ORF524 - Problem Set 1

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Problem 1

Let's consider the following optimization problem P:

$$\begin{aligned} \max_{x} \quad & 0 \\ \text{subject to} \quad & Ax \leq b \end{aligned} \tag{1}$$

and its dual D:

$$\begin{aligned} \min_{y \geq 0} \quad & y^T b \\ \text{subject to} \quad & y^T A = 0 \end{aligned} \tag{2}$$

By the duality theorem, both problems have the same optimal solution.

If 1. is feasible, then 0 is the optimal solution. If 2. is feasible, then the optimal value is negative. We conclude that the two systems cannot be feasible at the same time.

If 2. is infeasable, if and only if $N(A^T) \cap \{y \geq 0\} \cap \{b^T y < 0\} = \emptyset$, and using Problem 3 of the first assignement, this means that the dual D (and consequently the primal P) has an optimal solution, and 1 is then feasible.

Problem 2

- $$\begin{aligned} \max_{f_{u,v} \geq 0} \quad & \sum_{v:(s,v) \in E} f_{s,v} \\ \text{subject to} \quad & \forall (u,v) \in E \; f_{u,v} \leq w(u,v) \\ & \forall v \in V \setminus \{s,t\} \sum_{u:(u,v) \in E} f_{u,v} = \sum_{u:(v,u) \in E} f_{v,u} \end{aligned} \tag{3}$$

  If we set $w_{u,v}$ to 0 when $(u,v) \notin E$, we can rewrite the problem as:

  $$\begin{aligned} \max_{f_{u,v}} \quad & \sum_{v:(s,v) \in E} f_{s,v} \\ \text{subject to} \quad & \forall u, v \in V \; 0 \leq f_{u,v} \leq w(u,v) \\ & \forall v \in V \setminus \{s,t\} \sum_{u \in V} f_{u,v} = \sum_{u \in E} f_{v,u} \end{aligned} \tag{4}$$

  Or in vectorial form:

  $$\begin{aligned} \max_{f \in R^{|V|^2}, f \geq 0} \quad & c^T f \\ \text{subject to} \quad & 0 \leq f \leq w \\ & Af = 0 \end{aligned} \tag{5}$$

- Let's call g, u the dual variables (for the first and second constraint resp.)

  The dual can be written as

$$\max_{f\geq 0}\min_{g\geq 0,u} c^T f + g^T(w-f) + u^T A f$$

e.g.

$$\max_{f\geq 0}\min_{g\geq 0,u} (c - g + u^T A)^T f + g^T w$$

e.g.

$$\begin{aligned} \min_{g\geq 0,u} \quad & g^T w \\ \text{subject to} \quad & u^T A - g \geq c \end{aligned} \tag{6}$$

Problem 3

$$c = (2, 1, 3, 1, -3)$$

$$A = \left(\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & 2 & 0 & 4 & -3 \\ 1 & 1 & 0 & -3 & 4 \\ -1 & -3 & 3 & 0 & 0 \end{array}\right)$$

The first problem $\hat{A} := (A, I_3)$

$$\begin{aligned} \min_{x,y\geq 0} \quad & e^T y \\ \text{subject to} \quad & Ax + y = b \end{aligned} \tag{7}$$

$z = (x = 0, y = b)$ is a BFS

z	B	A_B	A_B^{-1}	$e^T x$	$\bar{e}$	j	d	θ
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}$	$6, 7, 8$	I_3	I_3	5	$\bar{e}_1 = -1$	1	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$	2
$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$	$1, 2, 8$	$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$	3	$\bar{e}_3 = -3$	3	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}$	1
$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$1, 2, 3$	–	–	0(optimal)	–	–	–	–

We found a BFS $x = (2, 0, 1, 0, 0)$ to our problem

x	B	A_B	A_B^{-1}	$c^T x$	$\bar{c}$	j	d	θ
$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$1, 3, 5$	$\begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 4 \\ -1 & 3 & 0 \end{pmatrix}$	$\frac{1}{21}\begin{pmatrix} 12 & 9 & 0 \\ 4 & 3 & 7 \\ -3 & 3 & 0 \end{pmatrix}$	7	$\bar{c}_2 = -\frac{8}{7}, \bar{c}_4 = -5$	4	$\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix}$	2
$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 2 \\ 2 \end{pmatrix}$	$3, 4, 5$	$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 0 & 4 \\ -3 & 3 & 0 \end{pmatrix}$	$\frac{1}{33}\begin{pmatrix} 12 & 9 & 0 \\ 4 & 3 & 7 \\ -3 & 3 & 0 \end{pmatrix}$	-3	$\hat{c}_1 = 5, \hat{c}_2 = \frac{47}{3}$	$-$	$-$	$-$

The optimal value is then -3.

Problem 4

- Let x_i be the number of times that the manager uses the process i, p_i the the benefit from selling the ouput of that process, and c_i the cost.

$$\begin{aligned} \max_{x\geq 0} \quad & \sum_i x_i(p_i - c_i) = x^T(p-c) \\ \text{subject to} \quad & 2x_1 + 2x_2 + 4x_3 \leq 8M \\ & 4x_1 + 2x_2 + 2x_3 \leq 5M \end{aligned} \tag{8}$$

which can be put to standard form:

$$\begin{aligned} \max_{x\geq 0} \quad & \sum_i x_i(p_i - c_i) = x^T(p-c) \\ \text{subject to} \quad & 2x_1 + 2x_2 + 4x_3 + x_4 = 8 \\ & 4x_1 + 2x_2 + 2x_3 + x_5 = 5 \end{aligned} \tag{9}$$

$$c = (111, 11, 100), p = (4, 1, 3)38 + (3, 1, 4)33 = (251, 71, 246)$$

$$A = \left(\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 2 & 2 & 4 & 1 & 0 \\ 4 & 2 & 2 & 0 & 1 \end{array} \right)$$

$$b = (5, 8)^T$$

$$w = p - c$$

We rewrite the program:

$$\begin{aligned} \max_{x\geq 0} \quad & x^T w \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{10}$$

Simplex We start with a $B = (1, 3)$ as a basis. we calculate

$$x_B = A_B^{-1} b = \frac{1}{6} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} (5, 8)^T = (\frac{1}{3}, \frac{11}{6})^T \geq 0$$

The reduced cost for 2, 4, 5: $\bar{c}_2 = -106/3, \bar{c}_4 = -76/3, c_5 = -77/3 \leq 0$

We have found the optimum for $x^* = (\frac{1}{3}, 0, \frac{11}{6}, 0, 0)^T$.

- The increase in price (that we call Δ) can change the optimal solution if one of the new reduced costs becomes positive.

  The only parameter that changes is w, it becomes $w' := w + \Delta(4, 1, 3, 0, 0)$ It is easy to see that $\bar{w}'_4$ and $\bar{w}'_5$ will not be affeced.

  Let's call $\bar{w}'$ the reduced cost after the price increase. $\bar{w}'_2 := \hat{w}'_2 - w'^T_B A_B^{-1} A_2 = \bar{w}_2 + \Delta(1 - (4, 3)A_B^{-1} A_2) = \bar{w}_2 + (1 - \frac{7}{3})\Delta = -106/3 - 4/3\Delta$ $\bar{w}'_2$ becomes positive when $\Delta < -\frac{3}{4} w_2$

- The new constraints are:

$$4x_1 + 3x_2 + 5x_3 \leq 14$$

  The optimal solution x^* verifies this constraint $4 * 1/3 + 3 * 0 + 5 * 11/6 = 63/6 \sim 10.5 < 14$

  So the optimal solution doesn't change

- Maximizing $U(r)$ is the same as maximizing $U(r)^2 = 2$ when U is positive

Problem 5