Main motivation: $max(X_t, Y_t)$. We want to know if it is semimartingale.

Claim: This Q is related to the following Q: Take a brownian motion B. We know that $\int_0^t 1_{B(s)=0} ds = 0$ for all t with probability 1. What is the right way of measuring the time spent at 0 by a BM.

Relation between the two questions: Suppose B_1, B_2 are independent standard BMs and we want to stdy $max(B_1, B_2)$

Note: $max(x, y) = \frac{x+y+|x-y|}{2}$, so we only need to analyse $|B_1 - B_2|$. $f(x) = |x|$, $f''(x) = 2\delta_0(x)$ Try to apply Ito: $|B(t)| = \int_0^t sgn(B_s) dB_s + \int_0^t \delta_0(B_s) ds$

Three natural ways of measuring time spent at 0 by BM:

- Take $\varepsilon > 0$, consider $\int_0^t 1_{|B_s|<\varepsilon} ds$, take the limit $\varepsilon \downarrow 0$ in some way.
- Define time spent at 0 as: $2(|B(t)| - \int_0^t sign(B_s) dB_s)$
- Recall that a BM is a limit of random walks, $B(t) = \lim_{\text{distribution}} \frac{S_{\lfloor Kt \rfloor}}{\sqrt{K}}$ Take $\lim_K \frac{1}{\sqrt{K}} \#\{i, S_i = 0, i \leq \lfloor Kt \rfloor\}$

Luckily $1 \iff 2 \iff 3$

Definition 1 (Local time)**.** *For a BM B and a point $a \in R$, call the almost sure limit $\lim_{\epsilon 0} \frac{1}{2\epsilon} \int_0^t 1_{|B_s - a| < \epsilon} ds$ the local time of B at a and write L_t^a. With probability 1 $(a, t) \to L_t^a$ is continuous.(Trotter '57)*

Remarks:

1. Need to justify the a.s limit and continuity.
2. $t \to L_t^a$ is non decreasing wp 1.
3. $B(R) \ni A \to \int_0^t 1_{B_s \in A} ds$ has $a \to L_t^a$ as its density.