

ORF526 - Problem Set 1

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Question 1

Let X and Y be the result of two independent coin tosses, and let

$$\begin{aligned}A_1 &= \{X = H\} \\A_2 &= \{Y = H\} \\A_3 &= \{X = Y\}\end{aligned}$$

Question 2

$$\begin{aligned}\mathbb{E}[X] &:= \sum_{n=1}^N X(\omega_n) p_n \\&= \sum_{n=1}^N [\operatorname{Re}(X)(\omega_n) + i \operatorname{Im}(X)(\omega_n)] p_n \\&= \sum_{n=1}^N \operatorname{Re}(X)(\omega_n) p_n + i \sum_{n=1}^N \operatorname{Im}(X)(\omega_n) p_n \\&= \mathbb{E}[\operatorname{Re}(X)] + i \mathbb{E}[\operatorname{Im}(X)]\end{aligned}$$

Question 3

$$(i) \Rightarrow (ii)$$

$$\begin{aligned}\mathbb{E}[f_1(X_1) \dots f_M(X_M)] &= \sum_{x_1, \dots, x_M} f_1(x_1) \dots f_M(x_M) \mathbb{P}(X_1 = x_1, \dots, X_M = x_M) \\&= \sum_{x_1, \dots, x_M} f_1(x_1) \dots f_M(x_M) \mathbb{P}(X_1 = x_1) \dots \mathbb{P}(X_M = x_M) && \text{(because of (i))} \\&= \sum_{x_1} f_1(x_1) \mathbb{P}(X_1 = x_1) \dots \sum_{x_M} f_M(x_M) \mathbb{P}(X_M = x_M) \\&= \mathbb{E}[f_1(X_1)] \dots \mathbb{E}[f_M(X_M)]\end{aligned}$$

$$(ii) \Rightarrow (iii)$$

Take $f_i(x) = e^{iu_i x}$

$$(iii) \Rightarrow (i)$$

By linearity we prove that the equality holds for polynomials of complex exponentials of the random variables too.

Let $\{x_1, \dots, x_n\}$ be the elements of Ω , and

$$f : \mathbb{C}^{n-1}[X] \longrightarrow \mathbb{C}^n$$

$$P \longrightarrow (P(e^{i\frac{x_i}{n}}))_i$$

where n is large enough so that $e^{i\frac{x_i}{n}}$ are all different.

f is linear and injective (two polynomials of degree $< n$ who agree on n points are equal), it is then a bijection (because $\dim(\mathbb{C}^{n-1}[X]) = \dim(\mathbb{C}^n)$). As a consequence, for each indicator function of the form 1_{x_i} there exists a polynomial $P_i(e^{i\frac{u}{n}}) = 1_{u=x_i}$, ie (i) is verified.

Question 4

Immediate using definition (ii)

Question 5

a) if X and Y are independent, then

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)] = \mathbb{E}[X - \mathbb{E}[X]]\mathbb{E}[Y - \mathbb{E}[Y]] = 0$$

b) Let X and ϵ be two independent uniform variables on $\{-1, 0, 1\}$ and $\{-1, 1\}$ respectively, then $\text{cov}(X, \epsilon X) = \mathbb{E}[\epsilon X^2] = \mathbb{E}[\epsilon]\mathbb{E}[X^2] = 0$, but $\mathbb{P}(X = 1, \epsilon X = 0) = 0 \neq \mathbb{P}(X = 1)\mathbb{P}(\epsilon X = 1) = \frac{1}{9}$

Question 6

A vector space $(V, +, \cdot, \mathbb{K})$ over a field \mathbb{K} verifies

For all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{K}$, then $u + v \in V$, $\lambda u \in V$ and

- $(V, +)$ is an Abelian group
- $\lambda(\mu u) = (\lambda\mu)u$.
- $(\lambda + \mu)u = \lambda u + \mu u$.
- $\lambda(u + v) = \lambda u + \lambda v$.
- $1u = u$.

examples: R^n , space of continuous functions from R to R , space of square matrices of dimension n ...

Question 7

a) By using symmetry, bilinearity and then symmetry

b) When $y = 0$ it is trivial. When $y \neq 0$ and $\lambda = \frac{\langle x, y \rangle}{\|y\|^2}$, $0 \leq \langle x - \lambda y, x - \lambda y \rangle = \frac{\|x\|^2\|y\|^2 - \langle x, y \rangle^2}{\|y\|^2}$

- c)
- Positive homogeneity is a result of Bilinearity.
 - Triangle inequality can be obtained by squaring both sides of the inequality and applying Cauchy-Schwartz.
 - Positive definiteness of the norm is a direct consequence of the Positive definiteness of the scalar product.

d) We can assume that X and Y are centred (adding a constant doesn't change the cov or var). Since the mapping $(X, Y) \longrightarrow \text{cov}(X, Y)$ is scalar product in the space of centered random variables on the finite probability space Ω , we then apply Cauchy-Schwarz.