

1 measure theory

Definition 1 (Sigma Algebra) \mathcal{F} σ -algebra:

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- $\cup_n A_n \in \mathcal{F}$

Definition 2 (Probability measure) *Probability measure*

- $\mathbb{P}(A) \in [0, 1]$
- $\mathbb{P}(\Omega) = 1$
- $A \cap B = \emptyset \rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Theorem 1 (Equivalence additive measure) *The following are equivalent for μ finitely additive measure:*

- μ σ -additive
- μ continuous from below / above / at 0.

Definition 3 (Monotone class theorem) *Monotone class $\mathcal{M} \subset \mathcal{P}(\Omega)$, and is closed under countable monotone unions and intersections.*

Theorem 2 (Monotone class theorem) *G an algebra, $\sigma(G) = M(G)$*

Theorem 3 ($\lambda - \pi$) *D is a Dynkin system if:*

- $\Omega \in D$
- $A \in D \Rightarrow A^c \in D$
- $A_1, \dots \in D$ pairwise disjoint, $\cup A_i \in D$

Equivalently

- $\Omega \in D$
- $A, B \in D; A \subset B \Rightarrow B \setminus A \in D$
- $A_1, \dots \in D$ increasing, $\cup A_i \in D$

*P -system: closed under finite intersection.
 $P \subset D \Rightarrow \sigma(P) \subset P$*

Theorem 4 (Sigma in out)

$$\sigma(f^{-1}(A) : A \in \epsilon) = \{f^{-1}(A) : A \in \sigma(\epsilon)\}$$

Definition 4 (Semi-ring) • $\emptyset \in S$

- $A \cap B \in S \forall A, B \in S$
- For all $A, B \in S$ there exist pairwise disjoint subset $C_1, \dots, C_n \in S$ such that $A \setminus B = \cup_{i \leq n} C_i$

Theorem 5 (Caratheodory's Extension Theorem) • *A measure μ on a semi-ring S can be extended to a measure on $\sigma(S)$.*

- If μ is σ -finite, the extension is unique.

Definition 5 (Consistence) • $\mathbb{P}^{i_1, \dots, i_n}[A_1 \times \dots \times A_n] = \mathbb{P}^{\pi(i_1), \dots, \pi(i_n)}[A_{\pi(1)} \times \dots \times A_{\pi(n)}]$

- $\mathbb{P}^{i_1, \dots, i_{n-1}}[A_1 \times \dots \times A_{n-1}] = \mathbb{P}^{i_1, \dots, i_n}[A_1 \times \dots \times A_{n-1} \times \mathbb{R}]$

Theorem 6 (Kolmogorov's Extension Theorem) *I non empty. $(\mathbb{P}^{i_1, \dots, i_n})_{i_1, \dots, i_n \in I}$ consistent family. There exists a unique probability measure on \mathbb{P} on $(\mathbb{R}^I, \mathbb{B}(\mathbb{R})^{\times I})$ such that*

$$\mathbb{P}[\{\omega \in \mathbb{R}^I : (\omega_{i_1}, \dots, \omega_{i_n}) \in B\}] = \mathbb{P}^{i_1, \dots, i_n}[B]$$

2 Integrals

Theorem 7 (Monotone Convergence) f_1, \dots be a pointwise non-decreasing sequence of non-negative valued measurable functions, set $\sup f_n = f$. Then f is measurable and $\lim_{k \rightarrow \infty} \int f_k d\mu = \int f d\mu$.

Theorem 8 (Fatou) Let f_1, f_2, f_3, \dots be a sequence of non-negative measurable functions. Define $f = \liminf_{n \rightarrow \infty} f_n$. Then f is measurable and $\int_S f d\mu \leq \liminf_n \int_S f_n d\mu$.

Theorem 9 (Dominated Convergence) g, f_1, f_2, \dots measurable functions such that $\int |g| < \infty$, $|f_n| \leq g \forall n$ a.s., $f_n \xrightarrow{a.s.} f$, then $\int |f| \leq \int |g| < \infty$ and $\lim \int |f_n - f| \rightarrow 0$, $\lim \int f_n \rightarrow \int f$

Theorem 10 (Fubini) μ_1, μ_2 are σ -finite.

- $\int_{\Omega_1 \times \Omega_2} |f| d(\mu_1 \times \mu_2) < \infty \Rightarrow \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int_{\Omega_2} f$
- $f \geq 0$ a.s. $\Rightarrow \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int_{\Omega_2} f$

Theorem 11 (Inequalities) • Holder: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \int |fg| \leq (\int |f|^p)^{\frac{1}{p}} (\int |g|^q)^{\frac{1}{q}}$

- Minkowsky: $\forall p \leq 0 \ ||f + g||_p \leq ||f||_p + ||g||_p$

Theorem 12 (Borel Cantelli) • $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n \geq m} A_n] = 0$

- (A_n) , independent, $\sum \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n \geq m} A_n] = 1$

3 Random Variables

Definition 6 (Uniform integrability) (X_i) u.i iff $\lim_n \sup_i \int_{|X_i| > c} |X_i| d\mathbb{P} = 0$ iff $\lim_n \sup_i \mathbb{E}[1_{|X_i| > c} |X_i|] = 0$

Theorem 13 (Characterisation) • $\forall i |X_i| \leq X \in L_1 \Rightarrow (X_i)$ uc

- uc iff:
 - $\sup E[|X_i|] < \infty$
 - $\forall \epsilon > 0, \exists \delta > 0 \forall A \mathbb{P}(A) < \delta \Rightarrow \forall i \int_A |X_i| < \epsilon$

Theorem 14 (L_1 Convergence) $X_i \xrightarrow{\mathbb{P}} X$, X_i uc. Then $X \in L_1$, $X_i \xrightarrow{L_1} X$

Theorem 15 (De la Valle-Pousson) X_i uc $\iff \exists \Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\frac{\Phi(x)}{x} \rightarrow \infty$ st $\sup \Phi |X_i| < \infty$. Φ can be assumed convex and non-decreasing.

Theorem 16 (Weak Law of large numbers) $X_i \in L_2$ uncorrelated, $E[X_i] = m$, $\sup E[X_i^2] < \infty$, then $\frac{\sum_i X_i}{n} \rightarrow m$ in L_2 .

Theorem 17 (Characteristic Function) • $|\Phi_X(u)| \leq \Phi_X(0) = 1$

- $\Phi_X(-u) = \overline{\Phi_X(u)}$
- $\Phi_X \in \mathbb{R} \iff X \stackrel{\mathbb{D}}{=} -X$
- Φ_x is uniformly continuous.
- $E[|X|^n] < \infty \Rightarrow \exists \Phi_X^k \forall k \leq n$, and $\Phi_X^k(u) = E[(iX)^k e^{iuX}]$, and $\Phi_X(u) = \sum_k \frac{(iu)^k}{k!} E[X^k] + \frac{(iu)^n}{n!} \mathcal{E}_n(u)$, with $\mathcal{E}_n \rightarrow_0 0$
- $\exists \Phi_X^{2k}(0) \Rightarrow E[X^{2k}] < \infty$
- Inversion Formula: $\frac{F_X(b) + F_X(b^-)}{2} - \frac{F_X(a) + F_X(a^-)}{2} = \lim_{c \rightarrow \infty} \frac{1}{2\pi} \int_{-c}^c \frac{e^{-iua} - e^{-iub}}{iu} \Phi_X(u) du$
- $\int_{\mathbb{R}} |\Phi_X| < \infty \Rightarrow f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \Phi_X(u) du$
- $X = (X_1, \dots, X_n)$ independent $\iff \Phi_X = \prod \Phi_{X_i}$

Theorem 18 (Continuity Theorem) • $X_n \xrightarrow{D} X \iff \Phi_{X_n} \rightarrow \Phi_X$

- $\Phi_{X_n} \rightarrow \Phi$ and Φ continuous at 0 then $\exists X$ $X_n \xrightarrow{D} X$
- $X_n \xrightarrow{D} X \iff F_n \xrightarrow{in C(F_X)} F_X$

Theorem 19 (LLN) X_i iid in L_1 , $\frac{\sum X_i}{n} \rightarrow E[X]$ as and in L_1

Theorem 20 (CLT) X_n iid $Var(X) = \sigma^2 < \infty$ then $\frac{1}{\sqrt{n}} \sum \frac{X_i - E[X]}{\sigma} \rightarrow \mathcal{N}(0, 1)$

1. *Continuous distributions.*

	Name	Parameters	Density $f_X(x)$	Ch. function $\varphi_X(t)$
1	Uniform	$a < b$	$\frac{1}{b-a} \mathbf{1}_{[a,b]}(x)$	$\frac{e^{-ita} - e^{-itb}}{it(b-a)}$
2	Symmetric Uniform	$a > 0$	$\frac{1}{2a} \mathbf{1}_{[-a,a]}(x)$	$\frac{\sin(at)}{at}$
3	Normal	$\mu \in \mathbb{R}, \sigma > 0$	$\frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$	$\exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$
4	Exponential	$\lambda > 0$	$\lambda \exp(-\lambda x) \mathbf{1}_{[0,\infty)}(x)$	$\frac{\lambda}{\lambda - it}$
5	Double Exponential	$\lambda > 0$	$\frac{1}{2} \lambda \exp(-\lambda x)$	$\frac{\lambda^2}{\lambda^2 + t^2}$
6	Cauchy	$\mu \in \mathbb{R}, \gamma > 0$	$\frac{\gamma}{\pi(\gamma^2 + (x-\mu)^2)}$	$\exp(i\mu t - \gamma t)$

2. *Discrete distributions.*

	Name	Parameters	Distribution μ_X	Ch. function $\varphi_X(t)$
7	Dirac	$c \in \mathbb{R}$	δ_c	$\exp(itc)$
8	Biased Coin-toss	$p \in (0, 1)$	$p\delta_1 + (1-p)\delta_{-1}$	$\cos(t) + (2p-1)i\sin(t)$
9	Geometric	$p \in (0, 1)$	$\sum_{n \in \mathbb{N}_0} p^n (1-p) \delta_n$	$\frac{1-p}{1-e^{it}p}$
10	Poisson	$\lambda > 0$	$\sum_{n \in \mathbb{N}_0} e^{-\lambda} \frac{\lambda^n}{n!} \delta_n, n \in \mathbb{N}_0$	$\exp(\lambda(e^{it} - 1))$

3. *A singular distribution.*

	Name	Ch. function $\varphi_X(t)$
11	Cantor	$e^{it/2} \prod_{k=1}^{\infty} \cos(\frac{t}{3^k})$

Figure 1: Distributions

4 Martingales

Theorem 21 (Radon-Nikodym) $\mu_2 \ll \mu_1 \Rightarrow \exists$ unique μ_1 -a.s $f = \frac{d\mu_2}{d\mu_1}$

Theorem 22 (Stopping times) • $X_n^\tau = X_0 + (V.X)_n$ is martingale because V is predictable.

- If τ bounded $E[X_0] = E[X_\tau]$
- $M \geq \tau \geq \sigma$ stopping times, then $E[X_\tau | F_\sigma] = X_\sigma$

Theorem 23 (Upcrossing inequality) X_n submartingale. $E[B_n(a, b)] \leq \frac{E[(X_n - a)^+]}{b - a}$

Theorem 24 (Convergence) • X_n submartingale, L_1 bounded, then there exists X_∞ such that $X_n \xrightarrow{a.s} X_\infty$, and $E[|X_\infty|] < \sup E[|X_n|]$

- a submartingale that is bounded above converges a.s.
- X_n ui submartingale, then there exists $X_\infty \in L_1$ such that $X_n \rightarrow X_\infty$ in L_1 . Moreover $E[X_\infty | F_n] \geq X_n$.
- (F_n) filtration, $E[X | F_n] \rightarrow E[X | \cup F_n]$ a.s and L_1 (because u.i.)
- X_i iid, $\mathcal{G} = \cup \sigma(X_n, \dots)$, then $\forall A \in \mathcal{G}$ $P(A) \in \{0, 1\}$ (because $1_A = E[1_A]$)
- (G_i) dec-filtration, $E[X | G_n] \rightarrow E[X | \cap G_n]$ as and in L_1 .

Theorem 25 (Doob Maximal inequality) X_n non-negative submartingale.

- $\forall \lambda > 0$, then $\lambda^p \mathbb{P}[\max_{k \leq n} X_k \geq \lambda] \leq E[X_n^p]$
- $|\max_{k \leq n} X_k|_p \leq \frac{p}{p-1} |X_n|_p$
- $|\max_{k \leq n} X_k|_1 \leq \frac{e}{e-1} (1 + |X_n \log(X_n)|_1)$

5 Markov

Theorem 26 (Markov property) • (X_n) markov (λ, P) . Conditional on $X_m = i$, X_{n+m} is markov (δ_i, P) independent of X_0, \dots, X_m .

- (X_n) markov (λ, P) . Conditional on $X_T = i$, X_{n+T} is markov (δ_i, P) independent of X_0, \dots, X_T .

Definition 7 (Some defs) • Communicating classes: I / \sim where $i \sim j \iff i \leftrightarrow j$

- C Closed $\iff i \in C, i \rightarrow j \Rightarrow j \in C$
- P irreducible $\iff \forall i, j, i \rightarrow j \iff$ there is only one communicating class.
- $H_i = \inf\{n \geq 0; X_n = i\}, T_i = \inf\{n \geq 1; X_n = i\}, V_i := \sum_n 1_{\{X_n = i\}}, f_i = P_i(T_i < \infty), m_i = E[T_i]$
- i is reccurent if $P_i(V_i = \infty) = 1 \iff f_i = 1 \iff \sum p_{ii}^{(n)} = \infty$, otherwise transient.
- i is positive recurrent $\iff m_i < \infty$
- $P_i(V_i \geq k + 1) = f_i^k$
- In a communicating class all states are transient or all are reccurent.
- recurrence \Rightarrow closed
- finite + closed \Rightarrow recurrent.
- P irreducible + recurrent $\Rightarrow P(T_j < \infty)$
- i aperiodic $\iff p_{ii}^n > 0$ for large n

Theorem 27 (Invariant Distribution) • I finite, if for some $i \in I$ $p_{ij}^{(n)} \rightarrow \pi_j \forall j \in I$ then π is an invariant distribution.

- if P irreducible and $\lambda \geq 0$ invariant, then $\lambda \in \{0, \infty, \mathbb{R}^n\}$
- $\gamma_i^k = E_k[\sum_{n=0}^{T_k-1} 1_{X_n=i}]$. If P irreducible and recurrent, then
– $\gamma_k^k = 1$

- γ^k is invariant
- $0 < \gamma^k < \infty$
- If P irreducible and λ invariant with $\lambda_k = 1$ then $\lambda \geq \gamma^k$. If P is recurrent, $\lambda = \gamma^k$.
- If P irreducible, every state is positive recurrent \iff state i is pos rec \iff P has invariant distribution π .
Moreover $\pi_i = 1/m_i$

Theorem 28 (Convergences) P transition matrix of an ergodic Markov chain (irreducible, aperiodic and positive recurrent), with invariant measure π , then for any initial distribution, $P(X_n = j) \rightarrow \pi_j$

Theorem 29 (Ergodic theorem) • P irreducible, then $\frac{V_n(n)}{n} \rightarrow \frac{1}{m_i}$ as. If P is irreducible and positive recurrent, for every bounded function: $\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \rightarrow \sum_i \pi_i f(i)$ a.s