Problem set 4, ORF523

Bachir El khadir

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1 Problem 1

Notations:

- c^* the solution to the problem. It exists because every subset of $\mathbb{R}$ has a supremum.
- $X = (x_1, x_2, x_3, x_4)$

$$f(X) = X^T \begin{pmatrix} 1 & -\frac{1}{2} & a & \frac{1}{10} \\ -\frac{1}{2} & 1 & b & c \\ a & b & 1 & -\frac{3}{10} \\ \frac{1}{10} & c & -\frac{3}{10} & 1 \end{pmatrix} X + \begin{pmatrix} 2 \\ -a \\ 0 \\ c \end{pmatrix}^T X := X^T S X + w^T X$$

- If S is not nonnegative semi definite, there exist X such that $X^T S X < 0$, and

$$f(\lambda X) = \lambda^2 X^T S X + \lambda w^T X \to_{\lambda\infty} -\infty$$

In order for the optimzation problem to be finite, we therefore need to have $S \geq 0$.

- If S positive definite, Let $\lambda_{\min}$ be its smallest eign value. Then $f(X) \geq \lambda_{\min}||X||^2 - ||w||||X|| \geq \lambda_{\min}(||X|| - \frac{1}{2\lambda_{\min}}||w||)^2 + cte$ is bounded from below.

Let c^+ be the solution to the following convex optimization problem: $\max_{a,b,c,S\geq 0} c$

```
n = 4
cvx_begin quiet
variable S(n,n) hermitian;
variable a;
variable b;
variable c;

maximize(c);
S == hermitian_semidefinite( n );
for i = 1:4
    S(i, i) == 1
end

S(1, 2) == -1/2;
S(1, 3) == a;
S(1, 4) == 1/10;
S(2, 3) == b;
```

```
S(2, 4) == c;
S(3, 4) == -3/10;
cvx_end

ans = [a b c]
```

-0.054543 -0.22906 0.81168

$0.8116 > c^+$ For all $c > c^+$, the optimization problem is not finite, so $c^* \leq c^+$.

Let $c^- = 0.8115$, we check with CVX that

$$S = \begin{pmatrix} 1 & -\frac{1}{2} & a^- & \frac{1}{10} \\ -\frac{1}{2} & 1 & b^- & c^- \\ a^- & b^- & 1 & -\frac{3}{10} \\ \frac{1}{10} & c^- & -\frac{3}{10} & 1 \end{pmatrix} > 0$$

with $a^- = -0.0682$, $b^- = -0.2216$ The problem for c^- is then finite, so $c^- <= c$.

Conclusion:

$$0.9116 \leq c^* \leq 0.9117$$

or

$$c^* \approx 0.911$$

2 Alternative way

$$S = \begin{pmatrix} 1 & -\frac{1}{2} & a & \frac{1}{10} \\ -\frac{1}{2} & 1 & b & c \\ a & b & 1 & -\frac{3}{10} \\ \frac{1}{10} & c & -\frac{3}{10} & 1 \end{pmatrix}$$

With a change of variable $x_3 \leftrightarrow x_4$, we can rewrite S:

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{10} & a \\ -\frac{1}{2} & 1 & c & b \\ \frac{1}{10} & c & 1 & -\frac{3}{10} \\ a & b & -\frac{3}{10} & 1 \end{pmatrix}$$

Note that $Q \geq 0 \iff S \geq 0$, and $Q > 0 \iff S > 0$.

In order of Q to be positive semi-definite, the following submatrix has to have a non-negative determinant:

$$Q_1 = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{10} \\ -\frac{1}{2} & 1 & c \\ \frac{1}{10} & c & 1 \end{pmatrix}$$

```
syms a b c;
S=[1,-1/2,a,1/10;-1/2,1,b,c;a,b,1,-3/10;1/10,c,-3/10,1];
Q_1 = [1 -1/2 1/10; -1/2 1 c; 1/10 c 1]
```

$det(Q_1) = g(c) = -c^2 - \frac{1}{10}c + \frac{37}{50}$

It has two roots $r_1 = \frac{-1-3\sqrt{33}}{10}, r_2 = \frac{-1+3\sqrt{33}}{10}$. Since the leadin coefficient is negative, the polynomial is non-negative iff $c \in [r_1, r_2]$.

Let's now check that for $c \in]r_1, r_2[$, there exist a, b that make $Q > 0$. Indeed, by using Sylvester criterion:

- $1 > 0$
- $$\det \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} = 1 - \frac{1}{4} > 0$$
- $det(Q_1) > 0$

It remains to show that there exist to show that there exist a, b that make the following determinant positive:

$$\begin{aligned}
P(a,b) &:= \begin{vmatrix} 1 & -\frac{1}{2} & \frac{1}{10} & a \\ -\frac{1}{2} & 1 & c & b \\ \frac{1}{10} & c & 1 & -\frac{3}{10} \\ a & b & -\frac{3}{10} & 1 \end{vmatrix} \\
&= a^2c^2 - (3b)/100 - c/10 - ab - (3ac)/10 - (3bc)/5 - a^2 - (99b^2)/100 - c^2 - (3a)/50 - (abc)/5 + 269/400 \\
&= (c^2-1)a^2 - \frac{99}{100}b^2 - (\frac{3}{10}c + \frac{3}{50})a - (\frac{3}{100} + \frac{3}{5}c)b - (\frac{c}{5}+1)ab - c^2 - c/10 + \frac{269}{400} \\
&= -\begin{pmatrix} a \\ b \end{pmatrix}^T \underbrace{\begin{pmatrix} 1-c^2 & \frac{c}{10}+\frac{1}{2} \\ \frac{c}{10}+\frac{1}{2} & \frac{99}{100} \end{pmatrix}}_{R} \begin{pmatrix} a \\ b \end{pmatrix} - \underbrace{\begin{pmatrix} \frac{3}{10}c + \frac{3}{50} \\ \frac{3}{100} + \frac{3}{5}c \end{pmatrix}}_{v} \begin{pmatrix} a \\ b \end{pmatrix} \underbrace{-c^2 - c/10 + \frac{269}{400}}_{\alpha} \\
&= \begin{pmatrix} a \\ b \end{pmatrix}^T R \begin{pmatrix} a \\ b \end{pmatrix} + v \begin{pmatrix} a \\ b \end{pmatrix} + \alpha
\end{aligned}$$

R is symmetric, By cholesky decomposition we can write it as $R = U^T U$

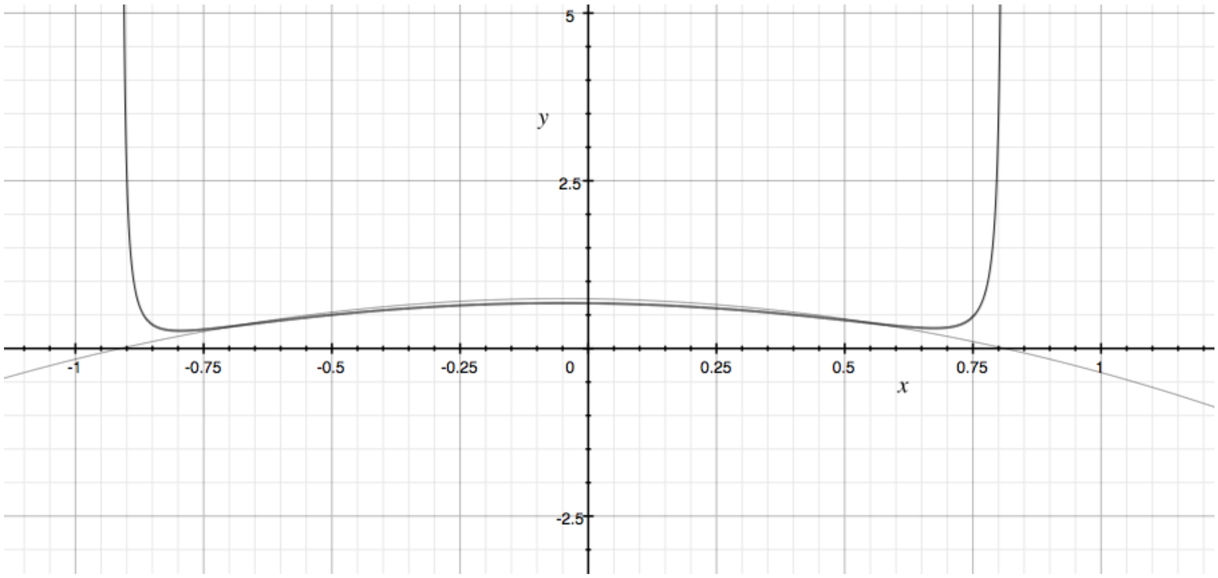
$$U = \begin{pmatrix} \sqrt{1-c^2} & \frac{c+5}{10\sqrt{1-c^2}} \\ 0 & \sqrt{\frac{g(c)}{1-c^2}} \end{pmatrix}$$

Let's do the change of variable $(x, y) = U(a, b)$:

$$\begin{aligned}
P(a,b) &= -\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} + (\underbrace{v^T U^{T^{-1}}}_{u^T}) \begin{pmatrix} x \\ y \end{pmatrix} + \alpha \\
&= -(x + \frac{u_1}{2})^2 - (y + \frac{u_2}{2})^2 + \alpha + \frac{u_1^2}{4} + \frac{u_2^2}{4}
\end{aligned}$$

```
assume(c < 1/2 & c > 0)
R = [(1-c^2), (c/10 + 1/2); (c/10+1/2), 99/100];
U = chol(R, 'real');
u = v'*(U')^(-1);
```

This polynomial can be non negative at some point if and only if $\alpha + \frac{u_1^2}{4} + \frac{u_2^2}{4} \geq 0$, we use matlab to calculate that expression, we plot it:



Which shows that it is always positive between the roots.
As a conclusion:

- for $r_1 < c < r_2$, there is a, b that make $S > 0$, and therefore the problem bounded.
- for $c > r_2$, all a, b make S not semi-definite, and therefore the problem not bounded.
- we conclude that $c^* = r_2 = \frac{-1+3\sqrt{33}}{10}$

3 Problem 2

1. Let $\|A\|_{\text{dual}} = \max_{\|X\|_{op}\leq 1}\langle Y, X\rangle$ and let's prove that $\|A\|_* = \|A\|_{\text{dual}}$

Lemma 3.1 *If $A = U\Lambda V^T$ be the SVD decomposition of A, then* $\|A\|_* = \text{Tr}(\Lambda)$

- Let $A = U\Lambda V^T$ be the SVD decomposition of A, then $\langle A, UV^T\rangle = \text{Tr}(V\Lambda U^T UV^T) = \text{Tr}(V\Lambda V^T) = \text{Tr}(\Lambda) = \|A\|_*$. Note that $\|UV^T\|_{op} = 1$ because UV^T is orthogonal. We have just proved that $\|A\|_* \leq \|A\|_{\text{dual}}$
- Let X be a matrix st $\|X\|_{op} \leq 1$, $\langle A, X\rangle = \text{Tr}(A^T X) = \text{Tr}(V\Lambda U^T X) = \text{Tr}(\Lambda U^T XV) = \sum \Lambda_{ii}(U^T XV)_{ii} = \sum \Lambda_{ii} \underbrace{u_i^T X v_i}_{\leq \|X\|_{op}} \leq \|X\|_{op}\|\Lambda\|_* \leq \|A\|_*$. so $\|A\|_* \geq \|A\|_{\text{dual}}$
- As a conclusion $\|A\|_* = \max_{\|X\|_{op}\leq 1}\langle Y, X\rangle$, and $\|.\|_{op}$ is the dual of $\|.\|_*$

Let's now prove that the nuclear norm is indeed a norm:

- If $\|A\|_* = 0$, then $\forall i \leq m \wedge n$ $\sigma_i(A) = 0$, If $U\Lambda V$ the SVD of A, then $\Lambda = 0$, and therefore $A = 0$.
- If $\alpha > 0$, $\alpha A = U(\alpha\Lambda)V^T$, and therefore $\|\alpha A\|_* = \text{Tr}(\alpha\Lambda) = \alpha\,\text{Tr}(\Lambda) = \alpha\|A\|_*$
- If $\alpha < 0$, $\alpha A = (-U)(-\alpha\Lambda)V^T$, we conclude in the same way as before.
- $\|A+B\|_* = \max_{\|X\|_{op}\leq 1} < A+B, X >= \max_{\|X\|_{op}\leq 1} < A, X > + < B, X >\leq \max_{\|X\|_{op}\leq 1} < A, X > + \max_{\|X\|_{op}\leq 1} < B, X >$ (Where we have used the fact that $\sup(S_1 + S_2) \leq \sup S_1 + \sup S_2$ for any two sets S_1, S_2), so $\|A+B\|_* \leq \|A\|_* + \|B\|_*$

1.

Let's first find unit sphere

$A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, Let λ_1, λ_2 be its eigen values.

$$\begin{aligned}
\|A\|_* = 1 &\iff |\lambda_1| + |\lambda_2| = 1 \\
&\iff \lambda_1^2 + \lambda_2^2 + 2|\lambda_1\lambda_2| = 1 \\
&\iff (\lambda_1 + \lambda_2)^2 + -2\lambda_1\lambda_2 + 2|\lambda_1\lambda_2| = 1 \\
&\iff \mathrm{Tr}(A)^2 + 2(|det(A)| - det(A)) = 1 \\
&\iff (\mathrm{Tr}(A)^2 = 1 \text{ and } \det(A) \geq 0) \text{ or } (\mathrm{Tr}(A)^2 - 4det(A) = 1 \text{ and } \det(A) \leq 0) \\
&\iff ((x+z)^2 = 1 \text{ and } xz \geq y^2) \text{ or } ((x+z)^2 - 4(xz - y^2) = 1 \text{ and } xz \leq y^2) \\
&\iff ((x+z)^2 = 1 \text{ and } xz \geq y^2) \text{ or } ((x-z)^2 + 4y^2 = 1 \text{ and } xz \leq y^2)
\end{aligned}$$

Let's do the linear change of variable

$$\begin{aligned}
u &= \frac{x+z}{\sqrt{2}} \\
v &= \sqrt{2}y \\
w &= \frac{x-z}{\sqrt{2}}
\end{aligned}$$

Which can also be written in matrix form as:

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} cos(-\frac{\pi}{4}) & -sin(-\frac{\pi}{4}) & 0 \\ sin(-\frac{\pi}{4}) & cos(-\frac{\pi}{4}) & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R} \begin{pmatrix} x \\ z \\ y \end{pmatrix}$$

The linear transformation R is then a rotation of $-\frac{\pi}{4}$ in the (X, Z) plane, and a scaling of $\frac{1}{\sqrt{2}}$ along the Y axis.

To find the shape of the unit cylinder, we work in the (u, v, w) space, and then we apply to inverse transformation of R.

Then $2xz = u^2 - w^2$, and

$$\begin{aligned}
\|A\|_* = 1 &\iff (2u^2 = 1 \text{ and } u^2 - w^2 \geq v^2) \text{ or } (2w^2 + 2v^2 = 1 \text{ and } u^2 - w^2 \leq v^2) \\
&\iff (u^2 = \frac{1}{2} \text{ and } u^2 \geq v^2 + w^2) \text{ or } (w^2 + v^2 = \frac{1}{2} \text{ and } u^2 \leq v^2 + w^2) \\
&\iff (u = \pm\frac{1}{2} \text{ and } \frac{1}{2} \geq v^2 + w^2) \text{ or } (w^2 + v^2 = \frac{1}{2} \text{ and } -\frac{1}{2} \leq u \leq \frac{1}{2})
\end{aligned}$$

$\{u = \pm\frac{1}{2} \text{ and } \frac{1}{2} \geq v^2 + w^2\}$ is two centered disks of radius $\frac{1}{\sqrt{2}}$ in the plane $u = \pm\frac{1}{2}$. $\{w^2 + v^2 = \frac{1}{2} \text{ and } -\frac{1}{2} \leq u \leq \frac{1}{2}\}$ is the lateral surface of the cylinder of radius $\frac{1}{\sqrt{2}}$ and axis u

Conclusion: In (u, v, w) space, the unit sphere $S(0, 1)$ is the basis and lateral surface of the cylinder with radius u, radius $\frac{1}{\sqrt{2}}$, and height 1.

Lemma 3.2 *The unit ball $B(0, 1)$ is the convex hull of the unit sphere $S(0, 1)$.*

$B(0, 1)$ is convex containing $S(0, 1)$. If $x, y \in S(0, 1)$, and $\alpha \in (0, 1)$, then $|\alpha x + (1-\alpha)y|_* \leq \alpha|x|_* + (1-\alpha)|y|_* \leq 1$ because of the triangular inequality.

$B(0, 1)$ is then a cylinder.

```python
import matplotlib
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import np

num_points = 30

# Construct cylinder
# base
x=np.linspace(-1,1,num_points)
z=np.linspace(-1,1,num_points)
X, Z=np.meshgrid(x,z)
Y=np.sqrt(1-X**2)

P = map(lambda u: u.ravel(), [X, Y, Z])
P[0] = np.concatenate((P[0], X.ravel()))
P[1] = np.concatenate((P[1], (-Y.ravel())))
P[2] = np.concatenate((P[2], Z.ravel()))

# lateral surface
t=np.linspace(0,1,num_points)
theta=np.linspace(0, 2*np.pi,num_points)
T, Theta = np.meshgrid(t, theta)

X = np.cos(Theta)*T
Y = np.sin(Theta)*T
Z = X*0 + 1
P[0] = np.concatenate((P[0], X.ravel()))
P[1] = np.concatenate((P[1], Y.ravel()))
P[2] = np.concatenate((P[2], Z.ravel()))

P[0] = np.concatenate((P[0], X.ravel()))
P[1] = np.concatenate((P[1], Y.ravel()))
P[2] = np.concatenate((P[2], -Z.ravel()))

# Inverse tranformation
R = np.array([[1, 0, 1], [0, 2, 0], [1,0,-1]]) / np.sqrt(2)
P = np.array(P)
P = np.dot(np.linalg.inv(R), P)

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.scatter(*P)
plt.xlabel('x')
plt.ylabel('y')
plt.zlabel('z')
plt.show()
fig.savefig('cylinder.png')
```

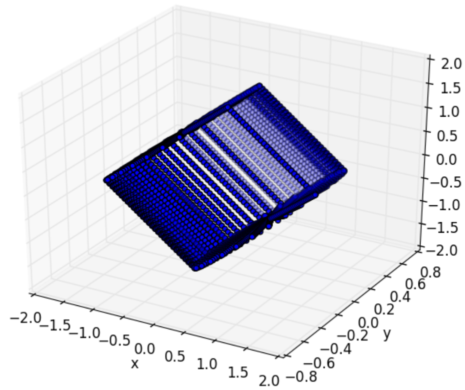


Figure 1: Shape of the unit nuclear ball

3.

Lemma 3.3 *For $Y \in \mathbb{R}^{n\times m}$*

$$||Y||_{op} \leq 1 \iff \begin{pmatrix} I_n & Y \\ Y^T & I_m \end{pmatrix} \geq 0$$

Proof: $||Y||_{op} \leq 1 \iff \forall x \in \mathbb{R}^{>} \leq 1 x^T Y^T Y x \leq x^T I_m x \iff Y^T I_n Y \leq I_m$ We conclude by Schur lemma.

Back to the problem, we can write the nuclear norm as a solution to an SDP:

$$
\begin{aligned}
.||X||_* = \max_{||Y||_{op}\leq 1} < X, Y > &= \max_{\begin{pmatrix} I_n & Y \\ Y^T & I_m \end{pmatrix} \geq 0} < X, Y > \\
&= \max_{\underbrace{\begin{pmatrix} I_n & Y \\ Y^T & I_m \end{pmatrix}}_{Z} \geq 0} < X, Y > \\
&= \frac{1}{2} \max_{Z\geq 0, Z\in C} < \underbrace{\begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}}_{X'}, Z >
\end{aligned}
$$

Where

- $C = \{Z \in \mathbb{R}^{(n+m)\times(n+m)} :< E_{ij}, Z >= 1_{i=j} \text{ for } (i,j) \in \mathcal{I}\}$ is a set defined by affine inequalities
- E_{ij} is the $(n+m)\times(n+m)$ matrix with 0 every where except on the entries (i,j) and (j,i) where it is equal to 1.
- $\mathcal{I} = \{(i,j) \in [1, n+m]^2, ((i \vee j) \leq n) \text{ or } (n < (i \wedge j))\}$

The feasible set $\{Z \geq 0, Z \in C\} \equiv \{Y : ||Y||_{op} \leq 1\}$ is the unit ball of a norm, so it is strictly feasible. The dual of this SDP can then be written as:

$\min_{\mu\in\mathbb{R}^{\mathcal{I}}, \sum_{(i,j)\in I} \mu_{ij} E_{ij} \geq X'} < \mu, b >$

Where $b_{ij} = 1_{ij}$

The feasible set of this program can be written as:

$$\{(\mu_1, \mu_2) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times m} \begin{pmatrix} \mu_1 & X \\ X^T & \mu_2 \end{pmatrix} \geq 0\}$$

This set also strictly feasible, indeed, since adding t times the identity matrix shift all the eigen values, for $t \in \mathbb{R}$ large enough, we have that:
$\begin{pmatrix} tI_n & X \\ X^T & tI_m \end{pmatrix} > 0$
This proves that the primal and dual are equal.

Now, we can rewrite the originial problem $\min_{\mathcal{A}(X)=c} |X|_*$ where $\mathcal{A}$ any linear functional as: $\min_{\mathcal{A}(X)=c} |X|_* = \min_{\mathcal{A}(X)=c} \max_{|Y|_{op}\leq 1} < X, Y >= \min_{X,\mu,\mathcal{A}(X)=c,\sum \mu_i A_i \geq X'} \mu^T b$ Where b, X' are the same as defined above. This is obviously an SDP.

4 Problem 3

1.

Let's first show the following:

Lemma 4.1 *a Matrix D is a distance matrix of somes points $x_1, \dots, x_n$ iff there exist $X \geq 0, X \in \mathbb{R}^{m\times m}$ such that $D_{ij} = X_{ii} + X_{jj} - 2X_{ij}$*

Indeed, if D is the distance matrix of $x_1, \dots, x_n$, let $X = (< x_i, x_j >)_{ij}$. Then:

- $D_{ij} = ||x_i - x_j||^2 =< x_i, x_i > + < x_j, x_j > -2 < x_i, x_j >= X_{ii} + X_{jj} - 2X_{ij}$.
- X is symmetric because $< ., . >$ is symmetric.
- Let $y \in \mathbb{R}^n$ $y^T X y = \sum_{ij} < x_i, x_j > y_i y_j = || \sum y_i x_i||_2^2 \geq 0$, so $X \geq 0$

  For the converse, let X a psd such that $D_{ij} = X_i i + X_j j - 2X_{ij}$. By Cholesky decomposition, let $M = \begin{pmatrix} m_1^T \\ \vdots \\ m_n^T \end{pmatrix} \in \mathbb{R}^{n\times n}$ s.t $X = MM^T$, eg $X_{ij} = m_i^T m_j$, so that $D_{ij} = ||m_i - m_j||^2$

  Using this Lemma, to solve the problem we have the check only if D can be written as $D_{ij} = X_{ii} + X_{jj} - 2X_{ij}$ for some matrix $X \geq 0$. e.g. check if the following SDP is feasible

$$\{X \geq 0; D_{ij} = X_{ii} + X_{jj} - 2X_{ij}\}$$

.

2. Take

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

It trivially verifies the triangular inequality.

Suppose it is a distance matrix, then by the lemma, there exist 4 points x_1, x_2, x_3, x_4 in $\mathbb{R}^4$ such that $||x_i - x_j|| = D_{ij}, i, j = 1, \dots, 4$

- $x_2, x_3, x_4 \in B(x_1, 1)$
- $|x_4 - x_3| = 2$, so $[x_4, x_3]$ is a diameter in $B(x_1, 1)$, so $x_1 \in \frac{x_3+x_4}{2}$
- $|x_2 - x_4| + |x_2 - x_3| = 1 + 1 = |x_4 - x_3|$, so $x_2 \in [x_3, x_4]$, so $x_2 = \frac{x_4 - x_3}{2} = x_1$
- But $|x_2 - x_1| = 1 \neq 0$, contradiction

  Concolusion: D is not a distance matrix.

5 Problem 4

1.

$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$A_2 A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

A_1, A_2 and A_2A_1 are triangular, so the eigen values are all in the diagonal, e.g $\rho(A_1) = \rho(A_2) = 0$, but , $\rho(A_2A_1) = 1$.

2.

The following program:

```
A1 = [-1 -1; -4 0] / 4;
A2 = [3 3; -2 1] / 4;

cvx_begin sdp
variable P(2, 2)
minimize(P(2, 1))
P >= 1
A1 * P * A1' <= P
A2 * P * A2' <= P
cvx_end

ans=P
```

proves that there exist $P \geq 0$ such that:

$A_1' P A_1 \leq P, A_2' P A_2 \leq P$

Lemma 5.1 *Let* $A, B \geq 0$, *If* $A \geq 0$, *then* $B'AB \geq 0$

Proof 5.2 *Indeed, for* $x \in \mathbb{R}^n$, $x'B'ABx = (Bx)'A(Bx) \geq 0$

Let Σ be a product of term of the form A_1 or A_2

Let's prove by induction on the size on the number of terms Σ, that $\Sigma' P \Sigma \leq P$.

Indeed, let Σ be of size $n+1$, without loss of generality $\Sigma = \Sigma_1 A_1$, with Σ_1 of size n.

Then $\Sigma' P \Sigma = A_1' \underbrace{\Sigma_1' P \Sigma_1}_{\leq P} A_1 \leq A_1' P A_1 \leq A_1$.

By lyapounouv theorem, Σ is then stable, or equivalenty $\{A_1, A_2\}$ is stable.