ORF524 - Problem Set 1

Bachir EL KHADIR

October 3, 2015

Question 1

1) Let $A_1, ..A_2 \in \Sigma$, so $\cup_i A_i = (\cap_i A_i^c)^c \in \Sigma$

2) $\Sigma \neq \emptyset$. Let $A \in \Sigma$, we have $\Omega = A \cup A^c$ and $\emptyset = A \cap A^c$ are both in Σ

3) Let Σ_A be the smallest algebra containing A. By definiton, we have $\Omega, \emptyset, A, A^c \in \Sigma_A$. Conversely, it's easy the see that $\{\emptyset, \Omega, A, A^c\}$ is an algebra

Question 2

$P(A) \geq 0$ because P is a measure. $P(A) = P(\Omega) - P(A^c) = 1 - P(A^c) \leq 0$

Question 3

1) $mathbbR = \cap_{n \in \mathbb{Z}}[n, n+1]$ and $\mu([n, n+1]) = 1$

2) If Ω is countable, then there exist a sequence $(a_i)_{i \in \mathbb{N}}$ such that $\Omega = \cup_i a_i$. Ω is then σ-finite because $\mu(a_i) = 1$. Conversly, if Ω is σ-finite, there Ω can be written as a union of countably many finite sets, Ω is then countable.

Question 4

1) Let's first suppose that $f \geq 0$.

   Let $\phi = \sum_i a_i 1_{A_i}$ be a simple function such that $0 \leq \phi \leq f$, then $\int \phi = \sum_i a_i P(A_i) = \sum_i a_i \sum_\omega 1_{A_i}(\omega) = \sum_\omega \phi(\omega)$

   $\int f d\mathbb{P} = \sup_{\phi \leq f} \int \phi = \sup_{\phi \leq f} \sum_\omega \phi(\omega) = \sum_\omega \sup_{\phi \leq f} \phi(\omega) = \sum_\omega f(\omega)$

   In the general case, $f = f^+ - f^-$, it's easy to apply the previous proof to f^+ and f^- (if well defined) and conclude because of linearity.

2) P_f is a probability measure because:

   - $P_f \geq 0$ becase $f \geq 0$.
   - P_f is σ-additive because if $\{A_i \subset \Omega, i \in \mathbb{N}\}$ is a set of disjoint sets, then $P_f(\cup_i A_i) = \sum_{\omega \in \cup_{A_i}} f(\omega) = \sum_i \sum_{\omega \in A_i} f(\omega) = \sum_i P(A_i)$. We could re-arrange the terms because they are all positive.
   - $P(\omega) = \sum f(\omega) = 1$