_B) = 0$

ex. where $\tilde{\mathbb{P}}(B^c) = 0$ but $\mathbb{P}(B^c) = \frac{1}{2} \neq 0$)

claim. if $Z>0$ then $\tilde{\mathbb{P}}(A)=0 \Longrightarrow \mathbb{P}(A)=0$

final measure: define the following

$\tilde{\mathbb{P}}(A)=\mathbb{E}\,\mathbb{1}_A Z$

where $\mathbb{E}Z=0$ and $Z>0$

so

$\tilde{\mathbb{P}}(A)=\mathbb{E}(\mathbb{1}_A Z)$ with $Z>0$ and $\mathbb{E}Z=1$

notation: if we define $\tilde{\mathbb{P}}(A)=\mathbb{E}(\mathbb{1}_A Z)$ then we write $d\tilde{\mathbb{P}}=Z\,d\mathbb{P}$

prop. $\tilde{\mathbb{E}}X=\mathbb{E}XZ$

$(\tilde{\mathbb{E}}X=\int_\Omega X\,d\tilde{\mathbb{P}}=\int_\Omega XZ\,d\mathbb{P}=\mathbb{E}(XZ))$ recall expectation $\mathbb{E}X=\int_\Omega X\,d\mathbb{P}$

if $X=\mathbb{1}_A$, $\tilde{\mathbb{E}}X=\tilde{\mathbb{P}}(A)=\mathbb{E}(\mathbb{1}_A Z)=\mathbb{E}XZ$

condition 2. of RNM:

to be a RNM, we need DS^i to be an mg $\forall i$

$D_t S_t^i \longrightarrow \tilde{\mathbb{P}}$ mg $\forall i \in \{1,\cdots,d\}$

$\Longleftrightarrow \tilde{\mathbb{E}}_s(D_t S_t^i)=D_s S_s^i$

thm. if X_t = wealth at time t of a self financing portfolio, then $D_t X_t$ is a $\tilde{\mathbb{P}}$ mg

further, $D_t X_t$ is also a $\tilde{\mathbb{P}}$ mg

say market has only 1 stock. want

$D_t X_t \longrightarrow \tilde{\mathbb{P}}$ mg, X self financing, $D_t S_t \longrightarrow \tilde{\mathbb{P}}$ mg

compute $d(D_t X_t)$

1. $dX_t=\Delta_t dS_t+R_t(X_t-\Delta_t S_t)dt$

2. $dD_t=-R_t D_t dt$

$d(D_t X_t)=X_t dD_t+D_t dX_t+\underbrace{\cancel{d[D,X]_t}}_{=0}$

$=X_t(-R_t D_t dt)+D_t(\Delta_t dS_t+R_t(X_t-\Delta_t S_t)dt)$

$=D_t\Delta_t dS_t-D_t R_t\Delta_t S_t dt$

$=D_t\Delta_t[dS_t-R_t S_t dt]$

Lecture 31 Notes

what we have: $d\tilde{P} = Z dP$, $\mathbb{E}Z = 0$, $Z > 0$ and $\tilde{\mathbb{E}}X = \mathbb{E}XZ$

last time: where $\tilde{\mathbb{P}}(A) = \mathbb{E}\,\mathbb{1}_A Z$

RNM - equivalent measure, $\underbrace{d\tilde{P} = Z dP}$, $\mathbb{E}Z = 1$, $Z > 0$

$\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbb{1}_A Z)$. $\tilde{\mathbb{E}}X = \mathbb{E}(XZ)$

for every $i \in \{1, \cdots, d\}$, $D_t S_t^i$ is a $\tilde{P}$-mg (S^i = price of ith stock)

thm:

discounted stock process is a mg under $\tilde{\mathbb{P}}$

$\downarrow$

fundamental requirement of RNM

if $\tilde{\mathbb{P}}$ is a RNM and X_t = wealth at time t of self-fin portfolio

then $D_t X_t$ is a $\tilde{P}$-mg

proof.

1. say $d = 1$

$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$

$dD_t = -R_t D_t dt$

product rule, compute $d(D_t X_t)$

$d(D_t X_t) = \Delta_t D_t (dS_t - R_t S_t dt)$ (*)

compute $d(D_t S_t) = D_t dS_t + S_t dD_t + d\underbrace{[D,S]_t}_{=0}$

$= D_t dS_t - R_t D_t S_t dt$

$= D_t(dS_t - R_t S_t dt)$ (**)

use ** in *

$\Rightarrow d(D_t X_t) = \Delta_t \underbrace{d(D_t S_t)}_{\tilde{\mathbb{P}}\text{-mg}}$

integral wrt $D_t S_t$ which is a $\tilde{\mathbb{P}}$mg

$\Rightarrow D_t X_t = X_0 + \int_0^t \Delta_s d(D_s S_s)$

$D_t X_t$ = integral of Δ_t wrt the $\tilde{\mathbb{P}}$-mg $D_t S_t$ $\Rightarrow$ $D_t X_t$ is a $\tilde{\mathbb{P}}$-mg

thm. risk neutral pricing formula. Say $\tilde{\mathbb{P}}$ is a RNM and a security pays V_T at time T

($V_T \to \mathcal{F}_T$-mb RV)

if the security is replicable, then the AFP at time t = wealth of replicating portfolio at time t

$= \frac{1}{D_t}\tilde{E}[D_T V_T]$

risk neutral pricing : say $\tilde{P}$ is a RNM, security pays V_T at time T

(V_T is $\mathcal{F}_T$ mb) $\Rightarrow V_t = \frac{1}{D_t}\tilde{E}[V_T D_T]$

proof.

V_t = AFP of security at time t

X_t = wealth of R. portfolio at time t

} know $V_t = X_t$

previous theorem, $D_t X_t$ is a $\tilde{P}$ mg

$\Rightarrow D_t X_t = \tilde{E}_t[D_T X_T] = \tilde{E}_t[D_T V_T]$

$D_t V_t = \tilde{E}_t[D_T V_T] \Rightarrow V_t = \frac{1}{D_t}\tilde{E}[D_T V_T]$

proof.

under $\tilde{P}$, $D_t X_t$ is a $\tilde{P}$ mg

$\Rightarrow D_t X_t = \tilde{E}_t[D_T X_T]$

by defn, $V_t = X_t$ Y_t

$\Rightarrow V_t = \frac{1}{D_t}\tilde{E}_t[D_T V_T]$

find risk neutral measure using Girsanov Theorem :

now find risk neutral measures from Girsanov

$W \to$ BM

$\tilde{W}_t = \int_0^t b_s ds + W_t$: $d\tilde{W}_t = b_t dt + dW_t$

fix $T > 0$, define $Z_T = \exp(-\int_0^T b_s dW_s - \frac{1}{2}\int_0^T b_s^2 ds)$

thm (Cameron, Martin, Girsanov) :

if Z is a mg. then fix $T>0$ and define $\tilde{P}$ by $d\tilde{P} = Z_T dP$

under $\tilde{P}$, $\tilde{W}$ is a BM up to time T

W is a BM

define $d\tilde{W}_t = b_t dt + d\tilde{W}_t$ $(W_t = \underline{\int_0^t b_s ds} + \tilde{W}_t)$

fix $T > 0$, define the following :

we add some finite first var process to W_t

$Z_T = \exp(-\int_0^T b_t dW_t - \frac{1}{2}\int_0^T b_t^2 dt)$

Girsanov.

if Z_T is a mg, then fix $T>0$ and define $\tilde{P}$ by $d\tilde{P} = Z_T dP$

under $\tilde{P}$, $\tilde{W}$ is a BM

$Z_T = \exp(-\int_0^T b_t dW_t - \frac{1}{2}\int_0^T b_t^2 dt)$

Lecture 32 Notes

$b_t \to$ adapted process

$\tilde{W}_t = \int_0^t b_s ds + W_t$ $Z_T = \exp(-\int_0^T b_t dW_t - \frac{1}{2}\int_0^T b_t^2 dt)$

$Z_t = \exp(-\int_0^t b_s dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds)$

fix $T>0$. define $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$

if Z_t is a martingale up to time T, then $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$ up to time T

$\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$, $W_t = \int_0^t b_t dt + \tilde{W}_t$

d-dim version : $W \to$ d-dim BM d-dimensional Girsanov :

$b_t \to$ d-dim adapted process $b_t \to$ d-dim process

$\tilde{W}_t = \int_0^t b_s ds + W_t$ ($\Longleftrightarrow$ $\tilde{W}_t^i = \int_0^t b_s^i ds + W_t^i$) $\forall i$, $\tilde{W}_t^i = \int_0^t b_t^i dt + W_t^i$

$Z_t = \exp(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds)$ now $Z_T = \exp(-\int_0^T b_t \cdot dW_t - \frac{1}{2}\int_0^T |b_t|^2 dt)$

$(\int_0^t b_s \cdot dW_s = \sum_{i=1}^d \int_0^t b_s^i dW_s^i)$ where $\int_0^t b_s \cdot dW_s = \sum_{i=1}^d \int_0^t b_s^i dW_s^i$

theorem. if Z_t is a martingale up to time T, then $\tilde{W}$ is a d-dim BM under $\tilde{\mathbb{P}}$ up to time T

note. for $\tilde{\mathbb{P}}$ to be a prob measure, need $Z_T \geq 0$

for equivalence, need $Z_T > 0$. true because $Z_T = \exp(\cdots)$

also need $\mathbb{E}Z_T = 1$ need $Z_T > 0$

proof. Z-mg $\Rightarrow \mathbb{E}Z_T = Z_0 = \exp(0) = 1$

check Z_t is a mg to check that Z_t is a mg, we compute by Itô

compute dZ_t using Itô, $dZ_t = -Z_t b_t^1 dW_t^1 - Z_t b_t^2 dW_t^2 = -Z_t b_t \cdot dW_t$

may not have $\mathbb{E}\int_0^t Z_s |b_s|^2 ds < \infty$

in general, Z_t may not be a martingale. it is a local mg

assume Girsanov $\to$ construct RNM

model: 1 stock : $dS_t = \alpha_t S_t dW_t + \sigma_t S_t dW_t$
Bank : interest rate R_t

find RNM : want $\tilde{\mathbb{P}}$ s.t. ① $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ ② $D_t S_t$ is a $\tilde{\mathbb{P}}$-mg

$dD_t = -R_t D_t\,dt$

compute $d(D_t S_t)$: start by computing $d(D_t S_t)$

$d(D_t S_t) = D_t\,dS_t + S_t\,dD_t + d[S,D]_t$

$= D_t(\alpha_t S_t\,dW_t + \sigma_t S_t\,dW_t) - R_t S_t D_t\,dt + 0$

$d(D_t S_t) = D_t S_t(\alpha - R_t)\,dt + \sigma_t D_t S_t\,dW_t$

scratch :

$d\tilde{W} = b\,dt + dW$

$Z = \exp(\ \ )$

find $\tilde{\mathbb{P}}$ so that $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$

$d(D_t S_t) = \sigma_t D_t S_t\left(\left(\frac{\alpha_t - R_t}{\sigma_t}\right)dt + dW_t\right)$ *

let $\tilde{W}$ be defined by $d\tilde{W}_t = \frac{\alpha_t - R_t}{\sigma_t}\,dt + dW_t$: $\Theta_t = \frac{\alpha_t - R_t}{\sigma_t}$ market price of risk

define $Z_T = \exp\left(-\int_0^T \Theta_s\,dW_s - \frac{1}{2}\int_0^T \Theta_s^2\,ds\right)$

assume Z is a mg up to time T

define $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = Z_T\,d\mathbb{P}$: girsanov $\Rightarrow$ $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$

$\Rightarrow$ by *, $\underline{\underline{d(D_t S_t) = \sigma_t D_t S_t\,d\tilde{W}_t}}$ $\Rightarrow$ $D_t S_t$ is a $\tilde{\mathbb{P}}$ mg ✓

market price of risk BOI

GBM case :

$\alpha_t = \alpha$. $R_t = r$. $\sigma_t = \sigma$

$S \to \mathrm{GBM}(\alpha, \sigma)$ under $\mathbb{P}$

$d\tilde{\mathbb{P}} = Z_T\,d\mathbb{P}$, where $Z_T = \exp\left(-\int_0^T \Theta_s\,dW_s - \frac{1}{2}\int_0^T \Theta_s^2\,ds\right) = \exp\left(\Theta W_t - \frac{1}{2}\Theta^2 T\right)$

security pays $g(S_T)$ at time T

compute AFP at time $t < T$

we know

$V_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T g(S_T)]$ ** $\qquad d\tilde{W} = \Theta\,dt + dW_t$

compute ** using the fact that

under $\tilde{\mathbb{P}} \to S = \mathrm{GBM}(r, \sigma)$

Lecture 33 Notes

RNM $\rightarrow D_tS_t^i$ is a $\tilde{\mathbb{P}}$ mg

market : 1 stock , $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

1 bank , R_t interest rate

$\Theta_t = \frac{\alpha_t - R_t}{\sigma_t}$ (market price of risk)

found RNM to be :

$d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$ where $Z_T = \underline{\exp\left(-\int_0^T \Theta_t dW_t - \frac{1}{2}\int_0^T \Theta_t^2 dW_t\right)}$

found that choosing $b_t = \Theta_t$ gives us a $\tilde{\mathbb{P}}$ such that D_tS_t is a $\tilde{\mathbb{P}}$ mg, where $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

price securities :

if a security is replicable , RNP $\Rightarrow$ AFP $= \underline{V_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T V_T]}$

will compute $\tilde{\mathbb{E}}$ as follows : to compute $\tilde{\mathbb{E}}$:

1. constructed $\tilde{\mathbb{P}}$ by Girsanov so know $\underline{\tilde{W} \text{ is a BM under } \tilde{\mathbb{P}}}$

know $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$

where $d\tilde{W}_t = \Theta_t dt + dW_t$

now we can price securities

$V_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[V_T D_T]$

2. write stock in terms of $\tilde{W}$

$d\tilde{W}_t = \Theta_t dt + dW_t$

now write the stock in terms of $\tilde{W}_t$

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t \quad *$$
$$= \alpha_t S_t dt + \sigma_t S_t\left(d\tilde{W}_t - \left(\frac{\alpha_t - R_t}{\sigma_t}\right)dt\right)$$
$$= R_t S_t dt + \sigma_t S_t d\tilde{W}_t \Rightarrow \text{beautiful.} \quad ** \checkmark$$

under $\tilde{\mathbb{P}}$, changed $\alpha_t \rightarrow R_t$ and $W_t \rightarrow \tilde{W}_t$ in equation for *

under $\tilde{\mathbb{P}}$, $\alpha_t \rightarrow R_t$ and $W_t \rightarrow \tilde{W}_t$

GBM case :

$\alpha_t = \alpha$, $R_t = r$, $\sigma_t = \sigma$ (all nonrandom constants)

$dS_t = \alpha S_t dt + \sigma S_t dW_t$

$dS_t = r S_t dt + \sigma S_t d\tilde{W}_t$ $\Rightarrow S_t \sim$ GBM under $\tilde{\mathbb{P}}$ with parameters $\underline{r \text{ and } \sigma}$

consider security that pays $g(S_T)$. price this security

RNP $\Rightarrow V_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t(D_T g(S_T))$ $\quad [D_t = e^{-rt}, \tau = T - t]$

$= e^{-r\tau}\tilde{\mathbb{E}}_t(g(S_T))$ ***

under RNP : $V_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[V_T D_T]$

$= e^{-r\tau}\tilde{\mathbb{E}}_t[g(S_T)]$

GBM formula :

$S_t \rightarrow$ GBM(α, σ) : $S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$ under $\tilde{\mathbb{P}}$:

$S_T = S_t \exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)\right)$

under $\tilde{\mathbb{P}}$: $S_t \to GBM(r,\sigma)$: $S_t = S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma\tilde{W}_t\right)$

$$\Rightarrow S_T = \underbrace{S_t}_{\mathcal{F}_t\text{ mb}}\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\underbrace{\tilde{W}_T-\tilde{W}_t}_{\perp\!\!\!\perp\ \mathcal{F}_t})\right)$$

so independence lemma

we write S_T in such a way that it is in terms of a $\mathcal{F}_t$ mb var as well as an $\perp\!\!\!\perp\ \mathcal{F}_t$ var

from ***

N(0,1) and indep

$$V_t = e^{-r\tau}\tilde{E}_t\left[g\left(S_t\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{\tau}\,\frac{(\tilde{W}_T-\tilde{W}_t)}{\sqrt{\tau}}\right)\right)\right]$$

$$= e^{-r\tau}\int_{-\infty}^{\infty} g\left(S_t\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y\right)\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \quad ****$$

prop. compute AFP of European call options when interest rate $= r$ and stock $= GBM(\alpha,\sigma)$

sol. use ****

$$V_t = e^{-r\tau}\int_{-\infty}^{\infty}\left[S_t\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y\right)-K\right]^+\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

let $S_t = x$

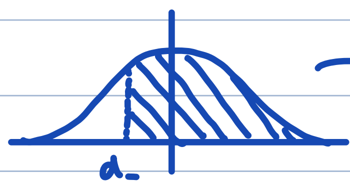
just do the same old, same old

$$x\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y\right)-K=0$$

$$\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y = \ln(K/x) \ ; \ y = -\frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau\right) = -d_-$$

$$\Rightarrow V_t = e^{-r\tau}\int_{-d_-}^{\infty}\left(S_t\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y\right)-K\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

$$= \int\_\_\_ - e^{-r\tau}K\underbrace{\int_{-d_-}^{\infty}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy}_{N(d_-)}$$



first term :

$$-\int_{-d_-}^{\infty} S_t e^{\left(\frac{-\sigma^2}{2}\tau+\sigma\sqrt{\tau}y-\frac{y^2}{2}\right)}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

$$-\int_{-d_-}^{\infty} S_t e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

$S_t N(d_+)$

$$dS_t = \alpha_t S_t dt + \sigma_t S_t \left[d\tilde{W}_t - \frac{\alpha_t - R_t}{\sigma_t} dt \right]$$

$$\left(d\tilde{W}_t = \Theta_t dt + dW_t \right)$$

$$= \alpha_t S_t dt + \sigma_t S_t d\tilde{W}_t - S_t (\alpha_t - R_t) dt$$

$$= R_t S_t dt + \sigma_t S_t d\tilde{W}_t$$

Lecture 34 Notes

RNP Formula: $V_t = \frac{1}{D_t}\tilde{E}_t[D_T V_T]$ (if security is replicable)

Girsanov: $d\tilde{P} = Z_T dP$, $Z_T = (\quad)$

$\hookrightarrow d\tilde{W} = \Theta dt + dW$ is a BM

if $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

then $dS_t = R_t S_t dt + \sigma_t S_t d\tilde{W}_t$

When a security is replicable:

thm (martingale rep thm) $W \to$ BM, $\mathcal{F}_t \to$ brownian filtration

$$\left.\begin{array}{l} M_t = M_0 + \int_0^t \sigma_s dW_s \\ \text{and } \mathbb{E}\int_0^t \sigma_s^2 ds < \infty \end{array}\right\} M \text{ is a mg and } \mathbb{E}M_t^2 = \mathbb{E}\int_0^t \sigma_s^2 ds < \infty$$

martingale representation theorem:

thm (MRT):

if M is a mg wrt $\mathcal{F}_t$ and $\underbrace{\mathbb{E}M_t^2 < \infty}_{\text{finite variance (bounded)}}$

if M is a mg wrt $\mathcal{F}_t$ and $\mathbb{E}M_t^2 < \infty$

then $\exists$ adapted process σ such that

then $\exists$ process σ_t such that $M_t = M_0 + \int_0^t \sigma_r dW_r$

$M_t = M_0 + \int_0^t \sigma_s dW_s$ and $\mathbb{E}\int_0^t \sigma_s^2 ds < \infty$

ie. $dM_t = \underbrace{\sigma_t dW_t}_{\text{that's wild.}}$

market $\begin{cases} \text{bank interest rate } R_t \\ 1 \text{ stock } S_t: dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t \end{cases}$

Say $\tilde{P}$ is a RNM. Know $(D_t S_t)$ is a $\tilde{P}$ mg

$\Rightarrow$ (have shown) X_t = wealth of self financing portfolio $\Rightarrow D_t X_t$ is $\tilde{P}$ mg

$\tilde{P}$ is a RNM. we know $D_t S_t$ and $D_t X_t$ are $\tilde{P}$ mgs

thm. 1

if X_t is any adapted process s.t. $D_t X_t$ is a $\tilde{P}$ mg

then X_t = wealth of a self financing portfolio $\searrow$ basically any X_t s.t. $D_t X_t$ is a $\tilde{P}$ mg is the wealth of a self replicating portfolio

thm 2.

If payoff of a security is $\mathcal{F}_T$ measurable (and matures at T) then the security is replicable

If the payoff of a security is $\mathcal{F}_T$ measurable $\Rightarrow$ the security is replicable

proof of thm 2.

let V_T = payoff of security at time T

If V_T is our payoff. we want a self replicating portfolio such that $X_T = V_T$

want a self financing portfolio (wealth X_t) with $X_T = V_T$

thm 1 $\Rightarrow$ only need to find X such that $D_t X_t$ is a $\tilde{\mathbb{P}}$ mg and $X_T = V_T$

scratch.

Simply need to find an X_t such that $X_T = V_T$ and $D_t X_t$ is a $\tilde{\mathbb{P}}$ mg

$D_t X_t$ a $\tilde{\mathbb{P}}$ mg $\Rightarrow D_t X_t = \tilde{\mathbb{E}}_t(D_T X_T) \Rightarrow X_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t(D_T X_T)$

choose $X_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T X_T] = \frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T V_T]$

need to show DX is $\tilde{\mathbb{P}}$ mg $\Leftrightarrow \tilde{\mathbb{E}}_s[D_t X_t] = D_s X_s$

we choose

$X_t = \frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T V_T]$

this is true because

$\tilde{\mathbb{E}}_s[D_t X_t] = \tilde{\mathbb{E}}_s[\tilde{\mathbb{E}}_t[D_T X_T]] = \tilde{\mathbb{E}}_s D_T V_T = D_s V_s$

show $D_t X_t$ is mg

$\mathbb{E}_s[D_t X_t] =$

$\mathbb{E}_s[\frac{1}{D_t}\tilde{\mathbb{E}}_t[D_T V_T]]$

proof of thm 1.

given $D_t X_t$ is a $\tilde{\mathbb{P}}$ mg. NTS X is self financing $\Leftrightarrow$

NTS $\exists \Delta_t$ st $dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$

$\Leftrightarrow dX_t - R_t X_t dt = \Delta_t(dS_t - R_t S_t dt)$ **

rearrange the self financing condition, then compute $D_t X_t$

compute $d(D_t X_t) = D_t dX_t + X_t dD_t + \underbrace{d[X, D]_t}_{=0}$ $\qquad (dD_t = -R_t D_t dt)$

compute $D_t X_t$

$= D_t dX_t - R_t D_t X_t dt = D_t(dX_t - R_t X_t dt)$

given $D_t X_t$ is a $\tilde{\mathbb{P}}$ mg, show that it is self financing

from ** NTF Δ_t so that

$\underbrace{D_t(dX_t - R_t X_t dt)}_{d(D_t X_t)} = \Delta_t D_t(dS_t - R_t S_t dt)$

$d(D_t X_t) = \Delta_t d(D_t S_t)$ ***

from self fin:

$dX_t - R_t X_t dt = \Delta_t(dS_t - R_t S_t dt)$

assumption $\Rightarrow DX$ is $\tilde{\mathbb{P}}$ mg

from $d(D_t X_t)$:

def of RNM $\Rightarrow DS$ is $\tilde{\mathbb{P}}$ mg

$d(D_t X_t) = D_t(dX_t - R_t X_t dt)$

$d(D_tX_t) = \Delta_t d(D_tS_t)$

MRT $\Rightarrow d(D_tX_t) = \gamma_t d\tilde{W}_t$

and $d(D_tS_t) = \delta_t d\tilde{W}_t$

$\Big\}$ by the MRT

Substitute in ***

$d(D_tX_t) = \Delta_t d(D_tS_t)$

$\gamma_t d\tilde{W}_t = \Delta_t \delta_t d\tilde{W}_t$

choose $\Delta_t = \gamma_t/\delta_t \Longrightarrow \gamma_t d\tilde{W}_t = \gamma_t/\cancel{\delta_t}\, \cancel{\delta_t} d\tilde{W}_t$ ✓

Substitute back and check that $dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$ QED

Therefore we have found a plausible Δ_t

multiple stocks, bank → interest rate R_t

m-stocks : $S^1, \cdots, S^m$

RNM.

$D_tS_t^i$ is a $\tilde{\mathbb{P}}$ mg $\forall i \in \{1, \cdots, m\}$

self financing :

1 stock → $dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt$

m-stocks → $dX_t = \sum_{i=1}^m \Delta_t^i dS_t^i + R_t(X_t - \sum_{i=1}^m \Delta_t^i S_t^i)dt$

Lecture 35 Notes

market → bank, interest rate R_t
→ m-stocks, prices $S^1, S^2, \cdots, S^m$

now we have m-stocks

last time → X_t = wealth of self financing portfolio

$dX_t = \Delta_t \cdot dS_t + R_t(X_t - \Delta_t \cdot S_t)\,dt$

the self financing condition now is

$dX_t = \Delta_t \cdot dS_t + R_t(X_t - \Delta_t \cdot S_t)\,dt$

Δ_t → $\mathbb{R}^m$ valued adapted process

Δ_t^i → # of shares in the ith stock held at time t

$\Delta_t \cdot dS_t = \sum_{i=1}^m \Delta_t^i S_t^i$; $\Delta_t \cdot S_t = \sum_{i=1}^m \Delta_t^i S_t^i$

where $\Delta_t \cdot dS_t = \sum_{i=1}^m \Delta_t^i dS_t^i$; $\Delta_t \cdot S_t = \sum_{i=1}^m \Delta_t^i S_t^i$

one model : $m=1$ → $dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t$

$m>1$: generalization

$$\sigma_t = \underbrace{\left(\sigma_t^{i,j}\right)}_{d} \Big]\, m$$

let W be a d-dim BM. σ_t = m × d dim matrix process

$$\sigma_t = \underbrace{\left(\sigma_t^{i,j}\right)}_{d} \Big]\, m$$

where $\sigma_t^{i,j}$ is an adapted process

then choose $\alpha_t = \begin{pmatrix} \alpha_t^1 \\ \vdots \\ \alpha_t^m \end{pmatrix}$

σ_t is a m×d matrix (m stocks)

then

$\alpha_t = \begin{pmatrix} \alpha_t^1 \\ \vdots \\ \alpha_t^m \end{pmatrix}$

stock prices evolve according to :

$dS_t^i = \alpha_t^i S_t^i\,dt + S_t^i \sum_{j=1}^d \sigma_t^{i,j}\,dW_t^j$ *

stock price now follows : $dS_t^i = \alpha_t^i S_t^i\,dt + S_t^i \sum_{j=1}^d \sigma_t^{i,j}\,dW_t^j$

goal : 1. find a RNM for this system

theorem.

exists a unique RNM iff exists unique solution to the following :

$\alpha - \vec{R} = \sigma\Theta$

consider a market with stock prices following *

there exists a unique RNM ⟺ the system of equations

$\alpha - \vec{R} = \sigma\Theta$ has a unique solution $(\Theta \in \mathbb{R}^d)$

$\vec{R} = \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} \in \mathbb{R}^m$ (R is interest rate) [Θ is the unknown]

where $R \in \mathbb{R}^m$

1D case : RNM, $d\tilde{W} = \Theta\,dt + dW_t$

girsanov, $Z_T = \exp(-\int_0^T \theta dW - \int_0^T \theta^2 dt)$, $d\tilde{P} = Z_T dP$

$\sigma\theta = \alpha - R$

want $\tilde{P}$ where S_t^i is a $\tilde{P}$ mg

pf. want $\tilde{P}$ where $D_t S_t^i$ is a $\tilde{P}$ mg $\forall i \in \{1, \cdots, m\}$

compute $d(D_t S_t^i) = D_t dS_t^i + S_t^i(-R_t D_t dt)$ always compute $d(D_t S_t^i)$

$= D_t(\alpha_t^i S_t^i dt + S_t^i \sum_j \sigma_t^{i,j} dW_j) - D_t R_t S_t^i dt$

$= \underline{D_t S_t^i((\alpha^i - R)dt + \sum_j \sigma_t^{i,j} dW^j)}$

now apply d-dim girsanov $Z_T = \exp(-\sum_j \int_0^T b^j dW^j - \frac{1}{2}\int_0^T |b_t|^2 dt)$

d-dim Girsanov

$d\tilde{W}_t^i = b_t^i dt + dW_t^i$; $Z_T = \exp(-\sum_j \int_0^T b^j dW^j - \frac{1}{2}\int_0^T |b_t|^2 dt)$

$d\tilde{P} = Z_T dP$ and $\tilde{W}$ is a BM under $\tilde{P}$ (up to time T)

assume unique solution

assumption. $\alpha - \vec{R} = \sigma\theta$ has a unique solution

$\iff \alpha_t^i - R_t = \sum_{j=1}^d \sigma_t^{i,j} \theta_t^j$ $\implies$ $\alpha_t^i - R_t = \sum_{j=1}^d \sigma_t^{i,j} \theta_t^j$

market price of risk system

$= D_t S_t^i (\sum_{j=1}^d \sigma_t^{i,j} \theta_t^j dt + \sum_{j=1}^d \sigma_t^{i,j} dW^j)$

$d(D_t S_t^i) = D_t S_t^i (\sum_{j=1}^d \sigma_t^{i,j} [\underline{\theta_t^j dt + dW_t^j}])$

can apply girsanov

apply Girsanov: $d\tilde{W}_t = \theta_t dt + dW$, $Z_T = \exp(-\int_0^T \theta dW - \int_0^T \theta^2 dt)$, $d\tilde{P} = Z_T dP$

know Girsanov $\implies$ $\tilde{W}$ is a BM under $\tilde{P}$ (up to time T) [if Z is a mg]

shows:

if MPR has a unique solution $\implies$ $\underline{d(D_t S_t^i) = D_t S_t^i \cdot \sum_{j=1}^d \sigma_t^{i,j} d\tilde{W}_t^j}$

$\implies$ $D_t S_t^i$ is a mg under $\tilde{P}$

unique solution implies $D_t S_t^i$ a $\tilde{P}$ mg

(also check converse) QED

fundamental theorem of asset pricing:

1. no arbitrage $\Longleftrightarrow$ existence of RNM

2. no arbitrage and completeness $\Longleftrightarrow$ existence and uniqueness of RNM

every security is replicable

Lecture 36 Notes

last time: m-stocks, $S^1 \cdots S^m$ interest rate R_t

$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma_t^{i,j} dW_t^j$ } $\sigma \to m \times d$ matrix

W is a d-dim BM $dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma_t^{i,j} dW_t^j$

σ is an $m \times d$ matrix

unique RNM $\iff$ $\alpha - \vec{R} = \sigma\Theta$ has unique sol

$\alpha - \vec{R}$ vector in $\mathbb{R}^m$ (known) $\alpha - \vec{R}$ is vector in $\mathbb{R}^m$

$\sigma \to m \times d$ matrix (known) MPR is a system of m equations and d unknowns

$\Theta \to$ vector in $\mathbb{R}^d$ (unknown)

Θ is vector in $\mathbb{R}^d$

usual situation:

$m > d \to$ no solutions (usually)

$m = d \to$ unique solution

$m < d \to$ infinitely many solutions

fundamental thm of asset pricing

existence of RNM $\iff$ no arbitrage

unique RNM $\iff$ complete + no arbitrage

$Ax = b \to x = A^{-1}b$

FTAP: existence of a RNM $\iff$ no arbitrage

existence and uniqueness of a RNM $\iff$ no arbitrage and completeness

last time: formula for RNM

claim: $dS_t^i = \underline{\alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma_t^{i,j} dW_t^j}$

under $\tilde{\mathbb{P}}$: $\tilde{W}$ is a BM

$dS_t^i = R_t S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma_t^{i,j} d\tilde{W}_t^j$

ex. $d=1, m=2$

$dS_t^1 = \alpha_t^1 S_t^1 dt + S_t^1 \sigma^1 dW_t$

$dS_t^2 = \alpha_t^2 S_t^2 dt + S_t^2 \sigma^2 dW_t$

When is the market AF and complete

($\Longleftrightarrow$ When does MPR system have a unique solution)

MPR system:

solve the equation

$\begin{pmatrix} \alpha^1 - R \\ \alpha^2 - R \end{pmatrix} = \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \theta$ where $\theta \in \mathbb{R}^1$

$\Longleftrightarrow \alpha^1 - R = \sigma^1 \theta, \ \alpha^2 - R = \sigma^2 \theta$ unique when $\dfrac{\alpha^1 - R}{\sigma^1} = \dfrac{\alpha^2 - R}{\sigma^2}$ *

market is unique and AF $\Longleftrightarrow \dfrac{\alpha^1 - R}{\sigma^1} = \dfrac{\alpha^2 - R}{\sigma^2}$

Say α^i, R, σ^i are constant (non-random)

Suppose * does not hold: $\dfrac{\alpha^1 - R}{\sigma^1} \neq \dfrac{\alpha^2 - R}{\sigma^2}$ **

Know $\exists$ arbitrage $\rightarrow$ find arbitrage

Choose trading strategy that holds Δ^1 shares of first stock 1 and Δ^2 shares of stock 2

$dX_t = \sum \Delta_i dS_t^i + R_t (X_t - \sum \Delta_t^i S_t^i) dt$ use self financing

$\quad = \sum \Delta_i (\alpha_t^i S_t^i dt + S_t^i \sigma_t^i dW_t^i) + R_t (X_t - \sum \Delta_t^i S_t^i) dt$

$\quad = R_t X_t dt + \sum (\alpha^i - R) \Delta_t^i S_t^i dt + \sum \Delta_t^i S_t^i \sigma_t^i dW_t$

choose: $\Delta_t^1 = 1$

$\Delta_t^2 = -\dfrac{S^1 \sigma^1}{S^2 \sigma^2}$ $\qquad S^1 \sigma^1 + S^2 \sigma^2 \Delta^2 = 0$

$\Rightarrow dX_t = R_t X_t dt + (\alpha_t^1 - R) S_t^1 dt + (\alpha_t^2 - R) S_t^2 \Delta_t^2 dt + 0$

$\quad = R X_t dt + \left[(\alpha_t^1 - R) S_t^1 - \dfrac{\sigma_t^1 S_t^1}{\sigma_t^2 \cancel{S_t^2}} (\alpha_t^2 - R) \cancel{S_t^2} dt \right.$

$dX_t = R X_t dt + S_t^1 \sigma_t^1 \left[\dfrac{\alpha_t^1 - R}{\sigma_t^1} - \dfrac{\alpha_t^2 - R}{\sigma_t^2} \right] dt$ ✓

Lecture 37 Notes

dividend paying stocks :

model : no dividends , no bank

$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

with dividends :

$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt$

decreases the stock price by dividend rate

dividends : assume dividends are paid continuously in time

say dividends are paid at rate A_t

stock price : $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt$

stock price decreases by A_t

RNM → without dividends : discounted stock price is a $\tilde{\mathbb{P}}$-mg

→ with dividends : discounted wealth of self financing portfolio with dividends reinvested is a $\tilde{\mathbb{P}}$-mg

with dividends, under RNM we want the discounted wealth of self financing portfolio with dividends reinvested is a $\tilde{\mathbb{P}}$

price securities : same RNP formula

$$V_t = \frac{1}{D_t} \tilde{E}_t[D_T V_T]$$

price securities w/ same pricing

1. compute RNM

$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt + \Delta_t A_t S_t dt$

$X_t \rightarrow$ wealth of a self financing portfolio. Δ_t shares, rest cash

$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)dt + \Delta_t A_t S_t dt$

dividends adds $\Delta_t A_t S_t dt$ term

RNM : want to make $D_t X_t$ a $\tilde{\mathbb{P}}$mg

want to make $D_t X_t$ a $\tilde{\mathbb{P}}$mg

compute $d(D_t X_t)$: recall $dD_t = -R_t D_t dt$

$$d(D_t X_t) = D_t dX_t + X_t(-R_t D_t dt) + 0$$

$$= D_t(\Delta_t dS_t + \cancel{R_t X_t} - R_t\Delta_t S_t)dt + \Delta_t A_t S_t dt - \cancel{R_t X_t} dt)$$

$$= D_t(\Delta_t(\alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt) - R_t \Delta_t S_t dt + \Delta_t A_t S_t dt)$$

$$= D_t \Delta_t S_t((\alpha_t - R_t)dt + \sigma_t dW_t)$$

$$= \sigma_t D_t \Delta_t S_t\left(\frac{(\alpha_t - R_t)}{\sigma_t} dt + dW_t\right) \quad *$$

→ when computing $d(D_t S_t)$, the A_t dividend terms cancel

let $\Theta_t = \dfrac{\alpha_t - R_t}{\sigma_t}$ (market price of risk)

Same as without dividends

set $d\tilde{W}_t = \Theta_t dt + dW_t \Longrightarrow$ Girsanov : $Z_T = \exp\left(-\int_0^T \Theta_t dW_t - \frac{1}{2}\int_0^T \Theta_t^2 dt\right)$

$d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$

know $\tilde{W}_t$ is a BM under $\tilde{\mathbb{P}}$ up to T

by * :

$d(D_tX_t)$ is mg under $\tilde{\mathbb{P}}$

$d(D_tX_t) = \sigma_t D_t \Delta_t S_t d\tilde{W}_t \Longrightarrow D_tX_t$ is a mg under $\tilde{\mathbb{P}}$

compute dS_t in terms of $\tilde{W}_t$

$$\begin{aligned} dS_t &= \alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt \\ &= \alpha_t S_t dt + \sigma_t S_t \left(d\tilde{W}_t - \frac{\alpha_t - R_t}{\sigma_t} dt \right) - A_t S_t dt \\ &= \underline{(R_t - A_t) S_t dt} + \sigma_t S_t d\tilde{W}_t \end{aligned}$$

changes mean return rate

price of european option :

assume $R_t = r$, $\sigma_t = \sigma$, $A_t = a$

pays $(S_T - K)^+$ at time T. compute AFP at time $t \le T$

RNP :

$V_t = \frac{1}{D_t}\tilde{E}_t[D_T V_T] = e^{-r\tau}\tilde{E}_t[(S_T-K)^+]$ **

note under $\tilde{\mathbb{P}}$, S is a GBM$(r-a, \sigma)$

note: BS formula

interest rate is r' then AFP of european call is

$c(t,S_t) = e^{-r'\tau}\tilde{E}_t(S_T-K)^+$ with $S \to$ GBM(r', σ) (BS)

know $c(t,x) = N(d_+) - Ke^{-r'\tau}N(d_-)$

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r' \pm \frac{\sigma^2}{2}\right)\tau\right]$$

$$** \Longrightarrow V_t = e^{-a\tau} \underbrace{e^{-(r-a)\tau}\tilde{E}_t[(S_T-K)^+]}_{\text{use BS with int. rate } r-a}$$

$V_t = e^{-at} c(t, S_t)$ where $c(t,x)$ given by (BS) with $r' = r - a$

using *

$$d(D_t X_t) = \sigma_t D_t \Delta_t S_t \Big(\underbrace{\frac{(\alpha_t - R_t)}{\sigma_t}}_{\theta_t} dt + dW_t \Big)$$

we know :

$$d\tilde{W}_t = \theta_t dt + dW_t$$

$$d(D_t X_t) = \sigma_t D_t \Delta_t S_t \, d\tilde{W}_t$$

Lecture 38 Notes

Girsanov: (d-dim version)

$d\tilde{W}_t = b_t dt + dW_t$ ($W \to$ d-dim BM, $b \to \mathbb{R}^d$ adapted process)

$Z_T = \exp\left(-\int_0^T b_s \cdot dW_s - \frac{1}{2}\int_0^T |b_s|^2 ds\right)$

$\left(b_s \cdot dW_s = \sum_{i=1}^d b_s^i dW_s\right)$

$d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$ then if Z is a mg $\to$ $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$ (up to time T)

lemma 1:

$\tilde{\mathbb{E}}_s X_t = \frac{1}{Z_s} \mathbb{E}_s (X_t Z_t)$

$\tilde{\mathbb{E}} X_t = \mathbb{E} X_t Z_T = \mathbb{E}\mathbb{E}_t X_t Z_T = \mathbb{E} X_t \mathbb{E}_t Z_T = \mathbb{E} X_t Z_t$

lemma 2:

M is a $\tilde{\mathbb{P}}$ mg $\Longleftrightarrow$ ZM is a $\mathbb{P}$ mg

proof.

Say ZM is a $\mathbb{P}$ mg. NTS M is a $\tilde{\mathbb{P}}$ mg

$\Longleftrightarrow$ $\tilde{\mathbb{E}}_s M_t = M_s$

$\tilde{\mathbb{E}}_s M_t = \frac{1}{Z_s} \mathbb{E}_s (Z_t M_t) = \frac{1}{Z_s} Z_s M_s = M_s$

other direction is the same

proof of Girsanov.

idea: NTS $\tilde{W}$ is a BM under $\tilde{\mathbb{P}}$

use Levy: 1. $\tilde{W}$ is a cts mg 2. $d[\tilde{W}^i, \tilde{W}^j]_t = \begin{cases} 0\,dt, & i \neq j \\ 1\,dt, & i = j \end{cases}$

check 2.

since dt terms don't change the joint QV $d[\tilde{W}^i, \tilde{W}^j]_t = d[W^i, W^j]_t = \begin{cases} 0\,dt, & i \neq j \\ 1\,dt, & i = j \end{cases}$

check 1.

use lemma 2, and check $(Z\tilde{W})$ is a $\mathbb{P}$ mg

compute $d(Z\tilde{W}) = 0\,dt + (\quad)\,dW$

compute dZ_t:

let $X^i = \int_0^t b_s^i dW^i$

$f(t,x) = \exp(-\sum_{i=1}^d x_i - \frac{1}{2}\int_0^t |b_s|^2 ds)$

$Z_t = f(t,X)$

$\partial_t f = \exp(\cdots)(-\frac{1}{2}|b_t|^2)$

$\partial_j f = \partial f/\partial x_j = \exp(\cdots)(-1)$

$\partial_j \partial_k f = \exp(\cdots)(-1)(-1) = \exp(\cdots)$

$d[X^i, X^j]_t = b_t^i b_t^j d[W^i, W^j]_t$

$$= \begin{cases} (b_t^i)^2 & i=j \\ 0 & i\neq j \end{cases}$$

$\Rightarrow dZ_t = \partial_t f dt + \sum_{i=1}^d \partial_i f dX_t^i + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \partial_i \partial_j f \, d[X^i, X^j]_t$

$= Z_t(-\frac{1}{2}|b_t|^2)dt - \sum_{i=1}^d Z_t b_t^i dW_t^i + \frac{1}{2}\sum_{i=1}^d Z_t (b_t^i)^2 dt$

$\Rightarrow dZ_t = -Z_t \sum_{i=1}^d b_t^i dW_t^i = -Z_t b_t \cdot dW_t$

$d(Z_t \tilde{W}_t^i) = Z_t d\tilde{W}_t^i + \tilde{W}_t^i dZ_t + d[Z, \tilde{W}^i]_t$

$d\tilde{W}_t^i = b_t^i dt + dW_t^i$

$d[Z, \tilde{W}]_t = -Z_t b_t^i d[W^i, W^i]_t = -Z_t b_t^i dt$

thus

$d(Z_t \tilde{W}_t^i) = Z_t(\cancel{b_t^i dt} + dW_t^i) + \tilde{W}_t^i(-Z_t b_t \cdot dW_t) - \cancel{Z_t b_t^i dt}$

$= Z_t dW_t^i + \tilde{W}_t^i(-Z_t b_t \cdot dW_t)$

$\Rightarrow Z\tilde{W}^i$ is a $\mathbb{P}$ mg

lemma 2 $\Rightarrow \tilde{W}^i$ is a $\tilde{\mathbb{P}}$ mg

$\Rightarrow$ cond ① in Levy $\Rightarrow \tilde{W}$ is a BM