Computing Flip Graphs of Highly Non-Convex Polygons

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What is a Flip Graph?

- Triangulations have wide applications in many areas of science.
- Flip graphs is a natural model that captures the idea of how to change one triangulation into another using a series of minimal changes.

Let P denote a finite set of points in $\mathbb{R}^2$ (thought of as vertices of a polygon),

Definition. *A triangulation of P is a planar graph with vertices P whose edges are maximal straight-lines embedded on $\mathbb{R}^2$.*

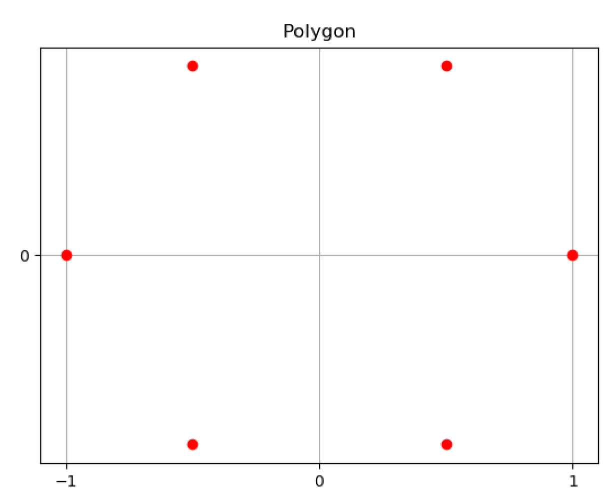

Fig. 1: 6 vertices of a regular Hexagon

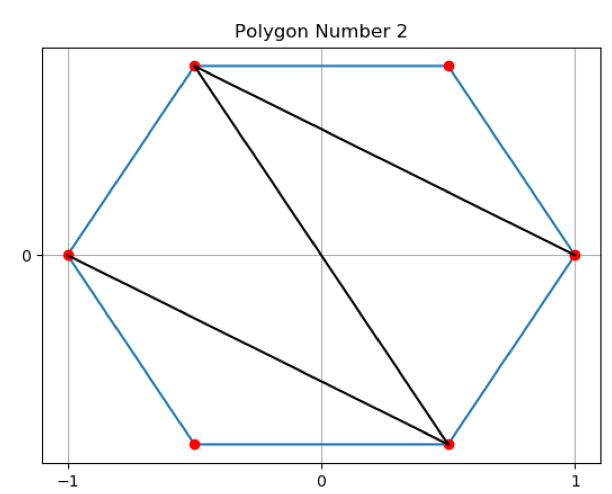

Fig. 2: Triangulation of the 6 points

Definition. *We define the flip graph $\mathrm{Flip}(P)$ of P as a simple undirected graph:*

- *The vertices of $\mathrm{Flip}(P)$ are all triangulations of P.*
- *Let ξ_0 and ξ_1 be two triangulations of P, there's an edge between ξ_0 and ξ_1 if ξ_0 can be changed into ξ_1 with a "flip":*

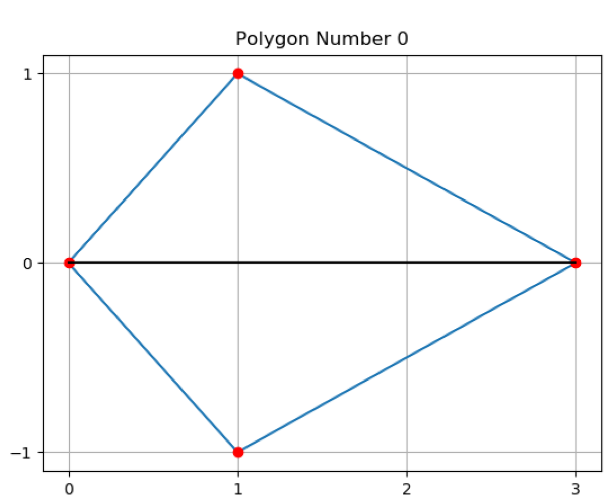

Fig. 3: 6 vertices of a regular Hexagon

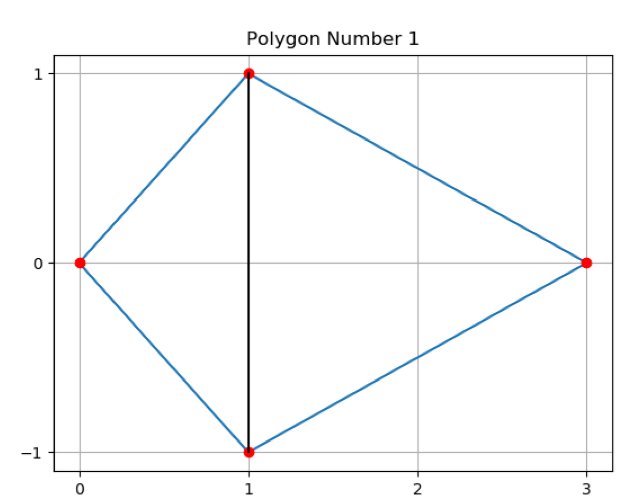

Fig. 4: Triangulation of the 6 points

For some examples of flip graphs:

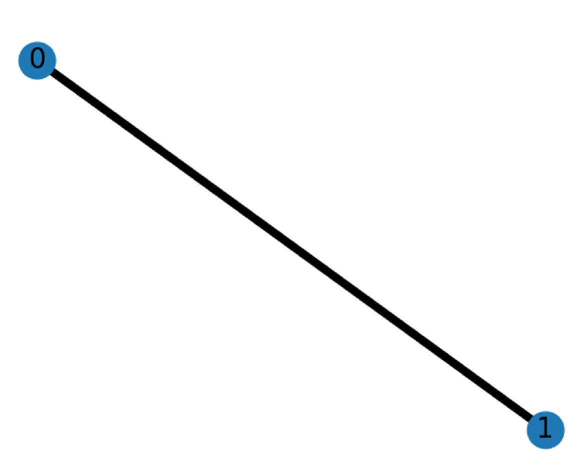

Fig. 5: Flip Graph of a Convex Quadrilateral

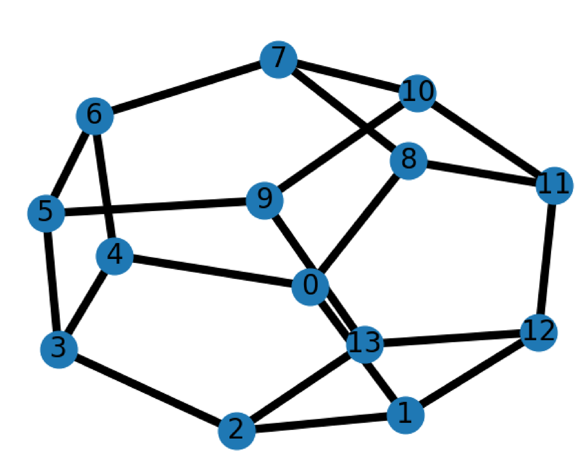

Fig. 6: Flip Graph of a Convex Hexagon

Flip graphs were considered by Lawson in [3] and Weingarten in [7]. A fundamental result in this study is proven by Lawson in [4]:

Theorem (Lawson [4])**.** *Flip graphs are always connected. Hence, any two triangulations of a finite point set may be related by a series of flips.*

Algorithmic Aspects of Flip Graphs

In [4], Lawson also considered the following application of flip graphs:

- Consider data of the form (x_i, y_i, z_i) thought of as values from the function $z = f(x, y)$.
- How can one infer certain properties of f based on these points?
- One approach may be to triangulate the data points (x_i, y_i) so that f may be defined within each polygon, but which triangulation is better than the other?
- We can find the most suitable triangulation by minimizing some cost R, which amounts to flipping a triangulation to converge to some minimum.

Convex vs Non-Convex Flip Graphs

Fact. *The flip graphs of any two convex polygon of order n are isomorphic.*

Flip graphs for convex polygons are thus very well-studied. In fact, they form the 1-skeleton of a class of polyhedra known as the "associahedra" [5].

The case for non-convex polygons varies a lot more. We do know that

Fact. *Let P be a non-convex polygon with n vertices, then the flip graph of P may be realized as a subgraph of the flip graph of the regular n-gon.*

Algebraic Connectivity of Flip Graphs

Definition. *Let G be a finite undirected simple graph, the algebraic connectivity $a(G)$ (or Fiedler value [1]) of G is the second smallest eigenvalue of the graph Laplacian of G, up to multiplicity.*

There are many motivating reasons on why $a(G)$ should be considered as a valid measure of connectivity, please see [1] and [2] for more details. We will list one motivating reason here:

Proposition. *G is connected if and only if $a(G) \neq 0$.*

We know that flip graphs are connected, a natural question is to quantify this notion of connectedness. The case for vertex and edge connectivity are quite clear:

Theorem (Wagner and Welzl [6])**.** *There exists some $n_0 \in$ such that for any flip graph $\mathrm{Flip}(P)$ with $|P| \geq n_0$, both the vertex and edge connectivity of $\mathrm{Flip}(P)$ are equal to the minimum degree δ of $\mathrm{Flip}(P)$, which is lower bounded by $\lceil \frac{|P|}{2} - 2 \rceil$.*

Question. *What about algebraic connectivity?*

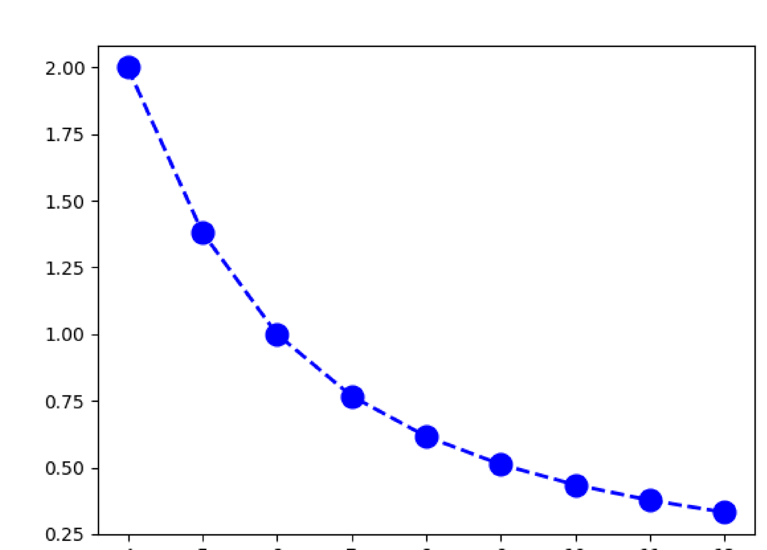

Fig. 7: Algberaic Connectivity of Convex n-gon

Empirical calculations suggest that:

Conjecture (Uniform Boundedness of $a(\mathrm{Flip}(P))$)**.** *There exists some universal constant $M > 0$ such that for any flip graph $G = \mathrm{Flip}(P)$ such that $a() \leq M$. Moreover, M can be chosen to be 2.*

Variational Approach

Another question we are interested in is how algebraic connectivity varies as we gradually deviate away from a convex polygon.

We consider the following guiding question:

Question. *Let P_n be an embedded regular n-gon and v be one of its vertices, suppose we perturb v towards the center of P_n, how does its Fiedler value vary over time?*

The function is clearly upper semicontinuous and piece-wise constant.

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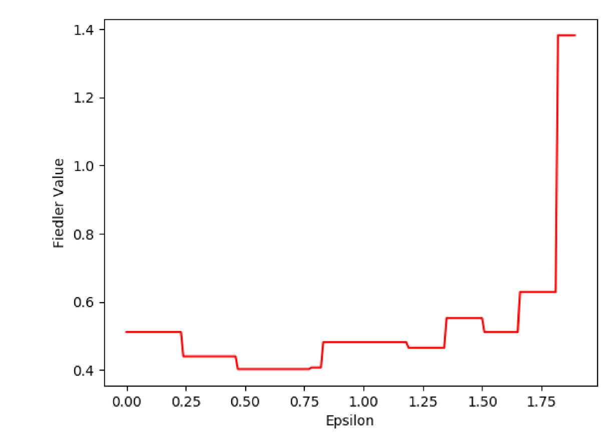

Fig. 8: Varying the Convex 9-gon

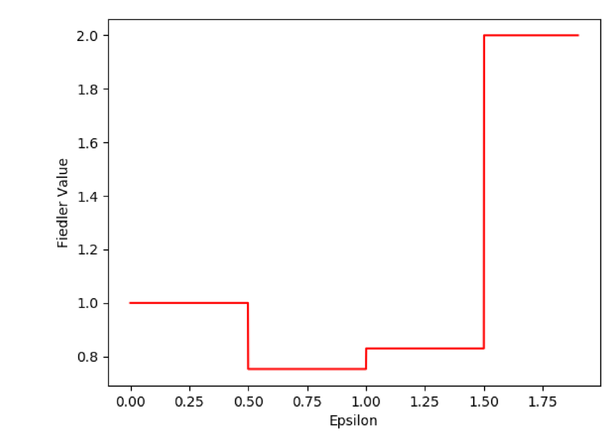

Fig. 9: Varying the Convex 6-gon

Clearly the function can only change finitely many times,

Conjecture. *The number of times the change can occur is $O(n^2)$.*

Question. *What is the limiting behavior of this function?*

Acknowledgements

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- DIMACS's Wokrshop on Modern Techniques in Graph Algorithms for hosting this event.
- Richard Schwartz for the question that motivated this project and his helpful guidance and conversations over this topic.

To see how flip graphs are generated and relevant computations around them, please check out our GitHub repository at the top left corner.

References

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