Convergence of the optimal polynomial solution to the optimal

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1 Notation

$$\begin{array}{ll} \text{maximize} & \langle c(t), x(t)\rangle \\ \text{subject to} & A(t)x(t) = b(t) \\ & x(t) \geq 0 \end{array} \tag{P_t}$$

$$\begin{array}{lll} \text{maximize} & \int_0^1 \langle c(t), x(t)\rangle \mathrm{d}t & \\ \text{subject to} & A(t)x(t) = b(t) & \forall t \in [0,1] \\ & x(t) \geq 0 & \end{array} \tag{P}$$

$\mathcal{P}_t = \{x \in \mathbb{R}^n,\ A(t)x \leq b\}$ the feasible region of (P_t).

Hypothesis 1.1 *The optimal value of P_t is finite for all $t \in [0,1]$.*

Let $x(t)$ be an optimal solution to (P_t).

2 Behavior of the solution

Theorem 2.1 *There exist $N > 0$, and $0 = t_1 < \ldots < t_N = 1$ such that, for every $t \in (t_i, t_{i+1})$, there exist $B \in \binom{[n]}{r}$*

- *$A_B(t)$ is invertible*
- *$x(t) = A_B(t)^{-1}b(t)$*

Lemma 2.2 *For all $B \in \binom{[n]}{r}$ $A_B(t)$ is either never invertible or always invertible except for finitely many $t \in [0,1]$.*

Lemma 2.3 *For all $B \in \binom{[n]}{r}$, if $A_B(t)$ is invertible for some t, then the point*

$$v_B(t) = A_B(t)^{-1}b(t)$$

changes feasibility finitely many times. When $v_B(t)$ is feasible, it is a vertex of the feasible region of P_t

Let $\mathcal{B} = \{B, \exists t\ A_B(t) \text{ is invertible}\}$.

Lemma 2.4 *We can always choose $x(t)$ to be of the form 2.3. Call $B(t) := B$ the optimal basis.*

To summarize, there is a (finite) partition of $[0,1]$ into intervals I_i such that in the interior of any I_i:

- For all $B \in \mathcal{B}$, $A_B(t)$ is invertible and $v_B(t)$ is either feasible or not. Let $\mathcal{F}_i = \{B \in \mathcal{B},\ v_B(t) \text{ is feasible on } \mathring{I}\}$.
- $x(t) = A_B(t)^{-1}b(t)$

Lemma 2.5 *We can choose $B(t)$ so that it changes finitely many times inside each I_i.*

Proof 2.6 *$B(t)$ change only if there exist $B, B' \in \mathcal{F}_i$ such that $\langle c(t), A_B^{-1}(t)b(t)\rangle = \langle c(t), A_{B'}^{-1}(t)b(t)\rangle$. $t \to \langle c(t), (A_B^{-1}(t) - A_{B'}^{-1}(t))b(t)\rangle$ is a rational fraction. If it is no identically zero, then it hits zero finitely many times.*

Remark 2.7 *Replace linear objective by convex objective.*

3 Approximation of the solution by a continuous function

Hypothesis 3.1 *P_t admits one feasible continuous solution f_0. e.g there exist a continuous function $f_0 : [0,1] \to \mathbb{R}^n$ such that $A(t)f_0(t) \leq b(t)$, $\forall t \in [0,1]$*

Theorem 3.2 *For every $\varepsilon > 0$, there exist a continuous function $f : [0,1] \to \mathbb{R}^n$ such that:*

- *$f(t)$ is feasible of all t, e.g $A(t)f(t) \leq b(t)$, $\forall t \in [0,1]$*
- *$\int_0^1 \langle c(t), x(t)\rangle - \int_0^1 \langle c(t), f(t)\rangle \leq \varepsilon$.*

Proof 3.3 *Theorem 2.1 proves the existence of a partition $[0,1] = \cup_1^n [t_i, t_{i+1})$ such that $x(t)$ is a continuous (in fact, a rational function).*

Define $I_i^\alpha = (t_i + \alpha, t_i - \alpha)$ for some $\alpha > 0$ that we are going to fix later on.
Let f^α be the function that:

- *is equal to $x(t)$ on every I_i^α.*
- *is equal to f_0 on all the t_i.*
- *interpolates linearly between $x(t)$ and $f_0(t)$ on $[t_i - \alpha, t_i + \alpha]$*

As $\alpha \to 0$, $f^\alpha(t) \to x(t)$ almost surely. Given that $|f^\alpha(t)| \leq |x(t)| + |f_0(t)|$, the Dominated convergence theorem gives $f^\alpha(t) \to_{L_1} x(t)$

4 Approximation of the solution by a polynomial solution

Hypothesis 4.1 *P_t admits one strictly feasible continuous solution f_0. e.g there exist $\beta > 0$ and a continuous function $f_0 : [0, 1] \to \mathbb{R}^n$ feasible for the following program*

$$\begin{array}{ll} \text{maximize} & \int_0^1 \langle c(t), x(t)\rangle \mathrm{d}t \\ \text{subject to} & A(t)x(t) \leq b(t) - \beta \quad \forall t \in [0, 1] \end{array} \tag{$P(\beta)$}$$

Idea 4.2 *We start with a continuous solution f that is near optimal to $P(\beta)$, we approximate it uniformally by a polynomial $p(t)$, if $p(t)$ is close enough to f, then p is feasible to P and near optimal.*

Lemma 4.3 *As $\beta \to 0$, the optimal value to $P(\beta)$ convergence to the optimal value of P.*

5 Finding the best polynomial solution

If we add the constraint that $x(t)$ is a polynomial of degree d in P, then the objective function $\int_0^1 \langle c(t), x(t)\rangle$ is a linear function in the coefficients of $x(t)$, and the constraint $A(t)x(t) \leq b(t)$ is equivalent to the polynomial $b(t) - A(t)x(t)$ being sum of square.

6 From LP to SDP

$$\begin{array}{ll} \text{maximize} & \int_0^1 \langle c(t), x(t)\rangle \mathrm{d}t \\ \text{subject to} & Q_0 + \sum x_i(t)Q_i(t) \succeq 0 \quad \forall t \in [0, 1] \\ & x(t) \geq 0 \end{array} \tag{SDP}$$

$$\begin{array}{ll} \text{maximize} & \int_0^1 \langle c(t), x(t)\rangle \mathrm{d}t \\ \text{subject to} & Q_0 + \sum x_i(t)Q_i(t) = \sum_{j=1}^{l} \alpha_j(t)B_j \quad \forall t \in [0, 1] \\ & \begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} \geq 0 \end{array} \tag{LP_l}$$

Theorem 6.1 *$LP_l \to_l SDP$.*