Problem set 7, ORF527

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1 8.3 (Steele)

(a)

u and v are C^∞

- $\partial_x\partial_x u = \partial_x\partial_y v = \partial_y\partial_x v$
- $\partial_y\partial_y u = -\partial_y\partial_x v$ Which proves that $\Delta u = 0$

  We conclude the same for v by considering the function $-if$

$\exp(z)) = e^x\cos(y) + ie^x\sin(y)$
$z\exp(z) = e^x(x+iy)(\cos(y)+i\sin(y)) = e^x(x\cos(y) - y\sin(y)) + ie^x(x\sin(y)+y\cos(y))$
We conclude that the following function of (x,y) are harmonic: $e^x\cos(y), e^x\sin(y), e^x(x\cos(y)-y\sin(y)), e^x(x\sin(y)+y\cos(y))$

(b)

$z^2 = (x^2-y^2)+i2xy$, which give that x^2-y^2, xy are both harmonic functions.
The process $Z_t = (B_t^1)^2 - (B_t^2)^2$ is a local martingal.
Let $\tau_\alpha = \inf\{t, B_t \in H(\alpha)\} = \inf\{t, Z_t = \alpha\}$
Let $\tau = \tau_1 \wedge \tau_5$ so that $Z_{t\wedge\tau}$ is bounded between 1 and 5, so it is a martingale.
Let's now prove that $\tau < \infty$. Indeed, $B_t \in B(0,\frac{1}{2}) \implies (B_t^1)^2 - (B_t^2)^2 \le (B_t^1)^2 + (B_t^2)^2 < 1 \implies \tau \le t$ so: $P(\tau<\infty) \ge P(\exists t > 0\ B_t \in B(0,\frac{1}{2})) = 1$
Now by dominated convergence theorem: $4 = E[Z_0] = E[Z_{t\wedge\tau}] \to E[Z_\tau] = P(\tau=\tau_1)\times 1 + P(\tau=\tau_5)\times 5$ which means that: $P(\tau=\tau_1) = 1 - P(\tau=\tau_5) = \frac{1}{4}$

2 8.4 (Steele)

(a) $(*) \iff f_t = -\frac{1}{2}f_{xx} \iff \phi'(t)\psi(x) = -\frac{1}{2}\phi(t)\psi''(x)$
$\phi = 0$ or $\psi = 0$ lead to trivial solutions.
Let's assume there exist t_0, x_0 such that $\phi(t_0) \neq 0, \psi(x_0) \neq 0$

$$(*) \iff \phi'(t) = -\frac{\psi''(x_0)}{2\psi(x_0)}\phi(t), \psi''(x) = -2\frac{\phi'(t_0)}{\phi(t_0)}\psi(x)$$

Which means $\phi(t) = ae^{bt}$, $\psi(x) = \alpha e^{wx} + \beta e^{-wx}$ By pluggin this function into the equation, we get $b = -\frac{w^2}{2}$
General solution:

$$f(t,x) = ae^{-\frac{w^2}{2}t}(\alpha e^{wx} + \beta e^{-wx})$$

(b)

$$\begin{aligned} M_t &= \sum_k \frac{(\alpha B_t - \frac{t}{2}\alpha^2)^k}{k!} \\ &= 1 + \alpha B_t - \frac{t}{2}\alpha^2 + \frac{1}{2}(\alpha B_t - \frac{t}{2}\alpha^2)^2 + \frac{1}{6}(\alpha B_t - \frac{t}{2}\alpha^2)^3 + \alpha^4 P(\alpha) && \text{(Where } P \text{ some polynomial)} \\ &= 1 + \alpha B_t + \alpha^2(\frac{B_t^2}{2} - \frac{t}{2}) + \alpha^3(\frac{1}{6}B_t^3 - \frac{1}{2}tB_t) + \alpha^4 Q(\alpha) \end{aligned}$$

Let $s < t$. On one hand:

$$E[M_t|F_s] = M_s = \sum_{k=1}^{\infty} \alpha^k H_k(t, B_t)$$

on the other hand:

$$\begin{aligned} E[M_t|F_s] &= E[\sum_{k=0}^{\infty} \alpha^k H_k(t, B_t)|F_s] \\ &= \sum_{k=0}^{\infty} \alpha^k E[H_k(t, B_t)|F_s] \qquad (*) \end{aligned}$$

This is valid for all α, which means that $E[H_k(t, B_t)|F_s] = H_k(s, B_s)$, and that $H_k(t, B_t)$ is a martingale.

To justify $(*)$, we use dominated convergence applied to the series $\sum_{k=0}^{\infty} \alpha^k H_k(t, B_t)$ Indeed, for $n \in \mathbb{N}$ and $\alpha \geq 0$: $|\sum_{k=0}^{\infty} \alpha^k H_k(t, B_t)| \leq \sum_{k=0}^{\infty} \alpha^k |H_k(t, B_t)|$ and notice that $\sum_{k=0}^{\infty} \alpha^k |H_k(t, B_t)| \leq \sum_k \frac{(\alpha|B_t| + \frac{t}{2}\alpha^2)^k}{k!} = \exp(\alpha|B_t| + \alpha^2 \frac{t}{2}) \in L_1$ which justify the swapping of $\sum$ and E.

3 8.5 (Steel)

a) $f(X) = (X^T X)^{-\frac{1}{2}}$ $\partial_x f = -\frac{1}{2} \frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -x f(x)^3$ $\partial_{xx} f = -f(x, y, z)^3 - 3x \partial_x f(x, y, z) f(x, y, z)^2 = -f(x, y, z)^3 + 3x^2 f(x, y, z)^5$ By symmetry of x, y and z:

- $\partial_{yy} f = -f^3 + 3y^2 f^5$
- $\partial_{zz} f = -f^3 + 3z^2 f^5$

So: $\Delta f = -3f^3 + 3(x^2 + y^2 + z^2) f^5 = -3f^3 + 3f^{-2} f^5 = 0$

Let $\tau_n = \inf\{t \geq 1, |B_t|_2 \leq \frac{1}{n}\}$

$P(|B_1| = 0) = P(\mathcal{N}(0, 1) = 0)^3 = 0$, so with probability one, $B_{t \wedge \tau_n} \notin B(0, \frac{1}{2n})$ f is harmonic on $\mathbb{R}^n \setminus B(0, \frac{1}{2n})$, so $f(B_{t \wedge \tau_n})$ is a local martingale, and so is $f(B_t)$

b) Since $\frac{1}{\sqrt{t}} B_t \sim (N_1, N_2, N_3) \sim \mathcal{N}(0, I_3)$:

$$\begin{aligned} E[M_t^2] &= E[\frac{1}{t(N_1^2 + N_2^2 + N_3^2)}] \\ &= \frac{1}{t} \frac{1}{\sqrt{(2\pi)^3}} \int \frac{1}{x^2 + y^2 + z^2} e^{-\frac{1}{2} x^2 + y^2 + z^2} dx dy dz \\ &= \frac{1}{t} \frac{1}{\sqrt{(2\pi)^3}} \int_{\theta \in [0, 2\pi], \phi \in [0, \pi]} \sin(\theta) d\theta d\phi \int_0^{\infty} \frac{1}{r^2} e^{-\frac{1}{2} r^2} r^2 dr \\ &= \frac{1}{t} \frac{1}{\sqrt{(2\pi)^3}} \int_{\theta \in [0, \pi], \phi \in [0, 2\pi]} d(-\cos(\theta)) d\phi \int_0^{\infty} e^{-\frac{1}{2} r^2} dr \\ &= \frac{1}{t} \frac{1}{\sqrt{(2\pi)^3}} 4\pi \sqrt{\frac{\pi}{2}} \\ &= \frac{1}{t} \end{aligned}$$

c) Assume M_t is martingale, by Jensen: $\frac{1}{t} = E[M_t^2] \geq (E[M_t])^2 \geq (E[M_1])^2$

By taking t to infinity, this leads to $E[M_1] = 0$, and since M_1 is non negative, to $M_1 = 0$ as. which contradicts the fact that $E[M_1^2] = 1$

4 Q2

a

$$f_\varepsilon'(x) = \begin{cases} 1 & \text{if } x > \varepsilon \\ -1 & \text{if } -x < -\varepsilon \\ \frac{x}{\varepsilon} & \text{if } |x| < \varepsilon \end{cases}$$

$$f_\varepsilon''(x) = \begin{cases} 0 & \text{if } |x| > \varepsilon \\ \frac{1}{\varepsilon} & \text{if } |x| < \varepsilon \end{cases} = \frac{1_{|x|<\varepsilon}}{\varepsilon}$$

Let's pretend that Ito formula works:

$$\begin{aligned} f_\varepsilon(W_t) &= f_\varepsilon(0) + \int_0^t f_\varepsilon'(W_s)dW_s + \frac{1}{2}\int_0^t f_\varepsilon''(W_s)ds \\ &= f_\varepsilon(0) + \int_0^t f_\varepsilon'(W_s)dW_s + \frac{1}{2\varepsilon}\int_0^t 1_{|W_s|<\varepsilon}ds \\ &= f_\varepsilon(0) + \int_0^t f_\varepsilon'(W_s)dW_s + \frac{1}{2\varepsilon}\lambda\{s \in [0,t], |W_s| < \varepsilon\} \end{aligned}$$

By Fubini-Tonelli: $E[\lambda\{s \in [0,t], W_s = \pm\varepsilon\}] = \int_0^t [P(W_s = \varepsilon) + P(W_s = \varepsilon)]ds = 0$ Therefore $\lambda\{s \in [0,t], |W_s| < \varepsilon\} = \lambda\{s \in [0,t], |W_s| \leq \varepsilon\}$ as.

Conclusion: $f(W_t) = \int_0^t f_\varepsilon'(W_s)dW_s + \frac{1}{2\varepsilon}\lambda\{s \in [0,t], |W_s| \leq \varepsilon\}$

b

$$\begin{aligned} \mathbb{E}[(f_\varepsilon'(W_s) - sign(W_s))dW_s)^2] &= \int_0^t E[(f_\varepsilon'(W_s) - sign(W_s))^2]ds \\ &\leq \int_0^t E[1_{|W_s|\leq\varepsilon}(1 + \frac{|W_s|}{\varepsilon})^2]ds \\ &\leq 4\int_0^t E[1_{|W_s|\leq\varepsilon}]ds \end{aligned}$$

By dominated convergence theorem, this quantity converges to 0.

From Ito formula:

$\frac{1}{2\varepsilon}\lambda\{s \in [0,t], |W_s| < \varepsilon\} = f_\varepsilon(W_t) - f_\varepsilon(0) - \int_0^t f_\varepsilon'(W_s)dW_s$

We can easily notice that $f_\varepsilon(x) \leq |x|$.

We have proven that:

- $f_\varepsilon(W_t) - f_\varepsilon(0) \to_{\varepsilon\downarrow 0} |W_t|$ almost surely. Since $|f_\varepsilon(W_t) - f_\varepsilon(0)| \leq |W_t| + \varepsilon \in L_2$, by dominated convergence theorem, the convergence holds in L_2
- $\int_0^t f_\varepsilon'(W_s)dW_s \to_{\varepsilon\downarrow 0} \int_0^t sign(W_s)dW_s$ in L_2

This proves that $\frac{1}{2\varepsilon}\lambda\{s \in [0,t], |W_s| \leq \varepsilon\}$ converges in L_2 to $L_t := |W_t| - \int_0^t sign(W_s)ds$

L_t is non-decreasing because if $u < v$, $L_v = \lim_\varepsilon \frac{1}{2\varepsilon}(\lambda\{s \in [0,u], |W_s| \leq \varepsilon\} + \lambda\{s \in [u,v], |W_s| \leq \varepsilon\} \geq L_u$

3

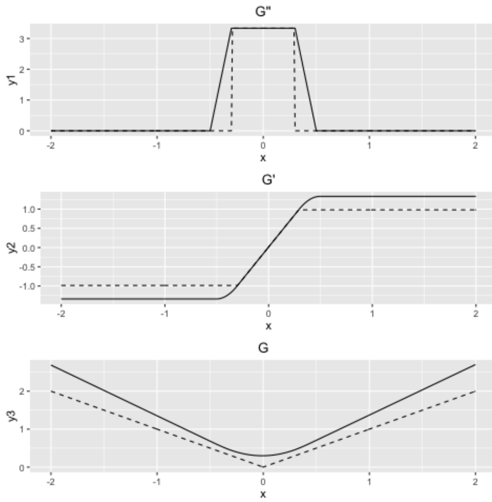

Figure 1: Approximation of f

Let $n \in \mathbb{N}^*$ Let $\varepsilon > 0$, $g \in C^0$ an approximation f''_ε , such that g is symmetric and:

$$g_n(x) = \begin{cases} 0 & \text{if } x \geq \varepsilon + \frac{1}{n} \\ \frac{1}{\varepsilon}\frac{x-\frac{1}{n}}{\varepsilon-\frac{1}{n}} & \text{when } \varepsilon \leq x \leq \frac{1}{n} + \varepsilon \\ f''_\varepsilon(x) & \text{when } x \leq \varepsilon \end{cases}$$

let's call $G_n(x) = \frac{1}{\varepsilon} + \int_0^x \int_0^u g_n(s) ds du \in C^2$

$G_n(W_t) = \underbrace{G_n(0)}_{\frac{1}{\varepsilon}} + \int_0^t G'_n(W_s) dW_s + \int_0^t g_n(W_s) ds$

It is clear that

- $g_n(x) \downarrow f''_\varepsilon(x)$
- $|G'_n(x) - f'_\varepsilon(x)| \leq \frac{1}{n}$, so that $1_{[0,t]}(G'_n(W_t) - f'_\varepsilon(W_t))$ converges to 0 in L_2.
- $G_n(x) \to f_\varepsilon(x)$

We can thus take limits in probability and get:
$f_\varepsilon(W_t) = \frac{1}{\varepsilon} + \int_0^t f'_\varepsilon(W_s) dW_s + \int_0^t f''_\varepsilon(W_s) ds$