ORF527 - Problem Set 2

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February 17, 2016

Q.1

$(W_{t_1}, \dots W_{t_n})$ and $(W'_{t_1}, \dots W'_{t_n})$ are both guassian with the same mean and covariance matrix, so they do have the same carateristic function and therefore the same distribution. So

$$\mathbb{E}f(W_{t_1}, \dots W_{t_n}) = \mathbb{E}f(W'_{t_1}, \dots W'_{t_n})$$

Q.2

$$\begin{aligned} t \le \tau &\iff \forall \varepsilon > 0\ \exists s \le t + \varepsilon\ X_s \in B \\ &\iff \exists s \le t\ X_s \in B \end{aligned}$$

The last equivalence follows from the following:

- $\Rightarrow$ Trivial
- $\Leftarrow$ Suppose $\forall n \in \mathbb{N}; \exists s_n \le t + \frac{1}{n}\ X_{s_n} \in B$ Without loss of generality, we can assume that s_n converges to s (by taking a subsequence) It is easy to see that $s \le t$, X that $X_{s_n} \to X_s$ (by continuity), and that $X_s \in B$ (because B is closed).

$$\begin{aligned} t \le \tau &\iff \exists s \le t\ X_s \in B \\ &\iff \inf_{s \le t} d(X_s, B) = 0 \end{aligned}$$

Where the last equivalence is due to the fact that when B is closed $x \in B \iff d(x, B) = 0$

Let's consider $f : x \to d(x, B)$. f is continuous, X_t is also continuous, therefore $\inf_{s \le t} f(X_s) = \inf_{s \le t, s \in \mathbb{Q}} f(X_s)$, and $\inf_{s \le t} f(X_s)$ is $\mathcal{F}_t$ measurable as a limit a countable number of $\mathcal{F}_t$-measurable functions.

Q.3

1. (a) $\{\tau \le t\} = \cap_{\varepsilon > 0} \underbrace{\{\exists s \le t + \varepsilon, X_s \in B\}}_{U_{t+\varepsilon}}$, $U_t = \{\exists s \le t, X_s \in B\} \in \mathcal{F}_t$:

    - $U_t = \cup_{s \le t}\{X_s \in B\} = \cup_{s \le t, s \in \mathbb{Q}}\{X_s \in B\}$. Indeed, Let $\omega \in \{X_s \in B\}$ for some $s \le t$. Since B is open there exist $\alpha > 0$ such that $\mathcal{B}(X_s(\omega), \alpha) \subset B$. By continuity of X_s, there exist $s' \in \mathbb{Q}$ smaller than t such that $|X_s(\omega) - X_{s'}(\omega)| < \alpha$, therefore $\omega \in \{X_{s'} \in B\}$
    - $\cup_{s \le t, s \in \mathbb{Q}}\{X_s \in B\} \in \mathcal{F}_t$ as a countable union of sets in $\mathcal{F}_t$.

    Let $a > 0$ $(U_{t+\varepsilon})_{\varepsilon > 0}$ is non-decreasing sequence. So $\cap_{\varepsilon > 0} U_{t+\varepsilon} \subseteq U_{t+a} \in \mathcal{F}_{t+a}$, Therefore $\cap_{\varepsilon>0} U_{t+\varepsilon} \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$ Since the filtration satisfies the usual conditions, $\{\tau \le t\} = \cap_{\varepsilon > 0} U_{t+\varepsilon} \in \mathcal{F}_t$ and τ is a stopping times.

2. $\Omega = \{\omega_1, \omega_2\}$, $X_t(\omega_1) = t$, $X_t(\omega_2) = -t$, $B = (0, \infty)$, $\tau = \inf\{t, X_t \in B\}$, $\{\tau \le 0\} = \{\omega_1\}$.

    $\mathcal{F}_0 = \{\emptyset, \Omega\}$ so $\{\tau \le 0\} \notin \mathcal{F}_0$

    si τ is not stopping time.

Q.4

- Without loss of generality we can assume $i \in J \Rightarrow i+t \in J$. $(W_i)_{i\in J}$ is a discrete markov chain. Let $A \in \mathcal{F}_\tau$, so that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$.

  Let's prove that $E[1_A f(W_{t+\tau})] = E[1_A \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}dx]$

$$
\begin{aligned}
E[1_A \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}dx] &= \sum_J E[1_{A,\tau=i} \int f(x)\frac{e^{-(x-W_i)^2/2t}}{\sqrt{2\pi t}}dx] && \text{Fubini for bounded r.v} \\
&= \sum_J E[1_{A,\tau=i} E[f(W_{i+t})|\mathcal{F}_i]] && \text{(Markov Property)} \\
&= \sum_J E[E[1_{A,\tau=i} f(W_{i+t})|\mathcal{F}_i]] && \text{(Because } A \in \mathcal{F}_\tau\text{)} \\
&= \sum_J E[E[1_{A,\tau=i} f(W_{\tau+t})|\mathcal{F}_i]] \\
&= \sum_J E[1_{A,\tau=i} f(W_{\tau+t})] \\
&= E[1_A f(W_{\tau+t})] && \text{Fubini for bounded r.v}
\end{aligned}
$$

  W_τ is $\mathcal{F}_\tau$-measurable. Indeed, $\{W_\tau \leq a, \tau \leq t\} = \cup_{j\in J, j\leq t}\{W_j \leq a, \tau = j\} \in \mathcal{F}_t$

  By fubini, $\int \frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$ is F_τ measurable as integral of $\mathcal{F}_\tau$-measurable function. So:

$$E[W_{t+\tau}|\mathcal{F}_\tau] = \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$$

- Let $\tau_k \downarrow \tau$ a sequence of discrete stopping times, then

$$E[f(W_{t+\tau_k})|\mathcal{F}_{\tau_k}] = \int f(x)\frac{e^{-(x-W_{\tau_k})^2/2t}}{\sqrt{2\pi t}}$$

  - By continuity of W_t, $f(x)\frac{e^{-(x-W_{\tau_k})^2/2t}}{\sqrt{2\pi t}} \to f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$ everywhere.
  - f is bounded, by dominated convergence theorem: $\int f(x)\frac{e^{-(x-W_{\tau_k})^2/2t}}{\sqrt{2\pi t}} \to \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$

  For $A \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_k}$ $E[f(W_{\tau+t})1_A] = \lim E[f(W_{\tau_k+t})1_A] = \lim \int f(x)\frac{e^{-(x-W_{\tau_k})^2/2t}}{\sqrt{2\pi t}} = \int f(x)\frac{e^{-(x-W_\tau)^2/2t}}{\sqrt{2\pi t}}$