MATH2410 HW 1

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I did the first 4 questions, and my .5 is the last question, my solution to Problem 5 is unbearable.

1 Problem 1

1.1 Part (a)

Prove that Antoine's necklace A is homeomorphic to the middle third Cantor set without using any results about the classification of perfect, totally disconnected metric spaces.

Proof. Let T be the original solid torus and $T_1, T_2, ..., T_\infty$ be each iteration of Antoine's Necklace. In particular, $A = T_\infty$. We wish to show that T_∞ is homeomorphic to the middle third Cantor Set.

Now let $x \in T_\infty$ be a point, then by Definition, it resides in each of $T_1, T_2, ...$ We will label the initial torus on T_1 from $0, ..., K-1$, then we define x_1 as the index of the torus in T_1 that x is on. Then on that torus, in T_2, it is replaced by a smaller copy of T_1 with the same index $0, ..., K-1$, then we will define x_2 as the index of the torus x is on with the smaller copy of T_1, etc.

We will repeat this process for all T_n, then this constructs a sequence of digits in base K:

$$x_1 x_2 ... x_n ...$$

This sequence is well-defined as we know no two distinct tori ever intersects with one another and exists as x is in the intersection of all T_n.

Now, let C^K be the metrizable Cantor Set obtained by cutting out K pieces at each iteration. Then we will label each interval from $0, ..., K-1$. Then a nearly identical argument as in the construction of Antoine's Necklace shows that for each $y \in C^K$, y can be identified with some well-defined sequence of digits in base K:

$$y_1 y_2 ... y_n ...$$

Now, we claim that this identication is unique for both T_∞ (resp. C^K) in the sense that any two points in T_∞ (resp. C^K) with the same sequence of digits must be the same point. Indeed, suppose $x, y \in T_\infty$ (resp. C^K) give the same sequence of digits but are different points,then $d(x, y) > 0$.

We know that the size of the each individual copy of tori (resp. interval) is decreasing at each iteration toward 0, so there will exist some $n \in \infty$ such that the size of each diameter of tori in T_n (resp. intervals in C^K_n) is less than $\frac{1}{100} \cdot d(x, y)$, but then there's no way that x and y can be in the same tori, as that would suggest that the diameter of each tori T_n (resp. intervals in C^K_n) is at least $d(x, y)$, hence a contradiction.

We also claim that the identfication from T_∞ (resp. C^K) to the sequence of digits in base K is surjective. Indeed, this just follows from the construction of both topological spaces as we go down a countably infinite number of steps and make a choice between K choices in each step.

Thus, we have established a well-defined bijection between T_∞ (resp. C^K) and the sequence of base K digits. Now, this gives a bijection $f : T_\infty \to C^K$ defined by the common digit representative each point in T_∞ and C^K share.

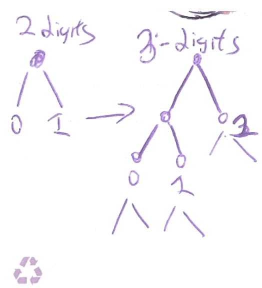
Now clearly f is continuous, this is because for any open ball $B_r(x)$ of radius r center at x in C^K, its preimage under this map is exactly the set of points around $f^{-1}(x) \in T_\infty$ whose digit representation is bounded by some finite radius in its base K representation, which would be an open ball. This argument works backward too, so f^{-1} is also continuous.

Thus, $f : T_\infty \to C^K$ is a valid homeomorphism, so T_∞ is homeomorphic to the Cantor Set obtained by cutting out K-pieces at each iteration.

Now, we note that the middle third Cantor Set is exactly when when we take $K = 2$, hence it suffices for us to show the homeomorphism $C^K \cong C^2$.

We claim that we can represent each point in C^2 as a sequence of base K digits.

Here's what we do, we can take K-pathes down the binary tree given by C^2 as our choices of digits. For example, if $K = 3$, we would associate the digits like this:



In general for $K > 3$, we associate the digits as the leaves of the balanced binary subtree with K leaf nodes starting at the top. We can repeat this process indefinitely as the leaves of the subtree is the parent of an infinite binary tree.

Thus, every point in C^2 can be represented as a sequence of base K digits. This is necessarily surjective since we forced the subtree in the iteration to have K leaf nodes. This is also injective by the metric argument as in the case of C^K.

Thus, we can define $g : C^2 \to C^K$ as the bijection given by the base K digit representation. The continuity of g and g^{-1} follows similarly as f, so we have shown a homeomorphism between C^2 and C^K. Thus, we conclude that

$$T_\infty \cong C^K \cong C^2$$

■

1.2 Part (b)

(Extra credit) Assuming any results you like in knot theory concerning the linking of loops in $\mathbb{R}^3$, prove that $\mathbb{R}^3 - A$ is not simply connected. This is the amazing property of Antoine's necklace: It is homeo to a Cantor set but it's complement in space is not simply connected.

♣♣♣ Mattie: [I read the first 3 chapters of Rolfsen's "Knots and Links" for my final project back in MATH 1410, so I vaguely remember the main idea behind this.]

Proof Attempt. Let T be the original solid torus and $T_1, T_2, ..., T_\infty$ be each iteration of Antoine's Necklace. We first note that the homomorphism inducded by the inclusion map:

$$i : \mathbb{R}^3 - T \to \mathbb{R}^3 - T_1$$

is injective. By an inductive argument on the base case above, we actually obtain a chain of injective homomorphisms:

$$\pi_1(\mathbb{R}^3 - T) \to \pi_1(\mathbb{R}^3 - T_1) \to \pi_1(\mathbb{R}^3 - T_2)...$$

What is $\pi_1(\mathbb{R}^3 - T)$? Well, we will first deformation retract the missing solid torus down into a circle, this is a well-defined continuous process and hence $\pi_1(\mathbb{R}^3 - T) \cong \pi_1(\mathbb{R}^3 - S^1)$, where S^1 is the unknot in $\mathbb{R}^3$.

Then, an application of the Wirtinger Presentation (probably an overkill), would show that the knot group of the unknot is exactly isomorphic to $\mathbb{Z}$.

Now, we claim that the inclusion homomorphism $\pi_1(\mathbb{R}^3 - T) \to \pi_1(\mathbb{R}^3 - T_\infty)$ is actually injective. Indeed, suppose a loop in $\mathbb{R}^3 - T$ is homotopic some point in $\mathbb{R}^3 - T_\infty$.

Since T_∞ is obtained by a chain of decreasing compact sets, we know that this means this loop is actually homotopic to some point in $\mathbb{R}^3 - T_n$ for some $n \in \mathbb{N}$, meaning that the map $\mathbb{R}^3 - T \to \mathbb{R}^3 - T_n$ is not injective. But this violates are chain of injective maps prior, so we have a contradiction.

Thus, we have that $\pi_1(\mathbb{R}^3 - T) \to \pi_1(\mathbb{R}^3 - T_\infty)$ is injective. Since $\pi_1(\mathbb{R}^3 - T)$ is isomorphic to $\mathbb{Z}$, $\pi_1(\mathbb{R}^3 - T_\infty)$ cannot be the trivial group and is thus not simply connected. ■

2 Problem 2

Without using any results about the classification of subgroups of free groups, prove that any finite index subgroup of a finitely generated free group is a free group.

Lemma 2.1. Let G be some finite graph, then any covering space X of G is also homeomorphic to a graph, with vertices and edges being the lifts of the vertices and edges of G

Proof. Let $p : X \to G$ be the covering space and $V(G), E(G)$ denote the vertices and edges of G respectively. We can then view G as the disjoint union of $V(G)$ and $E(G)$ quotienting some identification of end-points.

By the definition of covering maps, $p^{-1}(V(G))$ is a discrete set of points in X, which we will informally view as "vertices" of X.

Now for each edge $e \in E(G)$, consider its parametrization from an interval I_e into G, ie. the map $I_e \to G$, we then obtain a unique lift of $I_e \to X$ passing through each point in $p^{-1}(I_e(0))$. Doing this for all edges of G, we informally view these lifts as "edges" of X.

The topology given by the graph structure on X is homeomorphic to the original topology on X as the covering map $p : X \to G$ is a locla homeomorphism, so the two topologies would share the same topological basis with the lifting arguments constructed earlier. ∎

Proof of Question 2. Let F_n be a free group of rank n. Then consider the one-point union of n-circles, obtained by quotienting the product of n-S^1 at exactly one point as the following graph:

This is precisely a connected graph with 1 vertex and n-cycles, which we will call G.

A repeated application of Van Kampen's Theorem (or by induction on n) will show that $\pi_1(G) \cong F_n$.

Now, let H be a finite index subgroup of $F_n \cong \pi_1(G)$, then by the Galois correspondence of covering spaces, H is precisely isomorphic to the fundamental group of some covering space X_H over G with base-point $x_H \in X_H$.

By Lemma 2.1 above, we know that X_H is in fact a graph, so $\pi_1(X_H, x_h)$ is the fundamental group over some connected component of a graph, call it G'. It then suffice for us to show that the fundamental group of G' is free.

Indeed, we claim that G' contains some maximal tree. Indeed, let $T(G')$ be the collection of trees on G', then clearly the set of non-empty by including just the vertex singleton set. Now, consider the poset structure on $T(G')$ ordered by the subset relation, then every chain in this poset has a maximal element (the union of increasing order of trees) which is contained in $T(G')$. Thus, Zorn's Lemma gives the existence of maximal elements in $T(G')$.

Now, let T be a maximal tree of G', we claim that T is actually homotopic to a point, which implies that $\pi_1(G') \cong \pi_1(\frac{G'}{T})$. Indeed, let $x \in T$ be a base-point, it suffices for us to show that there exists a deformation retract of T onto

x. Indeed, for any $y \in T$, since T is a tree, we know there exist a unique path from y to x. Then we will consider the map, consider the map $F : T \times [0, 1] \to T$ where $F(y, 0) = y, F(y, 1) = x$ and $F(y, t)$ traces from y to x following the unique path given prior at times t. This map is thus clearly continuous and well-defined, and hence we have given a deformation retraction!

Thus, we have that $\pi_1(G') \cong \pi_1(\frac{G'}{T})$. What exactly is the space $\frac{G'}{T}$? Well, it would be a connected graph modulo its maximal tree, so in other words, $\frac{G'}{T}$ is exactly a one-point union of cycles. Since H was taken to be a finite index subgroup, we know that, the covering from X_H to G' was a $[F_n : H]-$fold covering map, meaning that $\frac{G'}{T}$ is a one-point union of finitely many cycles.

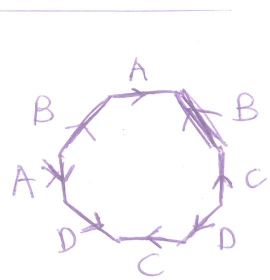
Then by our previous argument with Van Kampen's Theorem, $\pi_1(\frac{G'}{T})$ is exactly the free group of rank being the number of cycles. Thus, H is isomorphic to a free group of some finite rank. $\blacksquare$

3 Problem 3

Prove that the universal cover of a genus 2 surface can be realized as an increasing union $U = U_1 \cup U_2 \cup U_3 ...$ where

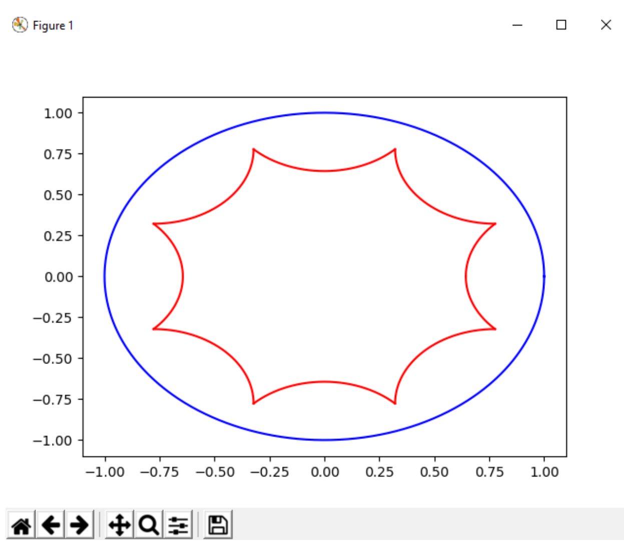
- U_k is simply connected for all k
- U_1 is a regular octagon
- U_{k+1} is obtained by gluing a regular octagon to U_k along some portions of the boundaries of these sets.

Proof. The proof for this question is going to be somewhat different. It is well-known (as in we proved this in 1410) that the following identication of the Octagon forms a Double Torus:



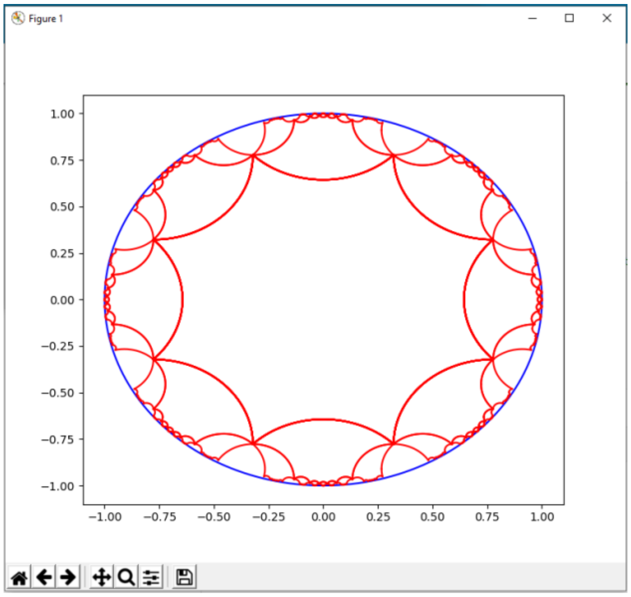
The intuition is to form a tiling of the Hyperbolic Plane with Octagons layer by layer, and the k-th layer would be precisely U_i. Now, I made a visualization of the tiling U_k at `https://github.com/maroon-scorch/Hyperbolic-Octagon` in Python, with an algorithm given in **main.py**. I will briefly explain the algorithm here, but more details are in the code file.

The goal is to construct an octagonal tiling of the hyperbolic plane where each vertex has 8 octagons meeting at it. The initial layer U_1 is going to a regular octagon at the center as follows:



Then in each iteration, we look at the octagons are the Outer Layer of the current U_k. Indeed, let Q be one of these octagons. Then for each edge on Q labeled from $s = 1, ..., 8$ and each vertex of Q from $i = 1, ..., 8$, we construct a new octagon Q' as follows, let $j = s - i + 1 \mod 8$, then we define the j-th vertex of the octagon Q' as the standard reflection of the i-th vertex of Q over the edge s. The new Octagon obtained would still be a regular Octagon, it just looks somewhat distorted under the hyperbolic metric.

For example, this is an example of the tiling with 3 layers:



You can think of this process intuitively as just attaching isometric regular octagons to the boundary of U_k to obtain U_{k+1}.

Finally, clearly U_k is simply connected for all k as we can think of it as a path-connected subset of the plane with no holes, so U_k is simply-connected.

It remains for us to show that the U constructed above is a universal cover of the double torus T. Indeed, define the covering map $p : U \to T$ as follows. For any point in the interior of any octagon on U, p will just map it to its correspondent position of the identified Octagon in the first figure of this proof. Moreover, for any point x in the interior of the octagon, we can find a small open ball such that $p^{-1}(x)$ is a disjoint union of open balls and gives a local homeomorphism.

For the edge identifications, we will identify U_1's edges the same as the first figure **in the sense** that if x, y are both identified on the edge labeled say a, then $p(x) = p(y)$. For each octagon attached to a boundary with a known identification, we note that there unique identification on the octagon on said boundary. Thus, this process extends to all of U.

Now, let v be the vertex of T (viewed as the identified octagon), there's only one vertex. Now for any point x on the edge that is not v, consider a small ball around x that does not touch v, then clearly its preimage is also a disjoint union of open balls since each edge is shared by 2 faces in U with the same identication on that edge.

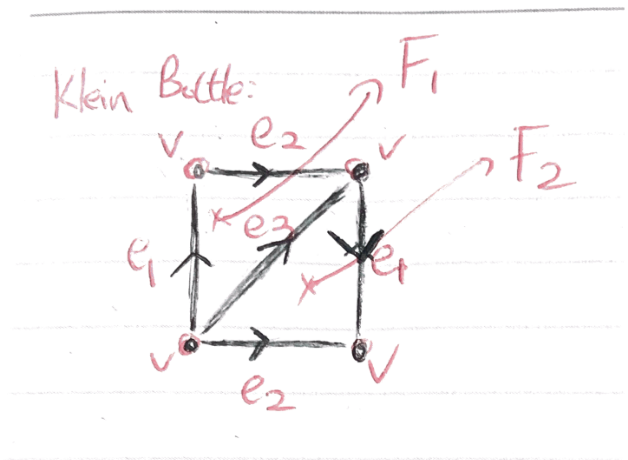
Finaly, $p^{-1}(v)$ is the most important one. Because p fails to be a local homeomorphism as v if we didn't pick the $(8, 8)$-tiling and force each vertex to be met by 8 octagons. However, since we did, then the preimage of a small ball around v in each octagon of U is 8 small filled arcs around the 8 corners, but since each vertex is met by 8 corners, each vertex of U has a small ball that's homeomorphic to the ball around v.

Thus, p is a valid covering map. Since U is simply connected, it follows that U is a covering space of p. ■

4 Problem 4

Put a Δ-complex structure on the Klein bottle K and compute its simplicial homology.

Proof. Consider the following Δ-complex structure on the Klein Bottle:



Let C_i be the formal $\mathbb{Z}$-sums of the i-dimensional objects on the given Δ-complex, then we note immediately that $C_i \cong \{0\}$ for all $i > 2$, hence we have the chain complex:

$$...0 \mapsto 0 \mapsto_{\partial_3} C_2 \mapsto_{\partial_2} C_1 \mapsto_{\partial_1} C_0 \mapsto_{\partial_0} 0$$

In particular this means that our homology group H_i is isomorphic to $\{0\}$ for all $i > 2$.

Moreover, $\{v\}$ is clearly a valid $\mathbb{Z}$-basis of C_0 and hence $C_0 \cong \mathbb{Z}$. Similarly, $\{e_1, e_2, e_3\}$ serve as a valid $\mathbb{Z}$-basis of C_1, and $\{F_1, F_2\}$ is a valid $\mathbb{Z}$-basis of C_2. Thus, $C_1 \cong \mathbb{Z}^3$ and $C_2 \cong \mathbb{Z}^2$.

What are the maps ∂_i between C_2, C_1, C_0 then? Well, they are exactly the maps that sends the generator to the respective sum (accounting orientation) of their boundaries. Specifically

$$\partial_2(F_1) = -e_1 - e_2 + e_3, \partial_2(F_2) = -e_1 + e_2 - e_3$$

$$\partial_1(e_1) = \partial_1(e_2) = \partial(e_3) = v - v = 0,$$

$$\partial_0(v) = 0$$

, in particular we note that both ∂_1 and ∂_0 are the zero maps. To compute the homology groups, we see that

$$H_0 \cong \frac{ker(\partial_0)}{Im(\partial_1)} \cong \frac{C_0}{\{0\}} \cong \mathbb{Z}$$

$$H_2 \cong \frac{ker(\partial_2)}{Im(\partial_3)} \cong ker(\partial_2)$$

We claim that ∂_2 is in fact an injective map, indeed, suppose we note that in general for $n, m \in \mathbb{Z}$,

$$\begin{aligned}\partial_2(nF_1 + mF_2) &= n\partial_2(F_1) + m\partial_2(F_2) \\ &= n \cdot (-e_1 - e_2 + e_3) + m \cdot (-e_1 + e_2 - e_3) \\ &= -e_1 \cdot (n + m) + e_2 \cdot (m - n) + e_3 \cdot (n - m)\end{aligned}$$

Thus, $\partial_2(nF_1 + mF_2) = 0$ if and only if $n + m = n - m = 0$, which implies that $n = m = 0$. Hence our map is injective, so we have that

$$H_2 \cong ker(\partial_2) \cong \{0\}$$

Now finally for H_1, we have that

$$H_1 \cong \frac{ker(\partial_1)}{Im(\partial_2)}$$

Since ∂_1 is the zero map, $ker(\partial_1) = C_1$ and is exactly formal sums of the form

$$e_1 \cdot a + e_2 \cdot b + e_3 \cdot c,\ a, b, c \in \mathbb{Z}$$

As shown earlier, the image of ∂_2 are exactly formal sums of the form

$$-e_1 \cdot (n+m) + e_2 \cdot (m-n) + e_3 \cdot (n-m),\ m, n \in \mathbb{Z}$$

Note that $\{(1,0),(1,1)\}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, so in particular when $(n,m) = (1,0)$, in the quotient world,

$$-e_1 - e_2 + e_3 = 0$$

When $(n,m) = (1,1)$, in the quotient world,

$$-e_1 \cdot 2 = 0$$

These two relations determine the entire quotient structure, thus

$$H_1 \cong \langle e_1, e_2, e_3 \mid e_3 = e_1 + e_2, 2e_1 = 0 \rangle$$

We note that since $e_3 = e_1 + e_2$, the generator e_3 is un-necessary so

$$H_1 \cong \langle e_1, e_2 \mid 2e_1 = 0 \rangle$$

So in other words H_1 is exactly the formal sum

$$H_1 \cong \langle e_1 | 2e_1 = 0 \rangle \oplus \langle e_2 | \rangle = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}$$

Thus, we conclude that

$$H_0 \cong \mathbb{Z}, H_1 \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \mathbb{Z}, H_i \cong \{0\}, i > 1$$

■

5 Problem 5

Suppose that X is a smooth compact manifold with a metric that makes it locally isometric to 3-dimensional Euclidean space. Prove that X is homeomorphic to a simplicial complex. (Note that topologically X need not be a 3-torus; there are some other topological types as well.)

Proof. Every smooth manifold has an essentially unique PL-structure, and therefore it is triangulable. ■

6 Problem 6

Without using any general results about triangulations of smooth com- pact manifolds, and without using any known explicit triangulation of CP^2, show that the complex projective plane CP^2 has a triangulation – i.e. is homeomorphic to a simplicial complex.

Proof. We can think of $\mathbb{CP}^2$ as the complex cube with identification being the anti-podal map on it, then there are 6-complex faces to the cube, which are identified oppositedly, so $\mathbb{CP}^2$ without loss as 3 complex faces identified together, call it B_1, B_2, B_3. Then in other words we have

$$\mathbb{CP}^2 = B_1 \cup B_2 \cup B_3$$

$B_1 := \{(z_1, z_2, z_3) | |z_1| \geq |z_2|, |z_1| \geq |z_3|\}$, and B_2, B_3 are defined similarly.

However, then B_1 is just homoemorphic to a bi-disk, which is a product of 2 unit-disks, moreover the intersection of each $B_i \cap B_j$ is exactly a solid torus. For example, take B_1 and B_2, then their intersection happens exactly when

$$|z_1| = |z_2| \geq |z_3|$$

So we have produced a trisection of $\mathbb{CP}^2$. Then we note that each bi-disks are clearly triangulable since they are products of unit-disks, which are triangulable. Each solid torus is triangulable, as we have shown in lecture. Thus, this gives a triangulation on $\mathbb{CP}^2$. ■