Computing Flip Graphs of Highly Non-Convex Polygons

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What is a Flip Graph?

- Triangulations have wide applications in many areas of science.
- Flip graphs is a natural model that captures the idea of how to change one triangulation into another using a series of minimal changes.

Let P denote a finite set of points in $\mathbb{R}^2$ (thought of as vertices of a polygon),

Definition. *A triangulation of P is a planar graph with vertices P whose edges are maximal straight-lines embedded on $\mathbb{R}^2$.*

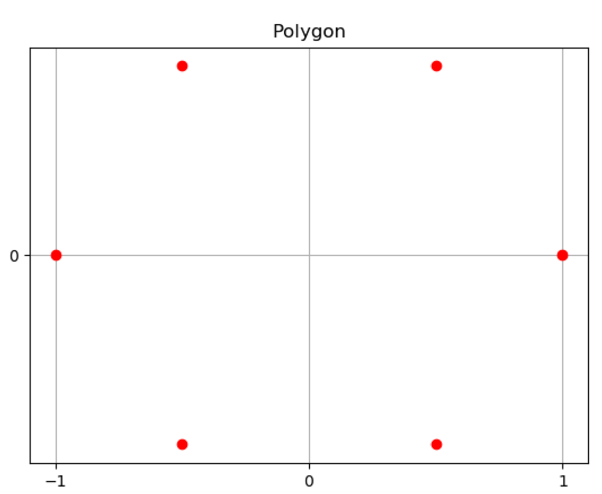

Fig. 1: 6 vertices of a regular Hexagon

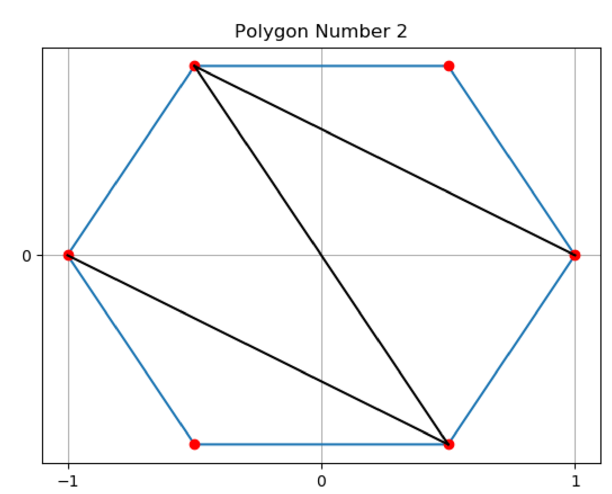

Fig. 2: Triangulation of the 6 points

Definition. *We define the flip graph $\mathrm{Flip}(P)$ of P as a simple undirected graph:*

- *The vertices of $\mathrm{Flip}(P)$ are all triangulations of P.*
- *Let ξ_0 and ξ_1 be two triangulations of P, there's an edge between ξ_0 and ξ_1 if ξ_0 can be changed into ξ_1 with a "flip":*

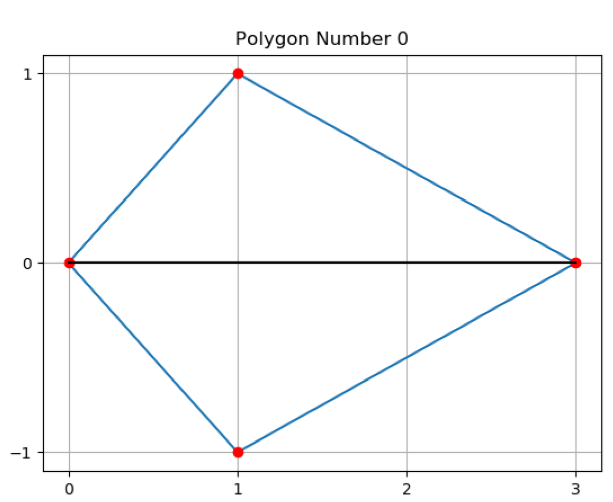

Fig. 3: 6 vertices of a regular Hexagon

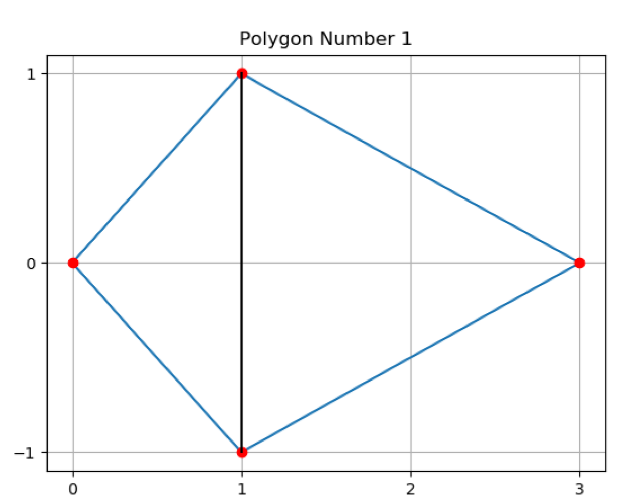

Fig. 4: Triangulation of the 6 points

For some examples of flip graphs:

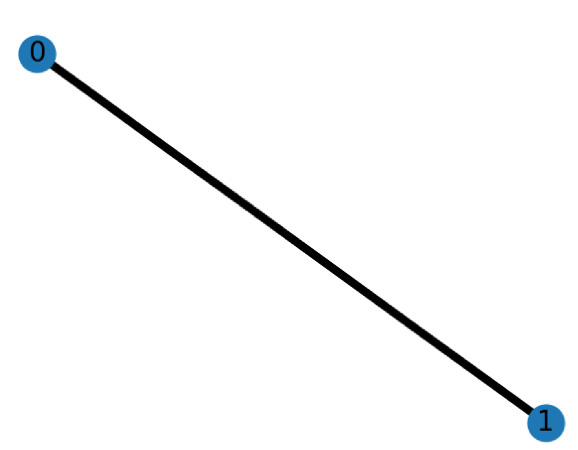

Fig. 5: Flip Graph of a Convex Quadrilateral

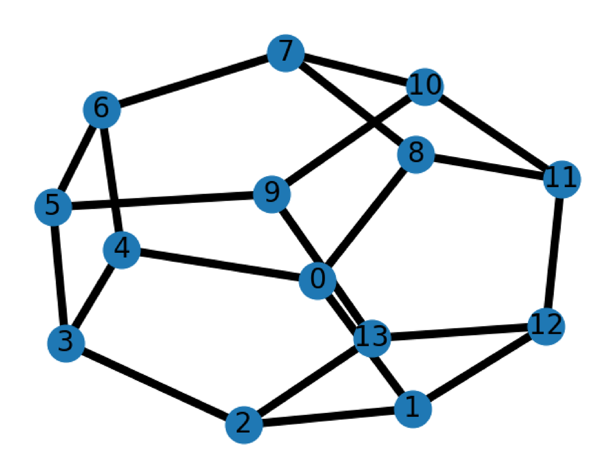

Fig. 6: Flip Graph of a Convex Hexagon

Flip graphs were considered by Lawson in [3] and Weingarten in [7]. A fundamental result in this study is proven by Lawson in [4]:

Theorem (Lawson [4]). *Flip graphs are always connected. Hence, any two triangulations of a finite point set may be related by a series of flips.*

Algorithmic Aspects of Flip Graphs

In [4], Lawson also considered the following application of flip graphs:

- Consider data of the form (x_i, y_i, z_i) thought of as values from the function $z = f(x, y)$.
- How can one infer certain properties of f based on these points?
- One approach may be to triangulate the data points (x_i, y_i) so that f may be defined within each polygon, but which triangulation is better than the other?
- We can find the most suitable triangulation by minimizing some cost R, which amounts to flipping a triangulation to converge to some minimum.

Convex vs Non-Convex Flip Graphs

Fact. *The flip graphs of any two convex polygon of order n are isomorphic.*

Flip graphs for convex polygons are thus very well-studied. In fact, they form the 1-skeleton of a class of polyhedra known as the "associahedra" [5].

The case for non-convex polygons varies a lot more. We do know that

Fact. *Let P be a non-convex polygon with n vertices, then the flip graph of P may be realized as a subgraph of the flip graph of the regular n-gon.*

Algebraic Connectivity of Flip Graphs

Definition. *Let G be a finite undirected simple graph, the algebraic connectivity $a(G)$ (or Fiedler value [1]) of G is the second smallest eigenvalue of the graph Laplacian of G, up to multiplicity.*

There are many motivating reasons on why $a(G)$ should be considered as a valid measure of connectivity, please see [1] and [2] for more details. We will list one motivating reason here:

Proposition. *G is connected if and only if $a(G) \neq 0$.*

We know that flip graphs are connected, a natural question is to quantify this notion of connectedness. The case for vertex and edge connectivity are quite clear:

Theorem (Wagner and Welzl [6]). *There exists some $n_0 \in \mathbb{N}$ such that for any flip graph $\mathrm{Flip}(P)$ with $|P| \geq n_0$, both the vertex and edge connectivity of $\mathrm{Flip}(P)$ are equal to the minimum degree δ of $\mathrm{Flip}(P)$, which is lower bounded by $\lceil \frac{|P|}{2} - 2 \rceil$.*

Question. *What about algebraic connectivity?*

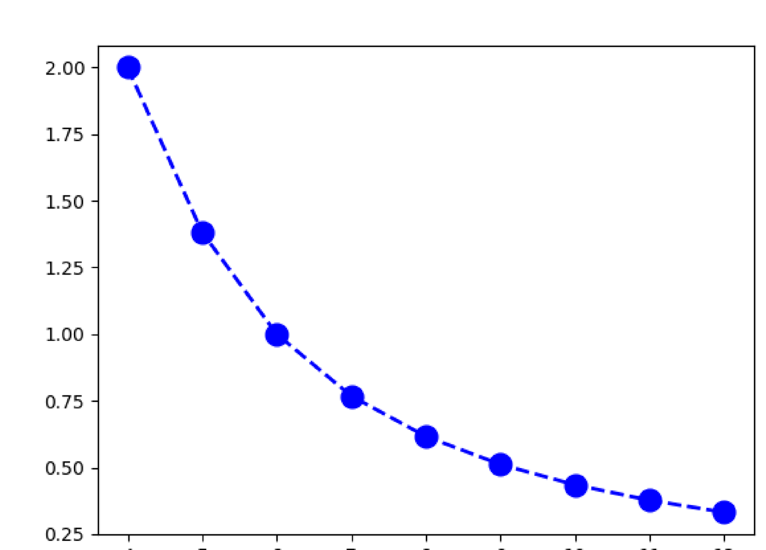

Fig. 7: Algberaic Connectivity of Convex n-gon

Empirical calculations suggest that:

Conjecture (Uniform Boundedness of $a(\mathrm{Flip}(P))$). *There exists some universal constant $M > 0$ such that for any flip graph $G = \mathrm{Flip}(P)$ such that $a(G) \leq M$. Moreover, M can be chosen to be 2.*

Variational Approach

Another question we are interested in is how algebraic connectivity varies as we gradually deviate away from a convex polygon.

We consider the following guiding question:

Question. *Let P_n be an embedded regular n-gon and v be one of its vertices, suppose we perturb v towards the center of P_n, how does its Fiedler value vary over time?*

The function is clearly upper semicontinuous and piece-wise constant.

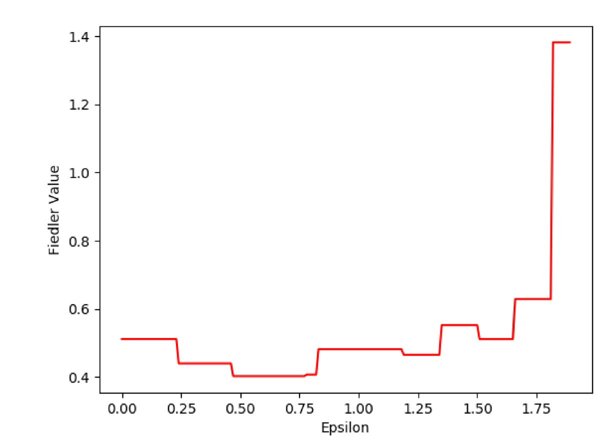

Fig. 8: Varying the Convex 9-gon

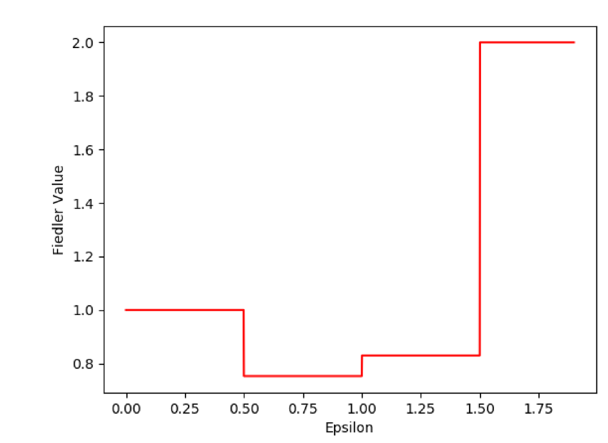

Fig. 9: Varying the Convex 6-gon

Clearly the function can only change finitely many times,

Conjecture. *The number of times the change can occur is $O(n^2)$.*

Question. *What is the limiting behavior of this function?*

Acknowledgements

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- DIMACS's Wokrshop on Modern Techniques in Graph Algorithms for hosting this event.
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To see how flip graphs are generated and relevant computations around them, please check out our GitHub repository at the top left corner.

References

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