ORF527 - Problem Set 2

Bachir EL KHADIR

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Q.1

Let $\varepsilon > 0, x \in [0,1]$.

- $\sup |f^n - f| \to_n 0$, let $n \in \mathbb{N}$ $\forall t \in [0,1]$ $|f^n(t) - f(t)| \leq \frac{\varepsilon}{3}$.
- f^n is continuous, let $\delta > 0$ such that $\forall y \in [0,1], |x-y| < \delta \Rightarrow |f^n(x) - f^n(y)| < \frac{\varepsilon}{3}$.
- Let $y \in [0,1]$ such that $|x - y| < \delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f^n(x)| + |f^n(x) - f^n(y)| + |f^n(y) - f(y)| \\ &\leq 3\frac{\varepsilon}{3} \\ &\leq \varepsilon \end{aligned}$$

Which conclude the proof.

Q.2

3.2

let $X \sim \mathcal{N}(0,1)$, and $\varepsilon \sim \mathcal{B}(-1, 1, \frac{1}{2})$ be two independant rv. And Let $Y = \varepsilon X$

- By symmetry of the distribution of X: $F_Y(y) = P(Y \leq y) = P(\varepsilon X \leq y) = E[P(\varepsilon X \leq y|\varepsilon)] = \frac{1}{2}P(X \leq y) + \frac{1}{2}P(-X \leq y) = P(X \leq x)$ so $Y \sim \mathcal{N}(0,1)$.
- $cov(X, Y) = E[X^2\varepsilon] = E[X^2]E[\varepsilon] = 0$
- X, Y are not independent. Indeed, Let $\alpha := \mathbb{P}(|X| > 0.5)$.
  - $P(|X| > 0.5, |Y| > 0.5) = P(|X| > 0.5) = \alpha$,
  - $P(|X| > 0.5)P(|Y| > 0.5) = P(|X| > 0.5)^2 = \alpha^2$

  But since $\alpha \notin \{0, 1\}$, $\alpha \neq \alpha^2$.

3.3

For a random variable X, let's call Φ_X its caracteristic function.

(a) For $t \in \mathbb{R}^n$, $\Phi_{AV}(t) = \mathbb{E}[e^{iV^TA^Tt}] = \Phi_V(A^Tt) = e^{(A\mu)^TV - \frac{1}{2}(t^TA\Sigma A^T)t}$, so $V \sim \mathcal{N}(A\mu, A\Sigma A^T)$

(b) By symmetry of the gaussisan distribution, $-Y$ has the same distribution as Y. So it suffices to show that the result holds for $X + Y$. By independence: $\Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) = \Phi_X(t)^2 = e^{i2\mu t - \frac{1}{2}2t^2}$, so $X + Y \sim \mathcal{N}(0, 2)$.

(c) If $cov(X,Y)=0$, the covariance matrix of the guassian process (X,Y) is the identity, therefore

$$\forall t=(t_1,t_2)\in\mathbb{R}^2\ \Phi_{(X,Y)}(t)=E[e^{i\mu_X t_1+i\mu_Y t_2-\frac{1}{2}\sigma_X^2 t_1^2--\frac{1}{2}\sigma_Y^2 t_2^2}]=\Phi_X(t_1)\Phi_Y(t_2)$$

Where $X\sim\mathcal{N}(\mu_X,\sigma_X^2), Y\sim\mathcal{N}(\mu_Y,\sigma_Y^2)$. Therefore X,Y are independent.

(d) Let $\alpha:=\frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}$. $(Y-\alpha X,X)$ is gaussian and $\operatorname{cov}(Y-\alpha X,X)=0$. So $Y-\alpha X\perp X$. Therefore:

$$\mu_{Y|X=x}=E[Y|X=x]=E[Y-\alpha X]+E[\alpha X|X=x]=\mu_Y-\alpha\mu_X+\alpha x=\mu_Y+\frac{\operatorname{cov}(X,Y)}{\operatorname{var}X}(x-\mu_X)$$

$$\sigma_Y^2=\operatorname{var}(Y|X=x)=\operatorname{var}(Y-\alpha X)+\underbrace{\operatorname{var}(\alpha X|X=x)}_{=0}=\operatorname{var}Y+\alpha^2\operatorname{var}X-2\alpha\operatorname{cov}(X,Y)=\sigma_Y^2-\frac{\operatorname{cov}(X,Y)^2}{\operatorname{var}X}$$

Q.3

(a) Since $\forall n>0,\Delta_n(1)=0,\ B_1=\sum_{n=0}^{\infty}\lambda_n Z_n\Delta_n(1)=\lambda_0 Z_0\Delta_0(1)$.

(b) If $s<t$, then $cov(B_s,B_t)=\operatorname{cov}(B_s-B_t,B_s)+\operatorname{var}(B_s)=s$.

$\operatorname{cov}(U_s,U_t)=\operatorname{cov}(B_s-sB_1,B_t-tB_1)=\operatorname{cov}(B_s,B_t)+ts\operatorname{cov}(B_1,B_1)-t\operatorname{cov}(B_s,B_1)-s(B_1,B_t)=s+ts-ts-ts=s-ts=s(1-t)$.

(c) For $s<t$, $cov(X_t,X_s)=g(t)g(s)h(s)$ should be equale to $s(1-t)$. We can take $g:t\to 1-t$, $h:t\to\frac{s}{1-s}$ defined on $[0,1)$