ORF527 - Problem Set 1

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Q.1

Let $\varepsilon > 0, x \in [0, 1]$.

- $\sup |f^n f| \to_n 0$, let $n \in \mathbb{N} \ \forall t \in [0,1] \ |f^n(t) f(t)| \le \frac{\varepsilon}{3}$.
- f^n is continuous, let $\delta > 0$ such that $\forall y \in [0,1], |x-y| < \delta \Rightarrow |f^n(x) f^n(y)| < \frac{\varepsilon}{3}$.
- Let $y \in [0,1]$ such that $|x-y| < \delta$

$$|f(x) - f(y)| \le |f(x) - f^n(x)| + |f^n(x) - f^n(y)| + |f^n(y) - f(y)|$$

$$\le 3\frac{\varepsilon}{3}$$

$$\le \varepsilon$$

Which conclude the proof.

Q.2

3.2

let $X \sim \mathcal{N}(0,1)$, and $\varepsilon \sim \mathcal{B}(-1,1,\frac{1}{2})$ be two independent rv. And Let $Y = \varepsilon X$

- By symmetry of the distribution of X: $F_Y(y) = P(Y \le y) = P(\varepsilon X \le y) = E[P(\varepsilon X \le y|\varepsilon)] = \frac{1}{2}P(X \le y) + \frac{1}{2}P(-X \le y) = P(X \le x)$ so $Y \sim \mathcal{N}(0,1)$.
- $cov(X,Y) = E[X^2\varepsilon] = E[X^2]E[\varepsilon] = 0$
- X, Y are not independent. Indeed, Let $\alpha := \mathbb{P}(|X| > 0.5)$.

$$-P(|X| > 0.5, |Y| > 0.5) = P(|X| > 0.5) = \alpha,$$

$$-P(|X|>0.5)P(|Y|>0.5)=P(|X|>0.5)^2=\alpha^2$$

But since $\alpha \notin \{0,1\}$, $\alpha \neq \alpha^2$.

3.3

For a random variable X, let's call Φ_X its caracteristic function.

- (a) For $t \in \mathbb{R}^n$, $\Phi_{AV}(t) = \mathbb{E}[e^{iV^TA^Tt}] = \Phi_V(A^Tt) = e^{(A\mu)^Tt \frac{1}{2}(t^TA\Sigma A^Tt)}$, so $V \sim \mathcal{N}(A\mu, A\Sigma A^T)$
- (b) By symmetry of the gaussisan distribution, -Y has the same distribution as Y. So it suffices to show that the result holds for X+Y. By independence: $\Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) = \Phi_X(t)^2 = e^{i2\mu t \frac{1}{2}2t^2}$, so $X+Y \sim \mathcal{N}(0,2)$.

(c) If cov(X,Y) = 0, the covariance matrix of the guassian process (X,Y) is diagonal, therefore

$$\forall t = (t_1, t_2) \in \mathbb{R}^2 \ \Phi_{(X,Y)}(t) = E[e^{i\mu_X t_1 + i\mu_Y t_2 - \frac{1}{2}\sigma_X^2 t_1^2 - \frac{1}{2}\sigma_Y^2 t_2^2}] = \Phi_X(t_1)\Phi_Y(t_2)$$

Where $X \sim \mathcal{N}(\mu_X, \sigma_X^2), Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Therefore X, Y are independent.

(d) We consider that the guaussian vector is non degenerate otherwise one of the variables is proportinal to the other.

Y|X=x is gaussian because it has the density $f_{Y|X=x}(y)=\frac{f(x,y)}{\int_R f(x,s)ds} \propto e^{-Q(y)}$ where is a quadratic form in y.

Let $\alpha := \frac{\text{cov}(X,Y)}{\text{var}(X)}$. $(Y - \alpha X, X)$ is gaussian and $\text{cov}(Y - \alpha X, X) = 0$. So $Y - \alpha X \perp X$. Therefore:

$$\mu_{Y|X=x} = E[Y|X=x] = E[Y-\alpha X] + E[\alpha X|X=x] = \mu_Y - \alpha \mu_X + \alpha x = \mu_Y + \frac{\text{cov}(X,Y)}{\text{var } X}(x-\mu_X)$$

$$\sigma_Y^2 = \text{var}(Y|X=x) = \text{var}(Y-\alpha X) + \underbrace{\text{var}(\alpha X|X=x)}_{=0} = \text{var}\,Y + \alpha^2 \, \text{var}\,X - 2\alpha \, \text{cov}(X,Y) = \sigma_Y^2 - \frac{\text{cov}(X,Y)^2}{\text{var}\,X}$$

Q.3

- (a) Since $\forall n > 0, \Delta_n(1) = 0$, $B_1 = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(1) = \lambda_0 Z_0 \Delta_0(1) = \lambda_0 Z_0$. But $B_t U_t = \lambda_0 Z_0 \Delta_0(t) = t\lambda_0 Z_0 = tB_t$, which proves the result.
- (b) If s < t, then $cov(B_s, B_t) = cov(B_s B_t, B_s) + var(B_s) = s$. $cov(U_s, U_t) = cov(B_s - sB_1, B_t - tB_1) = cov(B_s, B_t) + ts cov(B_1, B_1) - t cov(B_s, B_1) - s cov(B_1, B_t) = s + ts - ts - ts = s - ts = s(1 - t).$
- (c) For s < t, $cov(X_t, X_s) = g(t)g(s)(h(s) \land h(t))$, should be equal to s(1-t). We can take $g: t \to 1-t$, $h: s \to \frac{s}{1-s}$ defined on $(0, \frac{1}{2})$
- (d) U_t is a gaussian because for all $t_1 ldots t_n$, a linear combination of $(U_{t_1}, \ldots U_{t_n})$ is a linear combination of $(B_1, B_{t_1}, \ldots B_{t_n})$ which is guaussian. For the same reasons, $(U_{\frac{t}{1+t}})_{t>1}$ is a guaussian process, and so is Y_t .

For
$$s < t, \frac{s}{s+1} < \frac{t}{t+1}$$
: $cov(Y_t, Y_s) = (1+t)(1+s)cov(U_{\frac{t}{t+1}}, U_{\frac{s}{s+1}}) = (1+t)(1+s)\frac{s}{s+1}(1-\frac{t}{t+1}) = s$

 Y_t is a guaussian process with the same mean (0) and covariance as a Brownian Motion, therefore it has the same distribution as a Borwnian Motion.

In addition, Y_t is continuous as a composition of two continuous functions, therefore it is a brownian motion.