Wavelets in p-adic Analysis

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Motivation:

In class, given a scaling function φ and its associated wavelet ψ, we constructed the incremental spaces W_j as

$$W_j = \operatorname{span}\{\psi(2^j t - k) \ : \ k \in \mathbb{Z}\}$$

Question:

What if we replace 2 with a prime number p?

This observation is intricately related to a number system called the "p-adic numbers".

The p-adic norm

Let p be a prime number,

- For any non-zero $x \in \mathbb{Q}$, x may be written as

$$x = p^n \frac{a}{b}, \quad a \text{ and } b \text{ are coprime to } p$$

  We define the p-adic norm $|\bullet|_p : \mathbb{Q} \to \mathbb{R}$ as

$$|x|_p = \begin{cases} p^{-n}, \text{ if } x = p^n \frac{a}{b} \\ 0, \text{ if } x = 0 \end{cases}$$

- For example, when $p = 2$,

$$|3 - 1|_2 = \frac{1}{2} \text{ and } |1025 - 1|_2 = \frac{1}{2^{10}}$$

- In the 2-adic norm, 1 and 3 are very far away, but 1 and 1025 are really close.

The p-adic numbers

We define the p-adic numbers $\mathbb{Q}_p$ as the metric completion of $\mathbb{Q}$ with respect to $|\bullet|_p$. $\mathbb{Q}_p$ also extends $\mathbb{Q}$ as a field.

Although $\mathbb{R}$ and $\mathbb{Q}_p$ are both constructed from $\mathbb{Q}$, they are very different! For example,

- Every point in $\mathbb{Q}_p$ is disconnected from one another.
- The integers are actually bounded in $\mathbb{Q}_p$.

Why p-adic analysis?

Q: Why should we care about p-adic analysis?

All experimental and observational data we collect are actually rational numbers. Sometimes $\mathbb{Q}_p$ may be an appropriate extension than $\mathbb{R}$!

- In particle physics, there's a fundamental restriction on how small we can measure an object. This is known as the Planck Length.

  This discrete nature suggests that $\mathbb{Q}_p$ may be more intuitive as a geometrical interpretation of space-time (see [Vol87])
- Besides physics, p-adic analysis has found applications in Biological Systems, Stochastic Processes, Data Mining, and more. (see [DKK+17])

Where do Wavelets come in?

- In p-adic analysis, the analog of real fractional differentiations is defined using what's called a "Vladimirov operator" ([Vla88]):
- For $f \in L^2(\mathbb{Q}_p)$ and $\alpha > 0$

$$D^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x - y|_p^{1+\alpha}} d\mu(y)$$

- Wouldn't it be great if we can diagonalize this operator?

Where do Wavelets come in?

- In 2002, Kozyrev pioneered the field of p-adic wavelets by showing that there's a correspondence with the Haar Wavelet decomposition of $L^2(\mathbb{R}_+)$ and eigenvectors of the Vladimirov operator in $L^2(\mathbb{Q}_p)$ [Koz02].
- Hence, the theory of wavelets can be readily applied to p-adic fractional differentiations.
- We will spend the rest of this presentation discussing this result.

Extending Haar Wavelets[1]

In class, the Haar Wavelet ψ^H is given as the difference of two characteristic functions:

$$\psi^H(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1]}(x)$$

The construction here uses 2 characteristic functions, and when we extended this into a decomposition of $L^2(\mathbb{R}_+)$, we were considering multiplications by 2.

It turns out that the original Haar Wavelet is really good at interpreting the Vladimirov operator in $\mathbb{Q}_2$, but for $\mathbb{Q}_p$ in general, we wish to extend this a "p-Haar wavelet".

[1] The rest of the slides are all based on Kozyrev's original paper.

Extending Haar Wavelets

We define the p-Haar wavelet basis of $L^2(\mathbb{R}_+)$ as follows:

- For $1 \leq k \leq p-1$, the p-Haar wavelet function $\psi_k^{(p)}(x)$ is given as

$$\psi_k^{(p)}(x) = \sum_{\ell=0}^{p-1} e^{\frac{2\pi i k \ell}{p}} \chi_{[\frac{\ell}{p}, \frac{\ell+1}{p}]}(x)$$

- Consider the incremental spaces W_j given by

$$W_j = \operatorname{span}\{\psi_k^{(p)}(p^{-j}x - n) : n \in \mathbb{Z}_+, k = 1, ..., p-1\}$$

Then we in fact have that

$$\overline{\bigoplus_{j \in \mathbb{Z}} W_j} = L^2(\mathbb{R}_+)$$

When $p = 2$, this is exactly the Haar wavelet decomposition.

Example: $p = 3$

For $k = 1$, we obtain the following graphs of $\psi_1^{(3)}(x)$:

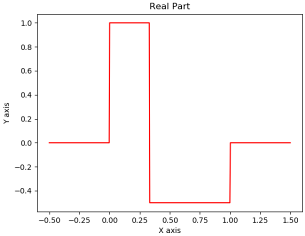

Figure: $\mathrm{Real}(\psi_1^{(3)})$

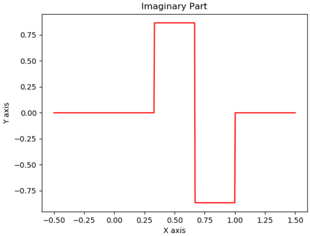

Figure: $\mathrm{Imag}(\psi_1^{(3)})$

Example: $p = 3$

For $k = 2$, we obtain the following graphs of $\psi_2^{(3)}(x)$:

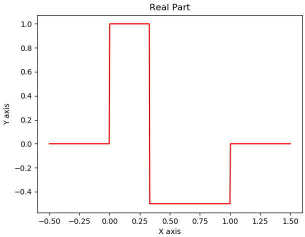

Figure: $\mathrm{Real}(\psi_2^{(3)})$

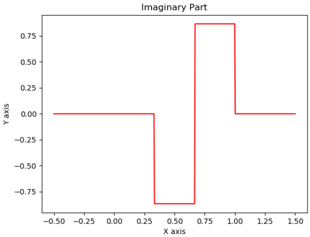

Figure: $\mathrm{Imag}(\psi_2^{(3)})$

The Monna map

Every non-zero element $x \in \mathbb{Q}_p$ may be represented uniquely as a "Laurent series" of the form

$$x = \sum_{i=k}^{\infty} a_i p^i, \quad a_i = 0, ..., p-1, k \in \mathbb{Z}$$

Definition:

Define $\rho : \mathbb{Q}_p \to \mathbb{R}_+$ as follows

$$\rho(\sum_{i=k}^{\infty} a_i p^i) = \sum_{i=k}^{\infty} a_i p^{-i-1}$$

This is called the Monna map or a p-adic change of variables.

We define the pullback of ρ as $\rho^* : L^2(\mathbb{R}_+) \to L^2(\mathbb{Q}_p)$ where

$$\rho^*(f) = f \circ \rho$$

Kozyrev's Main Result

Theorem [Kozyrev, [Koz02] and [KKS14]][2]

Let W_j be given by the p-Haar wavelets,

- $\{p^*(W_j)\}_{j \in \mathbb{Z}}$ forms an orthogonal decomposition of $L^2(\mathbb{Q}_p)$.
- The elements of this indicated basis are eigenvectors of the Vladimirov operator defined as

$$D^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x - y|_p^{1+\alpha}} d\mu(y)$$

such that

$$D^\alpha \rho^*(\psi_k^{(p)}(p^{-j}x - n)) = p^{\alpha(1-j)} \rho^*(\psi_k^{(p)}(p^{-j}x - n))$$

[2] The equalities here should be interpreted as an equality in L^2, as in it holds outside of a measure zero set

Kozyrev Wavelets

- This connection illustrates a link between the theory of wavelets and p-adic spectral analysis.
- The collection $\{p^*(W_j)\}_{j\in\mathbb{Z}}$ is thus aptly called the p-adic wavelet basis (or sometimes called the Kozyrev wavelet basis).

This only marked the beginning of the theory of p-adic wavelets:

- Generalizations to higher dimensions (ie. $\mathbb{Q}_p^d$) were later developed in [KS06].
- Methods of constructing other p-adic wavelets without using the Haar Wavelet were developed in [KS09] and [KS10].

Acknowledgements

The code used the generate the diagrams and my lecture notes for this course can be found on:

- `https://github.com/maroon-scorch/APMA1940Y-notes`

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