Combinatorics Notes

Megan Chang-Lee (and hopefully others)

Fall 2022

1 Lecture 1

Outline for the course:

- Posets (Lattices)
  - Holding info
  - Properties of info reflected in properties of the poset P
  - Combinatorial Equivalence of Structures

    Example (Hyperplane Arrangements). *$\mathcal{A} \subset \mathbb{R}^d$. A hyperplane arrangement $\mathcal{A}$ is a collection of subspaces of dim $d-1$.*
    subexample: If $d = 2$, then $\mathcal{A}$ is a collection of lines in $\mathbb{R}^2$.
    PICTURE: four lines intersecting in five two-fold intersection pts, two lines parallel
    Most of all, we're interested in the intersections of these hyperplanes.
    PICTURE 2: same combinatorial data (intersection data) with different slopes and geometry
- Graph Theory
- Matroids
- Combinatorics of Complexes
- Chip-Firing

Definition. *A **partially ordered set** (poset) $\mathcal{P}$ is a set P along with a binary relation $\leq$ on P that is*

- *Reflexive: $x \leq x$*
- *Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.*
- *Antisymmetric: if $x \leq y$ and $y \leq x$, then $x = y$.*

Notation. $[n] := \{1, 2, 3, \ldots, n\}$.

Example. *$\mathcal{P}_n$: The power set of $[n]$: where $S \leq T$ if $S \subset T$. We say this is "ordered by inclusion." (note: this is only a partial order, i.e. not every pair of sets is comparable)*

Example. *$\mathcal{B}_n$: The Boolean Lattice on $[n]$ (also called Boolean Algebra) oops write it out i zoned out*

with both one can verify that these do indeed form a partial order.

Example. *$\mathcal{D}_n$: All divisors of a positive integer n. The relation is $i \leq j$ if $i \mid j$.*

Example. *$\mathcal{L}_n(q)$: All subspaces of a vector space $V_n(q)$, where $V_n(q)$ is the n-dimensional vector space over $\mathbb{F}_q$. Our relation is given by $W_1 \leq W_2$ if $W_1 \subset W_2$. (another poset by inclusion)*

Example. *Π_n: All partitions of $[n]$. Where the relation is given by $\pi \leq \sigma$ if every block of π is contained in a block of σ. This is "ordered by refinement."*

Megan note: there's SO MUCH to say about this poset. We can have a conversation about it. :)

PICTURES: then we draw the Hasse diagrams, which idk how to do on LaTeX. :)))))

Definition. *We say $xHHHHy$ "y covers x" if $x \leq y$ and $\nexists$ z s.t. $x < z < y$.*

Megan note: anyone know how to do the covers symbol???????

some cool mentions of local finiteness and existence of a cover relation.

Something super far from being locally finite or having covers: $\mathbb{R}$ with the standard Euclidean metric. There are NO cover relations! Very neat.

Definition. *The **Hasse Diagram** of a poset $\mathcal{P}$ (and cover relations).*

Megan note: NEED HELP WITH PICTURES!

e.g. $\mathcal{B}_2$ labelled with either 12, 1, 2, and $\emptyset$ (subsets) or with 11, 10, 01, and 00 (binary strings)

e.g. $\mathcal{B}_3$ cube one ehhh too many labels

e.g. you can draw the Hasse diagram of $\mathcal{B}_4$ without too much fanfare. It gives a neat visulaization of the skeleton of a 4-cube.

Remark. *These are particularly nice Hasse diagrams. It's possible to get less nice ones. (e.g. no unique max or min, not being ranked)*

sadlife can't draw any of the cool pictures.

e.g. you can have isolated vertices in a Hasse diagram too

e.g. extreme example is an anti-chain, where all vertices are isolated. (draw in)

e.g. a chain is a path graph (draw in).

Definition. *A **chain** is a partial order such that every two elements are comparable.*

Definition. *An **antichain** is a poset such that no two elements are comparable.*

PICTURE: pick out some chains, contiguous and not from the $\mathcal{B}_3$ Hasse D

Qs one can ask: what do the subposets that are chains look like? What do the subposets that are antichains look like?

Important note: subposets don't have to be contiguous. (If we require contiguity, we call this a saturated chain) Will also study maximal chains. etc.

PICTURE: idk anything that's an antichain. maybe from $\mathcal{B}_3$ too if doesn't crowd picture too much. can be one elt vacuously. Can talk about maximal non maximal antichains. Etc.

2 Lecture 2

Carly starts class by redrawing and redefining the Boolean lattice $\mathcal{B}_n$.

TOOD: draw $\mathcal{B}_0$, the one-vertex graph, $\mathcal{B}_1$ the graph with two vertices one edge, $\mathcal{B}_2$, usual diamond, $\mathcal{B}_3$ etc etc

Remark. *We may fill in (that is, label the vertices of) these posets in with their corresponding binary strings.*

(Add these labels to drawings, where the bottom of the graph has 000)

Note that with these labelings, we can see the hypercubes of smaller dimension within these lattices. E.g. we achieve a $\mathcal{B}_2$ in the bottom square of $\mathcal{B}_3$ by forgetting the last index.

Carly recommends drawing a hypercube $\mathcal{B}_4$ if anyone ever lets you use their VR headset.

TODO: add a picture of $\mathcal{B}_4$. Maybe this live-texing is not that great for this class.

TODO: go in and add vertex labels

Remark. *A common phrase in combinatorics is a **combinatorial explostion**, a very visceral phrase that often refers to factorial growth (even faster than exponential growth).*

Back to posets: We have some small definitions to get back to!

Definition. *We say x and y are **comparable** if $x \leq y$ or $y \leq x$.*

Definition. *We say a poset $\mathcal{P}$ is a **chain** if all elements of $\mathcal{P}$ are comparable.*

Definition. *We say a poset $\mathcal{A}$ is an **antichain** if no two elements of $\mathcal{A}$ are comparable.*

Definition. *In a poset $\mathcal{P}$, the **interval** $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$.*

Definition. *A **locally finite** poset is a poset such that all intervals are finite. (These are determined by cover relations!)*

Remark. *Generally in this class, we'll only be looking at locally finite posets. Since these are determiend by their cover relations, the Hasse diagrams reflect all their structure (as Hasse diagrams encapsulate cover relations).*

And we can recall the example from last time – the reals with usual ordering.

Now to go on to our very first theorem of the course. :) But we first need to give some definitions first.

Definition. *The **Dilworth number** of a poset $\mathcal{P}$ is $d(\mathcal{P}) :=$ maximum size of an antichain in $\mathcal{P}$.*

Definition. *A **chain decomposition** of $\mathcal{P}$ is a decomposition of $\mathcal{P}$ into disjoint(!!!) chains: $\mathcal{P} = C_1 \cup C_2 \cup \cdots \cup C_m$.*

Theorem 1 (Dilworth's Theorem)**.** *For a finite poset $\mathcal{P}$, $d(P)$ is equal to the minimum size of a chain decomposition of $\mathcal{P}$ (minimum m).*

Note that if $\mathcal{P}$ is infinite, things become ill-defined. We might not have a notion of Dilworth's number e.g.

Proof. oops I zoned out for the intuition.

For the first direction, we wish to show that $d(P) \leq$ the size of the minimal decomposition. To show this, we note that for any chain decomposition, every element of a maximal antichain must be in a different chain. Thus, the number of chains is at least the number of elements in the antichain. That is, at least $d(P)$.

The converse direction is trickier. Now we wish to show that $d(P) \geq$ the size of the minimal decomposition. Let $d(P) = m$. We will show that there exists a chain decomposition with m chains. We'll do so by inducting on the size of $\mathcal{P}$.

For the base case, consider when $|\mathcal{P}| = 1$. There's not much to say here. We have $d(P) = 1 =$ the size of the minimal decomposition, so we're done.

For the inductive step, we may then assume that $|\mathcal{P}| > 1$. We split into two cases.

1. In the first case, assume there exists a maximal antichain $\mathcal{A}$ that does not contain all the maximal or mininal elements of $\mathcal{P}$. Let $\mathcal{P}^+ := \{x \in \mathcal{P} \mid x \geq a \text{ for some } a \in \mathcal{A}\}$. Similarly, let $\mathcal{P}^- := \{x \in \mathcal{P} \mid x \leq a \text{ for some } a\}$. (we may imagine the antichain as a band that goes across our Hasse diagram. This is just splitting $\mathcal{P}$ into everything above and below this band.)

   We now invoke the inductive hypothesis. Thus, we are able to split $\mathcal{P}^+, P^-$ into m chains. (Note that we have exactly m of these! Both $\mathcal{P}^+$ and $\mathcal{P}^-$ contain $\mathcal{A}$ that maximal chain, so the required number of chains won't go down.

   Now we glue the m chains of $\mathcal{P}^+$ to the m chains of $\mathcal{P}^-$ along $\mathcal{A}$. This gives the m chains of $\mathcal{P}$.

2. In the second case, suppose all maximal antichains contain all maximal or all minimal elements of $\mathcal{P}$. To get around this issue, let x be a minimal element of $\mathcal{P}$. Similarly, let y be a maximal element of $\mathcal{P}$. Furthermore, we take x and y such that $x \geq y$. (Note: if we weren't able to do this, this would imply the entire poset was an antichain, so we'd be done. Check for yourself!)

   Let $\mathcal{P}' = \mathcal{P} \setminus \{x, y\}$. Then $d(P) = m - 1$. This can't be m because we set x and y to be comparable. But it must be at least $m - 1$ since we only removed one element from top and bottom and it must contain all of the antichain but one of x and y.

   TODO: augh rephrase going too fast

   Then by the inductive hypothesis, $\mathcal{P}'$ may be decomposed into $m - 1$ chains. To decompose $\mathcal{P}$, all we must do is add in $\{x, y\}$.

□

The next theorem will be logically equivalent to Dilworth's Theore, which is wild! We'll be investigating the other direction (the coming theorem used to prove Diltworth's instead).

Example. *a poset with:*

and all the connections xcept the middle col of three dot rows

TODO: man you really gotta figure out how to get hte graphs in here. Someone help!!! Please add your notes from class!!! T.T

Then a maximal antichain would be either of the rows of three vertices

Question from Nick: Is the enumeration of chain decompositions something studied?

Carly's Answer: Perhaps in special posets. In general no, but there might be cool info. I didn't all the way process what she said as I was typing. Something about flag complexes. (Megan note: But there's so much cool stuff to be studied in combinatorics. Wouldn't be surprised if someone has looked at it!)

Example. *A foreshadowing example.*

I'm not gonna try to tex it in.

If we look for a maximal antichain, it's not going to be given by one of the levels like before. The maximal size is four!!! Pretty wild. B)

TODO: add picture and the chain decomp. Theorem tells us we can't do better than this! We can't be more effcient – we really did need all four chains.

This is a nice example of when the maximal antichains aren't always falling in a level of a poset. Moreover, levels of posets aren't even always definable!

Now on to graph matchings! Folks should ask me about this topic. I have a project I was working on (and couldn't make headway on) with matchings.

EDIT: nevermind! This is a broader definition than I had in mind! You can ask about it, but I might not know much.

Definition. *A **matching** M of two finite sets X and Y is a subset $M \subset X \times Y$ such that $(x, y) \neq (a, b) \implies x \neq a$ and $y \neq b$.*

Definition. *A **maximal** matching M is one such that $|M| = \min(|X|, |Y|)$.*

For an example of when a matching is not maximal, consider some $E \subset X \times Y$. Q: Does E have a maximum matching?

Example. *TODO: add in the picture of X and Y.*

The question of whether this subset E has a maximal matching is equivalently asked is there a matching M in E such that $|M| = 3$?

Answer: Yes! add in the picture with the matchings circled.

NOTE: in this example, we have only two possibilities: $\{e_1, e_3, e_4\}$ and $\{e_1, e_3, e_5\}$.

Example. *This example has no maximal matching. (also called perfect I think) One cannot exist because the second and third elements of X can only be paired with the same element of Y.*

Great Question from Nick: Do people study the minimal edges that have to be added to a given subset such that a maximal matching can then be achieved?

Carly's Answer: Yes! People care about this a lot. People ask this adversarially and "altruistically".

Megan note: oops I wasn't paying attention. What does she mean by these two terms?

Remark. *Problems of matchings have many applications in work/worker allocation. E.g. if a set of workers have certain skills and need to get a certain set of objectives completed.*

Megan note: I should get better at caring about the applications. This is an APMA class!

The following theorem is often used by med schools for residency pairings. Matching med students with their desired med schools. One can also factor in ranked orders rather than just pure unordered sets. But that becomes a different weighting sceme for the edges of such a graph, but then stability issues may come up. You end up with an unstable pairing. This is called the Stable Marriage Problem.

Definition. *For a subset S of a set X, the* ***neighborhood*** *of S is $N(S) := \{y \in Y \mid (x, y) \in E$ for some $x \in S\}$. (still in the context of sets X and Y.)*

Theorem 2 (Hall's Matching Theorem (also called Hall's Marriage Theorem))**.** *Let $E \subset X \times Y$. Suppose $|X| \leq |Y|$. Then E contains a maximal matching if and only if $|S| \leq |N(S)|\ \forall\ S \subset X$.*

Note: oops we erased the helpful example.

Definition.

3 Lecture 3

Carly has posted some great resources and information on Canvas that you can look at for reference!

4 Lecture 4

Megan was out with COVID this week. She would love help filling in these lectures!

5 Lecture 5

Back to lattices!

Definition (Lattice)**.** *A* ***lattice*** *is a poset with meets and joins. Recall the definitions of meet and join here:*

- *The meet of x and y, $x \wedge y$, is the greatest lower bound of x and y in the poset.*
- *The join of x and y, $x \vee y$, is the least upper bound of x and y in the poset.*

Carly draws some examples of joins and meets on the board. It's behind the podium for me and I'm still very clueless with pictures. Should figure that out this week.

Remark. *We think of $\wedge$ and $\vee$ as operations on the elements of our poset. They're associative, idempotent, commutative, etc.*

Definition. *A* ***meet semi-lattice*** *is a poset with well-defined meets but not necessarily joins, and analogously for joins. If we adjoin an element $\hat{1}$ (a unique maximal element) to a meet semi-lattice, we obtain a lattice. In a join semi-lattice, we can adjoin an element $\hat{0}$. With this extra element, we obtain a lattice.*

Definition. *A graded lattice is called* ***(upper) semi-modular*** *if the rank function satisfies*

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y).$$

Definition. *A lattice is* ***modular*** *if the rank function ρ satisfies*

$$\rho(x) + \rho(y) = \rho(x \wedge y) + \rho(x \vee y).$$

in the example from before, if we append a $\hat{1}$ and a $\hat{0}$, we obtain a lattice.

Definition. *An element $a \in \mathcal{P}$, with $\hat{0}, \hat{1} \in \mathcal{P}$, is called an **atom** if a covers the bottom element $\hat{0}$. (i.e. $a \geq \hat{0}$)*

Definition. *An element $a \in \mathcal{P}$, is called an **atom** if a is covered by the top element $\hat{1}$. i.e. $a \leq \hat{1}$.*

Definition. *$\mathcal{L}$ is **atomic** if every element of $\mathcal{L}$ can be written as a join of atoms. Respectively coatomic with coatoms.*

Example. *1. $\mathcal{B}_n$ and Π_n are both semi-modular and atomic. (In fact $\mathcal{B}_n$ is additionally modular, but we will want to focus on lattices that are specifically semi-modular and atomic.)*

2. $\mathcal{D}_n$ is not atomic but is modular.

Definition. *A lattice is **geometric** if it is both semi-modular and atomic.*

Remark. *We will discuss matroids later and how to construct them with geometric lattices.*

Definition. *$\mathcal{L}$ is **distributive** if*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

5.1 Towards the Fundamental Theorem of Finite Distributive Lattices

(There might be extra adjectives in there I forgot)

We need some definitions first!

Definition. *An **order ideal** of $\mathcal{P}$ is a subset $\mathcal{I} \subset \mathcal{P}$ such that if $x \in \mathcal{I}$ and $y \leq x$, then $y \in \mathcal{I}$. (It's "downward-closed.")*

We often use notation: $\mathcal{I} = \langle x_1, x_2, \ldots, x_n \rangle = \{y \mid y \leq x_i \text{ for some } i\}$.

Remark. *A minimal generating set of an order ideal is an antichain.*

Definition. *A **principal** order ideal is an order ideal generated by a single element. (so $n = 1$)*

Remark. *Reversing our inequalities ("looking up"), we instead come to the notion of filters.*

Now, let us consider all order ideals of a poset $\mathcal{P}$. We wish to impose an ordering on these ideals.

Definition. *$\mathcal{J}(P) =$ the poset of all order ideals of $\mathcal{P}$ ordered by containment.*

Example. *b] d]*
a] c]

with edge from a to b, b to c, c to d

All the antichains of this example are:

1. *a*
2. *c*
3. *b,a,c*

4. d,c

5. b,d,a,c

6. a,c

7. d,c,a

8. $\emptyset$, which we will call $\hat{0}$.

we obtain the following poset: highest level: abcd (covers both) 3rd: abc (covers ac) actually gen by $\langle b \rangle$, dca (covers dc) actually gen by $\langle ad \rangle$ 2nd: ac (covers both), dc = $\langle$drangle (covers c) 1st: a and c lowest level: $\hat{0}$

The result is $\mathcal{J}(\mathcal{P})$.

Claim. *$\mathcal{J}(\mathcal{P})$ is a distributive lattice.*

Proof. The meets $\wedge$ are intersections and the joins $\vee$ are unions. We already have distributivity of intersections and unions (by whatever whatever old set theory theorem), so we're done! (Isn't that slick?) □

Theorem 3 (Fundamental Theorem of Finite Distributive Lattices (FTFDL))**.** *Let $\mathcal{L}$ be a finite distributive lattice. Then there exists a unique $\mathcal{P}$ such that $\mathcal{L} \cong \mathcal{J}(\mathcal{P})$.*

The proof of this is in the Stanley book. It's a reasonable proof, but long.

proof sketch. Given some $\mathcal{L}$, what is $\mathcal{P}$? $\mathcal{P}$ will consist of something called join irreducibles. That is, elements that cannot be decomposed as the join of two other elements.

We could ask: what does it mean to be a join irreducible in $\mathcal{J}(\mathcal{P})$? We'll see that is means being a principal order ideal in $\mathcal{P}$. So $\mathcal{L} \cong \mathcal{J}(\text{irr}(\mathcal{L}))$.

□

Oooh it was great to see this in the example. I don't know how to insert that into this doc. But if you take all the join-irreducibles, the subposet with induced partial ordering is exactly the poset we started with!

One might want to make this last isomorphism what is on display instead, since this is a stronger statement:

Theorem 4 (Better FTFDL)**.**

$$\mathcal{L} \cong \mathcal{J}(\text{irr}(\mathcal{L})).$$

6 Lecture 6

Today we'll finish lattices today and start graphs at the end. People have all different levels of backgrounds on graph theory, so we'll lay down our preferred perspective, lay down some ground theory, and then jump to applying it to other topics we care about.

For this lecture, let $\mathcal{P}$ be a poset that's locally finite.

6.1 Definition of the Möbius Function

Definition. *The Möbius function of $\mathcal{P}$ is a function $\mu : \mathrm{INT}(\mathcal{P}) \to \mathbb{R}$, where $\mathrm{INT}(\mathcal{P})$ is the set of intervals of $\mathcal{P}$, we have that:*

1. *$\mu(x, x) = 1 \ \forall \ x \in \mathcal{P}$.*
2. *$\sum_{x \leq z \leq y} \mu(x, z) = 0 \ \forall \ x \neq y \in \mathcal{P}$. A consequence is that we also require:*
3. *$\mu(x, y) = -\sum_{x \leq z \leq y} \mu(x, z)$*

Example. *For example, take the diamond with four vertices. We'll label the bottom vertex $\hat{0}$.*

To define μ, we start from the bottom. By condition 1 above, we have that $\mu(\hat{0}, \hat{0}) = 1$. Then to satisfy condition 3, we also must have $\mu(\hat{0}, x) = -1$ for both vertices in the second row. And lastly, $\mu(\hat{0}, \hat{1}) = 1$, again by condition 3.

Example. *One can define a Möbius function on $\mathcal{B}_n$ as well, but I don't want to try to write it out here.*

In $\mathcal{B}_n$, we have that $\mu(x, y) = (-1)^{|x|-|y|}$.

So e.g. $\mu(1, 12) = -1$ bu this formula. Computing it out with the conditions, we also have that $\mu(1, 1) + \mu(1, 12) = 0$, so we have $1 + -1 = 0$ (recall condition 1 forces the first one to be 1), which means $\mu(1, 12) = -1$.

If we wanted to compute $\mu(1, 123)$ similarly, we could compute $\mu(1, 123) = -\mu(1, 1) - \mu(1, 12) - \mu(1, 13) = -1 + 1 + 1 = 1 = (-1)^{|123|-|1|}$.

Theorem 5 (Möbius Inversion Formula)**.** *Given two functions $f, g : \mathcal{P} \to \mathbb{R}$, then*

$$g(x) = \sum_{y \geq x} f(y) \iff f(x) = \sum_{y \geq x} g(y)\mu(x, y)$$

Remark. *An observation due to Mattie: If we replace the $\leq$ relation here with the divisors of a number, we recover the Möbius Inversion Formula as we know it in Number Theory.*

This also exists as a generalization of the inclusion-exclusion principle.

There's something called Finite Difference Calculus, and this has to do with that. Idk lol.

Proof. This will be a straight-forward Linear Algebra proof of the theorem. In Stanley, there's another proof of this using more machinery, but you should go through and look at that too. (It's a great book!)

Fix some $x_0 \in \mathcal{P}$. Consider the set of elements $y > x_0$. Call this set $\mathcal{P}^{x_0}$. Define now a matrix Z, a $|P^{x_0}| \times |P^{x_0}|$ matrix indexed by $\{y > x_0\}$. Define $Z_{i,j} := \begin{cases} 1 \text{ if } i \leq j \text{ in } \mathcal{P} \\ 0 \text{ else} \end{cases}$

Define as well another $|P^{x_0}| \times |P^{x_0}|$ matrix. This will carry the information of the Möbius formula. Let $M_{i,j} := \begin{cases} \mu(i, j) \text{ if } i \leq j \text{ in } \mathcal{P} \\ 0 \text{ else} \end{cases}$

Define row vectors $F := (f(z) \mid z \in \mathcal{P}^{\S'})$ and $G := (g(x) \mid z \in \mathcal{P}^{\S'})$. The theorem now can be rephrased as $G = F \cdot G$ if and only if $F = G \cdot M$. Thus, the theorem is true if $M = Z^{-1}$. So we require $M \cdot Z =_{|P^{x_0}|}$. We can check this row by row.

Computing out (row v) $\cdot$ (col w) $= \sum_{v \leq u \leq w} \mu(v, u) = \delta_{v,w}$ by definition of μ.

Technically, we've made the implicit assumption that $\mathcal{P}$ is a finite poset, since we have actual literal matrices. BUT! Stanley has the proof for the full locally-finite theorem statement. $\square$

6.2 Inclusion/Exclusion Formulae

Recall the inclusion/exclusion formula for two sets:

$$|A \cup B| = |A| + |B| - |A \cup B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cup B| - |A \cup C| - |B \cup C| + |A \cup B \cup C|$$

In general, we have a formula

$$|S_1 \cup S_2 \cup \cdots \cup S_k| = |S_1| + \cdots + |S_k| - \sum |S_i \cup S_j| + \sum_{|} S_i \cup S_j \cup S_k| - \ldots$$

Proof. Let $S_1, S_2, \ldots, S_n$ be finite sets. Let $\mathcal{P}$ = poset of intersections, ordered by inclusion. We'll include $\cup_i S_i = \hat{0}$ and $\cup_i S_i = \hat{1}$. For some $T \in \mathcal{P}$, define $f(T) := |\{x \in T, x \not\in T', \ \forall\ T' < T\}|$ and $g(T) := |T|$.

Now what we want to show is that $g(\hat{1}) = |U_i S_i| = \sum_{T \leq \hat{1}} f(T)$. We may note that $g(T) = \sum_{T' \leq T} f(T')$ and $f(\hat{1}) = 0$. By the Möbius Inversion Theorem, we have that

$$\begin{aligned}
f(\hat{1}) &= \sum_{T \leq \hat{1}} g(T)\mu(T, \hat{1}) \\
&= 0 \\
&= \sum_{T < \hat{1}} g(T)\mu(T, \hat{1}) + g(\hat{1})\mu(\hat{1}, \hat{1}) \\
&= 0
\end{aligned}$$

So we have that $g(\hat{1}) = -\sum_{T<\hat{1}} g(T)\mu(T, \hat{1})$. Note that $\mu(T, \hat{1}) = (-1)^N$, where N = the number of sets being intersected in T.

□

6.3 Incidence Algebras

Definition. *$I(\mathcal{P}, k)$ is the k-algebra of all functions $f : \mathrm{INT}(\mathcal{P}) \to k$ with convolution as multiplication and the usual pointwise function addition as addition.*

$$f \cdot g(x, y) := \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

And the identity will be given by $\delta(x, y) := \begin{cases} 1 & \textit{if } x = y \\ 0 & \textit{else} \end{cases}$.

Fact. *f has an inverse if and only if $f(x, x) \neq 0\ \forall\ x \in \mathcal{P}$.*

Definition. *The zeta function is given by* $\zeta(x, y) = \begin{cases} 1 & \textit{if } x \leq y \\ 0 & \textit{else} \end{cases}$

We'll also define $\zeta^k(x, y) \sum_{x=x_0 \leq x_1 \leq \cdots \leq x_k = y} 1 :=$ *the number of multichains of lenth k from x to y.*

Proposition 6.1. $(2-\zeta)^{-1}(x,y) =$ *the number of chains from* x *to* y. *Define* $(2-\zeta)(x,y) = \begin{cases} 1 \text{ if } x \leq y \\ -1 \text{ if } x < y \end{cases}$.

Proof. For a sketch, consider $(1 + (\zeta - 1) + (\zeta - 1)^2 + \cdots + (\zeta - 1)^\ell)$, where ℓ is the length of the largest chain. WTS: $(2-\zeta)$ multiplied by that whole polynomial is just 1. This can be done with two lines of proof. But just write it out.

$$\begin{aligned} (1 - (\zeta - 1)) \cdot (******) &= (1 - (\zeta - 1)^{\ell+1}) \\ &= 1 \end{aligned}$$

last equality happens because $(\zeta - 1)^{\ell+1} = 0$. □

7 Lecture ???

Remark. *Someone help me fill in the intermediary stuff hahahaha. I have the first lecture on matroids, but it is yet un-TeXed.*

Let's recall some stuff about matroids! In particular, let's define all the topics we need for the problemsets.

Today's topic: The Tutte Polynomial! :)

Definition. *Let's recall the equivalent definition of a matroid in terms of its flats:*

*The **flats** of a matroid are the closed subsets, i.e. those sets* S *where* $\bar{S} = (S) = S$.

*The **lattice of flats** is the poset of flats of a matroid ordered by containment. This is in correspondence with honest geometric lattices.*

7.1 Hyperplane Arrangements

Lattices of intersections of linear hyperplane arrangements

Definition. *Consider* $\mathcal{A} =$ *collection of (linear) hyperplanes in* $\mathbb{R}^n$. *For full clarity, a linear hyperplane is a codimension 1 linear subspace of* $\mathbb{R}^n$.

Note: the combinatorial study of hyperplane arrangements (linear or affine) tends to take place in $\mathbb{R}^n$ rather than $\mathbb{C}^n$. Topological study takes place more in $\mathbb{C}^n$. Megan note: I suppose this may be because chambers of an arrangement get messed up when promoting to $\mathbb{C}^n$.

Example. *In* $\mathbb{R}^2$, *a linear hyperplane arrangement is a collection of lines through the origin. In* $\mathbb{R}^3$, *a linear hyperplane arrangement is a collection of planes going through the origin. Say we take a collection of three planes, not collinear, all intersecting in* $\vec{0}$. *Regardless of what literal subspaces we take, this will yield the same combinatorial intersection data and hence will yield the same lattice.*

Definition. *The intersection lattice of hyperplanes is the poset of all intersections, ordered by reverse inclusion. With this ordering,* $V = \mathbb{R}^n$ *is a minimal element* $\hat{0}$, *since this is the intersection over the empty set and* $\vec{0}$ *is the maximal element* $\hat{1}$, *since these are all linear subspaces (and therefore contain* $\vec{0}$*). This gives us a geometric lattice.*

Example. *In our example of $\mathbb{R}^3$, the intersection lattice returns us precisely the Boolean lattice $\mathcal{B}_3$ (honestly I forget which subscript this should be).*

Since this yields a geometric lattice, we may associate to it the underlying matroid of $\mathcal{A}$.

Definition. *The* **underlying matroid $\mathcal{M}_\mathcal{A}$ of a hyperplane arrangement $\mathcal{A}$ is the representable(realizable) vector matroid of the normals of the hyperplanes in $\mathcal{A}$.**

Example. *A specific hyperplane arrangement is given by the* ***graphical arrangements****. Given a graph G, we have a hyperplane for each edge of G.*

For K_4 minus an edge (specifically edge 24), we have that the graphical matroid associated is the following set of hyperplanes in $\mathbb{R}^4$: $\{x_1 = x_2, x_1 = x_4, x_3 = x_4, x_1 = x_3\}$. We note that drawing the lattice of intersections of this arrangement is precisely the lattice of flats of the corresponding graphical matroid.

Definition. *Recall that the* ***characteristic polynomial*** *of a matroid is given by*

$$\chi_M(t) = \sum_{x \in \mathcal{L}(M)} \mu(\hat{0}, x) t^{\operatorname{rk} M - \operatorname{rk} x}$$

So we may define the characteristic polynomial to be precisely the characteristic polynomial for its associated matroid.

$$\chi_\mathcal{A}(t) = \chi_{\mathcal{M}_\mathcal{A}}(t).$$

Theorem 6 (Zaslavsky's Theorem)**.**

$$(-1)^n \chi_\mathcal{A}(-1) = r(\mathcal{A})$$

where $r(\mathcal{A})$ is the number of regions of $\mathbb{R}^n$ cut out by the arrangement $\mathcal{A}$.

Megan note: there are cool consequences of this theorem! For example, take a look at Theorem 5.17 in Stanley's *Hyperplane Arrangements* for a beautiful way to compute the characteristic polynomials of a special kind of sequence of (affine, not necessarily linear) arrangements.

Proof. Given a hyperplane $H \in \mathcal{A}$, consider the deletion $\mathcal{A} \setminus H$ and restriction $\mathcal{A}|_H$ (this is like the projection of $\mathcal{A}$ down to H). We use this to find recursions. In particular, one can show that the function $r(\mathcal{A})$ is additive across both deletion and restriction. That is,

$$r(\mathcal{A}) = r(A \setminus H) + r(\mathcal{A}|_H)$$

and we also have a distributive property:

$$\chi_\mathcal{A}(t) = \chi_{\mathcal{A} \setminus H}(t) - \chi_{\mathcal{A}|_H}(t)$$

∎

Later: Minors of a Matroid

7.2 Tutte Polynomial

Given a matroid $M = (E, \mathcal{I})$:

Definition. *For e not a coloop (an element in every basis), $M \setminus \{e\}$, the **deletion** of e is a matroid with ground set $E \setminus e$ and independent sets given by $\{I \in \mathcal{I} \mid e \not\subset I\}$.*

Note that if $A \subset E$, then $M \setminus A = (((M \setminus \{a_1\}) \setminus \{a_2\}) \setminus \ldots \setminus \{a_k\})$.

Definition. *The **restriction** of a matroid M is $M|_A = M \setminus (E \setminus A)$.*

Definition. *For e not a loop, we may define the **contraction** of M on e, M/e, is the matroid with groundset $E \setminus \{e\}$ and with independent sets $\{I \setminus \{e\} \mid e \in I \in \mathcal{I}\}$.*

Note again that if $A \subset E$, then $M/A = (((M/\{a_1\})/\{a_2\})/\ldots/\{a_k\})$.

Definition. *The **dual matroid** of a matroid M, $M^\perp$, is the matroid with the same ground set E and with bases given by $\mathscr{B}(M^\perp) = \{E \setminus B \mid B \in \mathscr{B}\}$. (here $\mathscr{B}$ denotes the bases of M itself.*

Definition. *The Tutte polynomial is the following (two-variable) polynomial associated to a matroid: $T_M(x, y)$ may be defined recursively as follows:*

1. *$T_M(x, y) = 1$ for $E = \emptyset$.*
2. *$T_M(x, y) = T_{M\setminus e}(x, y) + T_{M/e}(x, y)$ for e not a loop or coloop.*
3. *$T_M(x, y) = xT_{M\setminus e}(x, y)$ for e a coloop.*
4. *$T_M(x, y) = yT_{M/e}(x, y)$ for e a loop.*

Remark. *Any deletion-contraction invariant is somehow a specialization of the Tutte polynomial. This will be made more rigorous next time. This fact is beautiful and surprising! (though perhaps moreso if you see this in a different definition first)*

8 Lecture ???2

More matroids! :) Also Carly brought us leftover Halloween candy. (Snickers and Reese's, the two objectively best Halloween candies)

8.1 Minors of a Matroid

Suppose $M = (E, \mathcal{I})$ is a matroid. Suppose $X \subset E$. Let's define the restriction and the deletion of a matroid.

Definition. *The **restriction** of M to X is $\mathcal{I}_X := \{I \subset X \mid I \in \mathcal{I}\}$. This is equivalently the **deletion** of M by $E \setminus X$, $M \setminus (E \setminus X)$.*

Definition. *The **deletion** of M by a single element e is $M \setminus e := M\ |_{E\setminus e}$. And $\mathcal{I}_{E\setminus e} = \{I \subset E \setminus e \mid I \in \mathcal{I}\}$.*

Remark. *Note that if e is a coloop, $M \setminus e$ will have smaller rank then M.*

Definition. *For $M = (E, \mathscr{B})$, its dual M^* is the matroid whose bases are given by $\mathscr{B}^* := \{E \setminus B \mid B \in \mathcal{B}\}$. This is also denoted $M^\perp$. Note in particular that $(M^*)^* = M$.*

Moreover, the rank function of M^* is given by: $\mathrm{rk}^*(A) = |A| + \mathrm{rk}(E \setminus A) - \mathrm{rk}(E)$.

Good thing to point out: there are many cryptomorphic ways to define a matroid. One is in terms of a rank function! The above note about the dual rank function is not a definition but rather something one derives given the basis definition of the dual matroid.

Definition. *Let $M = (E, \mathcal{I})$ be a matroid and $X \subset E$). Then the* ***contraction*** *of M by X is $M/X := (M^* \setminus X)^*$.*

Example. *Refer to Mattie's great notes for what the graph looks like!*

Say we take the graphical matroid of G. Then e is a coloop, and f is a loop. Then M_G has bases $\{abe, ace, bce\}$, and f here is a loop. Thus, M_G^* has bases $\{cf, bf, af\}$.

Note: this is the graphical matroid associated to G's planar graph dual! (or various other graphs. Recall that the graphical matroid doesn't fully determine a planar graph).

Remark. *Kai asked a great question about when the graphical matroid will allow you to recover the graph fully up to isomorphism. Carly mentioned there exists something called Whitney's Three Theorem. This can allow you to prove the Torelli Theorem. MEGAN TODO: find references.*

Continuing our above example,

Example. *Then $M \setminus e = \{ab, ac, bc\}$ with f as a loop still. And note that $M^* \setminus e = \{cf, bf, af\}$ (we just get rid of e). Then, using $M^* \setminus e$ above, $M/e := (M^* \setminus e)^* = \{ab, ac, bc\}$ with f as a loop. (So here, coincidentally, $M/e = M \setminus e$).*

We may instead consider using f, our loop, rather than the coloop e.

We have $M \setminus f = \{abc, ace, bce\}$. To calculate M/f, as before let's compute $M^* \setminus f = \{c, b, a\}$ with e as a loop. Then $M/f = (M^* \setminus f)^* = \{abe, ace, bce\}$. Again, we have that $M \setminus f = M/f$.

Remark. *We only have $M/f = M \setminus f$ when we contract/delete by a loop or a coloop.*

Oops I spaced out and responded to emails for a bit.

8.2 Tuttue Polynomial

We may give two equivalent definitions:

Definition.

$$T_M(x, y) = \sum_{A \subset E} (x-1)^{\mathrm{rk}(E) - \mathrm{rk}(A)} (y-1)^{|A| - \mathrm{rk}(A)}$$

Equivalently,

Definition. *We may define the Tutte polynomial T_M recursively by:*

- *if e is not a loop or coloop:*

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y)$$

- *if e is a coloop:*

$$T_M(x, y) = x T_{M \setminus e}(x, y)$$

- *if e is a loop:*

$$T_M(x,y) = yT_{M/e}(x,y)$$

- *if $E = \emptyset$:*

$$T_M(x,y) = 1$$

Historical Fact from Carly: matroids were once called "combinatorial geometries," since once upon a time, they were simply used as a way to study the combinatorics of certain geometric phenomena.

Proposition 8.1. *Some cool properties of the Tutte polynomial that hold for any M:*

1. $T_M(1,1) =$ *the number of bases of M*

2. $T_M(2,1) =$ *the number of independent sets of M*

3. $T_M(1,2) =$ *the number of spanning sets of M.*

4. $T_M(x,y) = T_{M^*}(x,y)$ *(this may be a less magical statement than it appears. It's fairly straightforward to prove definitionally)*

Proposition 8.2. *For a graphical matroid M, we have moreover that:*

1. $T_M(2,0) =$ *the number of acyclic orientations*

2. $T_M(0,2) =$ *the number of totally cyclic orientations*

Proposition 8.3. *We may also rewrite the characteristic polynomial as a specialization of the Tutte polynomial:*

$$\chi_M(t) = (-1)^{\mathrm{rk}(M)} T_M(1-t, 0)$$

And we also have a way to write down the h-polynomial of the independent set complex of a matroid M.

$$T_M(x,1) = h - polynomial$$

Now, we will show that our two definitions of the tutte polynomial are equivalent.

Proof. We will do this by showing that the rank formulation satisfies recursion. We start with our first definition (the general one in terms of rank) and show that in each of our four defining cases, the definitions match up. (We'll do so using recursion, so we'll start with the smallest possible case of each)

The four base cases are computed out below:

1. First, consider the empty matroid, when $E = \emptyset$. Then $T_\emptyset(x,y) = (x-1)^0(y-1)^0 = 1$.

2. In the case when $M = e$ a single loop, $T_M(x,y) = (x-1)^0(y-1)^0 + (x-1)^0(y-1)^1 = 1 + y - 1 = y$.

3. In the case when $M = e$ is a single coloop, $T_M(x,y) = (x-1)^1(y-1)^0 + (x-1)^0(y-1)^0 = (x-1) + 1 = x$.

4. Lastly, in the case when e is neither a loop nor a coloop, $\mathrm{rk}_M(E \setminus e) = \mathrm{rk}_M(E)$. Moreover, $\mathrm{rk}_{M/e}(E \backslash e) = \mathrm{rk}(E) - 1$. Let's take our defining sum for the Tutte polynomial and decompose it as follows, by thinking of decomposing based on which subsets A that don't contain e (this corresponds to the first summand) and those that do (this corresponds to the second summand). We will show equivalence of the first with the deletion and the second with the contraction.

$$\begin{aligned}
\sum_{A \subset E} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} &= \sum_{A \subset E \backslash e} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} \\
&+ \sum_{A \subset E \backslash e} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A \cup e)}(y-1)^{|A \cup e|-\mathrm{rk}(A \cup e)} \\
&= \sum_{A \subset E \backslash e} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} \\
&+ \sum_{A \subset E \backslash e} (x-1)^{\mathrm{rk}_{M/e}(E \backslash e)+1-(\mathrm{rk}_{M/e}(A)+1)}(y-1)^{|A|+1-(\mathrm{rk}_{M/e}(A)+1)}
\end{aligned}$$

and we're done, using our equations about rank above to decompose the second summand.

□

9 Lecture ???3

We had some discussion about Sayles Hall before class started. Notably, there is an organ in the basement, which is apparently the largest extant organ of its type. (this sequence of words means nothing to Megan, but perhaps others can fill me in) cf. `https://en.wikipedia.org/wiki/Sayles_Memorial_Hall`

Carly also had an amusing story about Don Knud. Apparently he was only willing to travel if he could play an organ wherever he travelled to, so the way they got him to come to Brown was by offering him this Hutchings-Votey organ to play.

9.1 Matroids Continued

Definition. *Recall that a **circuit** of a matroid is a minimally dependent set.*

Remark. *There exists an axiom system for matroids in terms of circuits that one could show is equivalent to the basis one or the independent set one etc etc etc.*

Megan Q: Hmmmm how do you all think about matroids? I usually think about them in terms of independent sets. Also flats I guess. idk.

Example. *Mattie isn't here! T.T So I'll just try to get the minimal info down so one could reproduce this graph:*

Say we have K_4 minus an edge. Edges are labelled a, b, c, d, e. Then the circuits are $C = \{abe, cde, abcd\}$.

Note: though bases all have to have the same size, circuits don't all need to have the same size. (like flats)

Let's extend the notion of a circuit:

Definition. *Let M be a matroid on base set $E = x_1 < x_2 < \cdots < x_m$, a labeled and ordered groundset.*

*A **broken circuit** $C \setminus x$ is just a circuit C minus its largest element x.*

Example. *Going back to our example of K_4 minus an edge, we have that the set of broken circuits $BC = \{ab, cd, abc\}$.*

Finally we get to the definition we needed for the homework!

Definition. *The set of **non-broken circuits** is given by $NBC = \{S \subset E \mid S$ does not contain a broken circuit$\}$.*

Example. *For our example,*

$$NBC = \{\emptyset, a, b, c, d, e, ac, ad, ae, bc, bd, be, ce, de, ace, bce, bde\}$$

Remark. *This is often called the **non-broken circuit complex**, since this actually forms a simplicial complex. (i.e. that it satisfies I2 of the matroid independence matroids. It's closed under taking subsets.)*

Example. *For our NBC above, we have 1 subset of size 0, 5 of size 1, 8 of size 2, 4 of size 3.*

NOTICE!!! This is exactly the coefficients of the characteristic polynomial of M_G for our graph. (very wow, much tutte)

Theorem 7. *The coefficients of the characteristics polynomial of M count the number of non-broken circuits (NBC) by size, regardless of the ordering on E.*

Note that a set $X \in NCB$ if and only if $X \cup \{x_m\} \in NBC$, where x_m is the largest element of E. So there is some redundancy that this element gives. To this end, let us define $\overline{NBC}$.

Definition. *$\overline{NBC} = \{X \in NBC \mid x_m \not\in X\}$, where again x_m is the largest element of E.*

Megan needs more graph help:

there are pictures of NBC and $\overline{NBC}$. The simplex for NBC is a square with a central point, edges to each vertex. Then fill in corresponding faces. $\overline{NBC}$ has graph C_4.

9.2 Greedy Algorithms

Example. *Kruskal's Algorithm for spanning trees. The setup for this is: Say we have a graph G and some weight function $w : E(G) \to \mathbb{R}$ on the edges of G. The total weight of a tree T is simply: $w(T) = \sum_{e \in T} w(e)$. Our goal with this algorithm is to find a spanning tree T of minimal total weight.*

The algorithm may be given as follows:

- *Start with the empty tree $\emptyset$.*
- *Add a minimal weight edge as long as it doesn't form a cycle.*
- *Repeat process until we obtain a full spanning tree.*

Megan note: something I often get confused about is whether the edges have to be connected. They don't!!! This is important to guarantee minimality I guess.

More generally, what is a greedy algorithm?

Definition. *Some conventions we'll set: Let E be a finite set and $\mathcal{I}$ a collection of subsets of E such that*

- $\emptyset \in I$
- *If $I_1 \in \mathcal{I}$ and $I_2 \in \mathcal{I}$, then $I_2 \subset I_1$ implies $I_2 \in \mathcal{I}$.*

Suppose moreover that we have a weight function $w : E \to \mathbb{R}$. Here, as before, if $X \subset E$, then $w(X) = \sum_{e \in X} w(e)$.

Our goal is to find a maximal weight member of $\mathcal{I}$.

*Then a **greedy algorithm** is an algorithm as follows:*

- *Start with $X = \emptyset$.*
- *Add an element e of maximal weight such that $(X \cup \{e\}) \in \mathcal{I}$.*

Theorem 8. *$(E, \mathcal{I})$ is a matroid if and only if*

- *I_1: $\emptyset \in \mathcal{I}$*
- *I_2: if $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$.*
- *G: For any weight function $w : E \to \mathbb{R}$, the greedy algorithm outputs an optimal solution.*

Megan note: woahhhhhhhhh

Remark. *This tells us that, in a way, the greedy algorithm is a cryptomorphism for matroids. (like so many other things. e.g. rank functions. Did we talk about those in this class already?)*

Proof. We will now prove the cryptomorphism. First, let us assume M is a matroid (which we'll use the axioms I_1, I_2, I_3 for). Now let X be the output of the greedy algorithm. Let $X = \{x_1, x_2, \ldots, x_r\}$. We note that X will be a basis, since if not, the greedy algorithm would not have terminated (as the set would not have been maximal). Let Y be another basis of M.

Claim. *$w(X) \geq w(Y)$. We'll actually show that $w(x_i) \geq w(y_i)$ $\forall$ i. (where the subscript i is ordered by weight)*

To prove the claim, suppose for the sake of contradiction there is some k such that this fails. Let k be the maximal index such that $w(x_k) < w(y_k)$. Consider then the subsets $(x_1, x_2, \ldots, x_{k-1}) \subset X$ and $(y_1, y_2, \ldots, y_k) \subset Y$. By the basis exchange axiom, we have that $(x_1, x_2, \ldots, x_{k-1}) \cup y_n \in \mathcal{I}$.

(let's assume we have non-equal y_k and y_n) Since the y_i are ordered by weight as well, we know that $w(y_n) > w(y_k)$. And, by assumption, $w(y_k) > w(x_k)$, so $w(y_n) > w(x_k)$. This produces a contradiction, since then the greedy algorithm should have instead chosen y_n rather than x_k. Hence, our output X is optimal, so M is a greedy algorithm.

In the other direction, suppose that $(E, \mathcal{I})$ is a system that satisfies our greedy algorithm properties. We must show that it satisfies I_3, the basis exchange axiom.

Claim. *$\mathcal{I}$ satisfies I_3.*

Suppose for the sake of contradiction it fails I_3. In other words, there exists some $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ such that $I_1 \cup \{e\} \not\subseteq I$ for any $e \in I_2 \setminus I_1$.

Choose some ε such that

$$0 < (1+\varepsilon)|I_1 \setminus I_2| < |I_2 \setminus I_1|$$

We can define a weight function $w : E \to \mathbb{R}$ by

$$w(e) = \begin{cases} (1)2 \text{ if } e \in I_1 \cap I_2 \\ (2)\frac{1}{|I_1 - I_2|} \text{ if } e \in I_1 \setminus I_2 \\ (3)\frac{(1+\varepsilon)}{|I_2 \setminus I_1|} \text{ if } e \in I_2 \setminus I_1 \\ (4)0 \text{ else} \end{cases}$$

The Greedy Algorithm will pick all the elements from cases (1) and (2), but it cannot pick from case (3), so it will terminate to (4).

Let X be the output of the algorithm. Then

$$\begin{aligned} w(X) &= 2|I_1 \cap I_2| + (\frac{1}{|I_1 \setminus I_2|})(|I_1 \setminus I_2|) \\ &= 2|I_1 \cap I_2| + 1 \end{aligned}$$

Now consider

$$\begin{aligned} w(I_2) &= 2|I_1 \cap I_2| + (\frac{(1+\varepsilon)}{|I_2 \setminus I_1|})(|I_2 \setminus I_1|) \\ &= 2|I_1 \cap I_2| + 1 + \varepsilon \end{aligned}$$

Since I_2 is also a basis or contained in some basis, the greedy algorithm has failed to produce an optimal set. Contradiction! □

10 Lecture ???4

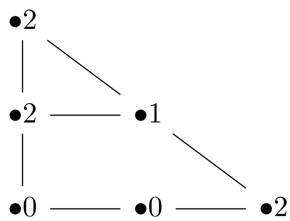
Today we'll be talking about chip-firing! This is a dynamical process. One way to describe chip-firing is as a discrete dynamical system.

Megan note: this will be good for those of us who spend all our time in pure math! We ought to get out of our comfort zones more often. (I am afraid of many of the words Carly is about to say)

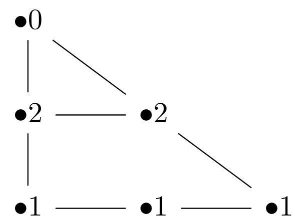
10.1 Chip-Firing Processes

To set some conventions, let G be a finite graph. The main idea is: We'll have configurations (i.e. $\mathbb{Z}$-assignments to vertices) and a dynamic (the evolution of the configurations). We will describe these first in example and pictures and then will make things rigorous.

Example. *helppppppp with the picture.*



Will go to



What's the process? If the number of chips at $v \geq \deg(v)$, then "give" 1 chip to each neighbor. Repeat this process.

Remark. *It's natural to ask ourselves the following two questions:*

1. *Does the order matter? (i.e. which vertex I start with)*
2. *Does it always stop? (will I terminate?)*

Let's get in to the dynamics!

Definition. *Let $G = (E, V)$ be a finite connected graph. A* ***chip configuration*** *is a tuple $c \in \mathbb{Z}_{\geq 0}^{|V|}$.*

Definition. *We say a vertex v can* ***legally fire*** *if $c_v \geq \deg(v)$.*

Definition. ***Firing at vertex*** *v explicitly means taking a configuration $c \in \mathbb{Z}_{\geq 0}^{|V|}$ to a configuration $c' = c - Le_v$, where L is the graph Laplacian.*

So we may address question 1. That is, does the order of firing matter?

Lemma (Locally Diamond Lemma)**.** *Given a configuration c, suppose two vertices v, w can legally fire. Let c^v and c^w be the resulting configurations after firing v or w respectively. Then there exists a configuration d such that d can be reached from both c^v and c^w in one firing move.*

Proof. Observe that if vertex w can fire (at some time t), then it can still fire after v has fired. Because firing v can only increase the value at w, so we will not break firability. Moreover, note that $d = c - Le_{c_v} - Le_{c_w} = c - Le_{c_w} - Le_{c_v}$. Thus, we have found some configuration d as desired.

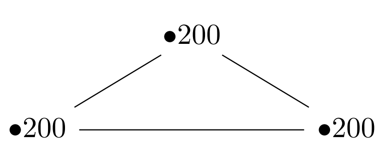
$$\begin{tikzcd}
 & \bullet c \arrow[dl, "\text{fire } v"] \arrow[dr, "\text{fire } w"] & \\
\bullet c^v \arrow[dr, "\text{fire } w"'] & & \bullet c^w \arrow[dl, "\text{fire } v"] \\
 & \bullet d &
\end{tikzcd}$$

$\square$

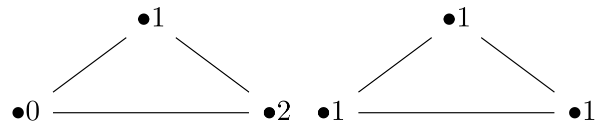
To prove termination, we will use this diamond lemma to show there is unique termination.

To answer Question 2: does this process always terminate? Nah.

Example. *Here's a silly example:*



Example. *Here's two with total weight 3! The one on the left never terminates, but the one on the right can (and already has).*



Lemma. *If an initial configuration c never terminates, then every vertex fires infinitely often in the process starting at c.*

Proof. Recall we're assuming connectedness and finiteness.

Since the initial configuration c never terminates, there must be some v that fires infinitely often. Each fire sends a chip to each neighbor. Thus, each neighbor of v will also have to fire infinitely often. So by connectedness of G, we are done. □

Lemma. *If from an initial configuration c, every vertex can legally fire at least once (during the process) then the process will never terminate.*

Proof. Suppose not. Then let v be the vertex that stopped firing first. But then all neighbors of v fire after v. Thus, v gains $\deg(v)$ chips and hence can fire. Contradiction! □

Theorem 9 (Three-Regime Theorem)**.** *Let $G = (V, E)$, $|V| = n$ and $|E| = m$. Let c be a configuration with N chips. Then,*

1. *$N > 2m - n$ gives an infinite firing.*
2. *$m \leq N \leq 2m - n$ gives both possibilities exist for any values.*
3. *$N < m$, then this terminates finitely.*

Proof. For (1), we may make a pidgeon-hole principle argument, since the total degree of the graph is $2m$ (twice the number of edges).

For (2), we assume $N \leq 2m - n$. Any initial configuration with $c_v > \deg(v)$ for each v will be stable. In the boundary case when $N = m$, we may refer to our triangle example from before. Then take an acyclic orientation AO of G. Let $c_v = \text{outdeg}_{AO}(v)$. Recall that in an acyclic orientation, there always exists a source s. Here, we will have $c_s = \deg(s)$, so s can fire. Firing will "turn around th edges" at the source. i.e. after firing, the orientation at each adjacent edge will flip. The resulting orientation will also be acyclic. Thus, we may repeat infinitely with this new acyclic orientation, so there will be a new source, etc.

Note: This implicitly makes use of the fact that, given an acyclic orientation, if one flips the orientation at the source, then one is left with another acyclic orientation.

For case (3), we assume $N < m$. We will associate chips to edges. In particular, for each edge, associate the first chip to traverse that edge. (we can realize this say by labeling this first chip by the edge). Since $N < m$, there is some edge that has no associated chip. This puts us back in the case of the second lemma, since there exists a vertex whose endpoints don't fire. Thus, the process terminates. □