The Promise of Finite Element Exterior Calculus

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Outline

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An 1D ODE

Consider the following 1D ODE problem of the form

$$-u''(x) = f(x), -1 < x < 1 \text{ and } u(-1) = u(1) = 0,$$

where we are given the information $f(x)$.

How might we approach to numerically solving this question?

An 1D ODE - First Approach

From Calculuus I, we learned one approach of by considering a small time-step Δx and divide $[-1,1]$ into segments of length Δx. In this case, we can write

$$u''(x) \approx \frac{u(x+\Delta x) - 2u(x) - u(x-\Delta x)}{\Delta x^2}$$

By writing $[-1,1]$ into segments of connecting points $-1 = x_0, x_1, ..., x_n = 1$ and $u_i := u(x_i)$. We can rewrite the ODE as

$$f(x_i) = u''(x_i) \approx \frac{u_{i+1} - 2u_i - u_{i-1}}{\Delta x^2}.$$

An 1D ODE - First Approach

Consider the equation:

$$f(x_i) = \frac{u_{i+1} - 2u_i - u_{i-1}}{\Delta x^2}.$$

Rearranging the terms, we see

$$f(x_i)\Delta x^2 = u_{i+1} - 2u_i - u_{i-1}.$$

This can be thought of as a linear equation in the variables $u_0, ..., u_n$. Doing this for each i gives a rather sparse matrix A such that

$$A \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} f(x_0)\Delta x^2 = 0 \\ f(x_1)\Delta x^2 \\ \vdots \\ f(x_n)\Delta x^2 = 0 \end{pmatrix}$$

Solving the linear equation gives the answers. There are variants of this method too, all based on the idea of the approximation:

$$u''(x) \approx \frac{u(x+\Delta x) - 2u(x) - u(x-\Delta x)}{\Delta x^2}$$

Such methods are examples of finite difference methods.

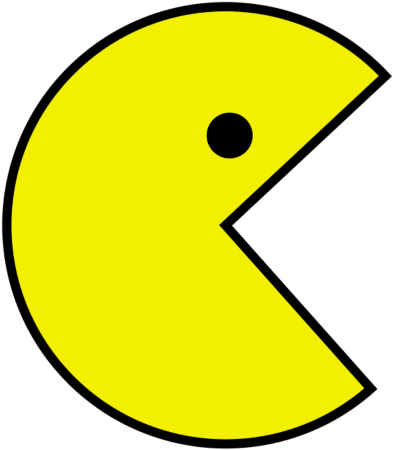
Historical Remarks

When I first took Numerical PDEs, this is what I was told:

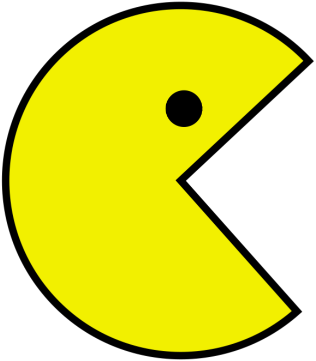
1. Finite difference methods came first. Up to about the 1940s or 1950s, the finite difference methods have been the standard.
2. The government also implemented a lot of code and machines strictly with the framework of finite differences methods.
3. But surely there are PDEs on higher dimensionald omains that were also needed to be solved, right?

Limitations

One issue with finite difference methods is that they are difficult to generalize over complicated domains. For example, consider the non-simply-connected Pac-Man domain:



Limitations



- Solving PDEs over complicated domains using finite differences usually involved setting up ad-hoc nodes on the domain.

A Non-Ad-Hoc Method?

1. In about 1970s, there are these methods called finite element methods (FEM) that came along.
2. The engineers were the first to realize that they are good and started using them.

Let us again consider the ODE:

$$-u''(x) = f(x), -1 < x < 1, u(-1) = u(1) = 0.$$

Question:

What are some solutions a computer can write that typically approximate this $u(x)$?

An 1D ODE - Second Approach

Question:

What are some solutions a computer can write that typically approximate this $u(x)$?

- For a computer, a typical choice would be piecewise polynomial functions drawn in disguise.
- Since u is C^2, but the approximate solution is not C^2, how do we know such function approximates u well enough?
- A typical way to measure this involves some kind of integral, and hence lead to the idea of a **weak formulation**.

Weak Formulation

Let $W_0^{1,2}(-1,1)$ be the space of L^2 functions on $[-1,1]$ whose first (weak) derivative is in L^2 and vanish on boundary.

Instead of solving for a strong solution, we seek a weak solution $u \in W_0^{1,2}(-1,1)$ in the sense that u satisfies

$$\int_{-1}^{1} u''(x)v(x)dx = \int_{-1}^{1} f(x)v(x)dx$$

for all $v(x) \in W_0^{1,2}(-1,1)$.

Note that by integration by parts, this is equivalent to

$$\int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} f(x)v(x)dx \quad \forall v \in W_0^{1,2}(-1,1).$$

Weak Formulation to Bilinear

$$\int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} f(x)v(x)dx \quad \forall v \in W_0^{1,2}(-1,1).$$

Bootstrapping some notations, we can define:

1. A Hilbert space $V = W_0^{1,2}(-1,1)$.
2. A bounded bilinear form $B : V \times V \to \mathbb{R}$ given by
$$B(u,v) := \int_{-1}^{1} u'(x)v'(x)dx.$$
3. A bounded linear form $F(v) : V \to \mathbb{R}$ given by
$$F(v) := \int_{-1}^{1} f(x)v(x)dx.$$

We can then formulate the equation as

$$B(u,v) = F(v) \quad \forall v \in V.$$

Trial Space

$$B(u, v) = F(v) \quad \forall v \in V.$$

1. We choose a finite-dimensional subspace $V_h \subseteq V$ (called the trial space) and limit out problem to solving:

$$B(u_h, v) = F(v) \quad \forall v \in V_h.$$

2. This now becomes a matrix algebra question. Let $\{v_i\}$ be a basis of V_h, we can write

$$B_{ij} = B(v_i, v_j) \text{ and } F_i = F(v_i).$$

The question now becomes solving

$$Bu_h = F.$$

Trial Space

For our specific ODE, one possible choice of V_h is to fix a list $-1 = x_0 < x_1 < ... < x_n = 1$, and define

$$V_h = \{v \in V \mid v|_{[x_i, x_{i+1}]} \text{ is affine}\}.$$

Generalization

This approach admits a generalization in the following sense. Consider the Poisson equation problem.

$$-\Delta u = f \text{ in } \Omega \subseteq \mathbb{R}^n, u \equiv 0 \text{ on } \partial\Omega.$$

In this case, we have

1. V is L^2-functions on Ω, zero on boundary, such that their first (weak) derivative is L^2.
2. $B : V \times V \to \mathbb{R}$ is

$$B(u, v) := \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v dx$$

3. $F : V \to \mathbb{R}$ defined by

$$F(v) := \int_\Omega f(x)v(x)dx.$$

Generalization

In this case, we can once again rewrite the problem as

$$B(u, v) = F(v), v \in V.$$

Fix a triangulation of Ω, the trial space V_h can be taken to be the subspace of V whose restriction to each n-simplex is polynomial, with some limitation on the degree.

From here, we can accordingly solve

$$B(u_h, v) = F(v), v \in V_h.$$

Outline

Maxwell's Equations

1. Gauss' Law: $\nabla \cdot E = \frac{\rho}{\epsilon_0}$
2. Magnetic Monopoles: $\nabla \cdot B = 0$
3. Faraday's Law: $\nabla \times E = -\frac{\partial B}{\partial t}$
4. Ampere-Maxwell Law: $\nabla \times H = J + \frac{\partial D}{\partial t}$

- One issue with classical FEM theory is that, although their numerical scheme does converge, it often does not converge to physical solutions when studying PDEs from physics!
- One such hard case is the Maxwell's equations.

Maxwell's Equations

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- Observe that many terms on here can be rewritten in the language of differential forms (ie. $\cdot$, $\times$).

A More Fundamental Question

There is a even more fundamental question here.

Question:

Are there connections between differential forms and PDEs?
Can we develop a suitable FEM theory for differential forms?

Let us first look at the case in 3D, where there is a nice interpretation with vector calculus.

Define the notations,

- $H(\mathrm{grad}) = \{u \in L^2(\Omega) \mid \mathrm{grad}(u) \in [L^2(\Omega)]^3\}$.
- $H(\mathrm{curl}) = \{\vec{u} \in [L^2(\Omega)]^3 \mid \mathrm{curl}\,\vec{u} \in [L^2(\Omega)]^3\}$
- $H(\mathrm{div}) = \{\vec{u} \in [L^2(\Omega)]^3 : \mathrm{div}\, u \in L^2(\Omega)\}$.

In this case we have a sequence of maps

$$0 \to H(\mathrm{grad}) \xrightarrow{grad} H(\mathrm{curl}) \xrightarrow{curl} H(\mathrm{div}) \xrightarrow{div} L^2(\Omega) \to 0.$$

Question

Is this a chain complex?

$$0 \to H(\text{grad}) \xrightarrow{grad} H(\text{curl}) \xrightarrow{curl} H(\text{div}) \xrightarrow{div} L^2(\Omega) \to 0.$$

1. Over the smooth world, we know from MATH 6000 that this certainly is true.
2. Over the L^2 world, one can still check this is a chain complex.

The Hodge Laplacian

For the ease of notation, we replace the diagram

$$0 \to H(\mathrm{grad}) \xrightarrow{grad} H(\mathrm{curl}) \xrightarrow{curl} H(\mathrm{div}) \xrightarrow{div} L^2(\Omega) \to 0.$$

with the diagram

$$0 \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} 0.$$

Observe that we also have a dual[1] complex of the form

$$0 \xleftarrow{d_0^*} V_0^* \xleftarrow{d_1^* = -\mathrm{div}} V_1^* \xleftarrow{d_2^* = \mathrm{curl}} V_2^* \xleftarrow{d_3^* = -grad} V_3^* \xleftarrow{d_4^*} 0,$$

where $V_1^* = H_0(\mathrm{div}), V_2^* = H_0(\mathrm{curl}), V_3^* = H_0(\mathrm{grad})$ (vanish on boundary in the appropriate sense).

[1] We have an honest dualization here because V_i is a Hilbert space

The Hodge Laplacian

$$0 \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} 0.$$
$$0 \xleftarrow{d_0^*} V_0^* \xleftarrow{d_1^* = -\operatorname{div}} V_1^* \xleftarrow{d_2^* = \operatorname{curl}} V_2^* \xleftarrow{d_3^* = -grad} V_3^* \xleftarrow{d_4^*} 0$$

Definition:

We define the k-th Hodge Laplacian as

$$L_k := d^{k-1} d_k^* + d_{k+1}^* d_k.$$

Note that domain of L_k is

$$D_k = \{u \in V^k \cap V_k^* : d^k u \in V_{k+1}^* \text{ and } d_k^* u \in V^{k-1}\}.$$

For $k = 0$, we hae that

$$L_0 = d^{-1}d_0^* + d_1^* d^0 = 0 - \operatorname{div}\operatorname{grad} = -\Delta.$$

$$D(L_0) = \{u \in H(\mathrm{grad}) \mid \mathrm{grad}(u) \in H_0(\mathrm{div}).$$

Suppose we want to solve

$$L_0 u = f.$$

This is the same as solving $-\Delta u = f$. Since $\mathrm{grad}(u) \in \mathrm{H}_0(\mathrm{div})$, this also imposes a boundary condition that

$$\operatorname{grad} u \cdot n = 0.$$

This is the **Neumann boundary condition**!

Examples of the Hodge Laplacian

Doing the same kind of exercises,

1. For $k = 3$, $L_3 u = f$ is the Poisson problem with Dirichlet boundary condition.
2. For $k = 1$, $L_1 u = f$ is the problem

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f$$

   with boundary conditions $u \cdot \eta = 0$ and $\operatorname{curl} u \times \eta = 0$.

3. ...

Thus, we see that many interesting PDE problems may be reformulated using the differential form interpretation above.

FEM on the Hodge Laplacian

Suppose we want to solve $L_k u = f$, how would we approach this? Our first step is to turn this into a weak formulation, ie:

$$\langle L_k u, v \rangle = \langle f, v \rangle$$

for all $v \in V^k \cap V_k^*$. Expanding out the definition of L_k and applying adjoint, we have

$$\begin{aligned} \langle L_k u, v \rangle &= \langle d^{k-1} d_k^* u, v \rangle + \langle d_{k+1}^* d^k u, v \rangle \\ &= \langle d_k^* u, d_k^* v \rangle + \langle d^k u, d^k v \rangle. \end{aligned}$$

Now we are finding $u \in V^k \cap V_k^*$

$$\langle d_k^* u, d_k^* v \rangle + \langle d^k u, d^k v \rangle = \langle f, v \rangle$$

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$$\langle d_k^* u, d_k^* v \rangle + \langle d^k u, d^k v \rangle = \langle f, v \rangle$$

Question:

If $f = 0$, is the choice of u unique?

Uniqueness

Let $k = 2$, and $u = v$ the equation here becomes

$$\langle \operatorname{curl}(u), \operatorname{curl}(u) \rangle + \langle \operatorname{div}(u), \operatorname{div}(u) \rangle = 0$$

Suppose $\Omega = B_1(0) - B_{1/2}(0)$. Define m such that

$$-\Delta m = 0 \text{ in } \Omega, m = \begin{cases} 0, & \text{on } \partial B_1(0) \\ 1, & \text{on } \partial B_{1/2}(0) \end{cases}.$$

Define $u = \operatorname{grad} m$. In this case, $\operatorname{div}(u) = \Delta m = 0$ and curl of gradient is zero. On the other hand, clearly u is non-zero. This gives a non-zero solution and is emblematic of the fact that $H^2(\Omega; \mathbb{R}) = \mathbb{R}$.

This means we need to adapt our FEM method to pay attention to the topology!

Definition:

The k-th harmonic forms[2] are

$$\mathcal{H}_k := \{p \in V^k \cap V_k^* : d^k p = 0 \text{ and } d_k^* p = 0\}.$$

We now seek to solve $u \in V^k \cap V_k^*$ such that $u \perp \mathcal{H}_k$ and

$$\langle d_k^* u, d_k^* v\rangle + \langle d^k u, d^k v\rangle = \langle f, v\rangle$$

[2] The curious reader should note $\mathcal{H}_{||}$ is isomorphic to $H^k(\Omega; \mathbb{R})$

Reformulation of the Question - I

The question above is equivalent to the following - Find $u \in V^k \cap V_k^*$ and $p \in \mathcal{H}_k$ such that

$$\langle d_k^* u, d_k^* v \rangle + \langle d^k u, d^k v \rangle + \langle p, v \rangle = \langle f, v \rangle,$$

$$\langle u, q \rangle = 0,$$

for any $v \in V^k \cap V_k^*$ and $q \in \mathcal{H}_k$.

Reformulation of the Question - II

The question above is equivalent to the following - Find $\sigma \in V^{k-1}, u \in V^k$ and $p \in \mathcal{H}_k$ such that

$$\langle \sigma, \tau \rangle - \langle u, d^{k-1}\tau \rangle = 0.$$

$$\langle d_k^* u, d_k^* v \rangle + \langle d^k u, d^k v \rangle + \langle p, v \rangle = \langle f, v \rangle,$$

$$\langle u, q \rangle = 0,$$

for any $\tau \in V^{k-1}$, any $v \in V^k$ and $q \in \mathcal{H}_k$. We are sort of "delooping" the dual spaces here.

Reformulation of the Question - III

The previous formulation admits a bilinear interpretation as

$$B((\sigma,u,p),(\tau,v,q)) = \langle \sigma,\tau\rangle - \langle u, d^{k-1}\tau\rangle + \langle d^{k-1}\sigma, v\rangle$$

$$+\langle d^k u, d^k v\rangle + \langle p, v\rangle - \langle u, q\rangle.$$

Thus, we have rephrased this to:

$$B((\sigma,u,p),(\tau,v,q)) = \langle f, v\rangle, \forall (\tau,v,q)\in\mathcal{V}.$$

What About the Trial Spaces?

We want the trial spaces to preserve the cohomology. Fix a triangulation τ_3 of Ω. For each 3-simplex $K \in \tau_3$, we first define the trial spaces on them.

- $V_h^0(K) = \{a + b \cdot x : a \in \mathbb{R}, b \in \mathbb{R}^3\}$ as the collection of affine functions. The dimension of $V_h^0(K)$ is 4 in this case.
- $V_h^1(K) = \{a + b \times x : a \in \mathbb{R}^3, b \in \mathbb{R}^3\}$. The dimension $V_h^1(K)$ is 6 in this case.
- $V_h^2(K) = \{a + bx : a \in \mathbb{R}^3, b \in \mathbb{R}\}$ (x is a scalar). The dimension of $V_h^2(K)$ is 4 in this case.
- $V_h^3(K) = \{a : a \in \mathbb{R}\}$. The dimension of $V_h^3(K)$ is 1 in this case.

Observe that the chain complex restricts to:

$$0 \longrightarrow V_h^0(K) \xrightarrow[d^0=grad]{} V_h^1(K) \xrightarrow[d^1=curl]{} V_h^2(K) \xrightarrow[d^2=div]{} V_h^3(K) \longrightarrow 0$$

What About the Trial Spaces?

From here we define the k-th trial space for τ_3 as

$$V_h^k := \{v \in V^k : v|_T \in V_h^k(T) \text{ for all } T \in \tau_3\}.$$

In this case we get natural chain complex:

$$0 \longrightarrow V_h^0 \xrightarrow[d^0=grad]{} V_h^1 \xrightarrow[d^1=curl]{} V_h^2 \xrightarrow[d^2=div]{} V_h^3 \longrightarrow 0$$

What About the Trial Space

Since our B is technically on the space

$$V = V^k \times V^{k-1} \times \mathcal{H}_k,$$

our trial space is then $V_h^k \times V_h^{k-1} \times \mathcal{H}_k$ (we are on a nice domain).

The Unifying Picture

Let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω:

$$0 \longrightarrow V_h^0 \longrightarrow V_h^1 \longrightarrow V_h^2 \longrightarrow V_h^3 \longrightarrow 0$$

$$0 \longrightarrow V^0 \longrightarrow V^1 \longrightarrow V^2 \longrightarrow V^3 \longrightarrow 0$$

$$0 \longrightarrow \Lambda^0(\Omega) \longrightarrow \Lambda^1(\Omega) \longrightarrow \Lambda^2(\Omega) \longrightarrow \Lambda^3(\Omega) \longrightarrow 0$$

Theorem:

The cohomology of the three columns are all isomorphic, via canonical maps between their chain complexes.

Outline

For General Domain

In general, for a good domain $\Omega \subseteq \mathbb{R}^n$, what we said above extends in a fairly straight-forward fashion, with the observations that.

1. The Hodge star operator $\star$ gives the L_2-norm on $\Lambda^k(\Omega)$ by

$$\langle \omega, \nu \rangle := \int_M \omega \wedge \star \nu$$

   and gives the definition of $L^2\Lambda^k$.
2. We again define $H\Lambda^k = \{\omega \in L^2\Lambda^k \mid d\omega \in L^2\Lambda^{k+1}\}$.
3. The chain complex is dualizable and admits the definition of Hodge Laplacian
4. The trial space is taken similarly.

Even More Generalizations

So far, we have only been discussing FEEC on L^2-de-Rham cohomologies. However, the original papers of FEEC actually did their theory on the generality of Hilbert complexes!!

Definition:

A Hilbert complex is a sequence

$$W^0 \xrightarrow{d^0} W^1 \xrightarrow{d^1} \dots$$

where W^k's are Hilbert spaces, d^k's are densely-defined closed linear operators such that $\mathrm{range}\, d^k \subseteq \mathrm{domain}\, d^{k+1}$ and $d^{k+1} \circ d^k = 0$ for all k.

Using other Hilbert complexes lead to other exciting applications in numerical PDEs.

Big Application of FEEC

What we talked about today is not just toy math either, it has been used to obtain significant results in the field.

Mass Conserving Mixed hp-FEM Approximations to Stokes Flow. Part I: Uniform Stability

Authors: **Mark Ainsworth** and **Charles Parker** | AUTHORS INFO & AFFILIATIONS

https://doi.org/10.1137/20M1359109

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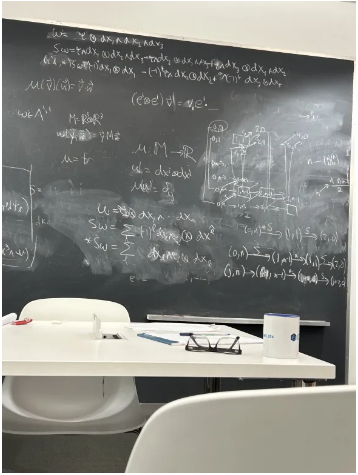
A Picture Taken on March 28th, 2024



PDE experts discussing algebraic topology and homological algebra in the ICERM lounge.

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