Problem set 3, ORF550

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1 Problem 3.13a

We want to show that:

$$Ent(Z) = \inf_{t>0} E[Z \log Z - Z \log t - Z + t]$$

Which is the same as:

$$-E[Z] \log E[Z] = \inf_{t>0} -E[Z] \log t - E[Z] + t$$

Or

$$\inf_{t>0} \frac{t}{E[Z]} - \log \frac{t}{E[Z]} = 1$$

Or

$$\inf_{u>0} u - \log u = 1$$

Which is true, because $f : u \to u - \log u$ is convex ($f''(u) = \frac{1}{u^2}$), and its first derivative ($f'(u) = 1 - \frac{1}{u}$) is 0 at 1.

2 Problem 3.20

a.

$$\begin{aligned} Ent_\nu X &= \inf_{t>0} E_\nu[X \log X - X \log t - X + t] \\ &= \inf_{t>0} E_\mu[(X \log X - X \log t - X + t)\frac{d\nu}{d\mu}] \\ &\le ?||\frac{d\nu}{d\mu}||_\infty \inf_{t>0} E_\mu[X \log X - X \log t - X + t] \\ &\le ||\frac{d\nu}{d\mu}||_\infty Ent_\mu X \end{aligned}$$

b.

$$\begin{aligned} \nu(\Gamma(\log f, f)) &= \mu(\frac{\Gamma(\log f, f)}{d\nu/d\mu}) \\ &\ge \frac{1}{\varepsilon}\mu(\Gamma(\log f, f)) \end{aligned}$$

c.

$\nu(dx) = \frac{1}{Z'} e^{-V(x)+x^2} \mu(dx)$, where $\mu \sim N(0, \sqrt{2})$

$$\frac{d\nu}{d\mu} \in [\frac{1}{Z'}e^{-b}, \frac{1}{Z'}e^{-a}]$$

So:

$$Ent_\nu f^2 \leq \frac{c}{Z'} e^{b-a} \nu(\Gamma(\log f^2, f^2)) = \frac{c}{Z'} e^{b-a} \nu((2f'/f).(2ff')) = \frac{4c}{Z'} e^{b-a} \nu(|f'|^2)$$

d.

$$\begin{aligned}
Var_\nu(f) &= \inf_{c\in\mathbb{R}} E_\nu[(f-c)^2] \\
&= \inf_{c\in\mathbb{R}} E_\mu[(f-c)^2 \frac{d\nu}{d\mu}] \\
&\leq \inf_{c\in\mathbb{R}} E_\mu[(f-c)^2]||\frac{d\nu}{d\mu}||_\infty \\
&\leq Var_\mu f||\frac{d\nu}{d\mu}||_\infty \\
&\leq c\delta\mu(\Gamma(f,f)) \\
&\leq c\delta\nu(\frac{\Gamma(f,f)}{d\nu/d\mu}) \\
&\leq \frac{c\delta}{\varepsilon}\nu(\Gamma(f,f))
\end{aligned}$$

3 Problem 4.2

Suppose $med(f)$ attained at x_0.

$A = \{f \leq med(f)\}$, $x_0 \in A$

$\mu(A) = \frac{1}{2}$

$|f - med(f)| = |f(x) - f(x_0)| \leq d(x, x_0)$, so $A_t \subseteq \{f - med(f) \geq t\}$

$= \{x : d(x, A)\} \subset \{x : f(x) - med(f) \geq t\}$

4 Problem 4.5

5 Problem 4.7

6 Problem 4.8