ORF524 - Problem Set 2

Bachir EL KHADIR

November 22, 2015

Problem 1

Let's consider the following optimization problem P:

$$\begin{aligned} \max_{x} \quad & 0 \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

and its dual D:

$$\begin{aligned} \min_{y \geq 0} \quad & y^T b \\ \text{subject to} \quad & y^T A = 0 \end{aligned}$$

If 1. is feasible, then 0 is the optimal solution. If 2. is feasible, then the optimal value of the dual problem is negative. By the duality theorem, we conclude that the two systems cannot be feasible at the same time.

If 1. is infeasable, it means that P is unfeasable, which means that D is either unbounded or unfeasble. But since 0 is a feasable solution for D, D must be unbounded, and there must exist y such that $y^T b < 0$, with $y \geq 0$ and $y^T A = 0$

Problem 2

•

$$\begin{aligned} \max_{f_{u,v} \geq 0} \quad & \sum_{v:(s,v)\in E} f_{s,v} \\ \text{subject to} \quad & \forall (u,v) \in E \; f_{u,v} \leq w(u,v) \\ & \forall v \in V \setminus \{s,t\} \sum_{u:(u,v)\in E} f_{u,v} = \sum_{u:(v,u)\in E} f_{v,u} \end{aligned}$$

If we set $w_{u,v}$ to 0 when $(u,v) \notin E$, we can rewrite the problem as:

$$\begin{aligned} \max_{f_{u,v}} \quad & \sum_{v:(s,v)\in E} f_{s,v} \\ \text{subject to} \quad & \forall u,v \in V \; 0 \leq f_{u,v} \leq w(u,v) \\ & \forall v \in V \setminus \{s,t\} \sum_{u\in V} f_{u,v} = \sum_{u\in E} f_{v,u} \end{aligned}$$

Or in vectorial form:

$$\begin{array}{ll} \max_{f\in R^{|V|^2}, f\geq 0} & c^T f \\ \text{subject to} & 0 \leq f \leq w \\ & Af = 0 \end{array}$$

The solution is:

$$f = \begin{pmatrix} 0 & 4 & 2 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$c^T f = 14$

- Let's call g, u the dual variables (for the first and second constraint resp.)

  The dual can be written as

$$\max_{f\geq 0} \min_{g\geq 0,u} c^T f + g^T(w-f) + u^T Af$$

e.g.

$$\max_{f\geq 0} \min_{g\geq 0,u} (c - g + u^T A)^T f + g^T w$$

e.g.

$$\begin{array}{ll} \min_{g\geq 0,u} & g^T w \\ \text{subject to} & g - u^T A \geq c \end{array}$$

e.g with $y = (w, g)$, $v = (0, w)$, $CC = (-c, 0)$

$$AA = \begin{pmatrix} A^T & -I_{n^2} \\ 0 & -I_{n^2} \end{pmatrix}$$

$$\begin{array}{ll} \min_{y} & y^T v \\ \text{subject to} & AAy \leq CC \end{array}$$

We obtain the same solution (14):

$$u = (0, -1, -1, -1, 0, -1, 0)$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
%% Primal problem ---------------------------------
n = 7;
A = [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
     0 0 0 0 0 0 0 0 1 0 0 0 0 0 -1 -1 -1 -1 -1 -1 -1 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0
    0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 -1 -1 -1 -1 -1 -1 -1
     0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0
     0 0 0 1 0 0 0 -1 -1 -1 -1 -1 -1 -1 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 -1 -1 -1 -1 -1 -1 -1 0 0 0 0 1 0 0
     0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0
     -1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];
A = reshape(A, n*n, n).';
c = [1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
     0 0 0 0 0 0 0];
W = [0 5 4 8 0 0 0 0 0 0 1 3 0 0 0 0 0 0 0 5 0 0 0 3 0 4 6 0 0 0 0 0 0 1    10 0 0 0 0 1
     0 6 0 0 0 0 0 1 0];
b = zeros(7, 1);
f = intlinprog(-c, 49, [], [], A, b, W*0, W);
c * f

%% Dual ---------------------------------
AA =  [A.' -eye(n*n); zeros(n*n, n) -eye(n*n)];
CC = - [c zeros(1, n*n)].';
V = [zeros(1, n) W] ;

g = intlinprog(V,52, AA, CC);
V * g
```

Problem 3

$$c = (2, 1, 3, 1, -3)$$

$$A = \begin{pmatrix} \begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & 2 & 0 & 4 & -3 \\ 1 & 1 & 0 & -3 & 4 \\ -1 & -3 & 3 & 0 & 0 \end{array} \end{pmatrix}$$

$\hat{A} := (A, I_3)$

First phase

$$\begin{array}{ll} \min\limits_{x,y\geq 0} & e^T y \\ \text{subject to} & Ax + y = b \end{array}$$

$z = (x = 0, y = b)$ is a BFS

z	B	A_B	A_B^{-1}	$e^T x$	$\bar{e}$	j	d	θ
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}$	$6,7,8$	I_3	I_3	5	$\bar{e}_1 = -1$	1	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$	2
$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$	$1,2,8$	$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$	3	$\bar{e}_3 = -3$	3	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}$	1
$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$1,2,3$	$-$	$-$	0(optimal)	$-$	$-$	$-$	$-$

We found a BFS $x = (2, 0, 1, 0, 0)$ to our problem

Second phase

x	B	A_B	A_B^{-1}	$c^T x$	$\bar{c}$	j	d	θ
$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$1,3,5$	$\begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 4 \\ -1 & 3 & 0 \end{pmatrix}$	$\frac{1}{21}\begin{pmatrix} 12 & 9 & 0 \\ 4 & 3 & 7 \\ -3 & 3 & 0 \end{pmatrix}$	7	$\bar{c}_2 = -\frac{8}{7}, \bar{c}_4 = -5$	4	$\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix}$	2
$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 2 \\ 2 \end{pmatrix}$	$3,4,5$	$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 0 & 4 \\ -3 & 3 & 0 \end{pmatrix}$	$\frac{1}{33}\begin{pmatrix} 12 & 9 & 0 \\ 4 & 3 & 7 \\ -3 & 3 & 0 \end{pmatrix}$	-3	$\hat{c}_1 = 5, \hat{c}_2 = \frac{47}{3}$	$-$	$-$	$-$

The optimal value is then -3 obtained for $x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 2 \\ 2 \end{pmatrix}$.

Problem 4

- Let x_i be the number of times that the manager uses the process i. The revenue for the manager is then $38(4x_1 + x_2 + 3x_2) + 33(3x_1 + x_2 + 4x_3) - 111x_1 - 11x_2 - 100x_3 = w^T x$ where $w = (140, 60, 146)$

$$\begin{array}{ll} \max\limits_{x \geq 0} & wx \\ \text{subject to} & 2x_1 + x_2 + 4x_3 \leq 8M \\ & 4x_1 + x_2 + 2x_3 \leq 5M \end{array}$$

which can be put to standard form:

$$\begin{array}{ll} \max\limits_{x \geq 0} & x^T w \\ \text{subject to} & 2x_1 + x_2 + 4x_3 + x_4 = 8 \\ & 4x_1 + x_2 + 2x_3 + x_5 = 5 \end{array}$$

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 2 & 1 & 4 & 1 & 0 \\ 4 & 1 & 2 & 0 & 1 \end{pmatrix} \end{array}$$

$$b = (5, 8)^T$$

We rewrite the program:

$$\begin{array}{ll} \max\limits_{x \geq 0} & x^T w \\ \text{subject to} & Ax = b \end{array}$$

Simplex tableau

x_1	x_2	x_3	x_4	x_5	p	
2	1	4	1	0	0	8
4	1	2	0	1	0	5
-140	-60	-146	0	0	1	0

x_1	x_2	x_3	x_4	x_5	p	
1/2	1/4	1	1/4	0	0	2
3	1/2	0	$-1/2$	1	0	1
-67	$-47/2$	0	73/2	0	1	292

x_1	x_2	x_3	x_4	x_5	p	
0	1/6	1	1/3	$-1/6$	0	11/6
1	1/6	0	$-1/6$	1/3	0	1/3
0	$-37/3$	0	76/3	67/3	1	943/3

x_1	x_2	x_3	x_4	x_5	p	
-1	0	1	1/2	$-1/2$	0	3/2
6	1	0	-1	2	0	2
74	0	0	13	47	1	339

The optimal BFS is then $x = (2, \frac{3}{2})$ corresponding to the basis $B = (2, 3)$ and the optimal solution 339

- The increase in price (that we call Δ) can change the optimal solution if one of the new reduced costs becomes positive.

  The only parameter that changes is w, it becomes $w' := w + \Delta(4, 1, 3, 0, 0)$ It is easy to see that $\bar{w}'_4$ and $\bar{w}'_5$ will not be affected.

$$\bar{w}_1 = -74$$

  Let's call $\bar{w}'$ the reduced cost after the price increase.

$$\bar{w}'_1 := w'_1 - w'^T_B A_B^{-1} A_1 = \bar{w}_1 + \Delta(4 - (1, 3)^T A_B^{-1} A_1) = \bar{w}_1 + \Delta$$

  $\bar{w}'_2$ becomes positive when $\Delta > -\bar{w}_1 = 74$,

- The new constraints are:

$$4x_1 + 3x_2 + 5x_3 \leq 14$$

  The optimal solution x^* verifies this constraint $4 * 0 + 3 * 2 + 5 * 3/2 = 13.5 < 14$

  So the optimal solution doesn't change since we are optimizing over a smaller set.

- Maximizing $U(r)$ is the same as maximizing $U(r)^2 = r$ when r is positive because U is increasing.

Problem 5

- Let's first prove that if we perturbate $a_{i,j}$, there is a basic feasible optimal solution that stays optimal. If that was not the case, it means that some BFS y wasn't optimal and become optimal after the perturbation. Said differently, the reduced costs corresponding to y were all positive and became non positive after each small perturbation. That is not possible because the reduced costs depend continuesly on A and therefore on $a_{i,j}$

  Indeed, if B the corresponding basis, $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$ seen as a function of $f(a_{i,j})$ is a rational fraction. Therefore if $\bar{c}_j$ is positive for $a_{i,j}$, there is an openset centered on $a_{i,j}$ where it stays positive.

  Let B be the basis corresponding to the BFS that stays optimal after the perturbation, then

  $V(A) = c^T A_B^{-1} b$ being linear in A, we have:

$$\frac{\partial}{\partial a_{i,j}} V(A) = c^T \frac{\partial}{\partial a_{i,j}} A_B^{-1} b = \begin{cases} -c^T A_B^{-1} E_{i,j} A_B^{-1} b & \text{if } j \in B \\ 0 & \text{if } j \notin B \end{cases}$$

  Where E_{ij} is the $|A| \times |A|$-matrix with only 0s except on the case (i, j) where there is 1. In the following we write only the entries of ∇V that are in the basis, ie $\nabla V := (\frac{\partial V}{\partial a_{ij}})_{i,j \in B}$

-

$$\begin{aligned} \nabla V_{i,j} &= -c_B^T A_B^{-1} E_{i,j} A_B^{-1} b && (1) \\ &= -c_B^T A_B^{-1} E_i E_j^T A_B^{-1} b && (2) \\ &= -(c_B^T A_B^{-1})_i (A_B^{-1} b)_j && (3) \end{aligned}$$

$$\nabla V = (c^T A_B^{-1})(A_B^{-1} b)^T$$

- – If the change in $a_{i,j}$ is for a number in a non-basic column, say A_N , then the original optimal solution is still feasible, the only change is to the reduced cost of $N-$th variable. Recompute $\bar{c}_N = c_N - c_B^T(A_B^{-1})^T A_N = f(a_{i,j})$, the solution doesn't change as long as $f(a_{ij}) >= 0$, we can solve this linear inequality to get the range.
  – If the change is for a number in a basic column, the solution doesn't change iff A_B^{-1} remains invertible, $A_B^{-1}b$ remains positive and all the reduced costs remain positive. eg one has to solve the following inequalities for a_{ij} (N is the set of columns not in B):

$$A_B^{-1}b \geq 0$$
$$\bar{c} = c_N - c_B^T(A_B^{-1})^T A_N \geq 0$$

Problem 6

Let $x_1, .., x_n$ is the set of BFS corresponding to the basis $B_1, ..B_n$ $\mathbb{P}(\exists i\, x_i \text{ degenrate}) \leq \sum_i \mathbb{P}(x_i \text{ degenrate})$

Let $i \in \{1, ..., n\}$, x_i is degenerate iff there exist a j such that $x_i^{(j)} = \sum_{k=1..n} {A_B^{-1}}_{i,k} b_k = 0$ $({A_B^{-1}}_{i,k})_k$ cannot be all 0 because A_B^{-1} is invertible. Let's suppose for example that ${A_B^{-1}}_{i,1} \neq 0$

But $\mathbb{P}(x_i^{(j)} = 0) = \mathbb{P}(b_1 = -\frac{1}{{A_B^{-1}}_{i,1}} \sum_{k>0} {A_B^{-1}}_{i,k} b_k) = \mathbb{E}\left[\mathbb{P}(b_1 = -\frac{1}{{A_B^{-1}}_{i,1}} \sum_{k>0} {A_B^{-1}}_{i,k} b_k | A, b_2, ...b_k)\right] = 0$ because b_1 has a density w.r.t. lebesgue measure.