ORF524 - Problem Set 4

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Problem 1

The KKT conditions we will need:

- $\exists \lambda_j, \geq 0, j = 1..n$ s.t $\nabla f(x^*) = -\sum_j \lambda_j \nabla g_j(x^*)$
- $\lambda_j g_j(x^*) = 0$ for $j = 1..n$

Let x be a feasible solution to the second optimization problem. We have that

$$\begin{aligned}
\nabla f(x^*)^T(x - x^*) &= -\sum_j \lambda_j \nabla g_j(x^*)^T(x - x^*) \\
&\geq -\sum_j \lambda_j g_j(x^*) && \text{by feasibility of } x \\
&\geq 0 && \text{By complementarity condition}
\end{aligned}$$

As a result, $f(x^*) + f(x^*)^T(x^* - x^*) \leq f(x^*) + f(x^*)^T(x - x^*)$, and since x^* trivially verifies the feasibility conditions of the second problem, x^* is a global optimal.

Problem 2

1. First order conditions:

$$0 = \nabla f = \begin{pmatrix} 4x + y - 6 \\ x + 2y + z - 7 \\ y + 2z - 8 \end{pmatrix}$$

Or

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$$

Which can be resolved:

$$X^* = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 6 \\ 6 \\ 17 \end{pmatrix}$$

2.

$$\nabla^2 f(x, y, z) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is symetric.

Let $X = (x, y, z)^T \in R^3$, $X^T \nabla^2 f X = 4x^2 + 2x^2 + 2z^2 + 2xy + 2yz = 2(x^2 + \frac{1}{2}y^2 + (x + \frac{1}{2}y)^2 + (z + \frac{1}{2}y)^2) \geq 0$
So $\nabla^2 f$ is positive semi-definite.

3. We proved that:

$$\nabla^2 f(x,y,z) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} > 0$$

So f is convexe, and every local minimum is also a global minimum. Since f admits only one local minimum X^*, it is the global minimum of f, and $\min f = f(\frac{6}{5}, \frac{6}{5}, \frac{17}{5}) = -30.4$

4.
```
cvx_begin
    variable x
    variable y
    variable z
    minimize(x^2 + 0.5 * y^2 + (x+0.5*y)^2 + (z+0.5*y)^2 -6*x - 7*y - 8*z - 9)
cvx_end
```

Problem 3

1. Let's consider the problem (P):

$$\max_{2\pi(x^2+y)\leq C, x\geq 0, y\geq 0} \pi xy$$

The feasible set is bounded and closed. The objective function is continuous. So it has an optimal solution (x^*, y^*). $x^*y^* \neq 0$ because $x = \sqrt{\frac{C}{4\pi}}, y = \frac{C}{4\pi}$ is a better feasible solution.

Let $h^* = \frac{x^*}{y^*}, r^* = x^*$. (r^*, h^*) is optimal because:

- It is feasible: $2\pi(r^{*2}, +r^*h^*) = 2\pi(x^{*2} + y^*) \leq C$, $r > 0, h > 0$
- If r, h another feasible solution, then $x = r, y = rh$ is feasible for the problem (P), and therefore: $\pi xy \leq \pi x^* y^*$, ie $\pi r^2 h \leq r^{*2} h^*$

2. the objective function $(r,h) \to r^2 + rh$ is not convexe. Indeed its hessian

$$\begin{pmatrix} 2h & 2r \\ 2r & 0 \end{pmatrix}$$

has negative derterminant$(-4r^2)$, so one of the eigen values have different signs, and therefore the objective function is neither convexe nor concave.

3. Let's look for local optimal solutions.

Let (r,h) be an optimal solution. We know that $r \neq 0$ because (ϵ, h) is a better feasible solution for $\epsilon > 0$ small enough so that $2\pi(\epsilon^2 + \epsilon h) < C$

The lagrangian is $\mathcal{L}(r,h,\lambda) = \pi r^2 h + \lambda(C - 2\pi r^2 - 2\pi rh)$

KKT:

- $0 = \frac{\partial}{\partial r}\mathcal{L} = 2\pi(rh - \lambda(2r+h)) \Rightarrow \lambda = \frac{rh}{2r+h}$
- $0 = \frac{\partial}{\partial h}\mathcal{L} = \pi(r^2 - \lambda 2r) \Rightarrow \lambda = \frac{r}{2} \neq 0$
- $\lambda = \frac{r}{2} = \frac{rh}{2r+h} \Rightarrow h = 2r$
- Complementary condition $\lambda \neq 0 \Rightarrow C = 2\pi(r^2 + rh) = 6\pi r^2 \Rightarrow r = \sqrt{\frac{C}{6\pi}}$

Conclusion: $r = \sqrt{\frac{C}{6\pi}}$, $h = \sqrt{\frac{2C}{3\pi}}$, $\pi r^2 h = \frac{C^{\frac{3}{2}}}{3\sqrt{6\pi}}$

Since there is only one local optimum, and the problem admits an global optimum, then the local optimum is also global.

Problem 4

1. $f(x) = -x$, $g(x) = x^2$ are both convexe but $fog(x) = -x^2$ is not convexe.

2. Let $\lambda \in [0,1]$, since g is convexe: $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$

   Since f is non-decreasing: $f(g(\lambda x + (1-\lambda)y)) \leq f(\lambda g(x) + (1-\lambda)g(y))$

   Since f is convexe: $f(\lambda g(x) + (1-\lambda)g(y)) \leq \lambda fog(x) + (1-\lambda)fog(y)$

   as a conclusion
   $$fog(\lambda x + (1-\lambda)y) \leq \lambda fog(x) + (1-\lambda)fog(y)$$

   and fog is convexe

3. Take $f(x) = -e^x$, $g(x) = x$, but $fog(x) = -e^x$ is not convexe.

4. Let's take $f = \frac{1}{1+e^{-x}}$.

   $f'(x) = \frac{e^x}{1+e^{-x}} > 0$

   $(xf)''(x) = \frac{e^x}{(1+e^{-x})^3}(2 + x - e^x(x-2)) \sim_{x+\infty} -xe^{2x}$

   f is increasing and non negative. $(xf)''$ goes to $-\infty$ as x grow larger, so there is an $x > 0$ where it is negative, and as a result xf is not convexe on R^+.