ORF526 - Problem Set 7

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Question 1

1. It is easy to check that C is symmetric and that:

$$C(s,t) = \begin{cases} \min(|s|,|t|) & \text{if } ts \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Let $(t_i)_1^n \in \mathbb{R}$, and $f_i = 1_{(0,t_i)}$ where $(0,t) = (t,0)$ if $t < 0$, then: $C(t_i,t_j) = \int_R f_i f_j$ which is a scalar product in L_2. C is definite positive semi-definite as a conclusion.

2. $C(t,s) = \min(t,s)$ when $t,s \geq 0$.

3.
   - $Var(B_0) = 0$ so $B_0 = 0$ as.
   - $B_t - B_s$ is normal because B_t is a guaussian process. $E[B_t - B_s] = 0$, and $Var(B_t - B_s) = Var(B_t) + Var(B_s) + 2Cov(B_t, B_s) = |t| + |s| + 2C(t,s) = |t-s|$
   - $Cov(B_t - B_s, B_u - B_v) = C(t,u) + C(s,v) - C(s,u) - C(t,v) = \frac{1}{2}(|t| + |u| - |t-u| + |s| + |v| - |s-v| - |s| - |u| + |s-u| - |t| - |v| + |t-v| = \frac{1}{2}(u - t + v - s + u - s + v - t) = 0$, and since the 2d process $B_t - B_s, B_u - B_v$ is guassian, its compenonets are independent.

Question 2

Let's call C_1 the function C definited on quesiton 1.

1. $C(u,v) = C_1(u_1,v_1)C_1(u_2,v_2) = \int_{R^2} 1_{(0,u_1)}(x)1_{(0,v_1)}(x)1_{(0,u_2)}(y)1_{(0,v_2)}(y)dxdy = \int_{R^2} 1_{(0,u_1)\times(0,u_2)}1_{(0,v_1)\times(0,v_2)} = < 1_{(0,u_1)\times(0,u_2)}, 1_{(0,v_1)\times(0,v_2)} >$

   So C is positive semi-definite.

2. $C(u,v) = \min(u_1,v_1)\min(u_2,v_2)$ when $u,v \geq 0$

3. if one compononet of u is 0, then $Var(X_u) = C(u_2,u_2)C(u_1,u_1) = 0$, ie $X_u = 0$ as.

4. $B_t = X_{(t,1)}$ is a guaussian process. $E[B_t] = 0$ and $Cov(B_t, B_s) = Cov(X_{(t,1)}, X_{(s,1)}) = C_1(t,s)$, so B_t is a two sided browninan motion

5. $Var(X_{(t,t)}) = C_1(t,t)^2 = |t|^2$

Question 3

Let's first show the following lemma: For every $X \in L_1$, there exist a sequence of simple function Z_n bounded by $|X|$ and converging to X. To show that, we write $X = X^+ - X^-$, and let Z_n^+ (resp. Z_n^-) a sequence of positive simple functions converging to X^+ (resp. X^-) from below (resp. above). And set $Z_n = Z_n^+ - Z_n^-$, which verifies the lemma.

1. X is $\mathcal{G}$ mesurable, and trivially verifies the definition of conditional probability, so $E[X|\mathcal{G}] = X$
2. $aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ is $\mathcal{G}$-measurable as sum of two functions that are $\mathcal{G}$-measurable, and if $A \in \mathcal{G}$:

$$\begin{aligned} E[(aE[X|\mathcal{G}] + bE[Y|\mathcal{G}])1_A] &= aE[E[X|\mathcal{G}]1_A] + bE[E[Y|\mathcal{G}]1_A] \\ &= aE[X1_A] + bE[Y1_A] && \text{because } A \text{ is } \mathcal{G}\text{-measurable} \\ &= E[(aX + bY)1_A] \end{aligned}$$

so $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$.

3. $E[X|\mathcal{G}] - E[Y|\mathcal{G}] = E[X - Y|\mathcal{G}]$ Let $H := E[X - Y|\mathcal{G}]$, and $A := \{H \leq 0\}$. A is $\mathcal{G}$-measurable and by positivity of the expectation: $0 \geq E[H1_A] = E[(X - Y)1_A] \geq 0$.

   Since $-H1_A \leq 0$ a.s and its expectation is 0, $H1_A = H^- = 0$ as, and therefore $H \geq 0$ as.

4. For $A \in H \subseteq \mathcal{G}$, $E[E[X|\mathcal{G}]|H]$ is H-measurable and :

$$\begin{aligned} E\left[1_A E[E[X|\mathcal{G}]|H]\right] &= E[1_A E[X|\mathcal{G}]] \\ &= E[1_A X] \end{aligned}$$

5. Let $A \in \mathcal{G}$, and prove that $E[1_A Y E[X|\mathcal{G}]] = E[1_A XY]$ If we denote $Z := 1_A Y$, this is equivalent to $E[ZE[X|\mathcal{G}]] = E[ZX]$.

   Z is $\mathcal{G}$-measurable and $|ZX| \leq |YX| \in L_1$

   - If Z is a simple function $\sum_{i=0..n} \alpha_i 1_{A_i}$, where $A_i \in \mathcal{G}$ for $i = 0..n$, then by linearity of the expectation:
   $$E[ZE[X|\mathcal{G}]] = \sum_i \alpha_i E[1_{A_i} E[X|\mathcal{G}]] = \sum_i \alpha_i E[1_{A_i} X] = E[ZX]$$
   - If X and Y are non-negative, Let Z_n be a sequence of non-negative simple $\mathcal{G}$-measurable functions s.t. $Z_n \uparrow Z$ and therefore $|Z_n X| \leq |ZX| \in L_1$. By monotnous convergence theorem:
   $$E[ZE[X|\mathcal{G}]] = \lim E[Z_n E[X|\mathcal{G}]] = \lim E[Z_n X] = E[ZX]$$
   - X now can be in L_1.
     We use h), to show that $|E[X|\mathcal{G}]| < E[|X||\mathcal{G}]$. (take $\phi : x \to |x|$)
     Let Z_n a sequence of simple functions converging to Z and bounded by $|Z|$. Then $|Z_n X| \leq |ZX| \in L_1$ and $|Z_n E[X|\mathcal{G}]| = |E[Z_n X|\mathcal{G}]| \leq E[|XZ||\mathcal{G}] \in L_1$ because $EE[|XZ||\mathcal{G}] = E[|XZ|] < \infty$.
     By dominated convergence theorem:
   $$E[ZE[X|\mathcal{G}]] = \lim E[Z_n E[X|\mathcal{G}]] = \lim E[Z_n X] = E[ZX]$$
   - If $Y \in L_1$, $Z = Z^+ - Z^-$, and by linearity
     $E[ZE[X|\mathcal{G}]] = E[Z^+ E[X|\mathcal{G}]] - E[Z^- E[X|\mathcal{G}]] = EE[XZ^+|\mathcal{G}] - EE[Z^- X|\mathcal{G}] = E[XZ^+] - E[XZ^-] = E[XZ]$

6. Let's first prove that if $A \in \mathcal{G}$, $E[X1_A] = E[X]E[1_A]$.
   (a) If X is an indicator function, then it follows from the definition of independence
   (b) If X is a simple function it follows from the linearity of the expectation.

(c) If Z_n a sequence of simple functions converging to X and uniformly bounded by an $|X|$, then by CVD:

$$E[X1_A] = \lim E[Z_n 1_A] = \lim E[Z_n]E[1_A] = \lim E[X]E[1_A]$$

So now we have:

$$E[1_A X] = E[1_A]E[X] = E[1_A E[X]]$$

$E[X]$ is a constant, so $\mathcal{G}$-measurable.

7.

$$E[X1_\emptyset] = 0 = E[X]E[1_\emptyset]$$

$$E[X1_\Omega] = E[X] = E[X]E[1_\Omega]$$

so X is independent of $\mathcal{G}$, and therefore $E[X|\mathcal{G}] = E[X]$.

8. If φ is affine $= ax + b$, then by linearity $E[\varphi(X)|\mathcal{G}] = \varphi(E[X|\mathcal{G}])$

If φ is convex not linear, we can write $\varphi = \sup \varphi_n = \sup_n a_n x + b_n$ where $a_n, b_n \in R$, then $\forall n\ E[\varphi(X)|\mathcal{G}] \geq E[\varphi_n(X)|\mathcal{G}] \geq \varphi_n(E[X|\mathcal{G}])$ as. Let Ω_n the set where this equality holds, so on $\Omega' := \cap_n \Omega_n$ we have that:

$$E[\varphi(X)|\mathcal{G}] \geq \sup_n \phi_n(E[X|\mathcal{G}]) = \varphi(E[X|\mathcal{G}]) \text{ on } \Omega'$$

and $P(\Omega') = 1 - P(\cup_n \Omega_n^c) \geq 1 - \sum_n P(\Omega_n^c) \geq 1$

Question 4

- $E[X_n|Y]$ is non-decreasing, let's call $L := \lim E[X_n|Y]$, and prove that $L = E[X|\mathcal{G}]$.

  Since $Y \leq X_n \uparrow X$, $Y \wedge n \leq X \wedge n \uparrow X$ and $E[Y|\mathcal{G}] \leq E[X_n|\mathcal{G}] \uparrow L$, by monotonous convergence theorem, for all $A \in \mathcal{G}$:

$$\begin{aligned} E[1_A L] &= \lim_n E[1_A E[X_n|\mathcal{G}]] \\ &= \lim_n E[1_A X_n] \\ &= E[1_A X] \\ &= \lim E[1_A (X \wedge k)] \\ &= \lim E[1_A E[X \wedge k]] \\ &= E[1_A E[X|\mathcal{G}]] \end{aligned}$$

  Let $B \in B(R)$, and for $a > 0$, $A = B \cap \{|L| < a, |E[X|\mathcal{G}]| < a\}$. And now we have:

$$0 = E[1_A(L - E[X|\mathcal{G}])] = E[1_B(L - E[X|\mathcal{G}])1_{|L|<a, |E[X|\mathcal{G}]|<a}]$$

  By taking B to be the set where $L - E[X|\mathcal{G}] > 0$ and then $L - E[X|\mathcal{G}] < 0$, we have that $(L - E[X|\mathcal{G}])1_{|L|<a,|E[X|\mathcal{G}]|<a} = 0$, and by taking a to ∞, we have that $L = E[X|\mathcal{G}]$.

- Let's define $L_k := \inf_{n \geq k} X_n \leq X_k$, so that

$$E[L_k|\mathcal{G}] \leq E[X_k|\mathcal{G}] \tag{1}$$

  But $Y \leq L_k \uparrow \liminf_n X_n$, by a) $E[L_k|\mathcal{G}] \uparrow_k E[\liminf_n X_n|\mathcal{G}]$, and by taking the $\liminf$ in the inequality 1 we have the result.

- X_n and $-X_n$ verify the conditions of the last quesiton, so:

$$\liminf E[-X_n|\mathcal{G}] \geq E[\liminf -X_n|\mathcal{G}] \Rightarrow \limsup E[X_n|\mathcal{G}] \leq E[\limsup X_n|\mathcal{G}]$$

$$\liminf E[X_n|\mathcal{G}] \geq E[\liminf X_n|\mathcal{G}]$$

as a result

$$E[\limsup X_n|\mathcal{G}] \geq \limsup E[X_n|\mathcal{G}] \geq \liminf E[X_n|\mathcal{G}] \geq E[\liminf X_n|\mathcal{G}]$$

Since $\limsup X_n = \liminf X_n = X$, we have the result.

Question 5

- $y \to p(y, A)$ is measurable because:

  1. $(x, y) \to f(x, y)$ is measurable since it is a density
  2. $N(y)$ is measurable by Fubini
  3. $y \to \int \frac{f(x,y)}{N(y)} 1_{0<N(y)<\infty} + (1 - 1_{0<N(y)<\infty})\varphi(x)$ is also measurable by Fubini

  $p(Y, A)$ is then $\sigma(Y)$-measurable.

- We have that:

$$N(y)f_y(x) = \begin{cases} N(y)\phi(x) = 0 & \text{if } N(y) = 0 \\ N(y)\phi(x) & \text{if } N(y) = \infty \\ f(x, y) & \text{otherwise} \end{cases}$$

  But since $N \in L_1$ (By fubini), the set $\{N = \infty\}$ is of measure 0, so $N(y)f_y(x) = 1_{N(y)\neq 0}f(x, y)$ a.s.
  Let $B \in B(R)$, all function integrated below are non negative, so:

$$\begin{aligned}
E[p(Y, A)1_{Y\in B}] &= \int_{R^2} p(y, A)1_{y\in B}f(x, y)dxdy \\
&= \int p(y, A)1_{y\in B}N(y)dy && \text{By Tonnelli} \\
&= \int_y 1_{y\in B} \int_x 1_{x\in A}N(y)f_y(x)dxdy \\
&= \int 1_{y\in B}1_{x\in A}1_{N(y)\neq 0}f(x, y)dxdy \\
&= \int 1_{y\in B}1_{x\in A}f(x, y)dxdy && \text{because if } N(y) = 0 \text{ then } \int_A f(x, y) = 0 \\
&= E[1_{Y\in B}1_{X\in A}] \\
&= E[1_{Y\in B}P(X \in A|Y)]
\end{aligned}$$

  Which prove the result.