Definition 1 (Total Variation). $TV[f,T] = \sup_{0<t_1<\ldots<t_n=T, n\geq 1} \sum_{k=1}^{n} |f(t_i) - f(t_{i-1})|$

Theorem 1 (TV of of a Brownian Motion). $(X_t)_t$ *BM* $\Rightarrow TV[X,T] = \infty$ *ass*

Proof. Eough to show that: $\lim_{n\infty} \sum_{k=1}^{2^n} |X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}| = \infty$

$$
\begin{aligned}
\mathbb{P}(\limsup_{n\infty} \sum_{k=1}^{2^n} |X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}|) < M) &\leq \liminf \mathbb{P}(\sum_{k=1}^{2^n} \underbrace{|X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}|}_{\mathcal{N}(0,T2^{-n})} < M) \\
&\leq \liminf \mathbb{P}(\frac{1}{2^n} \sum_{k=1}^{2^n} |\epsilon_k| < \frac{M}{\sqrt{T2^n}}) \\
&\to 0
\end{aligned}
$$

□

1 Continuous Time theory

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ filtered probability space.
(X_t) continuous time process

Definition 2 (Continuous - Adapted).

- *X is $\mathcal{F}_t$-adapted (nonanticipating) if X_t is $\mathcal{F}_t$-measurable $\forall t$*
- *X is continuous if $t \to X_t(\omega)$ is continuous $\forall \omega$.*
- *X is measurable if $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$-measurable.*
- *X is progressively measurable if $X : [0,T] \times \Omega \to \mathbb{R}; (t,\omega) \to X_t(\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}_t$-measurable.*

Beware: measurable and adapted $\not\Longrightarrow$ progressively measurable. But continuous + adapted $\Longrightarrow$ progressively measurable

Definition 3 (Usual conditions). *$(\mathcal{F}_t)_t$ is set to satisfy the* usual conditions *if*

a) $\forall B \subseteq A \in \mathcal{F}$ $P(A) = 0 \implies B \in \mathcal{F}_0$ (completeness)

b) $\mathcal{F}_t = \cup_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ (right-continuity)

Why?

- *(a) $\implies$: if (X_t) progressively measurable, B borel set in $\mathbb{R}$, $\tau = \inf\{t : X_t \in B\}$ is a stopping time.*

  But if (X_t) is continuous adapated, B closed, τ is always a stopping time. (doesn't need a)).

- *X BM, $\mathcal{F}_t^0$ the natural filtration. $\{t \to X_t$ has a local maximum at $t\} \in \mathcal{F}_{t^+}^0$ and not in $\mathcal{F}_t^0$. But one can prove $\mathcal{F}_t^0 = \mathcal{F}_{t^+}^0$ a.s (for every set $A \in \mathcal{F}_{t^+}^0$, there exist $B \in \mathcal{F}_t^0$ such that $\mathbb{P}(A \setminus B) = 0$).*

  $\mathcal{N} = \{B : B \subseteq A \in \mathcal{F}, \mathbb{P}(A) = 0\}$, $\mathcal{F}_t = \cap_{\varepsilon>0}(F_t^0 \vee N)$ is called the standard Brownian filtration, it satisfies usual conditions.

  b) is important if processes have jumps.

2 Continuous Time Martingale

Definition 4 (Martingale)**.** *(M_t) is a martingale if*

1. *M is adapted*
2. *$\mathbb{E}[|M_t|] < \infty \forall t$*
3. *$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \forall t \geq s$*

Definition 5 (Stopping time)**.** *A r.v $\tau : \Omega \to [0, \infty]$ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t \forall t$.*

Definition 6 (Stopped filtration)**.** *$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cup \{\tau \leq t\} \in \mathcal{F}_t \ \forall t\}$*

Theorem 2 (Optional Stopping)**.** *M is a **continuous** martingale. τ is a stopping time.*
Then $(M_{t\wedge\tau})_t$ is a continuous martingale

Proof. Step 1) Suppose $\tau \in \underbrace{\{t_1, t_2, \ldots\}}_{\Pi}$ as. Let's show that for $s < t$

(a) $M_{t\wedge\tau}$ is $\mathcal{F}_t$-measurable
(b) $\mathbb{E}|M_{t\wedge\tau}| < \infty$
(c) $\mathbb{E}(M_{t\wedge\tau}|F_s) = M_{s\wedge\tau}$

Without loss of generality let's assume that $s < t$ both in Π. Let $X_n = M_{t_n}$. Then X_n is a discrete time martingale with respect to the filtration $(\mathcal{G}_n = \mathcal{F}_{t_n})_n$.

Step 2) τ any stop time. Introduce discrete step times $\tau_n \downarrow \tau$, take limits. $\Pi_n := \{s, t, \infty, k2^{-n}, k \in \mathbb{Z}\}$ $\tau_n = \inf\{r \in \Pi_n r \geq \tau\}$ (We round up so it stays a stopping time)

Lemma 3. *τ_n is a stopping time beacause*

$$\{\tau_n \leq t\} = \{\tau \leq \underbrace{\max\{s \in \Pi_n : s \leq t\}}_{\leq t}\}$$

So:

a) $M_{t\wedge\tau} = \lim_n M_{t\wedge\tau_n}$ (because M is continuous) all $\mathcal{F}_t$-measurable

b) $\mathbb{E}|M_{t\wedge\tau}| = \mathbb{E}\liminf |M_{t\wedge\tau_n}| \underbrace{\leq}_{\text{fatou}} \liminf \mathbb{E} \underbrace{|M_{t\wedge\tau_n}|}_{\text{(submartingale)}} \leq E|M_t| < \infty$

c) $\mathbb{E}[M_{\tau_n\wedge t}|\mathcal{F}_s] = M_{\tau_n\wedge s}$ $M_{s\wedge\tau_n} \to M_{s\wedge\tau}$ as by continuity. **Claim:** $M_{s\wedge\tau_n} \to M_{s\wedge\tau}$ in L_1. Recall: $\mathbb{E}[M_s|\mathcal{F}_{s\wedge\tau_n}] = M_{s\wedge\tau_n}$ (matringale in discrete time) $\mathcal{F}_{s\wedge\tau_n} \downarrow$ as $n\to\infty$ so $M_{s\wedge\tau_n}$ is a reverse martingale, and any reverse martingale converges in L_1. As a result, $\mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s] \to \mathbb{E}[M_{t\wedge\tau}|\mathcal{F}_s]$ in L_1

□

Theorem 4 (Doob). *$(M_t)_t$ continuous mtg. Then*

- $\mathbb{P}[\sup_{0\leq t\leq T} M_t > x] \leq \frac{\mathbb{E}[|M_t|^p]}{x^p}$
- $\|\sup_{0\leq t\leq T} M_t\|_p \leq \frac{p}{p-1}\|M_T\|_p$

Proof. $\max_{0\leq k\leq 2^n} |M_{k2^{-n}T}| \uparrow \max_{0\leq t\leq T}|M_t|$ by continuity. Use Doob for discrete mtg + MCN. □

Theorem 5 (Mtg conv thm). *M_t continuous martingale.*

- *If L_1-bounded, then $M_t \to M_\infty$ as.*
- *If M_t are UI, then $M_t \to M_\infty$ in L_1.*
- *If L^p-bounded, $M_t \to M_\infty$ in L^p*

Proof.

- $N_T^n(a,b)$ # of up corssings of (a,b) on a grid $\Pi_n = \{\frac{k}{2^n}T, k \leq 2^n\}$(number of upcrossing of $(M_{t_i})_{t_i\in\Pi_n}$ $\mathbb{E}N_T^n(a,b) \leq \frac{\mathbb{E}(M_T-a)_+}{b-a}$, $(N_T^n(a,b))_n \uparrow N_T(a,b)$, by monotonuous convergence $\mathbb{E}N_T(a,b) \leq \frac{\mathbb{E}(M_T-a)_+}{b-a} \leq \frac{\mathbb{E}|M_t|+|a|}{b-a}$ By L_1 boundedness, $\mathbb{E}\mathbb{E}N_T(a,b) < C$. Let $T\to\infty$, (MON) $\mathbb{E}N_\infty(a,b) < \infty$

  $\mathbb{P}(M_t \not\to M_\infty) = \mathbb{P}(\exists a,b\in\mathbb{Q}, \liminf M_t < a < b < \limsup M_t) \leq \sum_{a<b\in\mathbb{Q}} \mathbb{P}(\liminf M_t < a < b < \limsup M_t) \leq\leq \sum_{a<b\in\mathbb{Q}} \mathbb{P}(N_\infty(a,b) = \infty) = 0$
- convergence as + UI $\implies$ convergence in L_1
- Doob's inequality + dominated convergence theorem.

□

Theorem 6 (Some classical BM martingales). *$(B_t)_t$ BM*

- *$(B_t)_t$ is a mtg*
- *$(B_t^2 - t)_t$ is a mtg*
- *$(e^{\sigma B_t - \frac{1}{2}\sigma^2 t})_t$ is a mtg*

Theorem 7 (Stooping time)**.** $\tau = \inf\{t, B_t = A \textit{ or } B_t = B\} = \inf\{t, B_t \in [A, B]\}$.

- $\mathbb{P}(\tau < \infty) = 1$
- $\mathbb{P}(B_\tau = A) = \frac{B}{A+B}$
- $\mathbb{E}\tau = AB$

Theorem 8 (Stooping time)**.** $\tau = \inf\{t, B_t = A\}$

- $\mathbb{P}(\tau < \infty) = 1$
- $\mathbb{E}\tau = \infty$
- $\mathbb{E}e^{-\lambda\tau} = e^{-|A|\sqrt{2\lambda}}$

Proof. For b): $e^{\lambda x} = 1 - x\int_0^\lambda e^{-\sigma x} d\sigma$ $\mathbb{E}e^{\lambda x} = 1 - \int_0^\lambda E[xe^{-\sigma x}] d\sigma$ (fubini) Take derivative at $\lambda = 0$. $\square$