ORF526 - Problem Set 1

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Question 1

Let X and Y be the result of two independant coin tosses, and let

$$
\begin{aligned}
A_1 &= \{X = H\} \\
A_2 &= \{Y = H\} \\
A_3 &= \{X = Y\}
\end{aligned}
$$

Question 2

$$
\begin{aligned}
\mathbb{E}[X] &:= \sum_{n=1}^{N} X(\omega_n) p_n \\
&= \sum_{n=1}^{N} \left[\mathrm{Re}(X)(\omega_n) + i \mathrm{Im}(X)(\omega_n) \right] p_n \\
&= \sum_{n=1}^{N} \mathrm{Re}(X)(\omega_n) p_n + i \sum_{n=1}^{N} \mathrm{Im}(X)(\omega_n) p_n \\
&= \mathbb{E}[\mathrm{Re}(X)] + i\, \mathbb{E}[\mathrm{Im}(X)]
\end{aligned}
$$

Question 3

$$(i) \Rightarrow (ii)$$

$$
\begin{aligned}
\mathbb{E}[f_1(X_1)...f_M(X_M)] &= \sum_{x_1,...,x_M} f_1(x_1)...f_M(x_M)\mathbb{P}(X_1 = x_1, ..., X_M = x_m) \\
&= \sum_{x_1,...,x_M} f_1(x_1)...f_M(x_M)\mathbb{P}(X_1 = x_1)...\mathbb{P}(X_M = x_M) \qquad \text{(because of}\,(i)) \\
&= \sum_{x_1} f_1(x_1)\mathbb{P}(X_1 = x_1)... \sum_{x_M} f_M(x_M)\mathbb{P}(X_M = x_M) \\
&= \mathbb{E}[f_1(X_1)]...\mathbb{E}[f_M(X_M)]
\end{aligned}
$$

$$(ii) \Rightarrow (iii)$$

Take $f_i(x) = e^{iu_i x}$

$$(iii) \Rightarrow (i)$$

Density of e^{iu}

Question 4

Immediate using definition (ii)

Question 5

a) if X and Y are independant, then $cov(X,Y) = \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right] = \mathbb{E}[X-\mathbb{E}[X]]\mathbb{E}[Y-\mathbb{E}[Y]] = 0$

b) Let X and ϵ be two independant uniform variables on $\{-1,1\}$, then $cov(X,\epsilon X) = \mathbb{E}[\epsilon X^2] = \mathbb{E}[\epsilon]\mathbb{E}[X^2] = 0$, but $\mathbb{P}(X=1,\epsilon X=1) = 0 \neq \mathbb{P}(X=1)\mathbb{P}(\epsilon X=1) = \frac{1}{4}$

Question 6

A vector space $(V,+,.,\mathbb{K})$ over a field $\mathbb{K}$ verifies

For all $u,v,w \in V$ and $\lambda,\mu \in \mathbb{K}$, then $u+v \in V$, $\lambda u \in V$ and

- $(V,+)$ is an Abelian group
- $\lambda(\mu u) = (\lambda\mu)u.$
- $(\lambda+\mu)u = \lambda u + \mu u.$
- $\lambda(u+v) = \lambda u + \lambda v.$
- $1u = u.$

Question 7

a) By using symetry, bilinearity and then symetry

b) When $y=0$ it is trivial. When $y \neq 0$ and $\lambda = \frac{<x,y>}{||y||^2}$, $0 \leq < x-\lambda y, x-\lambda y >= \frac{||x||^2||y||^2 - <x,y>^2}{||y||^2}$

c)
- Positive homogeneity is a result of Bilinearity.
- Triangle inequality can be obtained by squaring both sides of the inequality and applying Cauchy-Shwartz.
- Positive definiteness of the norm is a direct consquence of the Positive definiteness of the scalar product.

d) Cauchy-Shwartz