# ORF527 - Problem Set 2

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### Q.1

$(W_{t_1}, \dots W_{t_n})$ and $(W'_{t_1}, \dots W'_{t_n})$ are both guassian with the same mean and covariance matrix, so they do have the same carateristic function and therefore the same distribution. So

$$\mathbb{E}f(W_{t_1}, \dots W_{t_n}) = \mathbb{E}f(W'_{t_1}, \dots W'_{t_n})$$

### Q.2

Since $B$ is closed, $W_t \notin B \iff \exists \varepsilon > 0\ \forall y \in B(y, \varepsilon) W_y \notin B\ \{\tau \leq t\} =$

### Q.3

1. $\{\tau \leq t\} = \cap_{\varepsilon > 0} \underbrace{\{\exists s \leq t + \varepsilon, X_s \in B\}}_{U_{t+\varepsilon}}$, $U_t = \{\exists s \leq t, X_s \in B\} \in \mathcal{F}_t$:

   - $U_t = \cup_{s \leq t}\{X_s \in B\} = \cup_{s \leq t, s \in \mathbb{Q}\{X_s \in B\}}$. Indeed, Let $\omega \in \{X_s \in B\}$ for some $s \leq t$. Since $B$ is open there exist $\alpha > 0$ such that $\mathcal{B}(X_s(\omega), \alpha) \subset B$. By continuity of $X_s$, there exist $s' \in \mathbb{Q}$ smaller than $t$ such that $|X_s(\omega) - X_{s'}(\omega)| < \alpha$, therefore $\omega \in \{X_{s'} \in B\}$
   - $\cup_{s \leq t, s \in \mathbb{Q}}\{X_s \in B\} \in \mathcal{F}_t$ as a countable union of sets in $\mathcal{F}_t$.

### Q.4

Without loss of generality we can assume $i \in J \Rightarrow i + t \in J$. $(W_i)_{i \in J}$ is a discrete markov chain. Let $A \in \mathcal{F}_\tau$, so that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$.

$E[1_A f(W_{t+\tau})] = E[1_A \int f(x) \frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}} dx]$

$$\begin{aligned}
E[1_A \int f(x) \frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}} dx] &= \sum_J E[1_{A,\tau=i} \int f(x) \frac{e^{-(x-W_i)^2/2t}}{\sqrt{2\pi t}} dx] \\
&= \sum_J E[1_{A,\tau=i} E[f(W_{i+t})|\mathcal{F}_i]] && \text{(Markov Property)} \\
&= \sum_J E[E[1_{A,\tau=i} f(W_{i+t})|\mathcal{F}_i]] && \text{(Because } A \in \mathcal{F}_\tau\text{)} \\
&= \sum_J E[E[1_{A,\tau=i} f(W_{\tau+t})|\mathcal{F}_i]] \\
&= \sum_J E[1_{A,\tau=i} f(W_{\tau+t})] \\
&= E[1_A f(W_{\tau+t})]
\end{aligned}$$

Clearly, $\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$ is $W_\tau$-measurable. so:

$$E[W_{t+\tau}|\mathcal{F}_\tau] = \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$$