ORF526 - Problem Set 1

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$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, $\mathbb{P}$ is the restriction of the lebesgue measure on Ω. This is a probability space. Let's consider the sequence:

$$X_k = k1_{\{0<x<\frac{1}{k}\}}$$

- $\mathbb{E}[X_k] = 1$
- $X_k \to_{k\infty} 0$ a.s., because for all $x \in (0, 1)$, $X_k(x) = 0$ for all $k > \frac{1}{x}$

Question 1

- $\sup_k ||X_k||_1 = 1 < \infty$
- For any $C > 0$, for any $k > C$, $\mathbb{E}[|X_k|1_{\{X_k>C\}}] = \mathbb{E}[X_k] = 1$. Which means the sequence is not uniformly integrable.

Question 2

the (X_k) satisfy the conditions: $\sup E[|X_n|] = 1$, $X_n \to 0$ and $E[|X_n|] = 1$

Question 3

$\mathbb{E}(\liminf X_k) = \mathbb{E}(\lim X_k) = \mathbb{E}(0) = 0 < 1 = \lim_k \mathbb{E}(X_k) = \liminf \mathbb{E}(X_k)$

Question 4

Let's define: $\mu_1(A_1, ..., A_m) = \prod_i \mathbb{P}(X_i \in A_i)$, $\mu_2(A_1, ..., A_m) = \mathbb{P}(X_1 \in A_1, ..., X_m \in A_m)$

Let's prove by induction that for all $i = 0, ..m$:

$$\forall A_1, ...A_{i-1} \in B(R)\, \mu_1(A_1, ...A_{i-1}, (-\infty, x_i], ...(-\infty, x_n]) = \mu_2(A_1, ...A_{i-1}, (-\infty, x_i], ...(-\infty, x_n])$$

The propertie holds for $i = 0$.

Let's suppose the property holds for $i \leq m$

Let's define:

$$S(A_1, ...A_i) = \{A_{i+1} \in B(R) | \forall x_{i+2}...x_m \in \bar{\mathbb{R}}\, \mu_1(C) = \mu_2(C) \text{where } C = (A_1, ...A_{i+1}, (-\infty, x_{i+2}], ...(-\infty, x_n]\}$$

$S(A_1, ...A_i)$ is a Dynken system because:

- Let $A_{i+1} \in S(A_1, ...A_i)$, and $x_{i+2}...x_m \in \bar{\mathbb{R}}$, then

$$\begin{aligned}
\mu_1(A_1, ...A_{i+1}^C, (-\infty, x_{i+2}], ...(-\infty, x_n]) &= \prod_{k \leq i} P(X_k \in A_k) P(X_{i+1} \in A_{i+1}^c) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&= \prod_{k \leq i} P(X_k \in A_k)(1 - P(X_{i+1} \in A_{i+1})) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&= \prod_{k \leq i} P(X_k \in A_k) P(X_{i+1} \in (-\infty, \infty)) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&- \prod_{k \leq i} P(X_k \in A_k) P(X_{i+1} \in A_{i+1}) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&= P(\{X_k \in A_k, k < i\}, X_{i+1} \in \mathbb{R}, \{X_k \in (-\infty, x_k], k > i+1\}) \\
&- P(\{X_k \in A_k, k < i\}, X_{i+1} \in A_{i+1}, \{X_k \in (-\infty, x_k], k > i+1\}) \\
&= P(\{X_k \in A_k, k < i\}, X_{i+1} \in A_{i+1}^c, \{X_k \in (-\infty, x_k], k > i+1\}) \\
&= \mu_2(A_1, ...A_{i+1}^C, (-\infty, x_{i+2}], ...(-\infty, x_n])
\end{aligned}$$

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- Let $A_{i+1}^j \in S(A_1, ...A_i)$ a sequence of disjoint sets, and $x_{i+2}...x_m \in \bar{\mathbb{R}}$, and let's call $A = \underset{j}{\cup} A_{i+1}^k$ then

$$\begin{aligned}
\mu_1(A_1, ...A, (-\infty, x_{i+2}], ...(-\infty, x_n]) &= \prod_{k \leq i} P(X_k \in A_k) P(X_{i+1} \in A) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&= \prod_{k \leq i} P(X_k \in A_k)(\sum_j P(X_{i+1} \in A_{i+1}^j)) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&\text{(because disjoint)} \\
&= \sum_j \prod_{k \leq i} P(X_k \in A_k) P(X_{i+1} \in A_{i+1}^j) \prod_{k > i+1} P(X_k \in (-\infty, x_k]) \\
&= \sum_j P(\{X_k \in A_k, k < i\}, X_{i+1} \in A_{i+1}^j, \{X_k \in (-\infty, x_k], k > i+1\}) \\
&= \sum_j P(\{X_k \in A_k, k < i\}, X_{i+1} \in A, \{X_k \in (-\infty, x_k], k > i+1\}) \\
&\text{(because disjoint)} \\
&= \mu_2(A_1, ...A, (-\infty, x_{i+2}], ...(-\infty, x_n])
\end{aligned}$$

$S(A_1, ...A_i)$ contains the π-system $\{(-\infty, x_1 \times ... \times (-\infty, x_m] | \, x_1..., x_m \in \bar{\mathbb{R}}\}$ by induction hypothesis, so it contains the sigma algebra generated by them, which is $B(R)$. So $B(R) \subseteq S(A_1, ...A_i)$, $S(A_1, ...A_i) \subseteq B(R)$ trivially and therefore the induction hypothesis holds for $i+1$.

Question 5

1. $i \Rightarrow iii$

   Let $\epsilon > 0$,$A_n = \underset{m \geq n}{\cup} \{\omega, |X_n(\omega) - X(\omega)| > \epsilon\}$ and $A_\infty = \underset{n}{\cap} A_n$. A_n is a decreasing sequence.

   If $\omega \in A_\infty$, for infinitely many $m \in \mathbb{N}$, $|X_n(\omega) - X(\omega)| > \epsilon$. Which means that $\omega \in N$. Therefore $\mathbb{P}(A_\infty) \leq \mathbb{P}(N) = 0$

   By continuty from above:

$$\mathbb{P}(|X_n - X| > \epsilon) \leq \mathbb{P}(A_n) \to \mathbb{P}(A_\infty) = 0$$

2. $ii \Rightarrow iii$ By Markov inequality $\mathbb{P}(|X_n - X| > \epsilon) \leq \mathbb{P}(|X_n - X|^p > \epsilon^p) \leq \frac{E|X_n - X|^p}{\epsilon^{2p}} \to 0$

3. $iii \Rightarrow iv$

Lemma

For two rv X, Y and $a \in \mathbb{R}$:

$$\begin{aligned}
P(Y \leq a) &= P(Y \leq a,\ X \leq a+\varepsilon) + P(Y \leq a,\ X > a+\varepsilon) && (1)\\
&\leq P(X \leq a+\varepsilon) + P(Y - X \leq a - X,\ a - X < -\varepsilon) && (2)\\
&\leq P(X \leq a+\varepsilon) + P(Y - X < -\varepsilon) && (3)\\
&\leq P(X \leq a+\varepsilon) + P(Y - X < -\varepsilon) + P(Y - X > \varepsilon) && (4)\\
&= P(X \leq a+\varepsilon) + P(|Y - X| > \varepsilon) && (5)
\end{aligned}$$

Let F be the cdf of X, and a be a point of continuety of F.

Using the previous lemma:

$$P(X \leq a-\epsilon) - P(|X_n - X| > \epsilon) \leq P(X_n \leq a) \leq P(X_n \leq a+\epsilon) + P(|X_n - X| > \epsilon)$$

By taking the $\limsup$ and $\liminf$ $P(X \leq a-\epsilon) \leq \liminf P(X_n \leq a) \leq \limsup P(X_n \leq a) \leq P(X_n \leq a+\epsilon)$ And by continuity of F in a

$F(a) \leq \liminf P(X_n \leq a) \leq \limsup P(X_n \leq a) \leq F(a)$

And therefore $F_n(a) \to F(a)$. By using question 6, we have the convergence in distribution.

4. Let $\epsilon > 0$

$$A_n^\epsilon = \{|X_n - X| > \epsilon\}$$

For every n, there exist infinitely many m such that $P(A_m^{\frac{1}{n}}) \leq \frac{1}{2^n}$, and let $\phi(n)$ be one of the m such that $\phi(n) > \phi(n-1)$ when $n > 0$.

$$\sum_n \mathbb{P}(A_{\phi(n)}^{\frac{1}{n}}) \leq 1 < \infty$$

By Borel Cantelli, $\mathbb{P}(\limsup_n A_{\phi(n)}^{\frac{1}{n}}) = 0$

If $\omega \notin \limsup_n A_{\phi(n)}^{\frac{1}{n}}$, there is at most finitely many n s.t $\omega \in A_{\phi(n)}^{\frac{1}{n}}$. Which means that there exist $N > 0$ s.t $\forall n > N$ $|X_{\phi(n)}(\omega) - X(\omega)| \leq \frac{1}{n}$

We conclude that $X_{\phi(n)} \to X$ on $\left(\limsup_p A_{\phi(p)}^{\frac{1}{p}})\right)^c$, which is of measure 1.

Question 6

1. Every cdf is right continous and admits$F$ a left limit everywhere. (Let's call it $F(x-)$)

   A point of discontinuty is where $F(x-) \neq F(x)$.

   Let A be the set of discontinuties of F.

$$f : \begin{cases} A \longrightarrow & \mathbb{Q} \\ x \longrightarrow & \text{some arbitrary } r \in (F(x^-), F(x)) \end{cases}$$

   This application is an injection. So A is countable.

2. Let

$$f_k^a(x) = \begin{cases} 1 & \text{if } x \le a - \frac{1}{k} \\ 1 - k(x - (a - \frac{1}{k})) & \text{if } a - \frac{1}{k} < x \le a \\ 0 & \text{else} \end{cases}$$

$$g_k^a(x) = \begin{cases} 1 & \text{if } x \le a \\ 1 - k(x - a) & \text{if } a < x \le a + \frac{1}{k} \\ 0 & \text{else} \end{cases}$$

- f_k^a, g_k^a is continuous.
- $f_k^a \uparrow_k 1_{x \le a}$ pointwise.
- $g_k^a \downarrow_k 1_{x \le a}$ pointwise.
- f_k^a, g_k^a is positive and bounded.

Let x be a point of continuty of F.

$$E[f_k^x(X_n)] \le F_n(x) \le E[g_k^x(X_n)]$$

so by right continuity:

$$\limsup_n F_n(x) \ge \limsup_n E[f_k^x(X_n)] = F(x - \frac{1}{k}) \to_k F(x)$$

and by left continuty

$$\liminf_n F_n(x) \ge \liminf_n E[g_k^x(X_n)] = F(x + \frac{1}{k}) \to_k F(x)$$

We proved that:

$$F(x) \le \liminf_n F_n(x) \le \limsup_n F_n(x) \le F(x)$$

so the $F_n(x) \to F(x)$.

For the other direction: Let g be continuous and bounded.

Let first suppose that the support of g is compact. Then g is uniformly continuous. Given $\epsilon > 0$, there is $\delta > 0$ s.t $|x - y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon$. Let C be a closed interval containing the support of g. This interval being bounded, we can find a partition of C into intervals $(a_i, a_{i+1}]$, of size at most δ, and we can also assume that the boundaries of the intervals are points of continuity of F (the point of discontinuity being countable, we can avoid them)

We can construct a simple function $h = \sum_{i=1..p} g_i 1_{(a_i, a_{i+1}]}$, we chose arbitrarly a value g_i from the image of g on each interval), so we have that $\sup |h - g| < \epsilon$.

We can always rewrite h as $\sum_{i=1..r} h_i 1_{(-\infty, a_i]}$.

$$\lim_n E[h(X_n)] = \lim_n \sum_i h_i E[1_{(-\infty,a_i]}(X_n)] = \lim_n \sum_i h_i F_n(a_i) = \sum_i h_i F(a_i) = E[h(X)]$$

Let n be large enough so that $|E[h(X_n)] - E[h(X_n)]| < \epsilon$

$$\begin{aligned} |E[g(X_n)] - E[g(X)]| &\le |E[g(X_n) - h(X_n)]| + |E[h(X_n) - h(X)]| + |E[h(X) - g(X)]| && (6) \\ &\le 3\epsilon && (7) \end{aligned}$$

Which conclude the proof.

If g is only continous and bounded:

$X \neq \infty$ a.s, let $M > 0$ be so that $P(|X| > M) \leq \epsilon$

Let f be continous function equal to g on $[-M, M]$, equal to 0 when $|x| > M + \alpha$, and linear on the remaining intervals to make the function continous.

f has a compact support. Let n be large enough so that $|E[f(X_n) - f(X)]| < \epsilon$

$C = \sup g \geq \sup f$

$$\begin{aligned} |E[g(X_n) - f(X_n)]| &\leq |E[(g(X_n) - f(X_n))1_{|X_n|<M}]| && (8) \\ &+ |E[(g(X_n) - f(X_n))1_{|X_n|\geq M}]| && (9) \\ &= 0 + 2CP(M \leq |X_n|) && (10) \\ &\to_n 2CP(M \leq |X|) \leq 2C\epsilon && (11) \end{aligned}$$

So for n large enough, $|E[g(X_n) - f(X_n)]| \leq (2C + 1)\epsilon$.

Using the same calculations: $|E[g(X_n) - f(X_n)]| \leq 2C\epsilon$

$$\begin{aligned} |E[g(X_n)] - E[g(X)]| &\leq |E[g(X_n) - f(X_n)]| + |E[f(X_n) - f(X)]| + |E[f(X) - g(X)]| && (12) \\ &\leq (2C + 1)\epsilon + \epsilon + 2C\epsilon && (13) \\ &\leq C'\epsilon && (14) \end{aligned}$$

And we conclude the proof.