ORF526 - Problem Set 6

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Question 1

The pre image of open sets by a continous function is open. Let's call $\mathcal{O}$ the set of open sets.

We know that $\sigma(\mathcal{O}) = B(R^k)$.

Let's consider $B = \{A \in B(R^d), f^{-1}(A) \in B(R^k)\}$ we know that $\mathcal{O} \subseteq B$ because of the definition of continuity, so (using the fact that B is sigma algebra from a question from previous problem set) $B(R^k) = \sigma(\mathcal{O}) \subset B$, and therefore $B = B(R^k)$. Therefore f is measurable.

Question 2

a. f is positive, and integrates to one. Indeed, let I denote $\int_R e^{-\frac{x^2}{2}} dx = \int e^{-(\frac{x}{\sqrt{2}})^2} = \sqrt{2} \int_R e^{-u^2} du$.

$$\begin{aligned}
I^2 &= 8 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\, dx && \text{By Fubuni-Tonelli} && (1)\\
&= 8 \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\, dy \right) dx && && (2)\\
&= 8 \int_0^\infty \left(\int_0^\infty e^{-x^2(1+s^2)} x\, ds \right) dx && s \to \frac{y}{x} \text{ being a diffeomorphism} && (3)\\
&= 8 \int_0^\infty \left(\int_0^\infty e^{-x^2(1+s^2)} x\, dx \right) ds && && (4)\\
&= 8 \int_0^\infty \left[\frac{1}{-2(1+s^2)} e^{-x^2(1+s^2)} \right]_{x=0}^{x=\infty} ds && && (5)\\
&= 8 \left(\tfrac{1}{2} \int_0^\infty \frac{ds}{1+s^2} \right) && && (6)\\
&= 4 \Big[\arctan s \Big]_0^\infty && && (7)\\
&= 2\pi && && (8)
\end{aligned}$$

Since $I \geq 0$, $\int_R f = \frac{1}{\sqrt{2\pi}} I = 1$

b. $\frac{x^n f(x)}{x^2} \to_\infty 0$, so Z^n is integrable for all $n \in \mathbb{N}$

$E[Z^i] = 0$ by symetry of $x \to x^i f(x)$ for $i = 1, 3$.

$t \to x^2$ is a diffeomorphisme from R^{*+} to itself, $dx = \frac{dt}{2\sqrt{t}}$.

$$E[Z^2] = 2 \int_0^\infty x^2 f(x) dx = 2 \int t e^{-\frac{t}{2}} dt = -[e^{-\frac{t}{2}}]_0^\infty = 1$$

$$E[Z^4] = \int_R x^4 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = -\frac{1}{\sqrt{2\pi}} \left([x^3 e^{-\frac{x^2}{2}}]_{-\infty}^\infty + 3 \int_R x^2 e^{-\frac{x^2}{2}} \right) = 3E[Z^2] = 3$$

c.
- $E[|Z|] = \frac{2}{\sqrt{2\pi}} \int_0 z e^{-\frac{z^2}{2}} = [-\frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}]_0^\infty = \sqrt{\frac{2}{\pi}}$
- $E[|Z|^2] = Var(Z) = 1$
- $E[|Z|^3] = \frac{2}{\sqrt{2\pi}} \int_0 z^2 z e^{-\frac{z^2}{2}} = -\frac{2}{\sqrt{2\pi}} \int_0 z^2 (e^{-\frac{z^2}{2}})' = [-\frac{2}{\sqrt{2\pi}} z^2 e^{-\frac{z^2}{2}}]_0^\infty + \frac{4}{\sqrt{2\pi}} \int_0 z(e^{-\frac{z^2}{2}}) = \frac{4}{\sqrt{2\pi}}$
- $E[|Z|^4] = E[Z^4] = 3$

d. $E[\exp(aZ)] = \int e^{az} e^{-\frac{z^2}{2}} = \int e^{-\frac{(z+a)^2}{2}} e^{\frac{a^2}{2}} = e^{\frac{a^2}{2}}$

e. $Z = (Z_1, ...Z_n)$, by independence: $\Phi_Z(u) = \prod_i \Phi_{Z_i}(u_i) = (2\pi)^{-\frac{n}{2}} e^{-\frac{||u||^2}{2}}$
By linearity of E and bilinearity of Cov for centered rv:

$$E[X] = \mu + AE[Z] = \mu$$
$$Cov(X) = Cov(X - \mu) = Cov(AZ) = ACov(Z)A^T = AA^T$$
$$\Phi_X(u) = E[e^{iu^T X}] = E[e^{iu^T \mu} e^{iu^T AZ}] = e^{iu^T \mu} \Phi_Z(u^T A) = (2\pi)^{-\frac{n}{2}} e^{iu^T \mu - \frac{||u^T A||^2}{2}}$$

Question 3

let $X \sim \mathcal{N}(0, 1)$, and $\varepsilon \sim \mathcal{B}(-1, 1, \frac{1}{2})$ be two independant rv. And Let $Y = \varepsilon X$

By symmetry of the distribution of X:

$$F_Y(y) = P(Y \leq y) = P(\varepsilon X \leq y) = E[P(\varepsilon X \leq y|\varepsilon)] = \frac{1}{2}P(X \leq y) + \frac{1}{2}P(-X \leq y) = P(X \leq x)$$

so $Y \sim \mathcal{N}(0, 1)$.

(X, Y) is not normal because $(1, 1)(X, Y)^T = X+Y$ is not normal because $P(X+Y = 0) = P(\varepsilon = -1) = \frac{1}{2}$.

Question 4

a) Let $h : (x, y) \to (\sqrt{x^2 + y^2}, \phi(x, y))$, $h^{-1} : (r, \theta) \to (r\cos(\theta), r\sin(\theta)))$.
h is a diffeomorphisme from from $R^2 \setminus \{(0, 0)\}$ to $R^{+*} \times [0, 2\pi[$

$$\det(Dh^{-1}) = r$$

Since $\cos(\theta)^2 + \sin(\theta)^2 = 1$ we have that:

$$f_{X,Y}(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi} e^{-\frac{1}{2}r^2}$$

For g a continous bounded function, we have that:

$$\begin{aligned}
E[g(\sqrt{X^2 + Y^2}, \phi(X, Y))] &= \int g(\sqrt{x^2 + y^2}, \phi(x, y)) f_{X,Y}(x, y) dxdy \\
&= \int g(r, \theta) f_{X,Y}(r\cos(\theta), r\sin(\theta)) \frac{dr}{2\pi} d\theta \\
&= \int g(r, \theta) r e^{-\frac{1}{2}r^2} dr \frac{d\theta}{2\pi} \\
&= E[g(R, \Theta)]
\end{aligned}$$

Where (R, Θ) has a density $f(r, \theta) = re^{-\frac{1}{2}r^2} \frac{1}{2\pi} 1_{r>0} 1_{\theta \in [0, 2\pi)}$.

b) $X^2 + Y^2 \sim R^2$. Let g be bounded continuous. Using the change of variable $s = r^2$

$$E[g(X^2+Y^2)] = E[g(R^2)] = \int_{R^+} g(r^2) r e^{-\frac{1}{2}r^2} dr = \int_{R^+} g(s) \frac{1}{2} e^{-\frac{1}{2}s} ds$$

So $X^2 + Y^2 \sim \mathcal{E}xp(\frac{1}{2})$

c) $h(\sqrt{-2\log U} cos(2\pi V), \sqrt{-2\log U} sin(2\pi V)) = (-2\log U, 2\pi V) =: (A, B)$
$P(-2\log U < x) = P(U > e^{-\frac{x}{2}}) = (1 - e^{-\frac{x}{2}}) 1_{x>0}$
Since U, V are independent, (A, B) has the same distribution as (R, Θ).
Using a) we get that $(\sqrt{-2\log U} cos(2\pi V), \sqrt{-2\log U} sin(2\pi V))$ has the same distribution as X, Y because of the following: For g continuous bounded:

$$E[g(U,V)] = E[goh^{-1}(A,B)] = E[goh^{-1}(R,\Theta)] = E[g(X,Y)]$$

Question 5

1. For next assignment

2.

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, C^{-1} = \frac{1}{|C|} \begin{pmatrix} C_4 & -C_2 \\ -C_3 & C_1 \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let's call $u = y - \mu_x$

$$\begin{aligned}
N(y) &= \int_R f(x,y) dx && (9)\\
&= \int_R f(x + \mu_x, y) dx && (10)\\
&= \frac{1}{2\pi\sqrt{|C|}} \int \exp(-\frac{1}{2}(x,u) C^{-1} (x,u)^T) dx && (11)\\
&= \frac{1}{2\pi\sqrt{|C|}} \int \exp(-\frac{1}{2}[ax^2 + (b+c)xu + du^2) dx && (12)\\
&= \frac{1}{2\pi\sqrt{|C|}} \int \exp(-\frac{a}{2}[x^2 + (1 + \frac{b+c}{a})xu]) dx \exp(-\frac{1}{2}du^2) && (13)\\
&= \frac{1}{2\pi\sqrt{|C|}} \int \exp\left(-\frac{a}{2}[x + (1 + \frac{b+c}{2(a)})u]^2\right) dx \exp(-\frac{1}{2}du^2 + \frac{(b+c)^2}{8a}u^2) && (14)\\
&= \frac{1}{2\pi\sqrt{|C|}} \int \exp\left(-\frac{a}{2}x^2\right) dx \exp((-\frac{1}{2}d + \frac{(b+c)^2}{8a})u^2) && (15)\\
&= \frac{1}{\sqrt{2\pi|C|a}} \exp((-\frac{1}{2}d + \frac{(b+c)^2}{8a})u^2) && (16)\\
&= \frac{1}{\sqrt{2\pi C_4}} \exp((-\frac{C_1}{2|C|} + \frac{(C_2+C_3)^2}{8C_4|C|})u^2) && (17)\\
&= \frac{1}{\sqrt{2\pi C_4}} \exp(-\frac{1}{2|C|}(C_1 - \frac{(C_2+C_3)^2}{4C_4})(y - \mu_y)^2) && (18)
\end{aligned}$$

So

$$f_y(x) = \sqrt{\frac{C_4}{2\pi|C|}} \exp(-\frac{1}{2}((x,y)-\mu)C^{-1}((x,y)-\mu)^T + \frac{1}{2|C|}(C_1 - \frac{(C_2+C_3)^2}{4C_4})(y-\mu_y)^2)$$

Question 6

Since X is bounded. Let a be such that $|X| < a$. $c[X > x]$ is equal to 1 for x small enough and to 0 for x large enough. The integral is then well defined and equal to $\int_{-a}^{0} c[X > x] - c[\Omega] + \int_0^a c[X > x]$

1. Let $\Omega' := \{X \geq Y\}$.

   We know that $P(\Omega') = 1$ so for every measurable set A, $P(A \cup \Omega') = P(A)$.

   $c[X > x] \geq c[Y > x]$ by monotonicity of c because $\{Y > x\} \cap \Omega' \subseteq \{X > x\}$ and $P(Y > x) = P(\{Y > x\} \cap \Omega')$. And this true because:

   $P(Y > x) = P(\{Y > x\} \cap \Omega') + P(\{Y > x\} \cap \Omega'^c)$ and $P(\{Y > x\} \cap \Omega'^c) \leq P(\Omega'^c) = 0$

2. When $a = 0$ the result is trivial. When $a \neq 0$:

$$\int aXdc = \int^0 c[X > \frac{x}{a}]dx - c[\Omega] + \int_0 c[X > \frac{x}{a}] \tag{19}$$

$$= \int^0 c[X > u] - c[\Omega](adu) + \int_0 c[X > u]adu \qquad u = \frac{x}{a} \tag{20}$$

$$= a \int Xdc \tag{21}$$

3.

$$\begin{aligned}
\int (a + X)dc &= \int^0 c[X > x - a] - c[\Omega]dx + \int_0 c[X > x - a] \\
&= \int^{-a} (c[X > u] - c[\Omega])du + \int_{-a} c[X > u]du \ (u = x - a) \\
&= \int^0 (c[X > u] - c[\Omega])du - \int_{-a}^0 (c[X > u] - c[\Omega])du + \int_{-a}^0 c[X > u]du + \int_0 c[X > u]du \\
&= \int Xdc + \int_{-a}^0 c[\Omega]du \\
&= \int Xdc + ac[\Omega]
\end{aligned}$$