# ORF526 - Problem Set 3

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**Question 1**

a) $\cap_{n\in\mathbb{N}} A_n^c$

b) $\omega$ is in the event means it is in infinitely many $A_n$, which quivalent to saying if we take finitely many $A_m$, $\omega$ is still a subset of the union of the rest of the $A_n$: $\cap_{m\in\mathbb{N}} \cup_{m\le n} A_n$

c) That's the opposite of $b$), $(\cap_{m\in\mathbb{N}} \cup_{m\le n} A_n)^c = \cup_{m\in\mathbb{N}} \cap_{m\le n} A_n^c$

d) $\omega$ in the event means that it has to be in exactly two of the $A_i$, ie $\cap_{i,j\in\mathbb{N},i\neq j} (A_i \cap A_j \cap (\cup_{n\neq i,n\neq j} A_n^c))$

e) This event can be expressed as "$\Phi$ never occurs at even times", ie $\cap_{n\in\mathbb{N}} A_{2n}^c$

**Question 2**

- $\varepsilon \subseteq \sigma(\varepsilon)$, so $\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$. since the RHS is already a $\sigma$-algebra (showed in class), $\sigma\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$
- Let's note $B := \sigma\{f^{-1}(A) : A \in \varepsilon\}$, and $C := \{A : f^{-1}(A) \in B\}$.
  - $C$ is a $\sigma$-algebra containing $\varepsilon$, so $\sigma(\varepsilon) \subseteq C$
  - As a consequence, for every $A \in \sigma(\varepsilon)$, $A \in C$, ie $f^{-1}(A) \in B$.

  We have just proved that $\{f^{-1}(A) : A \in \sigma(\varepsilon)\} \subseteq B$

**Question 3**

- $X = \lim_n X_n = \lim_n \sup_{k\ge n} X_k$
- For $x \in \mathbb{R} \cup \{\infty\}$, $X \le x$ is quivalent to $\exists n \, \forall k \ge n \, X_k \le x$
- $\{X \le x\} = \cup_{n\in\mathbb{N}} \cap_{k\ge n} \{X_k \le x\}$ is then measuable.

**Question 4**

- Let's call $Q = P(\{1, .., n\})$. For $i = 1..n$:

$$A_i = \bigcup_{I\subseteq Q, i\in I} (\underset{k\in I}{\cap} A_k) \cap (\underset{k\in I^c}{\cap} A_k^c)$$

  This is true because every $\omega \in A_i$ is in some of the other $A_k$ ($k \in I$), and in the complement of the others ($k \notin I$).

  Note that this is a union of disjoint sets (because in the intersection of two of those sets there will be a $A_k$ and its complement).

Let's call $I_i := \{I \in Q : \sum_{i\in I} a_i = x_i\}$, ie the different possible combinations for the $A_i$ where $\omega \in \Omega$ can be so that its image by $f$ equals $x_i$. Note that $\omega$ can not be in any other set $A_i$, for $i \notin I_i$ because $a_i > 0$.

Written differently, $\{f = x_i\} = \bigcup_{I\in I_i} \underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c$. And as a result of the sets being disjoint:

$$\mu(f = x_i) = \sum_{I\in I_i} \mu(\underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c)$$

Note that any sum index by some $I \in Q$ in finite because $|I| \leq n$, and this we can rearrange the sums in any order.

$$\begin{aligned}
\sum_{i=1}^{m} x_i\mu(f = x_i) &= \sum_{i=1}^{m} x_i \sum_{I\in I_i} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1}^{m} \left( \sum_{I\in I_i} (\sum_{k\in I} a_k)\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \right) \\
&= \sum_{I\in Q} (\sum_{k\in I} a_k)\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{because } Q = \underset{k=1..m}{\cup} I_i \\
&= \sum_{i=1..n} \sum_{I\in Q, i\in I} a_i\mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{By rearranging the sum} \\
&= \sum_{i=1..n} a_i \sum_{I\in Q, i\in I} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1..n} a_i\mu(A_i)
\end{aligned}$$

- Let's first prove that if a set $A$ has measure 1, for all measurable sets $B$, $\mu(A \cap B) = \mu(B)$. This holds because

$$\mu(B) \geq \mu(A \cap B) = 1 - \mu(A^c \cup B^c) \geq 1 - \mu(B^c) = \mu(B)$$

Let $x \in \Omega$

$\mu(\{g = x\}) \geq \mu(\{g = x\} \cap \{f = x\}) = \mu(\{f = x\} \cap \{f = g\}) = \mu(\{f = x\})$ Symmetrically, we prove that $\mu(\{f = x\}) \geq \mu(\{g = x\})$, and thus this two quantities are equal.

This proves that $\sum_{x\in f(\Omega)} x\mu(f = x) = \sum_{x\in g(\Omega)} x\mu(g = x)$ by noting that $\mu(f = x) = 0$ for $x \notin f(\Omega)$.

And by the last question:

$$\sum_{i=1..n} a_i\mu(A_i) = \sum_{i=1..k} b_i\mu(B_i)$$

**Question 5**

If $h = gof$, for $A \in B(\mathbb{R})$, $h^{-1}(A) = f^{-1}(g^{-1}(A)) \in \sigma(f)$ because $g^{-1}(A) \in F$

For the other direction:

- Let's first suppose that $h := \sum_{i=1..n} a_i 1_{A_i}$ for $a_i > 0$ and $A_i \in \sigma(f)$. Let $h(\Omega_1) = \{x_1, ..., x_m\}$.

  We can write $h = \sum_{j=1..m} x_j 1_{\{h=x_j\}}$
  $$\{h = x_j\} = f^{-1}(A_j)$$
  $A_j$ are disjoint because for $i \neq j$, $f^{-1}(A_i) \cap f^{-1}(A_j) = f^{-1}(A_i \cap A_j) \subseteq \{h = x_i\} \cap \{h = x_j\} = \emptyset$

  We define $g(x) = x_i$ on $A_i$, and any other value outside.

  Let $\omega \in \Omega_1$, and $x_i = h(\omega)$, then $\omega \in f^{-1}(A_i)$, so $f(\omega) \in A_i$ and $gof(\omega) = x_i = h(\omega)$. So $h = gof$.

- Let $h$ be now just non negative.

  We can approximate $h$ point wise by a sequence of non negative simple functions $h_n$ measurable w.r.t the $\sigma$-algebra generated by $f$. We know now that $h_n$ can be written as $g_n of$.

  .By convergence of $h_n$, the sequence $g_n$ converges point wise in a set $C$ containing the image of $f$. With $C := \{x | \lim_n g_n(x) \text{ exists}\} = \{x | \text{limsup}_n g_n(x) \leq \text{liminf}_n g_n(x)\}$ is measurable.

  Let call $g$ the limit, and set $g$ to 0 outside $C$, then $h = gof$. and $g = (\limsup_n g_n) 1_C$ is measurable as the product of two measurable functions.

- Let $h$ be just measurable, $h = h^+ - h^- = g^+ of - g^- of = (g^+ - g^-) of$ with $g^+, g^-$ the corresponding $g$ for $h^+, h^-$

  If we consider the trivial $\sigma$-algebra $\{\mathbb{R}, \emptyset\}$ instead of Borel, and $f$ constant equal to 0, then any $h$ is measurable w.r.t the $\sigma$ algebra generated by $f$. If now $h$ is not constant, it cannot be written as $gof = g(0)$