Main motivation: $max(X_t, Y_t)$. We want to know if it is semimartingale.

Claim: This Q is related to the following Q: Take a brownian motion B. We know that $\int_0^t 1_{B(s)=0} ds = 0$ for all t with probability 1. What is the right way of measuring the time spent at 0 by a BM.

Relation between the two questions: Suppose B_1, B_2 are independent standard BMs and we want to stdy $max(B_1, B_2)$

Note: $max(x,y) = \frac{x+y+|x-y|}{2}$, so we only need to analyse $|B_1 - B_2|$. $f(x) = |x|$, $f''(x) = 2\delta_0(x)$ Try to apply Ito: $|B(t)| = \int_0^t sgn(B_s)dB_s + \int_0^t \delta_0(B_s)ds$

Three natural ways of measuring time spent at 0 by BM:

- Take $\varepsilon > 0$, consider $\int_0^t 1_{|B_s|<\varepsilon} ds$, take the limit $\varepsilon \downarrow 0$ in some way.
- Define time spent at 0 as: $2(|B(t)| - \int_0^t sign(B_s)dB_s)$
- Recall that a BM is a limit of random walks, $B(t) = \lim_{\text{distribution}} \frac{S_{\lfloor Kt \rfloor}}{\sqrt{K}}$ Take $\lim_K \frac{1}{\sqrt{K}} \#\{i, S_i = 0, i \leq \lfloor Kt \rfloor\}$

Luckily $1 \iff 2 \iff 3$

Definition 1 (Local time, def1)**.** *For a BM B and a point $a \in R$, call the almost sure limit $\lim_{\epsilon 0} \frac{1}{2\epsilon} \int_0^t 1_{|B_s - a| < \epsilon} ds$ the local time of B at a and write L_t^a. With probability 1 $(a,t) \to L_t^a$ is continuous.(Trotter '57)*

Remarks:

1. Need to justify the a.s limit and continuity.
2. $t \to L_t^a$ is non decreasing wp 1.
3. $B(R) \ni A \to \int_0^t 1_{B_s \in A} ds$ has $a \to L_t^a$ as its density.

Definition 2 (Local Time, def2)**.** *Define the local time L_t^a by $|B_t - a| - \int_0^t sgn(B_s - a)ds =: \frac{1}{2} L_t^a$*

Proof. Bump function $\rho(x) = ce^{\frac{1}{x^2-1}} 1_{|x|<1}$, $\int \rho = 1$, $\rho_n(x) = n\rho(n(x-a))$ $h_n(x) = \int_{-\infty}^x \rho_n(y)dy, h_n(x) = \int_{-\infty}^x \rho_n(y)dy$ Easy to check: $h_n(x) \to sgn(x-a), H_n(x) \to |x-a|$ Clear: $\rho_n \in C^\infty \Rightarrow$ Ito: $H_n(B_t) = H_n(B_0) + \int_0^t h_n(B_s)dB_s + \int_0^t \frac{1}{2}\rho_n(B_s)ds$

$n \to \infty$:

- $H_n(B_t) \to |B_t - a|$, as
- $\int h_n(B_s)dB_s \to \int_0^t sgn(B_s - a)dB_s$, in L_2?

$$
\begin{aligned}
E[\left(\int (h_n(B_s) - sgn(B_s - a))dB_s\right)^2] &= E[\int_0^t (h_n(B_s) - sgn(B_s - a))^2 ds] \\
&\leq E[\int_0^t 1_{|B_s - a| < \frac{1}{n}} ds] = \int_0^t \mathbb{P}(|B_s - a| \frac{1}{n})ds \\
&\underbrace{\to}_{DCT} 0
\end{aligned}
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- $\frac{1}{2}\int_0^t \frac{1}{2}\rho_n(B_s) = \int_{-\infty}^{\infty} \frac{1}{2}\rho_n(y)L_t^y dy$ (because L_t^a is a density) $\to L_t^a$ as

$H_n(B_t) = H_n(B_0) + \int_0^t h_n(B_s)dB_s + \int_0^t \frac{1}{2}\rho_n(B_s)ds$ Taking the limit in probability: $|B(t) - a| = |B(0) - a| + \int_0^t sgn(B_s - a)dB_s + L_t^a$ (Ito-Tanaka Formula) ☐

Theorem 1 (Def2 imply Def1)**.** *If we define L_t^a through Tanak's formula, it will be the density of the occupation time measure and joint continuity in (a,t)*

Proof. $H'(X) = \int_{-\infty}^{\infty} h(x)1_{y \leq x} dy$ $H(X) = \int_{-\infty}^{\infty} h(u)(x-u)^+ dy$ Note $f^+ = \frac{f+|f|}{2}$ $(B_t - a)^+ = \frac{1}{2}(B_t - a)^+ + \frac{1}{2}|B_t - a|$

Ito:

$$
\begin{aligned}
\frac{1}{2}\int_0^t h(B_s)d_s &= H(B_t) - H(B_0) - \int_0^t H'(B_s)dB_s \\
&=
\end{aligned}
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