Complex Function Theory

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These are lecture notes from **MATH 2250: Complex Function Theory** with Professor Sergei Treil at Brown University for the Fall 2022 semester, with some supplemental materials written by the note-taker herself. The most up-to-date version of the notes are maintained in the note repository.

These notes are taken by Mattie Ji with gracious help and input from the instructor of this course. If you find any mistakes in these notes, please feel free to direct them via email to me or send a pull request on GitHub.

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1 Lecture 1 - 09/07/2022

There are a lot of topics in Complex Analysis, it is the goal of the instructor to be a guide around these topics. There're many different treatments of Complex Analysis - some prefer Algebraic ways, some prefer pictorial ways, etc. Because of this, we'll not be strictly following the textbook. Sometimes we will present proofs that are closely aligned to the textbook, but sometimes we will deviate a lot from it, hence why notes are essential.

1.1 Review of Complex Analysis

Definition 1.1 (Complex Number)**.** A complex number z is denoted as $z = x + iy$, $x, y \in \mathbb{R}$ and i is the root satisfying $i^2 = -1$. The collection of all complex numbers is denoted as $\mathbb{C}$. We define addition and multiplication over $\mathbb{C}$ in the usual sense of algebra.

There's a close similarity between $x+iy \in \mathbb{C}$ and $(x, y) \in \mathbb{R}^2$. Concretely, when viewed as pairs over $\mathbb{R}^2$, the addition and multiplication of complex numbers becomes:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

Proposition 1.2. The complex numbers $\mathbb{C}$ form a field.

Proof. It turns out $\mathbb{C}$ is isomorphic to $\frac{\mathbb{R}[x]}{(x^2+1)}$ as commutative rings (would be nice to check that $\mathbb{C}$ is a commutative ring in the first place). Since x^2+1 is irreducible over $\mathbb{R}$, the ideal it generates is maximal, so $\mathbb{C}$ is a field. In particular, 1 correspond to 1 and x correspond to i in the quotient. ∎

Definition 1.3. Let $x + iy \in \mathbb{C}$, we refer to the **matrix representation** of $x + iy$ as:

$$x + iy \sim \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

In particular we have that

$$1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In particular, the representation is an isomorphism.

Definition 1.4. Let $z = x + iy \in \mathbb{C}$, we define the **complex conjugate** of $\mathbb{C}$ as $\overline{z} := x - iy$ and $|z|$ as the **complex norm** of $\sqrt{x^2 + y^2}$.

Remark 1.5. Usually, the explicitly construct the inverse of $z = x + iy \neq 0$, we have that

$$\frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

However, we note that with the isomorphism given by the definition above also gives us a matrix inverse as its determinant is $x^2 + y^2 \neq 0$

Definition 1.6. Let $z \in \mathbb{C}$ such that $|z| = 1$, then we can write

$$z = x + iy = \cos(\theta) + i\sin(\theta)$$

We refer to $\theta = \arg(z) + 2\pi n$, where $\arg(z)$ is the standard **argument** whose radian is within $[-\pi, \pi)$. This angle is sometimes called $Arg(z)$ and is called the **principal argument**.

Let $z \in \mathbb{C}$ with $|z| = 1$, then we can write

$$z \sim \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This is just the standard rotational matrix.
Now for an arbitrary non-zero $z \in \mathbb{C}$ whose norm need not be 1, we can write

$$z = |z|\frac{z}{|z|} = |z| \cdot (cos(\theta) + isin(\theta))$$

This is called the **polar representation** of $\mathbb{C}$.

Proposition 1.7. Let $z_1, z_2 \in \mathbb{C}$, then

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proof. To show the first, rewrite them in matrix and note their determinant is exactly the complex norm. To show the second, just use the polar coordinate representation and some trigonometry. ■

Corollary 1.8 (De Moivre's Formula)**.** $(\cos(\theta) + isin(\theta))^n = cos(n\theta) + isin(n\theta)$

Proof. This follows directly from the additivity of angles in complex multiplication. ■

Definition 1.9. Let $z = x + iy \in \mathbb{C}$, then $\Re(z) := x = \frac{z+\bar{z}}{2}$ and $\Im(z) := y = \frac{z-\bar{z}}{2}$ are the real and imaginary part of z respectively.

Theorem 1.10 (Euler's Identity)**.** $e^{i\theta} = cos(\theta) + isin(\theta)$

2 Lecture 2 - 09/09/2022

2.1 Differentiability

Definition 2.1. Let Ω be an open subset of $\mathbb{C}$, we say a function $f : \Omega \to \mathbb{C}$ is analytic on $\mathbb{C}$ if for any $z_0 \in \Omega$, there exists a neighborhood U where $z_0 \in U \subset \Omega$, such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \forall z \in U$$

Note that without loss of generality, we can assume $U = D_{z_0,\delta} = \{z : |z - z_0| < \delta\}$.

Remark 2.2. If we take $\Omega \subset \mathbb{R}$, then we say $f : \Omega \to \mathbb{R}$ is real-analytic if for all $x_0 \in \Omega$ locally

$$f(x) = \sum a_k (x - x_0)^k$$

If we take $\Omega \subset \mathbb{R}^2$, then we say $f : \Omega \to \mathbb{R}$ is real-analytic if for all $(x_0, y_0) \in \Omega$, there exist neighborhood U containing (x_0, y_0) such that

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n,k} (x - x_0)^n (y - y_0)^k$$

Note that real analytic function, when viewed as complex function, is not analytic!

The real miracle in the complex analysis is as follows:
If we consider real functions, then we have that

$$C_1 \supsetneq C_2 \supsetneq ... \supsetneq C^\infty \supsetneq \text{Real-Analytic Functions}$$

However, it turns out that in the Complex Case, complex differentiable functions are in fact analytic, so

$$C_1 = C_2 = ... = C^\infty = \text{Complex-Analytic Functions}$$

♣♣♣ Mattie: [Unless stated otherwise, we assume Ω to be open?]

Definition 2.3. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a complex-valued function. We say f is **complex-differentiable** at $z \in \Omega$, if the limit exists

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

f is sometimes also called **holomorphic** at z.

Definition 2.4. We denote $C_z^1(\Omega)$ as the set of functions f where $f(z)$ is differentiable on all of Ω and the map $z \mapsto f'(z)$ is continuous (continuous derivative).

2.2 Cauchy-Riemann Equations

Consider h_1 be the direction parallel to the real axis and h_2 be the direction parallel to the imaginary axis, then

$$\lim_{h_1 \to 0} \frac{f(z + h_1) - f(z)}{h_1} = \frac{\partial f}{\partial x} = f'(z) = \frac{1}{i} \frac{\partial f}{\partial y} = \lim_{h_2 \to 0} \frac{f(z + h_2) - f(z)}{h_2}$$

So in particular, we have that

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

This is known as the **Cauchy-Riemann Equation**.

Remark 2.5. Write $f = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then the Cauchy-Riemann Equation is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.6 (Complex Differentials). We define

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

For ease of notations, we will denote

$$\partial \coloneqq \frac{\partial}{\partial z}, \overline{\partial} \coloneqq \frac{\partial}{\partial \overline{z}}$$

Remark 2.7. In particular, this means that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$$

The definition above means that the Cauchy-Riemann Equations is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$

Proposition 2.8. Let f, g be complex dfferentiable functions, then

$$\partial(fg) = (\partial f)g + f(\partial g)$$

Remark 2.9. The $\frac{1}{2}$ coefficient for $\partial, \overline{\partial}$ is a **correcting** coefficient so that

$$\partial z = 1, \partial \overline{z} = 0$$

$$\partial z^n = nz^{n-1}$$

$$\overline{\partial} z = 0, \overline{\partial}\overline{z} = 1$$

$$\overline{\partial}\overline{z}^n = n\overline{z}^{n-1}$$

2.3 Complex Integrals and Cauchy's Integral Theorem

From Calculus, for a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Viewing f instead as a complex function, after some algebraic manipulations, you can show that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

, where $dz = dx + idy$ and $d\bar{z} = x - iy$.

Definition 2.10. A C^1-path is $\gamma : [a, b] \to \mathbb{C}$ where $\gamma \in C^1([a, b])$. If γ is furthermore injective and $\gamma'(t) \neq 0$ on $[a, b]$, then $\gamma([a, b])$ is a C^1-curve. (We require these two extra conditions to avoid spikes on the path)

Definition 2.11. Let $\Gamma = \gamma([a, b])$ be a C^1-curve, then

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Theorem 2.12 (First Cauchy's Theorem)**.** Let $f \in C_z^1(\Omega)$, let G be a bounded open set such that $cl(G) \subset \Omega$, and the boundary of G is C^1 or piece-wise C^1 (we will denote this as PC^1), then

$$\int_{\partial G} f(z)dz = 0$$

Theorem 2.13 (Stokes's Theorem)**.** Let G be an oriented smooth n-dimensional manifold with boundary and ω is a compactly support $(n-1)$-form on G, then

$$\int_{\partial G} w = \int_G d\omega$$

Proof of FIrst Cauchy's Theorem. In this case, G is just some subset of $\mathbb{C}$ and we define $\omega := f(z)dz$, then we note that

$$\begin{aligned} dw &= df \wedge dz \\ &= (\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}) \wedge dz \\ &= (\frac{\partial f}{\partial z}dz) \wedge dz && \text{Cauchy-Riemann Equation} \\ &= \frac{\partial f}{\partial z}(dz \wedge dz) \\ &= \frac{\partial f}{\partial z}(0) \\ &= 0 \end{aligned}$$

Then Stokes' Theorem tells us that

$$\int_{\partial G} w = \int_G d\omega = \int_G 0 = 0$$

■

3 Lecture 3 - 09/12/2022

3.1 Curves and Orientations

Definition 3.1 (Orientation of a Curve). Let $\Gamma \subset \mathbb{C}$ be a curve, and let $\gamma, \gamma_1 : [a, b] \to \mathbb{C}$ be the standard injective parameterization with $\gamma'(t) \neq 0, \gamma_1'(t) \neq 0$. Then we say γ and γ_1 have the same orientation if $(\gamma \circ \gamma_1^{-1})' > 0$.

Note that this is just the standard definition of orientation for 1-dimensional manifolds.
On a closed loop, the **positive** direction is given by the **left-leg rule**, meaning tracing along the curve, the left leg of the curve points into the region enclosed.
This turns out to align exactly with the definition of orientation inherited by the boundary manifold.

Definition 3.2. We say $f \in CR^1(\Omega)$ if $f \in C^1(\Omega)$ and $\frac{\partial}{\partial \bar{z}} f = 0$ on Ω.

Remark 3.3. It turns out that $CR^1 = C_z^1$, meaning that being in CR^1 is equivalent to being complex differentiable. The direction from $C_z^1 \implies CR^1$ is shown in the previous lecture, what about the other direction?

Definition 3.4 (Multivariable Differentiability). We say $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \Omega$ if there exists some $L \in M_{n\times m}(\mathbb{R})$ such that

$$f(x + h) = f(x) + L(h) + r_x(h)$$

, where $r_x(h)$ is sometimes denoted as $O(h)$ and $\lim_{h\to 0} \frac{r_x(h)}{||h||} = 0$

Proposition 3.5. If a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is CR^1, then f is C_z^1

Proof. Since f is CR^1, it is $C^1(\Omega)$, so f is differentiable when viewed as a function in $\mathbb{R}^2$. Write $f = u + iv$, then the differential of f is exactly the Jacobian matrix:

$$df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Then f satisfiying the CR-equations tells us that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

This means that df is just a scaled rotational matrix, so it's without loss a multiplication by complex numbers. Let a be the complex number representation of df.

Thus, we have that

$$f(z + h) = f(z) + a \cdot h + O(h)$$

Then we have that

$$\lim_{h\to 0} \frac{f(z + h) - f(z)}{h} = a$$

, so the complex derivative exists, so f is holomorphic on Ω. ■

When we say something is differentiable on an non-open set K, then this means there exists some open $\Omega \supset K$ such that it is differentiable on it, meaning there's some bigger open set this function is differentiable on.

3.2 Cauchy's Integral Formula

Theorem 3.6 (Cauchy Formula)**.** Let G be a bounded open set with boundary $\partial G \in PC^1$. Let $f \in C_z^1(cl(G)) = CR^1(cl(G))$. Then for all $z \in int(G)$,

$$f(z) = \frac{1}{2\pi i}\oint_{\partial G}\frac{f(\xi)}{\xi - z}d\xi$$

Proof. Let $\Omega \supset cl(G)$ be the function $f \in CR^1(\Omega)$. Now consider

$$g(z) = \frac{f(z)}{z - z_0}$$

Then we note that $\frac{1}{z-z_0} \in CR^1(\mathbb{C} \setminus \{z_0\})$. Therefore, $g(z) \in CR^1(\Omega \setminus \{z_0\})$.

Choose $\epsilon > 0$ small enough such that $D_{z_0,\epsilon}$, the disk of radius ϵ centered at z_0, does not intersect with ∂G. Now consier $G_\epsilon := G \setminus D_{z_0,\epsilon}$, then

$$\int_{\partial G_\epsilon}\frac{f(z)}{z - z_0}dz = 0 \forall \textbf{ small enough } \epsilon > 0$$

Now, we see that

$$\int_{\partial G_\epsilon}\frac{f(z)}{z - z_0} = \int_{\partial G}\frac{f(z)}{z - z_0} - \int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}$$

Taking the limit of ϵ on both sides to 0, so

$$0 = \int_{\partial G}\frac{f(z)}{z - z_0} - \lim_{\epsilon \to 0}\int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}dz$$

$$\int_{\partial G}\frac{f(z)}{z - z_0} = \lim_{\epsilon \to 0}\int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}dz$$

Since $f(z)$ is differentiable, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + O(z - z_0)$$

So we have that

$$\int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}dz = \int_{|z-z_0|=\epsilon}\frac{f(z_0)}{z - z_0}dz + \int_{|z-z_0|=\epsilon}f'(z_0)dz + \int_{|z-z_0|=\epsilon}\frac{O(z - z_0)}{z - z_0}dz$$

Parameterize $|z - z_0| = \epsilon$ by $z = z_0 + \epsilon e^{it}, t \in [0, 2\pi]$ and note that

$$\int_{|z-z_0|=\epsilon}\frac{1}{z - z_0}dz = 2\pi i$$

$$\int_{|z-z_0|=\epsilon}f'(z_0)dz = 0$$

$$|\int_{|z-z_0|=\epsilon}\frac{O(z - z_0)}{z - z_0}dz| \leq 2\pi\epsilon \max_{z \in |z-z_0|=\epsilon}|\frac{O(z - z_0)}{z - z_0}|$$

, which goes to zero as $\epsilon \to 0$. Then we have that

$$\int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}dz = 2\pi i f(z_0) + \int_{|z-z_0|=\epsilon}\frac{O(z - z_0)}{z - z_0}dz \lim_{\epsilon \to 0}\int_{\partial D_{z_0,\epsilon}}\frac{f(z)}{z - z_0}dz \qquad = 2\pi i f(z_0) + 0$$

Thus, we have that

$$\int_{\partial G}\frac{f(z)}{z - z_0} = 2\pi i f(z_0)$$

■

Now we have that

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

We will see that shows that we can take the derivative infinitely many times and that $f(z)$ is analytic.

4 Lecture 4 - 09/14/2022

4.1 Holomorphic Implies Infinitely Differentiable

Example 4.1. Let γ be a path from i to 2, then

$$\int_\gamma z^5 dz = \frac{1}{6} z^6 |_i^2$$

Definition 4.2. We say f has anti-derivative (otherwise known as **primitive**) if $F'(z) = f(z)$.

Proposition 4.3. If f is primitive, then

$$\int_\gamma f(z)dz = F(z_1) - F(z_0)$$

, where z_0 is the start of γ and z_1 is the end of γ. In particular if $z_0 = z_1$,

$$\int_\gamma f(z)dz = 0$$

Proof. Chain-Rule and Fundamental Theorem of Calculus ■

Recall Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

We claim that, taking the derivative of $f(z)$ gives

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{1}{(\xi - z)^2} f(\xi) d\xi$$

Taking the derivative again gives:

$$f''(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{2}{(\xi - z)^3} f(\xi) d\xi$$

Taking the derivative again gives:

$$f'''(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{2 \cdot 3}{(\xi - z)^4} f(\xi) d\xi$$

Using induction shows that

$$f^{(n)}(z) = \frac{1}{2\pi i} \oint_{\partial G} \frac{n!}{(\xi - z)^{n+1}} f(\xi) d\xi$$

This gives us the corollary:

Corollary 4.4. If f is holomorphic under the assumption of Cauchy's Integral Formula, then f is infinitely differentiable.

Proof. We note that for our claims about the derivative to work, we want to show that

$$\lim_{\Delta z \to 0} \int f(\xi) \frac{P(z + \Delta z, \xi)}{\Delta z} d\xi = \int \lim_{\Delta z \to 0} f(\xi) \frac{P(z + \Delta z, \xi) - P(z, \xi)}{\Delta z} d\xi$$

, ie. we can exchange the limit and the integral.

Then we have that

$$\int \lim_{\Delta z \to 0} f(\xi) \frac{P(z+\Delta z, \xi) - P(z,\xi)}{\Delta z} d\xi = \int f(\xi) \frac{\partial P}{\partial z}(z,\xi) d\xi$$

Indeed, we can exchange the limit and integral using **Dominated Convergence Theorem**. ■

Remark 4.5. Note that we were able to use the Dominated Convergence Theorem because of the following two reasons:

- Reason 1:

  Lemma 4.6. $L = \lim_{x \to x_0} f(x)$ exist if and only if for any sequence $\{x_n\}$ converging to x_0,

  $$\lim_{n \to \infty} f(x_n) = L$$

- Reason 2:

  Theorem 4.7 (Mean Value Estimates). $|P(z + \Delta z, \xi) - P(z, \xi)| \leq |\frac{\partial P}{\partial z}(z + \theta \Delta z, \xi)| \cdot |\Delta z|$, where $0 < \theta < 1$.

  We note that this inequality is actually uniformly bounded

  $$|\frac{\partial P}{\partial z}(z + \theta \Delta z, \xi)| \cdot |\Delta z| \leq M$$

  This is because the domain of $\frac{\partial P}{\partial z}$ is compact and $\frac{\partial P}{\partial z}$ is continuous.

  We also note that $|\partial G| < \infty$, so we can apply the Dominated Convergence Theorem.

Theorem 4.8 (Holomorphic Implies Analytic). Let $\Omega \supset G$, where Ω is open and G is a bounded open set where $\partial G \in PC^1$, and furthermore that $cl(G) \subset \Omega$. Let $z_0 \in G \subset \Omega$, then if $f \in CR^1(\Omega)$, then f is analytic in Ω. In particular,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all z where $|z - z_0| < dist(z_0, \partial G)$ and

$$a_n = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{f^{(n)}(z_0)}{n!}$$

Remark 4.9. In most textbook the neighborhood of power-series is over $|z - z_0| = R$ where $R < dist(z_0, \partial G)$

Proof of Holomorphic Implies Analytic. WLOG, we will assume $z_0 = 0$, because we can always shift the coordinate linearly.

How does the WLOG work, well, rewrite

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)}$$

, then we can just use the same reasoning as before. Trying to justify this takes a little bit of work.

Next, Cauchy's Integral Formula says

$$f(z) = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi$$

Now we note that

$$\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}}$$

We note that $|\frac{z}{\xi}| < 1$ since z is in the interior but ξ is on the boundary, so

$$\frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}} = \frac{1}{\xi}\sum_{n=0}^{\infty}\frac{z^n}{\xi^n}$$

So we have that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi &= \frac{1}{2\pi i}\int_{\partial G}\sum_{n=0}^{\infty}\frac{f(z)z^n}{\xi^{n+1}}d\xi \\
&= \sum_{n=0}^{\infty} z^n(\frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi^{n+1}}d\xi) && \text{ASSUMING We can switch integral and sum} \\
&= \sum_{n=0}^{\infty} z^n a_n
\end{aligned}$$

It remains for us to justify why we can switch this, we can usually use Fubini's Theorem or Dominated Convergence Theorem, but here we will just use a low-tech solution and note that, assuming z is fixed and $\xi \in \partial G$, then

$$\lim_{N\to\infty}\sum_{n=0}^{N}\frac{z^n}{\xi^{n+1}} = \frac{1}{\xi - z} \text{ converges uniformly}$$

, because $|\frac{z}{\xi}| \leq \frac{|z|}{dist(z_0,\partial G)} < 1$, so we have this uniform convergence. Thus, we can always switch the integral and the sum. ■

Corollary 4.10. Under the same setup, if f is analytic on z_0, then it converges in $D_{z,d}$ where $d = dist(z_0, \partial G)$, so the radius of convergence is at least as much as d.

Remark 4.11. In general, uniform convergence always means that you can switch the limit and the integration. Indeed, let S_n be a sequence of functions that uniformly converges to some function S, then we wish to show that

$$\lim_{n\to\infty}\int S_n = \int \lim_{n\to\infty} S$$

Indeed, since the convergence is uniform, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n > N$,

$$|S_n(\epsilon) - S(\epsilon)| < \epsilon$$

Taking the integrals gives then that

$$|\int_{\partial G} S_n - \int S| \leq \epsilon|\partial G|$$

Since this holds for any $\epsilon > 0$, the limit would thus converge.

We have proved that satisfying Cauchy-Riemann implies analytic, we will now show the converse.

Theorem 4.12 (Morera's Theorem)**.** If $f \in C^0(\Omega)$ and $\int_{\partial R} f(z)dz = 0$ for all sufficiently small rectnagles R where $cl(R) \subset \Omega$, then $f \in CR^1$.

Proof. We will prove this next lecture. ■

Theorem 4.13. $(\sum_{n=0}^{\infty} a_n(z-z_0)^n)' = \sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$, hence power series are holomorphic, and thus analytic functions are holomorphic.

Proof. You can usually do it by taking the partial sum and taking limit, but there's a trick to it! This is a direct corollary of Morera's Theorem, more in the next lecture. ■

5 Lecture 5 - 09/16/2022

5.1 Morera's Theorem and Analytic Implies Holomorphic

We require a rectangle R in the following theorem to be parallel to the x and y-axis.

Theorem 5.1 (Morera's Theorem)**.** Let $f \in C^0(\Omega)$, suppose for all sufficiently small rectangles R such that $cl(R) \subset \Omega$, we have that

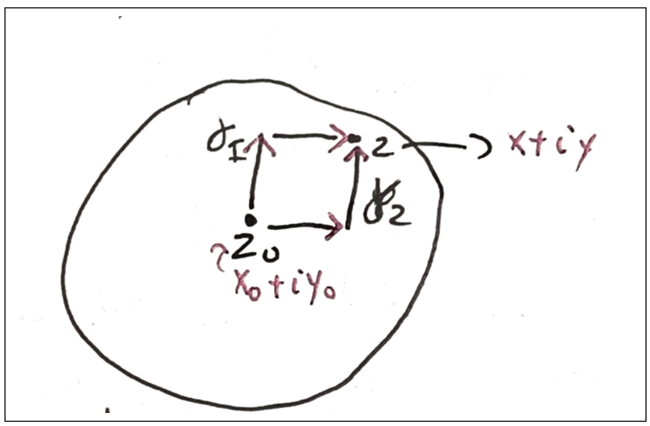
$$\int_{\partial R} f dz = 0 \; (*)$$

, then $f \in CR^1(\Omega)$ and hence holomorphic and analytic.

By **"sufficiently small"**, we mean that for all $z_0 \in \Omega$, there exists $\epsilon := \epsilon(z_0)$ such that, for all rectangles R where $cl(R) \subset D_{z_0,\epsilon}$, the condition in $(*)$ holds.

Proof. We first note that CR^1 is a **local property**, meaning that it suffices for us to check this in a neighborhood of each point. Thus, for some $r > 0$, we can without loss assume $\Omega = D_{z_0,r}$.

Now consider the diagram:



For all $z \in \Omega$, define $F(z) = \int_{z_0}^{z} f(\xi)d\xi := \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$. We note that $F(z)$ is well-defined as Cauchy's Theorem guarantees the path-independence.

We claim that $F \in CR^1(\Omega)$. Indeed, tracing γ_1 and using the Fundamental Theorem of Calculus tells us that

$$\frac{\partial F}{\partial x}(z) = f(z)$$

Similarly, tracing along γ_2, we can parameterize $F(z)$ as

$$F(z) = \int_{x_0}^{x} f(s + iy_0)ds + i\int_{y_0}^{y} f(x + it)dt$$

Then it follows again that

$$\frac{\partial F}{\partial y}(z) = if(z)$$

We can check that

$$\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = f(z) - f(z) = 0$$

, thus $F(z) \in CR^1(\Omega)$. Since $F(z)$ follows the Cauchy-Riemann Equations, we also know that

$$F'(z) = \frac{1}{2}(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}) = \frac{1}{2}(2f(z)) = f(z)$$

Since $F(z)$ is holomorphic, it is infinitely differentiable, so $f(z)$ is also holomorphic, so we are done. ∎

Corollary 5.2. If $f_n \in C_z^1(\Omega)$ and $\{f_n\}$ converges to f uniformly, then $f \in C_z^1(\Omega)$.

Proof. For all sufficiently small rectangle R within Ω, consider

$$\begin{aligned}
\int_{\partial R} f(z)dz &= \int_{\partial R} \lim_{n\to\infty} f_n dz \\
&= \lim_{n\to\infty} \int_{\partial R} f_n dz && \text{Uniform Convergence} \\
&= \lim_{n\to\infty} 0 && \text{Cauchy's Theorem} \\
&= 0
\end{aligned}$$

Thus, by Morera's Theorem, $f \in C_z^1(\Omega)$. ∎

Corollary 5.3. If $f_n \in C_z^1(\Omega)$ and $\{f_n\}$ converges to f uniformly, then $\{f_n'(z)\}$ converges to $f'(z)$ uniformly, and hence $\{f_n^k(z)\}$ converges to $f^k(z)$ uniformly.

Proof. It suffices for us to show this for the first derivative. Now for all $z \in \Omega$ and sufficiently small domain G, we have that

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\int_{\partial G} \frac{f(z)}{(\xi - z)^2} d\xi && \text{Cauchy's Formula for Derivatives} \\
&= \frac{1}{2\pi i}\int_{\partial G} \lim_{n\to\infty} \frac{f_n(z)}{(\xi - z)^2} d\xi \\
&= \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial G} \frac{f_n(z)}{(\xi - z)^2} d\xi && \text{Uniform Convergence} \\
&= \lim_{n\to\infty} f_n'(z) && \text{Cauchy's Formula for Derivatives}
\end{aligned}$$

Thus, we have that $\{f_n'(z)\}$ converges to $f'(z)$. Since the original convergence is uniform, this convergence is also uniform. ∎

Theorem 5.4 (Meta Theorem)**.** Let $\phi(z, \xi) : \mathbb{C} \times X \to \mathbb{C}$ where X is some parameterization space with measure $\mu(\xi)$ that is finite, let

$$f(z) = \int \phi(z, \xi) d\mu(\xi)$$

Suppose ϕ is CR^1 in the variable z and f is bounded, then $f(z)$ is analytic.

Proof. Since the measure μ is finite and the function is bounded, we can use Fubini's Theorem and see that, for all sufficiently small rectangles,

$$
\begin{aligned}
\int_R f(z)dz &= \int_R [\int \phi(z,\xi)d\mu(\xi)]dz \\
&= \int [\int_R \phi(z,\xi)dz]d\mu(\xi) && \text{Fubini's Theorem} \\
&= \int 0 d\mu(\xi) && \phi \text{ is } CR^1 \text{ in the variable } z \\
&= 0
\end{aligned}
$$

■

Corollary 5.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ defined in $D_{z_0,r}$, then $f \in CR^1(D_{z_0,r}) = C^1_z(D_{z_0,r})$

Proof. Let $z \in D(z_0, r)$, consider $r_1 > 0$ small enough that $cl(D(z, r_1)) \subset D(z_0, r)$, then by Heine-Cantor, we know that the partial sums

$$\sum_{k=0}^{N} a_k(z-z_0)^k \mapsto \sum_{0}^{\infty} a_n(z-z_0)^n \text{ converges uniformly in } D(z, r_1)$$

Each of the partial sum are complex differentiable, thus by Corollary above, we have that $f \in C^1_z(D_{z_0,r})$. ■

Theorem 5.6. $CR^1 = C^1_z =$ Analytic

Proof. We have already shown all the directions in previous lectures and this lecture. ■

5.2 Power Series

Definition 5.7. Let $f(z) = \sum a_k(z-z_0)^k$ be a power series. We define the radius of convergence of f as $R \in [0, \infty]$ such that for all z such that $|z - z_0| < R$, $f(z)$ diverges, and for all z such that $|z - z_0| > R$, $f(z)$ converges.

Remark 5.8. It turns out that

$$\frac{1}{R} := \lim \sup_{n\to\infty} |a_n|^{1/n}$$

Proposition 5.9. Let $f(z) = \sum a_k(z-z_0)^k$ be a power series with radius of convergence R, then $f(z)$ converges uniformly in any smaller disk of radius $r < R$ contained in $D_{z_0,R}$

Proof. We note that f is continuous on $D_{z_0,R}$, and the closure of any smaller disk is also contained in $D_{z_0,R}$, then apply Heine-Cantor. ■

Corollary 5.10. The following three power series:

- $\sum_{n=0}^{\infty} a_n z^n$
- $\sum_{n=0}^{\infty} n a_n z^{n-1}$
- $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$

all have the same radius of convergence.

Proof. Exercise. Hint: Note that

$$\lim_{n\to\infty} n^{1/n} = 1$$

■

Example 5.11. For sufficiently small $z \in \mathbb{C}$, consider

$$\frac{1}{1+sin(z)} = 1 - sin(z) + sin^2(z) + ...$$

, we can rewrite this into a power-series by noting that

$$sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

5.3 Liouville's Theorem

Theorem 5.12. Let $f \in Hol(\mathbb{C})$ be a holomorphic function on $\mathbb{C}$, and for all $z \in \mathbb{C}$ we have that $|f(z)| \leq M$ for some fixed $M \in \mathbb{C}$, then $f(z)$ is identically constant.

Proof. Using Cauchy's Formula for Derivatives, we note that for all $z \in \mathbb{C}$, for all $R > 0$,

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi - z| = R} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

Thus, we have that

$$\begin{aligned} |f'(z)| \leq \frac{1}{2\pi} \max_{\xi \in |\xi - z|} \left|\frac{f(\xi)}{(\xi - z)^2}\right| \cdot 2\pi R \\ &\leq \frac{1}{2\pi}\frac{M}{R^2} \cdot 2\pi R && \text{Since } f \text{ is bounded} \\ &= \frac{M}{R} \end{aligned}$$

Since this inequality holds for all $R > 0$, taking the limit as $R \to 0$, gives us that

$$|f'(z)| = 0$$

Hence $f'(z) = 0$, so

$$\frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial \overline{z}} = 0$$

So in other words

$$f_x - i f_y = 0, f_x + i f_y = 0 \implies f_x = 0, f_y = 0 \implies f(z) \text{ is constant}$$

■

Remark 5.13. While our Morera's Theorem applies to only triangles, up to a change of coordinate, the rectangle can really just be anything.

Remark 5.14. How to check say if $f(z) = |z|^2 = z\overline{z}$ is analytic or not? Well, we see that

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial z}{\partial \overline{z}}\overline{z} + z\frac{\partial \overline{z}}{\partial \overline{z}} = 0 + z(1) \neq 0$$

Thus, f is not holomorphic and hence not analytic.

Take another example, say $f(z) = |z| = (z\overline{z})^{1/2}$. This is real valued so we can use power rule and

$$\overline{\partial}(f) = \frac{1}{2}(z\overline{z})^{-1/2}z \neq 0$$

5.4 Appendix - Cauchy–Hadamard Theorem

In this section, we will discuss more about power series and convergence that was not covered during the lecture.

Theorem 5.15 (Cauchy–Hadamard Theorem)**.** Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Let R be finite, non-zero, satisfying

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| > R$. We call R the **radius of convergence** and $|z| < R$ the **disc of convergence**.

Proof. For any $\epsilon > 0$, let $\rho := \frac{1}{R}$.

If $|z| < R$, then there exist some $\epsilon > 0$ small enough that $|z| < \frac{1}{\rho+\epsilon} < \frac{1}{\rho} = R$. By the definition of $\limsup$ that for a sufficiently large n, for all $N > n$,

$$|a_N|^{1/N} < \rho + \frac{\epsilon}{2} \; (*)$$

$$\begin{aligned} |a_N z^N| &= |a_N^{1/N} z|^N \\ &< |a_N^{1/N}\frac{1}{\rho+\epsilon}|^N && \text{Since } |z| < \tfrac{1}{\rho+\epsilon} \\ &< |\frac{\rho+\frac{\epsilon}{2}}{\rho+\epsilon}|^N && \text{Using } (*) \end{aligned}$$

We note that $|\frac{\rho+\frac{\epsilon}{2}}{\rho+\epsilon}| < 1$ and $b_n = |\frac{\rho+\frac{\epsilon}{2}}{\rho+\epsilon}|^n$ forms a convergent geometric series. By the comparison test, we also have that $\sum_{n=N}^{\infty} a_n z^n$ converges, which implies that $\sum_{n=0}^{\infty} a_n z^n$ converges.

If $|z| > R$, then there exist some $\epsilon > 0$ small enough that $|z| > \frac{1}{\rho-\epsilon} > \frac{1}{\rho} = R$. By the definition of $\limsup$ that for a sufficiently large n, for all $N > n$,

$$|a_N|^{1/N} + \frac{\epsilon}{2} > \rho \; (*)$$

$$\begin{aligned} |a_N z^N| &= |a_N^{1/N} z|^N \\ &> |\frac{a_N^{1/N}}{\rho-\epsilon}|^N && \text{Since } |z| > \tfrac{1}{\rho-\epsilon} \\ &> |\frac{\rho-\frac{\epsilon}{2}}{\rho-\epsilon}|^N \end{aligned}$$

We note that $|\frac{\rho-\frac{\epsilon}{2}}{\rho-\epsilon}| > 1$ and $b_n = |\frac{\rho-\frac{\epsilon}{2}}{\rho-\epsilon}|^n$ forms a divergent geometric series, so the comparision test tells us that $\sum_{n=N}^{\infty} a_n z^n$ diverges, which implies that $\sum_{n=0}^{\infty} a_n z^n$ diverges. ∎

Corollary 5.16. Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$, and let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Then $g(z)$ has a radius of convergence that is also R.

Proof. Consider

$$\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} n^{1/n} |a_n^{1/n}|$$

It's a standard fact in real analysis that

$$\limsup_{n\to\infty} c_n b_n = \lim_{n\to\infty} c_n \limsup_{n\to\infty} b_n$$

if $\{c_n\}$ converges. We note that

$$\lim_{n\to\infty} n^{1/n} = 1$$

Since the limit exist

$$\limsup_{n\to\infty} n^{1/n} |a_n^{1/n}| = \limsup_{n\to\infty} |a_n^{1/n}| = \frac{1}{R}$$

∎

6 Lecture 6 - 09/19/2022

Previously, we have shown that a complex function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if it satisfies the Cauchy-Riemann Equations. Furthermore, if $f \in Hol(\Omega)$, $z_0 \in \Omega$, then we know f is analytic and locally there exists some bounded, open set G containing z_0 such that ∂G is PC^1 $cl(G) \subset \Omega$, and on G,

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k, |z - z_0| < r$$

$$a_n = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

Using the Cauchy's Formula for Derivatives, we note that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial G} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \implies a_n = \frac{f^{(n)}(z_0)}{n!}$$

6.1 Uniqueness Theorems

Theorem 6.1. Let $f \in Hol(\Omega)$, Ω be an open and connected subset of $\mathbb{C}$, and let $z_0 \in \Omega$ such that

$$f^{(n)}(z_0) = 0 \; \forall n \geq 0$$

Then $f(z) = 0$ on Ω, ie. f is identically zero on Ω.

Proof. We will prove this using **continuous induction**. Indeed, define A to be the set

$$A := \{z \in \Omega : f^{(n)}(z_0) = 0 \; \forall n \geq 0\} = \bigcap_{n \geq 0} \{z \in \Omega : f^{(n)}(z) = 0\}$$

We first note that A is closed since the preimage of $\{0\}$ is closed under continuous function and the intersection of closed sets are closed. Now clearly A is non-empty, since $z_0 \in A$. We now **claim that A is open**, then the connectedness of Ω would imply that $A = \Omega$.

Indeed, for all $z \in A$, since f is holomorphic, we can find some $\epsilon(z) > 0$ small enough that locally

$$f(\xi) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} (\xi - z)^n, \forall \xi \; s.t. \; |\xi - z| < \epsilon$$

However, since $z \in A$, we know that

$$f(\xi) = \sum_{n=0}^{\infty} \frac{0}{n!} (\xi - z)^n = 0$$

Thus we have that $\xi \in A$. Hence, $D_{z,\epsilon} \subset A$. Thus, A is also open. ■

Remark 6.2. The previous theorem shows that

$$f(z_0) = \sum a_n(z - z_0)^n, f^{(n)}(z_0) = 0 \; \forall n \iff a_n = 0 \; \forall n$$

Definition 6.3 (Limit Points). Let X be a topological space and $E \subset X$, we say $a \in X$ is a **accumulation/-cluster/limit point** of E if for all neighborhood U containing a, $E \cap (U \setminus \{a\}) \neq \emptyset$

Theorem 6.4 (Uniqueness Theorem)**.** Let $f \in Hol(\Omega)$ where Ω is open and connected. Suppose $E \subset \Omega$ such that $f(z) = 0$ for all $z \in E$ and E has some accumulation point z_0 in Ω, then $f(z)$ is identically zero on Ω.

Proof. The strategy is to apply the previous theorem. We first choose $r > 0$ small enough and consider the open disk $D_{z_0,r}$ to express f as a power-series around z_0. Now, let $\tilde{f}$ be f restricted to $D_{z_0,r}$, we first claim that

$$\tilde{f}(z_0) = 0$$

Indeed, since z_0 is a limit point of E, there exists a sequence of points $\{z_k\}$ in $E \cap D_{z_0,r}$ that converges to z_0. Since f is continuous, we have that

$$\tilde{f}(z_0) = \tilde{f}(\lim_{k\to\infty} z_k) = \lim_{k\to\infty} \tilde{f}(z_k) = \lim_{k\to\infty} 0 = 0$$

Now consider the power-series expression of f around z_0:

$$[f(z) = \sum_{n=0}^{\infty} a_n(z_0 - z)^n$$

Since $\tilde{f}(z_0) = 0$, we clearly have that $a_0 = 0$. Now we will define

$$f_1(z) := \frac{\tilde{f}(z)}{z - z_0} = \sum_{n=1}^{\infty} a_n(z - z_0)^{n-1}$$

♣♣♣ Mattie: [Division is fine here since $a_0 = 0$] and define $\tilde{f}_1(z)$ to be restricted to the same domain as before. Then clearly $f_1(z) = 0$ for all $z \in E \setminus \{z_0\}$, so it follows that $\tilde{f}_1(z_0) = 0$, so $a_1 = 0$.

Using induction, we can conclude that all of the a_n are 0, so $f^{(n)}(z_0) = 0$ for all $n \geq 0$. Apply the previous result and we are done! ■

Corollary 6.5 (Identity Theorem)**.** Let $f, g \in Hol(\Omega)$ where Ω is open and connected. Suppose $E \subset \Omega$ such that $f(z) = g(z)$ for all $z \in E$ and E has some accumulation point z_0 in Ω, then $f = g$ on Ω.

Proof. Define $h(z) := f(z) - g(z)$ and apply Theorem 6.4. ■

6.2 Analytic Extensions

The uniqueness theorem implies that there's only one unique way to extend certain functions.

Example 6.6. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$, then $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an analytic extension of e^x to the complex plane, and the identity theorem shows that it is in fact that only possible analytic extension.

Proof. Existence is already proven. For uniqueness, take $\Omega = \mathbb{C}$ and $E = \mathbb{R}$, $\mathbb{R}$ clearly has an accumulation point in $\mathbb{C}$, so we can apply the Identity Theorem. ■

Example 6.7. Similarly, one can also show that

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}...$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!}...$$

are the only unique extensions of $cos(x)$ and $sin(x)$, hence all trigonometric identities also hold for $z \in \mathbb{C}$.

Corollary 6.8. Let Ω be an open connected set, suppose $f \in Hol(\Omega)$ isn't identically zero on Ω, then the zeroes of f are isolated.

Proof. Suppose not, then there exists a sequence of roots $\{z_k\}$ converging to some limit point. Then take $E = \{z_1, ..., z_k, ...\}$, then the Uniqueness Theorem tells us that f is identically zero, so we have a contradiction. ■

Remark 6.9. While this statement says that roots inside Ω are isolated, it says NOTHING about the boundary of Ω.

6.3 Order of Zeroes

Definition 6.10. Let $f \in Hol(\Omega)$, we define $Z(f)$ as the zero locus of f:

$$Z(f) := f^{-1}(0)$$

Definition 6.11. Suppose $f \in Hol(\Omega)$ and $f(z_0) = 0$, rewrite f as a power series around z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, a_0 = 0$$

We say the **order of zero** on z_0 is $\min\{n : a_n \neq 0\} = \min\{n : f^{(n)}(z_0) \neq 0$

Example 6.12. $\sin(z)$ has zero of order 1 at all roots. $(z - z_0)^3$ has a zero of order 3 at z_0.

7 Lecture 7 - 09/21/2022

7.1 Revisiting Exponential

Definition 7.1. Let $z \in \mathbb{C}$, we define the complex exponential map $e^z : \mathbb{C} \to \mathbb{C}$ as

$$e^z := 1 + z + \frac{z^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Theorem 7.2 (Euler's Identity)**.** Let $x \in \mathbb{R}$, then

$$e^{ix} = cos(x) + isin(x)$$

Proof. We can show this via explicit computation, indeed,

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} + i\sum_{k=0}(-1)^k \frac{x^{2k+1}}{(2k+1)!} && \text{Separate even and odd degree terms} \\ &= cos(x) + isin(x) && \text{Taylor Series} \end{aligned}$$

■

Lemma 7.3. Facts from Analysis:

1. For every $r \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{r^n}{n!}$ converges absolutely
2. If $a_n \in \mathbb{C}$ and $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges
3. If $L_a = \sum_{n=0}^{\infty} a_n, L_b = \sum_{m=0}^{\infty} b_m$ converges absolutely, then

   $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

   converges to

   $$(\sum_{n=0}^{\infty} a_n) \cdot (\sum_{m=0}^{\infty} b_m)$$

This will helpful in proving some identities.

Proof. For (1), we will use the ratio test. Indeed, consider $a_n = \frac{r^n}{n!}$, then

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{r}{n+1}| = 0 < 1$$

Thus, the series converges absolutely.

For (2), let $b_n = \sum_{i=1}^{n} a_i, c_n = \sum_{i=1}^{n} |a_i|$. We note that the topology $\mathbb{C}$ is homeomorphic to the Euclidean topology on $\mathbb{R}^2$, so in particular $\mathbb{C}$ is a complete metric space, so a sequence is convergent if and only if it is Cauchy. We know c_n converges, so for every $\epsilon > 0$, there exists some N_c such that for all $i, j > N_c$.

$$|c_i - c_j| < \epsilon$$

It remains for us to show that b_n is also Cauchy. Indeed, if $i = j$, then we are done. Without loss, we will then assume $j > i$, then

$$\begin{aligned} |b_i - b_j| &= |\sum_{k=i+1}^{j} a_k| \\ &\leq \sum_{k=i+1}^{j} |a_k| && \text{Triangle's Inequality} \\ &= |c_i - c_j| \\ &< \epsilon \end{aligned}$$

Thus, $\{b_n\}$ is Cauchy and converges.

For (3), $(\sum_{n=0}^{\infty} a_n)\cdot(\sum_{m=0}^{\infty} b_m)$ certainly converges to $L_a \cdot L_b$. Now write $(\sum_{n=0}^{\infty} a_n)\cdot(\sum_{m=0}^{\infty} b_m) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m$, this corresponds exactly to the terms of $\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$. The exact combinatorics is left as details to the reader. ∎

Corollary 7.4. Let e^z be the complex exponential map, then

- e^z converges for all $z \in \mathbb{C}$
- $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$

Proof. For the first, let $a_n = \sum_{k=1}^{n} |\frac{z^k}{k!}|$. It suffices for us to prove that a_n converges as absolute convergence implies monotone convergence from Lemma 7.3(2). But we note that

$$|\frac{z^k}{k!}| = \frac{|z|^k}{k!}$$

and $|z|$ is a real number, so Lemma 7.3(1) tells us that a_n converges.

For the second, we note that

$$\begin{aligned} e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{z^k w^{n-k}}{n!} && \text{Binomial Theorem} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k}{k!}\frac{w^{n-k}}{(n-k)!} \\ &= (\sum_{n=0}^{\infty} \frac{(z)^n}{n!}) \cdot (\sum_{n=0}^{\infty} \frac{(w)^n}{n!}) && \text{By Lemma 7.3(3)} \\ &= e^z \cdot e^w \end{aligned}$$

∎

Proposition 7.5. We can also recover $\cos(z)$ and $\sin(z)$ from e^z as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

If $x \in \mathbb{R}$, we also have that

$$\cos(x) = \Re e^{ix}, \sin(x) = \Im e^{ix}$$

7.2 Complex Logarithms

As in the case of $\mathbb{R}$, we want to define $\log(z)$ as an inverse to e^z such that

$$e^{\log(z)} = z$$

The problem is that e^z is not actually injective, so there are multiple choices for $\log(z)$. Consequently, this will result in $\log(z)$ not being continuous on all of $\mathbb{C}$.

Question 7.6. Given z, can we find all solutions satifying $e^w = z$?

Answer. We will again leverage on Polar Coordinates. Indeed, write $z = re^{i\theta}, w = x + iy$, then we have that

$$re^{i\theta} = e^{x+iy} = e^x e^{iy}$$

Thus, $x = \log(r), y = \theta + 2\pi k$.

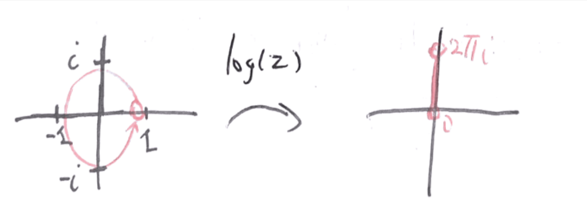
Thus, w is of the form $w_k = \log(r) + i(\theta + 2\pi k)$ ■

Definition 7.7 (Complex Logarithm). For $z \in \mathbb{C} \setminus \{0\}$, write $z = re^{i\theta}, \theta \in [-\pi, \pi]$. Then we define

$$\log(z) = \log(re^{i\theta}) := \log(r) + i\theta$$

θ is sometimes refered to as the **principal argument** of z and we denote $Arg(z) = \theta$.

Remark 7.8. Note that the complex logarithm $\log(z)$ is not continuous on all of $\mathbb{C}$:



As you trace around the unit circle, going back to $z = 1$ presents a problem.

Therefore, we can only define the complex logarithm up to a **branch cut** that's a radial line extending out from the origin. We will without loss choose this branch to be $(-\infty, 0]$

Proposition 7.9. $\log(z)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and in fact $\frac{d}{dz}\log(w) = \frac{1}{w}$

Proof. One way to do this is to use the Cauchy-Riemann equations in polar coordinates. We will however use the Inverse Function Theorem instead. Indeed, we note that on $\mathbb{C} \setminus (-\infty, 0]$, $z \mapsto e^z$, when viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, is a smooth function whose derivative is exactly

$$(e^z)' = e^x \begin{pmatrix} cos(y) & -sin(y) \\ sin(y) & cos(y) \end{pmatrix}$$

, which satisfies the matrix representation of complex numbers (hence we can extend the argument to holomorphic functions). Thus, the inverse function theorem tells us that $g(z) = \log(z)$ is also holomorphic and that

$$g'(f(x)) = (f'(x))^{-1}$$

This equation tells us that, locally for all $w = e^z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\frac{d}{dz}\log(w) = e^{-x} \begin{pmatrix} cos(y) & sin(y) \\ -sin(y) & cos(y) \end{pmatrix} = (e^z)^{-1} = \frac{1}{w}$$

Note that the branch cut does not prevent the existence of a local inverse, but it does prevent the existence of a global inverse on all of $\mathbb{C}$. ■

Remark 7.10. Let $z \in \mathbb{C} \setminus (-\infty, 0]$, then since $\frac{1}{z}$ is primitive on the path 1 to z, we have that:

$$\int_1^z \frac{d\xi}{\xi} = \log(z) - \log(1) = \log(z)$$

7.3 Homotopy Invariance of Integral

Let $f \in Hol(\Omega)$ and $\gamma : [a, b] \to \Gamma$ is continous and $\Gamma \subset \Omega$, then we can define the integral

$$\int_\gamma f(z)dz$$

Informally, we can define this because at each point on γ, we can find some disk to represent f as a power series with a primitive F, then we can split $[a, b]$ into a union of (not necessarily uniform) subintervals, then the integral γ can be approximated as a sum of the anti-derivatives F_k at each end-points.
We can formally justify this with what's called the Lebesgue's Number Lemma.

Lemma 7.11 (Lebesgue's Number Lemma)**.** Let K be a compact metric space and let $\{U_a\}$ be an open cover for K, then there exists some $\delta > 0$ (we call the **Lebesgue Number**) such that for all $x \in K$, there exists some a such that $B_{x,\delta} \subset U_a$

Proof. For all $x \in K$, since $\{U_a\}$ is an open cover of K, there exists some a such that $x \in U_a$. Since U_a is open, there exists some $r(x) > 0$ small enough such that $B_{x,2r(x)} \subset U_\alpha$.

Note that $\{B_{x,r(x)}\}$ running over all $x \in K$ is an open cover of K. Since K is compact, we in fact have a finite subcover:

$$B_{x_1,r_1}, ..., B_{x_n,r_n}$$

Choose $\delta = \min\{r_i\}$. Now for all $x \in K$, it belongs to one of the $B_{x_k,r_k} \subset U_a$.
Then we claim $x \in B_{x,\delta} \subset B_{x_k,2r_k} \subset U_a$. To do this, we need to verify $B_{x,\delta} \subset B_{x_k,2r_k}$. Indeed, for all $y \in B_{x,\delta}$, we have that

$$d(y, x_k) \leq d(y, x) + d(x, x_k) < \delta + r_k \leq 2r_k$$

This concludes the proof. ■

To apply this lemma in our context, we take $K = [a, b]$ and for all $s \in [a, b]$, we define $U_s = \gamma^{-1}(D_{\gamma(s),r(s)})$, where $r(s)$ is the radius of convergence of $\gamma(s)$ under the power-series. Then we can apply the Lebesgue's Number Lemma.

Question 7.12. Is the integral well-defined up to two different splitting?

Answer. Yes, the idea is that given two splitting on the interval, one can join them using a refinement by taking smaller intervaks of both of them and used the associated anti-derivative from either original splittings. To be more rigorous, let $\{I_k\}$ and $\{J_n\}$ be two different splittings of the interval $[0, 1]$. Now define

$$I_{k,n} \coloneqq I_k \cap J_n$$

Then we note that, by definition, the integral over the splitting $I_{k,n}$ would be of the form

$$\begin{aligned}\sum \int_{I_{k,n}} f(z)dz &= \sum_k \sum_n \int_{I_{k,n}} f(z)dz \\ &= \sum_k [\sum_n \int_{I_{k,n}} f(z)dz] \\ &= \sum_k \int_{I_k} f(z)dz\end{aligned}$$

Similarly, we can also show that

$$\sum \int_{I_{k,n}} f(z)dz = \sum_n \int_{J_n} f(z)dz$$

Thus, the two splittings yield the same integral. ∎

Definition 7.13. Let $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be continuous. We say γ_0 is **homotopy equivalent** to γ_1 if there exists a continuous function $\Gamma : [a, b] \times [0, 1] \to \Omega$ such that $\Gamma(s, 0) = \gamma_0(s)$ and $\gamma(s, 1) = \gamma_1(s)$ for all $s \in [a, b]$. We will also assume that either:

- All pathes are closed, for all t
$$\Gamma(a, t) = \Gamma(b, t)$$
- OR Endpoints are fixed
$$\Gamma(a, t) = \gamma_0(a) = \gamma_1(b)$$
$$\Gamma(b, t) = \gamma_0(b) = \gamma_1(b)$$

Theorem 7.14. If γ_0, γ_1 are homotopy equivalent in Ω and $f \in Hol(\Omega)$, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Remark 7.15. In a simply connected domain, any two pathes are homotopy equivalent. In otherwords, the path of integration is irrelevant.

8 Lecture 8 - 09/23/2022

8.1 Homotopy Invariance of Integrals - Continued

Theorem 8.1. If γ_0, γ_1 are homotopy equivalent in Ω and $f \in Hol(\Omega)$, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Proof. Let Γ be the homotopy equivalence function. For all $z_0 \in \Omega$, let D_{z_0} be a disc centered at z_0 small enough such that f can be expressed as a power-series on it, define

$$U_{z_0} = \Gamma^{-1}(D_{z_0})$$

Then $\{U_{z_0}\}$ will be an open cover of $R := [a,b] \times [0,1]$, which is a compact metric space. Apply Lebesgue's Number Lemma, we find $\delta > 0$ such that, we can split R into smaller rectangles R_k where each $diam(R_k) < \delta$.

The Lebesgue Number Lemma tells us that, for each R_k, $\Gamma(cl(R_k)) \subset D_z$ for some z. Since f is primitive on D_z, we have that

$$\int_{\Gamma|_{\partial R_k}} f(z)dz = 0$$

Putting the small rectangles together (Green's Theorem style), the inner edges cancel out, so we have that

$$\int_{\Gamma|_{\partial R}} f(z)dz = 0$$

Now consider the pathes $\ell_1 = \{a\} \times [0,1]$ and $\ell_2 = \{b\} \times [0,1]$, we claim that

$$\int_{\Gamma|_{\ell_1 + \ell_2}} f(z)dz = 0$$

Indeed, if endpoints are fixed, both paths are constant (stays at same point). If all path are closed, then this is the same path in opposite directions.

Thus, we only have to integrate over the horizontal pathes:

$$\int_{\partial R} f(z)dz = 0 = \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz$$

■

Definition 8.2. A set Ω is **simply connected** if any closed loop in Ω is homotopy equivalent to the constant path. This is equivalent to saying, for any pathes γ_0, γ_1, $\gamma_0(a) = \gamma_1(a) = z_0$, $\gamma_0(b) = \gamma_1(b) = z_1$, are homotopy equivalent.

Remark 8.3. If Ω is simply connected domain, then

$$\int_{z_0}^{z_1} f(z)dz$$

does not depend on the path specified.

Definition 8.4. Let Ω be simply connected and $f \in Hol(\Omega)$. Suppose $f(z) \neq 0$ for all $z \in \Omega$.

Fix some $z_0 \in \Omega$, and consider

$$a_0 \text{ such that } f(z_0) = e^{a_0}$$

We can find a_0 as

$$Re(a_0) = \log|f(z_0)|, Im(a_0) = arg\, f(z_0)$$

Then we define

$$\log f(z) := a_0 + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)} d\xi$$

Show that $\log f(z)$ aligns with the global definition of the complex logarithm.

Proof. Exercise. The idea is to denote $\varphi(z) = a_0 + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)} d\xi$ and show that

$$(f(z)e^{-y(z)})' = 0$$

and realize that their product must be 1. ■

♣♣♣ Mattie: [explain motivation later]

Definition 8.5. What is $f(z)^\alpha$? We define

$$f(z)^\alpha = e^{\alpha \log(f(z))}$$

This is defined when $f \in Hol(\Omega)$, $f(z) \neq 0$ on all of Ω, and Ω is simply connected.

In a simply connected domain, the branch of the logarithm, it is enough to be determined by which a_0 you choose.

8.2 Laurent Series

Suppose f is holomorphic on $\{z : a < |z - z_0| < A\}$, where we require $a \geq 0$ and $A \leq \infty$.

Theorem 8.6. If $f \in Hol\{z : a < |z - z_0| < A\}$, then f can be represented as

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

, where we have that

$$a_n = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} f(\xi)(\xi - z_0)^{-(n+1)} d\xi$$

, where r is between $a < r < A$ (Note that for $\mathbb{Z}_+$, this is the same as the Taylor Power Series)

Proof. Without loss of generality, we can shift this to $z_0 = 0$.

Take z such that $a < |z| < A$ and pick r, R such that

$$a < r < |z| < R < A$$

Then consider the set

$$G := \{z : r < |z| < R\}$$

Then Cauchy's Integral Formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi \\ &= \frac{1}{2\pi i}(\int_{|z|=R}\frac{f(\xi)}{\xi - z}d\xi - \int_{|z|=r}\frac{f(\xi)}{\xi - z}d\xi) \\ &= \frac{1}{2\pi i}(I_1 - I_2) \end{aligned}$$

For I_1, we note that $|z| < |\xi| = R$, so

$$\begin{aligned} I_1 &= \int_{|z|=R}\frac{f(\xi)}{\xi - z}d\xi \\ &= \int_{|z|=R}\frac{f(\xi)}{\xi}\frac{1}{1-\frac{z}{\xi}}d\xi \\ &= \int_{|z|=R} f(\xi)\sum_{k=0}^{\infty}\frac{z^k}{\xi^{k+1}} \end{aligned}$$

For I_2, we note that $|z| > |\xi| = r$, so

$$\begin{aligned} I_2 &= \int_{|z|=r}\frac{f(\xi)}{\xi - z}d\xi \\ &= \int_{|z|=r}\frac{-f(\xi)}{z}\frac{1}{1-\frac{\xi}{z}}d\xi \\ &= \int_{|z|=r} -f(\xi)\sum_{k=0}^{\infty}\frac{\xi^k}{z^{k+1}} \\ &= \int_{|z|=r} -f(\xi)\sum_{n=-\infty}^{-1}\frac{z^n}{\xi^{n+1}} \end{aligned}$$

Change of Variables

Since both series converge uniformly, we can integrate term by term and use Cauchy's Formula for Derivatives, rewriting both I_1 and I_2 out in series finishes the proof. ■

Now we will consider a particular case of the Laurent Series!

Definition 8.7. Let $f \in Hol(\Omega \setminus \{z_0\})$ where $z_0 \in \Omega$. In this case, we say f has a singularity at z_0.

Take δ small enough and $D_{z_0,\delta} \subset \Omega$, then the Laurent Series tells us

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$$

We say that

- f has a **removable singularity** at z_0 if $a_n = 0$ for all $n < 0$. In this case,

  $$f(z) = \sum_{n\geq 0} a_n(z - z_0)^n, f(z_0) = a_0$$

  An example of such function would be $f(z) = \frac{sin(z)}{z}$ has removable singularity at 0
- We say f has a **pole at** z_0 if $a_n \neq 0$ for finitely many $n < 0$. We say the **order of a pole** as

  $$\max\{n \geq 0 : a_{-n} = 0\}$$

- We say that f has an essential singularity if there exists infinitely many $n < 0$ such that $a_n \neq 0$.

  For example consider

  $$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

  This has an essential singularity at $z = 0$.

These are the singularities we care about.

9 Lecture 9 - 09/26/2022

9.1 Classification of Singularities

Recall, let $f \in Hol(\Omega \setminus \{z_0\})$, where $z_0 \in \Omega$, we can write f into a Laurent series

$$f(z) = \sum_{z \in \mathbb{Z}} a_n (z - z_0)^n, 0 < |z - z_0| < N$$

, and we say that z_0 is removable if $a_n = 0$ for all $n < 0$.

Theorem 9.1. z_0 is a removable singularity of f if and only if there exists a neighborhood $z_0 \in U$ such that f is bounded in $U \setminus \{z_0\}$.

Proof. If f has a removable singularity, we can write it locally as

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

which has a positive radius of convergence and hence bounded in a neighborhood of z_0.

Conversely, suppose f is bounded on $0 < |z - z_0| < \delta$. Then for all $n \geq 1$, we note that

$$a_{-n} = \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)(z - z_0)^{n-1} dz$$

As $n - 1 \geq 0$ and f is bounded, this integral is also bounded, so

$$|a_{-n}| \leq \frac{1}{2\pi} M(1)(2\pi r) = Mr$$

for all $0 < r < \delta$, we can shrink r down and have that $a_{-n} = 0$. ■

Remark 9.2. If $f(z) = o(\frac{1}{|z-z_0|})$, then f has a removable singularity at z_0.

We say f has a **pole at** z_0 if $a_n \neq 0$ for finitely many $n < 0$. We say the **order of a pole** as

$$\max\{n \geq 0 : a_{-n} = 0\}$$

Theorem 9.3. z_0 is a pole if and only if $\lim_{z \to z_0} |f(z)| = \infty$

Proof. Suppose z_0 is a pole of order m, then we can write

$$f(z) = \frac{g(z)}{(z - z_0)^m}, g(z_0) \neq 0$$

Hence we have that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} |\frac{g(z)}{(z - z_0)^m}| = \infty$$

Conversely, suppose $\lim_{z \to z_0} |f(z)| = \infty$, then we can find some $r > 0$ such that $|f(z)| > 1$ for all z in $0 < |z - z_0| < r$.

Now consider $h(z) = \frac{1}{f(z)}$ on $D_{z_0,r} \setminus \{z_0\}$, then $h(z) \in Hol(D_{z_0,r} \setminus \{z_0\})$ and $\lim_{z \to z_0} h(z) = 0$ - so we can extend h to the whole disk. Therefore it has to be the case that $|h(z)| < 1$, then by the previous theorem, this means

z_0 is a removable singularity for h, so in fact h is holomorphic on the whole disk.

Since $h(z_0) = 0$, we can find the smallest m such that $h(z) = (z - z_0)^m h_0(z)$ but $h_0(z_0) \neq 0$. Now consider $g = \frac{1}{h}$, then we have that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

Thus, z_0 is a pole of order m. ■

We say that f has an essential singularity if there exists infinitely many $n < 0$ such that $a_n \neq 0$.

Theorem 9.4 (Casorati-Weierstrass Theorem)**.** If f has an essential singularity at z_0, then for any neighborhood U of z_0, $U \subset \Omega$, $f(U)$ is **dense** in $\mathbb{C}$ (in other words, for all $w \in \mathbb{C}$, there exists a sequence of z_n such that $f(z_n) \to w$ as $z_n \to z_0$)

Proof. We will prove this using contradiction. Suppose there exists some neighborhood U of z_0 such that $f(U)$ is not dense. Then there exists some $a \in \mathbb{C}$ and $r > 0$ such that $D_{a,r} \cap f(U) = \emptyset$.

Now define $g(z) = \frac{1}{f(z)-a}$. Then clearly $g \in Hol(U)$ as the denominator is never 0. We also have that $|g| < \frac{1}{|f(z)-a|} < \frac{1}{r}$. Moreover, we note that $g(z) \neq 0$ in $U \setminus \{z_0\}$ because f is bounded. While $g(z_0)$ could be 0, we will choose the smallest m such that $g(z) = (z - z_0)^m g_0(z)$ and $g_0(z_0) \neq 0$.

Thus, $g_0(z) \neq 0$ on U. But we have that

$$f(z) = \frac{1}{g(z)} + a = \frac{1}{(z - z_0)^m g_0(z)} + a$$

, so f has a removable singularity or some pole at z_0, hence a contradiction. ■

Definition 9.5. Suppose f has a singularity at z_0 and write

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

Then we define the **residue of** f **at** z_0 as a_{-1} and denote is as $Res_{z_0}(f)$ or $res(f, z_0)$

Remark 9.6. If z_0 is a pole of order 1, then

$$res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

9.2 Cauchy's Residue Theorem

Theorem 9.7 (Cauchy's Residue Theorem)**.** Let G be a bounded domain with $\partial G \in PC^1$ and $\overline{G} \subset \Omega$. Let $z_1, ..., z_n \in G$ and $f \in Hol(\Omega \setminus \{z_1, ..., z_n\})$, then

$$\int_{\partial G} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

Proof. Let D_k be small disks around each z_k and consider $\tilde{G} = G \setminus \bigcup D_k$. Note that $f(z)$ is holomorphic on $\tilde{G}$, so

$$\partial_{\partial \tilde{G}} f(z) dz = 0$$

On the other hand, we also have that $\partial \tilde{G} = \partial G - \bigcup \partial D_k$, so in other words

$$\int_{\partial G} f(z) dz = \sum_k \int_{\partial D_k} f(z) dz$$

Write f as a Laurent Series, then every term except for the -1 term is primitive and thus evaluate to 0, so we are left with:

$$\int_{\partial G} f(z) dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

■

10 Lecture 10 - 09/28/2022

Clarification from last lecture: In the proof of the Casorati–Weierstrass theorem, we have that $g(z) \neq 0$ for all $z \in U \setminus \{z_0\}$, but $g(z_0) = 0$ is totally possible! Hence we write

$$f(z) = \frac{1}{g(z)} + a$$

, where f either has a pole or a removable singularity at z_0, giving the contradiction.

10.1 Computing Real Integrals with Residues

Example 10.1 (A Straight-forward Application). Let $s \in \mathbb{R}$, consider the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1} e^{isx} dx$$

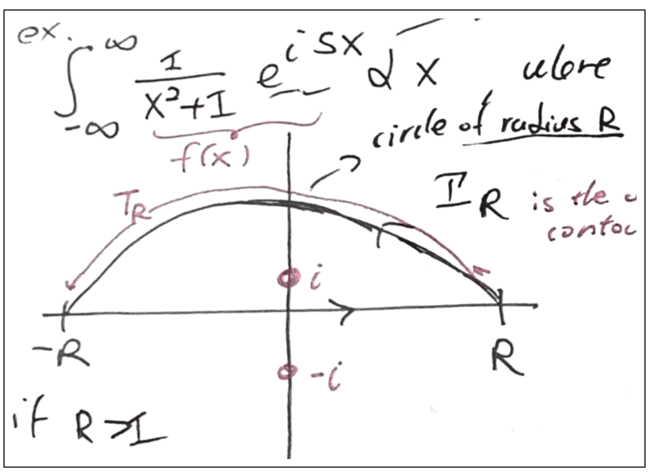
Let $f(z) = \frac{1}{x^2+1} e^{isx}$, then $I = \int_{\mathbb{R}} f(z)dz = \pi e^{-|s|}$

Remark 10.2. Why do we even care about integrals of this form? Well, they are really closed aligned with **inverse Fourier transforms**. In particular, you would calculate integrals of the form

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s} dx$$

Proof of Example. We will consider three cases $s = 0$, $s > 0$, and $s < 0$. For $s = 0$, this is obvious.

Now for $s > 0$, consider the following contour:



Then Cauchy's Residue Theorem tells us that

$$\int_{\Gamma_R} f(z)dz = 2\pi i Res(f, i)$$

i is a pole of order 1 for f, so

$$Res(f, i) = \lim_{z \to i} (z - i) f(z) = \frac{e^{-s}}{2i}$$

Thus, we have that

$$\int_{\Gamma_R} f(z)dz = \pi e^{-s}$$

Now take $R \to \infty$, we note that since f **is integrable** as $\int_{-\infty}^{\infty} |f(x)|dx < \infty$,

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} f(z)dz = I$$

So in other words,

$$I = \lim_{R\to\infty} [\int_{\Gamma_R} f(z)dz - \int_{T_R} f(z)dz] = \pi e^{-s} - \lim_{R\to\infty} \int_{T_R} f(z)dz$$

We want to show that the last term goes to 0, indeed

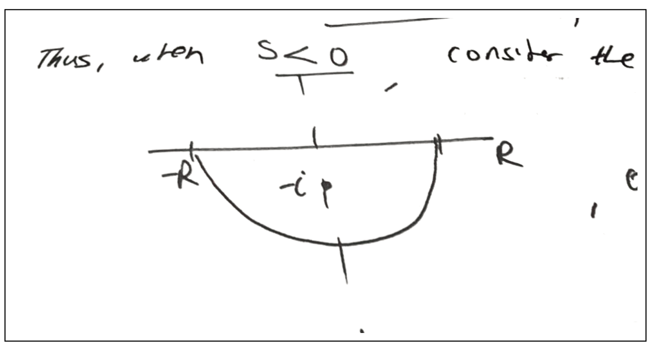
$$\int_{T_R} f(z)dz = \int_0^{\pi} \frac{e^{isRe^{it}}}{(Re^{it})^2+1} iRe^{it}dt \qquad \text{Let } z = Re^{it}$$

$$|\int_{T_R} f(z)dz| \le \frac{R}{R^2-1}\pi \qquad \text{Note that } e^{isRe^{it}} \text{ is bounded by 1 as } s > 0 \text{ and } 0 < t < \pi$$

Taking $R \to \infty$, the bound goes to 0, thus, we have that

$$I = \pi e^{-s}, s \ge 0$$

What about when $s < 0$? In this case, $\int_{T_R} f(z)dz$ would not actually go to 0, so instead, we need to consider the opposite contour:



, and we can similarly show the result is true. Alternatively, we could also have argued using symmetry.

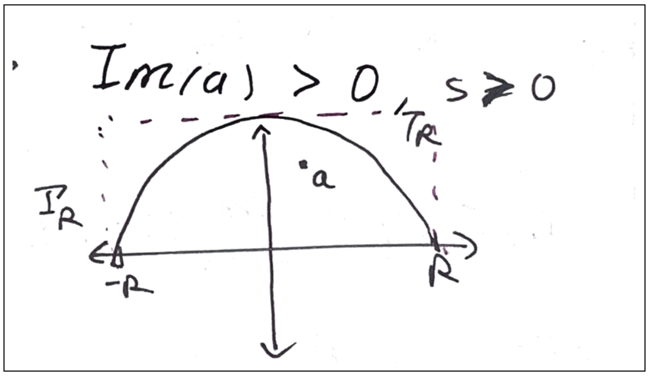
Note that while we proved this using a circle, we could have also used a rectangle instead. ■

Example 10.3 (A Not-So-Simple Application)**.** Let $s \in \mathbb{R}$ and $a \in \mathbb{C}$ such that $s > 0$ and $\Im(a) > 0$, consider the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{isx}}{x-a} dx$$

Let $f(z) = \frac{e^{isz}}{z-a}$, then what is I?

Consider the following contour:



where $R > |a|$, while we could use a rectangle, we will stick with the half-circle again. Then again Cauchys' Residue Theorem tells us that

$$\int_{\Gamma_R} f(z)dz = 2\pi i Res(f, a)$$

a is a pole of order 1 of f, so we have that

$$Res(f, a) = \lim_{z\to a}(z-a)\frac{e^{isz}}{z-a} = e^{isa}$$

However, we note that f is actually **NOT INTEGRABLE**, so the following limit need not exist

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

Howebver, if $\int_{T_R} f(z)dz = 0$, then the limit would exist, and we would have

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = 2\pi i e^{isa} = p.v\int_{-\infty}^{\infty} f(x)dx$$

However, we run into a problem with doing ML-esitmate on $\int_{T_R} f(z)dz = 0$, because it turns out it'd give us

$$|\int_{T_R} f(z)dz| \leq \frac{R}{R-|a|}\pi$$

, which does not converge to 0 as $R \to \infty$.

Fortunately, we do have a workaround:

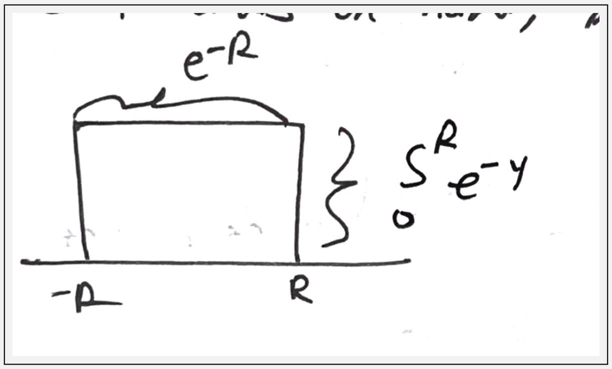
> **Lemma 10.4** (Jordan's Lemma). Let $C \in \mathbb{R}$ be some fixed constant, then
>
> $$\int_{T_R} |e^{iz}||dz| \leq C$$
>
> , where $|dz|$ is with respect to the Lebesgue Measure of the unit circle.

Now take $z = Re^{i\theta}$, then $dz = iRe^{i\theta}d\theta$, so we have that $|dz| = R|d\theta|$.

> **Corollary 10.5.** If $s > 0$, then
>
> $$\int_{T_R} |e^{iz}||dz| \leq C(s)$$

, where $C(s)$ is some constant dependent on s.

Remark 10.6. Jordan's Lemma for circles are generally hard to show, so most textbooks only prove it on a rectangle instead:



Now, using Jordan's Lemma, we have that

$$\begin{aligned} \left|\int_{T_R} \frac{e^{isz}}{z-a} dz\right| &\leq \frac{1}{R-|a|} \cdot \int_{T_R} |e^{isz}||dz| \\ &\leq \frac{C(s)}{R-|a|} \end{aligned}$$

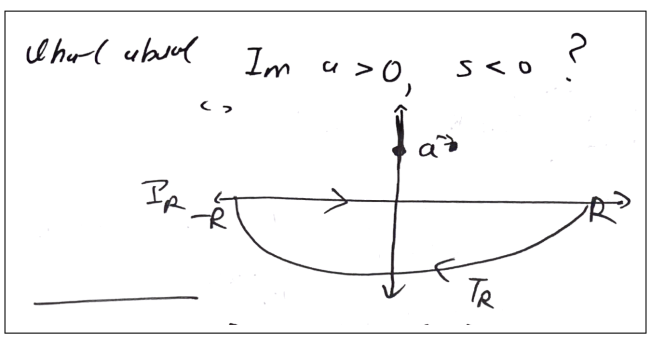
, so the limit goes to 0 as $R \to \infty$.

Question 10.7. What about if $\Im(a) < 0$ and $s > 0$?

In this case, we can either close the lower half or use a change of variables to get the same result.

Question 10.8. What about if $\Im(a) > 0$ but $s < 0$?

In this case, we will consider the contour



There are no singularities inside the contour so

$$\int_{\Gamma_R} f(z)dz = 0$$

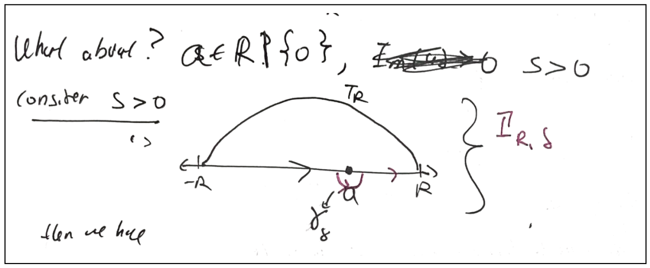
In addition, as $R \to \infty$, we also have that

$$\int_{T_R} f(z)dz = 0$$

Thus, the integral I is just 0.

Question 10.9. What if $a \in \mathbb{R} \setminus \{0\}$ and $s > 0$?

In this case, consider the contour:



In this, case we note that as we expand R and shrink δ, we have that

$$\int_{[-R,R]\setminus[a-\delta,a+\delta]} \mapsto p.v \int_{-\infty}^{\infty} f(z)dz$$

Cauchy's Residue Theorem tells us that

$$\int_{\Gamma_{R,\delta}} f(z)dz = 2\pi i Res(f,a) = 2\pi i e^{isa}$$

Jordan's Lemma tells us that

$$\lim_{R\to\infty} \int_{T_R} f(z)dz = 0$$

Finally, taking $\delta \to 0$ gives that

$$\int_{\gamma_\delta} f(z)dz = \pi i e^{isa}$$

Thus, we have that

$$p.v \int_{-\infty}^{\infty} f(z)dz = 2\pi i e^{isa} - \pi i e^{isa} = \pi i e^{isa}$$

11 Lecture 11 - 09/30/2022

11.1 Computing Real Integrals with Residues - Continued

Example 11.1. The Fresnel Integrals are the real and imaginary parts of the following integral:

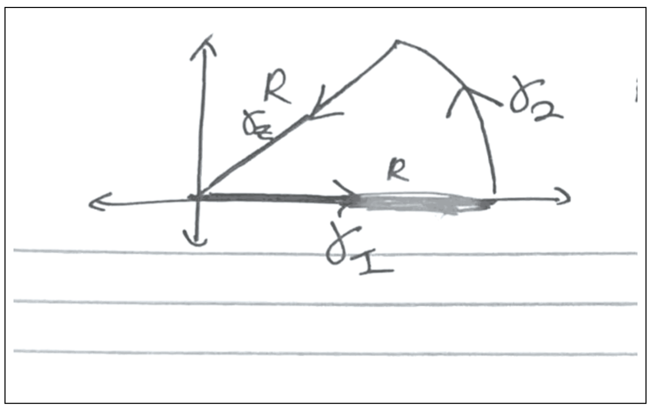
$$I = \int_0^\infty e^{-ix^2}\,dx$$

How does one compute I?

Answer. We will first look at the Gaussian Integral, which we are quite familiar with already:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Now for the Fresnel Integrals, consider the following contour:



We note that e^{-iz^2} is analytic over the entire region, so Cauchy's Integral Theorem tells us that

$$0 = \int_{\gamma_1} e^{-iz^2}\,dz + \int_{\gamma_2} e^{-iz^2}\,dz + \int_{\gamma_z} e^{-iz^2}\,dz$$

We note that as $R \to \infty$, a similar argument as before show that the integral over γ_2 vanishes. Thus, we have that

$$I = \lim_{R\to\infty} \int_{\gamma_1} e^{-iz^2}\,dz = -\lim_{R\to\infty} \int_{\gamma_z} e^{-iz^2}\,dz$$

We can parameterize γ_3 as $z = \frac{1+i}{\sqrt{2}}t$ from $t = R$ to $t = 0$, then

$$\begin{aligned}
\int_{\gamma_z} e^{-iz^2}\,dz &= \int_R^0 e^{-t^2}\frac{1+i}{\sqrt{2}}dt \\
&= -\frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}dt \\
\lim_{R\to\infty}\int_{\gamma_z} e^{-iz^2}\,dz &= -\frac{1+i}{\sqrt{2}}\int_0^\infty e^{-t^2} \\
&= -\frac{1+i}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2}
\end{aligned}$$

■

11.2 Argument Principle

Let G be a bounded domain and $\partial G \in PC^1$, and consider points $p_1, ..., p_m \in G$. Suppose $f \in Hol(cl(G) \setminus \{p_1, ..., p_m\})$, meaning that f is holomorphic in $\Omega \setminus \{p_1, ..., p_m\}$ where Ω is an open set containing $cl(G)$.

Suppose furthermore that $f(z)$ is not the constant zero function. We note that $cl(G)$ is compact, so the asusmption above implies that f has finitely many zeroes in $cl(G)$. Otherwise, if $f(z)$ has infinitely many zeroes, compact sets are sequentially compact in $\mathbb{C}$, so f would containing some zero that's not isolated, violating the Uniqueness Theorem. Let $z_1, ..., z_N$ be the zeroes of $f(z)$.

Now assume that $f(z) \neq 0$ for all $z \in \partial G$, the Argument Principle states that:

> **Theorem 11.2** (Argument Principle)**.** Let Z be the number of zeros inside G (counting order of the zero) and P be the number of poles inside G (counting order of the pole), then
>
> $$Z - P = \frac{1}{2\pi i}\int_{\partial G}\frac{f'(z)}{f(z)}dz$$

Note that when we say number of zeroes, we are always counting multiplicity.

Proof. We first note that $\frac{f'(z)}{f(z)}$ is not holomorphic on a given z if and only if z is one of the poles or the zeros of $f(z)$.

Since f is analytic, take $a \in cl(G)$, let $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$, then we note that

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g * z(} \\ &= \frac{m}{z-a} + \frac{g'(z)}{g(z)}\end{aligned}$$

Since $g(a) \neq 0$, we note that $\frac{g'(z)}{g(z)}$ is analytic in a neighborhood of a and can be represented as a power series, so

$$\frac{f'(z)}{f(z)} = m(z-a)^{-1} + \sum_{n=0}^{\infty} b_n (z-a)^n$$

Thus, in other words, m is the coefficient of the -1-th power term, hence

$$res(\frac{f'(z)}{f(z)}, a) = m$$

If a is one of the zeroes z_k, then m is exactly the order of the zero.

If a is one of the poles p_k, then m is given by factoring out the Laurent Series. Clearly m is exactly -1 times the order of the pole.

Thus, applying Residue's Theorem around the zeroes and the poles, we have that

$$Z - P = \frac{1}{2\pi i}\int_{\partial G}\frac{f'(z)}{f(z)}dz$$

■

> **Remark 11.3.** Why is this theorem called the **Argument Principle**? This is because $\frac{f'(z)}{f(z)}$ is actually primitive

given a chosen branch, and

$$\frac{f'(z)}{f(z)} = [\log f(z)]'$$

Thus, often we sometimes rewrite the Argument Principle as

$$Z - P = \frac{1}{2\pi i}\int_{\partial G} d\log(f(z))$$

We claim that in fact

$$\frac{1}{2\pi i}\int_{\partial G} d\log(f(z)) = \frac{1}{2\pi}\int_{\partial G} d(\arg f(z))$$

Proof. We first note that we can rewrite $f(z)$ in its polar coordinate form as

$$f(z) = r(z)e^{i\cdot arg(f(z))}$$

, then we have that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial G} d\log(f(z)) &= \frac{1}{2\pi i}\int_{\partial G} d(\log[r(z)e^{i\cdot arg(f(z))}]) \\
&= \frac{1}{2\pi i}\int_{\partial G} d(\log r(z)) + \frac{1}{2\pi i}\int_{\partial G} i\cdot d(arg(f(z))) \\
&= \frac{1}{2\pi i}\int_{\partial G} d(\log r(z)) + \frac{1}{2\pi}\int_{\partial G} d(\arg f(z))
\end{aligned}$$

It remains for us to show that $\frac{1}{2\pi i}\int_{\partial G} d(\log r(z))$ is actually 0. We first note that by the Argument Principle, this integral is an integer. Now, since $r(z)$ is a real valued function, $\int_{\partial G} d(\log r(z))$ is real, hence the integral is also a complex number.

The only number that is both complex and integer is 0.

Thus, we conclude that

$$\frac{1}{2\pi i}\int_{\partial G} d\log(f(z)) = \frac{1}{2\pi}\int_{\partial G} d(\arg f(z))$$

■

11.3 Rouche's Theorem

Rouche's Theorem is a standard corollary of the Argument Principle. Assume the same setup as before:

Suppose G is a bounded domain with $\partial G \in PC^1$ and $f, h \in Hol(cl(G))$:

Theorem 11.4 (Rouche's Theorem)**.** Suppose that $|h(z)| < |f(z)|$ for all $z \in \partial G$, then the number of zeroes of $f + h$ in G is the same as the number of zeroes of f in G.

Proof. Let $\varphi(z) = \frac{f(z)+h(z)}{f(z)}$. We claim that the number of zeroes of φ is the number of poles of φ (inside G). This is because the common zeroes of $f(z) + h(z)$ and $f(z)$ would cancel out, while the rest would match, so we claim we could without choose f, h such that they don't have any common zeroes.

Before we proceed with the rest of the proof, we will first justify why this is true. Now suppose $f(z) + h(z)$ and $f(z)$ have common zeroes $c_1, ..., c_m$ in G (counting multiplicity). We note that their common zeroes have to be finite since $cl(G)$ is compact, so having a infinite number of zeroes would imply that both functions here are identically zero (by the uniqueness theorem).

Now $f(z)+h(z)$ and $f(z)$ have common zeroes $c_1, ..., c_m$ if and only if $f(z), h(z)$ have common zeroes $c_1, ..., c_m$.

Now we claim that in general, for $g \in Hol(\Omega)$ and $z_0 \in \Omega$ such that $g(z_0) = 0$, $\frac{g(z)}{z-z_0}$ has a removable singularity at z_0 (this just comes from the fact that its neighborhoods are bounded). So we can without loss view the quotient with the limit filled in as a holomorphic function on Ω.

Applying this principle to $f(z)$ and $h(z)$, we have that

$$f(z) = \prod_{k=1}^{m}(z-c_k)f_0(z), h(z) = \prod_{k=1}^{m}(z-c_k)h_0(z)$$

, then the extraneous terms would cancel out but $f_0(r) \neq 0, h_0(r) \neq 0$ for any $r \in \{c_1, ..., c_m\}$.

Now, we write $\varphi(z) = 1 + \frac{h(z)}{f(z)}$, and note that

$$|\frac{h(z)}{f(z)}| < 1 \text{ on } \partial G$$

, so the values of $\varphi(z)$ are exactly values contained in $D_{1,1}$ (the open disk centered at 1 of radius 1) by Triangle's Inequality. In particular we note this means that $Re(\varphi(z)) > 0$.

Therefore, we note that $Log\varphi(z)$ ($z \in \partial G$) is well-defined as we avoided the branch cut at $(-\infty, 0]$. Thus, it is a well-defined anti-derivative where

$$(Log\varphi(z))' = \frac{\varphi'(z)}{\varphi(z)}$$

, hence we have that

$$\int_{\partial G} \frac{\varphi'(z)}{\varphi(z)} = 0$$

Then by the argument principle, the number of zeroes and φ is the same as the number of poles of φ in G. ∎

Corollary 11.5 (The Fundamental Theorem of Algebra). Let $p(z) = \sum_{k=0}^{n} a_k z^k$, where $a_n \neq 0$, then $p(z)$ has n complex roots (counting multiplicity).

Proof. Take $G = D_{0,R}$ where $R > 0$ is big enough that

$$|a_n|R^n > \sum_{k=0}^{n-1}|a_k|R^k$$

, we can find this R because

$$\lim_{R\to\infty} \frac{\sum_0^{n-1}|a_k|R^k}{|a_n|R^n} = 0$$

We note that on $z \in \partial D_{0,R}$, $|a_n z^n| > |p(z) - a_n z^n|$, so we can take $f(z) = a_n z^n$ and $h(z) = p(z) - a_n z^n$.

Then we note that $a_n z^n$ has n-roots, so Rouche's Theorem tells us that $p(z)$ has n-roots. ∎

11.4 Hurwitz Theorem

Definition 11.6. Let $\{f_n\}$ be a sequence of functions, we say $f_n \to f$ converes normally in G if for all compact $K \subset G$, f_n converges to f on K uniformly.

In this class, when we say f_n converges to f and f_n are holomorphic on G, then we always mean that the convergence is normal!

Proposition 11.7. If f_n are holomorphic functions on G that converges to f normally, then f is also holomorphic on G. Furthermore, we have that $f_n^{(k)}$ converges to $f^{(k)}$ normally.

Proof. Let R be a rectangle contained in G, since f_n converges to f normally, f_n converges to f uniformly on R, so we can switch the integral and limit to see that

$$0 = \lim_{n\to\infty} \int_{\partial R} f_n dz = \int_{\partial R} \lim_{n\to\infty} f(z)dz = \int_{\partial R} f(z)dz$$

, thus Morera's Theorem tells us that f is holomorphic on G.

Now for the convergence of $f_n^{(k)}$ to $f^{(k)}$, let K be a compact set in $G \subset \Omega$. For all $z \in K$, consider the disk D_z small enough that $cl(D_z) \subset \Omega$, then the collection $\{D_z, z \in K\}$ form an open cover of K. As K is compact, we can find a finite subcover $D_{z_1}, ..., D_{z_n}$.

We will without loss take $G = \bigcup_{i=1}^n D_{z_i}$ (as we only need to show uniform convergence on K). This is going to be a bounded domain with piecewise smooth boundary, so we can use Cauchy's Formula for Derivatives and see that

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial G} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} d\xi$$

We now note that as both K and ∂G are compact and disjoint,

$$dist(K, \partial G) = \inf_{(x,y)\in K\times\partial G} d(x, y) = \delta > 0$$

Thus, for any $z \in K, \xi \in \partial G$, $|z - \xi| \geq \delta$, so we have that

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{1}{2\pi} \frac{k!}{\delta^{k+1}} \cdot [\max ||f_n - f||] \cdot Len(\partial G)$$

Now since f_n converges to f uniformly on G, it converges in measure and hence $[\max ||f_n - f||]$ shrinks to 0 as $n \to \infty$. Thus, we have that

$$\lim_{n\to\infty} |f_n^{(k)}(z) - f^{(k)}(z)| = 0$$

, hence we have shown uniform convergence on K. ■

Definition 11.8 (Alternative Definition of Normal Convergence)**.** If for all disc D such that $cl(D) \subset \Omega$, f_n converges to f on D uniformly, then we say f_n converges to f normally in Ω.

Note that the two definition of f_n are equivalent.

Theorem 11.9 (Hurwitz's Theorem)**.** Suppose $f \in Hol(\Omega)$ and $z_0 \in \Omega$ is a zero order m.

Let $z_0 \in U$, where U is some bounded open set, and $\partial U \in PC^1$.

Now suppose for all $z \in cl(U) \setminus \{z_0\}$, $f(z) \neq 0$, and suppose $\{f_n\}$ converges to f normally on Ω, then there exists some $N \in \mathbb{N}$ such that for all $k > N$, f_k has exactly m zeroes in U.

Proof of Hurwitz's Theorem. We know that f_n converges uniformly to f on ∂U and f_n' converges uniformly to f' on ∂U. Since $f(z) \neq 0$ on ∂U, we note that

$$\frac{f_n'}{f} \to \frac{f'}{f} \text{ converges uniformly on } \partial U$$

Since the convergence is uniform, we also have that

$$\frac{1}{2\pi i}\int_{\partial U}\frac{f_n'(z)}{f_n(z)}dz \to \frac{1}{2\pi i}\int_{\partial U}\frac{f'(z)}{f(z)}dz$$

, but we note that the Argument Principle tells us that both values above are integers. Now pick some $\epsilon < \frac{1}{2}$, then there exists some N such that for all $n > N$,

$$|\frac{1}{2\pi i}\int_{\partial U}\frac{f_n'(z)}{f_n(z)}dz - \frac{1}{2\pi i}\int_{\partial U}\frac{f'(z)}{f(z)}dz| < \epsilon$$

, and hence they are the same integer. But we note that $f'(z)/f(z)$ has no zeroes and only a pole of order m while $f_n'(z)/f_n(z)$ has no poles but only zeroes.

Thus, $f_n(z)$ has m zeroes (counting multiplicity). ■

Remark 11.10. Note that Hurwitz's Theorem tells us that zeroes have to be appear gradually in the convergence.

This need not be true in the real case. For example, take $f_n(x) = x^2 + \frac{1}{n}$. This is not zeroes on $\mathbb{R}$. However, the limit is x^2 and has 2 zeroes at $x = 0$. So the zeroes can just pop up out of nowhere.

12 Lecture 12 - 10/03/2022

12.1 Remark: Normal Convergence

Note that the following facts holds on just in $\mathbb{C}$, but in $\mathbb{R}^n$ in general. Recall the two definitions of normal convergence:

Definition 12.1 (Equivalent Definitions of Normal Convergence)**.** Let $\{f_n\}$ be a sequence of functions, we say $f_n \to f$ converes normally in G if:

- (a) for all compact $K \subset G$, f_n converges to f on K uniformly.
- or, (b) if for all disc D such that $cl(D) \subset \Omega$, f_n converges to f on D uniformly.

The two definitions are in fact equivalent.

Proof. $(a) \implies (b)$ is obvious. Now, for $(b) \implies (a)$, suppose $K \subset \Omega$ is compact, then for all $z \in K$, we can choose D_z be an open disk centered at z small enough that $cl(D_z) \subset \Omega$.

Then the collection $\{D_z\}_{z \in K}$ form a clear open cover of K, so we can find a finite subcover, $D_{z_1}, ..., D_{z_n}$. f converges uniformly on each $cl(D_{z_i})$, so f converges uniformly on $\bigcup_{k=1}^{n} D_{z_i}$, which covers K. ■

How do we in general check for normal convergence? It seems like we will have to check uncountably many disks!

Proposition 12.2 (Strong Covering Property)**.** Consider the sequence $\{K_n\}_{n=1}^{\infty} \subset \Omega$ such that for all compact $K \subset \Omega$, there exists some N such that $K \subset \bigcup_{n=1}^{N} K_n$.

Then, f_n converges to f uniformly on all compact sets $K \subset \Omega$ if and only if f_n converges to f uniformly on each K_n.

Proof. The forward direction is obvious. Conversely, for any compact set K, find $N \in \mathbb{N}$ such that $K \subset \bigcup_{n=1}^{N} K_n$. Then the uniform convergence on each K_n implies uniform convergence on all finite unions on all of $\bigcup_{n=1}^{N} K_n$, hence the uniform convergence is held on K. ■

Remark 12.3. In practice, for open subset $\Omega \subset \mathbb{R}^d$, we would define each K_n as

$$K_n := \{x \in \Omega \mid dist(x, \Omega^c) > \frac{1}{n}, |x| \leq n\}$$

♣♣♣ Mattie: [Ask about this]

Definition 12.4. We say the collection $\{K_n\}_{n=1}^{\infty}$ is a **compact exhaustion** of Ω if

$$K_n \subset int(K_{n+1}) \text{ and } \bigcup_{n=1}^{\infty} K_n = \Omega$$

, note that the latter implies for any compact $K \subset \Omega$, there exist $N \in \mathbb{N}$ such that $K \subset \bigcup_{n=1}^{N} K_n$

12.2 Seminorms of Holomorphic Function

Definition 12.5. Let $Hol(\Omega)$ be the set of all holomorphic functions on Ω, given $f \in \Omega$, we define the norm

$$||f||_{C(K_n)} = |\sup_{z \in K_n} f(z)|$$

When K_n are points, this is called a semi-norm. $C(K_n)$ refers to the space of continuous real-valued functions on K_n.

Proposition 12.6. If K_n has the strong covering property, then f_n converges to f on compact sets uniformly if and only if $||f_n - f||_{C(K_n)} \to 0$ as $j \to \infty$.

Remark 12.7. An important object of study in functional analysis are what's called **Frechet Spaces**, whose topology is given by countably many semi-norms.

Definition 12.8. Let $f, g \in Hol(\Omega)$, we can define a metric on $Hol(\Omega)$ as

$$p(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||f - g||_{C(K_n)}}{1 + ||f - g||_{C(K_n)}}$$

One can verify that this is indeed a metric (more details in Conway's Book)

Proposition 12.9. f_n converges to f uniformly on compact sets if and only if $p(f_n, f) = 0$ as $n \to \infty$.

In practice, no one would actually compute the norm or fixes a compact exhaustion. However, there are great theoretical values in this metric.

Theorem 12.10. $(Hol(\Omega), p)$ is a complete metric space.

Proof. We first note that for a compact set K, $C(K)$ is a complete normed space (hence a Banach Space), so we observes that

- 1. $\lim_{k \to \infty} p(f_k, f) = 0$ if and only if $\lim_{k \to \infty} ||f - f_k||_{C(K_n)} \to \infty$ for all n.
- 2. $\{f_k\}$ is Cauchy with respect to p on $Hol(\Omega)$ if and only if f_k is Cauchy with respect to $|| \cdot ||_{C(K_n)}$ on $C(K_n)$ for every n

Now, every convergent sequence is Cauchy. Conversely, suppose $\{f_k\}$ is a Cauchy sequence, then since each $C(K_n)$ is a complete normed space, Hence on each K_n, $\{f_k\}$ converges uniformly to some limit F_n.

We claim that we can actually find a global function $f \in C(\Omega)$ such that $f|_{K_n} = F_n$. Indeed, we want to essentially glue each F_n together, which follows from the glueing lemma that for any two continuous function f on K_1 and g on K_2 (within $\mathbb{C}$ so Hausdorff) with non-empty intersection $K_1 \cap K_2$, we can extend f and g together into a larger continuous function.

Now, since f is the limit on each K_n, we have that $||f_k - f||_{C(K_n)} \to 0$ as $k \to \infty$, for all n. Hence we have from Observation 1 that $\lim_{k \to \infty} p(f_k, f) = 0$, so f is the limit of the Cauchy seuqnece in $Hol(\Omega)$.

It remains for us to show that $f \in Hol(\Omega)$. Indeed, we note that for any $K \subset \bigcup_1^N K_j$, f_n converges to f uniformly, hence by Morera's Theorem, we can get that $f \in Hol(\Omega)$. ■

12.3 Open Mapping Theorem and Inverse Function Theorem

Suppose $f \in Hol(z)$ (non-constant), $z_0 \in \Omega$, $f(z_0) = w_0$. Let G be a bounded domain with $z_0 \in G$ and $\partial G \in PC^1$.

Furthermore, for all $z \in \partial G$, suppose we have $|f(z) - w_0| \geq \delta > 0$.

Does such G always exist? Yes!
This is because the zeroes of the function $f(z) - w_0$ are isolated (as $f(z)$ is non-constant). Therefore, $f(z) - w_0$ has no zeroes in $cl(D_{z_0,r}) \setminus \{z_0\}$ for a sufficiently small r.

Let $w \in \mathbb{C}$ such that $|w - w_0| < \delta$, and let m be the multiplicity of the zero z_0 of $f(z) - w_0$

Lemma 12.11. Fix w as above, the fiber $f^{-1}(w)$ has m points counting multiplicity in G, meaning that the function $g(z) = f(z) - w$ has zeroes $z_1, \ldots, z_m$ (counting multiplicity)

Proof. We will apply Rouche's Theorem on this. Indeed, let $g_0(z) = f(z) - w_0$, then we note that

$$g(z) = f(z) - w = [f(z) - w_0] + [w - w_0]$$

By our setup above, we know that $|f - w_0| \geq \delta$ and $|w - w_0| < \delta$. So Rouche's Theorem tells us that both $f(z) - w_0$ and $g(z) = [f(z) - w_0] + [w - w_0]$ have the same number of zeroes, counting multiplicity.

m was defined to be exactly the multiplicity of z_0 at $f(z) - w_0$, as we chosen G small enough that $f(z) - w_0$ has no other zeroes within. ■

Corollary 12.12 (Open Mapping Theorem). If $f \in Hol(\Omega)$ be a non-constant function, then f is an open map, meaning that for all open sets $U \subset \Omega$, $f(U)$ is open.

Proof. The argument here will run similarly to how we had above.

Let $w_0 \in f(U)$, we want to show that there exist some open set $V \subset f(U)$ that contains w_0.

Since $w_0 \in f(U)$, we know there exist some point $z_0 \in U$ such that $f(z_0) = w_0$. Since U is open, we can find some radius $\epsilon > 0$ small enough that $cl(D_{z_0,\epsilon}) \subset U$.

Now consider the function $g(z) = f(z) - w_0$. Since g is holomorphic and non-constant, the uniqueness theorem for analytic function tells us that the zeros of $g(z)$ are isolated. Hence, we can choose $\epsilon > 0$ small enough that without loss $g(z)$ only has the root z_0 in $cl(D_{z_0,\epsilon})$.

Since $\partial D_{z_0,\epsilon}$ is compact, and $|g(z)|$ is continuous and positive, the extreme value theorem shows that there exist some minimum value $\delta > 0$ such that δ is the minimum of $|g(z)|$ for z on the boundary.

Now consider the disk $D_{w_0,\delta}$. For any $w \in D_{w_0,\delta}$, by Rouche's Theorem, write $g(z) = f(z) - w = [f(z) - w_0] + [w - w_0]$, then we note that $|g(z)| \geq \delta > |w - w_0|$, then this means that $f(z) - w_0$ and $g(z)$ have the same number of zeroes in the disk $D_{z_0,\epsilon}$.

In particular, this means that there exist some $z \in D_{z_0,\epsilon}$ such that $f(z) = w$. So in other words, $w \in f(U)$.

Hence, $D_{w_0,\delta} \subset f(U)$. In other words, $w_0 \in D_{w_0,\delta} \subset f(D_{z_0,r}) \subset U$, which implies that $f(U)$ is open. ■

Theorem 12.13 (Inverse Mapping Theorem)**.** Let $f \in Hol(\Omega)$, $z_0 \in \Omega$, and $f(z_0) = w_0$ but $f'(z_0) \neq 0$ (so the zero is simple). Let $G \subset \Omega$ be an open bounded set that contains z_0 with piecewise C^1 boundary.

Now suppose that, for all $z \in \partial G$, $|f(z) - w_0| \geq \delta$, then for all w such that $|w - w_0|$, there exists some unique function z such that $z = f^{-1}(w)$ and

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi$$

13 Lecture 13 - 10/05/2022

13.1 Inverse Mapping Theorem - Continued

Recall that per the setting of the Inverse Mapping Theorem:

- Let $f \in Hol(\Omega)$ and $z_0 \in \Omega$, $f(z_0) = w_0$ and $f'(z_0) \neq 0$
- G is a bounded domain such that $cl(G) \subset \Omega$ (in textbook we choose $D_{z_0,p}$) and $\partial G \in PC^1$
- $|f(\xi) - w_0| \geq \delta$ for all $\xi \in \partial G$
- $f(z) \neq w_0 \forall z \in G$ (possible because all zeroes of non-constant analytic functions are isolated)

Theorem 13.1 (Inverse Function Theorem)**.** Given the setup above, then for all w such that $|w - w_0| < \delta$, there exists a unique z in G such that $f(z) = w$ and

$$f^{-1}(w) = z = \frac{1}{2\pi i}\int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi$$

Proof. **Existence and uniquenss are given by Rouche's Theorem.** Indeed, by the Argument Principle, we note that

$$\#\text{ of zeroes of } f(z) - w \text{ in } G \; = \frac{1}{2\pi i}\int_{\partial G} \frac{f'(\xi)}{f(\xi) - w} d\xi$$

We note that we can rewrite

$$f(z) - w = [f(z) - w_0] + [w_0 - w]$$
$$|f(z) - w_0| \geq \delta, |w_0 - w| < \delta, \text{ on } \partial G$$

So Rouche's Theorem tells us that $f(z) - w$ and $f(z) - w_0$ has the same number of zeroes. But we note that we chose $f(z) - w_0$ to only have one zero, and it's a zero of order 1 by condition given, so in other words $f(z) - w$ has exactly 1 solution, hence uniqueness and existence are both proven.

For the formula given, we first note that the formula

$$\frac{\xi f'(\xi)}{f(\xi) - w}$$

has a unique singularity at $\xi = f^{-1}(w)$, which is a pole of order 1 by construction. We can calculate its residue as

$$\begin{aligned}
Res(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z) &= \lim_{\xi \to z} \frac{\xi f'(\xi)}{f(\xi) - w} \cdot (\xi - z) \\
&= z f'(z) \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - w} \\
&= z f'(z) \lim_{\xi \to z} \frac{\xi - z}{f(\xi) - f(z)} && w \text{ is defined to be } w = f(z) \\
&= z f'(z) \frac{1}{f'(z)} \\
&= z
\end{aligned}$$

Then apply Cauchy's Residue Theorem gives us

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial G} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi &= \frac{2\pi i}{2\pi i} Res(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z) \\
&= Res(\frac{\xi f'(\xi)}{f(\xi) - w}, \xi = z) \\
&= z \\
&= f^{-1}(w)
\end{aligned}$$

■

Note that the local solvability of f is just a standard result that can follow from Real Analysis too. However, the formula is more significant.

13.2 Winding Numbers

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous function such that $\gamma(a) = \gamma(b)$ (This is a C^0-closed-path). We will denote Γ as the image of γ in $\mathbb{C}$.

Without loss of generality, we can view γ as a function:

$$\gamma : \mathbb{T} \to \mathbb{C}$$

, where $\mathbb{T} := \frac{\mathbb{R}}{\mathbb{Z}}$ is the 1-dimensional torus given by an equivalence relation on $\mathbb{R}$, where we say $a \sim b$ if $a - b \in \mathbb{Z}$.

Definition 13.2. Let γ be as before and let $z \in \mathbb{C} \setminus \Gamma$, then we define the **winding number** as

$$w(\gamma, z) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z} d\xi$$

Proposition 13.3. $w(\gamma, z)$ is an integer valued function.

Proof. The proof is similar to the idea behind the Argument Principle, but we do not assume γ is C^1. We first use the Lebesgue Number Lemma to split Γ into small enough intervals such that $\frac{1}{\xi - z}$ locally has anti-derivatives on each interval (up to the choice of some branch):

$$log(\xi - z) + 2\pi ik, k \in \mathbb{Z}$$

As we move from one interval to another, we want to continue the branch we are previously on. We don't know what the branch is exactly, but this would form a telescoping series of primitivies, and the entire integral will evaluate to

$$\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z} d\xi &= \frac{1}{2\pi i} [log(\xi - z) + 2\pi ik_1 - log(\xi - z) - 2\pi ik_2] \\
&= \frac{2\pi i}{2\pi i}(k_1 - k_2) \\
&= k_1 - k_2 \in \mathbb{Z}
\end{aligned}$$

■

Proposition 13.4. $w(\gamma, z)$ measures the number of times (sign given by whether it's counter-clockwise or clockwise) γ wraps around z.

Proof Idea. We note that γ is homotopy equivalent to a circle, so the value of the integral doesn't change if we switch γ to a circle of some positive radius around z. The branch choice given in the previous proposition actually follows in a circle (when viewed as a Riemann Surface), so the loop on the circle does measure the numbers of times it wraps around z. This is an invariant around homotopy equivalence. ■

Definition 13.5. We say that γ is a **generalized path** if $\gamma : \mathbb{T}_1 \sqcup \mathbb{T}_2 ... \sqcup \mathbb{T}_n \to \mathbb{C}$ and γ restricted to each $\mathbb{T}_k$ is a C^0-closed-path. In other words, γ is intuitively the disjoint union of finitely many closed paths.

Remark 13.6. Let γ be a generalized path, and consider the function $z \mapsto W(\gamma, z)$. This is a well-defined function on $\mathbb{C} \setminus Im(\gamma)$ and is moreover analytic on the same domain (this follows from Morera's Theorem). Thus, analyticity implies continuity, and for $W(\gamma, z)$ to be a continuous integer value, this means that $W(\gamma, z)$ is constant on each connected component of $\mathbb{C} \setminus Im(\gamma)$. We sometimes call this the **index** of γ at z.

13.3 Generalized Residue Theorem

Theorem 13.7 (Generalized Residue Theorem)**.** Let γ be a generalized path in Ω, and say $f \in Hol(\Omega \setminus \{z_1, ..., z_n\})$, and suppose for all k, $z_k \notin Im(\gamma)$. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} w(\gamma, z_k) \cdot Res(f, z_k)$$

Before, we prove this theorem, we will introduce another theorem.

Theorem 13.8. Let γ be a generalized path in Ω, such that

$$W(\gamma, z) = 0, \text{ for all } z \notin \Omega$$

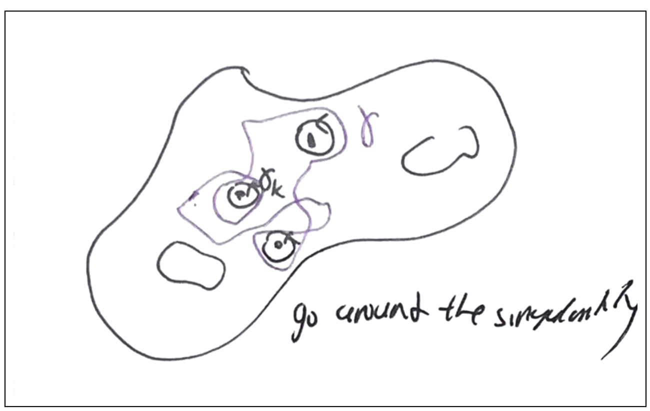
(Note that this is true for any simply connected domain Ω. If Ω has holes, then this is say that γ avoids these holes). Then we have that

$$\int_\gamma f(z)dz = 0, \text{ for all } f \in Hol(\Omega)$$

Proof. Textbook (to be added) ■

It turns out that Theorem 13.8 actually implied the Generalized Residue Theorem.

Proof of the Generalized Residue Theorem. Consider contours $\gamma_1, ..., \gamma_n$ be circles of small enough radius, each wrapping around $z_1, ..., z_n$ counterclock direction:



Then consider the path $\hat{\gamma}$ given by

$$\hat{\gamma} := \gamma - \bigcup_{k=1}^{n} W(\gamma, z_k)\gamma_k$$

Then we note that for all $z \not{i}n\Omega$, $W(\hat{\gamma}, z) = 0$ because the index of $gam\hat{m}a$ at these points is reduced to 0.

So Theorem 13.8 tells us that

$$\begin{aligned} 0 &= \int_{\hat{\gamma}} f(z)dz \\ &= \int_{\gamma} f(z)dz - \sum_{k=1}^{n} \int_{\gamma_k} W(\gamma, z_k) f(z)dz \\ &= \int_{\gamma} f(z)dz - [2\pi i \sum_{k=1}^{n} w(\gamma, z_k) \cdot Res(f, z_k)] \end{aligned}$$

■

14 Lecture 14 - 10/07/2022

14.1 Generalized Cauchy's Theorem

Last class, recall we discussed the Generalized Residue Theorem, whose proof relied on Theorem 13.8. Theorem 13.8 above is sometimes called the **Generalized Cauchy's Theorem**, not to be confused with the **Generalized Residue Theorem** that we proved last lecture.

Theorem 14.1. Let γ be a generalized closed path (ie. it is a union of finitely many closed paths) in Ω, such that for all $z \notin \Omega$,

$$w(\gamma, z) = 0$$

Then, we have that for all $f \in Hol(\Omega)$,

$$\int_\gamma f(z)dz = 0$$

Before proving this, we will first state a lemma:

Lemma 14.2. Let K be a compact subset of open Ω (ie. $K \subset \Omega$), then there exists some bounded open set G with $\partial G \in PC^1$ such that:

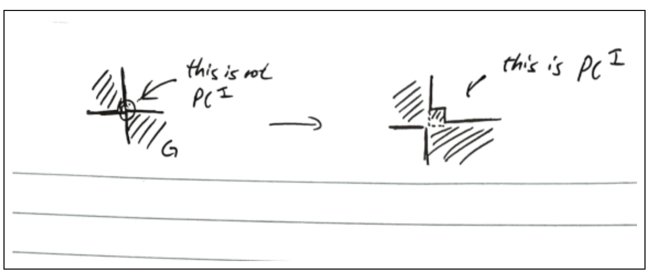
- $cl(G) \subset \Omega$
- $dist(K, G^c) \geq \delta > 0$

Proof. Since K is compact and Ω^c is closed, we know that $dist(K, \Omega^c)$ must be positive (or else they would intersect). So let's pick some δ such that $dist(K, \Omega^c) \geq 4\delta > 0$.

Now consider a δ-grid of Ω (split Ω into squares of side length δ), and consider

$$G := int(\bigcup_{Q \text{ is a square on the } \delta\text{-grid}, \; dist(K,Q)\leq\delta} Q)$$

Note, that we might create a G where it does have a self-intersection on the boundary, which would not make it PC^1, in this case, we would add a little square on the corner to "nudge" the boundary away:



Then we can check that

$$dist(K, G^c) = dist(K, \partial G) \geq \delta$$

If this is not true, then our G would have to be surrounded by other cubes whose distance to Q is less than δ, and hence we are not looking at the boundary of G. ■

Proof of Theorem 13.8. Recall that $f \in Hol(f)$, and consider

$$\int_\gamma f(z)dz = \int_\gamma \frac{1}{2\pi i}\int_{\partial G}\frac{f(\xi)}{\xi - z}d\xi dz \qquad \text{Since f is holomorphic, use Cauchy's Integral Formula}$$

$$= \int_{\partial G}\frac{-f(\xi)}{2\pi i}\int_\gamma \frac{1}{z-\xi}dzd\xi \qquad \text{Fubini's Theorem}$$

Now for any $\xi \in \Omega^c$, we claim that

$$\int_\gamma \frac{1}{z-\xi}dz = 0$$

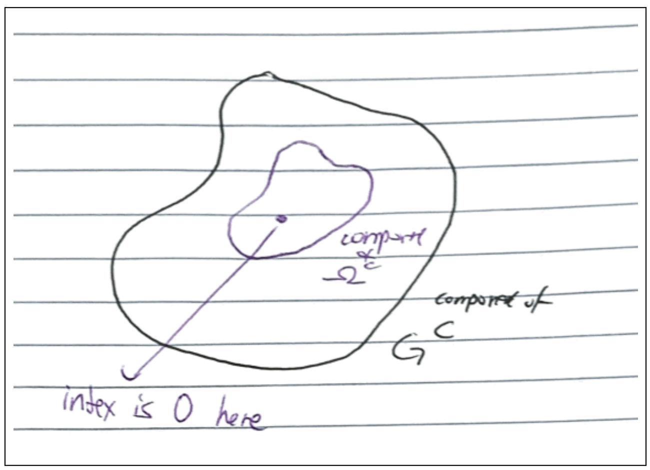
Since $G \subset \Omega$, we have that $\Omega^c \subset G^c$. Thus, any connected component of Ω^c is contained in some connected component of G^c.

If a bounded connected component C of G^c does not intersect with Ω^c, we just add it to G (take $G' = G \cup C$ to be the new G). So we can without loss assume that any bounded component of G^c intersect with Ω^c.

Thus, for all ξ in such bounded component of G^c,

$$\int \frac{1}{z-\xi}dz = 0$$

Indeed, consider the diagram:



Recall that by assumption, the index of any point in Ω^c is 0. Since the winding number is an analytic (thus continuous) function, this means that the index of every point on this one bounded component of G^c is identically zero.

Now, on unbounded connected components of G^c, we can't just add this to G since we still want G to be bounded. But we note that $W(\gamma, \xi) = 0$. This is because as $|\xi| \to \infty$, we can eventually find some ξ such that the loop γ does not enclose it. The fact that the entire component has index 0 follows from the fact that the winding number is a continuous function.

Thus, we have that

$$\int_\gamma f(z)dz = \int_{\partial G}\frac{-f(\xi)}{2\pi i}\cdot 0 d\xi = 0$$

■

> **Remark 14.3.** On the complex plane $\mathbb{C}$, we can add a point at infinity as the one-point compactification of $\mathbb{C}$, which we denote as $\hat{\mathbb{C}}$. This is called the **Riemann Sphere**.

Then on $\hat{\mathbb{C}}$, Ω is a simply connected domains if and only if $\hat{\mathbb{C}} \setminus \Omega$ is connected. The proof in the book is rather tedious and uses many details, but there's a standard topological proof of this using the Riemann Mapping Theorem, which we will discuss later.

Corollary 14.4. Let $\Omega \subsetneq \mathbb{C}$ be a bounded domain and $\mathbb{C} \setminus \Omega$ is connected, then for any $z_0 \in \mathbb{C} \setminus \Omega$, there exists a branch of $\log(z - z_0)$ in Ω.

Proof. Fix some $w_0 \in \Omega$ and take a_0 as one of the values of $\log(w_0 - z_0)$ (this is defined up to $2\pi i$, but we just choose one). Now we define the branch $\log z - z_0$ as

$$\log(z - z_0) = a_0 + \int_{w_0}^{z} \frac{d\xi}{\xi - z_0}$$

Note that our integral does not depend on the choice of pathes, as shown by the Generalized Cauchy's Theorem, so our branch is well-defined. ■

14.2 Jump Theorem for Cauchy Integrals

Let γ be a C^1-curve and suppose f be a C^1 compact-supported function on γ (denote this as $f \in C_c^1(\gamma)$, note this need not be analytic). We want f to be compactly supported to avoid any convergence issue.
There's a slight abuse of notation going on here, by $f \in C^1(\gamma)$, we actually mean that for $\gamma : [a, b] \to \mathbb{C}$, $f \circ \gamma$ is C^1.
Define the function

$$F(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi$$

We note that $F \in Hol(\gamma^c)$, now take some point $z_0 \in \gamma$, and consider

$$F_{\pm}(z_0) := \lim_{z \to z_0} F(z)$$

, where "+" is from inside and "−" is from outside.

Naively, we claim we could just exchange the limit and the integral, then

$$F_{\pm}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi$$

However, in general the integral is not integrable around z_0, so we next hope that we could converge it to some principal value:

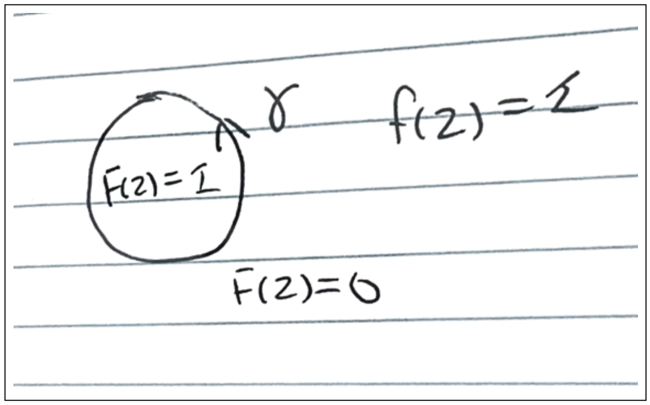
$$F_{\pm}(z_0) = \frac{1}{2\pi i} p.v. \int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi$$

, where we say that

$$p.v. \int_{\gamma} ... = \lim_{\delta \to 0} \int_{\gamma \setminus D_{z_0, \gamma}} ...$$

This, however, is also problematic! Indeed, consider the following counter-example where γ is a circle and $f(z) = 1$

is the constant 1 function:



, but $F(z)$ is the winding number and is hence 1 inside and 0 outside. So the limit does not even converge. The next best thing we get, is fortunately true. This is sometimes also called the **Jump Theorem**:

Theorem 14.5 (Plemelji-Sokhotsky formula)**.**

$$F_{\pm}(z_0) = \frac{1}{2\pi i} p.v. \int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi \pm \frac{1}{2} f(z_0)$$

Remark 14.6. Note that the theorem is actually also true for just curves (need not be closed). In this case, by inside and outside we mean the orientation given by the **left leg rule**.

Before we prove the theorem, we first make an observation:

Observation 14.7. ♣♣♣ Mattie: [Why??] We note that the theorem is actually a local theorem. If $f \in C^1(\gamma)$ and $f \equiv 0$ in a small neighborhood of z_0, then everything is trivial as

$$F_{\pm}(z_0) = \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z_0} d\xi$$

Therefore, we can without loss just assume $f \in C^1(D_{z_0,\delta_0})$.

Proof of Jump Theorem.

$$\int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi = \int_{\gamma} \frac{f(\xi) - f(z)}{\xi - z} d\xi + \int_{\gamma} \frac{f(z)}{\xi - z} d\xi$$

Note that as we take $z \to z_0$ on the left integral, we can apply the Dominated Convergence Theorem as $|\nabla f| \leq M$ in D_{z_0,δ_0} is bounded on compact set, and everything is compactly supported

$$\begin{aligned} \lim_{z \to z_0} \int_{\gamma} \frac{f(\xi) - f(z)}{\xi - z} d\xi &= \int_{\gamma} \lim_{z \to z_0} \frac{f(\xi) - f(z)}{\xi - z} d\xi && \text{Dominated Convergence Theorem} \\ &= \int_{\gamma} \frac{f(\xi) - f(z_0)}{\xi - z_0} d\xi \\ &= p.v. \int \frac{f(\xi)}{\xi - z_0} d\xi - p.v. \int_{\gamma} \frac{f(z_0)}{\xi - z_0} d\xi && \text{Existence of Principal Values left as Exercise} \end{aligned}$$

For the right integral, we will evaluate

$$\lim_{z \to z_0} \int_{\gamma} \frac{f(z)}{\xi - z} d\xi$$

We will finish the proof in the next lecture. ∎

15 Lecture 15 - 10/12/2022

15.1 Jump Theorem for Cauchy Integrals

Let $\gamma \in PC^1$ with image $\Gamma := \gamma([a, b]) \subset \mathbb{C}$, and let $f(z)$ be a continuous function on Γ.

Definition 15.1. The **Cauchy Integral** of $f(z)$ along γ is the function

$$F(\xi) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - \xi} dz, \forall \xi \in \mathbb{C} \setminus \Gamma$$

Note that as we take ξ to infinity, $F(\xi) = 0$. Furthermore, $F(\xi)$ is analytic on $\mathbb{C} \setminus \Gamma$ since it could be represented in power series by geometric expansion.

Example 15.2. If γ is a closed loop and $f(z)$ is identically 1, then the Cauchy integral of $f(z)$ around γ is exactly the winding number $F(\xi) = w(\gamma, \xi)$ of γ around ξ, which is either 1 if the point is inside or 0 if the point is outside.

Theorem 15.3 (Plemelj-Sokhotsky Theorem)**.** Let γ be a simple C^1-closed curve, and $f \in C^1_{\mathbb{C}}(\gamma)$. Note that the Cauchy Integral $F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi$ is undefined for any $z \in \gamma$ but is defined on the interior and exterior of the curve, which we will denote by F_+ for interior and F_- for exterior respectively.

Now take z_0, we have that

$$F_\pm(z_0) = \frac{1}{2\pi i} p.v. \int_\gamma \frac{f(\xi)}{\xi - z_0} d\xi \pm \frac{1}{2} f(z_0)$$

Proof. First we rewrite

$$\int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \int_\gamma \frac{f(\xi) - f(z)}{\xi - z} d\xi + \int_\gamma \frac{f(z)}{\xi - z} d\xi \ (*)$$

Now let $\delta > 0$ be chosen appropriately and consider the disk $D_{z_0,\delta}$ such that the domain of where f is homolomorphic contains the closure of the disk (recall f being holomorphic on compact γ means there exists some open set containing γ that f is holomorphic on). Then since ∇f is a continuous function on compact $\overline{D_{z_0,\delta}}$, $|\nabla f|$ is bounded on the the disk.

Thus, by Dominated Convergence Theorem, as we take the limit as $z \to z_0$:

$$\lim_{z \to z_0} \int_\gamma \frac{f(\xi) - f(z)}{\xi - z} d\xi = \int_\gamma \lim_{z \to z_0} \frac{f(\xi) - f(z)}{\xi - z} d\xi = p.v. \int_\gamma \frac{f(\xi)}{\xi - z_0} d\xi - p.v. \int_\gamma \frac{f(z_0)}{\xi - z_0} d\xi$$

There are two more things that we want to check:

- The principal value of $\int_\gamma \frac{1}{\xi - z_0} d\xi$ exists
- The following limit is true

$$\lim_{z \to z_0, \text{interior}} \int_\gamma \frac{1}{\xi - z} d\xi = p.v. \int_\gamma \frac{d\xi}{\xi - z_0} + \pi i$$

Then it follows from the limit and our prior computation that:

$$\begin{aligned} F_+(z_0) &= \frac{1}{2\pi i} \cdot [p.v. \int_\gamma \frac{f(\xi)}{\xi - z_0} d\xi - p.v. \int_\gamma \frac{f(z_0)}{\xi - z_0} d\xi] + \frac{1}{2\pi i} \cdot [p.v. \int_\gamma \frac{f(z_0) d\xi}{\xi - z_0} + f(z_0)\pi i] \\ &= \frac{1}{2\pi i} p.v. \int_\gamma \frac{f(\xi)}{\xi - z_0} d\xi + \frac{1}{2} f(z_0) \qquad \text{We can cancel the principal values because they exist} \end{aligned}$$

A similar argument will also prove the case for $F_-(z_0)$. It then remains for us to prove the two claims:

1. For existence, let $\gamma_\delta := \gamma \cap D_{z_0,\delta}$ and let $\gamma^\delta := \gamma \setminus \gamma_\delta$. In other words, the latter is γ with its path in the disk removed, then we have that
$$\int_\gamma \frac{d\xi}{\xi - z} = \int_{\gamma_\delta} \frac{d\xi}{\xi - z} + \int_{\gamma^\delta} \frac{d\xi}{\xi - z}$$
Now consider the following contour:

Then by Cauchy's Theorem
$$\int_{\gamma_\delta} - \int_C = 0$$
Hence we have that
$$\int_\gamma \frac{d\xi}{\xi - z} = \int_C \frac{d\xi}{\xi - z} + \int_{\gamma^\delta} \frac{d\xi}{\xi - z}$$
Then as we take the limit and exchange it with the integral using uniform convergence:
$$\lim_{z \to z_0, \text{inside}} \int_\gamma \frac{d\xi}{\xi - z} = \int_C \frac{d\xi}{\xi - z_0} + \int_{\gamma^\delta} \frac{d\xi}{\xi - z_0}$$
So the principal value exist (because the limit exist as the new contour we made it avoid z_0) and the value of the integral does not depend on δ.

2. Note that $\lim_{z \to z_0, \text{interior}} \int_C \frac{1}{\xi - z} d\xi = i \cdot \theta_\gamma$, where θ_γ is the angle of the arc made with C. Then as $\delta \to 0$, $\theta_\delta \to \pi$ as γ is continuous.

3. So we have that $\lim_{\delta \to 0} \int_C \frac{d\xi}{\xi - z_0} = \pi i$

4. We note that as $z \to z_0$, $\delta \to 0$, so
$$\lim_{z \to z_0} \int_\gamma \frac{d\xi}{\xi - z} = \lim_{z \to z_0} \int_C \frac{d\xi}{\xi - z_0} + \int_{\gamma^\delta} \frac{d\xi}{\xi - z_0} = \pi i + p.v. \int_\gamma \frac{d\xi}{\xi - z_0}$$

■

15.2 Conformal Mappings

In Real Analysis, we have often discussed the angle between two arbitrary vectors:

Definition 15.4. Consider two non-zero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, the **angle** between $\vec{u}$ and $\vec{v}$ is

$$\cos(\alpha) \coloneqq \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||},\ 0 \leq \alpha \leq \pi$$

In complex analysis, we can also define it in a similar way:

Definition 15.5. Let $z, w \in \mathbb{C}$ be two non-zero complex numbers, then

$$\text{angle}(z, w) \coloneqq \arg(z\overline{w})$$

Definition 15.6 (Conformal Map). Let $\Omega \subseteq \mathbb{R}^n$ be open, we say a map $f : \Omega \to \mathbb{R}^n$ is conformal at $x_0 \in \Omega$ if f preserves angles between tangent lines of directed curves through x_0 (not necessarily length).

Fact 15.7. If f is conformal at x_0 then the Jacobian of f at x_0 is an orthogonal matrix multiplied by some scalars.

In $\mathbb{R}^n$, $n > 2$, there's usually not a lot of choices for conformal maps - you only really have translations, rotation, flip, and occasionally "blow ups".

In $\mathbb{C}$ ($n = 2$), it turns out that the conformal mappings f are either orientation preserving or orientation reversing:

- If f is analytic whose $f'(z_0) \neq 0$, then f is conformal and preserves orietnation
- If f is anti-analytic (ie. $z \mapsto f(\overline{z})$ is analytic, or $f(z) = \sum a_k \overline{(z - z_0)}^k$) with non-zero derivative, then this preserves angles but reverses the orientation.

Definition 15.8. Let $f \in \text{Hol}(\Omega)$ and $f : \Omega \to G \subseteq \mathbb{C}$. We say that f is a **conformal map** from Ω to G if f is bijective in Ω.

Proposition 15.9. If $f \in \text{Hol}(\Omega)$ is injective, then $f'(z) \neq 0$ for all $z \in \Omega$.

Proof. Suppose there exist some $z_0 \in \Omega$ such that $f'(z_0) = 0$, then let's also denote $f(z_0) = w_0$. We could write out the Taylor expansion of f at $z = z_0$ as

$$f(z) = f(z_0) \cdot (z - z_0)^0 + \sum_{q=1}^{\infty} a_q (z - z_0)^q = w_0 + \sum_{q=1}^{\infty} a_q (z - z_0)^q = w_0 + (z - z_0)^k \cdot f_0(z)$$

where f_0 is obtained by factoring out $(z - z_0)$ as much as one could so that $f_0(z_0) \neq 0$. Note that since $f'(z_0) = 0$, z_0 is a zero of order 2, so we have that $k \geq 2$.

Then for $w \neq w_0$ such that $w_0 - w$ sufficiently small, we can write $f(z) - w = (f(z) - w_0) + (w_0 - w)$ such that $|f(z) - w_0| > |w_0 - w|$ on a circle centered at z_0.

Then we note that the number of zeroes $f(z) - w_0$ has exactly $k \geq 2$ inside the circle, so by Rouche's Theorem, $f(z) - w_0 + w_0 - w = f(z) - w$ has exactly two zeroes inside the circle.

There're two cases, either $f(z)$ has a zero of multiplicity at least 2 other than w_0, or $f(z)$ has at least two distinct points that get mapped to w, which contradicts injectivity.

Hence $f(z)-w$ has a zero of multiplicity two, but we can restrict our circle around z_0 such that $f'(z) \neq 0$ for all $z \neq z_0$ (since roots are isolated for non-constant analytic functions, and f clearly is not constant). Hence $(f(z)-w)' = f'(z)$ cannot be zero around here, so its zero has multiplicity less than 2, contradiction. ■

There's a very important theorem about conformal mapping that we will spend many lectures to build up to prove - the Riemann Mapping Theorem:

Theorem 15.10. Let $U \subseteq \mathbb{C}$ be a non-empty, simply-connected domain that is not the entire $\mathbb{C}$, then there exists a conformal map f from U to the unit disk $\mathbb{D}$. Furthermore, if we fix $z_0 \in U$ and require $f(z_0) = 0$ and $f'(z_0) > 0$ (derivative is real valued), then f is unqiue.

Remark 15.11. There's no analog of the Riemann Mapping Theorem for $n > 2$ in $\mathbb{R}^n$,

Definition 15.12. Let $f : \Omega \to \mathbb{C}$, we say f is univalent if f is holomorphic and injective on Ω. In particular, the univalent function f on Ω is a conformal map of Ω to $f(\Omega)$.

Definition 15.13 (Riemann Sphere)**.** We define the **Riemann Sphere** (or the "extended complex plane") as $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as the one point compactification of $\mathbb{C}$. Recall $\mathbb{C}$ is topologically equivalent to $\mathbb{R}^2$, then this process turns $\hat{\mathbb{C}}$ topologically equivalent to S^2.

16 Lecture 16 - 10/14/2022

16.1 Conformal automorphisms of the complex plane and the Riemann sphere

Theorem 16.1. Let $f : \mathbb{C} \to \mathbb{C}$, f is a conformal map if and only if $f(z) = az + b$, $a \neq 0$. In other words, the conformal automorphisms of $\mathbb{C}$ are all affine transformations.

Proof. Note that f is an entire function, so we can represent f as a power series globally, hence

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \ \forall z \in \mathbb{C}$$

Then consider

$$g(z) \coloneqq f(\frac{1}{z}) = \sum_{k=0}^{\infty} a_k z^{-k}, \forall z \in \mathbb{C} \setminus \{0\}$$

In other words, $g(z)$ has a singularity at 0. The question is then - **what type of singularity is it?**.

We first claim this is NOT an essential singularity. Indeed, suppose it is, then by Casorati–Weierstrass theorem, for any neighborhood U of 0, then $g(U)$ is dense in $\mathbb{C}$.

Let $U = D_{0,1/3}$ be the open disk centered at 0 of radius $1/3$, then $g(D_{0,1/3})$ is dense. Hence, for any open set V, $g(D_{0,1/3}) \cap V \neq \emptyset$. Now consider $V = g(D_{1,1/3})$, then we have that

$$g(D_{0,1/3}) \cap g(D_{1,1/3}) \neq \emptyset$$

In other words, there exist some $z_1 \in D_{0,1/3}$ and $z_2 \in D_{1,1/3}$ such that

$$g(z_1) = g(z_2)$$

But we know that f is conformal, so it has to be injective, hence g also has to be injective, so we have the contradiction.

Thus, $z = 0$ is either a pole or a removable singularity, so we can write

$$g(z) = \sum_{k=0}^{m} a_k z^{-k}, \text{ the rest of } a_i \text{ is } 0$$

Hence we have that

$$f(z) = \sum_{k=0}^{m} a_k z^k, \ m > 0$$

($m > 0$ because $m = 0$ implies f is a constant function) Hence, $f(z)$ is a complex polynomial.

If $\deg(f) > 1$, then $f(z)$ cannot be a bijection. If $f(z)$ has at least two distinct roots, we are done, otherwise, suppose $f(z)$ has the same roots r, then we write

$$f(z) = b(z - r)^m \ m > 1$$

$f(z)$ is injective if and only if z^m is injective on the complex plane, but we do have distinct roots of unity for $z^m - 1$.

Hence we conclude that

$$f(z) = az + b, a \neq 0$$

Converse is not hard to check. ■

Now we will move our discussion from the complex plane to the Riemann Sphere! This is an example of what's known as Riemann surfaces:

Definition 16.2 (Riemann Surface)**.** A Riemman surface is a $2d$-dimensional manifold M with a complex structure. In other words, M is a second countable, Hausdorff Space, with a coordinate atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that

$$\varphi_\alpha : U_\alpha \to \text{an open subset of } \mathbb{C} \text{ - is biholomorphic}$$

and $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a biholomorphic map.

Example 16.3. Consider the Riemann Sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the maps:

$$\varphi_1 : \hat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}, \varphi_1(z) = z$$

$$\varphi_2 : \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}, \varphi_1(z) = \frac{1}{z}$$

This turns the Riemann sphere into a Riemann surface.

Definition 16.4. We call a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{C}$$

such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ a **linear fractional transformation (LFT)**.

Theorem 16.5. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then f is conformal if and only if

$$f(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{C}$$

such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Proof. Suppose f is conformal, we will consider a few cases.

- If $f(\infty) = \infty$, then $f(\mathbb{C}) = \mathbb{C}$, so $f(z) = az + b$ which is a linear fractional transformation by considering $c = 0, d = 1$.
- If $f(\infty) = w_0 \in \mathbb{C}$, then consider

  $$g(z) := \frac{1}{f(z) - w_0}$$

  In this case we note that

  $$g(\infty) = \infty$$

  and g is conformal, hence $g(z) = az + b$. Now the question is to solve

  $$\frac{1}{f(z) - w_0} = az + b$$

  So we get that

  $$f(z) = \frac{1}{az+b} + w_0 = \frac{1 + (az+b) \cdot w_0}{az+b}$$

which is a valid Linear fractional transformation.

We can check that any LFT with determinant 0 cannot be conformal (it fails injectivity). Converse is also not hard to check. ∎

Proposition 16.6. Let f, g be Linear Fractional Transformations, then $f \circ g$ is a Linear Fractional Transformation.

Proof. We can prove this by doing a simple algebra exercise. Alternatively, we also note that the composition of conformal automorphisms are also conformal automorphisms, so the Theorem implies $f \circ g$ is a linear fractional transformation. ∎

16.2 Homogeneous coordinates and LFTs

Consider the space $\mathbb{C}^2 \setminus \{\vec{0}\}$, then this is a collection of vectors (z_1, z_2) excluding $(0, 0)$. We say that

$$(z_1, z_2) \sim (w_1, w_2) if \exists a \in \mathbb{C} \setminus \{0\}, a(z_1, z_2) = (w_1, w_2)$$

Now consider the quotient space obtained by this:

$$\frac{\mathbb{C}^2 \setminus \{\hat{0}\}}{\sim}$$

This is actually isomorphic to the Riemann Sphere $\hat{\mathbb{C}}$!

This gives an alternative representation of $\hat{\mathbb{C}}$. In particular, if $z_2 \neq 0$, we can identify

$$(z_1, z_2) \iff \frac{z_1}{z_2} \in \mathbb{C}$$

If $z_2 = 0$, then $z_1 \neq 0$, then (z_1, z_2) corresponds to $\infty \in \hat{\mathbb{C}}$.

This is an example of a **projective line**, so we alternatively denote this construction as $\mathbb{CP}^1$, and

$$\hat{\mathbb{C}} = \mathbb{CP}^1$$

Now consider $f(z) = \frac{az+b}{cz+d}$ on $\mathbb{CP}^1$. Write $z = z_1/z_2$. Then, this actually corresponds to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

which corresponds to the fraction

$$\frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{az_1 + bz_2}{cz_1 + dz_2} \cdot \frac{1/z_2}{1/z_2} = \frac{az + b}{cz + d}$$

We will not use this representation much, but it is used a lot in other areas of mathematics.

Definition 16.7. A generalized circle in $\mathbb{C}(\text{or}\hat{\mathbb{C}})$ is either a cirlce in $\mathbb{C}$ or a line (this is a circle if you add ∞ to it).

Theorem 16.8. Linear Fractional Transformation maps a generalized circle to a generalized circle.

Naive Proof Sketch. The naive proof is that any Linear Fraction Transformation is some composition of the following:

- $z \mapsto z + a$ (Translation)
- $z \mapsto bz$
- $z \mapsto \frac{1}{z}$

We can prove that all 3 types of transformation here map a generalized circle to a generalized circle. Specifically, the first two maps lines to lines and circles to circles, the only difficult part is the last map, which comes down to an exercise in Analytic Geometry. ■

We will instead prove this using an alternative method.

Proposition 16.9. Let $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ be distinct points, then there exists unique a Linear Fraction Transformation $S = S_{z_2,z_3,z_4}$ such that

$$S(z_2) = 1, S(z_3) = 0, S(z_4) = \infty$$

Proof. Suppose $z_2, z_3, z_4 \in \mathbb{C}$ (no infinity), then choose

$$S_{z_2,z_3,z_4}(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_4}$$

We note that if one of them is infinity, define

$$S_{\infty,z_3,z_4}(z) = \frac{z - z_3}{z - z_4}$$

$$S_{z_2,\infty,z_4}(z) = \frac{z_2 - z_4}{z - z_4}$$

$$S_{z_2,z_3,\infty}(z) = \frac{z - z_3}{z_2 - z_3}$$

For uniqueness, suppose there two LFTs S, T with the property given then $S^{-1} \circ T$ is a linear fractional transformation such that

$$S^{-1} \circ T(z_1) = z_1, S^{-1} \circ T(z_2) = z_2, S^{-1} \circ T(z_3) = z_3$$

We note that it's a well-known fact (which we will not prove) in Complex Analysis that any LFT with more than 2 fixed points is the identity, so we have that $S = T$. ■

Remark 16.10. Note that, since every Linear Fractional Transformation is bijective, this means that every Linear Fractional Transformation is of the form S_{z_2,z_3,z_4} for some $z_2, z_3, z_4 \in \mathbb{C}$, and we are saying the determination is unique. Hence there's roughly $\mathbb{C}^3$ choices of a LFT.

Definition 16.11. Let $z_1 \notin \{z_2, z_3, z_4\}$, then we call the Cross Ratio of z_1, z_2, z_3, z_4 as:

$$(z_1 : z_2 : z_3 : z_4) := S_{z_2,z_3,z_4}(z_1) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_4}$$

Theorem 16.12. If T is a LFT, then

$$(z_1 : z_2 : z_3 : z_4) = (Tz_1 : Tz_2 : Tz_3 : Tz_4)$$

Proof. Indeed, let $S := S_{z_2,z_3,z_4}$, then we see that

$$S \circ T^{-1}(Tz_2) = S(z_2) = 1$$

$$S \circ T^{-1}(Tz_3) = S(z_3) = 0$$

$$S \circ T^{-1}(Tz_4) = S(z_4) = \infty$$

Hence that we have by uniqueness that $ST^{-1} = S_{Tz_2,Tz_3,Tz_4}$. Then we have that

$$\begin{aligned}(Tz_1 : Tz_2 : Tz_3 : Tz_4) &= S_{Tz_2,Tz_3,Tz_4}(Tz_1) \\ &= ST^{-1}(Tz_1) \\ &= S(z_1) \\ &= (z_1 : z_2 : z_3 : z_4)\end{aligned}$$

■

Theorem 16.13. If S is a Linear Fractional Transformation, then the

$$\{z : S(z) \in \mathbb{R}\}$$

is a generalized circle.

Proof. We will prove this the next lecture! ■

It follows as an immediate corollary that

Corollary 16.14. The set $\{z : (z : z_2 : z_3 : z_4) \in \mathbb{R}\}$ is a generalized circle. Conversely, if C is a generalized circle, then pick $z_2, z_3, z_4 \in T$

$$C = \{z \in \mathbb{C} : (z : z_2 : z_3 : z_4) \in \mathbb{R}\}$$

Proof. The forward direction is clearly from Theorem 16.13. We will also prove the converse next lecture. ■

17 Lecture 17 - 10/17/2022

17.1 LFTs perserve Genralized Circles

Theorem 17.1. Let $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ be distinct, then

$$\{z \in \mathbb{C} | S_{z_2,z_3,z_4}(z) \in \mathbb{R}\}$$

is a generalized circle going through z_2, z_3, and z_4.

Proof. Write S as S_{z_2,z_3,z_4}, then since S is a LFT, we have that

$$S(z) = \frac{az+b}{cz+d}$$

We first note the set is clearly non-empty as it contains z_2, z_3, z_4.

Then we note that

$$\begin{aligned}
S(z) \in \mathbb{R} &\iff \frac{az+b}{cz+d} = \overline{(\frac{az+b}{cz+d})} && z \in \mathbb{C} \text{ is real iff it's fixed under conjugation} \\
&\iff \frac{az+b}{cz+d} = \frac{\overline{az}+\overline{b}}{\overline{cz}+\overline{d}} \\
&\iff (az+b)\overline{(cz+d)} = \overline{(az+b)}(cz+d) \\
&\iff (a\overline{c} - \overline{a}c) \cdot |z|^2 + (a\overline{d} - \overline{b}c)z + (b\overline{c} - \overline{a}d)\overline{z} + (b\overline{d} - \overline{b}d) = 0 \\
&\iff -i \cdot [(a\overline{c} - \overline{a}c) \cdot |z|^2 + (a\overline{d} - \overline{b}c)z + (b\overline{c} - \overline{a}d)\overline{z} + (b\overline{d} - \overline{b}d)] = 0
\end{aligned}$$

We note that both

$$-i \cdot [a\overline{c} - \overline{a}c] \text{ and } -i \cdot [b\overline{d} - \overline{b}d]$$

are fixed under conjugation and hence are real numbers, we will write them as B and C, and we see that

$$-i \cdot [a\overline{d} - \overline{b}c] \text{ and } -i \cdot [b\overline{c} - \overline{a}d]$$

are complex conjugates of one another, we will write them as $\overline{W}$ and W respectively.

So we have that the equation

$$S(z) = 0$$

is exactly the equation

$$B|z|^2 + \overline{W}z + W\overline{z} + C = 0$$

If $B = 0$, then this is the equation of a line. Otherwise, if $B \neq 0$, we can divide the equation by B and have

$$|z|^2 + \frac{\overline{W}}{B}z + \frac{W}{B}\overline{z} + C = 0$$

$\frac{W}{B}$ and $\overline{W}/B$ are still conjugates of each other, so we can write them as $\overline{A}$ and A respectively, so

$$|z|^2 + \overline{A}z + A\overline{z} + C = 0$$

Then we see that

$$|z|^2 + \overline{A}z + A\overline{z} + C = 0 \iff |z+A|^2 = |A|^2 - C$$

which is the equation of a circle containing z_1, z_2, z_3. Hence the entire set would have to occupy the circle. ■

Now we will prove that LFTs perserve generalized circles!

Proof of Theorem 16.8. Let f be a LFT and T be a generalized circle, choose any distinct $z_2, z_3, z_4 \in T$, then we have that

$$T = \{z \in \mathbb{C} \mid (z : z_2 : z_3 : z_4) \in \mathbb{R}\}$$

Applying f gives

$$\begin{aligned} f(T) &= \{w = f(z) \mid (z : z_2 : z_3 : z_4) \in \mathbb{R}\} \\ &= \{w = f(z) \mid (f(z) : f(z_2) : f(z_3) : f(z_4)) \in \mathbb{R}\} \\ &= \{w \in f(T) \mid (w : f(z_2) : f(z_3) : f(z_4)) \in \mathbb{R}\} \end{aligned}$$

which we proved is a circle above going through $f(z_2), f(z_3), f(z_4)$. ■

17.2 Symmetry of Generalized Circles

We note the symmetric point of z as z^*

Definition 17.2. For any $z_2, z_3, z_4 \in \mathbb{C}$, let $z \in \mathbb{C}$ be a point other than the first three, then the reflection of z - z^* - across the circle is the unique point satisfying:

$$(z^* : z_2 : z_3 : z_4) = \overline{(z : z_2 : z_3 : z_4)}$$

Example 17.3. When the circle is S^1,

$$\begin{aligned} (z^* : z_2 : z_3 : z_4) &= \overline{(z : z_2 : z_3 : z_4)} \\ &= (\overline{z} : \overline{z_2} : \overline{z_3} : \overline{z_4}) \\ &= (1/\overline{z} : 1/\overline{z_2} : 1/\overline{z_3} : 1/\overline{z_4}) \\ &= (1/\overline{z} : z_2 : z_3 : z_4) \end{aligned}$$

Hence $z^* = \frac{1}{\overline{z}}$. (For general circle of radius R at the origin, $z^* = \frac{R^2}{\overline{z}}$).

Theorem 17.4. Let z, z^* be symmetric points with respect to a generalized circle C going through z_2, z_3, z_4, and let f be a LFT, then $f(z)$ and $f(w)$ are symmetric with respect to $f(C)$.

Proof. It suffices for us to show that

$$(f(z^*) : f(z_2) : f(z_3) : f(z_4)) = \overline{(f(z) : f(z_2) : f(z_3) : f(z_4))}$$

Indeed, since LFT's perserve cross ratio, we have that

$$\begin{aligned} (f(z^*) : f(z_2) : f(z_3) : f(z_4)) &= (z^* : z_2 : z_3 : z_4) \\ &= \overline{(z : z_2 : z_3 : z_4)} && \text{Since } z \text{ and } z^* \text{ are symmetric} \\ &= \overline{(f(z) : f(z_2) : f(z_3) : f(z_4))} \end{aligned}$$

■

18 Lecture 18 - 10/19/2022

18.1 Schwarz Lemma

Lemma 18.1 (Maximum Modulus Principle)**.** Let Ω be a domain $f \in \text{Hol}(\Omega)$, $z_0 \in \Omega$, if $|f|$ obtains a local maximum or minimum at z_0, then $f(z)$ is identically constant.

Proof. Suppose f is non-constant and attains some maximum at z_0. Choose some $r > 0$ chosen such that $\overline{D_{z_0,r}}^{cl}$ is contained in the neighborhood given by the local maximum, then by the Open Mapping Theorem $f(D_{z_0,r})$ is open in $\mathbb{C}$.

Now since $f(z_0) \in f(D_{z_0,r})$, we know that it is contained in some disk $D_{z,\epsilon} \subseteq f(D_{z_0,r})$, then consider the radial line drawn from the origin to z_0, then this radial line must intersect points of $D_{z,\epsilon}$ after it intersects $f(z_0)$, but this would imply that there exist $w \in D_{z,\epsilon}$ such that $|w| > |f(z_0)|$. Since $w \in f(D_{z_0,r})$, this means there exist some $\xi \in D_{z_0,r}$ such that $f(\xi) = w$ so

$$|f(\xi)| = |w| > |f(z_0)|$$

Hence we have a contradiction.

For the case of minimum at z_0, we can obtain the same exactly argument to $1/f$ and then z_0 becomes a local maximum of $1/f$, which is identically constant by our prior argument. ∎

Corollary 18.2. If Ω is a bounded domain and $f \in \text{Hol}(\Omega) \cap C^0(\overline{\Omega}^{cl})$, then $|f|$ attains its maximum on $\partial\Omega$.

Proof. Since f is continuous on $\overline{\Omega}^{cl}$ (which is compact), $|f|$ attains a maximum value, then we apply the Maximum Modulus Principle. ∎

Lemma 18.3 (Schwarz Lemma)**.** Let $f \in \text{Hol}(\mathbb{D})$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and $f(0) = 0$, then

$$|f'(0)| \leq 1 \text{ and } |f(z)| \leq |z|$$

Moreover, if either $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some non-zero z_0, then

$$f(z) = \alpha z, \ |alpha| = 1$$

Proof. The idea is that, if we assume f is continuous on $\partial\mathbb{D}$, then we can extend such that $|f(z)| \leq 1$ on $\partial\mathbb{D}$, then clearly $|\frac{f(z)}{|z|}| = |f(z)| \leq 1$ on $\partial\mathbb{D}$.

Now we note $f(z)/z$ as a removable singularity at $z = 0$, hence it is holomorphic on all of $\partial\mathbb{D}$, since holomorphic function on bounded domains only obtain their maximums at the boundary, we have that $|\frac{f(z)}{|z|}| \leq 1$ on $\mathbb{D}$, then

$$|f(z)| \leq |z| \text{ on } \mathbb{D}$$

This also implies that $f'(0) \leq 1$. If not then

$$\lim_{z\to 0} |\frac{f(z)}{z}| = |f'(0)| > 1$$

But we have that

$$|f(z)/z| \leq 1 \forall z \in \mathbb{D}$$

It remains for us to show that this extension always exists! Indeed, consider $r < 1$, then the same argument as before implies that

$$|\frac{f(z)}{z}| \leq \frac{1}{r} \text{ for all } z \text{ such that } |z| \leq r$$

If we take the limit as r goes to 1, we will have the desired inequality.

Now if $|f(z_0)| = |z_0|$ for some non-zero $|z_0|$, then $|\frac{f(z_0)}{z_0}| = 1$, then the Maximum Modulus Principle tells us that $f(z)/z = \alpha$ for some $|\alpha| = 1$, hence we have that

$$f(z) = \alpha z$$

If instead $|f'(0)| = 1$, then define $zg(z) = f(z)$, then we have that $g(0) = \lim_{z\to 0} \frac{f(z)}{z} = f'(0)$, so $g(0)$ obtains a maximum at 0, so we again use the Maximum Modulus Principle. ■

18.2 Conformal Automorphisms of the Unit Disk

Theorem 18.4. Let $f : \mathbb{D} \to \mathbb{D}$, then f is conformal if and only if f is of the form

$$f(z) = \alpha \cdot \frac{z - z_0}{1 - \overline{z_0}z}, \ |\alpha| = 1, z_0 \in \mathbb{D}$$

Proof. Converse is quick to check. For the forward direction, we will first check this for all $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi(0) = 0$. Indeed, since the range of φ is restricted to the unit disk, Schwartz's Lemma tells us that

- $\varphi'(0) = a$ for some $|a| \leq 1$

Now let $\psi : \mathbb{D} \to \mathbb{D}$ be the inverse of φ, then the inverse function theorem tells us that

$$\psi'(0) = \frac{1}{a}$$

And the Schwartz's Lemma tells us that:

- $|\psi'(0)| = |\frac{1}{a}| \leq 1$

Since $|a| \leq 1$ and $|1/a| \leq 1$, we have that $|a| = 1$, so Schwartz's Lemma tells us that

$$\varphi(z) \equiv \alpha z, \text{ for some } |\alpha| = 1$$

Now suppose instead we have some $z_0 \in \mathbb{C}$ such that $\varphi(z_0) = 0$, then we can construct a Linear Fraction Transformation $\psi : \mathbb{D} \to \mathbb{D}$ that sends z_0 to 0, then $\varphi \circ \psi^{-1}$ sends 0 to 0 and is equivalent to αz, so we have that

$$\varphi \equiv (z \mapsto \alpha z) \circ \psi$$

Since the composition of LFTs are LFTs, φ is an LFT.

Since φ is bijective, there exist some $z_0 \in \mathbb{D}$ such that $f(z_0) = 0$.

Then the symmetric point to z_0 - $\frac{1}{\overline{z_0}}$ - is mapped to infinity by φ since φ perserves symmetry and $\varphi(z_0) = 0$ (the symmetric point of 0 is ∞). In other words, φ has a singularity at $z = \frac{1}{\overline{z_0}}$.

In other words, φ maps z_0 to 0 and $\frac{1}{\overline{z_0}}$ to ∞, so we can write

$$\varphi(z) = c \cdot \frac{z - z_0}{z - 1/\overline{z_0}}$$

For some constant c. What is c? Well, take $z = \frac{z_0}{|z_0|}$, then

$$|\varphi(\frac{z_0}{|z_0|})| = |\frac{|z_0| - 1}{1 - |z_0|}| = 1$$

Hence we have that

$$|c| \cdot |z_0| = 1$$

So we have that

$$\varphi(z) = \alpha \cdot \frac{z - z_0}{1 - \overline{z_0} z}, \ |\alpha| = 1$$

■

Example 18.5. Let $C_+ := \{z \in \mathbb{C} : Im(z) > 0\}$ be the upper half complex plane. What are all the conformal maps from C_+ to $\mathbb{D}$?

Well, take f to be a conformal map, then there exist some $z_0 \in C_+$ such that $f(z_0) = 0$, then since f perserves symmetry, the symmetric point of z_0 w.r.t C_+ - which is $\overline{z_0}$ - is mapped to infinity, hence we conjecture

$$\varphi(z) = \alpha \cdot \frac{z - z_0}{z - \overline{z_0}}, \ |\alpha| = 1$$

It turns out that these are the only conformal maps by Schwartz's Lemma.

19 Lecture 19 - 10/21/2022

19.1 Working with conformal maps

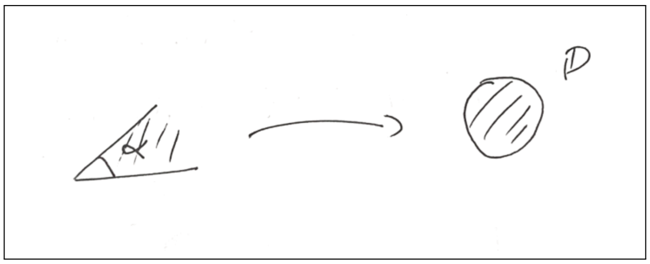
We have described all LFTs from $\mathbb{D}$ to $\mathbb{D}$ and from C_+ to $\mathbb{D}$. It turns out that, if there exists a conformal map $f : \Omega \to \mathbb{D}$, then we can describe all conformal maps to $\mathbb{D}$! In fact, they are of the form

$$g \circ f, g : \mathbb{D} \to \mathbb{D} \text{ is a conformal map}$$

This is because for any two conformal maps $f_1, f_2 : \Omega \to \mathbb{D}$, $f_1 \circ f_2^{-1}$ is a conformal automorphism of the unit disk.

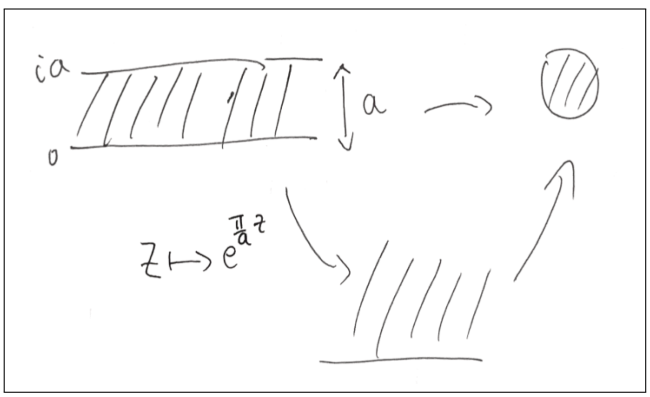
Now we will list some examples of conformal maps:

- How can we construct a map of the following:



  Then consider the map $z \mapsto z^{\pi/\alpha}$ from the trianlge, this will be a conformal map onto C_+! Then we can pick any map from C_+ to $\mathbb{D}$.

- Here's a map from the strip to $\mathbb{D}$:



Another useful map is **Zhukovsky's function:**

$$\varphi(z) = z + \frac{1}{z}$$

This is not an injective map! But the image of $\mathbb{D} \cup cl(\mathbb{D})^c$ is the complement of $[-2, 2]$ in $\mathbb{C}$. This is really important in aerodynamics.

There's also a conformal map from polygons into the unit disk. This is given by a formula called the **Cristoffel-Schwarz Formulas**.

19.2 Conformally invariant metric and Hyperbolic Geometry

Definition 19.1. A Riemannian Manifold is a smooth manifold M equipped with a metric tensor g_{jk} such that the quadratic form:

$$\sum_{j,k=1}^{n} g_{jk}x_jx_k > 0$$

and $g_{jk} = g_{kj}$. In other words, the metric tensor is a positive definite matrix.

Definition 19.2. Suppose $\gamma : [a, b] \to M$ is a C^1-path, then we define the **length** of γ as

$$\text{Length}\gamma = \int_a^b [\sum_{j,k=1}^{n} a_{j,k}(\gamma(t))\gamma_j^{'(t)\gamma_k^{'(t)]^{1/2}dt}}$$

Locally the metric tensor becomes quadratic form.

Definition 19.3. Let $x, y \in M$, then we define

$$d(x, y) = \inf\{\text{Length}\gamma \mid \gamma \text{ connects } x, y\}$$

Now for our set up in Complex Analysis, we will consider the 2-dimensional manifold $\mathbb{D}$. We want to find a "conformally invariant" Riemannian Metric on $\mathbb{D}$, in other words the metric tensor is preserved under conformal automorphisms on $\mathbb{D}$.

What are the LFTs on $\mathbb{D}$? We proved last lecture that they are of the form:

$$\varphi(z) = \alpha \cdot \frac{z - z_0}{1 - \overline{z_0}z}$$

We can use this to compute the metric tensor!

Computation. In 2-dimension, our metric tensor is really just a 2×2 matrix, so we only need to find 4 elements. A 2-dimensional quadratic form is either a disk or an ellipse, and ellipse are not perserved under LFTs (think rotation).

We will first compute the quadratic form at $z = x + iy = 0$ ($x = y = 0$), then since we know the quadratic form geometrically is a disk:

$$(g_{jk}(0))_{j,k=1}^{2} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we can write the metric tensor at $z = 0$ as $M(0, dx, dy)$ (at $z = 0$)

$$M(0, dx, dy) = a^2((dx)^2 + (dy)^2) = a^2|dz|^2$$

In particular, $dx = (\gamma_x^{'),dy=(\gamma_y^{')}}$. Now consider the Linear Transformation,

$$w = \frac{z - z_0}{1 - \overline{z_0}z}$$

Then by Conformal Invariance,

$$\begin{aligned} M(z_0, dz) &= M(0, dw) \\ &= a^2|dw|^2 \\ &= a^2|w'(z_0)|^2|dz|^2 && \text{Change of variables} \end{aligned}$$

In particular,

$$w'(z) = (\frac{z - z_0}{1 - \overline{z_0}z})' = ... = \frac{1 - |z_0|^2}{(1 - \overline{z_0}z)^2}$$

$$w'(z_0) = \frac{1}{1 - |z_0|^2}$$

Hence we have that

$$M(z_0, dz) = (\frac{a}{1 - |z_0|^2})^2 |dz|^2$$

It remains for us to find a suitable choice of a, we will choose $a = 2$. This is because

$$\frac{2}{1 - |z_0|^2} = \frac{2}{1 + |z_0|} \cdot \frac{1}{1 - |z_0|}$$

Note as we take the limit $|z_0| \to 1$, $\frac{2}{1+|z_0|}$ goes to 1 and $\frac{1}{1-|z_0|}$ is just the inverse of the distance to the boundary. ■

What we have just discovered is what's so called the **Hyperbolic Metric**!

Definition 19.4. Under the hyperbolic metric, take $\gamma : [a, b] \to \mathbb{D}$, then

$$\text{Length}(\gamma) = \int_a^b \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt$$

Example 19.5. Take $\gamma(t) = t$, $0 \leq t \leq r < 1$, then

$$\begin{aligned}
\text{Length}(\gamma) &= \int_0^r \frac{2dx}{1 - x^2} \\
&= \int_0^r (\frac{1}{1 - x} + \frac{1}{1 + x}) dx \\
&= \ln(1 + r) - \ln(1 - r) \\
&= \ln(\frac{1 + r}{1 - r})
\end{aligned}$$

It is a good exercise to show that $[0, r]$ is the shortest path from $z = 0$ to $z = r$! Then by Conformal Invariance (rotation in particular), we have that:

Proposition 19.6. Let $z \in \mathbb{D}$ with $|z| = r$, then

$$dist(0, z) = \ln(\frac{1 + r}{1 - r})$$

How do we compute the distance between two points in general?

Proposition 19.7. Let $z, w \in \mathbb{D}$, then

$$dist(z, w)$$

Proof. We will use conformal invariance! Consider

$$\varphi(\xi) = \frac{\xi - w}{1 - \overline{w}\xi} \cdot \alpha$$

Then we have that

- $\varphi(w) = 0$
- $\varphi(z) = \frac{z-w}{1-\overline{w}z} \cdot \alpha$

Now let

$$\rho(z,w) := |\frac{z-w}{1-\overline{w}z}|$$

We pick α such that $\varphi(z) = \rho(z,w)$, then

$$dist(z,w) = \ln(\frac{1+\rho(z,w)}{1-\rho(z,w)})$$

$\blacksquare$

Definition 19.8. The quantity

$$\rho(z,w) := |\frac{z-w}{1-\overline{w}z}|$$

is called the **pseudohyperbolic distance**. If $\rho(z,w)$ is small (ie. $|z-w| << (1-|z|), (1-|w|)$, it is approximately the same as the $dist(z,w)$.

In practice, what are the geodesics of the hyperbolic metric?

- Any diameter through 0 are geodesics
- All other geodesics are conformal images of the diameter, so they have to be circles that cross $\partial\mathbb{D}$ perpendicularly.

Remark 19.9. In the hyperbolic geometry on $\mathbb{D}$, the geodesics are called "lines". Take a link L and consider $z_0 \notin L$, then there exists infinitely many "lines" through z_0 that are parallel to L (ie. the parallel line need not be unique).

There's also a way to define Hyperbolic Geometry on the upper complex plane $\mathbb{C}_+$, in which we define

$$M(z,dz) = |\frac{1}{Im(z)}dz|^2$$

which is the inverse distance from point z to the boundary.

20 Lecture 20 - 10/24/2022

20.1 Why is the radius a geodesic?

Proposition 20.1. Let $r < 1$, the radius $[0, r]$ is the geodesic from $z = 0$ to $z = r$ in $\mathbb{D}$ under the hyperbolic geometry.

Proof. Let $\gamma : [a, b] \to \mathbb{D}$ be an arbitrary path such that $\gamma(a) = 0, \gamma(b) = r$. Write $\gamma(t) = (x(t), y(t))$, then we note that

$$\begin{aligned}
\text{Length}(\gamma) &= 2 \cdot \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt \\
&\geq 2 \cdot \int_a^b \frac{|x'(t)|}{1 - (x(t)^2 + y(t)^2)} dt \\
&\geq 2 \cdot \int_a^b \frac{|x'(t)|}{1 - x(t)^2} dt && \text{Since } x(t)^2 + y(t)^2 < 1\text{, denominator is larger here} \\
&\geq |2 \cdot \int_a^b \frac{x'(t)}{1 - x(t)^2} dt| \\
&= |2 \cdot \int_0^r \frac{dx}{1 - x^2}| && x = x(t), dx = x'(t)dt \\
&= \ln(\frac{1 + r}{1 - r})
\end{aligned}$$

■

20.2 Riemann Mapping Theorem on the Riemann Sphere

We will now state a stronger version of the Riemann Mapping Theorem than the one we encountered before. We will prove in the next lecture!

Theorem 20.2 (RMT on the Riemann Sphere)**.** Suppose $\Omega \subset \hat{\mathbb{C}}$ is open and connected such that $\hat{\mathbb{C}} \setminus \Omega$ is connected and non-trivial ($\hat{\mathbb{C}} \setminus \Omega$ is not a singleton set). Let $z_0 \in \Omega$, then there exists a unique conformal map $f : \Omega \to \mathbb{D}$ such that

$$f(z_0) = 0,\ f'(z_0) \in \mathbb{R},\ f'(z_0) > 0$$

Remark 20.3. If $\Omega = \mathbb{D}$, the first condition implies $f(z) = \alpha z$ by Schwartz's Lemma for some $|\alpha| = 1$, and the second and third condition implies $\alpha = 1$.

20.3 Normal Families and Montel's Theorem

Definition 20.4. Let Ω be open (not necessarily connected), then $\mathcal{F} \subset \text{Hol}(\Omega)$ is a **familiy of holomorphic function** on Ω. We say $\mathcal{F}$ is a **normal family** if for all compact $K \subseteq \Omega$, there exists some real constant $C_K < \infty$ such that

$$|f(z)| < C_K,\ \forall f \in \mathcal{F},\ \forall z \in K$$

Theorem 20.5 (Montel's Theorem)**.** If $\mathcal{F} \subseteq \text{Hol}(\Omega)$ is normal, then for any sequence of functions $\{f_n\}$ in $\mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that converges normally to some $f_0 \in \text{Hol}(\Omega)$. Note that f_0 need not be in $\mathcal{F}$.

Definition 20.6. If X is a metric space, then $A \subseteq X$ is **totally bounded** if for all $\epsilon > 0$, there exists a finite ϵ-net (disks of radius ϵ) that covers A.

Proposition 20.7. If $X = \mathbb{R}^n$, then $A \subseteq X$ is totally bounded if and only if it is bounded.

Proof. Suppose A is totally bounded, there exists balls $B_{x_i,\epsilon}$ from $i = 1, ..., N$ such that $A \subseteq \bigcup_{i=1}^{N} B_{x_i,\epsilon}$. Let $B = \bigcup_{i=1}^{N} B_{x_i,\epsilon}$, it suffices for us to show that B is bounded.

if $N = 1$, we are done. If $N > 1$, then consider

$$R = \max_{1 \leq i \leq N} d(x_1, x_i)$$

Now for any $x \in B$, we know $x \in B_{x_i,\epsilon}$ for some i, then

$$\begin{aligned} d(x_1, x) &\leq d(x, x_i) + d(x_i, x_1) && \text{Triangle's Inequality} \\ &\leq r + R \end{aligned}$$

Thus, A is bounded.

Suppose A is bounded, then there exist some $r > 0$ such that $A \subseteq D_{x,r}$ for some open ball centered at x of radius r. It suffices for us to show that $D_{x,r}$ is totally bounded. Indeed, for any $\epsilon > 0$, we can paritition $D_{x,r}$ into a finite union of $B_{x_i,\epsilon}$ because $D_{x,r}$ has finite volume, and each $B_{x_i,\epsilon}$ has finite volume, so

$$\frac{\text{vol}(D_{x,r})}{\text{vol}(B_{x_i,\epsilon})} < \infty$$

■

Note that the forward direction above never used anything special about X being a Euclidean space, so in a metric space in general, totally bounded implies bounded. The converse is however not true:

Example 20.8. Let $X = \mathbb{Z}$ be endowed with the discrete metric, ie. $d(x, y) = 1$ if $x = y$ and $d(x, y) = 0$ if $x \neq y$. Then X is certainly bounded, as the ball $B_{0,1.1}$ centered at 0 of radius 1.1 contains $\mathbb{Z}$. However, X is certainly not totally bounded, as any ϵ-net of X with $0 < \epsilon < 1$ cannot be finite.

Lemma 20.9. A metric space X is compact if and only if it is complete and totally bounded.

Proof. For the forward direction, suppose X is compact, then for all $\epsilon > 0$, consider the open cover

$$\bigcup_{x \in X} B_{x,\epsilon}$$

This is an open cover of X, then by compactness, there exists some finite subcover of $cl(A)$, which covers A. Hence A has an epsilon net.

Now we note X is compact if and only if it is sequentially compact. Let $\{x_i\}$ be a Cauchy sequence in X, We also know that $\{x_i\}$ has a convergent subsequence $\{x_{n_k}\}$ to some limit L, we claim this is also the limit of the sequence!

For all $\epsilon > 0$, since $\{x_i\}$ is Cauchy, there exists some N such that for $m, n > N$,

$$d(x_m, x_n) < \epsilon/2$$

We can choose N large enough some that for all elements of the subsequence that appeared later than x_N, such that for all $x_{n_k}, x_{n_{k+1}}, ...$

$$d(L, x_{n_{k+i}}) < \epsilon/2$$

Then we have that for all $n > N$

$$d(L, x_n) < d(L, x_{k+i}) + d(x_{k+i}, x_n) < \epsilon/2 + \epsilon/2$$

Thus, the Cauchy Seuquence converges.

Conversely, suppose X is complete and totally bounded, in analysis, we showed that totally bounded implies every sequence of X has a Cauchy subsequence, which converges since X is complete. ■

Proposition 20.10. Let X be a complete metric space, then the following are equivalent:

1. $A \subseteq X$ is totally bounded.
2. $cl(A)$ is compact.
3. For all sequence $x_n \in A$, there exists a convergent subsequence x_{n_k} to some element in $cl(A)$.

Proof. We note that a subset of metric space, For (1) $\implies$ (2), suppose A is totally bounded, then for any $\epsilon > 0$, we have that there exists $x_1, ..., x_n$ so that

$$A \subseteq \bigcup_{i=1}^{n} B_{x_i, \epsilon/2}$$

Then we see that

$$cl(A) \subseteq cl(\bigcup_{i=1}^{n} B_{x_i, \epsilon/2}) \subseteq \bigcup_{i=1}^{n} B_{x_i, \epsilon}$$

Thus, $cl(A)$ is also totally bounded. Since a closed subset of a complete metric space is complete, $cl(A)$ is totally bounded and complete and is hence compact.

For (2) $\implies$ (1), since $cl(A)$ is compact, it is totally bounded, hence A is totally bounded.

For (2) $\implies$ (3), we recall that the subset of a metric space is compact if and only if it is sequentially compact, hence $cl(A)$ is sequentially compact, so we are done.

For (3) $\implies$ (1), suppose A is not totally bounded for contradiction, then there exists some $\epsilon > 0$ such that A cannot be finitely covered by balls of radius ϵ. Now pick some point $x_1 \in A$, we will call $B_1 = D_{x_1, \epsilon}$. Then we pick $x_2 \in A - B_1$ and call $B_2 = D_{x_2, \epsilon}$, and so on. In each step, we make sure to pick some element that's not covered before.

We can repeat this process infinitely since A cannot be finitely covered, and obtain a collection of balls $\{B_i\}_{i=1}^{\infty}$.

Now the sequence $\{x_i\}$ has to have some convergent subsequence, but each x_i is chosen so that $d(x_i, x_j) \geq \epsilon$ for $i \neq j$, so this can never have a convergent subsequence. So we have a contradiction. ■

Theorem 20.11 (Arzela-Ascoli Theorem)**.** Suppose $\mathcal{F} \subseteq C(K, X)$, where $C(K, X)$ is the space of continuous functions from compact metric space K to metric space X, then $\mathcal{F}$ is totally bounded if and only if

1. $\bigcup_{f \in \mathcal{F}} \mathrm{im}(f)$ is totally bounded
2. $\mathcal{F}$ is equicontinuous, ie. for all $f \in \mathcal{F}$, for all $x_0 \in K$, for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d(x, x_0) < \delta \implies d(f(x_0), f(x)) < \epsilon$$

Proof. See Browder's Analysis. ∎

Now we will put out attention back to complex analysis:

Proof of Montel's Theorem. We will prove the forward direction first.

Let $K_n \subseteq K_{n+1} \subseteq ... \subseteq \Omega$ be a sequence of compact sets K_i such that for any compact set $K \subseteq \Omega$, there exists N such that $K \subseteq \bigcup_{i=1}^{N} K_i$ (In other words, take K_i to the compact exhaustion of Ω).

Consider $F_n \subseteq C(K_n, X)$ defined as $F_n \coloneqq \{f|_{K_n} \mid f \in \mathcal{F}\}$. We note that $C(K_n, X)$ is a complete metric space, and **we want to first show that F_n is totally bounded**. To do this, we will invoke the Arzela Ascoli's Theorem:

1. Since $\mathcal{F}$ is normal, we know that in particular there $|f(z)| \leq C_{k_n}$ for all $f \in \mathcal{F}$ and $z \in K_n$. Thus, the union of images of f is bounded and hence totally bounded.
2. Let $z_0 \in K_n$ and choose $r_n < dist(z_0, \partial K_{n+1})$, then take z such that $|z - z_0| < r_n$, then

$$\begin{aligned} |f(z) - f(z_0)| &\leq |\frac{1}{2\pi i} \int_{|\xi - z_0| = r_n} f(\xi)(\frac{1}{\xi - z} - \frac{1}{\xi - z_0}) d\xi| && \text{Cauchy's Integral Formula} \\ &\leq \frac{1}{2\pi} \int_{|\xi - z_0| = r_n} C_{K_n} |\frac{1}{\xi - z_0} - \frac{1}{\xi - z}||d\xi| \end{aligned}$$

We claim that as we take limit from $z \to z_0$, $|\frac{1}{\xi - z_0} - \frac{1}{\xi - z}|$ converges to 0 uniformly in ξ. Indeed, write

$$|\frac{1}{\xi - z} - \frac{1}{\xi - z_0}| = |\frac{z - z_0}{(\xi - z)(\xi - z_0)}| \leq 2 \cdot \frac{|z - z_0|}{r^2}$$

Thus, we have an uniform convergence, so F_n is equicontinuous.

Thus, we have shown that F_n is totally bounded. Since this is the norm of the function space $C(X_n, K)$, this means that any sequence of functions on F_n has a uniformly convergennt subsequence.

Now take a sequence of functions $f_1, ..., f_n, ...$ on $\mathcal{F}$, there exists a sequence of functions f_{n_k} that converges uniformly on K_1, write them as $f_1^1, f_2^1, f_3^1, ...$ For K_2, we take another subsequence $f_2^1, f_2^2, f_3^2, ...,$ etc.

Then the seuqence $\{f_k^k\}_{k=1}^{\infty}$ converges uniformly on all K_n! Hence, we have obtained normal convergence. ∎

21 Lecture 21 - 10/26/2022

This lecture referenced notes by https://math.berkeley.edu/~vvdatar/m185f16/notes/Riemann_Mapping.pdf.

21.1 Proof of the Riemann Mapping Theorem

Definition 21.1. A function $f : \Omega \to \mathbb{C}$ is **univalent** if f is holomorphic and injective.

Theorem 21.2 (Hurwitz's Theorem on limit of univalent functions)**.** Let $f_n : \Omega \to \mathbb{C}$ be a sequence of univalent functions that converges normally to some function f, then f is either injective or constant.

Proof. This is a corollary of the standard Hurwitz's Theorem, which we will recall:

Suppose $\{f_n\}$ is a sequence of holomorphic functions on Ω that converges uniformly to f that is not the constant zero function. if f has a zero of order m at some root z_0, then there exists some N such that for all $n > N$, f_n has a zero of order m at z_0

Now back to our question, if f is constant, then we are done. Otherwise, suppose f is non-constant, then we wish to show that f is injective.

Indeed, if f not injective, then there exists $z_1 \neq z_2 \in \Omega$ such that

$$f(z_1) = f(z_2)$$

Now consider $g = f - f(z_1)$, this is the normal limit of univalent functions $f_n - f(z_1)$ (as each f_n is univalent), then g has a zero of order at least 1 at $z = z_1$ and another zero of order at least 1 at $z = z_2$, then by Hurtwitz's Theorem, for sufficiently large N, for all $n > N$, $f_n(z_1) = f_n(z_2)$, so f_n is not injective, hence we have a contradiction. ■

Lemma 21.3. Let $\Omega \subseteq \mathbb{C}$ such that $\hat{\mathbb{C}} \setminus \Omega$ is connected, then for all closed paths γ in Ω and for all $a \notin \Omega$, the winding number $\text{wind}(\gamma, a) = 0$.

Proof. This statement is equivalent to the fact that Ω is simply connected! The winding number is a homotopy invariant, so in particular we can deformation retract every closed curse γ to some constant curve, which has winding number 0 for all $a \notin \Omega$. ■

Definition 21.4. Let $G \subseteq \hat{\mathbb{C}}$ be a domain, we say G has the **zero index property** if $\text{wind}(\gamma, a) = 0$ for all γ in G and $a \notin G$.

Remark 21.5. If G has the zero index property, then $\int_\gamma F(z)dz$ for all closed paths γ and $F \in \text{Hol}(G)$. Thus, there exists a branch of logairthms on G. All simply connected domains have the zero index property.

Now we are ready to prove the Riemann Mapping Theorem.

Theorem 21.6 (Riemann Mapping Theorem)**.** If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $z_0 \in \Omega$, then there exists a unique conformal map $f : \Omega \to \mathbb{D}$ such that

$$f(z_0) = 0, \ f'(z_0) > 0$$

Proof. We will split the proof into uniqueness and existence:

1. **Uniqueness:** Suppose there exists two conformal maps $f, g : \Omega \to \mathbb{D}$ such that

$$f(z_0) = g(z_0) = 0, f'(z_0) > 0, g'(z_0) > 1$$

Consider $\varphi = f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$, we have that

$$\varphi(0) = f \circ g^{-1}(z_0) = f(z_0) = 0$$

And by the Chain Rule and the Inverse Function Theorem:

$$\varphi'(0) = \frac{f'(z_0)}{g'(z_0)} > 0$$

Thus, by Schwartz's Lemma, we have that $\varphi(z) = \alpha z$ for some $|\alpha| = 1$, but the derivative of φ at $z = 0$ must be positive and real, so $\alpha = 1$.

Thus, $f \circ g^{-1}$ is the identity map, so $f = g$.

2. **Existence:** Fix some $z_0 \in \Omega$, and define $\mathcal{F}$ as

$$\mathcal{F} := \{f \in \text{Hol}(\mathcal{F}) \mid f \text{ is univalent}, f(z_0) = 0,\ f'(z_0) > 0\}$$

We claim that **$\mathcal{F}$ is a normal family**. Indeed, for any compact subset $K \subseteq \Omega$, for all $f \in \mathcal{F}$, we know that

$$f(K) \subseteq f(\Omega) \subseteq \mathbb{D}$$

Since $\mathbb{D}$ is bounded uniformly by 1, we have that $|f(z)| < 1$ for all $z \in K$. Hence $\mathcal{F}$ is a normal family!.

Now we claim that **the following supremum is finite**

$$M := \sup_{g \in \mathcal{F}} g'(z_0)$$

Indeed, take some disk $D_{z_0,r}$ such that its closure is contained in Ω. We note that by Cauchy's Formula for Derivatives:

$$\begin{aligned} |g'(z_0)| &= |\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{g(z)}{(z-z_0)^2} dz| \\ &\leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{\sup_{|z-z_0|=r} |g(z)|}{r^2} \end{aligned}$$

Since $cl(D_{z_0,r})$ is compact, we know $\sup_{|z-z_0|=r} |g(z)|$ is bounded, hence $|g'(z_0)|$ is bounded. Thus, the set $\{g'(z_0) \mid g \in \mathcal{F}\}$ has an upperbound, so its supremum exists.

We will export the proof that $\mathcal{F} \neq \emptyset$ as a proposition following this. If $\mathcal{F} \neq \emptyset$, then we claim that **the function $f \in \mathcal{F}$ maximizing $f'(z_0)$ is a conformal map**! Indeed, consider the sequence (this exists as we took M to be the supremum):

$$f_n \in \mathcal{F} \mid f_n'(z_0) \geq M - 2^{-n}\}$$

By **Montel's Theorem**, this sequence has a convergence subsequence f_{n_k} to the limit say f. Then we also clearly have that $f'(z_0) = M > 0$, $f(z_0) = 0$, and that $|f(z)| \leq 1$ for all $z \in \Omega$. We know by **Hurwitz's Theorem on limit of univalent functions**, f is necessarily also a univalent function.

We also need to **show that** $f(\Omega) = \Omega$. Indeed, suppose not, then we pick some element $a \in \mathbb{D} \setminus f(\Omega)$, then we want to constrcut some $g \in \mathcal{F}$ such that $|g'(z_0)| > |f'(z_0)|$.

Indeed, consider

$$\psi_a(z) = \frac{a - z}{1 - \overline{a} z}, g_1(z) = \sqrt{\psi_a \circ f(z)}$$

We note that the square root on g_1 always has a holomorphic branch as $\psi \circ F(z) = 0$ only if f is surjective on a, and that Ω is simply connected. So we define g_1 as the branch

$$g_1(z) = e^{1/2 \cdot \log[\psi_a \circ f(z)]}$$

Now consider

$$g(z) := \psi_{g_1(z_0)} \circ g_1(z)$$

We see that g is univalent as both ψ and g_1 are univalent, moreover

$$g(z_0) = \psi_{g_1(z_0)}(g_1(z_0)) = 0$$

It remains for us to show that $|g'(z_0)| > |f'(z_0)|$. Indeed, we note that

$$f = \psi_a^{-1} \circ s \circ \psi_{g(z_0)}^{-1} \circ g$$

where $s : z \mapsto z^2$ is the squaring function. We will denote $\phi := \psi_a^{-1} \circ s \circ \psi_{g(z_0)}^{-1}$ and note that this is a function from $\mathbb{D} \to \mathbb{D}$. Furthermore

$$\phi(0) = \psi_a^{-1}(s(g(z_0)) = \psi_a^{-1}(\psi_a(f(z_0))) = f(z_0) = 0$$

Hence, by Schwarz's Lemma, we have that $|\phi(z)| \leq |z|$ and $|\phi'(0)| \leq 1$.

We claim that $|\phi'(0)| < 1$. Indeed, if $|\phi'(0)| = 1$, then Schwarz's Lemma says $\phi(z) = \alpha z$ is univalent, but this would imply s is univalent, so we have a contradiction.

Thus, by Chain Rule

$$f'(z_0) = \phi'(g(z_0)) \cdot g'(z_0) = \phi'(0) \cdot g'(z_0)$$

Hence

$$|f'(z_0)| \leq |\phi'(0)| \cdot |g'(z_0)| < |g'(z_0)|$$

(Note we can WLOG scale g by an appropriate constant of modulus 1 such that $g'(z_0)$ is real, so $g \in \mathcal{F}$).

■

Proposition 21.7. $\mathcal{F} \neq \emptyset$

Proof. **It remains for us to show that $\mathcal{F}$ exists**. Note that for this family to exist, it **suffices for us to find any univalent function** $f : \Omega \to \mathbb{C}$ such that $f(z_0) = 0$. This is because $\alpha f'(z_0) \neq 0$ as f is already univalent, and we can multiply f by an appropriate constant of modulus 1 such that $\alpha f \in \mathcal{F}$.

If Ω is bounded, then choose $\epsilon > 0$ small enough that $f(z) = \epsilon(z - z_0)$ has image contained in $\mathbb{D}$. Then $f(z_0) = 0$ and $f'(z_0) = \epsilon > 0$, so we are done.

If Ω is unbounded but not dense in $\mathbb{C}$, there exists some $a \in \mathbb{C}$ such that there exist some $r > 0$ such that $D_{a,r} \cap \Omega = \Omega$. Now consider the function

$$\varphi(z) = \frac{1}{z - a}$$

This is univalent on any domain not containing a and $\varphi(\Omega) \subseteq D_{0,1/r}$ is bounded, so we can scale appropriately.

Finally, we have the most general case of Ω - that it is just simply connected, we still know that $\hat{\mathbb{C}} \setminus \Omega$ is connected. We can assume that both $0 \notin \Omega$ and $\infty \in \Omega$ (for any $a \notin \Omega$ and $b \in \Omega$, take the conformal map $\varphi(z) = \frac{z-a}{z-b}$. If $b = \infty$, take $\varphi(z) = z - a$).

Now since Ω is simply connected and does not contain 0, there exists a branch of $\log(z)$ in Ω. Now consider

$$\varphi : \Omega \to \mathbb{C}, \; z \mapsto e^{0.5 \log(z)}$$

We claim this is injective and hence univalent. Indeed, suppose there exist z_1, z_2 such that $\varphi(z_1) = \varphi(z_2) = w$. Then $z_1 = w^2 = z_2$.

Now we claim that if $w \in \varphi(\Omega)$, then $-w \notin \varphi(\Omega)$. Indeed, if they both exist, then there exists distinct $z_1, z_2 \in \Omega$ such that $w = \varphi(z_1), -w = \varphi(z_2)$. Then we again have that

$$z_1 = w^2 = (-w)^2 = z_2$$

So we have a contradiction.

Thus, we have that if a disk $D_{a,r}$ is contained in $\varphi(\Omega)$, then $-D_{a,r} \cap \varphi(\Omega) = \emptyset$. Hence, $\varphi(\Omega)$ is not dense, so we have reduced this to the previous case. ■

> **Remark 21.8.** What happens if we take $\Omega = \mathbb{C}$? We claim that such function cannot exist! Indeed, suppose there exist a conformal map $f : \mathbb{C} \to \mathbb{D}$, then by Liouville's Theorem, f is a bounded entire function and is thus constant, so we have a contradiction.

22 Lecture 22 - 10/28/2022

22.1 Elements of Geometric Function Theory

Geometric function theory is the study of univalent functions!

Definition 22.1. Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function. We can without loss assume $f(0) = 0, f'(0) = 1$ by replacing f by $af + b$ with appropriate constants $a, b \in \mathbb{C}$. This class of functions are called **Schlicht** functions.

Theorem 22.2 (Bieberbach Conjecture - De Branges Theorem)**.** Suppose f is Schlicht and write $f(z) = \sum_{n\geq 1} a_n z^n$ with $a_1 = 1$, (by normalization we always have $a_0 = 0, a_1 = 1$), then $|a_n| \leq n$ for all $n \geq 2$.

This was a long time conjecture it was proven in 1985. A lot of PhD got their dissertation from find a good ϵ such that

$$|a_n| \leq (1 + \epsilon) \cdot n$$

De Branges was famous for making false proofs, but no one believed in him when he proved it, so he had to go to the Soviet Union to show his proof is correct. Professor Treil was actually a graduate student at the time and saw his lecture!

We will prove this theorem for $n = 2$. To do this, we will use what's so called the Area Principle:

Theorem 22.3 (Area Principle)**.** Let $f(z) = z + \sum_{k\geq 1} a_k z^{-k}$ be a univalent function in $\{z \in \mathbb{C} \mid |z| > 1\}$, then

$$\sum_{k\geq 1} |a_k|^2 k \leq 1$$

We will first show the following Lemma, which we actually proved as an exercise!

Lemma 22.4. Let G be a bounded domain such that $\partial G \in PC^1$, then

$$\text{Area}(G) = \frac{1}{2i}\int_{\partial G} \overline{z} dz$$

Proof. We will explicitly compute this using Stoke's Theorem:

$$\int_{\partial G} w = \int_G dw$$

Expliciting computing the integral gives us:

$$\begin{aligned}
\int_{\partial G} \overline{z} dz &= \int_G d(\overline{z} dz) \\
&= \int_G d(\overline{z}) \wedge dz \\
&= \int_G 1 d\overline{z} \wedge dz \\
&= \int_G (dx - idy) \wedge (dx + idy) \\
&= \int_G idx \wedge dy - idy \wedge dx
\end{aligned}$$

$$\begin{aligned}
&= \int_G 2i dx \wedge dy \\
&= 2i \int_G dx \wedge dy \\
&= 2i \text{Area}(G)
\end{aligned}$$

∎

Proof of the Area Principle. Let $r > 1$, let G_r the the domain bounded by $f(re^{it}), t \in (0, 2\pi]$. Then we have that

$$\begin{aligned}
0 &\leq \text{Area}(G_r) \\
&= \frac{1}{2i} \int_{\partial G_r} \overline{z} dz \\
&= \frac{1}{2i} \int_0^{2\pi} \overline{re^{it} + \sum_{k \geq 1} \overline{a_k} r^{-k} e^{-ikt}} \cdot (re^{it} + \sum_{k \geq 1} a_k r^{-k} e^{-ikt})' dt \\
&= \frac{1}{2i} \int_0^{2\pi} (re^{-it} + \sum_{k \geq 1} \overline{a_k} r^{-k} e^{ikt}) \cdot (re^{it} + \sum_{k \geq 1} a_k r^{-k} e^{-ikt})' dt \\
&= \frac{1}{2i} \int_0^{2\pi} (re^{-it} + \sum_{k \geq 1} \overline{a_k} r^{-k} e^{ikt}) \cdot (ire^{it} - \sum_{k \geq 1} ika_k r^{-k} e^{-ikt}) dt \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} (re^{-it} + \sum_{k \geq 1} \overline{a_k} r^{-k} e^{ikt}) \cdot (re^{it} - \sum_{k \geq 1} ka_k r^{-k} e^{-ikt}) dt
\end{aligned}$$

Note that $\int_0^{2\pi} e^{ikt} e^{-int} dt = 0$ whenever $k \neq n$ and $\int_0^{2\pi} e^{ikt} e^{-int} dt = 2\pi$ when $k = n$

$$\text{Area}(G_r) = \frac{2\pi}{2} (r^2 - \sum_{k \geq 1} |a_k|^2 r^{-2k})$$

Since the orientation of re^{it} gives $+$ orientation of the boundary, we have that

$$\sum_{k \geq 1} |a_k|^2 r^{-2k} \leq r^2$$

Take the limit as $r \to 1$ on both side, since the values are both positive, by Monotone Convergence Theorem, we can interchange summation and limit, we have

$$\sum_{k \geq 1} |a_k|^2 \leq 1$$

∎

Now we are ready to prove Bieberbach conjecture for $n = 2$:

Proof. Suppose $f(z) = z + \sum_{n \geq 2} a_n z^n$ is an univalent function, then

$$f(z^2) = z^2 + \sum_{n \geq 2} a_n z^{2n} = z^2 (1 + \sum_{n \geq 2} a_n z^{2n-2})$$

Then write $g(z) = 1 + \sum_{n \geq 2} a_n z^{2n-2}$ and we have that

$$f(z^2) = z^2 g(z)$$

Now we note that $g(z) \neq 0$ for all $z \in \mathbb{D}$ and is univalent, so we can take a logairhmic branch of $g(z)$, then we define

$$\psi(z) = z\sqrt{g(z)}$$

satisfying

$$\psi(z)^2 = f(z^2)$$

Now we note that from Calculus, we have the expansion

$$\sqrt{1+z} = 1 + \frac{z}{2} + ...$$

So we have that

$$\psi(z) = z(1 + \frac{a_2}{2}z^2 + ...)$$

Now define

$$F(z) := \frac{1}{\psi(1/z)} = z(1 + \frac{a_2}{z}z^{-2} + ...)^{-1}$$

What is the inverse of $1 + \frac{a_2}{z}z^{-2} + ...$? Recall that

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ..., |x| < 1$$

So we have that

$$\begin{aligned} F(z) &= z(1 - \frac{a_2}{2}z^2 + ...) \\ &= z - \frac{a_2}{2}z^{-1} + \sum_{k\geq 2} b_k z^{-k} && b_2 = 0 \text{ here} \end{aligned}$$

Applying Area Principle gives on $F(z)$ gives us

$$|\frac{a_2}{2}|^2 + \sum k|b_k|^2 \leq 1 \implies |\frac{a_2}{2}|^2 \leq 1$$

Hence, we have that

$$|a_2| \leq 2$$

■

> **Remark 22.5.** What's interesting about De Brange's Theorem is that the critical case is known. In particular,
>
> $$f(z) = \frac{z}{(1-z)^2}$$
>
> then $a_k = k$. It is a fun exercise to show that this f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is the complex plane without a ray. All the critical functions are of the form
>
> $$f(z) = \frac{z}{(1-\alpha z)^2}, \ |\alpha| = 1$$

22.2 Köbe 1/4 Theorem

> **Theorem 22.6** (Köbe 1/4 Theorem)**.** If f is a Schlicht function, then $D_{0,1/4} \subseteq f(\mathbb{D})$.

Proof. Let $w_0 \notin f(\mathbb{D})$, we note w_0 exists because $\mathbb{C}$ is not conformally equivalent with $\mathbb{D}$. Then consider

$$\varphi(z) = \frac{f(z)}{1 - \frac{1}{w_0}f(z)}$$

In particular, φ is univalent and $\varphi(0) = 0$. Then, decomposing numerator and denominator of $f(z)$ into series gives

$$\begin{aligned}\varphi(z) &= (z + \sum_{k\geq 2} a_k z^k) \cdot [1 - \frac{1}{w_0}(z + \sum_{k\geq 2} a_k z^k)]^{-1} \\ &= (z + \sum_{k\geq 2} a_k z^k) \cdot [1 + \frac{z}{w_0} + O(z^2)] && (1-x)^{-1} = 1 + x + x^2 + ... \\ &= z + (a_z + \frac{1}{w_0}) + ...\end{aligned}$$

In other words, φ is a Schicht function! So by DeBrange's Theorem when $n = 2$,

$$|a_2 + \frac{1}{w_0}| \leq 2$$

By Triangle's Inequality,

$$|\frac{1}{w_0}| \leq |-a_2| + |a_2 + \frac{1}{w_0}|$$

Both terms on the RHS are bounded by 2 by DeBrange's Theorem, hence

$$\frac{1}{w_0}| \leq 2 + 2 = 4$$

So in other words,

$$|w_0| \geq \frac{1}{4}$$

■

There's another theorem surrounding Schlicht functions that we will state without proof:

Theorem 22.7 (Köbe Distortion Theorem)**.** Suppose f is a Schlicht function, then

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

23 Lecture 23 - 10/31/2022

23.1 Runge's Theorem

In this lecture, we will prove Runge's Theorem. (Fun Fact: The Runge–Kutta methods in Numerical Analysis comes from the same mathematician Carl Runge)

Theorem 23.1 (Runge's Theorem)**.** Let K be compact, $f \in Hol(K)$ (recall this means that there exists some open Ω containing K where $f \in Hol(\Omega)$), then

1. There exist a sequence of rational functions $\{f_n\}_{n=1}^{\infty}$ with poles NOT in K such that $\{f_n\}$ converges to f uniformly on K
2. Let $\{\mathcal{O}_k\}$ be connected components of $\mathbb{C} \setminus K$ (Note that the collection $\{\mathcal{O}_k\}$ is countable by separability and denseness of $\mathbb{R}^2$), fix $z_k \in \mathcal{O}_k$ for each k, then one can choose the rational $\{f_n\}_{n=1}^{\infty}$ as before such that they only have poles at z_k.

Proof. We will first prove (1):

Idea: The idea is to find some bounded open G with $\partial G \in PC^1$ such that

$$K \subseteq G \subseteq cl(G)\Omega$$

Then using Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi$$

We can actually approximate the integral using Riemann Sums to get

$$f(z) \sim \sum \frac{f(\xi_k)}{\xi_k - z}(\xi_k - \xi_{k-1})$$

, which would be get us a rational approximation, but showing the uniform convergence is a bit tedious. We will however generalize this idea topologically, as shown below.

Let $\gamma = \partial G$, we want to divide γ into finitely many non-overlapping (intersect at most at vertices) arcs $\{\gamma_k\}$ such that each γ_k is contained in D_{λ_k,δ_k} such that $D_{\lambda_k,2\delta_k} \subseteq \mathbb{C} \setminus K$. There are two ways to find the δ we want:

1. In practice, recall we usually construct our desired G using rectangular grids, so its boundary is just some straight-lines. Then, fix $\delta < dist(\gamma, K)/2$, then we can split each interval to be less than δ
2. Using some abuse of notation, write $\gamma : [a, b]$
$to\Omega$, then we can cover the image of γ by open disks $D_{z,\delta}$ using the Lebesgue Number Lemma, which will also give our desired result.

Now, using the arcs $\{\gamma_k\}$, we have that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \sum_{1}^{N} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\xi)}{\xi - z} dz$$

, where for simplicity we will write $\varphi_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\xi)}{\xi - z} dz$, so we have that

$$f(z) = \sum_{1}^{N} \varphi_k(z)$$

Note that $\varphi_k(z)$ is analytic for $|z - \lambda_k| > \varphi_k$ and furthermore that $\varphi_k(z) \to 0$ as $z \to \infty$. Thus, we can write each $\varphi_k(z)$ as the Laurent Series

$$\varphi_k(z) = \sum_{j \geq 1} a_j^k (z - \lambda_k)^{-j}$$

, which converges uniformly on compact subsets on $(D_{\lambda_k, \delta_k})^c$, and hence on K.

Now take $\epsilon > 0$, we can approximate each φ_k by finite sum

$$\sum_{j=1}^{N_k} a_j^k (z - \lambda_k)^{-j}$$

, up to ϵ/N. Hence we have that

$$|f(z) - \sum_{k=1}^{N} \sum_{j=1}^{N_k} a_j^k (z - \lambda_k)^{-j}| < \epsilon$$

The inner function $\sum_{k=1}^{N} \sum_{j=1}^{N_k} a_j^k (z - \lambda_k)^{-j}|$ is rational, he nce we have a rational approximation that converges uniformly. This concludes the proof of (1).

We will now proce (2):

<u>Idea:</u> Consider $z_k, \lambda \in \mathcal{O}_k$, ie. that they belong in the same connected component. Let f be some rational function with pole at z_k, then our goal is to show that there's a rational approximation of f with poles on λ, using a continuous induction.

We will first introduce a short lemma:

Lemma 23.2. Let $\mathcal{O}$ be open and connected, let K be a compact subset of $\mathbb{C} \setminus \mathcal{O}$. Suppose $a \in \mathcal{O}$, $\lambda \in \mathbb{C}$ such that

$$|a - \lambda| < dist(a, \mathbb{C} \setminus \mathcal{O})$$

, then any rational (resp. proper rational) function with pole at λ can be approximated by rational (resp. proper rational) functions uniformly with pole at a.

Proof of Lemma. The proof for proper rational function follows similarly to the case of rational functions. Let δ be some number between

$$|a - \lambda| < \delta < dist(a, \mathbb{C} \setminus \mathcal{O})$$

, and suppose f is a rational function with pole at λ.

In particular, f is analytic at $z : |z - a| > \delta$, hence f can be written as

$$\sum_{k \in \mathbb{Z}} a_k (z - a)^k$$

, which converges uniformly on compact subsets of $\{z : |z-a| > \delta\}$. Since K is a compact subset of $\{z : |z-a| > \delta\}$, we have a uniform convergence on K. ∎

Now back to our setup, let $z_k \in \mathcal{O}_k$. Define

$$A := \{\lambda \in \mathcal{O}_k | \text{ any rational function with pole at } \lambda \text{ can be approximated by functions with poles at } z_k\}$$

Clearly $z_k \in A$. Now, we claim A is open. Indeed, for any $a \in A$, we know there exist $\delta > 0$, such that for all $|\lambda - a| < \delta$, any rational function with pole at λ can be approximated by finite sum $\sum \frac{c_k}{(z-a)^k}$. Then we know that

$$|f_{\text{pole at } \lambda} - f_{\text{pole at } a}| < \frac{\epsilon}{2} \text{ , by Lemma}$$

$$|f_{\text{pole at } z_k} - f_{\text{pole at } a}| < \frac{\epsilon}{2} \text{ , by Definition of A}$$

Hence,

$$|f_{\text{pole at } z_k} - f_{\text{pole at } \lambda}| < \epsilon$$

Now we claim that A is also closed. Indeed, let $\lambda \notin A$ and take $\delta < \frac{dist(\lambda,\mathcal{O}^c}{)}4$, then for all a such that $|a - \lambda| < \delta$, $D_{a,\delta} \subseteq \mathcal{O}_k$. Now suppose for contradiction that $a \in A$, then our Lemma implies that $\lambda \in A$, which is false. Hence $a \notin A$. Thus, A is closed.

Since $\mathcal{O}_k$ is connected and A is a non-empty clopen subset of $\mathcal{O}_k$, we have that $A = \mathcal{O}_k$. ■

> **Remark 23.3.** Note that a nearly identical argument shows that Runge's Theorem holds for proper rational functions too.

24 Lecture 24 - 11/02/2022

24.1 Mergelyan's Theorem

Theorem 24.1 (Mergelyan's Theorem)**.** Let $K \subseteq \mathbb{C}$ be compact such that $\mathbb{C} \setminus K$ is connected, and let $f \in \text{Hol}(int(K)) \cap C(K)$, then f can be uniformly approximated by polynomials.

Proof. Recall from Runge's Theorem, if K is compact, and A is a set containing at least one element from every connected component of $\mathbb{C} \setminus K$, then there exists an uniform rational approximation of f with only poles at A.

Let $K \subseteq D_{0,R}$ be compact as before and f be analytic on K. Let $a = \infty$, then Runge's Theorem tells us that there exist a sequence of rational functions f_i that only has pole at $z = \infty$ because there's only one connected component with $\mathbb{C} \setminus K$. Since each f_i only has poles at infinity, it has no poles on all of $\mathbb{C}$, so they are really just polynomials. ■

Example 24.2 (Connectedness is essential)**.** We claim that $f(z) = 1/z$ cannot be uniformly approximated by polynomials on the annulus $1/2 \leq |z| \leq 1$.

Suppose for contradictions that such sequence of polynomials does exist, call it $\{f_n(z)\}_{z=1}^{\infty}$, and it converges uniformly to $f(z) = 1/z$.

We note that each of $f_n(z)$ is a polynomial and is hence an entire function on $\mathbb{C}$, hence Cauchy's Theorem tells us that

$$\int_{|z|=3/4} f_n(z)dz = 0$$

However, we know from lecture that the contour integral of $1/z$ of a circle centered at the origin is the following:

$$\int_{|z|=3/4} f(z)dz = \lim_{|z|=3/4} \frac{1}{z} dz = 2\pi i$$

However, since $\{f_n(z)\}$ converges uniformly to $f(z)$, we know from lecture that we can exchange the limit and the integral in the sense that

$$\lim_{n\to\infty} \int_{|z|=3/4} f_n(z)dz = \int_{|z|=3/4} \lim_{n\to\infty} f_n(z)dz = \int_{|z|=3/4} f(z)dz = 2\pi i$$

On the other hand, we also know that $\int_{|z|=3/4} f_n(z)dz = 0$ for all n, so

$$\lim_{n\to\infty} \int_{|z|=3/4} f_n(z)dz = 0$$

So we have that

$$0 = 2\pi i$$

which is clearly not true. Hence we have a contradiction.

Remark 24.3. In general, if K is compact, then $\mathbb{C} \setminus K$ may have infinitely many connected components. Suppose $f \in \text{Hol}(K)$, then by definition there exists some open set Ω containing K that f is holomorphic on, then Ω contains all but finitely many connected components of $\mathbb{C} \setminus K$.

24.2 Swiss Cheese

The following counter-example is what's called a "Swiss Cheese"

Proposition 24.4. There exists a compact K with non-empty interior such that there exists $f \in C(K)$ such that it cannot be approximated by rational functions with poles off K.

Proof. Let $K = \{z \mid |z| \leq 1\} \setminus \bigcup D_k$, where $cl(D_k) = cl(D_{z_k,r_k})$ are each disjoint with $\sum r_k < \infty$ and $\sum r_k^2 << 1/2$ (we can take $r_k < 2^{-k}$. Then this is a compact set (closed minus open is closed) with non-empty interior.

Let $\{w_k\}$ be a countable dense subset of $cl(\mathbb{D})$ (take $\mathbb{Q}^2 \cap \mathbb{C}$, this is dense in $cl(\mathbb{D})$). We choose $z_1 = w_1$, $r_1 = 1/2$, and $z_2 = w_K$, where K is the smallest integer such that $w_k \notin cl(\overline{z_1, r_1})$, then we choose $r_2 \leq 2^{-1/2}$ such that $cl(D_2) \cap cl(D_1) = \emptyset$. We can do this repeatedly to create disks $D_1, D_2, \ldots$.

Now take $f(z) = \overline{z}$ and let $K_m := cl(D) \setminus \bigcup_{i=1}^m D_i$. Suppose there esxists a uniform rational approximation of f on K so in particular, f_n uniformly approximates f on ∂K_m, then

$$\begin{aligned}
\int_{\partial K_m} f(z)dz &= \int_{|z|=1} f(z)dz - \sum_{i=1}^m \int_{\partial D_i} f(z)dz \\
&= 2\pi i - \sum_{i=1}^m \int_{\partial D_i} f(z)dz \\
&= 2\pi i - \sum_{i=1}^m 2\pi i r_i^2 \\
&= 2\pi i(1 - \sum_{i=1}^m r_i^2) \\
&\geq 2\pi i(1/2) && \text{Since we construct } \sum r_i^2 \leq 1/2
\end{aligned}$$

So we have that

$$\frac{1}{2\pi i}\int_{\partial K_m} f(z)dz \geq \frac{1}{2}$$

$$\lim_{m\to\infty} \inf \frac{1}{2\pi i}\int_{\partial K_m} \geq \frac{1}{2}$$

On the other hand, if we take f_n to be rational functions with poles in $\mathbb{C} \setminus K$, then there exists N (depending on n) such that $f_n \in \mathrm{Hol}(K_m)$ for all $m > N$.

If $|f_n - f| < \epsilon$ on K, then

$$|\int_{\partial K_m} f_n - f| \leq \epsilon 2\pi(1 + \sum_1^\infty r_k)$$

We can choose ϵ such that

$$\epsilon(1 + \sum r_k) < \frac{1}{4}$$

But this means that

$$\begin{aligned}
\frac{1}{2\pi}|\int_{\partial K_m} f(z)dz| &= \frac{1}{2\pi}|\int_{\partial K_m} f(z) - f_n(z)dz| && \text{Since } f_n \in \mathrm{Hol}(K_m) \\
&\leq \frac{1}{4}
\end{aligned}$$

But we also had that

$$[\frac{1}{2\pi i}\int_{\partial K_m} f(z)dz \geq \frac{1}{2}$$

So we have a contradiction. ■

25 Lecture 25 - 11/04/2022

25.1 Mittag Lefler Problem

Let Ω be a domain, and consider $\{z_k\}$ accumulating to the boundary, let p_k be polynomials of the form

$$p_k(z) = \sum_{j=1}^{n_k} a_j^{(k)} z^k$$

The question is - can we find a meromorphic f such that at each z_k, the "principal part" of the Laurent Series of f is $p_k(\frac{1}{2-z_k}) = \sum_{j=1}^{n_k} a_j^k (z-z_k)^{-j}$.

In other words, we want that $f - p_k(\frac{1}{z-z_k})$ is analytic at each z_k (so z_k is a removable singularity at $f - p_k(\frac{1}{z-z_k})$).

Example 25.1. Consider f with simple poles at $k \in \mathbb{N}$ with residue 1, our guess is that

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{z-k}$$

The problem with this is that $f(z)$ does NOT converge!

26 Lecture 26 - 11/07/2022

27 Lecture 27 - 11/09/2022

28 Lecture 28 - 11/11/2022

29 Lecture 29 - 11/14/2022

29.1 Harmonicity

In this lecture, we will continue finishing the proof that wMVP$_2$ $\implies$ Harmonic. First, we will prove the following theorem that will be vital in showing this implication:

Theorem 29.1 (Global Maximum Principle)**.** Suppose $u \in \text{wMVP}_2(\Omega) \cap C(\Omega)$, where Ω is a domain. If u has a local minimum at $z_0 \in \Omega$, then $u(z) \equiv u(z_0)$ is identifically constant.

Proof. We will prove this using a continuous induction! Indeed, let

$$A := \{z \in \Omega \mid u(z) = u(z_0)\} = u^{-1}(\{u(z_0)\})$$

Clearly $z_0 \in A$ and A is closed. Now to show that A is open, let $z \in A$.

Since $z \in A$, we know that $u(z) = u(z_0) \geq u(\xi)$ for all $\xi \in \Omega$, hence u has a global (so local) maximum at z.

Then, by the Local Maximum Principle, there exists some $r > 0$ such that $u(\xi) = u(z)$ for all $\xi \in D_{z,r} \subset A$. So A is open.

Since Ω is connected, we conclude that $A = \Omega$ ■

Theorem 29.2. Let Ω be a domain, suppose $u \in \text{wMVP}_2(\Omega) \cap C(\Omega)$, then $u \in \text{Harm}(\Omega)$.

Proof. It suffices for us to verify this on disks $D_{z_0,R}$ such that $cl(D_{z_0,R}) \subset \Omega$. Indeed, let φ be the function

$$\varphi := u - v$$

, where $v \in \text{Harm}(D_{z_0,R}) \cap C(\overline{D_{z_0,R}}^{cl})$ such that

$$v|_{\partial D_{z_0,R}} = u|_{\partial D_{z_0,R}}$$

We can construct v by taking boundary values of u and take its Poisson extension on $D_{z_0,R}$. We claim that it is actually the case that $\varphi \equiv 0$.

First we note that

$$\pm\varphi \in \text{wMVP}_2(D_{z_0,R}) \cap C(\overline{D_{z_0,R}}^{cl})$$
$$\pm\varphi \in \text{wMVI}_2(D_{z_0,R})$$

Let M be the maximum (this exists as the closure of the disk is compact),

$$M := \max_{z \in \overline{D_{z_0,R}}^{cl}} \varphi(z)$$

Suppose $M > 0$, then M is not attained at the boundary because φ is identically 0 on the boundary, but this means that M is attained at $z_1 \in D_{z_0,R}$. By the Gloval Maximum Principle, $\varphi(z) \equiv M$ on $D_{z_0,R}$. Then by continuity this means that φ attains M on the boundary, which is a contradiction!

If we replace φ by $-\varphi$, we also find that M cannot be negative.

But this means that $M = 0$, so the Global Maximum Principle tells us that φ is identically 0, so we are done. ■

29.2 Reflection Principles

There're some particularly useful corollaries of the theorem, known as the reflection principle!

Corollary 29.3 (Reflection Principle for Harmonic Functions)**.** Let $\Omega = \Omega^*$, where Ω^* denote the complex conjugation of Ω. In other words, Ω is a **symmetric domain**.

Define $\Omega^+ := \Omega \cap \mathbb{C}_+ = \{z \in \Omega : Im(z) > 0\}$ and $\Omega^- := \Omega \cap \mathbb{C}_-$ similarly, let $u \in$ Harm(Ω^+) such that for all $\xi \in \mathbb{R} \cap \Omega)$, $\lim_{z\to\xi} u(z) = 0$, then the following function

$$\tilde{u}(z) := \begin{cases} u(z), \ z \in \Omega^+ \\ -u(\overline{z}), \ z \in \Omega^- \\ 0, \ z \in \mathbb{R} \cap \Omega \end{cases}$$

Then, $\tilde{u} \in$ Harm(Ω)

Proof. It suffices for us to show that $\tilde{u}$ is continuous and wMVP (doesn't matter if it's 1 or 2) over Ω, then our prior theorems imply that $\tilde{u}$ is harmonic over Ω.

Clearly, $\tilde{u} \in C(\Omega)$ by the Glueing Lemma.

Now for wMVP, if $z_0 \in \Omega^+ \cup \Omega^-$, then wMVP holds by harmonicity of u and $-u$. Now, if $z_0 \in \mathbb{R} \cap \Omega$, then we note that

$$\frac{1}{\pi r^2} \int_{D_{z_0,r}} \tilde{u(z)} dA(z) = 0$$

by symmetry, so $\tilde{u}$ is **wMVP** on z_0. ∎

Our reflection principle can also be shown for analytic functions!

Theorem 29.4 (Reflection Principle for Analytic Functions)**.** Define $\Omega = \Omega^*$, Ω^+, and Ω^- as before, and suppose $f \in$ Hol(Ω^+) such that for all $\xi \in \mathbb{R} \cap \Omega$,

$$f(\xi) := \lim_{z\to\xi} f(z) \in \mathbb{R}$$

Then the following function

$$\tilde{f}(z) := \begin{cases} f(z), \ z \in \Omega^+ \\ \overline{f(\overline{z})}, \ z \in \Omega^- \\ f(z), \ z \in \mathbb{R} \cap \Omega \end{cases}$$

, then $\tilde{f} \in$ Hol(Ω)

Proof. We can show this using Morera's Theorem. In fact, recall in a previous homework exercise, we had that:

Exercise 29.5. Let L be a line in the complex plane. Suppose $f(z)$ is a continuous complex-valued function on a domain D that is analytic on $D \setminus L$. Show that $f(z)$ is analytic on D.

∎

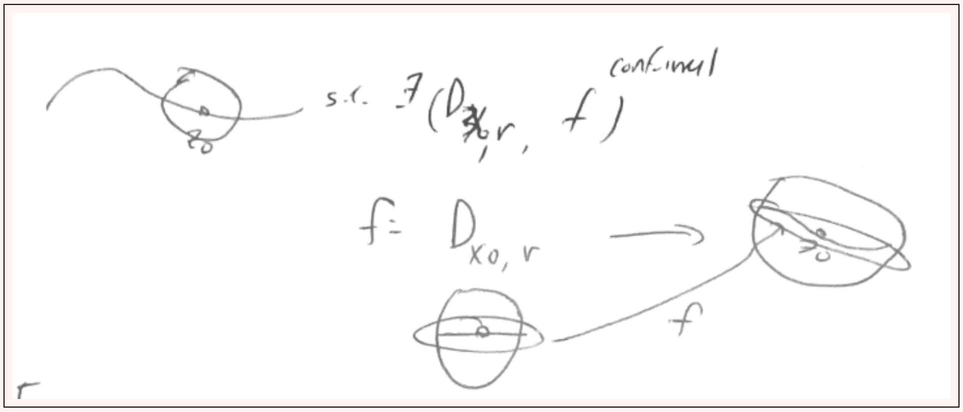
Remark 29.6. The conclusion of the reflection principle for analytic functions is still true if we only assume that

$$\lim_{z\to\xi} Im(f(z)) = 0$$

We can also replace the line in both reflection principles with a circle! For harmonic functions, this is because harmonic maps are invariant under conformal maps.

While reflection about lines and circles are all global extensions, we also have a notion of local extensions in what's known as **analytic curves**

Definition 29.7. γ is an **analytic curve** if locally it is a continuous image of a diameter of a disc, ie:

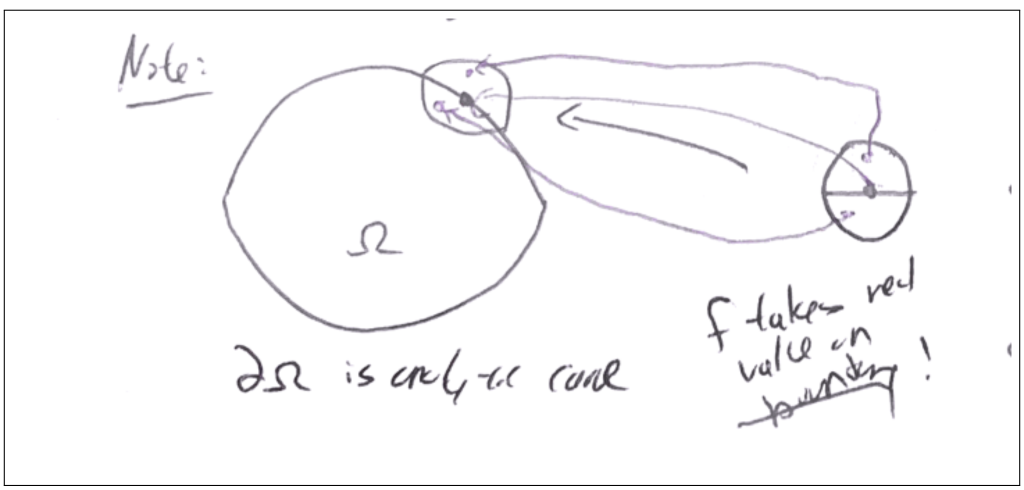


For all $z_0 \in \Gamma$, there exist some neighborhood N around z_0 and $(D_{x_0,r}, f)$ such that f maps the diameter of $D_{x_0,r}$ to $N \cap \gamma$.

Example 29.8. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an analytic curve.

The main idea behind reflection across analytic curves is as follows:

Suppose Ω is some domain where $\partial\Omega$ is an analytic curve:



The symmetry in the disk induces a symmetry near $\partial\Omega$, so we can extend it locally and use the reflection principle.

Proposition 29.9. Harmonic functions are real-analytic, meaning that the function is locally representable in power series as

$$(x - x_0)^n \cdot (y - y_0)^m$$

Or equivalently as power-series:

$$(z - z_0)^k \overline{(z - z_0)}^\ell$$

Proof. Recall the Poisson Formula lets you write a harmonic function $u(z)$ as

$$u(z) = \int_{\mathbb{T}} f(\xi) \frac{d\xi}{2\pi} + \sum_{k>0} z^k \int f(\xi) \overline{\xi}^k \frac{|d\xi|}{2\pi} + \sum_{k>0} \overline{z}^k \int f(\xi) \xi^k \frac{|d\xi|}{2\pi}$$

Applying a change of coordinates, we can reconstruct the power series based on harmonic values od disks. ■

30 Lecture 30 - 11/16/2022

30.1 Harmonic Conjugates

Definition 30.1. Let Ω be a domain and real-valued $u \in \text{Harm}(\Omega)$, suppose there exist real-valued $v \in \text{Harm}(\Omega)$ such that

$$u + iv \in \text{Hol}(\Omega)$$

then v is called the **harmonic conjugate** of u and we denote $\tilde{u} := v$. In particular, note that $\tilde{v} = -u$. Note that all complex conjugates of u differ by a constant.

Theorem 30.2. If Ω is a simply connected domain, then harmonic conjugates always exist.

Proof. Let $u \in \text{Harm}(\Omega)$, then by the Cauchy-Riemann Equation, it suffices to find $v \in \text{Harm}(\Omega)$ satisfying:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

Or equivalently that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Now fix $z_0 \in \Omega$, then we claim that

$$v(\xi) = a + \int_{z_0}^{\xi} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is our desired function. Note that v is well-defined since Ω is simply connected. Now clearly $u + iv$ is holomorphic, so it remains for us to show that v is harmonic:

$$\begin{aligned} d(dv) &= d(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) \\ &= -\frac{\partial^2 u}{\partial y^2}dy \wedge dx + \frac{\partial^2 u}{\partial x^2}dx \wedge dy \\ &= (\Delta u)dx \wedge dy \\ &= 0 && u \text{ is harmonic} \end{aligned}$$

■

Remark 30.3. The assumption that Ω is simply connected is essential. Consider the following example:

$$u(z) = \ln|z| \text{ on } r < |z| < R$$

We claim that u does not have any harmonic conjugates.

Proof. Suppose a conjugate v does exist, then

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Let r' be between r and R and $C_{r'}$ be the circle centered at origin of radius r', since $C_{r'}$ is a closed loop, endpoints match, so

$$\int_{C_{r'}} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0$$

However, evaluating the integral explicitly gives us that

$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \neq 0$$

■

> **Question 30.4.** Let $\mathbb{D}$ be the unit disk. Suppose $u \in \text{Harm}(\mathbb{D}) \cap C(\overline{\mathbb{D}}^{cl})$, then we know u contains a harmonic conjugate v. How can we describe $u + iv$ explicitly?

Idea. The idea is to find $S(z) \in \text{Hol}(\mathbb{D})$ such that

$$P(z) = \text{Re}(S(z))$$

, where $P(z)$ is the Poisson Formula. Then we have that

$$f(z) = \int_{\mathbb{T}} S(z\overline{\xi})u(\xi)\frac{|d\xi|}{2\pi}$$

Note that f is analytic by Morera's Theorem and

$$\text{Re}(f(z)) = \int_{\mathbb{T}} P(z\overline{\xi})u(\xi)\frac{|d\xi|}{2\pi} = u(\xi)$$

So, how do we find this desired $S(z)$? Recall that

$$P(z) = 1 + \sum_{k\geq 1} z^k + \sum_{k\geq 1} \overline{z}^k$$

Then choosing

$$\begin{aligned} S(z) &\coloneqq 1 + 2\sum_{k\geq 1} z^k \\ &= 1 + \frac{2z}{1-z} && \text{Domain of } S(z) \text{ is } \mathbb{D} \\ &= \frac{1+z}{1-z} \end{aligned}$$

will suffice. $S(z)$ is sometimes called the **Schwartz Kernel**.

Now let $Q(z) = \text{Im}(S(z))$, then

$$v(z) = \text{Im}(f(z)) = \int_{\mathbb{T}} Q(z\xi)u(\xi)\frac{|d\xi|}{2\pi}$$

So we have that

$$\begin{aligned} f(z) &= u(z) + iv(z) \\ &= \int_{\mathbb{T}} u(\xi)\frac{1+z\overline{\xi}}{1-z\overline{\xi}}\frac{|d\xi|}{2\pi} \\ &= \int_{\mathbb{T}} u(\xi)\frac{\xi+z}{\xi-z}\frac{|d\xi|}{2\pi} \end{aligned}$$

■

30.2 Reflection Principle for Analytic Functions

Let $\Omega = \Omega^*$ be a symmetric domain and $f \in \text{Hol}(\Omega^+)$ such that

$$\lim_{z \to \xi} \text{Im}(f(z)) = 0,\ \forall \xi \in \mathbb{R} \cap \Omega$$

(recall previously we assumed that the limit to the real part exists)
Let $f = u + iv$. Recall the reflection principle allowed us to extend v harmonically to

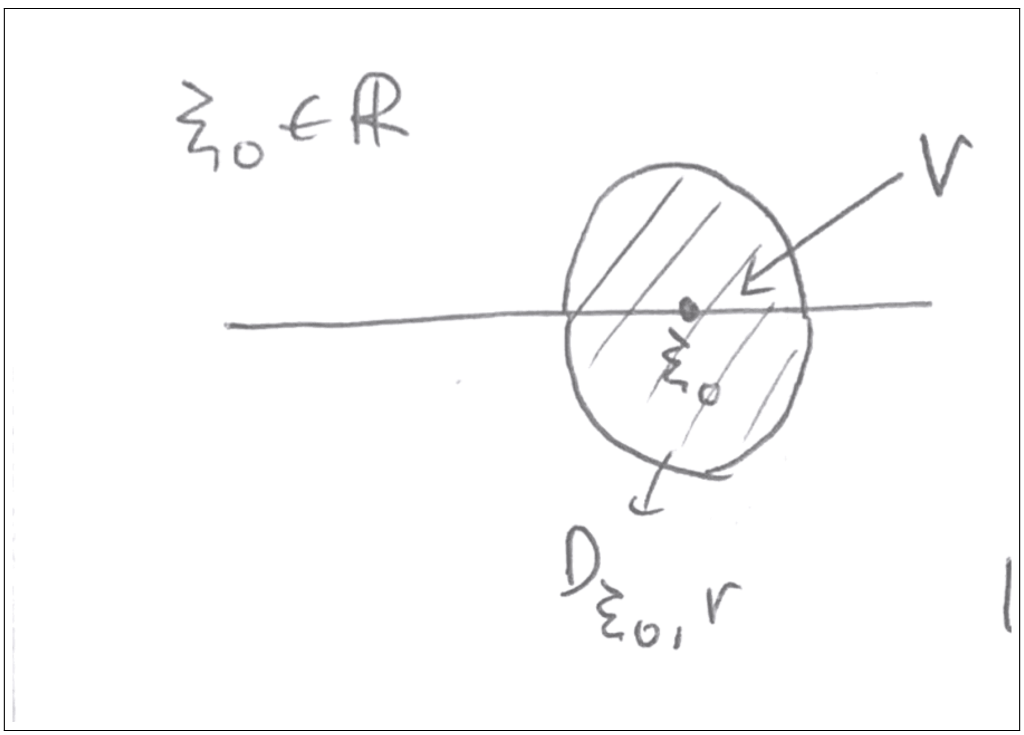
$$\tilde{v}(z) = \begin{cases} v(z),\ z \in \Omega^+ \\ 0,\ z \in \mathbb{R} \cap \Omega \\ -u(\overline{z}),\ z \in \Omega^- \end{cases}$$

We can extend u to $\tilde{u}$ on $\Omega \setminus \mathbb{R}$ as

$$\tilde{u}(z) = \begin{cases} u(z),\ z \in \Omega^+ \\ u(\overline{z}),\ z \in \Omega^- \end{cases}$$

Altogether, this extends f to a function $\tilde{f} \in \text{Hol}(\Omega \setminus \mathbb{R})$.

For a particular $\xi_0 \in \mathbb{R}$, consider the diagram:



Let u, v be harmonic on the upper-half disk as before, and let $\tilde{v}$ be the extension of v and v' be the harmonic conjugate, then

$$-v' + i\tilde{v} \in \text{Hol}(D_{\xi_0, r})$$

We can choose v' to be symmetric ($v'(z) = v'(\overline{z})$ (Follows from the Schwartz Kernel Formula), then it follows that

$$-v'(z) = u(z) + C, \forall z \in D^+_{\xi_0, r}$$

30.3 Liouville's Theorem for Harmonic Functions

Theorem 30.5 (Liouville's Theorem for Harmonic Functions)**.** Let $u \in \text{Harm}(\mathbb{C})$, if for all $z \in \mathbb{C}$, $|u(z)| \leq C < \infty$, then $u(z)$ is identically constant.

The proof of this theorem relies on the following lemma:

Theorem 30.6 (Harnack's Inequality)**.** If $u \in \text{Harm}(D_{0,R})$ and $u \geq 0$ is non-negative, then for any $|z| \leq r < R$,

$$\frac{R-r}{R+r}u(0) \leq u(z) \leq \frac{R+r}{R-r}u(0)$$

Proof of Harnack's Inequality. If $R = 1$, then consider $u \in \text{Harm}(\mathbb{D}) \cap C(\overline{D}^{cl})$, then we can write

$$u(z) = \int_{\mathbb{T}} \frac{1-|z\bar{\xi}|^2}{|1-z\xi|^2}u(\xi)\frac{|d\xi|}{2\pi}$$

, where we know that ξ takes values from $\mathbb{T}$ the unit circle, so

$$\begin{aligned}
\frac{1-|z\bar{\xi}|^2}{|1-z\xi|^2} &\leq \frac{(1-|z|)(1+|z|)}{(1-|z|)^2} \\
&= \frac{1+|z|}{1-|z|} \\
&\leq \frac{1+r}{1-r}\frac{1-|z\bar{\xi}|^2}{|1-z\xi|^2} && \geq \frac{(1-|z|)(1+|z|)}{(1+|z|)^2} && \text{Since } |z| \text{ is monotonic} \\
&= \frac{1-|z|}{1+|z|} \\
&\geq \frac{1-r}{1+r} && && \text{Since } |z| \text{ is monotonic}
\end{aligned}$$

So we have the inequality

$$\frac{R-1}{R+1}u(0) \leq u(z) \leq \frac{R+1}{R-1}u(0)$$

Now for a general R, consider

$$u_a(z) = u(az), a < R$$

Then we will eventually get the inequality that

$$\frac{a-r}{a+r}u(0) \leq u(z) \leq \frac{a+r}{a-r}u(0)$$

Then we can take the limit as $a \to R$. ■

Now we are ready to prove Liouville's Theorem for Harmonic Functions:

Proof. Let $|z| \leq r$, we can without loss replace u by $u + C$ so that u is non-negative (we can do this because u is bounded). So Harnack's Inequality gives us that

$$\frac{R-r}{R+r}u(0) \leq u(z) \leq \frac{R+r}{R-r}u(0)$$

Taking $r \to \infty$ on both sides gives us that

$$u(0) \leq u(z) \leq u(0),\ \forall z \in \mathbb{C}$$

■

31 Lecture 31 - 11/18/2022

32 Lecture 32 - 11/21/2022

33 Lecture 33 - 11/23/2022

34 Lecture 34 - 11/28/2022

35 Lecture 35 - 11/30/2022

36 Lecture 36 - 12/2/2022

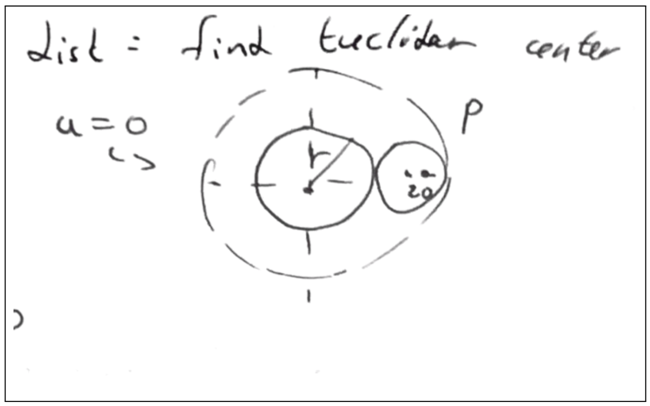
37 Lecture 37 - 12/5/2022

37.1 Common Mistakes on Midterm

You are allowed to resubmit for the midterm! Please either send an email with the resubmitted programs or send a regrade request on Gradescope. Midterm Resubmission to due this Friday by **midnight**.

There were two common mistakes a lot of people had on the midterm:

1. **Hyperbolic Disks:** There were some confusions on finding the Euclidean center of the disk. The idea is to first solve the question in the case where the hyperbolic center is $a = 0$:



   When $a = 0$, we have that
   $$\rho = \ln(\frac{1+r}{1-r})$$
   , and the Euclidean center coincide with the hyperbolic center. Now consider when $a = z_0$, then the following Mobius Transformation:
   $$z \mapsto \frac{z - z_0}{1 - \overline{z_0}z}$$
   maps hyperbolic disk to hyperbolic disk at the center. But note that z_0 is not the Euclidean center of its disks!

   It remains for us to find the Euclidean center of the disk, there are two ways to do this:

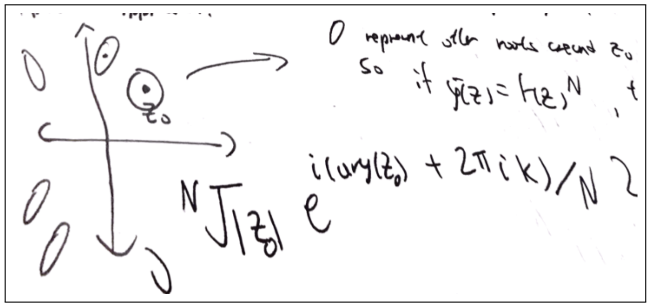
   - Find 2 points on the diameter opposite to each other
   - Linear Fractional Transformation preserves symmetry, so Euclidean centers are symmetric to ∞, then we can find the inverse image of infinity, abd this map goes to $\frac{1}{\overline{z_0}}$

2. f^N analytic $\implies$ f$analytic : The common approach is first fix$z$_0$ and write
   $$f(z)^N - f(z_0)^N = (f(z) - f(z_0)) \cdot (....)$$
   Then the claim is that if $f(z_0) \neq 0$, then $f(z)^N - f(z_0)^N$ be differentiable gives differentiability on $f(z) - f(z_0)$.

   In the case where $f(z_0) = 0$, we prove that the roots of f are isolated and are in fact removable singularities.

   There's another approach to this: Let U be some domain around z_0 small enough, then the N-th root of

z_0 creates exactly N regions whose N-th power goes to U:



In other words, if $g(z) = f(z)^N$, there exist analytic branches of $f(z) = g(z)^{1/N}$. We then claim that $f(z)$ must be in exactly ONE of these branches!

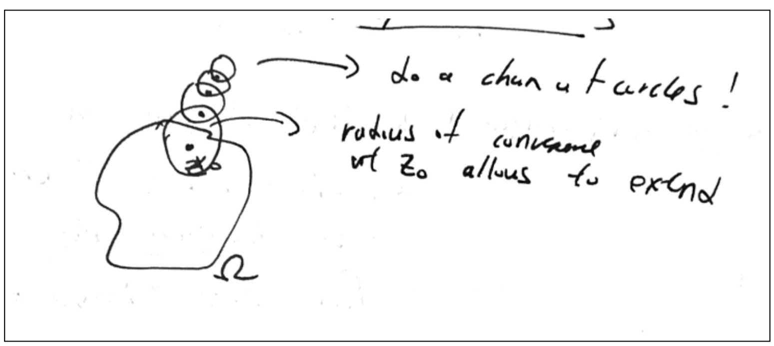
This claim follows from the fact that each branch is connected, f is continuous, and the continuous image of f has to be connected.

37.2 Analytic Continuation

Previously we have seen that functions like $\sqrt{1-z^2}, \log(z)$ are not well-defined on the complex plane, as they are multi-valued functions. However, there's a way to make them well-defined by representing them on **Riemann surfaces** instead.

One notion to introduce Riemann Surfaces uses the idea of Analytic Continuation.

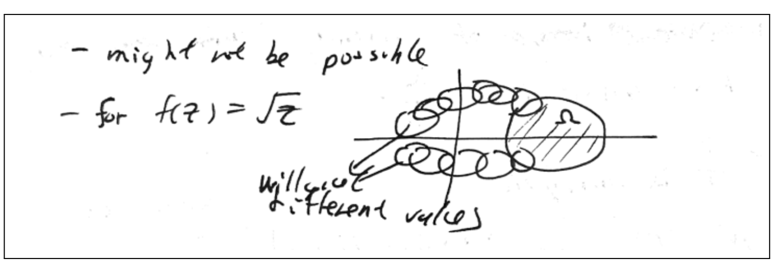
There's a naive approach to analytic continuation. Let $f \in Hol(\Omega)$ and consider some $z_0 \in \Omega$ as follows:



The idea is if the local power series at z_0 has radius of convergence greater than $d(z_0, \partial\Omega)$, then we could extend f out of Ω. If we can repeatedly do this, we can construct a chain of balls out of the boundary.

However, there are two problems with this approach:

1. You are not always guaranteed that you can construct this chain of balls.
2. Sometimes your extension may not match up with one another. For example take $f = \sqrt{z}$ with Ω being on the right half-plane:



If you take a chain of circles going from the clock-wise and counter-clockwise direction, they WILL not give the same value when they meet.

Therefore, we need a more standard way to discuss analytic continuation, and this leads to the idea of analytic continuation along a path.

Definition 37.1 (Analytic Continuation Along a Path)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a path, we say that an **analytic continuation exists along** γ if there exist a family of (f_s, U_s), $s \in [a, b]$ such that

$$\gamma(s) \in U_s, f_s \in \text{Hol}(U_s)$$

and for all $t_0 \in [a, b]$, there exist $\delta > 0$ such that

$$|t - t_0| < \delta \implies \begin{cases} 1.\ U_t \cap U_{t_0} \neq \emptyset \\ 2.\ f_t \equiv f_{t_0} \text{ on } U_t \cap U_{t_0} \end{cases}$$

Remark 37.2. In practice, once we have the family of functions, we will simplify f_t as

$$f_t(z) = \sum_0^\infty a_k(t)(z - \gamma(t))^k$$

as a power series and take U_t as a disk instead.

Proposition 37.3. Let $\gamma : [a, b] \to \mathbb{C}$ be a path, an **analytic continuation exists along** γ if and only if there exists a finite division of $[a, b]$ into intervals $I_1, \ldots, I_n$ and family $(f_i, U_i)_{i=1}^n$ such that

- $\gamma(I_k) \subseteq U_k$
- $U_k \cap U_{k+1} \neq \emptyset$
- $f_k \equiv f_{k+1}$ on $U_k \cap U_{k+1}$

Proof. Converse is straight-forward, for all $t \in [a, b]$, just choose g_t to be f_k if $t \in I_k$, then we have (g_t, U_t) to be a valid family.

For the forward, direction, consider U_t for each $t \in [a, b]$ and

$$\bigcup_{t \in [a,b]} \gamma^{-1}(U_t) \text{ is a cover of } [a, b]$$

Then the Lebesgue Number's Lemma implies that there exists some $\alpha > 0$ such that for any $E \subseteq [a, b]$ where $diam(E) < \alpha$, there exist U_t such that $E \subset \varphi^{-1}(U_t)$.

Then we split intervals of $[a, b]$ to intervals $I_1, \ldots, I_n$ each less than α in length. Then for $I_k = [a_{k-1}, a_k]$ we have that $I_k \subseteq \gamma^{-1}(U_{t_k})$, hence $\gamma(I_k) \subseteq U_{t_k}$.

Then the family (f_{t_k}, U_{t_k}) is our desired finite division. ■

Proposition 37.4. Analytic Continuation along a path is independent of the choice of splitting. In other words, the analytic continuation is unique.

Proof. This follows from the Uniquenss Theorem of analytic functions. Suppose we have a finite splitting I_k, J_j of $[a, b]$, then we can consider the splitting $\{I_k \cap I_j\}$ of $[a, b]$. This is a finer splitting than both given, then we apply the Uniqueness Theorem. ■

Definition 37.5 (Germs of Analytic Functions)**.** Let $z_0 \in \mathbb{C}$, then this consider the set of the form:

$$S = \{(f, U) \mid f \in \text{Hol}(U), U \text{ is a neighborhood of } z_0\}$$

We can define an equivalence relation as follows, where $(f, U) \sim (g, V)$ if there exists some neighborhood $W \subseteq U \cap V$ containing z_0 such that $f \equiv g$ on W.

This equivalence class is called the **germ of analytic functions at** z_0, which we denote an element of this class as $[f]_{z_0}$ and call it the **germ of** f **at** z_0.

Remark 37.6. If U, V are connected and convex domains, we can without loss take $W = U \cap V$. Analytic continuation being unique is essentially saying that our germs coincides along the path.

37.3 Monodromy's Theorem

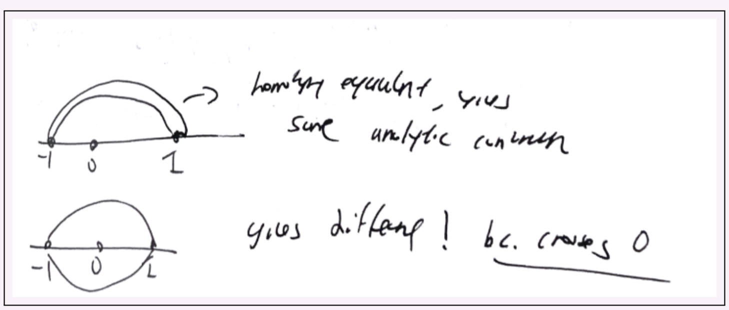
Theorem 37.7 (Monodromy's Theorem)**.** Suppose $\gamma_0, \gamma_1 : [a, b] \to \mathbb{C}$ are homotopy pathes with same start and end points, meaning there exist a continuous function $\Gamma(t, s) : [a, b] \to [0, 1] \to \mathbb{C}$ such that

$$\Gamma(t, 0) = \gamma_0(t), \Gamma(t, 1) = \gamma_1(t), \Gamma(a, s) = z_0, \Gamma(b, s) = z_1$$

Define $\gamma_s(t) := \Gamma(t, s)$ for all $s \in [a, b]$. Suppose there exists analytic continuation along all of $\gamma_s(t)$ for some analytic function f, then the analytic continuation of $[f]_{\gamma_s(b)}$ is independent on s!

Idea. If two analytic continuations are "close enough", they are the same. ■

Example 37.8. Take $f = \sqrt{z}$ on the principle branch, consider the two following ways to draw pathes from $\gamma_0(a) = 1$ to $\gamma_0(b) = -1$:



The pathes in the top figure are homotopy equivalent, the pathes in the bottom figure are not.

38 Lecture 38 - 12/7/2022

38.1 Sheaf of germs of analytic functions

Recall in last lecture, we set up the discussion of germs in the following context:

Fix $z_0 \in \mathbb{C}$, then this consider the set of the form:

$$S = \{(f, U) \mid f \in \text{Hol}(U), U \text{ is a neighborhod of } z_0\}$$

We can define an equivalence relation as follows, where $(f, U) \sim (g, V)$ if there exists some neighborhood $W \subseteq U \cap V$ containing z_0 such that $f \equiv g$ on W.

This equivalence class is called the **germ of analytic functions at** z_0, which we denote an element of this class as (f, z_0) and call it the **germ of** f **at** z_0.

A question arises from this is - what happens when we consider the set of all germs of analytic functions?

Definition 38.1. We denote $\sigma(\mathbb{C})$ as the set of all germs of analytic functions (f, z_0) for all $z_0 \in \mathbb{C}$. We can define a topology on $\sigma(\mathbb{C})$ as follows:

- For all $(f, z_0) = [f, z_0, U] \in \sigma(\mathbb{C})$. For any $V \subseteq U$, we denote

$$\tilde{V} := \{[f, z, U] : z \in V\}$$

  Then the base at (f, z_0) is the collection of following open sets:

$$\{\tilde{V} \mid V \text{ is an element of the base of Euclidean topology of } \mathbb{C} \text{ on } z_0\}$$

- Repeating this for all $(f, z_0) \in \sigma(\mathbb{C})$ and unioning all the collections forms a topological basis of $\sigma(\mathbb{C})$!

Equivalently, a subset $\Omega \subset \sigma(\mathbb{C})$ is open if for all $[f, z_0, U] \in \Omega$, we have that for all $z \in U$, $[f, z, U] \in \Omega$.

We call $\sigma(\mathbb{C})$ equipped with this topological structure - **the sheaf of germs of analytic functions**.

Remark 38.2. Consider the projection map $\pi : \sigma(\mathbb{C}) \to \mathbb{C}$ such that

$$\pi([f, z, U]) = z$$

This is locally a biholomorphism to $\mathbb{D}$ on any given $\tilde{V}$ containing $[f, z, U]$. The resulting topology on $\sigma(\mathbb{C})$ is Hausdorff and Locally Euclidean, but it is clearly not 2nd countable, as there are uncountably many choices of disks of different radius centered at z_0.

38.2 Riemann Surface of a Germ

Definition 38.3. The Riemann Surface of a germ $[f, z_0, U]$ is the connected component of $\sigma(\mathbb{C})$ containing $[f, z_0, U]$. Note that since connected implies path-connected in a locally Euclidean space, we can equivalently define this to be the path component of $\sigma(\mathbb{C})$ containing $[f, z_0, U]$.

What is the analytic continuation in the context of Riemann Surfaces? Recall our definition of analytic continuation in the usual context:

Let $\gamma : [a,b] \to \mathbb{C}$ be a path, we say that an **analytic continuation exists along** γ if there exist a family of (f_s, U_s), $s \in [a,b]$ such that

$$\gamma(s) \in U_s, f_s \in \text{Hol}(U_s)$$

and for all $t_0 \in [a,b]$, there exist $\delta > 0$ such that

$$|t - t_0| < \delta \implies \begin{cases} 1.\ U_t \cap U_{t_0} \neq \emptyset \\ 2.\ f_t \equiv f_{t_0} \text{ on } U_t \cap U_{t_0} \end{cases}$$

Consider $\gamma : [a,b] \to \mathbb{C}$ for some $\gamma(a) = z_0$, then consider a lift $\tilde{\gamma} : [a,b] \to \sigma(\mathbb{C})$ such that the following diagram commutes:

$$\begin{tikzcd} \sigma(\mathbb{C}) \arrow[r, "\pi"] & \mathbb{C} \\ {[a,b]} \arrow[u, "\exists!\tilde{\gamma}"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

where we essentially want

$$\pi(\tilde{\gamma}(t)) = \gamma(t),\ \forall t \in [a,b]$$

Then **our lift $\tilde{\gamma}$ encodes the family of functions required in the definition of analytic continuation along a path!** If we fix what $\tilde{\gamma}(a) = (f, z_0, U)$ is, then this is the analytic continuation with respect to the germ (f, z_0)!

Theorem 38.4. Every path γ's lift $\tilde{\gamma}$ to $\sigma(\mathbb{C})$, such that $\tilde{\gamma}(a) = (f, z_0, U)$ is fixed, (if it exists) is unique.

Proof. Suppose there exists $\gamma', \gamma'' : [a,b] \to \sigma(\mathbb{C})$ that are both possible lifts. Consider the set

$$S = \{t \in [a,b] \mid \gamma'(t) = \gamma''(t)\}$$

By definition we have that $\gamma'(a) = \gamma''(a)$, so $a \in S$.

Now for any $t \in S$, then recall π is a local biholomorphism (and is in particular locally bijective), so locally around $\gamma'(t) = \gamma''(t)$, there exists some neighborhood $(t-\epsilon, t+\epsilon)$ such that $\gamma' \equiv \gamma''$. Thus, $(t-\epsilon, t+\epsilon) \subseteq S$. Hence S is open.

Now, for any $t \notin S$, the fact that π is a local biholomorphism (in particular π would be a 2-to-1 here) means that there's a neighborhood of t such that γ' and γ'' cannot agree on. Hence S^c is open, so S is closed.

Since S is non-empty, open and closed, and $S \subseteq [a,b]$, we have that $S = [a,b]$. ■