ORF524 - Problem Set 2

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Problem 1

$\min \epsilon$		$2x_1$ $+\mu x_1$	$-6x_2$ $+\mu x_2$	$0x_3$ $+\mu x_3$
w_1	$-1+\mu$	1	1	1
w_2	$1+\mu$	-2	1	-1

Feasibility and optimality conditions: $-6+\mu \geq 0$, $-1+\mu \geq 0$. $\Rightarrow \mu \geq 6$
for $\mu = 6$

$\min \epsilon$		$8x_1$	$0x_2$	$6x_3$
w_1	5	1	1	1
w_2	7	-2	1	-1

Entering basis: x_2. The ratio test fails $\Rightarrow$ The problem is unbounded.
If we go back to the original problem:

- It is feasible. ($x^* = (0, 0, 1)$ is a feasible solution for example)
- It is unbounded, because $\forall x_2 > 0\, x^* + x_2 e_2$ is feasible, and $c^T(x^* + x_2 e_2) = -6x_2 \to -\infty$

Problem 2

Let A be the col player and B the row player.
The payoff matrix is: (we set the value of a nickel to 1)

$$M = \begin{pmatrix} N & -D \\ -N & D \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

The game can be written as

$$\max_{a_1+a_2=1} \min_{b_1+b_2=1} b^T M a$$

Or

$$\min_{b_1+b_2=1} \max_{a_1+a_2=1} b^T M a$$

And the equivalent LP form:
$\max_{v,a} v$ subject to: $v \leq e_i M a$, $a_1 + a_2 = 1$, $a \geq 0$

$$\begin{aligned} b^T M a &= (b_1 - b_2, -2(b_1 - b_2))a && (1) \\ &= (b_1 - b_2)(1, -2)a && (2) \\ &= (b_1 - b_2)(a_1 - 2a_2) && (3) \\ &= (2b_1 - 1)(3a_1 - 2) && (4) \\ &:= f(a_1, b_1) && (5) \end{aligned}$$

Using the equiavlent LP form, we have that:

$$a_1^* = \arg\max_{a_1} [\min(f(a_1, 0), f(a_1, 1))] = \arg\max_{a_1} \min(3a_1 - 2, -(3a_1, 2)) = \arg\max_{a_1} -|3a_1 - 2| = \frac{2}{3}$$

$$b_1^* = \arg\min_{b_1} [\max(f(0, b_1), f(1, b_1))] = \arg\min_{b_1} \max(2b_1 - 1, -(2b_1 - 1)) = \arg\min_{b_1} |2b_1 - 1| = \frac{1}{2}$$

The Nash equilibrium is thus attained for $a = (\frac{2}{3}, \frac{1}{3}), b = (\frac{1}{2}, \frac{1}{2})$

Problem 3

By strong duality, if the (P) has a solution x, so does (D). (let's call it p)

Let's assume that x, the solution to (P), is unique and non degenerate, and prove that the solution to (D) is unique. If x is unique, x is a BFS. Let B be the associated basis. By complementary slackness, we have that $(p^T A_B - c_B^T)x_b = 0$, since $x_b > 0$: $p = (A_B^T)^{-1}c_B$.

Let's now prove that: p degenrate and x unique $\Rightarrow$ x degenrate.

If p is degenerate, there exist an entry that is 0, meaning that the reduced cost of one of the non basic variables in the primal problem is 0, and we can enter this variable without changing the objective function.As a result, we get two different basis B_1, B_2 that are associated with optimal solutions. If we assume the uniqueness of x, they must be both associated to x. Let $i \in B_1 \setminus B_2$, $x_i = 0$, so x is degenerate.

Conclusion: The solution to (P) is unique and non degenerate $\Rightarrow$ The solution to (D) is unique and non degenerate.

Problem 4

1. Let x and y be in the solution set, and $\lambda \in [0, 1]$, then $A(\lambda x + (1 - \lambda y)) = \lambda b + (1 - \lambda)b = b$, and $c^T(\lambda x + (1 - \lambda)y) = \lambda c^T x + (1 - \lambda)c^T y = \lambda v^* + (1 - \lambda)v^* = \lambda c^T x + (1 - \lambda)c^T y = v^*$, so the solution set is convex.

2. Let's consider the problem:

$$\max_{x \geq 0} e_j^T x \text{ s.t. } Ax = b, c^T x \leq v^*$$

Its dual is

$$\min_{p,z \geq 0} b^T p + zv^* \text{ s.t. } e_j - A^T p \leq zc$$

can also be written as :

$$\max_{p,z \geq 0} b^T p - zv^* \text{ s.t. } e_j + A^T p \leq zc$$

We know from the question that the primal problem has an optimal value of 0, and so does the dual. Let (p, z) be a solution for the dual.

(a) If $z \neq 0$:
And let's note $u = \frac{p}{z}$. We have that:
$b^T u - v^* = 0$, and $\frac{e_j}{z} + A^T u \leq c$ which means that u is optimal for (D), and

$$A_j^T u \leq c_j - \frac{e_j}{z} < c_j$$

(b) If $z = 0$, we have the existence of p such that $b^T p = 0$ and $A^T p + e_j \leq 0$
$A^T(p^* + p) \leq c - e_j$, and $b^T(p^* + p) = v^*$ we can take $u = p^* + p$ to be a solution to (P) that satisfies $A_j^T u < c_j$.