ORF524 - Problem Set 4

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Problem 1

1.
$$\hat{\beta} = (X^TX)^{-1}X^T(X\beta + \eta) = \beta + (X^TX)^{-1}X^T\eta$$

$$(X^TX)(\hat{\beta} - \beta) = X^T\eta$$

So

$$\hat{\beta} - \beta \sim \mathcal{N}(0, (X^TX)^{-1}X^T((X^TX)^{-1}X^T)^T) \sim \mathcal{N}(0, (X^TX)^{-1})$$

$X^TX > 0$ so is $(X^TX)^{-1}$. There exist an orthogonal matrix U, and $D = diag(\lambda_1, ..., \lambda_n) > 0$ such that $(X^TX)^{-1} = UDU^T = (U\sqrt{D}U^T)^2$, where $\sqrt{D} = diag(\sqrt{\lambda_1}, ...\sqrt{\lambda_n})$. Let $L := U\sqrt{D}U^T = L^T$.

$$\hat{\beta} - \beta \sim L\mathcal{N}(0, I)$$

$$L^{-1}(\hat{\beta} - \beta) \sim \mathcal{N}(0, I)$$

$$P(||L^{-1}(\hat{\beta} - \beta)||^2 \leq z) = P(\chi_n^2 \leq z) = \Phi(z)$$

$$S(\hat{\beta}) = \{\hat{\beta} + u : ||L^{-1}u|| \leq z\}$$
$$z = \Phi^{-1}(1 - \alpha)$$

2. $||Ax||_2 \leq ||A||_2||x||_2 \leq n||A||_2||x||_\infty$

$$\{\beta : ||A(\beta - \hat{\beta})||_2 < r\} \subseteq \{\beta : ||\beta - \hat{\beta}||_\infty < \frac{r}{n||A||_2}\}$$

So

$$P(\{\beta : ||\beta - \hat{\beta}||_\infty < \frac{r}{n||A||_2}\}) \geq 1 - \alpha$$

3.

4. By taking the conditional probability in X and then summing over all possible values of X, we prove that the confidence sets still of level $1 - \alpha$.

$$P^{H_0}(T(x) = 0) = E_X^{H_0}[P^{H_0}(T(x) = 0|X)] \geq 1 - \alpha$$

Problem 2

$$\hat{\beta} = \beta + \frac{\eta}{n}$$

The procedure chooses the estimator of $S := \text{supp}(\beta)$, order the component of $|\hat{\beta}|$ in descending order $|\hat{\beta}_{j_1}| \geq ... \geq |\hat{\beta}_{j_n}|$, and choose the set of k indices $\hat{S} = \{j_1, ...j_k\}$.

The probability of success is

$$\begin{aligned} P(\hat{S} = S) &= P(\forall\, i \notin S, j \in S\, |\hat{\beta}_i| \leq |\hat{\beta}_j|) \\ &\geq P(\forall\, i \notin S, j \in S\, |\eta_i| \leq |\eta_j + n\beta_j|) \\ &\geq P(\forall\, i \notin S, j \in S\, |\eta_i| \leq |\eta_j| - n\kappa) \\ &\geq P(\forall\, i \notin S, j \in S\, \eta_i \leq \eta_j + n\kappa) \\ &\geq P(\mathcal{N}(0,1) - \mathcal{N}(0,1) \leq n\kappa) \\ &\geq \Phi(\frac{n\kappa}{\sqrt{2}}) \end{aligned}$$

For this quantity to be bigger than $1-\alpha$, n must be bigger than $\frac{\sqrt{2}\Phi^{-1}(1-\alpha)}{\kappa}$.

Problem 3

- (NB: $\mathbb{R}$ is a $(1-\alpha)$-confidence set for θ.)
  We look for an confidence set of the form $\hat{S}_\lambda = [\lambda \max x_i, \infty)$
  $P(\theta \in \hat{S}_\lambda) = P(\max x_i \leq \frac{\theta}{\lambda}) = (\frac{1}{\theta}\frac{\theta}{\lambda})^n = \lambda^{-n}$
  For $\hat{S}_\lambda$ to be a $(1-\alpha)$-confidence set for θ, $\lambda = (1-\alpha)^{-\frac{1}{n}}$.
- $P(m \in [x_{(1),\infty})) = P(x_{(1)} \leq F^{-1}(\frac{1}{2})) = 1 - P(\forall\, i\, x_i > F^{-1}(\frac{1}{2})) = 1 - (1 - F(F^{-1}(\frac{1}{2})))^n = 1 - \frac{1}{2}^n$
  $P(m \in (-\infty, x_{(n)}]) = 1 - P(\forall i\, x_i \leq m) = 1 - \frac{1}{2}^n$
  $A := [x_{(1)}, \infty)$, $B := (-\infty, x_{(n)}]$, $A \setminus B = (x_{(n)}, \infty)$, $B \setminus A = (-\infty, x_{(1)})$
  $P(m \in [x_{(1)}, x_{(n)}]) = P(m \in A \cap B) = 1 - P(m \in A^c \cup B^c) = 1 - P(m \in A \setminus B) - P(m \in B \setminus A) = 1 - \frac{1}{2}^{n-1}$

Problem 4

$$\begin{aligned} \beta_T(\mu) = E^\mu[T(x)] &= P^\mu(|\mathcal{N}(\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}, 1)| > z_{\frac{\alpha}{2}}) && (1) \\ &= 1 - P(-\frac{\sqrt{n}}{\sigma}(\mu-\mu_0) - z_{\frac{\alpha}{2}} < N(0,1) < -\frac{\sqrt{n}}{\sigma}(\mu-\mu_0) + z_{\frac{\alpha}{2}})) && (2) \\ &= 1 + \phi(-\frac{\sqrt{n}}{\sigma}(\mu-\mu_0) - z_{\frac{\alpha}{2}}) - \phi(-\frac{\sqrt{n}}{\sigma}(\mu-\mu_0) + z_{\frac{\alpha}{2}}) && (3) \end{aligned}$$

$$\beta_T(\mu) \to_n 1$$

Problem 5

- For $p < p_0$, $P_p(T(x) = 0) \geq P_{p_0}(T(x) = 0) = P(\mathcal{B}(n, p_0) \geq nc_\alpha) = 1 - F(nc_\alpha)$
  $1 - F(nc_\alpha) = 1 - \alpha \iff c_\alpha = \frac{F^{-1}(\alpha)}{n}$
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Problem 6

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$$P^\theta(\theta \in S(x)) = P^\theta(T_\theta(x) = 0) = 1 - \alpha$$

– $H_0 : (\mu, \sigma) = (\mu_0, \sigma_0),\ H_1 : (\mu, \sigma) \neq (\mu_0, \sigma_0)$
  $T_{\mu_0,\sigma_0}(x) = 1_{\{\sqrt{n}|\bar{x}-\mu_0|/\sigma_0 > f(\alpha)\}}$

$$S(x) = \{\theta | \sqrt{n}|\bar{x} - \mu_0|/\sigma_0 < f(\alpha)\}$$

  is a $(1-\alpha)$-confidence set

Problem 7

1. If $T(x)$ a test of

Problem 8

– Let's consider the test

$$T(x) = 1_{\{\sqrt{n}|\bar{x}-\bar{y}|>t\}}$$

  Under H_0, $P(T(x) = 1) = P(|\mathcal{N}(0,1)| > t) = 1-(\Phi(t)-\Phi(-t)) = 1-2(\Phi(t)-\Phi(0)) = 2(1-\Phi(t))$.
  In order for the test to be of size $1-\alpha$, $t = \Phi^{-1}(1 - \frac{1-\alpha}{2}) = \Phi^{-1}(\frac{1+\alpha}{2})$

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	$T_<$	T	$T_>$
$t + \mu_0$	$-\sqrt{n}\Phi^{-1}(1+\alpha)$	$\sqrt{n}\Phi^{-1}(\frac{1+\alpha}{2})$	$\sqrt{n}\Phi^{-1}(1+\alpha)$
β			

Problem 9

– $E[g(X)] = \int g(x)\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})dx = \int g(\sigma(u+\mu))f(u)du = E[g(\sigma(Y+\mu)]$
  For $g = \text{id}, (\text{id} - E[X])^2$,

$$E[X] = E[\sigma(Y+\mu)] = \sigma(\mu + \mu_Y)$$
$$Var(X) = Var(\sigma(Y+\mu)) = \sigma^2\sigma_Y^2$$

– $Z = \frac{\bar{x}-\mu}{\sigma}$

Problem 10

$$\begin{aligned} P^{H_0}(T(x) = 1) &= E_l[P(X > \theta_0 + z_\alpha\sigma_l)|l] && X \sim \mathcal{N}_l(\theta_0, \sigma_l^2) && (4)\\ &= E_l[P(\theta_0 + \sigma_l N > \theta_0 + z_\alpha\sigma_l)|l] && N \sim \mathcal{N}_l(0,1) && (5)\\ &= 1 - \Phi(z_\alpha) = \alpha && && (6) \end{aligned}$$

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\begin{aligned}
P^{H_0}(T'(x)=1) &= P(l=1) + P(l=2)P(x > \theta_0 + z_{\frac{\alpha-p}{1-p}}\sigma_2 | l=2) && (7)\\
&= p + (1-p)P(\theta_0 + \sigma_2 N > \theta_0 + z_{\frac{\alpha-p}{1-p}}\sigma_2) && (8)\\
&= p + (1-p)\Phi(z_{\frac{\alpha-p}{1-p}}) && (9)\\
&= p + (1-p)(1 - \frac{\alpha-p}{1-p}) && (10)\\
&= 1 - (\alpha - p) \geq 1 - \alpha && (11)
\end{aligned}
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