ORF526 - Problem Set 3

Bachir EL KHADIR

October 3, 2015

bls

Question 1

a) $\cap_{n\in\mathbb{N}} A_n^c$

b) $\cap_{m\in\mathbb{N}} \cup_{m\leq n} A_n$

c) $(\cap_{m\in\mathbb{N}} \cup_{m\leq n} A_n)^c = ..$

d) $\cap_{i,j\in\mathbb{N},i\neq j} \left(A_i \cap A_j \cap (\cup_{n\neq i,n\neq j} A_j^c)\right)$

e) This event can be expressed as "Φ nevr occurs at even times", ie $\cap_{n\in\mathbb{N}} A_{2n}^c$

Question 2

- $\varepsilon \subseteq \sigma(\varepsilon)$, so $\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$. since the RHS is already a σ-algebra (showed in class), $\sigma\{f^{-1}(A) : A \in \varepsilon\} \subseteq \{f^{-1}(A) : A \in \sigma(\varepsilon)\}$
- Let's note $B := \sigma\{f^{-1}(A) : A \in \varepsilon\}$, and $C := \{A : f^{-1}(A) \in B\}$.
  - C is a σ-algebra containing ε, so $\sigma(\varepsilon) \subseteq C$
  - As a consequence, for every $A \in \sigma(\varepsilon)$, $A \in C$, ie $f^{-1}(A) \in B$.

  We have just proved that $\{f^{-1}(A) : A \in \sigma(\varepsilon)\} \subseteq B$

Question 3

- $X = \lim_n X_n = \lim_n \sup_{k\geq n} X_k$
- $X \leq x$ eq $\exists n \, \forall k \, \geq n X_k \leq x$
- $\{X \leq x\} = \cup_{n\in\mathbb{N}} \cap_{k\geq n} \{X_k \leq x\}$

Question 4

- Let's call $Q = P(\{1, .., n\})$. For $i = 1..n$:

$$A_i = \bigcup_{I\subseteq Q, i\in I} (\bigcap_{k\in I} A_k) \cap (\bigcap_{k\in I^c} A_k^c)$$

  Note that this a union of disjoint sets.

  Let's call $I_i := \{I \in Q : \sum_{i\in I} a_i = x_i\}$, ie the different possible combinations for the A_i where $\omega \in \Omega$ can be so that its image by f equals x_i.

Written differently, $\{f = x_i\} = \bigcup_{I \in I_i} \underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c$. And as result of the sets being disjoint:

$$\mu(f = x_i) = \sum_{I\in I_i} \mu(\underset{k\in I_i}{\cap} A_k \cap \underset{k\in I_i^c}{\cap} A_k^c)$$

Note that any sum index by some $I \in Q$ in finite because $|I| \leq n$, and this we can rearrange the sums in any order.

$$\begin{aligned}
\sum_{i=1}^{m} x_i \mu(f = x_i) &= \sum_{i=1}^{m} x_i \sum_{I\in I_i} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1}^{m} \left(\sum_{I\in I_i} (\sum_{k\in I} a_k) \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \right) \\
&= \sum_{I\in Q} (\sum_{k\in I} a_k) \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{because } Q = \underset{k=1..m}{\cup} I_i \\
&= \sum_{i=1..n} \sum_{I\in Q, i\in I} a_i \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) && \text{By rearranging the sum} \\
&= \sum_{i=1..n} a_i \sum_{I\in Q, i\in I} \mu(\underset{k\in I}{\cap} A_k \cap \underset{k\in I^c}{\cap} A_k^c) \\
&= \sum_{i=1..n} a_i \mu(A_i)
\end{aligned}$$

- Let's first prove that if a set A has measure 1, for all measurable sets B, $\mu(A \cap B) = \mu(B)$. This holds because

$$\mu(B) \geq \mu(A \cap B) = 1 - \mu(A^c \cup B^c) \geq 1 - \mu(B^c) = \mu(B)$$

Let's now prove that $f(\omega) = g(\omega)$. Let x in the set on the left

$\mu(\{g = x\}) \geq \mu(\{g = x\} \cap \{f = x\}) = \mu(\{f = x\} \cap \{f = g\}) = \mu(\{f = x\})$ Symmetrically, we prove that $\mu(\{f = x\}) \geq \mu(\{g = x\})$, and thus this two quantities are equal.

This proves that in the sum $\sum_{x\in f(\Omega)} x\mu(f = x)$ there is a term that is non zero, $\mu(\{f = x\}) = \mu(\{g = x\}) \neq 0$, and $x \in g(\Omega)$. Since all quantities are positive, this means, $\sum_{x\in f(\Omega)} x\mu(f = x) \leq \sum_{x\in g(\Omega)} x\mu(g = x)$, and by symmetry: $\sum_{x\in f(\Omega)} x\mu(f = x) = \sum_{x\in g(\Omega)} x\mu(g = x)$