A Brief Introduction to Radiation Transport

M. I. Ortega

Department of Nuclear Engineering
University of California, Berkeley

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Outline

Particle interactions are incorporated through the concept of cross sections. Neutrons have a few interactions of interest in radiation transport: capture, fission, elastic, and inelastic scattering.

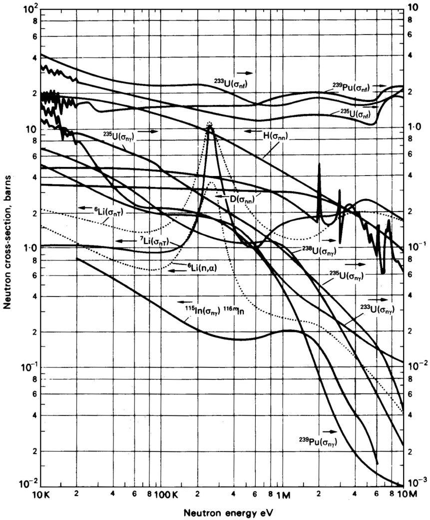
- Neutron Cross Sections:
  - Absorption:

$$\sigma_a(E) = \sigma_c(E) + \sigma_f(E) \tag{1}$$

  - Scattering (elastic and inelastic):

$$\sigma_s(E) = \sigma_n(E) + \sigma_{n'}(E) + \sum_{i=2}^{N} \sigma_{i,n}(E) \tag{2}$$

Nuclear Cross Sections
Example of Neutron Cross Sections



Photons have three interactions of interest in radiation transport: photoelectric, Compton scattering, pair production.

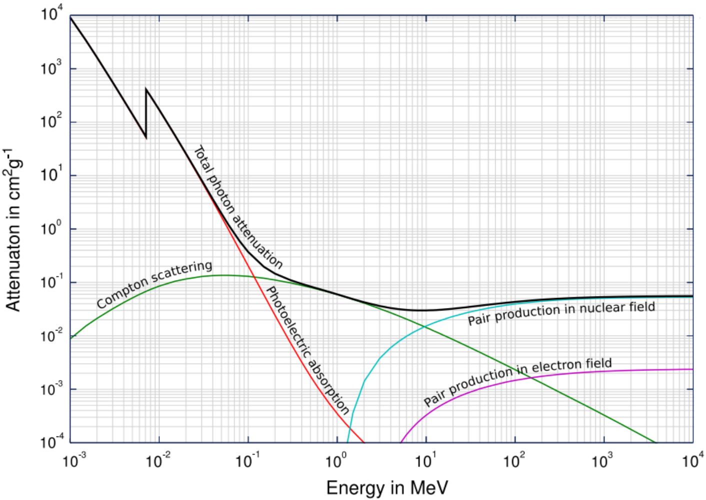
- Photon Cross Sections
  - Absorption:

$$\sigma_a(E) = \sigma_{pe}(E) \tag{3}$$

  - Scattering:

$$\sigma_s(E) = \sigma_{cs}(E) + \sigma_{pp}(E) \tag{4}$$

Nuclear Cross Sections
Example of Photon Cross Section



Scattered Particle Distributions

Express probability distribution laws governing scattered particle emission. The scattering cross section tells us the probability a scattered particle will be emitted in some specific direction with some specific energy:

$$\sigma_s(\vec{r}, E \to E', \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{\Omega}}')dE'd\hat{\mathbf{\Omega}}' \tag{5}$$

Fission Neutron Distributions

Two quantities need to be defined in fission:

- $\nu(E)$ = mean number of fission neutrons produced in a fission caused by a neutron with energy E.
- $\chi(E)dE$ = probability that a fission neutron will have an energy dE about E.

$$\int_0^\infty \chi(E)dE = 1 \tag{6}$$

Particle Interactions
Assumptions

In the derivation of the TE, we assume the following:

- Particles are considered as points.
- Particles travel in straight lines between point collisions (neutral!).
- Particle-particle interactions can be neglected.
- Collisions are instantenous (some exceptions like delayed neutron emission).
- Material properties are assumed to be isotropic.
- Nuclei properties and material properties are assumed constant and known (can be relaxed).
- Only expected or mean particle density distribution is considered.

Definitions

Angular Flux

Let the particle density distribution in a six-dimensional phase space (space, direction, energy, and time) be $N(\vec{r}, \hat{\mathbf{\Omega}}, E, t)$, we define the neutron angular flux as

$$\psi(\vec{r}, \hat{\mathbf{\Omega}}, E, t) = vN(\vec{r}, \hat{\mathbf{\Omega}}, E, t) \tag{7}$$

which is convenient for the calculation of reaction rates. Usually reaction rates are independent of direction, so we define the scalar flux:

$$\phi(\vec{r}, \hat{\mathbf{\Omega}}, E, t) = \int_{4\pi} \psi(\vec{r}, \hat{\mathbf{\Omega}}, E, t). \tag{8}$$

The Transport Equation

The TE is merely a balance equation (source - losses):

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(\vec{r},\hat{\mathbf{\Omega}},E,t)+\hat{\mathbf{\Omega}}\cdot\nabla\psi(\vec{r},\hat{\mathbf{\Omega}},E,t) + \sigma(\vec{r},E)\psi(\vec{r},\hat{\mathbf{\Omega}},E,t)=q(\vec{r},\hat{\mathbf{\Omega}},E,t), \quad (9)$$

where $q(\vec{r},\hat{\mathbf{\Omega}},E,t)$ includes both the external and fission source.

We need to impose boundary and initial conditions!

Boundary Conditions

Our Connection to the Real World

- We can impose an incoming angular flux. If no angular flux is imposed, the boundary condition is called vacuum.

$$\psi(\vec{r}, \hat{\mathbf{\Omega}}, E, t) = \tilde{\psi}(\vec{r}, \hat{\mathbf{\Omega}}, E, t), \quad \hat{\mathbf{\Omega}} \cdot \hat{n} < 0, \vec{r} \in \Gamma \tag{10}$$

- Albedo (reflective) boundary condition:

$$\psi(\vec{r}, \hat{\mathbf{\Omega}}, E, t) = \alpha(E)\psi(\vec{r}, \hat{\mathbf{\Omega}}', E, t), \quad \hat{\mathbf{\Omega}} \cdot \hat{n} < 0, \vec{r} \in \Gamma \tag{11}$$

These are usually the two boundary conditions of interest in applications.

- A problem of special interest is the criticality condition.
- Criticality is defined as a system capable of sustaining a chain reaction without an external source indefinitely. That is, there is a time-independent solution to the problem.
- In most circumstances, a problem under consideration is not exactly critical (no time-independent solution exists). We must reformulate the TE into an eigenvalue problem to determine how subcritical or supercritical the problem really is.
- Reformulation of the TE into an eigenvalue problem is not a unique process (that is, we can create an eigenvalue problem any way we want to).

No guarantee non-trivial solution exists for the following problem:

$$\left[\hat{\boldsymbol{\Omega}} \cdot \nabla + \sigma(\vec{r}, E)\right] \psi(\vec{r}, \hat{\boldsymbol{\Omega}}, E) = \int dE' \int d\hat{\boldsymbol{\Omega}}' \sigma_s(\vec{r}, E' \to E, \hat{\boldsymbol{\Omega}}' \cdot \hat{\boldsymbol{\Omega}}) \psi(\vec{r}, E', \hat{\boldsymbol{\Omega}}') + \chi(E) \int dE' \nu \sigma_f(\vec{r}, E') \int d\hat{\boldsymbol{\Omega}}' \psi(\vec{r}, E', \hat{\boldsymbol{\Omega}}'). \quad (12)$$

We "force" a solution to exist. We consider two formulations: the time-absorption and multiplication eigenvalues.

Assume an asymptotic solution to the previous equation in the form

$$\psi(\vec{r}, E, \hat{\boldsymbol{\Omega}}, t) = \psi_\alpha(\vec{r}, E, \hat{\boldsymbol{\Omega}}) e^{\alpha t}. \tag{13}$$

In general there will be a spectrum of eigenvalues for which there are solutions. At long times only nonnegative solutions will predominate, so we can define criticality as

$$\mathrm{Re}\ \alpha_0 \begin{cases} > & 0 \quad \textit{supercritical}, \\ = & 0 \quad \textit{critical}, \\ < & 0 \quad \textit{subcritical}. \end{cases} \tag{14}$$

Assume you can adjust ν, the average number of neutrons emitted in fission, to obtain a time-indenpendent solution. Our fission source term becomes:

$$\frac{\chi(E)}{k} \int dE' \nu \sigma_f(\vec{r}, E') \int d\hat{\boldsymbol{\Omega}}' \psi(\vec{r}, E', \hat{\boldsymbol{\Omega}}'). \tag{15}$$

We can define critcality as

$$k \begin{cases} > & 1 \quad \textit{supercritical}, \\ = & 1 \quad \textit{critical}, \\ < & 1 \quad \textit{subcritical}. \end{cases} \tag{16}$$

Energy Discretization

We divide energy into groups and define a corresponding angular flux in each group,

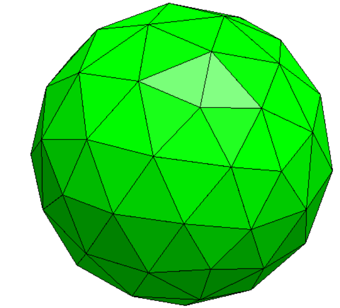
$$\psi_g(\vec{r}, \hat{\mathbf{\Omega}}) = \int_g dE \psi(\vec{r}, \hat{\mathbf{\Omega}}, E), \quad \int_g dE = \int_{E_g}^{E_{g-1}} dE \tag{17}$$

such that

$$\int_0^\infty dE' = \sum_{g'=1}^{G} \int_{g'} dE'. \tag{18}$$

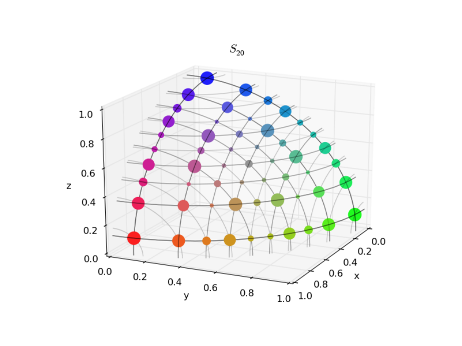
Spatial Discretization

Derivatives are expressed using either the finite difference or finite element method.



Angular Discretization

The Discrete Ordinate Method is used to discretize angle. The TE equation is solved along specific directions and a quadrature formula is used to reconstruct the angular flux.



The Transport Sweep Algorithm

The TE in Matrix Form

We can rewrite the discretized TE equation into matrix form:

$$\mathbf{H}\psi = \chi \mathbf{f}^T \int d\hat{\mathbf{\Omega}}\psi + \mathbf{q}^e \tag{19}$$

and recast it into the multiplication eigenvalue formulation:

$$\mathbf{H}\psi = \frac{1}{k}\chi \mathbf{f}^T \int d\hat{\mathbf{\Omega}}\psi + \mathbf{q}^e. \tag{20}$$

We define a spatial fission neutron distribution source (scalar):

$$F(\vec{r}) = \mathbf{f}(\vec{r})^T \int d\hat{\mathbf{\Omega}}\psi(\vec{r},\hat{\mathbf{\Omega}}). \tag{21}$$

We rewrite our problem in terms of this distribution and doing some manipulations we obtain

$$AF = kF \tag{22}$$

where define the scalar multigroup transport operator as

$$A \equiv \mathbf{f}^T \int d\hat{\mathbf{\Omega}} \mathbf{H}^{-1} \chi. \tag{23}$$

This is an eigenvalue problem that can be solved using the power method.

The Power Method

Updating the Eigenvector and Eigenvalue

We determine the eigenvector and eigenvalue using the power method. Assume the problem has a largest eigenvalue, the eigenvalue is positive, and is real and unique, we can obtain the better approximation to the eigenpair using the relationships

$$F_{i+1} = \frac{1}{k_i} A F_i, \tag{24}$$

and

$$k_{i+1} = k_i \frac{\int dV \ wAF_i}{\int dV \ wF_i}. \tag{25}$$

Suppose there are distinct eigenvalue problem solutions:

$$AF^{\ell} = \kappa_{\ell} F^{\ell}, \quad \ell = 1, 2, \ldots, \tag{26}$$

where

$$k = \kappa_1 > \kappa_2 > \kappa_3 \ldots . \tag{27}$$

Suppose we guess a linear combination of these solutions

$$F_0 = \sum_{\ell} \alpha_{\ell} F^{\ell}. \tag{28}$$

If we continue to apply on operator (algorithm) say n times, we obtain the following:

$$A^n F_0 = \sum_{\ell} \alpha_{\ell} \kappa_{\ell}^n F^{\ell} = \alpha_1 \kappa_1^n F^1 + \sum_{\ell > 1} \alpha_{\ell} \kappa_{\ell}^n F^{\ell}. \tag{29}$$

Factoring out κ_1^n (which is the value we are interested in) we see that

$$A^n F_0 \to \alpha_1 \kappa_1^n F^1, \quad n \to \infty. \tag{30}$$

since $(\kappa_\ell / \kappa_1)^n$ goes to zero as n goes to zero due to the ordering of the eigenvalues.

Inverting H

We Haven't Solved Anything Yet...

Hidden in the math is the fact we need to invert **H**. How do we do this?

- Iteratively! Matrix too large and process too expensive to invert directly.
- Gauss-Seidel, Jacobi, many other methods!
- A lot of research in acceleration (preconditioning) of problems. (Me hopefully)

Comparisons to Monte Carlo

Why Not Just Monte Carlo All the Time?

- Advantages
  - No random error. Error is caused by discretization and data only.
  - Fast! Matrix iterative methods are well studied and are less expensive than random sampling.
  - Give global solutions (solution defined at every point).
  - No need for variance reduction. We are solving an equation and not simulating particles.
- Disadvantage
  - Difficult to represent complex geometries (requires mesh).
  - Energy treatment is not continuous (multigroup approximation).
  - Difficult to parallelize (can be done however).