ORF526 - Problem Set 4

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Question 1

1. Let $\mathbb{F}$ be a field which is either $\mathbb{R}$ or $\mathbb{C}$. A normed vector space over $\mathbb{F}$ is a pair $(V, ||\cdot||)$ where V is a vector space over $\mathbb{F}$ and $||\cdot||: V \to \mathbb{R}$ is a function such that
   (a) $||v|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$ in V (*positive definiteness*)
   (b) $||\lambda v|| = |\lambda|||v||$
   for all $v \in V$ and all $\lambda \in \mathbb{F}$
   (c) $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in V$ (the *triangle inequality*)
2. Inner product space is a vector space with an inner product
3. A metric space M is a set with a distance. It is called complete if every Cauchy sequence of points in M has a limit that is also in M.
4. A Banach space is a vector space X over the field R of real numbers, or over the field C of complex numbers, which is equipped with a norm and which is complete with respect to that norm.
5. A Hilbert space is a vector space H with an inner product $< f, g >$ such that the norm defined by $||f|| = \sqrt{< f, f >}$ turns H into a complete metric space.

Question 2

In the following we use these properties: For A, B two measurable sets:

1. if $A \subseteq B$, $\mu(B \setminus A) = \mu(B) - \mu(A)$
2. $\mu(A \cup B) = \mu(A \cup (B \setminus A \cap B)) = \mu(A) + \mu(B) - \mu(A \cap B)$

- $$(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus \Big((-\infty, b_1] \times (-\infty, a_2] \cup (-\infty, a_1] \times (-\infty, b_2]\Big)$$

$$\begin{aligned}
\mu(a_1, b_1] \times (a_2, b_2] &= \mu(-\infty, b_1] \times (-\infty, b_2] - \mu\Big((-\infty, b_1] \times (-\infty, a_2] \cup (-\infty, a_1] \times (-\infty, b_2]\Big) \\
&= \mu(-\infty, b_1] \times (-\infty, b_2] - \mu(-\infty, b_1] \times (-\infty, a_2] - \mu((-\infty, a_1] \times (-\infty, b_2]) \\
&+ \mu\Big((-\infty, b_1] \times (-\infty, a_2] \cap (-\infty, a_1] \times (-\infty, b_2]\Big) \\
&= F(b_1, b_2) - F(b_1, a_2) - F(b_2, a_1) + F(a_1, a_2)
\end{aligned}$$

- The sequence x_k is decreasing so the following intersection is decreasing:

$$(-\infty, x_1] \times (-\infty, x_2] = \bigcap_{k\in\mathbb{N}} (-\infty, x_1^k] \times (-\infty, x_1^k]$$

  By continuity from above $F(x_k) \to F(x)$

- $$\mathbb{R} = \underset{k\in\mathbb{N}}{\cup} (-\infty, x_1^k] \times (-\infty, x_2^k]$$

  The union is increasing because x_k is increasing, by continuity from below we have the equality.

- If any of the x_i^k go to $-\infty$, we have the following decreasing intersection:

$$\emptyset = \underset{k}{\cap} (-\infty, x_1^k) \times (-\infty, x_2^k)$$

  We conclude by continuety at 0.

- By monotonicity of the measure $(-\infty, x_1] \times (-\infty, x_2] \subseteq (-\infty, y_1] \times (-\infty, x_2]$ so $F(x_1, x_2) \leq F(y_1, x_2)$ By symetry of (x_1, y_1) and (x_2, y_2), we prove the other inequality.

- Let G be function definied on R^2 such that

$$G(x,y) = \begin{cases} 1 & \text{when } x, y \geq 0 \text{ and } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

  $G(x,y)$ is non decreasing in x and y, but

$$G(1,1) - G(1,0) - G(0,1) + G(0,0) = 1 - 1 - 1 + 0 = -1$$

Question 3

Let's write f and g as: $f = \sum_i a_i 1_{A_i}$, $g = \sum_k b_k 1_{B_k}$, $f + g = \sum_i a_i 1_{A_i} + \sum_k b_k 1_{B_k}$

$$\int (f+g) = \sum_i a_i \mu(A_i) + \sum_k b_k \mu(B_k) = \int f + \int g$$

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General case: If $f, g \geq 0$ Let $(f_n)_n$ and$(g_n)_n$ two sequences of simple functions
$0 \leq f_n \uparrow f$,
$0 \leq g_n \uparrow g$
Then $0 \leq f_n + g_n \uparrow f + g$ By monotone convergence theorem:

$$\int f + g = \lim_n \int f_n + g_n = \lim_n \int f_n + \lim_n \int g_n = \int f + \int g$$

If f and g measurable, we write $f = f^+ - f^-$ and $g = g^+ - g^-$ and we apply the precedent result.

Question 4

- If $f = \sum a_i 1_{A_i}$ a simple function, then $cf = \sum (ca_i) 1_{A_I}$, $\int cf = \sum ca_i \mu(A_i) = c \sum a_i \mu(A_i) = c \int f$.

  If $f \geq 0$ If f_n a sequence of non negative increasing simple function converging to f, then (cf_n) is an monotonous sequence converging to cf, and therefore by monotnous convergence, $\int cf = \lim \int cf_n = c \lim \int f_n = c \int f$.

  General case $f = f^+ - f^-$, so by linearity

$$\int cf = \int cf^+ + \int (-c) f^- = c(\int f^+ - \int f^-) = c \int f$$

- $h := g - f$, $\mu h < 0 = 0$

$$\begin{aligned}\int g - \int f &= \int (g - f)\\ &= \int h\\ &= \int_{\{h \geq 0\}} h + \int_{\{h<0\}} h\\ &= \int_{\{h \geq 0\}} h^+ + \int_{\{h<0\}} h\end{aligned}$$

It is easy to see that the integral of constant functions on a set of measure 0 is 0 , by linearity it extends to simple functions, and it holds for measurable positive functions because of the definition of the integral as the sup of a set that contains only 0 and for measurable function by linearity. Therefore $\int_{\{h<0\}} h = 0$

Moreover, the integral of positive functions is definied as the sup of the integral of positive simple functions, and that integral is positive as the sum of positive terms. So $\int_{\{h \geq 0\}} h^+ \geq 0$

We conclude that $\int g - \int f \geq 0$.

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$$\int |f| = \int_{f=0} |f| + \int_{f \neq 0} |f| = \int_{f \neq 0} |f| = 0$$

$$\{f \neq 0\} = \underset{n \geq 1}{\cup} \{|f| \geq \frac{1}{n}\}$$

Let's call $A_n := \{|f| \geq \frac{1}{n}\}$ for $n \geq 1$, so that A_n is increasing. By using the last question:

$$0 \geq \int_{f \neq 0} |f| \geq \int_{\{f \neq 0\} \cap A_n} |f| + \int_{A_n} |f| \geq \int_{A_n} |f| \geq \frac{\mu A_n}{n} \geq 0$$

so $\mu A_n = 0$. and subsequently by continuty from below

$$\mu\{f \neq 0\} = 0$$

Question 5

μ is a measure because:

- $\mu_f(\emptyset) = \mu f^{-1}(\emptyset) = \mu \emptyset = 0$
- $\mu_f(B^c) = \mu f^{-1}(B^c) = \mu(f^{-1}B)^c = 1 - \mu(f^{-1}B) = 1 - \mu_f(B)$
- if $\{B_k | k \in \mathbb{N}\}$ a set of pairwise disjoint sets, so is$\{f^{-1}B_k | k \in \mathbb{N}\}$ and therefore

$$\mu_f(\underset{k}{\cup} B_k) = \mu(f^{-1} \underset{k}{\cup} B_k) = \mu(\underset{k}{\cup} f^{-1} B_k) = \sum_k \mu_f(B_k)$$

If g is simple, eg $g = \sum a_i 1_{A_i}$: $gof = \sum a_i 1_{f^{-1}(A_i)}$

$$\int_\Omega gof \mathrm{d}\mu = \sum_{\mathrm{i}} \mathrm{a_i}\mu(\mathrm{f^{-1}A_i}) = \sum_{\mathrm{i}} \mathrm{a_i}\mu_{\mathrm{f}}(\mathrm{A_i}) = \int_{\mathrm{E}} \mathrm{g}\mathrm{d}\mu_{\mathrm{f}}$$

If $g \geq 0$, and $0 \leq g_n \uparrow g$, $g_n of \uparrow gof$ by monotone convergence:

$$\int gof \mathrm{d}\mu = \lim \int \mathrm{g_n of} = \lim \int \mathrm{g_n d}\mu_{\mathrm{f}} = \int \mathrm{gd}\mu_{\mathrm{f}}$$

If g is measurable such that the integral exist, we write $g = g^+ - g^-$,$gof = g^+of - g^-of$, one of the integrals of g^+, g^- is finite, and we have the result by linearity of the integral.

Question 6

1. Let $(x_k)_k$ be a sequence s.t. $x_k \downarrow x$, then by continuety from above of the probability measure: $F(X \leq x) = P(\bigcap_k \{X \leq x_k\}) = \lim_k P(\{X \leq x_k\}) = \lim_K F(x_k)$ and therefore F is right continuous.

   First we notice that by definition of q and F, for every u and $\epsilon > 0$:

$$F(q(u) + \epsilon) > u \geq F(q(u) - \epsilon)$$

   q is non-decreasing, so the right limit exists at every point u. Let's note it $q(u^+)$. Let's suppose $q(u^+) > q(u)$ and note $\alpha = q(u^+) - q(u)$. Let (u_n) a decreasing sequence s.t $u_n \downarrow u$ so that $q(u_n) \downarrow q(u)$ For every $\epsilon > 0$ we have:$u_n \geq F(q(u_n) - \epsilon)$

   For $\epsilon = \frac{\alpha}{2}$, we have

$$u_n \geq F(q(u_n) - \epsilon) \geq F(q(u^+) - \epsilon) \geq F(q(u) + \epsilon) > u$$

   and by going to the limit:

$$u \geq F(q(u) + \epsilon) > u$$

   Contradiction. q is right continous.

2. Let $g_n(x) = x$, by question 5:

$$\begin{aligned}\mathbb{E}[X] &= \int_\Omega goX(\omega)\mathrm{dP} \\ &= \int_{\mathbb{R}} g \, \mathrm{d}\mu_{\mathrm{X}}\end{aligned}$$

   Let's consider the application from the Borel to $\mathbb{R}^+$: $\alpha : A \to \int_{\mathbb{R}} 1_A \mathrm{dF_X}$.

   Since F_X is right continous, we can prove that α is a measure that agrees with μ_X on every interval (so they agree on a semi-ring that generates the Borel σ-algebra), so by unicity garanteed by Carathodory extension theorem for σ-finite measures , $\mu_x = \alpha$. so that

$$E[X] = \int_R \mathrm{gd}\mu_{\mathrm{X}} = \int_{\mathrm{R}} \mathrm{gdF_X}$$

3. Let's first prove the statement of bounded functions. Let X be integrable such that $|X| < a$

$$
\begin{aligned}
\int_{-\infty}^{0}(P[X>x]-1)\mathrm{dx} &= -\int_{-a}^{0}F(x)\mathrm{dx}\\
&= \int_{-a}^{0}x\mathrm{dF(x)}-\mathrm{F(0)*0-F(-a)a}\\
&= \int_{-a}^{0}x\mathrm{dF(x)} \qquad\qquad = \int_{-\infty}^{0}x\mathrm{dF(x)}
\end{aligned}
$$

$$
\begin{aligned}
\int_{0}^{\infty}P[X>x]\mathrm{dx} &= \int_{0}^{a}1-F(x)\mathrm{dx}\\
&= -\int_{0}^{a}x\mathrm{d(1-F)(x)+a(1-F(a))}\\
&= \int_{0}^{a}x\mathrm{dF(x)}\\
&= \int_{0}^{\infty}x\mathrm{dF(x)}
\end{aligned}
$$

F being constant for $x>a$ and $x<-a$, the Stieltjes sum and there for the $\int$ is 0 on those interval. By summing both:

$$
\int_{-\infty}^{0}(P[X>x]-1)\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X>x]dx}=\int_{-\infty}^{\infty}\mathrm{xdF(x)=E[X]}
$$

Let X be just integrable now, $X=X^{+}-X^{-}$, and X^{+}, X^{-} are both integrable. If we prove the statement for non negative functions, then we can conclude by linearity because $E[X]=E[X^{+}]-E[X^{-}]$ and when the integrals exist we have:

$$
\begin{aligned}
\int_{-\infty}^{0}(P[X>x]-1)\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X>x]dx} &= \int_{-\infty}^{0}(P[-X^{-}>x]-1)\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X^{+}>x]dx}\\
&= -\int_{0}^{\infty}P[X^{-}\geq x]\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X^{+}>x]dx}\\
&= -\int_{0}^{\infty}P[X^{-}>x]\mathrm{dx}-\int_{0}^{\infty}\mathrm{P[X^{-}=x]dx}+\int_{0}^{\infty}\mathrm{P[X^{+}>x]dx}\\
&= -\int_{0}^{\infty}P[X^{-}\geq x]\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X^{+}>x]dx}\\
&= -\int_{0}^{\infty}P[X^{-}>x]\mathrm{dx}+\int_{0}^{\infty}\mathrm{P[X^{+}>x]dx}\\
&\text{because } P[X^{-}=x] \text{ is non zero on at most a countable set}
\end{aligned}
$$

Let's now suppose $X\geq 0$ Let $X_n := X1_{|X|<n}$, so $X_n\uparrow X$ which is integrable.

$$
\begin{aligned}
E[X] &= \lim_n E[|X|1_{|X|<n}] \qquad\qquad \text{by monotone convergence}\\
&= \int_{0}^{\infty}P[X_n>x]\mathrm{dx}\\
&= \int_{0}^{\infty}P[X_n>x]1_{x\geq 0}\mathrm{dx}\\
&= \lim_n\int_{\mathbb{R}}P[X_n>x]
\end{aligned}
$$

$$
\begin{aligned}
\phi_n &:= P[X_n > x] \\
&\leq P[X > x] && \text{because } X_n \text{ is increasing} \\
&\leq (1 - F(X \leq x))
\end{aligned}
$$

and we have by right continuety of F and the fact that $X_n \uparrow X$ that: $\phi_n \uparrow P[X > x]$ By monotne convergence we can swap limit and integral, and we have the equality.

$E[X] = \lim_n \int_{\mathbb{R}} P[X_n > x] = \lim_n \int \phi_n = \int P[X > x]$

4. Let's show that

$$\{u \in (0,1) : u < F(x)\} \subseteq \{u \in (0,1) : q(u) \leq x\} \subseteq \{u \in (0,1) : u \leq F(x)\}$$

   Let's note that sets A, B, C (it's clear that they are all measurable sets)

   Let's prove $A \subset B$. Let $u \in A$, then $u < F(x)$. Let $y \in \mathbb{R}$ s.t $F(y) \leq u$, then $F(y) < F(x)$, then $y < x$ because F is non-decreasing. By taking the sup, $q(u) \leq x$.

   Let's first prove that $F(q(u)) \geq u$. Let $\epsilon > 0$ we know that $F(q(u) + \epsilon) \geq u$, because otherwise $q(u) + \epsilon \in \{x | F(x) \leq u\}$ and $q(u) + \epsilon$ is greater than the sup of this set. We now take the limit since F is right continous, and we have $F(q(u)) \geq u$.

   Let's now prove that $B \subseteq C$ Let $u \in C$, ie $F(x) \geq u$, so $q(F(x)) \leq q(u)$ so $x \leq q(u)$.

   We conclude by noting that $\mu(C) = \mu(A)$ because $\mu(C \setminus A) = \mu(\{F(x)\}) = 0$. And therefore $\mu(A) = \mu(B) = \mu(C)$

   But $\mu(C) = \mu((0, F(x)]) = F(x)$

5. The distribution of X and q are the same, so the distribution of $F(X)$ and $F(q)$ are the same.

   In the case where F is increasing, F is a bijection and $F(q)$ is the identity of $(0,1)$, so $F(q) \sim \mathcal{U}\text{nif}(0,1)$