ORF524 - Problem Set 4

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Problem 1

KKT conditions

- $\exists \lambda_j, \geq 0, j = 1..n$ s.t $\nabla f(x^*) = -\sum_j \lambda_j \nabla g_j(x^*)$
- $g_j(x^*) \leq 0$ for $j = 1..n$

Let x be a feasible solution to the second optimization problem. We have that

$$\begin{aligned}
\nabla f(x^*)^T(x - x^*) &= -\sum_j \lambda_j \nabla g_j(x^*)^T(x - x^*) \\
&\geq -\sum_j g_j(x^*) && \text{by feasibility of } x \\
&\geq 0
\end{aligned}$$

As a result, $f(x^*) + f(x^*)^T(x^* - x^*) \leq f(x^*) + f(x^*)^T(x - x^*)$, and since x^* trivially verifies the feasibility conditions of the second problem, x^* is a global optimal.

Problem 2

1. First order conditions:

$$0 = \nabla f = \begin{pmatrix} 4x + y - 6 \\ x + 2y + z - 7 \\ y + 2z - 8 \end{pmatrix}$$

   Or

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$$

   Which can be resolved:

$$X^* = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 6 \\ 6 \\ 17 \end{pmatrix}$$

2.

$$\nabla^2 f(x, y, z) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

3.
$$\begin{aligned}
f(x,y,z) &= x^2 + \frac{1}{2}y^2 + (x+\frac{1}{2}y)^2 + (z+\frac{1}{2}y)^2 - 6x - 7y - 8z - 9 \\
&= x^2 - 6x + \frac{1}{2}(y^2 - 6y) + (x+\frac{1}{2}y)^2 + ((z+\frac{1}{2}y)^2 - 8(z+\frac{1}{2}y)) - 9 \\
&= (x-3)^2 + \frac{1}{2}(y-3)^2 + (x+\frac{1}{2}y)^2 + (z+\frac{1}{2}y-4)^2 - 9 - 9 - \frac{1}{2}9 - 16 \\
&= (x-3)^2 + \frac{1}{2}(y-3)^2 + (x+\frac{1}{2}y)^2 + (z+\frac{1}{2}y-4)^2 - 38.5 \geq -38.5
\end{aligned}$$

f is a continous function bounded from below, and is coercive, it has a global minimum, which is also a local minimum. Since f admits only one local minimum X^*, it is a global minimum, and $\min f = f(\frac{6}{5}, \frac{6}{5}, \frac{17}{5}) = -30.4$

4.
```
cvx_begin
    variable x
    variable y
    variable z
    minimize(x^2 + 0.5 * y^2 + (x+0.5*y)^2 + (z+0.5*y)^2 -6*x - 7*y - 8*z - 9)
cvx_end
```

Problem 3

1. Let's consider the problem (P):

$$\max_{2\pi(x^2+y)\leq C, x\geq 0, y\geq 0} \pi x y$$

The feasible set is bounded and closed. The objective function is continuous. So it has an optimal solution (x^*, y^*). $x^* y^* \neq 0$ because $\frac{\sqrt{C}}{2}\frac{C}{2} > 0$.

Let $h^* = \frac{x^*}{y^*}, r^* = x^*$. (r^*, h^*) is optimal because:

- It is feasible: $2\pi(r^{*2}, +r^* h^*) = 2\pi(x^{*2} + y^*) \leq C$, $r > 0, h > 0$
- If r, h another feasible solution, then $x = r, y = rh$ is feasible for the problem (P), and therefore: $\pi x y \leq \pi x^* y^*$, ie $\pi r^2 h \leq r^{*2} h^*$

2. We can rewrite the problem into an equivalent problem by taking the log of the objective function (with the convention $\log(0) = -\infty$, which makes log increasing in R^+).

$$\max_{2\pi(r^2+rh)\leq C} \log(\pi) + 2\log(r) + \log(h)$$

And this is a convexe problem because:

- the objective function in concave as the sum of two concave functions
- the constraint function $(r,h) \to r^2 + rh$ is convexe because its hessian

$$\begin{pmatrix} 2 & 2r \\ 2r & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

has positive derterminant (1), so the two eigen values are of the same sign, and their sum is positive (equal to the trace: 4)

3. The lagrangian $\mathcal{L}(r,h,\lambda) = \pi r^2 h + \lambda(C - 2\pi r^2 - 2\pi rh)$

   KKT (we know that $rh \neq 0$ from previous question):

   - $0 = \frac{\partial}{\partial r}\mathcal{L} = 2\pi(rh - \lambda(2r+h)) \Rightarrow \lambda = \frac{rh}{2r+h}$
   - $0 = \frac{\partial}{\partial h}\mathcal{L} = \pi(r^2 - \lambda 2r) \Rightarrow \lambda = \frac{r}{2} \neq 0$
   - $\lambda = \frac{r}{2} = \frac{rh}{2r+h} \Rightarrow h = 2r$
   - Complementary condition $\lambda \neq 0 \Rightarrow C = 2\pi(r^2 + rh) = 6\pi r^2 \Rightarrow r = \sqrt{\frac{C}{6\pi}}$

   Conclusion: $r = \sqrt{\frac{C}{6\pi}}$, $h = \sqrt{\frac{2C}{3\pi}}$.

Problem 4

1. $f = -\log$, $g = \exp$ are both convexe but $fog(x) = -x$ is concave

2. Let $\lambda \in [0,1]$, since g is convexe: $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$

   Since f is non-decreasing: $f(g(\lambda x + (1-\lambda)y)) \leq f(\lambda g(x) + (1-\lambda)g(y))$

   Since f is convexe: $f(\lambda g(x) + (1-\lambda)g(y)) \leq \lambda fog(x) + (1-\lambda)fog(y)$

   as a conclusion

   $$fog(\lambda x + (1-\lambda)y) \leq \lambda fog(x) + (1-\lambda)fog(y)$$

   and fog is convexe

3. if f concave and non-increasing, $-f$ is convexe and non-decreasing, therefore $-fog$ is convexe, and fog is concave.

4. Let's take $f = 1 - e^{-x}$. f is increasing and non negative. $f' = e^{-x}$, $f'' = -f'$

   $(xf)'' = xf'' + f' = e^{-x}(1-x) < 0$ when $x > 1$, so xf is not convexe on R^+.