Problem set 5, ORF523

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<2016-04-02 Sat>

1 Problem 1

Notation $E_{ij} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})_{k,l}$ the matrix with all 0 except in (i,j) and (j,i)

$$\begin{aligned} -\nu(G) = \min_X \quad & Tr(X(-J)) \\ \text{subject to} \quad & X \geq 0 \\ & Tr(XI_n) = 1 && (:\alpha) \\ & Tr(E_{ij}X) = 0 \ \forall (i,j) \in E, i < j && (:\lambda_{ij}) \end{aligned}$$

has for dual:

$$\begin{aligned} \max_{\alpha, \lambda_{ij} \in \mathbb{R}} \quad & \alpha \\ \text{subject to} \quad & \alpha I + \sum_{(i,j)\in E, i<j} \lambda_{ij} E_{ij} \leq -J \end{aligned}$$

Both are strictly feasible:

- for the primal, take $X = \frac{I_n}{n}$
- For the dual, take $\alpha = -2$, $\lambda_{ij} = 0$

Which proves that the dual and primal are equal. Taking $\beta = -\alpha$, we can write that as:

$$\begin{aligned} \nu(G) = \min_{\alpha, \lambda_{ij} \in \mathbb{R}} \quad & \beta \\ \text{subject to} \quad & -\beta I + \sum_{(i,j)\in E, i<j} \lambda_{ij} E_{ij} \leq -J \end{aligned}$$

Note that the $(1,1)$ entry of $-\beta I + \sum_{(i,j)\in E} \lambda_{ij} E_{ij} + J$: $1 - \beta$ shoud be negative, so we can ammend to the constraints that $\beta \geq 1$

$$\begin{aligned} -\beta I + \sum_{(i,j)\in E, i<j} \lambda_{ij} E_{ij} \leq -J \iff & \beta(I - \sum_{(i,j)\in E}, i<j \frac{\lambda_{ij}}{\beta} E_{ij}) \succeq J \\ \iff & I - \sum_{(i,j)\in E, i<j} \frac{\lambda_{ij}}{\beta} E_{ij} \succeq \frac{1}{\beta} 11^T \\ \iff & \begin{pmatrix} I - \sum_{(i,j)\in E, i<j} \frac{\lambda_{ij}}{\beta} E_{ij} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \begin{matrix} 1 & \dots & 1 \end{matrix} & \beta \end{pmatrix} \succeq 0 \quad \text{(By Schur Lemma bc } \beta > 0) \end{aligned}$$

Let's note this big matrix Z. It is clear that a matrix $Z \in S^{(n+1)}$ is of this form iff it verifies the constraints of the following optimization problem:

$$\begin{array}{ll} \text{min} & Z_{n+1,n+1} \\ \text{subject to} & Z \succeq 0 \\ & Z_{i,n+1} = Z_{ii} = 1 \\ & Z_{i,j} = 0 \forall \{i,j\} \in \bar{E} \end{array}$$

And this quantity is then equal to $\vartheta(G)$.
Let's now prove the inequality (2).
Let $C = \chi(\bar{G})$.
By definition, there exist a partition of V: $\{V_1, \dots, V_C\}$ such that V_i is a clique for all $i \leq C$

- Define $1_{V_i} \in \mathbb{R}^n$ to be the indicator function of the set V_i, and note that $1 = \sum_{i \leq C} V_i$
- Define $z_i = \begin{pmatrix} 1_{V_i} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$. Note that:

$$z_i z_i^T = \begin{pmatrix} 1_{V_i} 1_{V_i}^T & 1_{V_i} \\ 1_{V_i}^T & 1 \end{pmatrix}$$

- Define

$$Z = \sum_i z_i z_i^T = \begin{pmatrix} \sum 1_{V_i} 1_{V_i}^T & 1 \\ 1^T & C \end{pmatrix}$$

  Z is positive semidefinite because it is a sum of psd terms $z_i z_i^T$
- $(1_{V_i} 1_{V_i}^T)_{kl} = (e_k^T 1_{V_i})(e_l^T 1_{V_i}) = 1_{V_i}(k) 1_{V_i}(l)$. If $(k,l) \in \bar{E}$, then the k^{th} node and the j^{th} node cannot be in the same V_i, and therfore $(1_{V_i} 1_{V_i}^T)_{kl} = 0$
- If $k = l$, all the terms in $\sum_i (1_{V_i} 1_{V_i}^T)_{kl}$ are zero except for the i for which the k^{th} node is in V_i, in which case it is equal to one.

As a conclusion, Z verifies all constraints of the dual, and $Z_{n+1,n+1} = C = \chi(\bar{G})$, so

$$\chi(\bar{G}) \geq \vartheta(G)$$

2
Consider $G = C_5$.
Using CVX to calculate $\vartheta(G)$

```
n = 5
J = ones(n, n);
cvx_begin sdp
variable X(n, n) symmetric;
maximize(trace(X*J))
X >= 0
X(5, 1) == 0
for i=1:4
    X(i, i+1) == 0
end
trace(X) == 1
cvx_end
ans=cvx_optval
```

```
2.2361
```

$2 < \vartheta(G) < 3$

- $\vartheta(G) \notin \mathbb{N}$
- $\alpha(G), \chi(\bar{G}) \in \mathbb{N}$

  No inequality can thus be tight.

2 Q2

1

Consider

$$\begin{array}{lll} \min & Z_{n+1,n+1} & \text{(P(G))} \\ \text{subject to} & Z \succeq 0 & \\ & Z_{i,n+1} = Z_{ii} = 1 & \\ & Z_{i,j} = 0 \forall \{i,j\} \in \bar{E} & \end{array}$$

Write the lagrangian:

$$\begin{aligned} \mathcal{L}(Z,Y) &= Z_{n+1,n+1} + \sum_{i=1}^{n} 2Y_{i,n+1}(Z_{i,n+1} - 1) + Y_{ii}(Z_{ii} - 1) + \sum_{ij \in \bar{E}, i<j} Y_{ij} Z_{ij} \\ &= -\sum_i Y_{ii} + 2Y_{i,n+1} + \langle \underbrace{E_{n+1,n+1} + Y_{i,n+1}E_{i,n+1} + Y_{ii}E_{ii} + \sum_{ij \in \bar{E}, i<j} Y_{ij}E_{ij}}_{Y}, Z \rangle \end{aligned}$$

Dual:

$$\begin{array}{lll} \max & \sum_{i \le n} -Y_{ii} - 2Y_{i,n+1} & (D_1(G)) \\ \text{subject to} & Y \succeq 0 & \\ & Y_{n+1,n+1} = 1 & \\ & Y_{i,j} = 0 \forall \{i,j\} \in E & \end{array}$$

$P(G)$ and $D(G)$ are both strictly feasible (Consider I_{n+1} and $E_{n+1,n+1}$ respectively). So theire optimal values are **attained and are equal**.

Let Y be a feasible solution, write:

$$Y = \begin{pmatrix} Y' & y \\ y^T & 1 \end{pmatrix}$$

Note that by Schur's lemma:

$$Y \succeq 0 \iff Y' \succeq yy^T \iff Y' \succeq (-y)(-y)^T$$

So we can replace $Y_{i,n+1}$ by $-Y_{i,n+1}$ without affecting the optimal value:

$$
\begin{aligned}
\max \quad & \sum_{i\leq n} -Y_{ii} + 2Y_{i,n+1} && (D_2(G))\\
\text{subject to} \quad & Y \succeq 0\\
& Y_{n+1,n+1} = 1\\
& Y_{i,j} = 0 \forall \{i,j\} \in E
\end{aligned}
$$

Let's now prove that this is equivalent to the following problem:

$$
\begin{aligned}
\max \quad & \sum_{i\leq n} Y_{ii} && (D_3(G))\\
\text{subject to} \quad & Y \succeq 0\\
& Y_{n+1,i} = Y_{ii}\\
& Y_{n+1,n+1} = 1\\
& Y_{i,j} = 0 \forall \{i,j\} \in E
\end{aligned}
$$

- If Y is feasible to $D_3(G)$, then it is also feasible to $D_2(G)$. Moreover in that case, $\sum_i -Y_{ii} + 2Y_{i,n+1} = \sum_i Y_{ii}$, so $D_2(G) \geq D_3(G)$
- Let Y be an optimal solution to $D_2(G)$ (we proved that it exists), note $\gamma = \sum_{i\leq n} -Y_{ii} + 2Y_{i,n+1} = D_2(G)$. Argue by contradiction that that $Y_{n+1,j} - Y_{jj} \neq 0$.
  - Note $a = \sum_i Y_{ii} - 2Y_{i,n+1}$.
  - Note by Y' the matrix obtained from Y by multiplying the j^{th} row/column of Y by $s \in \mathbb{R}$.
  - $Y' = \text{diag}(1 \ldots \underbrace{s}_{j} \ldots 1) Y \,\text{diag}(1 \ldots \underbrace{s}_{j} \ldots 1) \succeq 0$, and we can see that Y' is feasible in $D_2(G)$.
  - Noting that $Y'_{jj} = s^2 Y_{jj}, Y'_{j,n+1} = sY_{j,n+1}$, the objective value of Y' in $D_2(G)$ is:

  $$\sum_{i\leq n} -Y'_{ii} + 2Y'_{i,n+1} - \gamma = -(s^2-1)Y_{jj} + 2(s-1)Y_{j,n+1} = -s^2 Y_{jj} + 2sY_{j,n+1} + Y_{jj} - 2Y_{j,n+1}$$

  The descriminant of the last equation in s is $\Delta = 4[Y_{j,n+1}^2 + Y_{jj}(Y_{jj} - 2Y_{j,n+1})] = 8Y_{j,n+1}(Y_{j,n+1} - Y_{j,n+1})$ Note that by looking at a 2×2 leading minor, $Y_{jj} \geq Y_{n+1,j}^2$, and since $Y_{jj} \neq Y_{n+1,j}$ they cannot be both equal to 0, so $Y_{jj} > 0$. As a result, $\Delta > 0$, meaning there exist an s that makes the objective value increase. Absurd.

We have that showed that Y verifies the satisfiability conditions of $D_3(G)$, and therefore $D_3(G) = \sum_i -Y_{ii} + 2Y_{i,n+1} = \sum_i Y_{ii} \leq D_2(G)$ Which completes the proof of the hint.
Let $Y \in S^{n+1\times n+1}$ be a feasible solution to this problem. Let $x := (Y_{ii})_{i\leq n}$.
By consider the 1×1 and 2×2 minors:

$$
\begin{aligned}
Y \succeq 0 &\implies \begin{vmatrix} Y_{ii} & Y_{ii} \\ Y_{ii} & 1 \end{vmatrix} \geq 0, Y_{ii} \geq 0\\
&\implies Y_{ii} - Y_{ii}^2 \geq 0, Y_{ii} \geq 0\\
&\implies 0 \leq Y_{ii} \leq 1
\end{aligned}
$$

We have just proved that $0 \leq x \leq 1$

Let $\{i_1, \ldots, i_k\}$ a clique in the graph. Let $I = \{i_1, \ldots, i_k, n+1\}$, and consider the principal minor:

$$\det Y_{I,I} = \det \begin{vmatrix} Y_{i_1 i_1} & \ldots & 0 & Y_{i_1 i_1} \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & Y_{i_k i_k} & Y_{i_k i_k} \\ Y_{i_1 i_1} & \ldots & Y_{i_k i_k} & 1 \end{vmatrix} = \det \begin{vmatrix} x_{i_1} & \ldots & 0 & x_{i_1} \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & x_{i_k} & x_{i_k} \\ x_{i_1} & \ldots & x_{i_k} & 1 \end{vmatrix} \geq 0$$

To calculate this determinant, substract the sum of the first n rows from the last one to get a triangular matrix:

$$\det Y_{I,I} = \det \begin{vmatrix} x_{i_1} & \ldots & 0 & x_{i_1} \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & x_{i_k} & x_{i_k} \\ 0 & \ldots & 0 & 1 - x_{i_1} - \ldots x_{i_k} \end{vmatrix} = x_{i_1} \ldots x_{i_k}(1 - x_{i_1} - \ldots - x_{i_k})$$

Which means that $x_{i_1} + \ldots + x_{i_k} \leq 1$, eg x respects all the clique inequalities.
As a result:

$$\sum_{i=1}^{n} Y_{ii} = \sum_{i=1}^{n} x_i \leq \eta_{LP}^{(k)}$$

Taking the sup over feasible Y: $\vartheta(G) \leq \eta_{LP}^{(k)}$
2

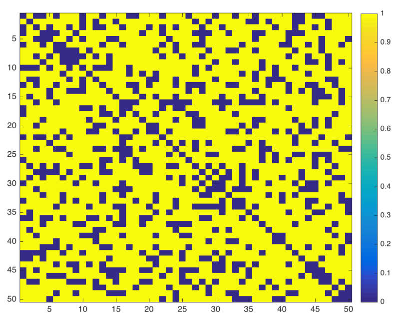

Figure 1: G Adjacency matrix

2.1 $\vartheta(G)$

```
n = 50
J = ones(n, n);

cvx_begin sdp
variable X(n, n) symmetric;
maximize(trace(X*J))
X >= 0
for i=1:n
    for j=1:i
        if G(i, j) == 1
            X(i, j) == 0
        end
    end
end
```

```
trace(X) == 1
cvx_end

ans=cvx_optval
```

```
5
```

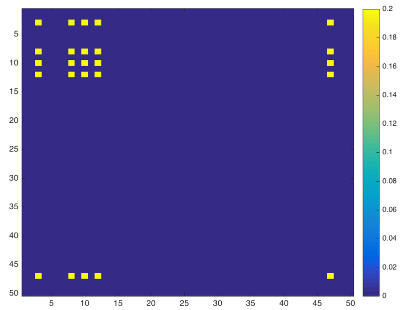

Figure 2: X optimal solution

Note that the resulting X is of rank 1, so it can be decomposed into $X = xx^T$. We check that $V_x = \{i, x_i \neq 0\}$ represents indeed a stable set.

```
[v,e] = eigs(full(X),1);
stableset = find(abs(v) > 0.01)
ans=stableset'
```

3	8	10	12	47

```
G(stableset, stableset)
```

Table 1: Subgraph of the nodes in the stableset

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Let's assume that there exist another stable set of size 5.

This would mean that there exist $v \in V_x$ such that imposing $x_j = 0$ (eg $X_{jj} = 0$) would not change α. Let's check:

```
  n = 50
  J = ones(n, n);
  opt = [stableset, zeros(5, 1)]
  for vi=1:5
      v = stableset(vi)
      cvx_begin sdp
      variable Y(n, n) symmetric;
```

```
        variable optvalue;
        maximize(trace(Y*J))
        Y >= 0
        for i=1:n
            for j=1:i
                if G(i, j) == 1
                    Y(i, j) == 0
                end
            end
        end
        Y(v,v) == 0
        trace(Y) == 1
        optvalue == trace(Y*J)
        cvx_end
        opt(vi, 2) = optvalue
    end
ans=opt
```

Table 2: Lovazs

Node removed	Lovasz of the subgraph
3	4.4463
8	4.5191
10	4.512
12	4.5586
47	4.4771

Since Lovasz number ϑ is an upper bound on α, This proves that any stable set not containing one of the nodes in V_x is of size less than 5.

We have just proved uniqueness of the stable set.

2.2 μ^{LP}

k = 2

```
cvx_begin
variable x(n)
maximize(sum(x))
for i=2:n
    for j=1:(i-1)
        if G(i, j) == 1
            x(i) + x(j) \langle = 1
        end
    end
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

50

k = 3

```
cvx_begin
variable x(n)
maximize(sum(x))
for i=2:n
    for j=1:(i-1)
        if G(i, j) == 1
            x(i) + x(j) <= 1
        end
        for r=1:(j-1)
            if G(i, j) +  G(j, r) + G(r, i) == 3
                x(i) + x(j) + x(r) <= 1
            end
        end
    end
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

16.667

k = 4

```
M = 50
cvx_begin
  variable x(n)
  maximize(sum(x))
  for i=2:M
      for j=1:(i-1)
          if G(i, j) == 0
              continue
          end
          x(i) + x(j) <= 1
          for r=1:(j-1)
              if G(j, r) == 0 || G(r, i) == 0
                  continue
              end
              x(i) + x(j) + x(r) <= 1
              for p =1:(r-1)
                  if G(i, p) == 0 || G(j, p) == 0 || G(r, p) == 0
                      continue
                  end
                  x(i) + x(j) + x(r) + x(p) <= 1
              end
          end
      end
```

```
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

```
12.5
```

3 Problem 3

1. Let $(a,b),(u,v)$ be two nodes in $G_A \otimes G_B$ The two nodes are connected if and only if:

- $A_{au}=1, A_{bv}=1$
- $a=u, A_{bv}=1$
- $A_{au}=1, b=v$

This can be summerised as $(A_{au}+\delta_{au})(A_{bv}+\delta_{bv})-\delta_{au}\delta_{bv}=1$

So the adjacency matrix of $G_A \otimes G_B$ is $(A+I_n)\otimes(B+I_m)-I_{nm}$.

Where $\otimes$ denote the Kronecker product: $(A\otimes B)_{p(r-1)+v,q(s-1)+w}=A_{rs}B_{vw}$

2.

$$5=\alpha(G)\leq\Theta(G)\leq\vartheta(G)=5$$

so $\Theta(G)=5$

4 Problem 4

1. (1) is equivalent to

$$\begin{cases} x^TAy &= \max_{\tilde{x}\in\Delta_m}\tilde{x}^TAy \\ x^TBy &= \max_{\tilde{y}\in\Delta_n}x^TB\tilde{y} \end{cases}$$

Consider the first problem:

$$\max_{\tilde{x}\in\Delta_m}\tilde{x}^TAy$$

This is an LP whose feasible region $\Delta_m=conv(e_i, i=1\dots m)$ is compact, so the maximum is attained in one of the extreme points e_{i_0}. Therefore

$$x^TAy=\max_{\tilde{x}\in\Delta_m}\tilde{x}^TAy \iff x^TAy=e_{i_0}^TAy=\max_i e_i^TAy \iff x^TAy\geq e_i^TAy\forall i$$

Same argument applies for y so that:

$$x^TBy=\max_{\tilde{y}\in\Delta_n}x^TB\tilde{y} \iff x^TAy\geq x^TBe_i\forall i$$

So:

$$(1) \iff \begin{cases} x^TAy &\geq e_i^TAy\ \forall i=1\dots m \\ x^TBy &\geq x^TAe_i\ \forall i=1\dots n \end{cases}$$

2.

$x\in\Delta_m, y\in\Delta_n$

Note $z = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$M = zz^T = \begin{pmatrix} xx^T & xy^T & x \\ yx^T & yy^T & y \\ x^T & y^T & 1 \end{pmatrix}$$

Note that

- $M \succeq 0$
- $rank(M) = 1$
- $M_{n+m:n+m+1,1:n} = x \in \Delta_n$
- $M_{n+m:n+m+1,n+1:n+m} = y \in \Delta_m$
- $M_{n+m+1,n+m+1} = 1$

Now to express the fact that (x, y) is a Nash equilibrium (*):

- $tr(M_{n+1:n+m,1:n}A) = tr(yx^T A) = tr(x^T Ay) \geq tr(e_i^T Ay) \geq tr(e_i^T AM_{n+m:n+m+1,n+1:n+m})$
- $tr(M_{n+1:n+m,1:n}B) \geq tr(M_{n+m:n+m+1,1:n}Ae_i)$

Now let $M \in S^{n+m}$, verifying all the previous conditions. Then by cholesky, there existe a vector $z \in \mathbb{R}^{n+m+1}$, such that: $M = zz^T$

- Let's decompose $z := \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+m+1}$, so that $M = zz^T = \begin{pmatrix} xx^T & xy^T & \alpha x \\ yx^T & yy^T & \alpha y \\ \alpha x^T & \alpha y^T & \alpha^2 \end{pmatrix}$
- $1 = M_{n+m+1,n+m+1} = \alpha^2 \implies \alpha = \pm 1$
- $M_{n+m:n+m+1,1:n} = \alpha x \in \Delta_n$
- Similarly: $\alpha y \in \Delta_m$
- If $\alpha = -1$, we can always change z to $-z$ without loss of generality to make $x, y \geq 0$ and therefore $x \in \Delta_m, y \in \Delta_n$
- (x, y) naturally represent a Nash equilibrium due to (*).

PSD relaxation:

- $M \succeq 0$
- $M_{n+m:n+m+1,1:n} \in \Delta_n$
- $M_{n+m:n+m+1,n+1:n+m} \in \Delta_m$
- $M_{n+m+1,n+m+1} = 1$

Moreover, we can add the following constraints:

- The sum of the columns of $M_{1:n,1:n} = xx^T = (x_i x_j)_{ij}$ is equal to x. (This is true because $x \in \Delta_n$)
- Similarly the sum of the columns of $M_{n+1:n+m,n+1:n+m}$ is equal to y.

- The sum of the columns of $M_{n+1:2n,1:n} = yx^T$ is equal to y, and the sum of the rows is equal to x.
- $M \geq 0$

```
A = [345 78 97 355 264 528;
     310 52 483 385 541 276;
     236 248 445 243 7 80;
     64 23 290 226 157 426;
     292 129 300 116 628 580;
     477 317 342 58 152 106]

B = [404 183 215 531 232 31;
     79 624 442 145 277 182;
     421 619 1 271 477 456;
     561 364 423 539 96 147;
     632 546 528 580 388 229;
     279 112 198 97 172 94]

n = length(A)
In = eye(n, n);

cvx_begin sdp
variable M(2*n+1, 2*n+1) symmetric;
variables x(n) y(n);
variables yx(n, n) xx(n, n) yy(n, n);

maximize(trace(yx * A))

M >= 0
M(2*n+1, 2*n+1) == 1
for i=1:(2*n+1)
    for j=1:(2*n+1)
        M(i, j) >= 0
    end
end

% sub-blocks of M
x == M(1:n, 2*n+1)
y == M(n + (1:n), 2*n+1)
xx == M(1:n, 1:n)
yy == M(n + (1:n), n + (1:n))
yx == M(n + (1:n), 1:n)

% x and y in the simplex
sum(x) == 1
sum(y) == 1

% Additional constraints
x == sum(xx)'
y == sum(yy)'
```

```
x == sum(yx')'
y == sum(yx)'

% Nash equilibrium constraint
for i=1:n
    ei = In(1:n, i);
    trace(yx * A) >= trace(ei' * A * y)
    trace(yx * B) >= trace(x' * B * ei)
end

cvx_end

ans=cvx_optval;
```

```
443.13
```

Which proves that the score of the first player cannot exceed 434