ORF526 - Problem Set 9

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December 3, 2015

Question 1

Because it is predictable we have that $E[M_{n+1}|F_n] = M_{n+1}$, and because it is a martingale $E[M_{n+1}|F_n] = M_n$. Therefore M_n is constant equal to M_0.

Question 2

Let $M_n = X_0 + \sum_{i=0}^{n-1} X_{i+1} - E[X_{i+1}|F_i]$, $A_n = X_n - M_n = \sum_{i=0}^{n-1} E[X_{i+1}|F_i] - X_i$ and as a convention $A_0 = 0$.
Then

- (M_n) is an (F_n) martingale because
  - It is (F_n)-adapted: For all n, M_n for $i = 0...n-1$, X_{i+1} and $E[X_{i+1}|F_i]$ are F_n measurable.
  - $M_{n+1} - M_n = X_{n+1} - E[X_{n+1}|F_n]$, so

$$E[M_{n+1}|F_n] - M_n = E[M_{n+1} - M_n|F_n] = E[X_{n+1}|F_n] - E[E[X_{n+1}|F_n]|F_n] = 0$$

- (A_n) is a non-decreasing predictable process because:
  - (A_n) is predictable because for $i < n$, X_i and $E[X_{i+1}|F_i$ are $F_i \subset F_n$ measurable
  - (A_n) is non-decreasing: $A_{n+1} - A_n = E[X_{n+1}|F_n] - X_n \geq 0$ because X_n is a submartingale.

The decomposition is unique, because if there exist an other decomposition $X_n = M'_n + A'_n$ with the same properties then: $M_n - M'_n = A_n - A'_n$, which a marintgale (as the difference of two martingales), and predictable (as the difference of two predictable processes). By question 1, this sequence is constant equal to $A_0 - A'_0 = 0$

Question 3

Let $n, p \in \mathbb{N}$,
By the itereated expectation:

$$E[M_{n+i+1}M_{n+i}] = E[E[M_{n+i+1}|F_n]M_{n+i}] = E[M_n^2]$$

So:

$$||M_{n+p} - M_n||_2^2 = E[|M_{n+p} - M_n|^2] = E[M_{n+p}^2] + E[M_n^2] - 2E[M_{n+p}M_n] = E[M_{n+p}^2] - E[M_n^2]$$

M_n^2 is a submartingale, so $E[M_n^2]$ is non-decreasing, and since it is bounded, it converges and therefore $E[M_{n+p}^2] - E[M_n^2] \to_{n,p} 0$ c/c: $||M_{n+p} - M_n||_2 \to_{n,p} 0$, and (M_n) is a cauchy sequence.

Question 4

1. M_n is L^p bounded and $p > 1$, so (M_n) is uniformly integrable, and therefore: M_n

Question 5

$B_n = \sum_i B_{i+1} - B_i = \sum_i Y_i$
Let F_n be the filtration generated by B_n

1. B_n is F_n adapted, so is $B_n^2 - n$

2. B_n^2 is L_1, and

$$\begin{aligned}
E[B_{n+1}^2 - (n+1)|B_n] &= E[(B_{n+1} - B_n + B_n)^2 - (n+1)] \\
&= E[(B_{n+1} - B_n)^2|B_n] + E[B_n^2|B_n] + 2E[(B_{n+1} - B_n)B_n|F_n] - (n+1) \\
&= E[\mathcal{N}(0,1)^2] + B_n^2 - (n+1) \\
&(\text{because } B_{n+1} - B_n \sim \mathcal{N}(0,1) \text{ and is independent from } B_n) \\
&= B_n^2 - n
\end{aligned}$$

   so $B_n^2 - n$ is a martingale.

3. where $Y_i = B_{i+1} - B_i \sim \mathcal{N}(0,1)$ are iid

$$E[\exp(\sigma B_n - \frac{1}{2}\sigma n^2)] =$$

Question 6

1.

$$\begin{aligned}
a\log(b) \leq alog(a) + \frac{b}{e} &\iff \frac{a}{b}\log(\frac{b}{a}) \leq \frac{1}{e} \\
&\iff \frac{\log(x)}{x} \leq \frac{\log(e)}{e} \qquad\qquad (x = \frac{b}{a}) \\
&\iff f(x) \leq f(e)
\end{aligned}$$

   Where $f : x \to \frac{\log(x)}{x}$ for $x > 0$, $f'(x) = \frac{1-\log(x)}{x^2} = -\frac{\log(\frac{x}{e})}{x^2}$ is positive when $x \leq e$ and negative otherwise, so f has a global maximum in $x = e$.

Question 8

By indepedence of the X_i, $E[X_1|S_n, S_{n+1}...] = E[X_1|S_n, X_{n+1}, ...] = E[X_1|S_n]$ By symmetry, $E[X_i|S_n] = E[X_n|S_n]$, and therefore: $S_n = E[S_n|S_n] = \sum_{i=1}^n E[X_i|S_n] = nE[X_1|S_n]$, ie $E[X_1|S_n, S_{n+1}...] = \frac{S_n}{n}$