Problem set 5, ORF523

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1 Problem 1

Notation $E_{ij} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})_{k,l}$ the matrix with all 0 except in (i,j) and (j,i)

$$
\begin{aligned}
-\nu(G) = &\min_X && Tr(X(-J)) && \\
&\text{subject to} && X \geq 0 && \\
& && Tr(XI_n) = 1 && (:\alpha) \\
& && Tr(E_{ij}X) = 0 \ \forall (i,j) \in E, i < j && (:\lambda_{ij})
\end{aligned}
$$

has for dual:

$$
\begin{aligned}
&\max_{\alpha, \lambda_{ij} \in \mathbb{R}} && \alpha \\
&\text{subject to} && \alpha I + \sum_{(i,j) \in E} \lambda_{ij} E_{ij} \leq -J
\end{aligned}
$$

Or:

$$
\begin{aligned}
&\max_{\alpha, \lambda_{ij} \in \mathbb{R}} && \alpha \\
&\text{subject to} && \alpha I + \sum_{(i,j) \in E, i<j} \lambda_{ij} E_{ij} \leq -J
\end{aligned}
$$

Both are strictly feasible:

- for the primal, take $X = \frac{I_n}{n}$
- For the dual, take $\alpha = -2$, $\lambda_{ij} = 0$

Which proves that the dual and primal are equal. Taking $\beta = -\alpha$, we can write that:

$$
\begin{aligned}
\nu(G) = &\min_{\alpha, \lambda_{ij} \in \mathbb{R}} && \beta \\
&\text{subject to} && -\beta I + \sum_{(i,j) \in E} \lambda_{ij} E_{ij} \leq -J
\end{aligned}
$$

Note that the $(1,1)$ entry of $-\beta I + \sum_{(i,j)\in E} \lambda_{ij} E_{ij} + J$: $1 - \beta$ shoud be negative, so we can ammend to the constraints that $\beta \geq 1$

$$
\begin{aligned}
-\beta I + \sum_{(i,j)\in E} \lambda_{ij} E_{ij} \leq -J &\iff \beta(I - \sum_{(i,j)\in E} \frac{\lambda_{ij}}{\beta} E_{ij}) \geq J \\
&\iff I - \sum_{(i,j)\in E} \frac{\lambda_{ij}}{\beta} E_{ij}) \geq \frac{1}{\beta} 11^T \\
&\iff \begin{pmatrix} I - \sum_{(i,j)\in E} \frac{\lambda_{ij}}{\beta} E_{ij} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \begin{matrix} 1 & \dots & 1 \end{matrix} & \beta \end{pmatrix} \geq 0 \qquad \text{(By Schur Lemma bc } \beta > 0)
\end{aligned}
$$

Let's note this big matrix Z. It is clear that a matrix $Z \in S^{(n+1)\otimes(n+1)}$ is of this form iff it verifies the constraints of the following optimization problem:

$$
\begin{aligned}
\min_{\alpha, \lambda_{ij} \in \mathbb{R}} \quad & Z_{n+1,n+1} \\
\text{subject to} \quad & Z \geq 0 \\
& Z_{i,n+1} = Z_{ii} = 0 \\
& Z_{i,j} = 0 \forall \{i,j\} \in \bar{E}
\end{aligned}
$$

Which is then equal to $\vartheta(G)$

Let $C = \chi(\bar{G})$ By definition, there exist a partition of V: $\{V_1, \dots, V_C\}$ such that V_i is a clique for all $i \leq C$

- Define $1_{V_i} \in \mathbb{R}^n$ to be the indicator function of the set V_i, and note that $1 = \sum_{i \leq C} V_i$
- Define $z_i = \begin{pmatrix} 1_{V_i} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$. Note that:

$$
z_i z_i^T = \begin{pmatrix} 1_{V_i} 1_{V_i}^T & 1_{V_i} \\ 1_{V_i}^T & 1 \end{pmatrix}
$$

- Define

$$
Z = \sum_i z_i z_i^T = \begin{pmatrix} \sum 1_{V_i} 1_{V_i}^T & 1 \\ 1^T & C \end{pmatrix}
$$

. Z is positive semidefinite because it is a sum of psd terms $z_i z_i^T$

- $(1_{V_i} 1_{V_i}^T)_{kl} = (e_k^T 1_{V_i})(e_l^T 1_{V_i}) = 1_{V_i}(k) 1_{V_i}(l)$. If $(k,l) \in \bar{E}$, then the k^{th} node and the j^{th} node cannot be in the same V_i, and therfore $(1_{V_i} 1_{V_i}^T)_{kl} = 0$
- If $k = l$, all the terms in $\sum_i (1_{V_i} 1_{V_i}^T)_{kl}$ are zero except for the i for which the k^{th} node is in V_i, in which case it is equal to one.

As a conclusion, Z verifies all constraints of the dual, and $Z_{n+1,n+1} = C = \chi(\bar{G})$, so

$$
\chi(\bar{G}) \geq \vartheta(G)
$$

2

Consider $G = C_5$.

Using CVX to calculate $\vartheta(G)$

```
n = 5
J = ones(n, n);
cvx_begin sdp
variable X(n, n) symmetric;
maximize(trace(X*J))
X >= 0
X(5, 1) == 0
for i=1:4
    X(i, i+1) == 0
end
trace(X) == 1
cvx_end
ans=cvx_optval
```

```
2.2361
```

$2 < \vartheta(G) < 3$

- $\vartheta(G) \notin \mathbb{N}$
- $\alpha(G), \chi(\bar{G}) \in \mathbb{N}$

  No inequality can thus be tight.

2 Q2

1 ————-Duality—————

Let $x \in [0,1]^n$ such that $C_1, \dots C_k$

Consider

$$Y = \begin{pmatrix} x_1 & & & & x_1 \\ & x_2 & & 0 & x_2 \\ & & \ddots & & \vdots \\ & 0 & & x_n & x_n \\ x_1 & x_2 & \dots & x_n & 1 \end{pmatrix}$$

2

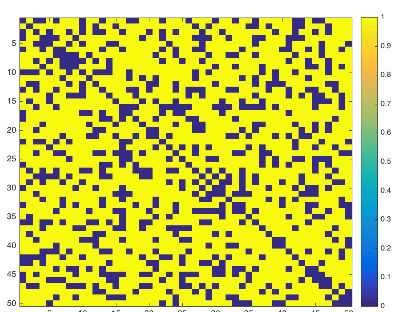

Figure 1: G Adjacency matrix

2.1 $\vartheta(G)$

```
n = 50
J = ones(n, n);

cvx_begin sdp
variable X(n, n) symmetric;
maximize(trace(X*J))
X >= 0
for i=1:n
    for j=1:i
        if G(i, j) == 1
            X(i, j) == 0
        end
    end
end
trace(X) == 1
cvx_end

ans=cvx_optval
```

5

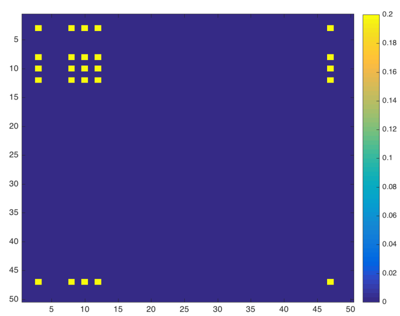

Figure 2: X optimal solution

Note that the resulting X is of rank 1, so it can be decomposed into $X = xx^T$. We check that $V_x = \{i, x_i \neq 0\}$ represents indeed a stable set.

```
[v,e] = eigs(full(X),1);
stableset = find(abs(v) > 0.01)
ans=stableset'
```

3 8 10 12 47

```
G(stableset, stableset)
```

Table 1: Subgraph of the nodes in the stableset

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Let's assume that there exist another stable set of size 5 V_y.
This would mean that there exist $v \in V_x$ such that imposing $X_{jj} = 0$ would not change α. Let's check:

```
  n = 50
  J = ones(n, n);
  opt = [stableset, zeros(5, 1)]
  for vi=1:5
      v = stableset(vi)
      cvx_begin sdp
      variable Y(n, n) symmetric;
      variable optvalue;
      maximize(trace(Y*J))
      Y >= 0
      for i=1:n
          for j=1:i
              if G(i, j) == 1
                  Y(i, j) == 0
              end
          end
      end
      Y(v,v) == 0
      trace(Y) == 1
      optvalue == trace(Y*J)
      cvx_end
      opt(vi, 2) = optvalue
  end
ans=opt
```

Table 2: Lovazs

Node removed	Lovasz of the subgraph
3	4.4463
8	4.5191
10	4.512
12	4.5586
47	4.4771

Since Lovasz number ϑ is an upper bound on α, This proves that any stable set not containing one of the nodes in V_x is of size less than 5.

We have just proved uniqueness of the stable set.

2.2 μ^{LP}

k = 2

```
cvx_begin
variable x(n)
maximize(sum(x))
for i=2:n
    for j=1:(i-1)
        if G(i, j) == 1
            x(i) + x(j) <= 1
        end
    end
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

25

k = 3

```
cvx_begin
variable x(n)
maximize(sum(x))
for i=2:n
    for j=1:(i-1)
        if G(i, j) == 1
            x(i) + x(j) <= 1
        end
        for r=1:(j-1)
            if G(i, j) +  G(j, r) + G(r, i) == 3
                x(i) + x(j) + x(r) <= 1
            end
        end
    end
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

16.667

k = 4

```
M = 50
cvx_begin
  variable x(n)
  maximize(sum(x))
  for i=2:M
      for j=1:(i-1)
```

```
        if G(i, j) == 0
            continue
        end
        x(i) + x(j) <= 1
        for r=1:(j-1)
            if G(j, r) == 0 || G(r, i) == 0
                continue
            end
            x(i) + x(j) + x(r) <= 1
            for p =1:(r-1)
                if G(i, p) == 0 || G(j, p) == 0 || G(r, p) == 0
                    continue
                end
                x(i) + x(j) + x(r) + x(p) <= 1
            end
        end
    end
end
0 <= x <= 1
cvx_end

ans=cvx_optval
```

```
12.5
```

3 Problem 3

1. Let $(a, b), (u, v)$ be two nodes in $G_A \otimes G_B$ The two nodes are connected if and only if:

- $A_{au} = 1, A_{bv} = 1$
- $a = u, A_{bv} = 1$
- $A_{au} = 1, b = v$

This can be summerised as $(A_{au} + \delta_{au})(A_{bv} + \delta_{bv}) - \delta_{au}\delta_{bv} = 1$

So the adjacency matrix of $G_A \otimes G_B$ is $(A + I_n) \otimes (B + I_m) - I_{nm}$.

Where $\otimes$ denote the Kronecker product: $(A \otimes B)_{p(r-1)+v, q(s-1)+w} = A_{rs}B_{vw}$

2.

4 Problem 4

1. (1) is equivalent to

$$\begin{cases} x^T Ay & = \max_{\tilde{x} \in \Delta_m} \tilde{x}^T Ay \\ x^T By & = \max_{\tilde{y} \in \Delta_n} x^T B\tilde{y} \end{cases}$$

Consider the first problem:

$$\max_{\tilde{x} \in \Delta_m} \tilde{x}^T Ay$$

This is an LP whose feasible region $\Delta_m = conv(e_i, i = 1 \ldots m)$ is compact, so the maximum is attained in one of the extreme points e_{i_0}. Therefore

$$x^T Ay = \max_{\tilde{x}\in\Delta_m} \tilde{x}^T Ay \iff x^T Ay = e_{i_0}^T Ay = \max_i e_i^T Ay \iff x^T Ay \geq e_i^T Ay \forall i$$

Same argument applies for y so that:

$$x^T By = \max_{\tilde{y}\in\Delta_n} x^T B\tilde{y} \iff x^T Ay \geq x^T Be_i \forall i$$

So:

$$(1) \iff \begin{cases} x^T Ay & \geq e_i^T Ay \ \forall i = 1 \dots m \\ x^T By & \geq x^T Ae_i \ \forall i = 1 \dots n \end{cases}$$

2.

$x \in \Delta_m, y \in \Delta_n$

Note $z = \begin{pmatrix} x \\ y \end{pmatrix}, u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $M = zz^T = \begin{pmatrix} xx^T & xy^T \\ yx^T & yy^T \end{pmatrix}$

Note that

- $M \succeq 0$
- $rank(M) = 1$
- $u^T Mu = 1^T xx^T 1 = (1^T x)^2 = 1$, similarly $v^T Mv = 1$
- $M_{(i+m),j} = x_i y_j \geq 0$

Now let $M \in S^{n+m}$, verifying all the previous conditions. Then by cholesky, there existe a vector $z \in \mathbb{R}^{n+m}$, such that: $M = zz^T$

- Let's decompose $z := \begin{pmatrix} x \\ y \end{pmatrix}$
- $1 = u^T Mu \implies \sum_i x_i = \pm 1$, similarly, $\sum y_j = \pm 1$
- $M_{(i+m),j} = x_i y_j \geq 0 \implies x_i, y_j$ all share the same sign.
- We can always change x and/or y to $-x, -y$ to make $x, y \geq 0$ and therefore $\sum_i x_i = \sum_j y_j = 1$, e.g $x \in \Delta_m, y \in \Delta_n$
- $Mu = \begin{pmatrix} xx^T 1 \\ yx^T 1 \end{pmatrix} = \begin{pmatrix} (x^T 1)x \\ (x^T 1)y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
- $x = \underbrace{\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}}_{J_1} Mu$
- $y = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}}_{J_2} Mu$
- $yx^T = M_{n:n+m,1:n}$

The constraint of (2) can be formulated as follow:

- $$x^T Ay = tr(yx^T A) = tr(M_{n:n+m,1:n} A)$$

- $e_i^T Ay = e_i^T AJ_2Mu = tr(ue_i^T AJ_2M)$
- $x^T By = tr(M_{n:n+m,1:n}B)$
- $x^T Be_i = (J_1Mu)^T Be_i = u^T MJ_1Be_i = tr(MJ_1Be_iu^T)$

In conclusion:

- $M \succeq 0$
- $rank(M) = 1$
- $M_{i+m,j} \geq 0$
- $tr(Muu^T) = 1$
- $tr(M_{n:n+m,1:n}A) \geq tr(Mue_i^T AJ_2)$
- $tr(M_{n:n+m,1:n}B) \geq tr(MJ_1Be_iu^T)$