ORF526 - Problem Set 2

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Question 1

- $$\mathbb{E}[X] = \mathbb{E}[\sum \epsilon_i] = \sum \mathbb{E}[\epsilon_i] = n\mathbb{E}[\epsilon] = np$$
$$Var[X] = Var[\sum \epsilon_i] = \sum Var[\epsilon_i] = nVar[\epsilon] = n\mathbb{E}[(X-p)^2] = n[p(1-p)^2 + (1-p)p^2] = np(1-p)$$

- $$\mathbb{E}[X] = \sum_{n\geq 1} ne^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda}\sum_{n\geq 0}\frac{\lambda^{n+1}}{n!} = \lambda$$
$$Var[X] = \mathbb{E}[X^2] - \lambda^2 = \sum_{n\geq 1} n^2 e^{-\lambda}\frac{\lambda^n}{n!} - \lambda^2 = e^{-\lambda}\lambda\sum_{n\geq 1}(n+1)\frac{\lambda^n}{n!} - \lambda^2 = \lambda(1+\lambda) - \lambda^2 = \lambda$$
$$\sum_n (n+1)\frac{x^n}{n!} = \left(\frac{d}{dx}x\sum_n \frac{x^n}{n!}\right)_{|x=\lambda} = \left(\frac{d}{dx}xe^x\right)_{|x=\lambda} = e^\lambda(1+\lambda)$$

Question 2

- 0
- $\mathbb{P}(X=k) = (1-p)^{k-1}p$ geometric
- $\mathbb{E}[X] = \sum_{k\geq 1} k(1-p)^{k-1}p = p\sum_{k\geq 0}(k+1)(1-p)^k = p\left(\frac{d}{dx}\sum_k x^{k+1}\right)_{|x=1-p} = p\left(\frac{d}{dx}(\frac{1}{1-x}-1)\right)_{|x=1-p} = p\left(\frac{1}{1-(1-p)}\right)^2 = \frac{1}{p}$
$$Var[X] = \frac{1-p}{p^2}$$
- $$\mathbb{E}[2^X] = \sum 2^n p(1-p)^{n-1} = \frac{p}{1-p}\sum_n (2(1-p))^n = \begin{cases} \infty & \text{if } p \leq \frac{1}{2} \\ \frac{p}{(1-p)(2p-1)} & \text{else} \end{cases}$$

Question 3

Let's note $Y = \mathbb{E}[X|\mathcal{F}]$

(i)
  - Y takes finite number of values $I := Y(\Omega) = \{x_1, ..., x_r\}$
  - Y is constant on the A_i
  - As a consequence, either all element of A_j are equal to x_i, or no one is. ie $A_j \cap Y^-1(x_i) = \emptyset$ or $A_j \subset Y^{-1}(x_i)$
  - Since $\Omega = \cap_j A_j$, $Y^{-1}(x_i) = \cap A_j \cup Y^{-1}(x_i) = \cap_{Y(A_j)=\{x_i\}} A_j$ is measurable

(ii) if $\mathcal{F} = \{\emptyset, \Omega\}$, $\mathbb{E}[X|\mathcal{F}] = \frac{\sum_{\omega_n \in \Omega} X(\omega_n)p_n}{P(\Omega} = \mathbb{E}[X]$

(iii) if X if $\mathcal{F}$-measurable, X is constant on the A_i, otherwise there exist two elements w_n, w_m such that $X(w_n) \neq X(w_m)$, but then $X^-1(X(w_n)) \cap A_i$ is an element if $\mathcal{F}$ and a strict subset of A_i whose measure is not 0. for $\omega \in A_m$:

$$\mathbb{E}[X|\mathcal{F}](\omega) = X(\omega)\frac{\sum_{\omega_n \in A_m} p_n}{P(A_m)} = X(\omega)$$

(iv) This is true because Y is a positive linear combination of the $X(\omega_n) \geq 0$

(v)

$$\mathbb{E}[XY + Z|\mathcal{F}] = \mathbb{E}[XY|\mathcal{F}] + \mathbb{E}[Z|\mathcal{F}] \text{ by linearity of } \mathbb{E}[]$$

Let $\omega \in A_m$, X is constant on A_m, so

$$\mathbb{E}[X|\mathcal{F}](\omega) = X(\omega)\frac{\sum_{\omega_n \in A_m} Y(w_n)p_n}{P(A_m)} = X(\omega)\mathbb{E}[Y|\mathcal{F}](\omega)$$

(vi) $\mathbb{E}[X|\mathcal{F}]$ is $\mathcal{G}$-measurable (by (i)), we conclude by (iii).

(vii) Independance of $\mathcal{F}$

(viii) $Y\mathcal{F}-$measurable is equivalent to Y being constant on each A_i: Let's note $Y(A_i)$ this value. Choosing Y is equivalent to choosing the vector $(Y(A_1), ..., Y(A_M)) \in \mathbb{R}^M$

Minimizating this quantity

$$\mathbb{E}[(X-Y)^2] = \sum_{A_i} \sum_{\omega_n \in A_i} (X(\omega_n) - Y(A_i))^2 p_n$$

is therefore equivalent to solving the following minimization program:

$$\begin{aligned}
\min_{a_1, \dots a_M \in \mathbb{R}^M} \sum_{A_i} \sum_{\omega_n \in A_i} (X(\omega_n) - a_i)^2 p_n &= \sum_{A_i} \min_{a_i \in \mathbb{R}} \sum_{\omega_n \in A_i} (X(\omega_n) - a_i)^2 p_n \\
&= \sum_{A_i} \min_{a_i \in \mathbb{R}} \sum_{\omega_n \in A_i} (X(\omega_n) - a_i)^2 p_n \\
&= \sum_{A_i} \min_{a_i \in \mathbb{R}} f_i(a_i)
\end{aligned}$$

$f : a_i \mapsto \sum_{\omega_n \in A_i} (X(\omega_n) - a_i)^2 p_n$ is a strictly convex function (as positive sun of convex functions), it has an unique absolute minimum a_i that verifies $f'(a_i) = 2\sum_{\omega_n \in A_i}(a_i - X(\omega_n))p_n = 0$, ie $a_i = \mathbb{E}[X|\mathcal{F}](A_i)$

Question 4

It is clear that $(\Omega, \hat{\mathcal{F}}, \hat{\mu})$ is a measure space.
Let $D \in \hat{\mathcal{F}}$, and Let $A, B, C \subset \Omega$ such that $A \subset B \in \mathcal{F}$ and $C \in \mathcal{F}$ such that $D = A \cup C$ and $\mu(B) = 0$.
Let's also suppose that $\hat{\mu}(D) = \mu(C) = 0$.
Let $E \subset D$, and let's show that $E \in \hat{F}$. We can write $E = (E \cap A) \cup (E \cap C) \subset B \cup C = (B \cup C) \cup \emptyset$.

Question 5

$$1) \Rightarrow 2)$$

$(E_i)_i$ sequence of increasing sets, as a convention $E_{-1} = \emptyset$

$$\mu(\cup E_i) = \mu(\cup(E_i \setminus E_{i-1})) = \sum_i \mu(E_i \setminus E_{i-1}) = \sum_i \mu(E_i) - \mu(E_{i-1})$$

$$\mu(E_j) = \sum_{i \leq j} \mu(E_i) - \mu(E_{i-1})$$

so

$$\mu(\cup E_i) - \mu(E_j) = \sum_i \mu(E_i \setminus E_{i-1}) = \sum_{i>j} \mu(E_i) - \mu(E_{i-1}) \to 0$$

(as the rest of a convergent series)

$$2) \Rightarrow 3)$$

Let $(E_i)_i$ be a sequence of decreasing sets, as a convention, $E_{-1} = \Omega$

$$\begin{aligned}
\mu(\cap E_j) &= \mu(\Omega \setminus \cup E_j^c) \\
&= \mu(\Omega) - \mu(\cup E_j^c) && \text{by } \sigma\text{-linearity} \\
&= \mu(\Omega) - \lim \mu(E_j^c) && \text{by coninuity from above} \\
&= \lim \mu(\Omega) - \mu(E_j^c) \\
&= \lim \mu(\Omega \setminus E_j^c) && \text{by } \sigma\text{-linearity} \\
&= \lim \mu(E_j)
\end{aligned}$$

$$3) \Rightarrow 4)$$

Question 6

Let's suppose $1) - 3)$

- $1')$ is trivially verified
- For $2')$, we have $B \setminus A = B \cap A^c = (B^c \cap A)^c \in \mathcal{D}$ (because $A \subset B \Rightarrow A \cap B^c = \emptyset$)
- For $3')$, $\cup A_n = \cup A_n \setminus A_{n-1} = \cup(A_n \cap A_{n-1}^c) \in \mathcal{D}$

Let's suppose $1') - 3')$

- $1)$ is trivially verified
- For $3)$, $A^c = \Omega \setminus A \in \mathcal{D}$
- For $2)$, Let's first show that it is true for a finite number of sets.
  For two disjoint sets $A_1 \cup A_2 = (A_2^c \setminus A_1)^c \in \mathcal{D}$ by $2')$
  For $n > 2$ sets $A_1, ...A_n$ pairwise disjoint, we write $\cup A_i = A_1 \cup \cup_{i>1} A_i$. Since A_1 and $\cup_{i>1} A_i$ are disjoint, we can apply the same proof as in the case $n = 2$.
  General case: $\cup A_n = \cup(\cup_{i \leq n} A_n) \in \mathcal{D}$ by $3')$