Definition 1 (Total Variation)**.** $TV[f,T] = \sup_{0<t_1<\ldots<t_n=T, n\geq 1} \sum_{k=1}^n |f(t_i) - f(t_{i-1})|$

Theorem 1 (TV of of a Brownian Motion)**.** $(X_t)_t$ *BM* $\Rightarrow TV[X,T] = \infty$ *ass*

Proof. Eough to show that: $\lim_{n\infty} \sum_{k=1}^{2^n} |X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}| = \infty$

$$
\begin{aligned}
\mathbb{P}(\limsup_{n\infty} \sum_{k=1}^{2^n} |X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}|) < M) &\leq \liminf \mathbb{P}(\sum_{k=1}^{2^n} \underbrace{|X_{Tk2^{-n}} - X_{T(k-1)2^{-n}}|}_{\mathcal{N}(0,T2^{-n})} < M) \\
&\leq \liminf \mathbb{P}(\frac{1}{2^n} \sum_{k=1}^{2^n} |\epsilon_k| < \frac{M}{\sqrt{T2^n}}) \\
&\to 0
\end{aligned}
$$

□

1 Continuous Time theory

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ filtered probability space.
(X_t) continuous time process

Definition 2 (Continuous - Adapted)**.**

- *X is $\mathcal{F}_t$-adapted (nonanticipating) if X_t is $\mathcal{F}_t$-measurable $\forall t$*
- *X is continuous if $t \to X_t(\omega)$ is continuous $\forall \omega$.*
- *X is measurable if $(t,\omega) \to X_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$-measurable.*
- *X is progressively measurable if $X : [0,T] \times \Omega \to \mathbb{R}; (t,\omega) \to X_t(\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}_t$-measurable.*

Beware: measurable and adapted $\not\Longrightarrow$ progressively measurable. But continuous + adapted $\Longrightarrow$ progressively measurable

Definition 3 (Usual conditions)**.** *$(\mathcal{F}_t)_t$ is set to satisfy the* usual conditions *if*

a) $\forall B \subseteq A \in \mathcal{F}$ $P(A) = 0 \implies B \in \mathcal{F}_0$ (completeness)

b) $\mathcal{F}_t = \cup_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ (right-continuity)

Why?

- *(a) $\implies$: if (X_t) progressively measurable, B borel set in $\mathbb{R}$, $\tau = \inf\{t : X_t \in B\}$ is a stopping time.*

  But if (X_t) is continuous adapated, B closed, τ is always a stopping time. (doesn't need a)).

- *X BM, $\mathcal{F}_t^0$ the natural filtration. $\{t \to X_t$ has a local maximum at $t\} \in \mathcal{F}_{t^+}^0$ and not in $\mathcal{F}_t^0$. But one can prove $\mathcal{F}_t^0 = \mathcal{F}_{t^+}^0$ a.s (for every set $A \in \mathcal{F}_{t^+}^0$, there exist $B \in \mathcal{F}_t^0$ such that $\mathbb{P}(A \setminus B) = 0$).*

  $\mathcal{N} = \{B : B \subseteq A \in \mathcal{F}, \mathbb{P}(A) = 0\}$, $\mathcal{F}_t = \cap_{\varepsilon>0}(F_t^0 \vee N)$ is called the standard Brownian filtration, it satisfies usual conditions.

  b) is important if processes have jumps.

2 Continuous Time Martingale

Definition 4 (Martingale)**.** *(M_t) is a martingale if*

1. *M is adapted*
2. *$\mathbb{E}[|M_t|] < \infty \forall t$*
3. *$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \forall t \geq s$*

Definition 5 (Stopping time)**.** *A r.v $\tau : \Omega \to [0, \infty]$ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t \forall t$.*

Definition 6 (Stopped filtration)**.** *$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cup \{\tau \leq t\} \in \mathcal{F}_t \ \forall t\}$*

Theorem 2 (Optional Stopping)**.** *M is a **continuous** martingale. τ is a stopping time. Then $(M_{t\wedge\tau})_t$ is a continuous martingale*