ORF526 - Problem Set 1

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$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P}$ is the restriction of the lebesgue measure on Ω. This is a probability space. Let's consider the sequence:

$$X_k = k1_{\{0<x<\frac{1}{k}\}}$$

- $\mathbb{E}[X_k] = 1$
- $X_k \to_{k\infty} 0$ a.s., because for all $x \in (0,1)$, $X_k(x) = 0$ for all $k > \frac{1}{x}$

Question 1

- $\sup_k ||X_k||_1 = 1 < \infty$
- For any $C > 0$, for any $k > C$, $\mathbb{E}[|X_k|1_{\{X_k>C\}}] = \mathbb{E}[X_k] = 1$. Which means the sequence is not uniformly integrable.

Question 2

the (X_k) satisfy the conditions

Question 3

$\mathbb{E}(\liminf X_k) = \mathbb{E}(\lim X_k) = \mathbb{E}(0) = 0 < 1 = \lim_k \mathbb{E}(X_k) = \liminf \mathbb{E}(X_k)$

Question 4

Let's define
$\mu_1(A_1, ..., A_m) = \prod_i \mathbb{P}(X_i \in A_i)$
$\mu_2(A_1, ..., A_m) = \mathbb{P}(X_1 \in A_1, ..., X_m \in A_m)$
μ_1 is a measure (as the product measure of $\mathbb{P}oX^{-1}$)
μ_2 agrees with μ_1 on sets of the form $(-\infty, x_1] \times ... \times (-\infty, x_m]$
Since μ_1 is σ-finite (it's a probability measure), by Cathedory extension theorem, $\mu_1 = \mu_2$ on ...

Question 5

1. $i \Rightarrow iii$

   Let $\epsilon > 0$,$A_n = \underset{m\geq n}{\cup}\{\omega, |X_n(\omega) - X(\omega)| > \epsilon\}$ and $A_\infty = \underset{n}{\cap} A_n$ A_n is a decreasing sequence.

   If $\omega \in A_\infty$, for infinitely many $m \in \mathbb{N}$, $|X_n(\omega) - X(\omega)| > \epsilon$. Which means that $\omega \in N$. Therefore $\mathbb{P}(A_\infty) \leq \mathbb{P}(N) = 0$

   By continuity from above:

   $$\mathbb{P}(|X_n - X| > \epsilon) \leq \mathbb{P}(A_n) \to \mathbb{P}(A_\infty) = 0$$

2. $ii \Rightarrow iii$ By Markov inequality $\mathbb{P}(|X_n - X| > \epsilon) \leq \frac{E|X_n - X|}{\epsilon} \to 0$

3. $iii \Rightarrow iv$

4. blabla

Question 6

1. Every cdf is right continous and admits$F$ a left limit everywhere. (Let's call it $F(x-)$)

   A point of discontinuty if where $F(x-) \neq F(x)$.

   Let A be the set of discontinuties of F.

$$f : \begin{cases} A \to & \mathbb{Q} \\ x \to & \text{some arbitrary } r \in (F(x^-), F(x)) \end{cases}$$

   This application is an injection. So A is countable.

2.