ORF523 - Problem Set 1

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Problem 1

1. (a) Let λ be an eigen value of A^TA corresponding to an eigen vector $u \neq 0$, then $0 \leq ||Au||^2 = u^TA^TAu = \lambda||u||^2$, therefore $\lambda > 0$.

   (b) Let λ be an eigen value of A corresponding to an eigen vector u, the $A^TAu = A(Au) = \lambda^2 u$, so λ^2 is an eigen value of A^TA. Since A has n eigen values (accounting for multplicity), the eigen values of A^TA are exactly the squares of the eigen values of A, and therefore the singular values of A are the absolute values of the eigen values of A.

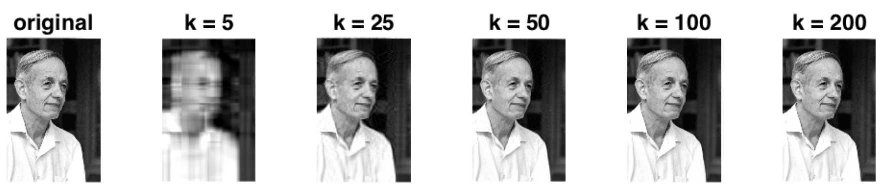
   (c)
$$u_i^T u_j = u_i^T \frac{A^TAu_j}{\lambda_j} = \frac{u_i^TA^TA}{\lambda_j} u_j$$
   Since $\lambda_i \neq \lambda_j$, $u_i^Tu_j = 0$

2. - The L_2 norm for vectors is unitarly invariant: Let O unitary matrix and X a vector, then $||OX||^2 = X^TO^TOX = X^TX = ||X||^2$.
   - Since O is invertible, the application $S \to S, X \to OX$, where S is the L_2 sphere, is a bijection. So $\{x, ||x||_2 = 1\} = \{Ox, ||x||_2 = 1\}$
   - The L_2 norm for matrices is unitarly invariant. If A a matrix, then $||AO|| = \max_{||x||_2=1} ||AOx|| = \max_{||Ox||_2=1} ||AOx|| = \max_{||y||_2=1} ||Ay|| = ||A||$ and $||OA|| = \max_{||x||_2=1} ||OAx|| = \max_{||x||_2=1} ||Ax|| = ||A||$.
   - Let B be a matrix of rank at most k. $||A - B|| = ||U(\Sigma - U^TBV)V^T|| = ||\Sigma - U^TBV||$ Let's do the change of variable $C = U^TBV$, and since U and V are invertible, that $rank(B) = rank(C)$.
     $rank(B) \leq k \Rightarrow null(C) \geq n - k$, therefore $\{e_1, \ldots, e_{k+1}\} \cap null(C) \neq \{0\}$ (otherwise the dimension of the sum will exceed n). Let $u = \sum_{i \leq k+1} \alpha_i e_i$ in that intersection, and assume wihtout loss of generality that $|u| = \sum_{i \leq k+1} \alpha_i^2 = 1$.
     $|(\Sigma - C)u|^2 = |\sum_{i \leq k+1} \alpha_i e_i|^2 = \sum_{i \leq k+1} \alpha_i^2 \sigma_i \geq \sigma_{k+1}$, so $|\sigma - C|_2 \geq \sigma_{k+1}$.
     $|\Sigma - \Sigma^{(k)}| = |diag(0, \ldots, \sigma_{k+1}, \ldots)| = \sigma_{k+1}$
     and since $rank(\Sigma^{(k)}) = k$, it is the argmin. Doing the inverse the change of variable $B = UCV^T = U\Sigma^{(k)}V^T$.
     But $U\Sigma^{(k)}V^T = U^{(k)}\Sigma^{(k)}{V^{(k)}}^T$. To prove that we consider the column vectors of V: $(v_1, ..., v_n)$, and we have that:
$$U^{(k)}\Sigma^{(k)}{V^{(k)}}^T v_i = \left\{ \begin{array}{cl} U^{(k)}(\sigma_i e_i) = \sigma_i u_i & \text{if } i \leq k \\ 0 = U\Sigma^{(k)}e_i & \text{if } i > k \end{array} \right\} = U\Sigma^{(k)}Vv_i$$
     Which proves the result.

3. We report here the result with precision 10^{-2}:

k	$\|\|A - A_{(k)}\|\|_F$
25	50.24
50	31.25
100	18.07
200	9.96

4. To store $A^{(k)}$ we need only to store $\Sigma^{(k)}$ (k parameters), $U^{(k)}, V^{(k)}$ (each of them has only at most k column not set to zero, so they take $nk + mk$), while A takes nm numbers to store. In conclusion, we save $(nm - k(n + m + 1))$.



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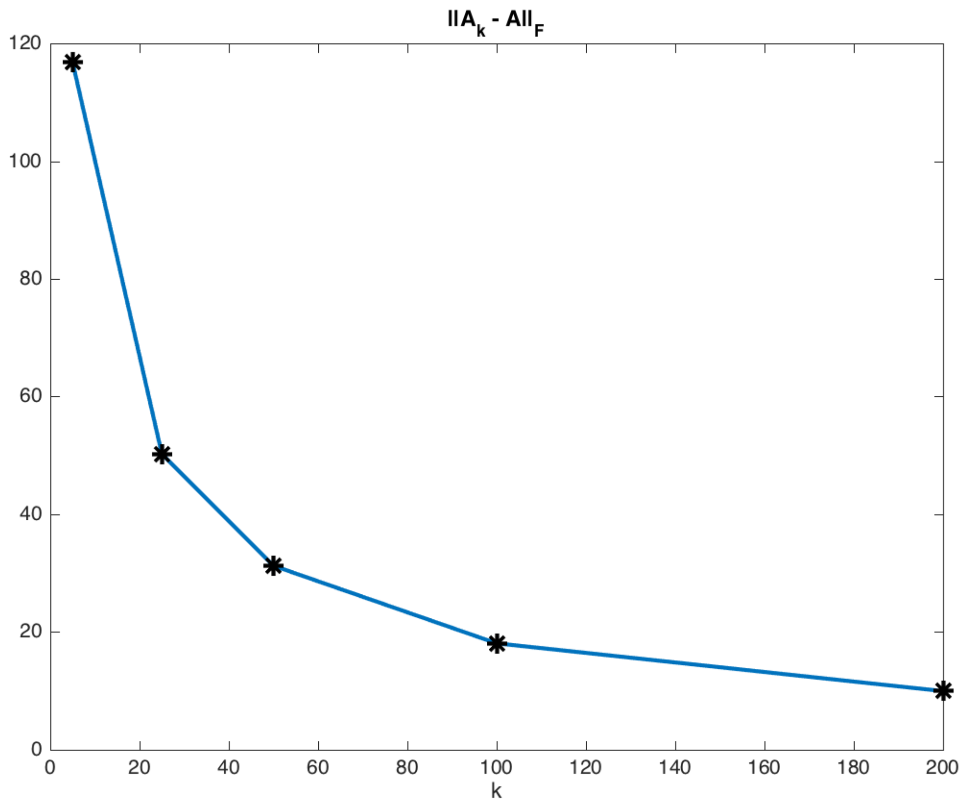

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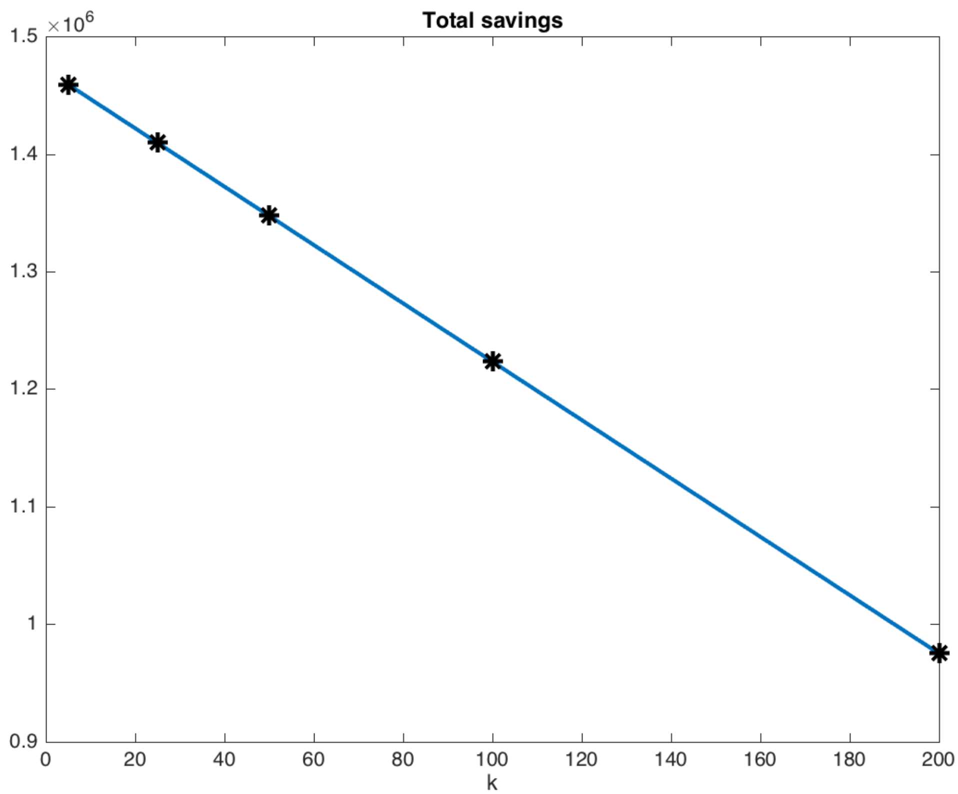

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  for $k = 200$, the total saving is 975100.

5. Comparing the original and the compressed image for $k = 200$. You can't tell the difference.

Code

```
%cd Documents/Princeton/ORF523/hw1/

%% Load Image
A=imread('nash.jpg');
A=im2double (A) ;
A=rgb2gray(A) ;
[m,n] = size(A);

%% SVD decomposition
[U,S,V] = svd(A);

%% arrays initialization
k_range = [5 25 50 100 200];
diff_norm = zeros('like', k_range);
total_savings = zeros('like', k_range);

%% Plot the original image
subplot(1,length(k_range)+1, 1);
imshow(A);
title('original')

for i = 1:length(k_range)
    k = k_range(i);
```

```
    Uk = U; Sk = S; Vk = V;
    Sk(k+1:end, k+1:end) = 0;
    Uk(:, k+1:end) = 0;
    Vk(:, k+1:end) = 0;
    Ak = Uk * Sk * Vk';
    diff_norm(i) = norm(A - Ak,'fro');
    total_savings(i) = prod(size(A)) ... % size of A
    - k*m ...% size of Uk
    - k*n ... % size of Vk
    - k; % size of Sigma

    subplot(1,length(k_range)+1, i+1);
    imshow(Ak);
    title(['k = ' int2str(k)])
end
print('compressed','-dpng')

%% Plot ||A_k - A||_F
figure
plot(k_range, diff_norm, ...
     '-*',...
     'LineWidth',2,...
     'MarkerEdgeColor','k',...
     'MarkerFaceColor',[.49 1 .63],...
     'MarkerSize',10)
title('||A_k - A||_F')
xlabel('k')
print('diffnorm','-dpng')

%% Plot total savings
figure
plot(k_range, total_savings,...
     '-*',...
     'LineWidth',2,...
     'MarkerEdgeColor','k',...
     'MarkerFaceColor',[.49 1 .63],...
     'MarkerSize',10)

title('Total savings')
xlabel('k')
print('totalsavings','-dpng')

%% Plot original and compressed for k = 200
figure
subplot(1,2, 1);
imshow(A);
title('original')
subplot(1,2, 2);
imshow(Ak);
```

```
title('k = 200')
print('compare','-dpng')
```

Problem 2

1. Let $u = x_1 + x_2$, it is easy to verify that $f(x_1, x_2) = \underbrace{\frac{u^2}{2} - 2u}_{g(u)} + \underbrace{-2x_2^2 + 3x_2 + \frac{1}{3}x_2^3}_{h(x_2)} = g(u) + h(x_2)$

   Since this transformation is a diffeomorphisme, it preserves neighbourhoods, and therefore it preserves local minimizers/maximizers.

   A point (u, x_2) is a local maximizer (resp. minimizer) of $g(u) + h(x_2)$ if and only if u is a local maximizer (resp. minimizer) of g and x_2 is local maximizer (resp. minimizer) of h.

   - $g(u) = \frac{1}{2}(u-2)^2 - 2$ has one local(and global) minimum $u = 2$
   - $h'(x_2) = x_2^2 - 4x_2 + 3 = (x_2 - 1)(x_2 - 3)$, $h''(x_2) = 2x_2 - 4$

   The candidates are $x_2 = 1$ and $x_2 = 3$. Since $h''(1) = -2 < 0, h''(3) = 2 > 0$, 1 is local maximum and 3 is a local minimum. f canot have a local maximum because g doesn't have one.

   In conclusion, $(u, x_2) = (2, 3)$ is the only local optimizer (it is a local min) of $g(u) + h(x_2)$, and $(x_1 = -5, x_2 = 3)$ is the only local optimizer (minimum) of f.

2. By Taylor expansion, since f is a polynomial of order 2: $f(x) - f(\bar{x}) = \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(x - \bar{x})(x - \bar{x})$

   a) ($\Rightarrow$) Let $\bar{x}$ be a local min, then $\nabla f(\bar{x}) = 0$. If $\nabla^2 f$ is not positive semi-definite, let $d \in \mathbb{R}^n d^T \nabla f(\bar{x}) =: -\lambda < 0$, therefore $f(x + \alpha d) - f(x) = -\alpha^2 \lambda < 0$, and for any neighbourhood of $\bar{x}$, there exist α such that $\bar{x} + \alpha d$ is in that neighbourhood, and we have a contradiction.

   ($\Leftarrow$) In this case: $f(x) - f(\bar{x}) = \frac{1}{2}(x - \bar{x})\nabla^2 f(x - \bar{x}) \geq 0$, which proves that $\bar{x}$ is local min.

   b) ($\Rightarrow$) In this case: $f(x) - f(\bar{x}) = \frac{1}{2}(x - \bar{x})\nabla^2 f(x - \bar{x}) > 0$, which proves that $\bar{x}$ is strict local min.

   ($\Leftarrow$) In this case: $f(x) - f(\bar{x}) = \frac{1}{2}(x - \bar{x})\nabla^2 f(x - \bar{x}) > 0$, which proves that $\bar{x}$ is strict local min.

   Counterexamples: $f : \mathbb{R} \to \mathbb{R}, x \to x^3$, $f'(0) = 0, f''(0) = 0$, but 0 is not a local min. $f : \mathbb{R} \to \mathbb{R}, x \to x^4$, $f'(0) = 0, f''(0) = 0$, but 0 is a strict local min.

3. using the chain rule: $g'(\alpha) = \frac{d}{||d||}\nabla f(x + \alpha \frac{d}{||d||})$, $g'(0) = \frac{d}{||d||}\nabla f(x)$.

   Using Cauchy-Shwarz, $|\frac{d}{||d||}\nabla f(x)| \leq ||\nabla f(x)||$, so $\frac{-\nabla f(x)}{||\nabla f(x)||}\nabla f(x) \leq \frac{d}{||d||}\nabla f(x) \leq \frac{\nabla f(x)}{||\nabla f(x)||}\nabla f(x)$ Which proves the result.

Problem 3

$Q > 0$, so it admist an eigenvalue decomposition and all the eigen values are positive, let $Q = U\Sigma U^T$ be that decomposition, with $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$, and Let $\sqrt{\Sigma} = diag(\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_n})$, then $Q = \underbrace{U\sqrt{\Sigma}U^T}_{\sqrt{Q}} U\sqrt{\Sigma}U^T = \sqrt{Q}^2$

$f(x) = \sqrt{x^T \sqrt{Q} \sqrt{Q} x} = \sqrt{||\sqrt{Q}x||^2} = ||\sqrt{Q}x||$

1. f is a norm because:

   - $$f(\lambda x) = \sqrt{(\lambda x)^T Q (\lambda x)} = \sqrt{\lambda^2} f(x) = |\lambda| f(x)$$

   - $$f(x+y) = ||\sqrt{Q}x + \sqrt{Q}y|| \leq ||\sqrt{Q}x|| + ||\sqrt{Q}y|| \leq f(x) + f(y)$$

   - $$f(x) = 0 \iff \sqrt{Q}x = 0 \iff x = 0$$

   By Riesz Representation theorem, we indentify a vector x with the linear form $y \to x^T y$. Let g be the dual norm of f, then

   $$\begin{aligned} g(x) &= \sup_{y \neq 0} \frac{x^T y}{f(y)} \\ &= \sup_{u \neq 0} \frac{x^T \sqrt{Q}^{-1} u}{||u||} && (y \to u = \sqrt{Q}y \text{ is a bijection}) \\ &= \sup_{u \neq 0} x^T \sqrt{Q}^{-1} \frac{u}{||u||} \\ &= x^T \sqrt{Q}^{-1} \frac{(\sqrt{Q}^{-1} x)}{||\sqrt{Q}^{-1} x||} && \text{(Cauchy shwarz, like in Problem 2)} \\ &= \frac{x^T Q^{-1} x}{\sqrt{x^T Q^{-1} x}} \\ &= \sqrt{x^T Q x^{-1}} \end{aligned}$$

3. $A^T A \geq 0$, Let $U \Sigma U^T$ be an eigen value decomposition where $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$, and $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$.

   $$\begin{aligned} ||A||_2 &= \sup_{||x||=1} x^T (A^T A) x \\ &= \sup_{||x||=1} ||\sqrt{A^T A} x|| \\ &= ||\sqrt{A^T A}||_2 \\ &= ||\Sigma||_2 \\ &= \sup_{||x||=1} x^T (\Sigma) x \\ &= \sup_{||x||=1} \sum_i x_i^2 \sigma_i \leq \sum_i x_i^2 \sigma_1 \\ &= \sigma_1 \\ &= ||\Sigma e_1|| \end{aligned}$$

   So $||A||_2 = \sigma_1 = \sqrt{\lambda_{\max}(A^T A)}$