

Problem set 2, ORF550

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1 Problem 2.11

a. By polarization identity

$$\mathcal{E}(P_t f, P_t g) = \frac{1}{4} [\mathcal{E}(P_t(f+g), P_t(f+g)) - \mathcal{E}(P_t(f-g), P_t(f-g))]$$

Since

$$\text{var}_\mu(P_t f) = 2 \int_0^\infty \mathcal{E}(P_t f, P_t f) dt$$

Taking the integral on both sides:

$$\int_0^\infty \mathcal{E}(P_t f, P_t g) dt = \frac{1}{8} [\text{var}_\mu(f+g) - \text{var}_\mu(f-g)]$$

e.g:

$$\int_0^\infty \mathcal{E}(P_t f, P_t g) dt = \frac{1}{2} \text{cov}_\mu(f, g)$$

b. $\text{cov}_\mu(f, g) = \int_0^\infty$

c. $dY_t = -Y_t dt + dB_t$

Y_t is a reversible ergodic Markov process, with:

- Stationary measure $\mu = N(0, I)$
 - Semi-group $P_t f(Y) = E[f(e^{-t}Y + \sqrt{1 - e^{-2t}}\epsilon)]$, $\epsilon \sim \mathcal{N}(0, I)$. $P_t f \geq 0$ when $f \geq 0$
 - Generator $\mathcal{E}(f, g) = \mu \langle \nabla f, \nabla g \rangle$
- Let $f_x(Y) = f(\Sigma^{\frac{1}{2}}Y)$, so that :
- $\nabla f_x(Y) = \Sigma^{\frac{1}{2}} \nabla f(X)$
 - $\nabla P_t f_x(Y) = e^{-t} \Sigma^{\frac{1}{2}} P_t \nabla f(X)$
 - f is coordinate-wise non-decreasing, so $\nabla f(X) \geq 0$, and therefore $P_t \nabla f(X) \geq 0$
 - The same applies to g .
 - $\Sigma \geq 0$, so $u' \Sigma u \geq 0$ whenever $u \geq 0$

$$\begin{aligned} \text{cov}(f(X), g(X)) &= \text{cov}(f_x(Y), g_x(Y)) \\ &= \int_0^\infty \mathcal{E}(P_t f_x, P_t g_x) dt \\ &= \int_0^\infty E[\nabla P_t f_x(Y) \cdot \nabla P_t g_x(Y)] dt \\ &= \int_0^\infty e^{-2t} E[P_t \nabla f(X)' \Sigma P_t \nabla g(X)] dt \\ &\geq 0 \end{aligned}$$

2 Problem 3.1

a. X σ^2 sub-gaussian, so $\psi(\lambda) = \log E[e^{\lambda(X-E[X])}] \leq \frac{\lambda^2 \sigma^2}{2}$

Because $e^x \geq 1 + \frac{1}{2}x^2$ when $x \geq 0$, we have that: $e^{\frac{\lambda^2 \sigma^2}{2}} \geq E[e^{|\lambda(X-E[X])|}] \geq 1 + \frac{1}{2}E[(\lambda(X-E[X]))^2] \geq 1 + \frac{\lambda^2}{2}\text{var}(X)$

Which proves that $\text{var}(X) \leq \frac{e^{\lambda^2/2\sigma^2}-1}{\lambda^2/2} \rightarrow_{\lambda} \sigma^2$

b.

$$\Phi(|X|) = \Phi(0) + \int_0^{|X|} \Phi'(t)dt = \Phi(0) + \int_0^\infty \Phi'(t)1_{t \leq |X|}dt$$

We conclude by taking the expectation on both sides. The swapping of E and \int is justified by Fubini for non-negative functions.

c. X σ^2 sub-gaussian, so: $P(X \geq t) \wedge P(X \geq -t) \leq e^{-t^2/2\sigma^2}$

$P(|X| \geq t) \leq P(X \geq t) + P(X \geq -t) \leq 2e^{-t^2/2\sigma^2}$

d.

$$\begin{aligned} E[e^{X^2/6\sigma^2}] &= 1 + \frac{1}{3}\sigma^2 \int_0^\infty t e^{t^2/6\sigma^2} P(|X| \geq t) dt \\ &\leq 1 + \frac{2}{3}\sigma^2 \int_0^\infty t e^{t^2/6\sigma^2 - t^2/2\sigma^2} dt \\ &\leq 1 + \int_0^\infty (2t) \frac{1}{3}\sigma^2 e^{-\frac{t^2}{3\sigma^2}} dt \\ &\leq 1 + \int_0^\infty \frac{d}{dt} - e^{-\frac{t^2}{3\sigma^2}} dt \\ &\leq 2 \end{aligned}$$

e. Let's prove the two inequalities in the hint

$$e^u \leq 1 + \frac{1}{2}u^2 e^{|u|} \quad (1)$$

- When $u \leq 0$, it is trivially verified.
- When $u \geq 0$, this is can be proven using the following:

$$e^u = 1 + u + \frac{1}{2}u^2 \underbrace{\sum_{k=0}^\infty u^k \frac{2}{(k+2)!}}_{\leq \frac{1}{k!}} \leq 1 + \frac{u^2}{2}e^u$$

$$|\lambda x| \leq \frac{a\lambda^2}{2} + \frac{x^2}{2a} \quad (2)$$

This follows from:

$$\begin{aligned} 0 &\leq (\sqrt{a}\lambda + \frac{1}{\sqrt{a}}x)^2 = \frac{a\lambda^2}{2} + \frac{x^2}{2a} - \lambda x \\ 0 &\leq (\sqrt{a}\lambda - \frac{1}{\sqrt{a}}x)^2 = \frac{a\lambda^2}{2} + \frac{x^2}{2a} + \lambda x \end{aligned}$$

We now show the X is subgaussian:

$$E[e^{\lambda X}] \leq 1 + \frac{\lambda^2}{2} E[X^2 e^{|\lambda X|}] \quad (\text{by1})$$

$$\leq 1 + \frac{\lambda^2}{2} e^{a \frac{\lambda^2}{2}} E[X^2 e^{\frac{X^2}{2a}}] \quad (\text{by2})$$

$$\leq 1 + \frac{\lambda^2}{2b} e^{\frac{a}{2} \lambda^2} E[e^{(\frac{1}{2a} + b) X^2}] \quad (\text{by??})$$

$$\leq 1 + \frac{\lambda^2}{1/a - 1/(3\sigma^2)} e^{a \frac{\lambda^2}{2}} \quad (\text{by2})$$

$$\leq 1 + \frac{\sigma^2 \lambda^2}{1/k - 1/3} e^{k \sigma^2 \frac{\lambda^2}{2}} \quad (a = k \sigma^2)$$

$$\leq 1 + 6 \sigma^2 \lambda^2 e^{\sigma^2 \lambda^2} \quad (k = 2)$$

$$\leq 1 + (e^{6 \sigma^2 \lambda^2} - 1) e^{\sigma^2 \lambda^2}$$

$$\leq 1 - e^{\sigma^2 \lambda^2} + e^{7 \sigma^2 \lambda^2}$$

$$\leq e^{7 \sigma^2 \lambda^2}$$

f.

$$E[X^{2q}] = 2q \int_0^\infty t^{2q-1} P(|X| \geq t) dt \leq 4q \int_0^\infty t^{2q-1} e^{-t^2/2\sigma^2} dt = (4\sigma^2)^q q!$$

g.

Fubini for non negative functions:

$$\begin{aligned} E[e^{X^2/8\sigma^2}] &= \sum_q \frac{1}{((8\sigma^2)^q q!)} E[X^{2q}] \\ &\leq \sum_q \frac{1}{((8\sigma^2)^q q!)} (4\sigma^2)^q q! \\ &\leq \sum_q \frac{1}{2^q} \\ &= 2 \end{aligned}$$

3 Problem 3.7

$$f(\varepsilon_1, \dots, \varepsilon_1) = \sup_{t \in T} \sum_k t_k \varepsilon_k$$

McDiarmid's inequality gives the following variance proxy for the subgaussian property:

$$\sigma^2 = \frac{1}{4} \sum_k \|D_k f\|_\infty^2$$

$$D_k f = \sup_{\varepsilon'_k, \varepsilon_k} \sup_{t', t \in T} \sum_{i \neq k} \varepsilon_i (t_i - t'_i) + \varepsilon_k t_k - \varepsilon'_k t'_k \geq \sup_{\varepsilon'_k, \varepsilon_k} \sup_{t', t \in T} (\varepsilon_k - \varepsilon'_k) t_k \geq 2 \sup t_k$$

So at best $\|D_k f\|_\infty = 4 \sup t_k^2$, and

$$\sigma^2 = \sum_k \sup_{t \in T} t_k^2$$

Now, pick T the set of vectors that have one component equal to 1, and the rest equal to 0, then

$$\sum_{k=1}^n \sup_T t_k^2 = n$$

while

$$\sup_T \sum_{k=1}^n t_k^2 = 1$$

4 Problem 3.8

$$f(X_1, \dots, X_n) = \sup_{C \in \mathcal{C}} \left| \frac{\#\{X_k \in C\}}{n} - \mu(C) \right| = \sup_{C \in \mathcal{C}} \left| \frac{\sum_i (1_{X_i \in C} - \mu(C))}{n} \right| = \left| \frac{\sum_{i \neq k} (1_{X_i \in C} - \mu(C))}{n} + \frac{1_{X_k \in C} - \mu(C)}{n} \right|$$

$$\begin{aligned} D_k f &= \sup_{X_k, X'_k} \left\{ \sup_C \left| \frac{\sum_{i \neq k} (1_{X_i \in C} - \mu(C))}{n} + \frac{1_{X_k \in C} - \mu(C)}{n} \right| - \sup_{C'} \left| \frac{\sum_{i \neq k} (1_{X_i \in C'} - \mu(C'))}{n} + \frac{1_{X'_k \in C'} - \mu(C')}{n} \right| \right\} \\ &\leq \sup_{X_k, X'_k} \sup_C \left| \frac{\sum_{i \neq k} (1_{X_i \in C} - \mu(C))}{n} \right| + \sup_C \left| \frac{1_{X_k \in C} - \mu(C)}{n} \right| - \sup_{C'} \left| \frac{\sum_{i \neq k} (1_{X_i \in C'} - \mu(C'))}{n} \right| + \sup_{C'} \left| \frac{1_{X'_k \in C'} - \mu(C')}{n} \right| \\ &\leq \frac{2}{n} \end{aligned}$$

$$\|D_k f\|_\infty = 2 \frac{1}{n}$$

as a result Z_n is subgaussian with proxy variance $\sigma^2 \leq \frac{1}{n}$, therefore ...

5 Problem 3.10

6 Problem 3.14

$$\begin{aligned} D_i^- f &= \sup_{\varepsilon'_i} f(\varepsilon_1, \dots, \varepsilon_n) - f(\varepsilon_1, \dots, \varepsilon_n) \\ &= \sup_{\varepsilon'_i} \sup_{t \in T} \sum_{k \neq i} \varepsilon_k t_k + \varepsilon_i t_i - \sup_{t' \in T} \sum_{k \neq i} \varepsilon_k t'_k + \varepsilon'_i t'_i \\ &= \sup_{\varepsilon'_i} \sup_{t \in T} \sum_{k \neq i} \varepsilon_k t_k + \varepsilon_i t_i - \sup_{t' \in T} \sum_{k \neq i} \varepsilon_k t'_k + \varepsilon'_i t'_i \end{aligned}$$