ORF526 - Problem Set 7

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Question 1

1. It is easy to check that C is symmetric and that:

$$C(s,t) = \begin{cases} \min(|s|,|t|) & \text{if } ts \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Let $(t_i)_1^n \in \mathbb{R}$, and $f_i = 1_{(0,t_i)}$ where $(0,t) = (t,0)$ if $t < 0$, then: $C(t_i,t_j) = \int_R f_i f_j$ which is a scalar product in L_2. C is definite positive semi-definite as a conclusion.

2. $C(t,s) = \min(t,s)$ when $t, s \geq 0$.

3.
   - $Var(B_0) = 0$ so $B_0 = 0$ as.
   - $B_t - B_s$ is normal because B_t is a guaussian process. $E[B_t - B_s] = 0$, and $Var(B_t - B_s) = Var(B_t) + Var(B_s) + 2Cov(B_t, B_s) = |t| + |s| + 2C(t,s) = |t-s|$
   - $Cov(B_t - B_s, B_u - B_v) = C(t,u) + C(s,v) - C(s,u) - C(t,v) = \frac{1}{2}(|t| + |u| - |t-u| + |s| + |v| - |s-v| - |s| - |u| + |s-u| - |t| - |v| + |t-v| = \frac{1}{2}(u - t + v - s + u - s + v - t) = 0$, and since the 2d process $B_t - B_s, B_u - B_v$ is guassian, its compenonets are independent.

Question 2

Let's call C_1 the function C definited on quesiton 1.

1. $C(u,v) = C_1(u_1,v_1)C_1(u_2,v_2) = \int_{R^2} 1_{(0,u_1)}(x)1_{(0,v_1)}(x)1_{(0,u_2)}(y)1_{(0,v_2)}(y)dxdy = \int_{R^2} 1_{(0,u_1)\times(0,u_2)}1_{(0,v_1)\times(0,v_2)} = < 1_{(0,u_1)\times(0,u_2)}, 1_{(0,v_1)\times(0,v_2)} >$

   So C is positive semi-definite.

2. $C(u,v) = \min(u_1,v_1)\min(u_2,v_2)$ when $u, v \geq 0$

3. if one compononet of u is 0, then $Var(X_u) = C(u_2,u_2)C(u_1,u_1) = 0$, ie $X_u = 0$ as.

4. $B_t = X_{(t,1)}$ is a guaussian process. $E[B_t] = 0$ and $Cov(B_t, B_s) = Cov(X_{(t,1)}, X_{(s,1)}) = C_1(t,s)$, so B_t is a two sided browninan motion

5. $Var(X_{(t,t)}) = C_1(t,t)^2 = |t|^2$

Question 3

Let's first show the following lemma: For every $X \in L_1$, there exist a sequence of simple function Z_n bounded by $|X|$ and converging to X. To show that, we write $X = X^+ - X^-$, and let Z_n^+ (resp. Z_n^-) a sequence of positive simple functions converging to X^+ (resp. X^-) from below (resp. above). And set $Z_n = Z_n^+ - Z_n^-$, which verifies the lemma.

1. X is G mesurable, and trivially verifies the definition of conditional probability, so $E[X|G] = X$

2. $aE[X|G] + bE[Y|G]$ is G-measurable as sum of two functions that are G-measurable, and if $A \in G$:

$$\begin{aligned} E[(aE[X|G] + bE[Y|G])1_A] &= aE[E[X|G]1_A] + bE[E[Y|G])1_A] \\ &= aE[E[X1_A|G]] + bE[E[Y1_A|G]] \qquad \text{because } A \text{ is } G\text{-measurable} \\ &= aE[X1_A] + bE[Y1_A] \\ &= E[(aX + bY)1_A] \end{aligned}$$

so $E[aX + bY|G] = aE[X|G] + bE[Y|G]$.

3. $E[X|G] - E[Y|G] = E[X - Y|G]$ Let $H := E[X - Y|G]$, and $A := \{H \leq 0\}$. A is G-measurable and by positivity of the expectency: $0 \geq E[H1_A] = E[(X - Y)1_A] \geq 0$.

   Since $-H1_A \leq 0$ a.s and its expectency is 0, $H1_A = H^- = 0$ as, and therefore $H \geq 0$ as.

4. For $A \in H \subseteq G$, $E[E[X|G]|H]$ is H-measurable and :

$$\begin{aligned} E\left[1_A E[E[X|G]|H]\right] &= E[1_A E[X|G]] \\ &= E[1_A X] \end{aligned}$$

5. Let $A \in G$, and prove that $E[1_A Y E[X|G]] = E[1_A XY]$ If we denote $Z := 1_A Y$, this is equivalent to $E[ZE[X|G]] = E[ZX]$.

   Z is G-measurable and $|ZX| \leq |YX| \in L_1$

   - If Z is a simple function $\sum_{i=0..n} \alpha_i 1_{A_i}$, where $A_i \in G$ for $i = 0..n$, then by linearity of the expectation:
   $$E[ZE[X|G]] = \sum_i \alpha_i E[1_{A_i} E[X|G]] = \sum_i \alpha_i E[1_{A_i} X] = E[ZX]$$
   - If X and Y are non-negative, Let Z_n be a sequence of non-negative simple G-measurable functions s.t. $Z_n \uparrow Z$ and therefore $|Z_n X| \leq |ZX| \in L_1$. By monotnous convergence theorem:
   $$E[ZE[X|G]] = \lim E[Z_n E[X|G]] = \lim E[Z_n X] = E[ZX]$$
   - X now can be in L_1.
   We use h), to show that $|E[X|G]| < E[|X||G]$. (take $\phi : x \to |x|$)
   Let Z_n a sequence of simple functions converging to Z and bounded by $|Z|$. Then $|Z_n X| \leq |ZX| \in L_1$ and $|Z_n E[X|G]| = |E[Z_n X|G]| \leq E[|XZ||G] \in L_1$ because $EE[|XZ||G] = E[|XZ|] < \infty$.
   By dominated convergence theorem:
   $$E[ZE[X|G]] = \lim E[Z_n E[X|G]] = \lim E[Z_n X] = E[ZX]$$
   - If $Y \in L_1$, $Z = Z^+ - Z^-$, and by linearity
   $$E[ZE[X|G]] = E[Z^+ E[X|G]] - E[Z^- E[X|G]] = EE[XZ^+|G] - EE[Z^- X|G] = E[XZ^+] - E[XZ^-] = E[X$$

6. Let's first prove that if $A \in G$, $E[X1_A] = E[X]E[1_A]$.

   (a) If X is an indicator function, then it follows from the definition of independence

   (b) If X is a simple function it follows from the linearity of the expectation.

(c) If Z_n a sequence of simple functions converging to X and uniformly bounded by an $|X|$, then by CVD:

$$E[X1_A] = \lim E[Z_n 1_A] = \lim E[Z_n]E[1_A] = \lim E[X]E[1_A]$$

So now we have:

$$E[1_A X] = E[1_A]E[X] = E[1_A E[X]]$$

$E[X]$ is a constant, so G-measurable.

7.

$$E[X1_\emptyset] = 0 = E[X]E[1_\emptyset]$$
$$E[X1_\Omega] = E[X] = E[X]E[1_\Omega]$$

so X is independent of G, and therefore $E[X|G] = E[X]$.

8. If φ is affine $= ax + b$, then by linearity $E[\varphi(X)|G] = \varphi(E[X|G])$

If φ is convex not linear, we can write $\varphi = \sup \varphi_n = \sup_n a_n x + b_n$ where $a_n, b_n \in R$, then $\forall n\, E[\varphi(X)|G] \geq E[\varphi_n(X)|G] \geq \varphi_n(E[X|G])$ as. Let Ω_n the set where this equality holds, so on $\Omega' := \cap_n \Omega_n$ we have that:

$$E[\varphi(X)|G] \geq \sup_n \phi_n(E[X|G]) = \varphi(E[X|G]) \text{ on } \Omega'$$

and $P(\Omega') = 1 - P(\cup_n \Omega_n^c) \geq 1 - \sum_n P(\Omega_n^c) \geq 1$

Question 4

- $E[X_n|Y]$ is non-decreasing, let's call $L := \lim E[X_n|Y]$, and prove that $L = E[X|G]$.

  Since $Y \leq X_n \uparrow X$, $Y \wedge n \leq X \wedge n \uparrow X$ and $E[Y|G] \leq E[X_n|G] \uparrow L$, by monotonous convergence theorem, for all $A \in G$:

$$\begin{aligned} E[1_A L] &= \lim_n E[1_A E[X_n|G]] \\ &= \lim_n E[1_A X_n] \\ &= E[1_A X] \\ &= \lim E[1_A (X \wedge k)] \\ &= \lim E[1_A E[X \wedge k]] \\ &= E[1_A E[X|G]] \end{aligned}$$

  Let's note $H := L - E[X|G]$ which is G-measurable because L and $E[X|G]$ are both G-measurable, and we have $E[1_{H<0} H] = 0$, so $H = 0$ as, ie $L = E[X|G]$

- Let's define $L_k := \inf_{n \geq k} X_n \leq X_k$, so that

$$E[L_k|G] \leq E[X_k|G] \tag{1}$$

  But $Y \leq L_k \uparrow \liminf_n X_n$, by a) $E[L_k|G] \uparrow_k E[\liminf_n X_n]$, and by taking the $\liminf$ in the inequality 1 we have the result.

- X_n and $-X_n$ verify the conditions of the last quesiton, so:

$$\liminf E[-X_n|G] \geq E[\liminf -X_n|G] \Rightarrow \limsup E[X_n|G] \leq E[\limsup X_n|G]$$

$$\liminf E[X_n|G] \geq E[\liminf X_n|G]$$

as a result

$$E[\limsup X_n|G] \geq \limsup E[X_n|G] \geq \liminf E[X_n|G] \geq E[\liminf X_n|G]$$

Since $\limsup X_n = \liminf X_n = X$, we have the result.

Question 5

- $y \to p(y, A)$ is measurable because:

  1. $(x, y) \to f(x, y)$ is measurable since it is a density
  2. $N(y)$ is measurable by Fubini
  3. $y \to \int \frac{f(x,y)}{N(y)} 1_{0<N(y)<\infty} + (1 - 1_{0<N(y)<\infty})\varphi(x)$ is also measurable by Fubini

  $p(Y, A)$ is then $\sigma(Y)$-measurable.

- We have that:

$$N(y)f_y(x) = \begin{cases} \phi(x) & \text{if } N(y) = 0 \\ \phi(x) & \text{if } N(y) = \infty \\ f(x,y) & \text{otherwise} \end{cases}$$

  But since $N \in L_1$, the set $\{N = \infty\}$ is of measure 0, so $N(y)f_y(x) = 1_{N(y)\neq 0}f(x,y) + 1_{N(y)=0}\phi(x)$ a.s.
  Let $B \in B(R)$, all function integrated below are non negative, so:

$$\begin{aligned} E[p(Y,A)1_{Y\in B}] &= \int_{R^2} p(y,A)1_{y\in B}f(x,y)dxdy \\ &= \int p(y,A)1_{y\in B}N(y)dy && \text{By Tonnelli} \\ &= \int_y 1_{y\in B}\int_x 1_{x\in A}N(y)f_y(x)dxdy \\ &= \int 1_{y\in B}1_{x\in A}1_{N(y)\neq 0}f(x,y)dxdy \\ &= \int 1_{y\in B}1_{x\in A}f(x,y)dxdy && \text{because if } N(y)=0 \text{ then } \int_A f(x,y) = 0 \\ &= E[1_{Y\in B}1_{X\in A}] \end{aligned}$$