ORF526 - Problem Set 3

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Question 1

a) $\cap_{n\in\mathbb{N}} A_n^c$

b) $\cap_{m\in\mathbb{N}} \cup_{m\le n} A_n$

c) That's the opposite of b), $(\cap_{m\in\mathbb{N}} \cup_{m\le n} A_n)^c = \cup_{m\in\mathbb{N}} \cap_{m\le n} A_n^c$

d) ω has to be in exactly two of the A_i, ie $\cap_{i,j\in\mathbb{N}, i\neq j} (A_i \cap A_j \cap (\cup_{n\neq i, n\neq j} A_n^c))$

e) This event can be expressed as "Φ nevr occurs at even times", ie $\cap_{n\in\mathbb{N}} A_{2n}^c$

Question 2

- $\varepsilon \subseteq \sigma(\varepsilon)$, so $\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$. since the RHS is already a σ-algebra (showed in class), $\sigma\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$
- Let's note $B := \sigma\{f^{-1}(A) : A \in \varepsilon\}$, and $C := \{A : f^{-1}(A) \in B\}$.
  - C is a σ-algebra containing ε, so $\sigma(\varepsilon) \subseteq C$
  - As a consequence, for every $A \in \sigma(\varepsilon)$, $A \in C$, ie $f^{-1}(A) \in B$.

  We have just proved that $\{f^{-1}(A) : A \in \sigma(\varepsilon)\} \subseteq B$

Question 3

- $X = \lim_n X_n = \lim_n \sup_{k\ge n} X_k$
- For $x \in \mathbb{R} \cup \{\infty\}$, $X \le x$ is quivalent to $\exists n \, \forall k \ge n \, X_k \le x$
- $\{X \le x\} = \cup_{n\in\mathbb{N}} \cap_{k\ge n} \{X_k \le x\}$ is then measuable.

Question 4

- Let's call $Q = P(\{1, .., n\})$. For $i = 1..n$:

  $$A_i = \bigcup_{I\subseteq Q, i\in I} (\underset{k\in I}{\cap} A_k) \cap (\underset{k\in I^c}{\cap} A_k^c)$$

  Note that this a union of disjoint sets.

  Let's call $I_i := \{I \in Q : \sum_{i\in I} a_i = x_i\}$, ie the different possible combinations for the A_i where $\omega \in \Omega$ can be so that its image by f equals x_i. Note that ω can not be in any other set A_i, for $i \notin I_i$ bacause $a_i > 0$.

Written differently, $\{f = x_i\} = \bigcup_{I\in I_i} \underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c$. And as result of the sets being disjoint:

$$\mu(f = x_i) = \sum_{I\in I_i} \mu(\underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c)$$

Note that any sum index by some $I \in Q$ in finite because $|I| \leq n$, and this we can rearrange the sums in any order.

$$\begin{aligned}
\sum_{i=1}^{m} x_i\mu(f = x_i) &= \sum_{i=1}^{m} x_i \sum_{I\in I_i} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1}^{m} \left(\sum_{I\in I_i} (\sum_{k\in I} a_k)\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \right) \\
&= \sum_{I\in Q} (\sum_{k\in I} a_k)\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{because } Q = \underset{k=1..m}{\cup} I_i \\
&= \sum_{i=1..n} \sum_{I\in Q, i\in I} a_i\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{By rearranging the sum} \\
&= \sum_{i=1..n} a_i \sum_{I\in Q, i\in I} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1..n} a_i\mu(A_i)
\end{aligned}$$

- Let's first prove that if a set A has measure 1, for all measurable sets B, $\mu(A \cap B) = \mu(B)$. This holds because

$$\mu(B) \geq \mu(A \cap B) = 1 - \mu(A^c \cup B^c) \geq 1 - \mu(B^c) = \mu(B)$$

Let's now prove that $f(\omega) = g(\omega)$. Let x in the set on the left

$\mu(\{g = x\}) \geq \mu(\{g = x\} \cap \{f = x\}) = \mu(\{f = x\} \cap \{f = g\}) = \mu(\{f = x\})$ Symmetrically, we prove that $\mu(\{f = x\}) \geq \mu(\{g = x\})$, and thus this two quantities are equal.

This proves that in the sum $\sum_{x\in f(\Omega)} x\mu(f = x)$ there is a term that is non zero, $\mu(\{f = x\}) = \mu(\{g = x\}) \neq 0$, and $x \in g(\Omega)$. Since all quantities are positive, this means, $\sum_{x\in f(\Omega)} x\mu(f = x) \leq \sum_{x\in g(\Omega)} x\mu(g = x)$, and by symmetry: $\sum_{x\in f(\Omega)} x\mu(f = x) = \sum_{x\in g(\Omega)} x\mu(g = x)$

Question 5

Let's first suppose that $h := \sum_{i=1..n} a_i 1_{A_i}$ for $a_i > 0$ and $A_i \in \sigma(f)$. Let $h(\Omega_1) = \{x_1, ..., x_m\}$.

We can write $h = \sum_{j=1..m} x_j 1_{\{h=x_j\}}$

$$\{h = x_j\} = f^{-1}(A_j)$$

A_j are disjoint because for $i \neq j$, $f^{-1}(A_i \cap A_j) \subseteq \{h = x_i\} \cap \{h = x_j\} = \emptyset$

We define $g(x) = x_i$ on A_i, and any other value outside.

Let $\omega \in \Omega_1$, and $x_i = h(\omega)$, then $\omega \in f^{-1}(A_i)$, so $f(\omega) \in A_i$ and $gof(\omega) = x_i = h(\omega)$. So $h = gof$.

Let h be now just non negative.

We can approximate h point wise by a sequence of non negative simple functions h_n measurable w.r.t the σ-algebra generated by f. We know now that h_n can be written as $g_n o f$.

By convergence of h_n, the sequence g_n converges point wise in the image of f. Let call g the limit, and set g to 0 outside the image of f, then $h = gof$. and $g = (\limsup_n g_n) 1_{f(\Omega_1)}$ is measurable as the product of two measurable functions.

Let h be just measurable, $h = h^+ - h^- = g^+ of - g^- of = (g^+ - g^-) of$.

If we consider the trivial σ-algebra $\{\mathbb{R}, \emptyset\}$ instead of Borel, and f constant equal to 0, then any h is measurable w.r.t the σ algebra generated by f. If now h is not constant, it cannot be written as $gof = g(0)$