Basics of Linear Algebra: Vectors, Matrices, and More

1 Vectors

Definition 1 (Vector)**.** A vector is an ordered list of numbers (components) representing magnitude and direction in a space. In $\mathbb{R}^n$, a vector $\mathbf{v}$ is written as $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

1.1 Vector Operations

- **Addition**: For vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the sum is $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.
- **Scalar Multiplication**: For a scalar c and vector $\mathbf{v}$, $c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$.
- **Dot Product**: For vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the dot product is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

Example 1. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $c = 2$. Compute:

1. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2+1 \\ 3+(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
2. $c\mathbf{u} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.
3. $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 3 \cdot (-1) = 2 - 3 = -1$.

2 Matrices

Definition 2 (Matrix). A matrix is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix has m rows and n columns, denoted $A = [a_{ij}]$, where a_{ij} is the element in the i-th row and j-th column.

2.1 Matrix Operations

- **Addition**: For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, $A + B = [a_{ij} + b_{ij}]$.
- **Scalar Multiplication**: For a scalar c, $cA = [ca_{ij}]$.
- **Matrix Multiplication**: For an $m \times n$ matrix A and an $n \times p$ matrix B, the product AB is an $m \times p$ matrix with entries $(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.
- **Transpose**: The transpose of an $m \times n$ matrix A, denoted A^T, is an $n \times m$ matrix where $(A^T)_{ij} = a_{ji}$.

2.2 Matrix Properties

- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^T)^T = A$
- If A is invertible, $(A^{-1})^T = (A^T)^{-1}$.

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, and $c = 3$. Compute:

1. $A + B = \begin{bmatrix} 1+0 & 2+1 \\ 3+2 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.
2. $cA = 3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$.
3. $AB = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 2 & 1 \cdot 1 + 2 \cdot (-1) \\ 3 \cdot 0 + 4 \cdot 2 & 3 \cdot 1 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix}$.
4. $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

3 Determinants

Definition 3 (Determinant)**.** The determinant is a scalar value associated with a square matrix. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$.

3.1 Determinant Properties

- $\det(A^T) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- If A is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Scaling a row by c scales the determinant by c.
- If two rows are swapped, the determinant changes sign.

Example 3. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, compute $\det(A)$:

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2.$$

4 Eigenvalues and Eigenvectors

Definition 4 (Eigenvalues and Eigenvectors)**.** For a square matrix A, a scalar λ is an eigenvalue if there exists a non-zero vector $\mathbf{v}$ (eigenvector) such that $A\mathbf{v} = \lambda\mathbf{v}$.

4.1 Intuition

Eigenvectors represent directions that are only scaled (not rotated) by the matrix transformation. Eigenvalues indicate the scaling factor. They are crucial in applications like stability analysis, quantum mechanics, and data compression.

4.2 Computation

To find eigenvalues, solve the characteristic equation $\det(A - \lambda I) = 0$. For each eigenvalue λ, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvector $\mathbf{v}$.

Example 4. For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, find eigenvalues and eigenvectors:

1. Form $A - \lambda I = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$.
2. Compute the characteristic polynomial:

$$\det(A - \lambda I) = (3 - \lambda)(3 - \lambda) - 1 \cdot 1 = (3 - \lambda)^2 - 1 = \lambda^2 - 6\lambda + 8.$$

3. Solve $\lambda^2 - 6\lambda + 8 = 0 \implies (\lambda - 4)(\lambda - 2) = 0 \implies \lambda = 2, 4$.

4. For $\lambda = 2$:
$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Solving $x + y = 0$, we get $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

5. For $\lambda = 4$:
$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \implies \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Solving $-x + y = 0$, we get $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5 Practice Problems

Problem 1. Given $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, compute:

1. $\mathbf{u} + \mathbf{v}$
2. $2\mathbf{u} - 3\mathbf{v}$
3. $\mathbf{u} \cdot \mathbf{v}$

Problem 2. For matrices $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, compute:

1. $A + B$
2. AB
3. A^T

Problem 3. Compute the determinant of $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

Problem 4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.