Problem set 5, ORF527

Bachir El khadir

<2016-03-04 Fri>

1 TODO Q1 (7.2 in Steel)

A convex function ϕ is a sup of affine function below it $\ ax + b \. Let τ_n be a localizing sequence of (X_t), so that $X_{t\wedge\tau_n}$ is bounded, and since ϕ is continuous, $Y_{t\wedge\tau_n} = \phi(X_{t\wedge n})$ is bounded. So for $s < t$, $E[Y_{t\wedge\tau_n}|F_s] = E[\phi(X_{t\wedge\tau_n})|F_s] \underbrace{\leq}_{\text{Jensen}} \phi(E[X_{t\wedge\tau_n}|F_s]) = \phi(Y_{s\wedge n})$ $\phi = X$

2 TODO Q1 (7.3)

blablba

3 Q2

a. By Ito isometry and linearity:

$$E[(\int_0^T X_s^n dW_s - \int_0^T X_s dW_s)^2] = E[\int_0^T (X_s^n - X_s)^2 ds] \to 0$$

b. Let τ_n be a localizing sequence. By markov inequality:

$$P(|\int_0^T X_{t\wedge\tau_n} dW_t| \geq \varepsilon) \leq \frac{E[|\int_0^T X_{t\wedge\tau_n} dW_t|^2]}{\varepsilon^2}$$

By Ito:

$$P(|\int_0^T X_{t\wedge\tau_n} dW_t| \geq \varepsilon) \leq \frac{E[\int_0^T X_{t\wedge\tau_n}^2 dt]}{\varepsilon^2}$$

c. By b.

$$P(|\int_0^T (X_t - X_t^n) dW_t|^2 > \varepsilon) \leq P(\int_0^T (X_t - X_t^n)^2 dt \geq \varepsilon) + \frac{N}{\varepsilon^2}$$

Taking the $\limsup$ with respect to n:

$$\limsup_n P(|\int_0^T (X_t - X_t^n) dW_t|^2 > \varepsilon) \leq \frac{N}{\varepsilon^2}$$

And thus for all $N > 0$. We conclude by taking the $N \to 0$.

d. Let $X \in \mathcal{H}^{loc}[0,T]$, let τ_n be a localizing sequence, so that $X 1_{[0,\tau_n]} \in \mathcal{H}[0,T]$. Since $\mathcal{H}_0[0,T]$ is dense in $\mathcal{H}[0,T]$ with respect to the $L_2(\Omega \times [0,T])$ norm, there exist a sequence $X_n \in \mathcal{H}_0$ such that: $E[\int_0^T (X 1_{[0,\tau_n]}(s) - X_n(s))^2 ds] \to_n 0$.

$P(\int_0^T (X(s) 1_{[0,\tau_n]}(s) - X_n(s))^2 ds > \varepsilon) \leq \frac{E[\int_0^T (X 1_{[0,\tau_n]}(s) - X_n(s))^2 ds]}{\varepsilon} \to_n 0$

So $\int_0^T (X(s) 1_{[0,\tau_n]}(s) - X_n(s))^2$ converges to 0 in probability. We also know that $\int_0^T X^2(s) 1_{[0,\tau_n]}(s) ds \to \int_0^T X^2(s) ds$ alsmost surely, and thus in probability.

Now, $\int_0^T (X(s) - X_n(s))^2 ds \leq \int_0^T (X(s) - X(s) 1_{[0,\tau_n]}(s))^2 + \int_0^T (X 1_{[0,\tau_n]}(s) - X_n(s))^2 \xrightarrow{P} 0$, which gives the result.

Using c., We can definie the integral of $X \in \mathcal{H}^{loc}[0,T]$ as the limit in probability of a sequence of simple function that converge to X in the sense of c.

To prove that such construction is sound, we have to prove that the limit doesn't depend on the sequence of simple functions chosen, or equivalenty, that by linearity that if $X_n \xrightarrow{c.} 0$, $\int_0^T X_n dW_t \xrightarrow{P} 0$.

4 TODO Q3

a. $d(e^t W_t) = e^t W_t dt + e^t dW_t$ b. $fx \to \frac{1}{1+x^2}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f''(x) = \frac{-2(1+x^2)^2+8x^2(1+x^2)}{(1+x^2)^4} = \frac{-2+6x^2}{(1+x^2)^3}$

$$d(1+W_t^2)^{-1} = \frac{-2W_t}{(1+W_t^2)^2} dW_t + \frac{-1+3W_t^2}{(1+W_t^2)^3} dt$$

c.

$$f : (t,x) \to e\alpha x + \sigma tcos(\int_0^t \sqrt{|x|}$$