ORF526 - Problem Set 4

Bachir EL KHADIR

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Question 1

1. Let $\mathbb{F}$ be a field which is either $\mathbb{R}$ or $\mathbb{C}$. A normed vector space over $\mathbb{F}$ is a pair $(V, ||\cdot||)$ where V is a vector space over $\mathbb{F}$ and $||\cdot||\colon V \to \mathbb{R}$ is a function such that
   (a) $||v|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$ in V (*positive definiteness*)
   (b) $||\lambda v|| = |\lambda| ||v||$
   for all $v \in V$ and all $\lambda \in \mathbb{F}$
   (c) $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in V$ (the *triangle inequality*)
2. Inner product space
3. A metric space M is called complete if every Cauchy sequence of points in M has a limit that is also in M or, alternatively, if every Cauchy sequence in M converges in M.
4. A Banach space is a vector space X over the field R of real numbers, or over the field C of complex numbers, which is equipped with a norm and which is complete with respect to that norm.
5. A Hilbert space is a vector space H with an inner product $< f, g >$ such that the norm defined by $||f|| = \sqrt{< f, f >}$ turns H into a complete metric space. If the metric defined by the norm is not complete, then H is instead known as an inner product space.

Question 2

- $$(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus \Big((-\infty, b_1] \times (-\infty, a_2] \cup (-\infty, a_1] \times (-\infty, b_2]\Big)$$

$$\begin{aligned}
\mu(a_1, b_1] \times (a_2, b_2] &= \mu(-\infty, b_1] \times (-\infty, b_2] - \mu\Big((-\infty, b_1] \times (-\infty, a_2] \cup (-\infty, a_1] \times (-\infty, b_2]\Big) \\
&= \mu(-\infty, b_1] \times (-\infty, b_2] - \mu(-\infty, b_1] \times (-\infty, a_2] - \mu((-\infty, a_1] \times (-\infty, b_2]) \\
&+ \mu\Big((-\infty, b_1] \times (-\infty, a_2] \cap (-\infty, a_1] \times (-\infty, b_2]\Big) \\
&= F(b_1, b_2) - F(b_1, a_2) - F(b_2, a_1) + F(a_1, a_2)
\end{aligned}$$

- The following intersection is decreasing:

$$(-\infty, x_1] \times (-\infty, x_2] = \bigcap_{k \in \mathbb{N}} (-\infty, x_1^k] \times (-\infty, x_1^k]$$

By continuity from above $F(x_k) \to F(x)$

- $$\mathbb{R} = \underset{k\in\mathbb{N}}{\cup}(-\infty, x_1^k] \times (-\infty, x_2^k]$$

  The union is increasing, by continuity from below we have the equality.

- $(-\infty, x_1] \times (-\infty, x_2] \subseteq (-\infty, y_1] \times (-\infty, x_2]$ so $F(x_1, x_2) \leq F(y_1, x_2)$

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Question 3

Let's write f and g as: $f = \sum_i a_i 1_{A_i}$, $g = \sum_k b_k 1_{B_k}$

$$\int (f+g) = \sum_i a_i \mu(A_i) + \sum_k \mu(B_k) = \int f + \int g$$

Question 4

- If $f = \sum a_i 1_{A_i}$ a simple function, then $cf = \sum (ca_i) 1_{A_I}$, $\int cf = \sum ca_i \mu(A_i) = c \sum a_i \mu(A_i) = c \int f$.

  If f_n a sequence of increasing simple function converging to f, then (cf_n) is an monotonous sequence converging to cf, and therefore by monotnous convergence, $\int cf = \lim \int cf_n = c \lim \int f_n = c \int f$.

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Question 5

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  - $\mu_f(\emptyset) = \mu f^{-1}(\emptyset) = \mu\emptyset = 0$
  - $\mu_f(B^c) = \mu f^{-1}(B^c) = \mu(f^{-1}B)^c = 1 - \mu(f^{-1}B) = 1 - \mu_f(B)$
  - if $\{B_k | k \in \mathbb{N}\}$ a set of pairwise disjoint sets, so is$\{f^{-1}B_k | k \in \mathbb{N}\}$ and therefore

    $$\mu_f(\underset{k}{\cup} B_k) = ...$$

- If g is simple, eg $g = \sum a_i 1_{A_i}$: $gof = \sum a_i 1_{f^{-1}(A_i)}$

  $\int_\Omega gof \mathrm{d}\mu = \sum_\mathrm{i} \mathrm{a_i} \mu(\mathrm{f^{-1}A_i}) = \sum_\mathrm{i} \mathrm{a_i} \mu_\mathrm{f}(\mathrm{A_i}) = \int_\mathrm{E} \mathrm{gd}\mu_\mathrm{f}$

  If $g_n \to g$, $g_n of \to gof$ so:

  $$\int gof = \lim \int g_n of = \lim \int g_n \mathrm{d}\mu_\mathrm{f} = \int \mathrm{gd}\mu_\mathrm{f}$$

Question 6