Picard Method for Enclosing ODEs with Uncertain Initial Values

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We present an algorithm for enclosing solutions of ordinary differential equation (ODE) initial value problems (IVPs) with uncertain initial values:

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{y}(0) \in \boldsymbol{A} \tag{1}$$

where $\boldsymbol{A} \in \mathbb{I}^n$ is a vector of intervals, seen as a box-shaped subset of $\mathbb{R}^n$ and $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz vector field and a solution $\boldsymbol{y}$ has the type $[0, b] \to \mathbb{R}^n$.

Unlike other methods for enclosing solutions of ODE IVPs with uncertain initial values [3, 2, 4], our algorithm does not assume that the field is differentiable. Also, as with Edalat and Pattinson's method, our algorithm is simple and the proof of its soundness and convergence is closely related to the usual proof of the Picard-Lindelöf theorem.

If $\boldsymbol{A} = \boldsymbol{a}$ is a singleton interval, the equation has a unique solution, which we denote $\boldsymbol{y}_{\boldsymbol{a}}$. In this case, we utilize the interval Picard operator introduced by Edalat and Pattinson [1]:

$$\begin{aligned} P_{\boldsymbol{F},\boldsymbol{a}} &: ([0, b] \to \mathbb{I}^n) \to ([0, b] \to \mathbb{I}^n) \\ P_{\boldsymbol{F},\boldsymbol{a}}(\boldsymbol{Y})(t) &= \boldsymbol{a} + \int_0^t \boldsymbol{F}(\boldsymbol{Y}(s))\,\mathrm{d}s \end{aligned} \tag{2}$$

where $\boldsymbol{Y}\colon [0, b] \to \mathbb{I}^n$ is an interval function and $\boldsymbol{F}\colon ([0, b] \to \mathbb{I}^n) \to ([0, b] \to \mathbb{I}^n)$ is a interval function extension of the vector field $\boldsymbol{f}$.

Theorem 1 (Edalat & Pattinson 2007 [1], interval Picard theorem).
If $\boldsymbol{F}$ is a Lipschitz interval extension of the field $\boldsymbol{f}$ in equation (1), $P_{\boldsymbol{F},\boldsymbol{a}}$ is inclusion isotone and $P_{\boldsymbol{F},\boldsymbol{a}}(\boldsymbol{Y}_0) \subseteq \boldsymbol{Y}_0$, then

$$\boldsymbol{Y}_0 \supseteq P_{\boldsymbol{F},\boldsymbol{a}}(\boldsymbol{Y}_0) \supseteq P^2_{\boldsymbol{F},\boldsymbol{a}}(\boldsymbol{Y}_0) \supseteq \ldots \qquad \textit{and} \qquad \bigcap_{i=0}^{\infty} P^i_{\boldsymbol{F},\boldsymbol{a}}(\boldsymbol{Y}_0) = \boldsymbol{y}_{\boldsymbol{a}}.$$

We demonstrate on a simple example that this theorem cannot be generalized to uncertain initial value $\boldsymbol{A}$ by simply replacing $\boldsymbol{a}$ with $\boldsymbol{A}$. We instead parametrize this operator so that it can be applied on IVPs without the need to exponentially bisect the space of initial value uncertainty:

$$\begin{aligned} P_{\boldsymbol{F}} &: (\boldsymbol{A} \times [0, b] \to \mathbb{I}^n) \to (\boldsymbol{A} \times [0, b] \to \mathbb{I}^n) \\ P_{\boldsymbol{F}}(\boldsymbol{Y})(\boldsymbol{a}, t) &= \boldsymbol{a} + \int_0^t \boldsymbol{F}(\boldsymbol{Y}(\boldsymbol{a}, s))\,\mathrm{d}s \end{aligned} \tag{3}$$

Theorem 2 (Interval Picard theorem for flows).
If $P_{\boldsymbol{F}}(\boldsymbol{Y}_0) \subseteq \boldsymbol{Y}_0$, then

$$\boldsymbol{Y}_0 \supseteq P_{\boldsymbol{F}}(\boldsymbol{Y}_0) \supseteq P^2_{\boldsymbol{F}}(\boldsymbol{Y}_0) \supseteq \ldots \qquad \textit{and} \qquad \bigcap_{i=0}^{\infty} P^i_{\boldsymbol{F}}(\boldsymbol{Y}_0) = \boldsymbol{y}_{\boldsymbol{A}}.$$

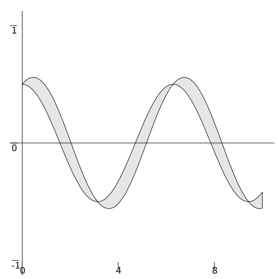
where $\boldsymbol{y}_{\boldsymbol{A}}$ is the flow of the ODE in (1), defined by $\boldsymbol{y}_{\boldsymbol{A}}(\boldsymbol{a}, t) = \boldsymbol{y}_{\boldsymbol{a}}(t)$ for all $\boldsymbol{a} \in \boldsymbol{A}$.

Switching from $P_{F,A}$ to the parametrized operator P_F is beneficial not only for solving IVPs with uncertain initial value, but also for piece-wise solving IVPs with exact initial value. This is because the initial value for each new step has a small level of uncertainty resulting from the imprecision of the enclosures for the preceding time step.

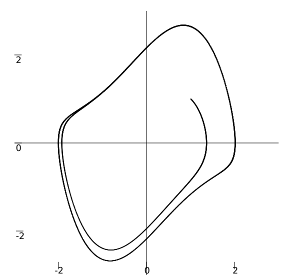
We propose an ODE solving algorithm **enclose-flow** based on a direct implementation of the parametrized interval Picard operator P_F. This algorithm relies on *function interval arithmetic*. A function interval is a pair of functions of type $D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. A function interval can be also seen as a function of type $D \to \mathbb{I}$, which appears in equations (2) and (3). A function interval arithmetic makes function intervals first-class objects and provides operations such as point-wise addition or integration, which facilitates computing the operator P_F.

We do not discuss the implementation of function interval arithmetic but formulate a set of requirements for the arithmetic. These requirements imply the safety of **enclose-flow**. By extending the requirements we obtain the concept of *exact function interval arithmetic*. We prove that with an exact arithmetic, **enclose-flow** can produce enclosures that converge to the exact flow y_A.

We demonstrate the feasibility of the approach using one of our implementations of function interval arithmetic. (Available freely from github.org/michalkonecny/aern) In particular, we produce very tight enclosures for a spring mass system with initial value uncertainty, an ODE featuring an absolute value, and a version of the Van der Pol system. Moreover, a version of our method, using a different function interval arithmetic, is included in the Acumen tool and language for modeling and rigorous simulation of hybrid dynamical systems. (Available freely from www.acumen-language.org)



Spring mass with uncertain initial speed



Van der Pol system

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