Complex Function Theory

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These are lecture notes from **MATH 2250: Complex Function Theory** with Professor Sergei Treil at Brown University for the Fall 2022 semester, with some supplemental materials written by the note-taker herself. The most up-to-date version of the notes are maintained in the note repository.

These notes are taken by Mattie Ji with gracious help and input from the instructor of this course. If you find any mistakes in these notes, please feel free to direct them via email to me or send a pull request on GitHub.

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Technically, this is a course on "functions of ONE complex variable".

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1 Lecture 1 - 9/7/2022

There are a lot of topics in Complex Analysis, it is the goal of the instructor to be a guide around these topics. There're many different treatments of Complex Analysis - some prefer Algebraic ways, some prefer pictorial ways, etc. Because of this, we'll not be strictly following the textbook. Sometimes we will present proofs that are closely aligned to the textbook, but sometimes we will deviate a lot from it, hence why notes are essential.

1.1 Review of Complex Analysis

Definition 1.1 (Complex Number)**.** A complex number z is denoted as $z = x + iy$, $x, y \in \mathbb{R}$ and i is the root satisfying $i^2 = -1$. The collection of all complex numbers is denoted as $\mathbb{C}$. We define addition and multiplication over $\mathbb{C}$ in the usual sense of algebra.

There's a close similarity between $x+iy \in \mathbb{C}$ and $(x, y) \in \mathbb{R}^2$. Concretely, when viewed as pairs over $\mathbb{R}^2$, the addition and multiplication of complex numbers becomes:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

Proposition 1.2. The complex numbers $\mathbb{C}$ form a field.

Proof. It turns out $\mathbb{C}$ is isomorphic to $\frac{\mathbb{R}[x]}{(x^2+1)}$ as commutative rings (would be nice to check that $\mathbb{C}$ is a commutative ring in the first place). Since x^2+1 is irreducible over $\mathbb{R}$, the ideal it generates is maximal, so $\mathbb{C}$ is a field. In particular, 1 correspond to 1 and x correspond to i in the quotient. ∎

Definition 1.3. Let $x + iy \in \mathbb{C}$, we refer to the **matrix representation** of $x + iy$ as:

$$x + iy \sim \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

In particular we have that

$$1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In particular, the representation is an isomorphism.

Definition 1.4. Let $z = x + iy \in \mathbb{C}$, we define the **complex conjugate** of $\mathbb{C}$ as $\overline{z} := x - iy$ and $|z|$ as the **complex norm** of $\sqrt{x^2 + y^2}$.

Remark 1.5. Usually, the explicitly construct the inverse of $z = x + iy \neq 0$, we have that

$$\frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

However, we note that with the isomorphism given by the definition above also gives us a matrix inverse as its determinant is $x^2 + y^2 \neq 0$

Definition 1.6. Let $z \in \mathbb{C}$ such that $|z| = 1$, then we can write

$$z = x + iy = \cos(\theta) + i\sin(\theta)$$

We refer to $\theta = \arg(z) + 2\pi n$, where $\arg(z)$ is the standard **argument** whose radian is within $[-\pi, \pi)$. This angle is sometimes called $Arg(z)$ and is called the **principal argument**.

Let $z \in \mathbb{C}$ with $|z| = 1$, then we can write

$$z \sim \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This is just the standard rotational matrix.
Now for an arbitrary non-zero $z \in \mathbb{C}$ whose norm need not be 1, we can write

$$z = |z|\frac{z}{|z|} = |z| \cdot (cos(\theta) + isin(\theta))$$

This is called the **polar representation** of $\mathbb{C}$.

Proposition 1.7. Let $z_1, z_2 \in \mathbb{C}$, then

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proof. To show the first, rewrite them in matrix and note their determinant is exactly the complex norm. To show the second, just use the polar coordinate representation and some trigonometry. ■

Corollary 1.8 (De Moivre's Formula)**.** $(\cos(\theta) + isin(\theta))^n = cos(n\theta) + isin(n\theta)$

Proof. This follows directly from the additivity of angles in complex multiplication. ■

Definition 1.9. Let $z = x + iy \in \mathbb{C}$, then $\Re(z) := x = \frac{z+\bar{z}}{2}$ and $\Im(z) := y = \frac{z-\bar{z}}{2}$ are the real and imaginary part of z respectively.

Theorem 1.10 (Euler's Identity)**.** $e^{i\theta} = cos(\theta) + isin(\theta)$

2 Lecture 2

2.1 Differentiability

Definition 2.1. Let Ω be an open subset of $\mathbb{C}$, we say a function $f : \Omega \to \mathbb{C}$ is analytic on $\mathbb{C}$ if for any $z_0 \in \Omega$, there exists a neighborhood U where $z_0 \in U \subset \Omega$, such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \forall z \in U$$

Note that without loss of generality, we can assume $U = D_{z_0,\delta} = \{z : |z - z_0| < \delta\}$.

Remark 2.2. If we take $\Omega \subset \mathbb{R}$, then we say $f : \Omega \to \mathbb{R}$ is real-analytic if for all $x_0 \in \Omega$ locally

$$f(x) = \sum a)k(x - x_0)^k$$

If we take $\Omega \subset \mathbb{R}^2$, then we say $f : \Omega \to \mathbb{R}$ is real-analytic if for all $(x_0, y_0) \in \Omega$, there exist neighborhood U containing (x_0, y_0) such that

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n,k} (x - x_0)^n (y - y_0)^k$$

Note that real analytic function, when viewed as complex function, is not analytic!

The real miracle in the complex analysis is as follows:
If we consider real functions, then we have that

$$C_1 \supsetneq C_2 \supsetneq \ldots \supsetneq C^{\infty} \supsetneq \text{Real-Analytic Functions}$$

However, it turns out that in the Complex Case, complex differentiable functions are in fact analytic, so

$$C_1 = C_2 = \ldots = C^{\infty} = \text{Complex-Analytic Functions}$$

♣♣♣ Mattie: [Unless stated otherwise, we assume Ω to be open?]

Definition 2.3. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a complex-valued function. We say f is **complex-differentiable** at $z \in \Omega$, if the limit exists

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

f is sometimes also called **holomorphic** at z.

Definition 2.4. We denote $C^1_z(\Omega)$ as the set of functions f where $f(z)$ is differentiable on all of Ω and the map $z \mapsto f'(z)$ is continuous (continuous derivative).

2.2 Cauchy-Riemann Equations

Consider h_1 be the direction parallel to the real axis and h_2 be the direction parallel to the imaginary axis, then

$$\lim_{h_1 \to 0} \frac{f(z + h_1) - f(z)}{h_1} = \frac{\partial f}{\partial x} = f'(z) = \frac{1}{i} \frac{\partial f}{\partial y} = \lim_{h_2 \to 0} \frac{f(z + h_2) - f(z)}{h_2}$$

So in particular, we have that

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

This is known as the **Cauchy-Riemann Equation**.

Remark 2.5. Write $f = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then the Cauchy-Riemann Equation is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.6 (Complex Differentials). We define

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

For ease of notations, we will denote

$$\partial \coloneqq \frac{\partial}{\partial z}, \overline{\partial} \coloneqq \frac{\partial}{\partial \overline{z}}$$

Remark 2.7. In particular, this means that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$$

The definition above means that the Cauchy-Riemann Equations is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$

Proposition 2.8. Let f, g be complex dfferentiable functions, then

$$\partial(fg) = (\partial f)g + f(\partial g)$$

Remark 2.9. The $\frac{1}{2}$ coefficient for $\partial, \overline{\partial}$ is a **correcting** coefficient so that

$$\partial z = 1, \partial \overline{z} = 0$$

$$\partial z^n = nz^{n-1}$$

$$\overline{\partial} z = 0, \overline{\partial}\overline{z} = 1$$

$$\overline{\partial}\overline{z}^n = n\overline{z}^{n-1}$$

2.3 Complex Integrals and Cauchy's Integral Theorem

From Calculus, for a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Viewing f instead as a complex function, after some algebraic manipulations, you can show that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

, where $dz = dx + idy$ and $d\bar{z} = x - iy$.

Definition 2.10. A C^1-path is $\gamma : [a, b] \to \mathbb{C}$ where $\gamma \in C^1([a, b])$. If γ is furthermore injective and $\gamma'(t) \neq 0$ on $[a, b]$, then $\gamma([a, b])$ is a C^1-curve. (We require these two extra conditions to avoid spikes on the path)

Definition 2.11. Let $\Gamma = \gamma([a, b])$ be a C^1-curve, then

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Theorem 2.12 (First Cauchy's Theorem)**.** Let $f \in C_z^1(\Omega)$, let G be a bounded open set such that $cl(G) \subset \Omega$, and the boundary of G is C^1 or piece-wise C^1 (we will denote this as PC^1), then

$$\int_{\partial G} f(z)dz = 0$$

Theorem 2.13 (Stokes's Theorem)**.** Let G be an oriented smooth n-dimensional manifold with boundary and ω is a compactly support $(n-1)$-form on G, then

$$\int_{\partial G} w = \int_G d\omega$$

Proof of FIrst Cauchy's Theorem. In this case, G is just some subset of $\mathbb{C}$ and we define $\omega := f(z)dz$, then we note that

$$\begin{aligned}
dw &= df \wedge dz \\
&= (\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}) \wedge dz \\
&= (\frac{\partial f}{\partial z}dz) \wedge dz && \text{Cauchy-Riemann Equation} \\
&= \frac{\partial f}{\partial z}(dz \wedge dz) \\
&= \frac{\partial f}{\partial z}(0) \\
&= 0
\end{aligned}$$

Then Stokes' Theorem tells us that

$$\int_{\partial G} w = \int_G d\omega = \int_G 0 = 0$$

■

3 Lecture 3

3.1 Curves and Orientations

Definition 3.1 (Orientation of a Curve). Let $\Gamma \subset \mathbb{C}$ be a curve, and let $\gamma, \gamma_1 : [a, b] \to \mathbb{C}$ be the standard injective parameterization with $\gamma'(t) \neq 0, \gamma_1'(t) \neq 0$. Then we say γ and γ_1 have the same orientation if $(\gamma \circ \gamma_1^{-1})' > 0$.

Note that this is just the standard definition of orientation for 1-dimensional manifolds.
On a closed loop, the **positive** direction is given by the **left-leg rule**, meaning tracing along the curve, the left leg of the curve points into the region enclosed.
This turns out to align exactly with the definition of orientation inherited by the boundary manifold.

Definition 3.2. We say $f \in CR^1(\Omega)$ if $f \in C^1(\Omega)$ and $\frac{\partial}{\partial \bar{z}} f = 0$ on Ω.

Remark 3.3. It turns out that $CR^1 = C_z^1$, meaning that being in CR^1 is equivalent to being complex differentiable. The direction from $C_z^1 \implies CR^1$ is shown in the previous lecture, what about the other direction?

Definition 3.4 (Multivariable Differentiability). We say $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \Omega$ if there exists some $L \in M_{n \times m}(\mathbb{R})$ such that

$$f(x + h) = f(x) + L(h) + r_x(h)$$

, where $r_x(h)$ is sometimes denoted as $O(h)$ and $\lim_{h \to 0} \frac{r_x(h)}{||h||} = 0$

Proposition 3.5. If a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is CR^1, then f is C_z^1

Proof. Since f is CR^1, it is $C^1(\Omega)$, so f is differentiable when viewed as a function in $\mathbb{R}^2$. Write $f = u + iv$, then the differential of f is exactly the Jacobian matrix:

$$df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Then f satisfiying the CR-equations tells us that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

This means that df is just a scaled rotational matrix, so it's without loss a multiplication by complex numbers. Let a be the complex number representation of df.

Thus, we have that

$$f(z + h) = f(z) + a \cdot h + O(h)$$

Then we have that

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = a$$

, so the complex derivative exists, so f is holomorphic on Ω. ∎

When we say something is differentiable on an non-open set K, then this means there exists some open $\Omega \supset K$ such that it is differentiable on it, meaning there's some bigger open set this function is differentiable on.

3.2 Cauchy's Integral Formula

Theorem 3.6 (Cauchy Formula)**.** Let G be a bounded open set with boundary $\partial G \in PC^1$. Let $f \in C_z^1(cl(G)) = CR^1(cl(G))$. Then for all $z \in int(G)$,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

Proof. Let $\Omega \supset cl(G)$ be the function $f \in CR^1(\Omega)$. Now consider

$$g(z) = \frac{f(z)}{z - z_0}$$

Then we note that $\frac{1}{z-z_0} \in CR^1(\mathbb{C} \setminus \{z_0\})$. Therefore, $g(z) \in CR^1(\Omega \setminus \{z_0\})$.

Choose $\epsilon > 0$ small enough such that $D_{z_0,\epsilon}$, the disk of radius ϵ centered at z_0, does not intersect with ∂G. Now consier $G_\epsilon := G \setminus D_{z_0,\epsilon}$, then

$$\int_{\partial G_\epsilon} \frac{f(z)}{z - z_0} dz = 0 \forall \textbf{ small enough } \epsilon > 0$$

Now, we see that

$$\int_{\partial G_\epsilon} \frac{f(z)}{z - z_0} = \int_{\partial G} \frac{f(z)}{z - z_0} - \int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0}$$

Taking the limit of ϵ on both sides to 0, so

$$0 = \int_{\partial G} \frac{f(z)}{z - z_0} - \lim_{\epsilon \to 0} \int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0} dz$$

$$\int_{\partial G} \frac{f(z)}{z - z_0} = \lim_{\epsilon \to 0} \int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0} dz$$

Since $f(z)$ is differentiable, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + O(z - z_0)$$

So we have that

$$\int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0} dz = \int_{|z-z_0|=\epsilon} \frac{f(z_0)}{z - z_0} dz + \int_{|z-z_0|=\epsilon} f'(z_0) dz + \int_{|z-z_0|=\epsilon} \frac{O(z - z_0)}{z - z_0} dz$$

Parameterize $|z - z_0| = \epsilon$ by $z = z_0 + \epsilon e^{it}, t \in [0, 2\pi]$ and note that

$$\int_{|z-z_0|=\epsilon} \frac{1}{z - z_0} dz = 2\pi i$$

$$\int_{|z-z_0|=\epsilon} f'(z_0) dz = 0$$

$$|\int_{|z-z_0|=\epsilon} \frac{O(z - z_0)}{z - z_0} dz| \leq 2\pi\epsilon \max_{z \in |z-z_0|=\epsilon} |\frac{O(z - z_0)}{z - z_0}|$$

, which goes to zero as $\epsilon \to 0$. Then we have that

$$\int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) + \int_{|z-z_0|=\epsilon} \frac{O(z - z_0)}{z - z_0} dz \lim_{\epsilon \to 0} \int_{\partial D_{z_0,\epsilon}} \frac{f(z)}{z - z_0} dz \qquad = 2\pi i f(z_0) + 0$$

Thus, we have that

$$\int_{\partial G} \frac{f(z)}{z - z_0} = 2\pi i f(z_0)$$

∎

Now we have that

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

We will see that shows that we can take the derivative infinitely many times and that $f(z)$ is analytic.

4 Lecture 4

4.1 Holomorphic Implies Infinitely Differentiable

Example 4.1. Let γ be a path from i to 2, then

$$\int_\gamma z^5 dz = \frac{1}{6} z^6 |_i^2$$

Definition 4.2. We say f has anti-derivative (otherwise known as **primitive**) if $F'(z) = f(z)$.

Proposition 4.3. If f is primitive, then

$$\int_\gamma f(z)dz = F(z_1) - F(z_0)$$

, where z_0 is the start of γ and z_1 is the end of γ. In particular if $z_0 = z_1$,

$$\int_\gamma f(z)dz = 0$$

Proof. Chain-Rule and Fundamental Theorem of Calculus ∎

Recall Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

We claim that, taking the derivative of $f(z)$ gives

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{1}{(\xi - z)^2} f(\xi) d\xi$$

Taking the derivative again gives:

$$f''(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{2}{(\xi - z)^3} f(\xi) d\xi$$

Taking the derivative again gives:

$$f'''(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{2 \cdot 3}{(\xi - z)^4} f(\xi) d\xi$$

Using induction shows that

$$f^{(n)}(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{n!}{(\xi - z)^{n+1}} f(\xi) d\xi$$

This gives us the corollary:

Corollary 4.4. If f is holomorphic under the assumption of Cauchy's Integral Formula, then f is infinitely differentiable.

Proof. We note that for our claims about the derivative to work, we want to show that

$$\lim_{\Delta z \to 0} \int f(\xi) \frac{P(z + \Delta z, \xi)}{\Delta z} d\xi = \int \lim_{\Delta z \to 0} f(\xi) \frac{P(z + \Delta z, \xi) - P(z, \xi)}{\Delta z} d\xi$$

, ie. we can exchange the limit and the integral.

Then we have that
$$\int \lim_{\Delta z\to 0} f(\xi)\frac{P(z+\Delta z,\xi)-P(z,\xi)}{\Delta z}d\xi = \int f(\xi)\frac{\partial P}{\partial z}(z,\xi)d\xi$$
Indeed, we can exchange the limit and integral using **Dominated Convergence Theorem**. ■

Remark 4.5. Note that we were able to use the Dominated Convergence Theorem because of the following two reasons:

- Reason 1:

  Lemma 4.6. $L = \lim_{x\to x_0} f(x)$ exist if and only if for any sequence $\{x_n\}$ converging to x_0,
  $$\lim_{n\to\infty} f(x_n) = L$$

- Reason 2:

  Theorem 4.7 (Mean Value Estimates). $|P(z+\Delta z,\xi) - P(z,\xi)| \le |\frac{\partial P}{\partial z}(z+\theta\Delta z,\xi)|\cdot|\Delta z|$, where $0<\theta<1$.

  We note that this inequality is actually uniformly bounded
  $$|\frac{\partial P}{\partial z}(z+\theta\Delta z,\xi)|\cdot|\Delta z| \le M$$
  This is because the domain of $\frac{\partial P}{\partial z}$ is compact and $\frac{\partial P}{\partial z}$ is continuous.

  We also note that $|\partial G| < \infty$, so we can apply the Dominated Convergence Theorem.

Theorem 4.8 (Holomorphic Implies Analytic). Let $\Omega \supset G$, where Ω is open and G is a bounded open set where $\partial G \in PC^1$, and furthermore that $cl(G)\subset\Omega$. Let $z_0 \in G \subset \Omega$, then if $f \in CR^1(\Omega)$, then f is analytic in Ω. In particular,
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
for all z where $|z-z_0| < dist(z_0,\partial G)$ and
$$a_n = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{(\xi-z)^{n+1}}d\xi = \frac{f^{(n)}(z_0)}{n!}$$

Remark 4.9. In most textbook the neighborhood of power-series is over $|z-z_0| = R$ where $R < dist(z_0,\partial G)$

Proof of Holomorphic Implies Analytic. WLOG, we will assume $z_0 = 0$, because we can always shift the coordinate linearly.

How does the WLOG work, well, rewrite
$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)-(z-z_0)}$$

, then we can just use the same reasoning as before. Trying to justify this takes a little bit of work.

Next, Cauchy's Integral Formula says

$$f(z) = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi$$

Now we note that

$$\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}}$$

We note that $|\frac{z}{\xi}| < 1$ since z is in the interior but ξ is on the boundary, so

$$\frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}} = \frac{1}{\xi}\sum_{n=0}^{\infty}\frac{z^n}{\xi^n}$$

So we have that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi &= \frac{1}{2\pi i}\int_{\partial G}\sum_{n=0}^{\infty}\frac{f(z)z^n}{\xi^{n+1}}d\xi \\ &= \sum_{n=0}^{\infty}z^n\left(\frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi^{n+1}}d\xi\right) && \text{ASSUMING We can switch integral and sum} \\ &= \sum_{n=0}^{\infty}z^n a_n\end{aligned}$$

It remains for us to justify why we can switch this, we can usually use Fubini's Theorem or Dominated Convergence Theorem, but here we will just use a low-tech solution and note that, assuming z is fixed and $\xi \in \partial G$, then

$$\lim_{N\to\infty}\sum_{n=0}^{N}\frac{z^n}{\xi^{n+1}} = \frac{1}{\xi - z} \text{ converges uniformly}$$

, because $|\frac{z}{\xi}| \leq \frac{|z|}{dist(z_0, \partial G)} < 1$, so we have this uniform convergence. Thus, we can always switch the integral and the sum. ∎

Corollary 4.10. Under the same setup, if f is analytic on z_0, then it converges in $D_{z,d}$ where $d = dist(z_0, \partial G)$, so the radius of convergence is at least as much as d.

Remark 4.11. In general, uniform convergence always means that you can switch the limit and the integration. Indeed, let S_n be a sequence of functions that uniformly converges to some function S, then we wish to show that

$$\lim_{n\to\infty}\int S_n = \int \lim_{n\to\infty} S$$

Indeed, since the convergence is uniform, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n > N$,

$$|S_n(\epsilon) - S(\epsilon)| < \epsilon$$

Taking the integrals gives then that

$$|\int_{\partial G} S_n - \int S| \leq \epsilon|\partial G|$$

Since this holds for any $\epsilon > 0$, the limit would thus converge.

We have proved that satisfying Cauchy-Riemann implies analytic, we will now show the converse.

Theorem 4.12 (Morera's Theorem)**.** If $f \in C^0(\Omega)$ and $\int_{\partial R} f(z)dz = 0$ for all sufficiently small rectnagles R where $cl(R) \subset \Omega$, then $f \in CR^1$.

Proof. We will prove this next lecture. ■

Theorem 4.13. $(\sum_{n=0}^{\infty} a_n(z-z_0)^n)' = \sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$, hence power series are holomorphic, and thus analytic functions are holomorphic.

Proof. You can usually do it by taking the partial sum and taking limit, but there's a trick to it! This is a direct corollary of Morera's Theorem, more in the next lecture. ■

5 Lecture 5

5.1 Morera's Theorem and Analytic Implies Holomorphic

We require a rectangle R in the following theorem to be parallel to the x and y-axis.

Theorem 5.1 (Morera's Theorem). Let $f \in C^0(\Omega)$, suppose for all sufficiently small rectangles R such that $cl(R) \subset \Omega$, we have that

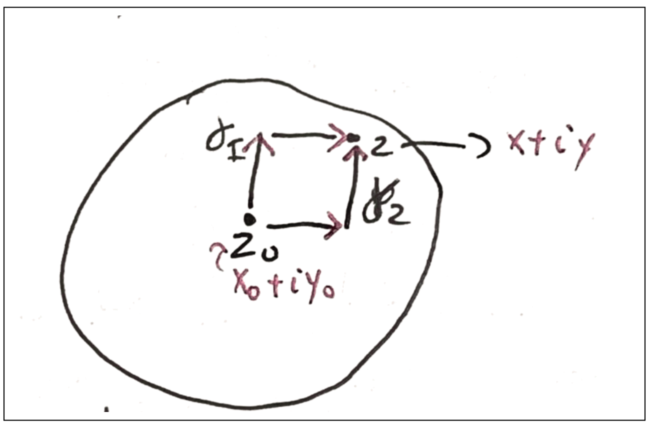
$$\int_{\partial R} f dz = 0 \, (*)$$

, then $f \in CR^1(\Omega)$ and hence holomorphic and analytic.

By **"sufficiently small"**, we mean that for all $z_0 \in \Omega$, there exists $\epsilon := \epsilon(z_0)$ such that, for all rectangles R where $cl(R) \subset D_{z_0,\epsilon}$, the condition in $(*)$ holds.

Proof. We first note that CR^1 is a **local property**, meaning that it suffices for us to check this in a neighborhood of each point. Thus, for some $r > 0$, we can without loss assume $\Omega = D_{z_0,r}$.

Now consider the diagram:



For all $z \in \Omega$, define $F(z) = \int_{z_0}^{z} f(\xi) d\xi := \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$. We note that $F(z)$ is well-defined as Cauchy's Theorem guarantees the path-independence.

We claim that $F \in CR^1(\Omega)$. Indeed, tracing γ_1 and using the Fundamental Theorem of Calculus tells us that

$$\frac{\partial F}{\partial x}(z) = f(z)$$

Similarly, tracing along γ_2, we can parameterize $F(z)$ as

$$F(z) = \int_{x_0}^{x} f(s + iy_0) ds + i \int_{y_0}^{y} f(x + it) dt$$

Then it follows again that

$$\frac{\partial F}{\partial y}(z) = i f(z)$$

We can check that

$$\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} = f(z) - f(z) = 0$$

, thus $F(z) \in CR^1(\Omega)$. Since $F(z)$ follows the Cauchy-Riemann Equations, we also know that

$$F'(z) = \frac{1}{2}(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}) = \frac{1}{2}(2f(z)) = f(z)$$

Since $F(z)$ is holomorphic, it is infinitely differentiable, so $f(z)$ is also holomorphic, so we are done. ■

Corollary 5.2. If $f_n \in C_z^1(\Omega)$ and $\{f_n\}$ converges to f uniformly, then $f \in C_z^1(\Omega)$.

Proof. For all sufficiently small rectangle R within Ω, consider

$$\begin{aligned}
\int_{\partial R} f(z)dz &= \int_{\partial R} \lim_{n\to\infty} f_n dz \\
&= \lim_{n\to\infty} \int_{\partial R} f_n dz && \text{Uniform Convergence} \\
&= \lim_{n\to\infty} 0 && \text{Cauchy's Theorem} \\
&= 0
\end{aligned}$$

Thus, by Morera's Theorem, $f \in C_z^1(\Omega)$. ■

Corollary 5.3. If $f_n \in C_z^1(\Omega)$ and $\{f_n\}$ converges to f uniformly, then $\{f_n'(z)\}$ converges to $f'(z)$ uniformly, and hence $\{f_n^k(z)\}$ converges to $f^k(z)$ uniformly.

Proof. It suffices for us to show this for the first derivative. Now for all $z \in \Omega$ and sufficiently small domain G, we have that

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\int_{\partial G} \frac{f(z)}{(\xi - z)^2} d\xi && \text{Cauchy's Formula for Derivatives} \\
&= \frac{1}{2\pi i}\int_{\partial G} \lim_{n\to\infty} \frac{f_n(z)}{(\xi - z)^2} d\xi \\
&= \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial G} \frac{f_n(z)}{(\xi - z)^2} d\xi && \text{Uniform Convergence} \\
&= \lim_{n\to\infty} f_n'(z) && \text{Cauchy's Formula for Derivatives}
\end{aligned}$$

Thus, we have that $\{f_n'(z)\}$ converges to $f'(z)$. Since the original convergence is uniform, this convergence is also uniform. ■

Theorem 5.4 (Meta Theorem)**.** Let $\phi(z, \xi) : \mathbb{C} \times X \to \mathbb{C}$ where X is some parameterization space with measure $\mu(\xi)$ that is finite, let

$$f(z) = \int \phi(z, \xi) d\mu(\xi)$$

Suppose ϕ is CR^1 in the variable z and f is bounded, then $f(z)$ is analytic.

Proof. Since the measure μ is finite and the function is bounded, we can use Fubini's Theorem and see that, for all sufficiently small rectangles,

$$\begin{aligned} \int_R f(z)dz &= \int_R [\int \phi(z,\xi)d\mu(\xi)]dz \\ &= \int [\int_R \phi(z,\xi)dz]d\mu(\xi) && \text{Fubini's Theorem} \\ &= \int 0 d\mu(\xi) && \phi \text{ is } CR^1 \text{ in the variable } z \\ &= 0 \end{aligned}$$

■

Corollary 5.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ defined in $D_{z_0,r}$, then $f \in CR^1(D_{z_0,r}) = C_z^1(D_{z_0,r})$

Proof. Let $z \in D(z_0, r)$, consider $r_1 > 0$ small enough that $cl(D(z, r_1)) \subset D(z_0, r)$, then by Heine-Cantor, we know that the partial sums

$$\sum_{k=0}^{N} a_k(z-z_0)^k \mapsto \sum_{0}^{\infty} a_n(z-z_0)^n \text{ converges uniformly in } D(z, r_1)$$

Each of the partial sum are complex differentiable, thus by Corollary above, we have that $f \in C_z^1(D_{z_0,r})$. ■

Theorem 5.6. $CR^1 = C_z^1 =$ Analytic

Proof. We have already shown all the directions in previous lectures and this lecture. ■

5.2 Power Series

Definition 5.7. Let $f(z) = \sum a_k(z-z_0)^k$ be a power series. We define the radius of convergence of f as $R \in [0, \infty]$ such that for all z such that $|z - z_0| < R$, $f(z)$ diverges, and for all z such that $|z - z_0| > R$, $f(z)$ converges.

Remark 5.8. It turns out that

$$\frac{1}{R} := \lim \sup_{n\to\infty} |a_n|^{1/n}$$

Proposition 5.9. Let $f(z) = \sum a_k(z-z_0)^k$ be a power series with radius of convergence R, then $f(z)$ converges uniformly in any smaller disk of radius $r < R$ contained in $D_{z_0,R}$

Proof. We note that f is continuous on $D_{z_0,R}$, and the closure of any smaller disk is also contained in $D_{z_0,R}$, then apply Heine-Cantor. ■

Corollary 5.10. The following three power series:

- $\sum_{n=0}^{\infty} a_n z^n$
- $\sum_{n=0}^{\infty} n a_n z^{n-1}$
- $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$

all have the same radius of convergence.

Proof. Exercise. Hint: Note that

$$\lim_{n\to\infty} n^{1/n} = 1$$

■

Example 5.11. For sufficiently small $z \in \mathbb{C}$, consider

$$\frac{1}{1+sin(z)} = 1 - sin(z) + sin^2(z) + ...$$

, we can rewrite this into a power-series by noting that

$$sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

5.3 Liouville's Theorem

Theorem 5.12. Let $f \in Hol(\mathbb{C})$ be a holomorphic function on $\mathbb{C}$, and for all $z \in \mathbb{C}$ we have that $|f(z)| \leq M$ for some fixed $M \in \mathbb{C}$, then $f(z)$ is identically constant.

Proof. Using Cauchy's Formula for Derivatives, we note that for all $z \in \mathbb{C}$, for all $R > 0$,

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi - z| = R} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

Thus, we have that

$$\begin{aligned} |f'(z)| \leq \frac{1}{2\pi} \max_{\xi \in |\xi - z|} \left|\frac{f(\xi)}{(\xi-z)^2}\right| \cdot 2\pi R \\ &\leq \frac{1}{2\pi}\frac{M}{R^2} \cdot 2\pi R && \text{Since } f \text{ is bounded} \\ &= \frac{M}{R} \end{aligned}$$

Since this inequality holds for all $R > 0$, taking the limit as $R \to 0$, gives us that

$$|f'(z)| = 0$$

Hence $f'(z) = 0$, so

$$\frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial \bar{z}} = 0$$

So in other words

$$f_x - if_y = 0, f_x + if_y = 0 \implies f_x = 0, f_y = 0 \implies f(z) \text{ is constant}$$

■

Remark 5.13. While our Morera's Theorem applies to only triangles, up to a change of coordinate, the rectangle can really just be anything.

Remark 5.14. How to check say if $f(z) = |z|^2 = z\overline{z}$ is analytic or not? Well, we see that

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial z}{\partial \overline{z}}\overline{z} + z\frac{\partial \overline{z}}{\partial \overline{z}} = 0 + z(1) \neq 0$$

Thus, f is not holomorphic and hence not analytic.

Take another example, say $f(z) = |z| = (z\overline{z})^{1/2}$. This is real valued so we can use power rule and

$$\overline{\partial}(f) = \frac{1}{2}(z\overline{z})^{-1/2}z \neq 0$$

5.4 Appendix - Cauchy–Hadamard Theorem

In this section, we will discuss more about power series and convergence that was not covered during the lecture.

Theorem 5.15 (Cauchy–Hadamard Theorem)**.** Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Let R be finite, non-zero, satisfying

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| > R$. We call R the **radius of convergence** and $|z| < R$ the **disc of convergence**.

Proof. For any $\epsilon > 0$, let $\rho := \frac{1}{R}$.

If $|z| < R$, then there exist some $\epsilon > 0$ small enough that $|z| < \frac{1}{\rho+\epsilon} < \frac{1}{\rho} = R$. By the definition of $\limsup$ that for a sufficiently large n, for all $N > n$,

$$|a_N|^{1/N} < \rho + \frac{\epsilon}{2} \; (*)$$

$$\begin{aligned} |a_N z^N| &= |a_N^{1/N} z|^N \\ &< |a_N^{1/N} \frac{1}{\rho+\epsilon}|^N && \text{Since } |z| < \tfrac{1}{\rho+\epsilon} \\ &< |\frac{\rho + \frac{\epsilon}{2}}{\rho+\epsilon}|^N && \text{Using } (*) \end{aligned}$$

We note that $|\frac{\rho+\frac{\epsilon}{2}}{\rho+\epsilon}| < 1$ and $b_n = |\frac{\rho+\frac{\epsilon}{2}}{\rho+\epsilon}|^n$ forms a convergent geometric series. By the comparison test, we also have that $\sum_{n=N}^{\infty} a_n z^n$ converges, which implies that $\sum_{n=0}^{\infty} a_n z^n$ converges.

If $|z| > R$, then there exist some $\epsilon > 0$ small enough that $|z| > \frac{1}{\rho-\epsilon} > \frac{1}{\rho} = R$. By the definition of $\limsup$ that for a sufficiently large n, for all $N > n$,

$$|a_N|^{1/N} + \frac{\epsilon}{2} > \rho \; (*)$$

$$\begin{aligned} |a_N z^N| &= |a_N^{1/N} z|^N \\ &> |\frac{a_N^{1/N}}{\rho-\epsilon}|^N && \text{Since } |z| > \tfrac{1}{\rho-\epsilon} \\ &> |\frac{\rho - \frac{\epsilon}{2}}{\rho-\epsilon}|^N \end{aligned}$$

We note that $|\frac{\rho-\frac{\epsilon}{2}}{\rho-\epsilon}| > 1$ and $b_n = |\frac{\rho-\frac{\epsilon}{2}}{\rho-\epsilon}|^n$ forms a divergent geometric series, so the comparision test tells us that $\sum_{n=N}^{\infty} a_n z^n$ diverges, which implies that $\sum_{n=0}^{\infty} a_n z^n$ diverges. ■

Corollary 5.16. Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$, and let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Then $g(z)$ has a radius of convergence that is also R.

Proof. Consider

$$\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} n^{1/n}|a_n^{1/n}|$$

It's a standard fact in real analysis that

$$\limsup_{n\to\infty} c_n b_n = \lim_{n\to\infty} c_n \limsup_{n\to\infty} b_n$$

if $\{c_n\}$ converges. We note that

$$\lim_{n\to\infty} n^{1/n} = 1$$

Since the limit exist

$$\limsup_{n\to\infty} n^{1/n}|a_n^{1/n}| = \limsup_{n\to\infty} |a_n^{1/n}| = \frac{1}{R}$$

■

6 Lecture 6

Previously, we have shown that a complex function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if it satisfies the Cauchy-Riemann Equations. Furthermore, if $f \in Hol(\Omega)$, $z_0 \in \Omega$, then we know f is analytic and locally there exists some bounded, open set G containing z_0 such that ∂G is PC^1 $cl(G) \subset \Omega$, and on G,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, |z - z_0| < r$$

$$a_n = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

Using the Cauchy's Formula for Derivatives, we note that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \implies a_n = \frac{f^{(n)}(z_0)}{n!}$$

6.1 Uniqueness Theorems

Theorem 6.1. Let $f \in Hol(\Omega)$, Ω be an open and connected subset of $\mathbb{C}$, and let $z_0 \in \Omega$ such that

$$f^{(n)}(z_0) = 0 \ \forall n \geq 0$$

Then $f(z) = 0$ on Ω, ie. f is identically zero on Ω.

Proof. We will prove this using **continuous induction**. Indeed, define A to be the set

$$A \coloneqq \{z \in \Omega : f^{(n)}(z_0) = 0 \ \forall n \geq 0\} = \bigcap_{n \geq 0} \{z \in \Omega : f^{(n)}(z) = 0\}$$

We first note that A is closed since the preimage of $\{0\}$ is closed under continuous function and the intersection of closed sets are closed. Now clearly A is non-empty, since $z_0 \in A$. We now **claim that A is open**, then the connectedness of Ω would imply that $A = \Omega$.

Indeed, for all $z \in A$, since f is holomorphic, we can find some $\epsilon(z) > 0$ small enough that locally

$$f(\xi) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} (\xi - z)^n, \forall \xi \ s.t. \ |\xi - z| < \epsilon$$

However, since $z \in A$, we know that

$$f(\xi) = \sum_{n=0}^{\infty} \frac{0}{n!} (\xi - z)^n = 0$$

Thus we have that $\xi \in A$. Hence, $D_{z,\epsilon} \subset A$. Thus, A is also open. ∎

Remark 6.2. The previous theorem shows that

$$f(z_0) = \sum a_n (z - z_0)^n, f^{(n)}(z_0) = 0 \ \forall n \iff a_n = 0 \ \forall n$$

Definition 6.3 (Limit Points). Let X be a topological space and $E \subset X$, we say $a \in X$ is a **accumulation/-cluster/limit point** of E if for all neighborhood U containing a, $E \cap (U \setminus \{a\}) \neq \emptyset$

Theorem 6.4 (Uniqueness Theorem)**.** Let $f \in Hol(\Omega)$ where Ω is open and connected. Suppose $E \subset \Omega$ such that $f(z) = 0$ for all $z \in E$ and E has some accumulation point z_0 in Ω, then $f(z)$ is identically zero on Ω.

Proof. The strategy is to apply the previous theorem. We first choose $r > 0$ small enough and consider the open disk $D_{z_0,r}$ to express f as a power-series around z_0. Now, let $\tilde{f}$ be f restricted to $D_{z_0,r}$, we first claim that

$$\tilde{f}(z_0) = 0$$

Indeed, since z_0 is a limit point of E, there exists a sequence of points $\{z_k\}$ in $E \cap D_{z_0,r}$ that converges to z_0. Since f is continuous, we have that

$$\tilde{f}(z_0) = \tilde{f}(\lim_{k\to\infty} z_k) = \lim_{k\to\infty} \tilde{f}(z_k) = \lim_{k\to\infty} 0 = 0$$

Now consider the power-series expression of f around z_0:

$$[f(z) = \sum_{n=0}^{\infty} a_n(z_0 - z)^n$$

Since $\tilde{f}(z_0) = 0$, we clearly have that $a_0 = 0$. Now we will define

$$f_1(z) \coloneqq \frac{\tilde{f}(z)}{z - z_0} = \sum_{n=1}^{\infty} a_n(z - z_0)^{n-1}$$

♣♣♣ Mattie: [Division is fine here since $a_0 = 0$] and define $\tilde{f}_1(z)$ to be restricted to the same domain as before. Then clearly $f_1(z) = 0$ for all $z \in E \setminus \{z_0\}$, so it follows that $\tilde{f}_1(z_0) = 0$, so $a_1 = 0$.

Using induction, we can conclude that all of the a_n are 0, so $f^{(n)}(z_0) = 0$ for all $n \geq 0$. Apply the previous result and we are done! ■

Corollary 6.5 (Identity Theorem)**.** Let $f, g \in Hol(\Omega)$ where Ω is open and connected. Suppose $E \subset \Omega$ such that $f(z) = g(z)$ for all $z \in E$ and E has some accumulation point z_0 in Ω, then $f = g$ on Ω.

Proof. Define $h(z) \coloneqq f(z) - g(z)$ and apply Theorem 6.4. ■

6.2 Analytic Extensions

The uniqueness theorem implies that there's only one unique way to extend certain functions.

Example 6.6. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$, then $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an analytic extension of e^x to the complex plane, and the identity theorem shows that it is in fact that only possible analytic extension.

Proof. Existence is already proven. For uniqueness, take $\Omega = \mathbb{C}$ and $E = \mathbb{R}$, $\mathbb{R}$ clearly has an accumulation point in $\mathbb{C}$, so we can apply the Identity Theorem. ■

Example 6.7. Similarly, one can also show that

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}...$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!}...$$

are the only unique extensions of $cos(x)$ and $sin(x)$, hence all trigonometric identities also hold for $z \in \mathbb{C}$.

Corollary 6.8. Let Ω be an open connected set, suppose $f \in Hol(\Omega)$ isn't identically zero on Ω, then the zeroes of f are isolated.

Proof. Suppose not, then there exists a sequence of roots $\{z_k\}$ converging to some limit point. Then take $E = \{z_1, ..., z_k, ...\}$, then the Uniqueness Theorem tells us that f is identically zero, so we have a contradiction. ■

Remark 6.9. While this statement says that roots inside Ω are isolated, it says NOTHING about the boundary of Ω.

6.3 Order of Zeroes

Definition 6.10. Let $f \in Hol(\Omega)$, we define $Z(f)$ as the zero locus of f:

$$Z(f) := f^{-1}(0)$$

Definition 6.11. Suppose $f \in Hol(\Omega)$ and $f(z_0) = 0$, rewrite f as a power series around z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, a_0 = 0$$

We say the **order of zero** on z_0 is $\min\{n : a_n \neq 0\} = \min\{n : f^{(n)}(z_0) \neq 0$

Example 6.12. $\sin(z)$ has zero of order 1 at all roots. $(z - z_0)^3$ has a zero of order 3 at z_0.

7 Lecture 7

7.1 Revisiting Exponential

Definition 7.1. Let $z \in \mathbb{C}$, we define the complex exponential map $e^z : \mathbb{C} \to \mathbb{C}$ as

$$e^z := 1 + z + \frac{z^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Theorem 7.2 (Euler's Identity). Let $x \in \mathbb{R}$, then

$$e^{ix} = cos(x) + isin(x)$$

Proof. We can show this via explicit computation, indeed,

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + i \sum_{k=0} (-1)^k \frac{x^{2k+1}}{(2k+1)!} && \text{Separate even and odd degree terms} \\
&= cos(x) + isin(x) && \text{Taylor Series}
\end{aligned}$$

■

Lemma 7.3. Facts from Analysis:

1. For every $r \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{r^n}{n!}$ converges absolutely
2. If $a_n \in \mathbb{C}$ and $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges
3. If $L_a = \sum_{n=0}^{\infty} a_n, L_b = \sum_{m=0}^{\infty} b_m$ converges absolutely, then

   $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

   converges to

   $$(\sum_{n=0}^{\infty} a_n) \cdot (\sum_{m=0}^{\infty} b_m)$$

This will helpful in proving some identities.

Proof. For (1), we will use the ratio test. Indeed, consider $a_n = \frac{r^n}{n!}$, then

$$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \lim_{n \to \infty} |\frac{r}{n+1}| = 0 < 1$$

Thus, the series converges absolutely.

For (2), let $b_n = \sum_{i=1}^{n} a_i, c_n = \sum_{i=1}^{n} |a_i|$. We note that the topology $\mathbb{C}$ is homeomorphic to the Euclidean topology on $\mathbb{R}^2$, so in particular $\mathbb{C}$ is a complete metric space, so a sequence is convergent if and only if it is Cauchy. We know c_n converges, so for every $\epsilon > 0$, there exists some N_c such that for all $i, j > N_c$.

$$|c_i - c_j| < \epsilon$$

It remains for us to show that b_n is also Cauchy. Indeed, if $i = j$, then we are done. Without loss, we will then assume $j > i$, then

$$\begin{aligned}
|b_i - b_j| &= |\sum_{k=i+1}^{j} a_k| \\
&\leq \sum_{k=i+1}^{j} |a_k| && \text{Triangle's Inequality} \\
&= |c_i - c_j| \\
&< \epsilon
\end{aligned}$$

Thus, $\{b_n\}$ is Cauchy and converges.

For (3), $(\sum_{n=0}^{\infty} a_n)\cdot(\sum_{m=0}^{\infty} b_m)$ certainly converges to $L_a \cdot L_b$. Now write $(\sum_{n=0}^{\infty} a_n)\cdot(\sum_{m=0}^{\infty} b_m) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m$, this corresponds exactly to the terms of $\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$. The exact combinatorics is left as details to the reader. ∎

Corollary 7.4. Let e^z be the complex exponential map, then

- e^z converges for all $z \in \mathbb{C}$
- $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$

Proof. For the first, let $a_n = \sum_{k=1}^{n} |\frac{z^k}{k!}|$. It suffices for us to prove that a_n converges as absolute convergence implies monotone convergence from Lemma 7.3(2). But we note that

$$|\frac{z^k}{k!}| = \frac{|z|^k}{k!}$$

and $|z|$ is a real number, so Lemma 7.3(1) tells us that a_n converges.

For the second, we note that

$$\begin{aligned}
e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{z^k w^{n-k}}{n!} && \text{Binomial Theorem} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\
&= (\sum_{n=0}^{\infty} \frac{(z)^n}{n!}) \cdot (\sum_{n=0}^{\infty} \frac{(w)^n}{n!}) && \text{By Lemma 7.3(3)} \\
&= e^z \cdot e^w
\end{aligned}$$

∎

Proposition 7.5. We can also recover $\cos(z)$ and $\sin(z)$ from e^z as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

If $x \in \mathbb{R}$, we also have that

$$\cos(x) = \Re e^{ix}, \sin(x) = \Im e^{ix}$$

7.2 Complex Logarithms

As in the case of $\mathbb{R}$, we want to define $\log(z)$ as an inverse to e^z such that

$$e^{\log(z)} = z$$

The problem is that e^z is not actually injective, so there are multiple choices for $\log(z)$. Consequently, this will result in $\log(z)$ not being continuous on all of $\mathbb{C}$.

Question 7.6. Given z, can we find all solutions satifying $e^w = z$?

Answer. We will again leverage on Polar Coordinates. Indeed, write $z = re^{i\theta}, w = x + iy$, then we have that

$$re^{i\theta} = e^{x+iy} = e^x e^{iy}$$

Thus, $x = \log(r), y = \theta + 2\pi k$.

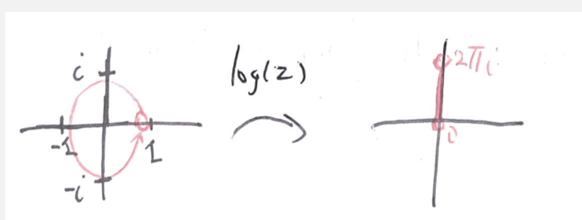
Thus, w is of the form $w_k = \log(r) + i(\theta + 2\pi k)$ ■

Definition 7.7 (Complex Logarithm). For $z \in \mathbb{C} \setminus \{0\}$, write $z = re^{i\theta}, \theta \in [-\pi, \pi]$. Then we define

$$\log(z) = \log(re^{i\theta}) := \log(r) + i\theta$$

θ is sometimes refered to as the **principal argument** of z and we denote $Arg(z) = \theta$.

Remark 7.8. Note that the complex logarithm $\log(z)$ is not continuous on all of $\mathbb{C}$:



As you trace around the unit circle, going back to $z = 1$ presents a problem.

Therefore, we can only define the complex logarithm up to a **branch cut** that's a radial line extending out from the origin. We will without loss choose this branch to be $(-\infty, 0]$

Proposition 7.9. $\log(z)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and in fact $\frac{d}{dz}\log(w) = \frac{1}{w}$

Proof. One way to do this is to use the Cauchy-Riemann equations in polar coordinates. We will however use the Inverse Function Theorem instead. Indeed, we note that on $\mathbb{C} \setminus (-\infty, 0]$, $z \mapsto e^z$, when viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, is a smooth function whose derivative is exactly

$$(e^z)' = e^x \begin{pmatrix} cos(y) & -sin(y) \\ sin(y) & cos(y) \end{pmatrix}$$

, which satisfies the matrix representation of complex numbers (hence we can extend the argument to holomorphic functions). Thus, the inverse function theorem tells us that $g(z) = \log(z)$ is also holomorphic and that

$$g'(f(x)) = (f'(x))^{-1}$$

This equation tells us that, locally for all $w = e^z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\frac{d}{dz} \log(w) = e^{-x} \begin{pmatrix} cos(y) & sin(y) \\ -sin(y) & cos(y) \end{pmatrix} = (e^z)^{-1} = \frac{1}{w}$$

Note that the branch cut does not prevent the existence of a local inverse, but it does prevent the existence of a global inverse on all of $\mathbb{C}$. ■

Remark 7.10. Let $z \in \mathbb{C} \setminus (-\infty, 0]$, then since $\frac{1}{z}$ is primitive on the path 1 to z, we have that:

$$\int_1^z \frac{d\xi}{\xi} = \log(z) - \log(1) = \log(z)$$

7.3 Homotopy Invariance of Integral

Let $f \in Hol(\Omega)$ and $\gamma : [a, b] \to \Gamma$ is continous and $\Gamma \subset \Omega$, then we can define the integral

$$\int_\gamma f(z)dz$$

Informally, we can define this because at each point on γ, we can find some disk to represent f as a power series with a primitive F, then we can split $[a, b]$ into a union of (not necessarily uniform) subintervals, then the integral γ can be approximated as a sum of the anti-derivatives F_k at each end-points.
We can formally justify this with what's called the Lebesgue's Number Lemma.

Lemma 7.11 (Lebesgue's Number Lemma)**.** Let K be a compact metric space and let $\{U_a\}$ be an open cover for K, then there exists some $\delta > 0$ (we call the **Lebesgue Number**) such that for all $x \in K$, there exists some a such that $B_{x,\delta} \subset U_a$

Proof. For all $x \in K$, since $\{U_a\}$ is an open cover of K, there exists some a such that $x \in U_a$. Since U_a is open, there exists some $r(x) > 0$ small enough such that $B_{x,2r(x)} \subset U_\alpha$.

Note that $\{B_{x,r(x)}\}$ running over all $x \in K$ is an open cover of K. Since K is compact, we in fact have a finite subcover:

$$B_{x_1,r_1}, ..., B_{x_n,r_n}$$

Choose $\delta = \min\{r_i\}$. Now for all $x \in K$, it belongs to one of the $B_{x_k,r_k} \subset U_a$.
Then we claim $x \in B_{x,\delta} \subset B_{x_k,2r_k} \subset U_a$. To do this, we need to verify $B_{x,\delta} \subset B_{x_k,2r_k}$. Indeed, for all $y \in B_{x,\delta}$, we have that

$$d(y, x_k) \leq d(y, x) + d(x, x_k) < \delta + r_k \leq 2r_k$$

This concludes the proof. ■

To apply this lemma in our context, we take $K = [a, b]$ and for all $s \in [a, b]$, we define $U_s = \gamma^{-1}(D_{\gamma(s),r(s)})$, where $r(s)$ is the radius of convergence of $\gamma(s)$ under the power-series. Then we can apply the Lebesgue's Number Lemma.

Question 7.12. Is the integral well-defined up to two different splitting?

Answer. Yes, the idea is that given two splitting on the interval, one can join them using a refinement by taking smaller intervaks of both of them and used the associated anti-derivative from either original splittings. To be more rigorous, let $\{I_k\}$ and $\{J_n\}$ be two different splittings of the interval $[0,1]$. Now define

$$I_{k,n} \coloneqq I_k \cap J_n$$

Then we note that, by definition, the integral over the splitting $I_{k,n}$ would be of the form

$$\begin{aligned}\sum \int_{I_{k,n}} f(z)dz &= \sum_k \sum_n \int_{I_{k,n}} f(z)dz \\ &= \sum_k [\sum_n \int_{I_{k,n}} f(z)dz] \\ &= \sum_k \int_{I_k} f(z)dz\end{aligned}$$

Similarly, we can also show that

$$\sum \int_{I_{k,n}} f(z)dz = \sum_n \int_{J_n} f(z)dz$$

Thus, the two splittings yield the same integral. ■

Definition 7.13. Let $\gamma_0, \gamma_1 : [a,b] \to \Omega$ be continuous. We say γ_0 is **homotopy equivalent** to γ_1 if there exists a continuous function $\Gamma : [a,b] \times [0,1] \to \Omega$ such that $\Gamma(s,0) = \gamma_0(s)$ and $\gamma(s,1) = \gamma_1(s)$ for all $s \in [a,b]$. We will also assume that either:

- All pathes are closed, for all t
$$\Gamma(a,t) = \Gamma(b,t)$$
- OR Endpoints are fixed
$$\Gamma(a,t) = \gamma_0(a) = \gamma_1(b)$$
$$\Gamma(b,t) = \gamma_0(b) = \gamma_1(b)$$

Theorem 7.14. If γ_0, γ_1 are homotopy equivalent in Ω and $f \in Hol(\Omega)$, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Remark 7.15. In a simply connected domain, any two pathes are homotopy equivalent. In otherwords, the path of integration is irrelevant.

8 Lecture 8

8.1 Homotopy Invariance of Integrals - Continued

Theorem 8.1. If γ_0, γ_1 are homotopy equivalent in Ω and $f \in Hol(\Omega)$, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Proof. Let Γ be the homotopy equivalence function. For all $z_0 \in \Omega$, let D_{z_0} be a disc centered at z_0 small enough such that f can be expressed as a power-series on it, define

$$U_{z_0} = \Gamma^{-1}(D_{z_0})$$

Then $\{U_{z_0}\}$ will be an open cover of $R := [a, b] \times [0, 1]$, which is a compact metric space. Apply Lebesgue's Number Lemma, we find $\delta > 0$ such that, we can split R into smaller rectangles R_k where each $diam(R_k) < \delta$.

The Lebesgue Number Lemma tells us that, for each R_k, $\Gamma(cl(R_k)) \subset D_z$ for some z. Since f is primitive on D_z, we have that

$$\int_{\Gamma|_{\partial R_k}} f(z)dz = 0$$

Putting the small rectangles together (Green's Theorem style), the inner edges cancel out, so we have that

$$\int_{\Gamma|_{\partial R}} f(z)dz = 0$$

Now consider the pathes $\ell_1 = \{a\} \times [0, 1]$ and $\ell_2 = \{b\} \times [0, 1]$, we claim that

$$\int_{\Gamma|_{\ell_1 + \ell_2}} f(z)dz = 0$$

Indeed, if endpoints are fixed, both paths are constant (stays at same point). If all path are closed, then this is the same path in opposite directions.

Thus, we only have to integrate over the horizontal pathes:

$$\int_{\partial R} f(z)dz = 0 = \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz$$

■

Definition 8.2. A set Ω is **simply connected** if any closed loop in Ω is homotopy equivalent to the constant path. This is equivalent to saying, for any pathes γ_0, γ_1, $\gamma_0(a) = \gamma_1(a) = z_0$, $\gamma_0(b) = \gamma_1(b) = z_1$, are homotopy equivalent.

Remark 8.3. If Ω is simply connected domain, then

$$\int_{z_0}^{z_1} f(z)dz$$

does not depend on the path specified.

Definition 8.4. Let Ω be simply connected and $f \in Hol(\Omega)$. Suppose $f(z) \neq 0$ for all $z \in \Omega$.

Fix some $z_0 \in \Omega$, and consider

$$a_0 \text{ such that } f(z_0) = e^{a_0}$$

We can find a_0 as

$$Re(a_0) = \log |f(z_0)|, Im(a_0) = arg\ f(z_0)$$

Then we define

$$\log f(z) := a_0 + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)} d\xi$$

Show that $\log f(z)$ aligns with the global definition of the complex logarithm.

Proof. Exercise. The idea is to denote $\varphi(z) = a_0 + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)} d\xi$ and show that

$$(f(z)e^{-y(z)})' = 0$$

and realize that their product must be 1. ■

♣♣♣ Mattie: [explain motivation later]

Definition 8.5. What is $f(z)^\alpha$? We define

$$f(z)^\alpha = e^{\alpha \log(f(z))}$$

This is defined when $f \in Hol(\Omega)$, $f(z) \neq 0$ on all of Ω, and Ω is simply connected.

In a simply connected domain, the branch of the logarithm, it is enough to be determined by which a_0 you choose.

8.2 Laurent Series

Suppose f is holomorphic on $\{z : a < |z - z_0| < A\}$, where we require $a \geq 0$ and $A \leq \infty$.

Theorem 8.6. If $f \in Hol\{z : a < |z - z_0| < A\}$, then f can be represented as

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

, where we have that

$$a_n = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} f(\xi)(\xi - z_0)^{-(n+1)} d\xi$$

, where r is between $a < r < A$ (Note that for $\mathbb{Z}_+$, this is the same as the Taylor Power Series)

Proof. Without loss of generality, we can shift this to $z_0 = 0$.

Take z such that $a < |z| < A$ and pick r, R such that

$$a < r < |z| < R < A$$

Then consider the set

$$G := \{z : r < |z| < R\}$$

Then Cauchy's Integral Formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi \\ &= \frac{1}{2\pi i}(\int_{|z|=R}\frac{f(\xi)}{\xi - z}d\xi - \int_{|z|=r}\frac{f(\xi)}{\xi - z}d\xi) \\ &= \frac{1}{2\pi i}(I_1 - I_2) \end{aligned}$$

For I_1, we note that $|z| < |\xi| = R$, so

$$\begin{aligned} I_1 &= \int_{|z|=R}\frac{f(\xi)}{\xi - z}d\xi \\ &= \int_{|z|=R}\frac{f(\xi)}{\xi}\frac{1}{1-\frac{z}{\xi}}d\xi \\ &= \int_{|z|=R} f(\xi)\sum_{k=0}^{\infty}\frac{z^k}{\xi^{k+1}} \end{aligned}$$

For I_2, we note that $|z| > |\xi| = r$, so

$$\begin{aligned} I_2 &= \int_{|z|=r}\frac{f(\xi)}{\xi - z}d\xi \\ &= \int_{|z|=r}\frac{-f(\xi)}{z}\frac{1}{1-\frac{\xi}{z}}d\xi \\ &= \int_{|z|=r} -f(\xi)\sum_{k=0}^{\infty}\frac{\xi^k}{z^{k+1}} \\ &= \int_{|z|=r} -f(\xi)\sum_{n=-\infty}^{-1}\frac{z^n}{\xi^{n+1}} \end{aligned}$$

Change of Variables

Since both series converge uniformly, we can integrate term by term and use Cauchy's Formula for Derivatives, rewriting both I_1 and I_2 out in series finishes the proof. ■

Now we will consider a particular case of the Laurent Series!

Definition 8.7. Let $f \in Hol(\Omega \setminus \{z_0\})$ where $z_0 \in \Omega$. In this case, we say f has a singularity at z_0.

Take δ small enough and $D_{z_0,\delta} \subset \Omega$, then the Laurent Series tells us

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$$

We say that

- f has a **removable singularity** at z_0 if $a_n = 0$ for all $n < 0$. In this case,

  $$f(z) = \sum_{n\geq 0} a_n(z - z_0)^n, f(z_0) = a_0$$

  An example of such function would be $f(z) = \frac{sin(z)}{z}$ has removable singularity at 0

- We say f has a **pole at** z_0 if $a_n \neq 0$ for finitely many $n < 0$. We say the **order of a pole** as

  $$\max\{n \geq 0 : a_{-n} = 0\}$$

- We say that f has an essential singularity if there exists infinitely many $n < 0$ such that $a_n \neq 0$.

  For example consider
  $$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$
  This has an essential singularity at $z = 0$.

These are the singularities we care about.

9 Lecture 9

9.1 Classification of Singularities

Recall, let $f \in Hol(\Omega \setminus \{z_0\})$, where $z_0 \in \Omega$, we can write f into a Laurent series

$$f(z) = \sum_{z \in \mathbb{Z}} a_n (z - z_0)^n, 0 < |z - z_0| < N$$

, and we say that z_0 is removable if $a_n = 0$ for all $n < 0$.

Theorem 9.1. z_0 is a removable singularity of f if and only if there exists a neighborhood $z_0 \in U$ such that f is bounded in $U \setminus \{z_0\}$.

Proof. If f has a removable singularity, we can write it locally as

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

which has a positive radius of convergence and hence bounded in a neighborhood of z_0.

Conversely, suppose f is bounded on $0 < |z - z_0| < \delta$. Then for all $n \geq 1$, we note that

$$a_{-n} = \frac{1}{2\pi i} \int_{|z - z_0| = r} f(z)(z - z_0)^{n-1} dz$$

As $n - 1 \geq 0$ and f is bounded, this integral is also bounded, so

$$|a_{-n}| \leq \frac{1}{2\pi} M(1)(2\pi r) = Mr$$

for all $0 < r < \delta$, we can shrink r down and have that $a_{-n} = 0$. ■

Remark 9.2. If $f(z) = o(\frac{1}{|z - z_0|})$, then f has a removable singularity at z_0.

We say f has a **pole at** z_0 if $a_n \neq 0$ for finitely many $n < 0$. We say the **order of a pole** as

$$\max\{n \geq 0 : a_{-n} = 0\}$$

Theorem 9.3. z_0 is a pole if and only if $\lim_{z \to z_0} |f(z)| = \infty$

Proof. Suppose z_0 is a pole of order m, then we can write

$$f(z) = \frac{g(z)}{(z - z_0)^m}, g(z_0) \neq 0$$

Hence we have that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} |\frac{g(z)}{(z - z_0)^m}| = \infty$$

Conversely, suppose $\lim_{z \to z_0} |f(z)| = \infty$, then we can find some $r > 0$ such that $|f(z)| > 1$ for all z in $0 < |z - z_0| < r$.

Now consider $h(z) = \frac{1}{f(z)}$ on $D_{z_0,r} \setminus \{z_0\}$, then $h(z) \in Hol(D_{z_0,r} \setminus \{z_0\})$ and $\lim_{z \to z_0} h(z) = 0$ - so we can extend h to the whole disk. Therefore it has to be the case that $|h(z)| < 1$, then by the previous theorem, this means

z_0 is a removable singularity for h, so in fact h is holomorphic on the whole disk.

Since $h(z_0) = 0$, we can find the smallest m such that $h(z) = (z - z_0)^m h_0(z)$ but $h_0(z_0) \neq 0$. Now consider $g = \frac{1}{h}$, then we have that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

Thus, z_0 is a pole of order m. ■

We say that f has an essential singularity if there exists infinitely many $n < 0$ such that $a_n \neq 0$.

Theorem 9.4 (Casorati-Weierstrass Theorem)**.** If f has an essential singularity at z_0, then for any neighborhood U of z_0, $U \subset \Omega$, $f(U)$ is **dense** in $\mathbb{C}$ (in other words, for all $w \in \mathbb{C}$, there exists a sequence of z_n such that $f(z_n) \to w$ as $z_n \to z_0$)

Proof. We will prove this using contradiction. Suppose there exists some neighborhood U of z_0 such that $f(U)$ is not dense. Then there exists some $a \in \mathbb{C}$ and $r > 0$ such that $D_{a,r} \cap f(U) = \emptyset$.

Now define $g(z) = \frac{1}{f(z)-a}$. Then clearly $g \in Hol(U)$ as the denominator is never 0. We also have that $|g| < \frac{1}{|f(z)-a|} < \frac{1}{r}$. Moreover, we note that $g(z) \neq 0$ in $U \setminus \{z_0\}$ because f is bounded. While $g(z_0)$ could be 0, we will choose the smallest m such that $g(z) = (z - z_0)^m g_0(z)$ and $g_0(z_0) \neq 0$.

Thus, $g_0(z) \neq 0$ on U. But we have that

$$f(z) = \frac{1}{g(z)} + a = \frac{1}{(z - z_0)^m g_0(z)} + a$$

, so f has a removable singularity or some pole at z_0, hence a contradiction. ■

Definition 9.5. Suppose f has a singularity at z_0 and write

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n$$

Then we define the **residue of** f **at** z_0 as a_{-1} and denote is as $Res_{z_0}(f)$ or $res(f, z_0)$

Remark 9.6. If z_0 is a pole of order 1, then

$$res(f, z_0) = \lim_{z\to z_0} (z - z_0)f(z)$$

9.2 Cauchy's Residue Theorem

Theorem 9.7 (Cauchy's Residue Theorem)**.** Let G be a bounded domain with $\partial G \in PC^1$ and $\overline{G} \subset \Omega$. Let $z_1, ..., z_n \in G$ and $f \in Hol(\Omega \setminus \{z_1, ..., z_n\})$, then

$$\int_{\partial G} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

Proof. Let D_k be small disks around each z_k and consider $\tilde{G} = G \setminus \bigcup D_k$. Note that $f(z)$ is holomorphic on $\tilde{G}$, so

$$\partial_{\partial \tilde{G}} f(z) dz = 0$$

On the other hand, we also have that $\partial \tilde{G} = \partial G - \bigcup \partial D_k$, so in other words

$$\int_{\partial G} f(z) dz = \sum_k \int_{\partial D_k} f(z) dz$$

Write f as a Laurent Series, then every term except for the -1 term is primitive and thus evaluate to 0, so we are left with:

$$\int_{\partial G} f(z) dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

■

10 Lecture 10

Clarification from last lecture: In the proof of the Casorati–Weierstrass theorem, we have that $g(z) \neq 0$ for all $z \in U \setminus \{z_0\}$, but $g(z_0) = 0$ is totally possible! Hence we write

$$f(z) = \frac{1}{g(z)} + a$$

, where f either has a pole or a removable singularity at z_0, giving the contradiction.

10.1 Computing Real Integrals with Residues

Example 10.1 (A Straight-forward Application)**.** Let $s \in \mathbb{R}$, consider the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1} e^{isx} dx$$

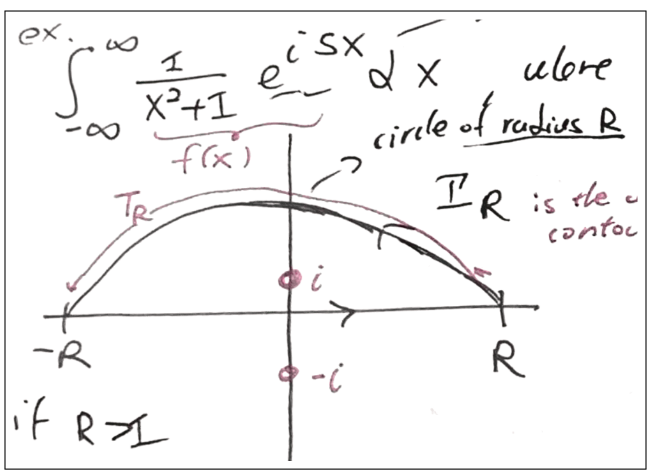
Let $f(z) = \frac{1}{x^2+1} e^{isx}$, then $I = \int_{\mathbb{R}} f(z)dz = \pi e^{-|s|}$

Remark 10.2. Why do we even care about integrals of this form? Well, they are really closed aligned with **inverse Fourier transforms**. In particular, you would calculate integrals of the form

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s} dx$$

Proof of Example. We will consider three cases $s = 0$, $s > 0$, and $s < 0$. For $s = 0$, this is obvious.

Now for $s > 0$, consider the following contour:



Then Cauchy's Residue Theorem tells us that

$$\int_{\Gamma_R} f(z)dz = 2\pi i Res(f, i)$$

i is a pole of order 1 for f, so

$$Res(f, i) = \lim_{z \to i} (z - i) f(z) = \frac{e^{-s}}{2i}$$

Thus, we have that

$$\int_{\Gamma_R} f(z)dz = \pi e^{-s}$$

Now take $R \to \infty$, we note that since f **is integrable** as $\int_{-\infty}^{\infty} |f(x)|dx < \infty$,

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} f(z)dz = I$$

So in other words,

$$I = \lim_{R\to\infty} [\int_{\Gamma_R} f(z)dz - \int_{T_R} f(z)dz] = \pi e^{-s} - \lim_{R\to\infty} \int_{T_R} f(z)dz$$

We want to show that the last term goes to 0, indeed

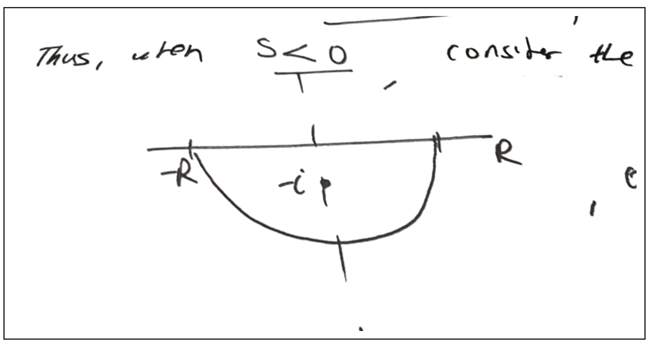
$$\int_{T_R} f(z)dz = \int_0^{\pi} \frac{e^{isRe^{it}}}{(Re^{it})^2+1} iRe^{it}dt \qquad \text{Let } z = Re^{it}$$

$$|\int_{T_R} f(z)dz| \le \frac{R}{R^2-1}\pi \qquad \text{Note that } e^{isRe^{it}} \text{ is bounded by 1 as } s > 0 \text{ and } 0 < t < \pi$$

Taking $R \to \infty$, the bound goes to 0, thus, we have that

$$I = \pi e^{-s}, s \ge 0$$

What about when $s < 0$? In this case, $\int_{T_R} f(z)dz$ would not actually go to 0, so instead, we need to consider the opposite contour:



, and we can similarly show the result is true. Alternatively, we could also have argued using symmetry.

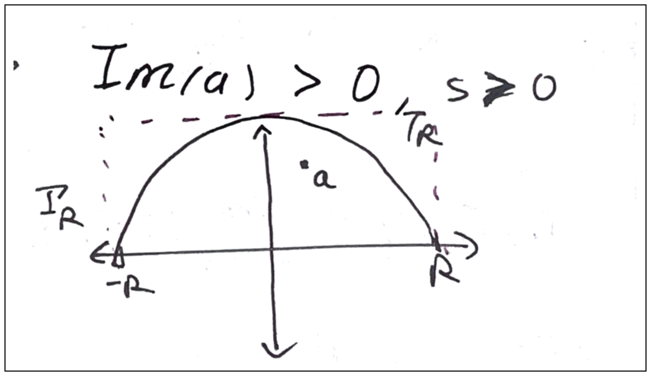
Note that while we proved this using a circle, we could have also used a rectangle instead. ■

Example 10.3 (A Not-So-Simple Application)**.** Let $s \in \mathbb{R}$ and $a \in \mathbb{C}$ such that $s > 0$ and $\Im(a) > 0$, consider the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{isx}}{x-a}dx$$

Let $f(z) = \frac{e^{isz}}{z-a}$, then what is I?

Consider the following contour:



where $R > |a|$, while we could use a rectangle, we will stick with the half-circle again. Then again Cauchys' Residue Theorem tells us that

$$\int_{\Gamma_R} f(z)dz = 2\pi i Res(f, a)$$

a is a pole of order 1 of f, so we have that

$$Res(f, a) = \lim_{z \to a}(z - a)\frac{e^{isz}}{z - a} = e^{isa}$$

However, we note that f is actually **NOT INTEGRABLE**, so the following limit need not exist

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$

Howebver, if $\int_{T_R} f(z)dz = 0$, then the limit would exist, and we would have

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)dx = 2\pi i e^{isa} = p.v \int_{-\infty}^{\infty} f(x)dx$$

However, we run into a problem with doing ML-esitmate on $\int_{T_R} f(z)dz = 0$, because it turns out it'd give us

$$|\int_{T_R} f(z)dz| \leq \frac{R}{R - |a|}\pi$$

, which does not converge to 0 as $R \to \infty$.

Fortunately, we do have a workaround:

Lemma 10.4 (Jordan's Lemma)**.** Let $C \in \mathbb{R}$ be some fixed constant, then

$$\int_{T_R} |e^{iz}||dz| \leq C$$

, where $|dz|$ is with respect to the Lebesgue Measure of the unit circle.

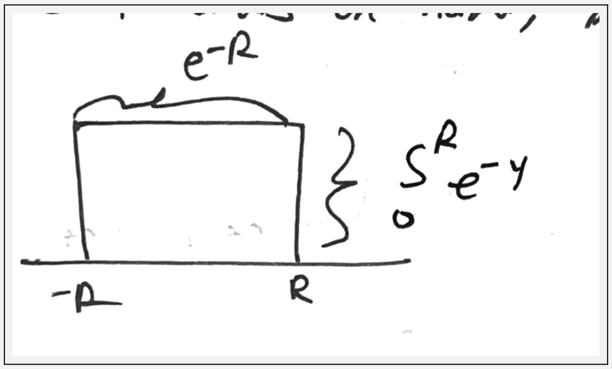
Now take $z = Re^{i\theta}$, then $dz = iRe^{i\theta}d\theta$, so we have that $|dz| = R|d\theta|$.

Corollary 10.5. If $s > 0$, then

$$\int_{T_R} |e^{iz}||dz| \leq C(s)$$

, where $C(s)$ is some constant dependent on s.

Remark 10.6. Jordan's Lemma for circles are generally hard to show, so most textbooks only prove it on a rectangle instead:



Now, using Jordan's Lemma, we have that

$$
\begin{aligned}
\left| \int_{T_R} \frac{e^{isz}}{z-a} dz \right| &\leq \frac{1}{R-|a|} \cdot \int_{T_R} |e^{isz}||dz| \\
&\leq \frac{C(s)}{R-|a|}
\end{aligned}
$$

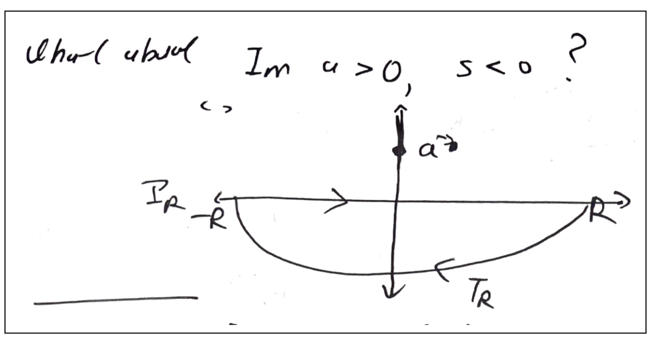
, so the limit goes to 0 as $R \to \infty$.

Question 10.7. What about if $\Im(a) < 0$ and $s > 0$?

In this case, we can either close the lower half or use a change of variables to get the same result.

Question 10.8. What about if $\Im(a) > 0$ but $s < 0$?

In this case, we will consider the contour



There are no singularities inside the contour so

$$\int_{\Gamma_R} f(z)dz = 0$$

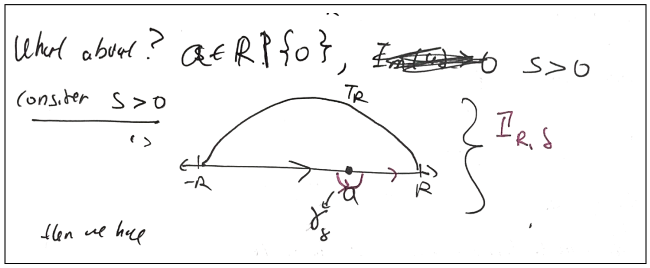
In addition, as $R \to \infty$, we also have that

$$\int_{T_R} f(z)dz = 0$$

Thus, the integral I is just 0.

Question 10.9. What if $a \in \mathbb{R} \setminus \{0\}$ and $s > 0$?

In this case, consider the contour:



In this, case we note that as we expand R and shrink δ, we have that

$$\int_{[-R,R]\setminus[a-\delta,a+\delta]} \mapsto p.v \int_{-\infty}^{\infty} f(z)dz$$

Cauchy's Residue Theorem tells us that

$$\int_{\Gamma_{R,\delta}} f(z)dz = 2\pi i Res(f,a) = 2\pi i e^{isa}$$

Jordan's Lemma tells us that

$$\lim_{R\to\infty} \int_{T_R} f(z)dz = 0$$

Finally, taking $\delta \to 0$ gives that

$$\int_{\gamma_\delta} f(z)dz = \pi i e^{isa}$$

Thus, we have that

$$p.v \int_{-\infty}^{\infty} f(z)dz = 2\pi i e^{isa} - \pi i e^{isa} = \pi i e^{isa}$$

11 Lecture 11

11.1 Computing Real Integrals with Residues - Continued

Example 11.1. The Fresnel Integrals are the real and imaginary parts of the following integral:

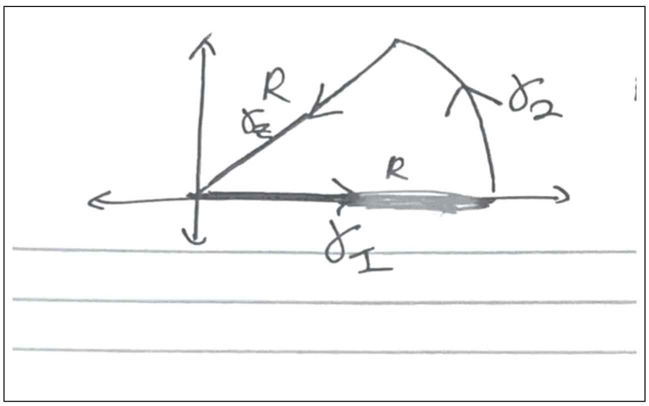
$$I = \int_0^\infty e^{-ix^2}\,dx$$

How does one compute I?

Answer. We will first look at the Gaussian Integral, which we are quite familiar with already:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Now for the Fresnel Integrals, consider the following contour:



We note that e^{-iz^2} is analytic over the entire region, so Cauchy's Integral Theorem tells us that

$$0 = \int_{\gamma_1} e^{-iz^2}\,dz + \int_{\gamma_2} e^{-iz^2}\,dz + \int_{\gamma_z} e^{-iz^2}\,dz$$

We note that as $R \to \infty$, a similar argument as before show that the integral over γ_2 vanishes. Thus, we have that

$$I = \lim_{R\to\infty} \int_{\gamma_1} e^{-iz^2}\,dz = -\lim_{R\to\infty} \int_{\gamma_z} e^{-iz^2}\,dz$$

We can parameterize γ_3 as $z = \frac{1+i}{\sqrt{2}}t$ from $t = R$ to $t = 0$, then

$$\begin{aligned}
\int_{\gamma_z} e^{-iz^2}\,dz &= \int_R^0 e^{-t^2}\frac{1+i}{\sqrt{2}}dt \\
&= -\frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}\,dt \\
\lim_{R\to\infty} \int_{\gamma_z} e^{-iz^2}\,dz &= -\frac{1+i}{\sqrt{2}}\int_0^\infty e^{-t^2} \\
&= -\frac{1+i}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}
\end{aligned}$$

∎

11.2 Argument Principle

Let G be a bounded domain and $\partial G \in PC^1$, and consider points $p_1, ..., p_m \in G$. Suppose $f \in Hol(cl(G) \setminus \{p_1, ..., p_m\})$, meaning that f is holomorphic in $\Omega \setminus \{p_1, ..., p_m\}$ where Ω is an open set containing $cl(G)$.

Suppose furthermore that $f(z)$ is not the constant zero function. We note that $cl(G)$ is compact, so the asusmption above implies that f has finitely many zeroes in $cl(G)$. Otherwise, if $f(z)$ has infinitely many zeroes, compact sets are sequentially compact in $\mathbb{C}$, so f would containing some zero that's not isolated, violating the Uniqueness Theorem. Let $z_1, ..., z_N$ be the zeroes of $f(z)$.

Now assume that $f(z) \neq 0$ for all $z \in \partial G$, the Argument Principle states that:

Theorem 11.2 (Argument Principle)**.** Let Z be the number of zeros inside G (counting order of the zero) and P be the number of poles inside G (counting order of the pole), then

$$Z - P = \frac{1}{2\pi i} \int_{\partial G} \frac{f'(z)}{f(z)} dz$$

Note that when we say number of zeroes, we are always counting multiplicity.

Proof. We first note that $\frac{f'(z)}{f(z)}$ is not holomorphic on a given z if and only if z is one of the poles or the zeros of $f(z)$.

Since f is analytic, take $a \in cl(G)$, let $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$, then we note that

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g * z(} \\ &= \frac{m}{z-a} + \frac{g'(z)}{g(z)} \end{aligned}$$

Since $g(a) \neq 0$, we note that $\frac{g'(z)}{g(z)}$ is analytic in a neighborhood of a and can be represented as a power series, so

$$\frac{f'(z)}{f(z)} = m(z-a)^{-1} + \sum_{n=0}^{\infty} b_n (z-a)^n$$

Thus, in other words, m is the coefficient of the -1-th power term, hence

$$res(\frac{f'(z)}{f(z)}, a) = m$$

If a is one of the zeroes z_k, then m is exactly the order of the zero.

If a is one of the poles p_k, then m is given by factoring out the Laurent Series. Clearly m is exactly -1 times the order of the pole.

Thus, applying Residue's Theorem around the zeroes and the poles, we have that

$$Z - P = \frac{1}{2\pi i} \int_{\partial G} \frac{f'(z)}{f(z)} dz$$

■

Remark 11.3. Why is this theorem called the **Argument Principle**? This is because $\frac{f'(z)}{f(z)}$ is actually primitive given a chosen branch, and

$$\frac{f'(z)}{f(z)} = [\log f(z)]'$$

Thus, often we sometimes rewrite the Argument Principle as

$$Z - P = \frac{1}{2\pi i} \int_{\partial G} d\log(f(z))$$

We claim that in fact

$$\frac{1}{2\pi i} \int_{\partial G} d\log(f(z)) = \frac{1}{2\pi} \int_{\partial G} d(\arg f(z))$$

Proof. We first note that we can rewrite $f(z)$ in its polar coordinate form as

$$f(z) = r(z)e^{i \cdot arg(f(z))}$$

, then we have that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial G} d\log(f(z)) &= \frac{1}{2\pi i} \int_{\partial G} d(\log[r(z)e^{i \cdot arg(f(z))}]) \\ &= \frac{1}{2\pi i} \int_{\partial G} d(\log r(z)) + \frac{1}{2\pi i} \int_{\partial G} i \cdot d(arg(f(z))) \\ &= \frac{1}{2\pi i} \int_{\partial G} d(\log r(z)) + \frac{1}{2\pi} \int_{\partial G} d(\arg f(z)) \end{aligned}$$

It remains for us to show that $\frac{1}{2\pi i} \int_{\partial G} d(\log r(z))$ is actually 0. We first note that by the Argument Principle, this integral is an integer. Now, since $r(z)$ is a real valued function, $\int_{\partial G} d(\log r(z))$ is real, hence the integral is also a complex number.

The only number that is both complex and integer is 0.

Thus, we conclude that

$$\frac{1}{2\pi i} \int_{\partial G} d\log(f(z)) = \frac{1}{2\pi} \int_{\partial G} d(\arg f(z))$$

∎

11.3 Rouche's Theorem

Rouche's Theorem is a standard corollary of the Argument Principle. Assume the same setup as before:

Suppose G is a bounded domain with $\partial G \in PC^1$ and $f, h \in Hol(cl(G))$:

Theorem 11.4 (Rouche's Theorem)**.** Suppose that $|h(z)| < |f(z)|$ for all $z \in \partial G$, then the number of zeroes of $f + h$ in G is the same as the number of zeroes of f in G.

Proof. Let $\varphi(z) = \frac{f(z)+h(z)}{f(z)}$. We claim that the number of zeroes of φ is the number of poles of φ (inside G). This is because the common zeroes of $f(z) + h(z)$ and $f(z)$ would cancel out, while the rest would match, so we claim we could without choose f, h such that they don't have any common zeroes.

Before we proceed with the rest of the proof, we will first justify why this is true. Now suppose $f(z) + h(z)$ and $f(z)$ have common zeroes $c_1, ..., c_m$ in G (counting multiplicity). We note that their common zeroes have to be finite since $cl(G)$ is compact, so having a infinite number of zeroes would imply that both functions here are identically zero (by the uniqueness theorem).

Now $f(z)+h(z)$ and $f(z)$ have common zeroes $c_1, ..., c_m$ if and only if $f(z), h(z)$ have common zeroes $c_1, ..., c_m$.

Now we claim that in general, for $g \in Hol(\Omega)$ and $z_0 \in \Omega$ such that $g(z_0) = 0$, $\frac{g(z)}{z-z_0}$ has a removable singularity at z_0 (this just comes from the fact that its neighborhoods are bounded). So we can without loss view the quotient with the limit filled in as a holomorphic function on Ω.

Applying this principle to $f(z)$ and $h(z)$, we have that

$$f(z) = \prod_{k=1}^{m}(z-c_k)f_0(z), h(z) = \prod_{k=1}^{m}(z-c_k)h_0(z)$$

, then the extraneous terms would cancel out but $f_0(r) \neq 0, h_0(r) \neq 0$ for any $r \in \{c_1, ..., c_m\}$.

Now, we write $\varphi(z) = 1 + \frac{h(z)}{f(z)}$, and note that

$$|\frac{h(z)}{f(z)}| < 1 \text{ on } \partial G$$

, so the values of $\varphi(z)$ are exactly values contained in $D_{1,1}$ (the open disk centered at 1 of radius 1) by Triangle's Inequality. In particular we note this means that $Re(\varphi(z)) > 0$.

Therefore, we note that $Log\varphi(z)$ ($z \in \partial G$) is well-defined as we avoided the branch cut at $(-\infty, 0]$. Thus, it is a well-defined anti-derivative where

$$(Log\varphi(z))' = \frac{\varphi'(z)}{\varphi(z)}$$

, hence we have that

$$\int_{\partial G} \frac{\varphi'(z)}{\varphi(z)} = 0$$

Then by the argument principle, the number of zeroes and φ is the same as the number of poles of φ in G. ■

Corollary 11.5 (The Fundamental Theorem of Algebra)**.** Let $p(z) = \sum_{k=0}^{n} a_k z^k$, where $a_n \neq 0$, then $p(z)$ has n complex roots (counting multiplicity).

Proof. Take $G = D_{0,R}$ where $R > 0$ is big enough that

$$|a_n|R^n > \sum_{k=0}^{n-1} |a_k|R^k$$

, we can find this R because

$$\lim_{R\to\infty} \frac{\sum_0^{n-1} |a_k|R^k}{|a_n|R^n} = 0$$

We note that on $z \in \partial D_{0,R}$, $|a_n z^n| > |p(z) - a_n z^n|$, so we can take $f(z) = a_n z^n$ and $h(z) = p(z) - a_n z^n$.

Then we note that $a_n z^n$ has n-roots, so Rouche's Theorem tells us that $p(z)$ has n-roots. ■

11.4 Hurwitz Theorem

Definition 11.6. Let $\{f_n\}$ be a sequence of functions, we say $f_n \to f$ converes normally in G if for all compact $K \subset G$, f_n converges to f on K uniformly.

In this class, when we say f_n converges to f and f_n are holomorphic on G, then we always mean that the convergence is normal!

Proposition 11.7. If f_n are holomorphic functions on G that converges to f normally, then f is also holomorphic on G. Furthermore, we have that $f_n^{(k)}$ converges to $f^{(k)}$ normally.

Proof. Let R be a rectangle contained in G, since f_n converges to f normally, f_n converges to f uniformly on R, so we can switch the integral and limit to see that

$$0 = \lim_{n\to\infty} \int_{\partial R} f_n dz = \int_{\partial R} \lim_{n\to\infty} f(z)dz = \int_{\partial R} f(z)dz$$

, thus Morera's Theorem tells us that f is holomorphic on G.

Now for the convergence of $f_n^{(k)}$ to $f^{(k)}$, let K be a compact set in $G \subset \Omega$. For all $z \in K$, consider the disk D_z small enough that $cl(D_z) \subset \Omega$, then the collection $\{D_z, z \in K\}$ form an open cover of K. As K is compact, we can find a finite subcover $D_{z_1}, ..., D_{z_n}$.

We will without loss take $G = \bigcup_{i=1}^{n} D_{z_i}$ (as we only need to show uniform convergence on K). This is going to be a bounded domain with piecewise smooth boundary, so we can use Cauchy's Formula for Derivatives and see that

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial G} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} d\xi$$

We now note that as both K and ∂G are compact and disjoint,

$$dist(K, \partial G) = \inf_{(x,y)\in K\times\partial G} d(x, y) = \delta > 0$$

Thus, for any $z \in K, \xi \in \partial G$, $|z - \xi| \geq \delta$, so we have that

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{1}{2\pi} \frac{k!}{\delta^{k+1}} \cdot [\max ||f_n - f||] \cdot Len(\partial G)$$

Now since f_n converges to f uniformly on G, it converges in measure and hence $[\max ||f_n - f||]$ shrinks to 0 as $n \to \infty$. Thus, we have that

$$\lim_{n\to\infty} |f_n^{(k)}(z) - f^{(k)}(z)| = 0$$

, hence we have shown uniform convergence on K. ∎

Definition 11.8 (Alternative Definition of Normal Convergence). If for all disc D such that $cl(D) \subset \Omega$, f_n converges to f on D uniformly, then we say f_n converges to f normally in Ω.

Note that the two definition of f_n are equivalent.

Theorem 11.9 (Hurwitz's Theorem). Suppose $f \in Hol(\Omega)$ and $z_0 \in \Omega$ is a zero order m.

Let $z_0 \in U$, where U is some bounded open set, and $\partial U \in PC^1$.

Now suppose for all $z \in cl(U) \setminus \{z_0\}$, $f(z) \neq 0$, and suppose $\{f_n\}$ converges to f normally on Ω, then there exists some $N \in \mathbb{N}$ such that for all $k > N$, f_k has exactly m zeroes in U.

Proof of Hurwitz's Theorem. We know that f_n converges uniformly to f on ∂U and f_n' converges uniformly to f' on ∂U. Since $f(z) \neq 0$ on ∂U, we note that

$$\frac{f_n'}{f} \to \frac{f'}{f} \text{ converges uniformly on } \partial U$$

Since the convergence is uniform, we also have that

$$\frac{1}{2\pi i}\int_{\partial U}\frac{f_n'(z)}{f_n(z)}dz \to \frac{1}{2\pi i}\int_{\partial U}\frac{f'(z)}{f(z)}dz$$

, but we note that the Argument Principle tells us that both values above are integers. Now pick some $\epsilon < \frac{1}{2}$, then there exists some N such that for all $n > N$,

$$|\frac{1}{2\pi i}\int_{\partial U}\frac{f_n'(z)}{f_n(z)}dz - \frac{1}{2\pi i}\int_{\partial U}\frac{f'(z)}{f(z)}dz| < \epsilon$$

, and hence they are the same integer. But we note that $f'(z)/f(z)$ has no zeroes and only a pole of order m while $f_n'(z)/f_n(z)$ has no poles but only zeroes.

Thus, $f_n(z)$ has m zeroes (counting multiplicity). ■

> **Remark 11.10.** Note that Hurwitz's Theorem tells us that zeroes have to be appear gradually in the convergence.
>
> This need not be true in the real case. For example, take $f_n(x) = x^2 + \frac{1}{n}$. This is not zeroes on $\mathbb{R}$. However, the limit is x^2 and has 2 zeroes at $x = 0$. So the zeroes can just pop up out of nowhere.

12 Lecture 12

12.1 Remark: Normal Convergence

Note that the following facts holds on just in $\mathbb{C}$, but in $\mathbb{R}^n$ in general. Recall the two definitions of normal convergence:

Definition 12.1 (Equivalent Definitions of Normal Convergence)**.** Let $\{f_n\}$ be a sequence of functions, we say $f_n \to f$ converes normally in G if:

- (a) for all compact $K \subset G$, f_n converges to f on K uniformly.
- or, (b) if for all disc D such that $cl(D) \subset \Omega$, f_n converges to f on D uniformly.

The two definitions are in fact equivalent.

Proof. $(a) \implies (b)$ is obvious. Now, for $(b) \implies (a)$, suppose $K \subset \Omega$ is compact, then for all $z \in K$, we can choose D_z be an open disk centered at z small enough that $cl(D_z) \subset \Omega$.

Then the collection $\{D_z\}_{z \in K}$ form a clear open cover of K, so we can find a finite subcover, $D_{z_1}, ..., D_{z_n}$. f converges uniformly on each $cl(D_{z_i})$, so f converges uniformly on $\bigcup_{k=1}^{n} D_{z_i}$, which covers K. ■

How do we in general check for normal convergence? It seems like we will have to check uncountably many disks!

Proposition 12.2 (Strong Covering Property)**.** Consider the sequence $\{K_n\}_{n=1}^{\infty} \subset \Omega$ such that for all compact $K \subset \Omega$, there exists some N such that $K \subset \bigcup_{n=1}^{N} K_n$.

Then, f_n converges to f uniformly on all compact sets $K \subset \Omega$ if and only if f_n converges to f uniformly on each K_n.

Proof. The forward direction is obvious. Conversely, for any compact set K, find $N \in \mathbb{N}$ such that $K \subset \bigcup_{n=1}^{N} K_n$. Then the uniform convergence on each K_n implies uniform convergence on all finite unions on all of $\bigcup_{n=1}^{N} K_n$, hence the uniform convergence is held on K. ■

Remark 12.3. In practice, for open subset $\Omega \subset \mathbb{R}^d$, we would define each K_n as

$$K_n := \{x \in \Omega \mid dist(x, \Omega^c) > \frac{1}{n}, |x| \leq n\}$$

♣♣♣ Mattie: [Ask about this]

Definition 12.4. We say the collection $\{K_n\}_{n=1}^{\infty}$ is a **compact exhaustion** of Ω if

$$K_n \subset int(K_{n+1}) \text{ and } \bigcup_{n=1}^{\infty} K_n = \Omega$$

, note that the latter implies for any compact $K \subset \Omega$, there exist $N \in \mathbb{N}$ such that $K \subset \bigcup_{n=1}^{N} K_n$

12.2 Seminorms of Holomorphic Function

Definition 12.5. Let $Hol(\Omega)$ be the set of all holomorphic functions on Ω, given $f \in \Omega$, we define the norm

$$||f||_{C(K_n)} = |\sup_{z \in K_n} f(z)|$$

When K_n are points, this is called a semi-norm. $C(K_n)$ refers to the space of continuous real-valued functions on K_n.

Proposition 12.6. If K_n has the strong covering property, then f_n converges to f on compact sets uniformly if and only if $||f_n - f||_{C(K_n)} \to 0$ as $j \to \infty$.

Remark 12.7. An important object of study in functional analysis are what's called **Frechet Spaces**, whose topology is given by countably many semi-norms.

Definition 12.8. Let $f, g \in Hol(\Omega)$, we can define a metric on $Hol(\Omega)$ as

$$p(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||f-g||_{C(K_n)}}{1 + ||f-g||_{C(K_n)}}$$

One can verify that this is indeed a metric (more details in Conway's Book)

Proposition 12.9. f_n converges to f uniformly on compact sets if and only if $p(f_n, f) = 0$ as $n \to \infty$.

In practice, no one would actually compute the norm or fixes a compact exhaustion. However, there are great theoretical values in this metric.

Theorem 12.10. $(Hol(\Omega), p)$ is a complete metric space.

Proof. We first note that for a compact set K, $C(K)$ is a complete normed space (hence a Banach Space), so we observes that

- 1. $\lim_{k\to\infty} p(f_k, f) = 0$ if and only if $\lim_{k\to\infty} ||f - f_k||_{C(K_n)} \to \infty$ for all n.
- 2. $\{f_k\}$ is Cauchy with respect to p on $Hol(\Omega)$ if and only if f_k is Cauchy with respect to $||\cdot||_{C(K_n)}$ on $C(K_n)$ for every n

Now, every convergent sequence is Cauchy. Conversely, suppose $\{f_k\}$ is a Cauchy sequence, then since each $C(K_n)$ is a complete normed space, Hence on each K_n, $\{f_k\}$ converges uniformly to some limit F_n.

We claim that we can actually find a global function $f \in C(\Omega)$ such that $f|_{K_n} = F_n$. Indeed, we want to essentially glue each F_n together, which follows from the glueing lemma that for any two continuous function f on K_1 and g on K_2 (within $\mathbb{C}$ so Hausdorff) with non-empty intersection $K_1 \cap K_2$, we can extend f and g together into a larger continuous function.

Now, since f is the limit on each K_n, we have that $||f_k - f||_{C(K_n)} \to 0$ as $k \to \infty$, for all n. Hence we have from Observation 1 that $\lim_{k\to\infty} p(f_k, f) = 0$, so f is the limit of the Cauchy seuqnece in $Hol(\Omega)$.

It remains for us to show that $f \in Hol(\Omega)$. Indeed, we note that for any $K \subset \bigcup_1^N K_j$, f_n converges to f uniformly, hence by Morera's Theorem, we can get that $f \in Hol(\Omega)$. ■

12.3 Open Mapping Theorem and Inverse Function Theorem

Suppose $f \in Hol(z)$ (non-constant), $z_0 \in \Omega$, $f(z_0) = w_0$. Let G be a bounded domain with $z_0 \in G$ and $\partial G \in PC^1$.

Furthermore, for all $z \in \partial G$, suppose we have $|f(z) - w_0| \geq \delta > 0$.

Does such G always exist? Yes!
This is because the zeroes of the function $f(z) - w_0$ are isolated (as $f(z)$ is non-constant). Therefore, $f(z) - w_0$ has no zeroes in $cl(D_{z_0,r}) \setminus \{z_0\}$ for a sufficiently small r.

Let $w \in \mathbb{C}$ such that $|w - w_0| < \delta$, and let m be the multiplicity of the zero z_0 of $f(z) - w_0$

Lemma 12.11. Fix w as above, the fiber $f^{-1}(w)$ has m points counting multiplicity in G, meaning that the function $g(z) = f(z) - w$ has zeroes $z_1, ..., z_m$ (counting multiplicity)

Proof. We will apply Rouche's Theorem on this. Indeed, let $g_0(z) = f(z) - w_0$, then we note that

$$g(z) = f(z) - w = [f(z) - w_0] + [w - w_0]$$

By our setup above, we know that $|f - w_0| \geq \delta$ and $|w - w_0| < \delta$. So Rouche's Theorem tells us that both $f(z) - w_0$ and $g(z) = [f(z) - w_0] + [w - w_0]$ have the same number of zeroes, counting multiplicity.

m was defined to be exactly the multiplicity of z_0 at $f(z) - w_0$, as we chosen G small enough that $f(z) - w_0$ has no other zeroes within. ■

Corollary 12.12 (Open Mapping Theorem)**.** If $f \in Hol(\Omega)$ be a non-constant function, then f is an open map, meaning that for all open sets $U \subset \Omega$, $f(U)$ is open.

Proof. The argument here will run similarly to how we had above.

Let $w_0 \in f(U)$, we want to show that there exist some open set $V \subset f(U)$ that contains w_0.

Since $w_0 \in f(U)$, we know there exist some point $z_0 \in U$ such that $f(z_0) = w_0$. Since U is open, we can find some radius $\epsilon > 0$ small enough that $cl(D_{z_0,\epsilon}) \subset U$.

Now consider the function $g(z) = f(z) - w_0$. Since g is holomorphic and non-constant, the uniqueness theorem for analytic function tells us that the zeros of $g(z)$ are isolated. Hence, we can choose $\epsilon > 0$ small enough that without loss $g(z)$ only has the root z_0 in $cl(D_{z_0,\epsilon})$.

Since $\partial D_{z_0,\epsilon}$ is compact, and $|g(z)|$ is continuous and positive, the extreme value theorem shows that there exist some minimum value $\delta > 0$ such that δ is the minimum of $|g(z)|$ for z on the boundary.

Now consider the disk $D_{w_0,\delta}$. For any $w \in D_{w_0,\delta}$, by Rouche's Theorem, write $g(z) = f(z) - w = [f(z) - w_0] + [w - w_0]$, then we note that $|g(z)| \geq \delta > |w - w_0|$, then this means that $f(z) - w_0$ and $g(z)$ have the same number of zeroes in the disk $D_{z_0,\epsilon}$.

In particular, this means that there exist some $z \in D_{z_0,\epsilon}$ such that $f(z) = w$. So in other words, $w \in f(U)$.

Hence, $D_{w_0,\delta} \subset f(U)$. In other words, $w_0 \in D_{w_0,\delta} \subset f(D_{z_0,r}) \subset U$, which implies that $f(U)$ is open. ■

Theorem 12.13 (Inverse Mapping Theorem). Let $f \in Hol(\Omega)$, $z_0 \in \Omega$, and $f(z_0) = w_0$ but $f'(z_0) \neq 0$ (so the zero is simple). Let $G \subset \Omega$ be an open bounded set that contains z_0 with piecewise C^1 boundary.

Now suppose that, for all $z \in \partial G$, $|f(z) - w_0| \geq \delta$, then for all w such that $|w - w_0|$, there exists some unique function z such that $z = f^{-1}(w)$ and

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi$$

13 Lecture 13

13.1 Inverse Mapping Theorem - Continued

Recall that per the setting of the Inverse Mapping Theorem:

- Let $f \in Hol(\Omega)$ and $z_0 \in \Omega$, $f(z_0) = w_0$ and $f'(z_0) \neq 0$
- G is a bounded domain such that $cl(G) \subset \Omega$ (in textbook we choose $D_{z_0,p}$) and $\partial G \in PC^1$
- $|f(\xi) - w_0| \geq \delta$ for all $\xi \in \partial G$
- $f(z) \neq w_0 \forall z \in G$ (possible because all zeroes of non-constant analytic functions are isolated)

Theorem 13.1 (Inverse Function Theorem)**.** Given the setup above, then for all w such that $|w - w_0| < \delta$, there exists a unique z in G such that $f(z) = w$ and

$$f^{-1}(w) = z = \frac{1}{2\pi i} \int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi$$

Proof. **Existence and uniqueness are given by Rouche's Theorem.** Indeed, by the Argument Principle, we note that

$$\# \text{ of zeroes of } f(z) - w \text{ in } G \; = \frac{1}{2\pi i} \int_{\partial G} \frac{f'(\xi)}{f(\xi) - w} d\xi$$

We note that we can rewrite

$$f(z) - w = [f(z) - w_0] + [w_0 - w]$$
$$|f(z) - w_0| \geq \delta, |w_0 - w| < \delta, \text{ on } \partial G$$

So Rouche's Theorem tells us that $f(z) - w$ and $f(z) - w_0$ has the same number of zeroes. But we note that we chose $f(z) - w_0$ to only have one zero, and it's a zero of order 1 by condition given, so in other words $f(z) - w$ has exactly 1 solution, hence uniqueness and existence are both proven.

For the formula given, we first note that the formula

$$\frac{\xi f'(\xi)}{f(\xi) - w}$$

has a unique singularity at $\xi = f^{-1}(w)$, which is a pole of order 1 by construction. We can calculate its residue as

$$\begin{aligned}
Res\left(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z\right) &= \lim_{\xi \to z} \frac{\xi f'(\xi)}{f(\xi) - w} \cdot (\xi - z) \\
&= z f'(z) \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - w} \\
&= z f'(z) \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - f(z)} && w \text{ is defined to be } w = f(z) \\
&= z f'(z) \frac{1}{f'(z)} \\
&= z
\end{aligned}$$

Then apply Cauchy's Residue Theorem gives us

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi &= \frac{2\pi i}{2\pi i} Res\left(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z\right) \\
&= Res\left(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z\right) \\
&= z \\
&= f^{-1}(w)
\end{aligned}$$

■

Note that the local solvability of f is just a standard result that can follow from Real Analysis too. However, the formula is more significant.

13.2 Winding Numbers

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous function such that $\gamma(a) = \gamma(b)$ (This is a C^0-closed-path). We will denote Γ as the image of γ in $\mathbb{C}$.

Without loss of generality, we can view γ as a function:

$$\gamma : \mathbb{T} \to \mathbb{C}$$

, where $\mathbb{T} := \frac{\mathbb{R}}{\mathbb{Z}}$ is the 1-dimensional torus given by an equivalence relation on $\mathbb{R}$, where we say $a \sim b$ if $a - b \in \mathbb{Z}$.

Definition 13.2. Let γ be as before and let $z \in \mathbb{C} \setminus \Gamma$, then we define the **winding number** as

$$w(\gamma, z) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z} d\xi$$

Proposition 13.3. $w(\gamma, z)$ is an integer valued function.

Proof. The proof is similar to the idea behind the Argument Principle, but we do not assume γ is C^1. We first use the Lebesgue Number Lemma to split Γ into small enough intervals such that $\frac{1}{\xi - z}$ locally has anti-derivatives on each interval (up to the choice of some branch):

$$log(\xi - z) + 2\pi ik, k \in \mathbb{Z}$$

As we move from one interval to another, we want to continue the branch we are previously on. We don't know what the branch is exactly, but this would form a telescoping series of primitivies, and the entire integral will evaluate to

$$\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z} d\xi &= \frac{1}{2\pi i} [log(\xi - z) + 2\pi ik_1 - log(\xi - z) - 2\pi ik_2] \\
&= \frac{2\pi i}{2\pi i}(k_1 - k_2) \\
&= k_1 - k_2 \in \mathbb{Z}
\end{aligned}$$

■

Proposition 13.4. $w(\gamma, z)$ measures the number of times (sign given by whether it's counter-clockwise or clockwise) γ wraps around z.

Proof Idea. We note that γ is homotopy equivalent to a circle, so the value of the integral doesn't change if we switch γ to a circle of some positive radius around z. The branch choice given in the previous proposition actually follows in a circle (when viewed as a Riemann Surface), so the loop on the circle does measure the numbers of times it wraps around z. This is an invariant around homotopy equivalence. ■

Definition 13.5. We say that γ is a **generalized path** if $\gamma : \mathbb{T}_1 \sqcup \mathbb{T}_2 ... \sqcup \mathbb{T}_n \to \mathbb{C}$ and γ restricted to each $\mathbb{T}_k$ is a C^0-closed-path. In other words, γ is intuitively the disjoint union of finitely many closed paths.

> **Remark 13.6.** Let γ be a generalized path, and consider the function $z \mapsto W(\gamma, z)$. This is a well-defined function on $\mathbb{C} \setminus Im(\gamma)$ and is moreover analytic on the same domain (this follows from Morera's Theorem). Thus, analyticity implies continuity, and for $W(\gamma, z)$ to be a continuous integer value, this means that $W(\gamma, z)$ is constant on each connected component of $\mathbb{C} \setminus Im(\gamma)$. We sometimes call this the **index** of γ at z.

13.3 Generalized Residue Theorem

> **Theorem 13.7** (Generalized Residue Theorem)**.** Let γ be a generalized path in Ω, and say $f \in Hol(\Omega \setminus \{z_1, ..., z_n\})$, and suppose for all k, $z_k \notin Im(\gamma)$. Then
>
> $$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} w(\gamma, z_k) \cdot Res(f, z_k)$$

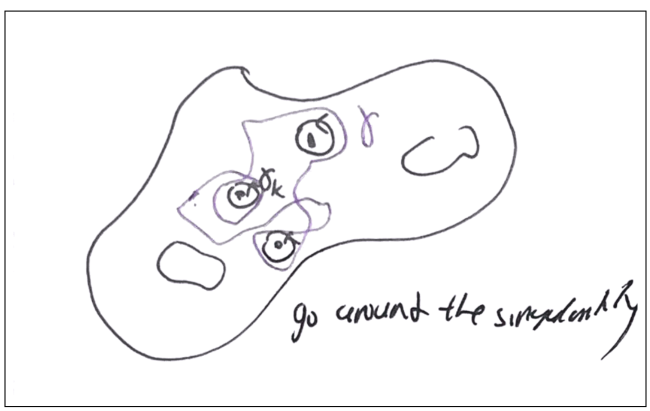
Before, we prove this theorem, we will introduce another theorem.

> **Theorem 13.8.** Let γ be a generalized path in Ω, such that
>
> $$W(\gamma, z) = 0, \text{ for all } z \notin \Omega$$
>
> (Note that this is true for any simply connected domain Ω. If Ω has holes, then this is say that γ avoids these holes). Then we have that
>
> $$\int_\gamma f(z)dz = 0, \text{ for all } f \in Hol(\Omega)$$

Proof. Textbook (to be added) ■

It turns out that Theorem 13.8 actually implied the Generalized Residue Theorem.

Proof of the Generalized Residue Theorem. Consider contours $\gamma_1, ..., \gamma_n$ be circles of small enough radius, each wrapping around $z_1, ..., z_n$ counterclock direction:



Then consider the path $\hat{\gamma}$ given by

$$\hat{\gamma} := \gamma - \bigcup_{k=1}^{n} W(\gamma, z_k)\gamma_k$$

Then we note that for all $z \not{i}n\Omega$, $W(\hat{\gamma}, z) = 0$ because the index of $gam\hat{m}a$ at these points is reduced to 0.

So Theorem 13.8 tells us that

$$\begin{aligned}
0 &= \int_{\hat{\gamma}} f(z)dz \\
&= \int_{\gamma} f(z)dz - \sum_{k=1}^{n} \int_{\gamma_k} W(\gamma, z_k) f(z)dz \\
&= \int_{\gamma} f(z)dz - [2\pi i \sum_{k=1}^{n} w(\gamma, z_k) \cdot Res(f, z_k)]
\end{aligned}$$

■