$$X = \underbrace{M}_{\text{local martingale}} + \underbrace{A}_{\text{bounded variation process}}$$

Ito: $f \in \mathcal{C}^2, df(X_t) = f'(X_s)dX_s + \frac{1}{2}f''(X_s)d < M >_s$

1 Basic concepts of SPT

Starting point: semimartingale market models, ie:

$$dB(t) = r(t)B(t)dt \tag{1}$$

$$dX_i(t) = X_i(t)\left(b_i(t)dt + \sum_\nu \sigma_{i,\nu} dW_\mu(t)\right) \tag{2}$$

Here:

- $B(t)$ is the value of the bank accound if we start from 1 dollar today.
- $X_i(t)$ stands for the price of one share of stock of company i.
- $r(t)$ is the short rate.
- $b_i(t)$ rate of return of stock i.
- $\sigma_{i,\nu}(t)$ volatility of stock i with respect to W_ν.

Theorem 1 (Solutions)**.** *(1) and (2) admist solutions (as long as we know the ?)* $B(t) = e^{\int_0^t r_s ds}$

$$X_i(t) = X_i(0)\exp(\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s))$$

where

$$\gamma_i(t) = b_i(t) - \frac{1}{2}a_{ii}(t) = b_i(t) - \frac{1}{2}\sum_{\mu=1}^d \sigma_{i\mu}(t)$$

is called the growth rate.

Proof.

- $e^{\int_0^t r(s)ds}$ is a process of bounded variations. ($\int_0^t r(s)ds = \int_0^t r(s)^+ds - r(s)^- ds$) By Ito's formula for the semi martingale $\int_0^t r(s)ds$ and $f = \exp$ $\mathrm{d}\mathrm{e}^{\int_0^t \mathrm{r(s)ds}} = \mathrm{e}^{\int_0^t \mathrm{r(s)ds}}\mathrm{d}(\int_0^t \mathrm{r(s)ds}) = \mathrm{e}^{\int_0^t \mathrm{r(s)ds}}\mathrm{r(t)dt}$.
-
$$X_i(t) = X_i(0)e^{\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)}$$

$$d\,log(X_i(t)) = d(\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)) = \gamma_i(t)dt + \Sigma_{\nu=1}^d \sigma_{i,\nu}(t)dW_\nu(t)$$

$$\begin{aligned} d\,log(X_i(t)) &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\frac{1}{X_i(t)^2}\underbrace{X_i(t)^2 \sum \sigma_{i\mu}^2(t)dt}_{d<X_i>(t)} \\ &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\sum \sigma_{i\mu}^2(t)dt \end{aligned}$$

□

Remak 1 (growth rate)**.**

$$\frac{1}{T}\log X_i(t) - \frac{1}{T}\int_0^T \gamma_i(t)dt \to 0$$

Whenever σ does not grow too fast in T.

Proof.

$$\frac{1}{T}\log X_i(t) - \frac{1}{T}\int_0^T \gamma_i(t)dt = \frac{1}{T}\int_0^T \sum_\nu \gamma_{i\nu}(t)dW_\nu(t)$$

□

Theorem 2 (Time change martingale)**.** *Every stochastic integral* $I_t = \sum \int h_\nu dW_\nu(s)$ *can be written as a time change of a brownian motion* β *where*

$$\beta(s) = I_{\tau_s}$$

$$\tau_s = \inf\{t : \int_0^t \sum h_\nu(s)^2 ds\}$$

$I_t = \beta(< I >_t)$

2 Class Portfolios old theory

Definition 1 (Portfolios)**.** *Fix a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *such that* B, X_i, r, b, σ *are adapated to it. A portfolio* $\Pi(t) = (\Pi_1(t), \ldots, \Pi_n(t))$*is a bounded progressively measurable process with respect to* $(\mathcal{F}_t)_t$ *such that:*

$$\sum_i \Pi_i(t) = 1 \ \forall t$$

We Π *call long-only portfolio if* $\pi_i(t) \geq 0 \forall i$

Definition 2 (Progessively measurable)**.** $\Pi(t)$ *measurable with respect to* $\cup_{s<t}\mathcal{F}_s$

Example 1.

- *Equal weigted portfolio:* $\Pi_1(t) = \ldots = \Pi_n(t) = \frac{1}{n}$.
- *Market portfolio: Suppose company* i *has* $N_i(t)$ *shares at time* t $\Pi_i(t) = \frac{X_i(t)V_i(t)}{\sum X_j(t)V_j(t)}$

Assumption: All portfolios Π are self financing($\iff$ we immediately re investing all gain from traind). Mathematically, the portfolio value $V^{(\pi)}(t) = \sum \Pi_i(t)X_i(t)$ satisfies the equation $\frac{dV^\pi(t)}{V_i^{pi}(t)} = \sum_i \pi_i(t)\frac{dX_i(t)}{X_i(t)}$.

Theorem 3. *Has an explicit solution*

$$V^{(\pi)}(t) = V^{(\pi)}(0)\exp(\int_0^t \gamma_\pi(u)du + \int_0^t \sum_\nu \sigma_{\pi\nu}(u)dW_\nu(u))$$

$\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \gamma^*_\pi(t)$ $\gamma^*_\pi(t) = \frac{1}{2}(\sum \pi_i(t)a_{ii}(t) - \sum_{i,j}\pi_i(t)\pi_j(t)a_{i,j}(t))$

$$\sigma_{\pi\nu}(t) = \sum_i \pi_i(t)\sigma_{i\mu}(t)$$

Definition 3 (Portfolio)**.**

- *Classical portfolios:*

$$\zeta(t) = (\underbrace{\zeta_i(t)}_{(\#\ of\ share)})_i$$

- *Self financing condition: portfolio value* $V(t) = \zeta.X$ *satisfies* $dV = \zeta.dX$
- *in SPT, we wwwwant to think about weights.* $\Pi_i(t) = \frac{\zeta_i(t)X_i(t)}{\zeta.X}$
- *It only make sens to think of* V *in relative terms:*

$$\frac{dV^{(\pi)}(t)}{V^\pi(t)} = \sum_i \pi_i(t)\frac{dX_i(t)}{X_i(t)}$$

Theorem 4. *Has an explicit solution*

$$V^{(\pi)}(t) = V^{(\pi)}(0)\exp(\int_0^t \gamma_\pi(u)du + \int_0^t \sum_\nu \sigma_{\pi\nu}(u)dW_\nu(u))$$

$$\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \underbrace{\gamma^*_\pi(t)}_{excess\ growth\ rate}$$

$$\gamma^*_\pi(t) = \frac{1}{2}(\sum \pi_i(t)a_{ii}(t) - \sum_{i,j}\pi_i(t)\pi_j(t)a_{i,j}(t))$$

$$\sigma_{\pi\nu}(t) = \sum_i \pi_i(t)\sigma_{i\mu}(t)$$

We can prove that $\frac{1}{T}\log(V^\pi(t)) - \frac{1}{T}\int_0^T \gamma^\pi(u)du \to 0$

Remak 2 (Market portfolios and market weights)**.** ***Disclaimer:*** *Fron now on, think of* $X_i(t)$ *as the market capitalization of company* i *(# shares . price per share).*

2.0.1 The market portfolio

Recall: the market portfolio has weights $\pi_i(t) = \frac{X_i(t)}{\sum X_j} = \mu_i(t)$. For the market portfolio:

$$\frac{1}{T}\int_0^T \gamma^\mu du = \frac{1}{T}\int_0^T \sum \gamma_i(u)\mu_i(u)du + \frac{1}{T}\int_0^T \gamma_\mu^*(u)du$$

If in the original model for X_i the coefficients only depend on the μ_is: $b_i(t) = \bar{b}_i.\mu, \sigma_{i\nu}(t) = \bar{\sigma}_{i\nu}.\mu$ then we are taking the average of a function on μ:

$$\frac{1}{T}\int_0^T f(\mu_1(t),\dots,\mu_n(t))dt$$

$\mu \to \int_0^T f(\mu(t))dt$ is aclled an additive functional. To understand market portfolio:

- Need to understand how μ begaves in the real world.
- Select a class of models compatible with that.
- Study the assymptotics of the additive functional, which will give us the asymptotic growth of market portfolio.

Main observation (Fernholz): rank the market weights: $\mu_{(1)} \geq \ldots \geq \mu_{(n)}$

- the curve $\log k \to \log \mu_{(k)}(t)$ is very stable over time.
- shape is close to linear (weights decay poly)
- $\implies$ look for models of $(\mu_1(t),\dots,\mu_n(t))$ so that $(\mu_{(1)}(t),\dots,\mu_{(n)}(t))$ is stochastically stable. e.g. there exist an initial distribution of $(\mu_{(1)}(0),\dots,\mu_{(n)}(0)) \overset{d}{=} (\mu_{(1)}(t),\dots,\mu_{(n)}(t))$ Such a distribution is a called a stationary / invariant distribution of the process.
- Simplest model of this kind: first model of Fernholz.

Definition 4 (First order model)**.** *Fix parameters $b_1,\dots,b_n$ and $\sigma_1,\dots,\sigma_n > 0$. Define the evolution of capitalizations:*

$$dX_i(t) = \sum_{k=1}^n 1_{\{X_i(t)=X_{(k)}(t)\}} b_k + \sum_{k=1}^n 1_{\{X_i(t)=X_{(k)}(t)\}} \sigma_k dW_i(t)$$

Warning: *Not so easy to make sens of the solution. We know:*

- *There exist a unique weak solution:*
  - *given a probability space on which $W_1,\dots,W_n$ are defined, I can find a larger probability space on which there are processes $X_1,\dots X_n$ solving the equation.*
  - *No matter how I do it, the law $(X_1,\dots,X_n)$ will be the same.(Bass Pardoux '87)*
- *There exist a unique strong solution if no more than 2 X_i's collide $\iff$ $k \to \sigma_k^2$ is concave. (Ichiba Karatzav, Misha '15)*

Goal: Derive a SDE for the ranked caps $X_{(1)}(t) \leq \ldots \leq X_{()}(t)$

Theorem 5 (Only two processse)**.** *$X = M^X + A^X, Y = M^Y + A^Y$ semi martingales. Then* $\max(X,Y)$ *and* $\min(X,Y)$ *are semi martingales.*

$$\begin{cases} d\max(X,Y)_t = 1_{\{\max=X\}}dX + 1_{\{\max=Y\}}dY + \frac{1}{2}dL_0^{\max(X,Y)-\min(X,Y)}(t) \\ d\min(X,Y)_t = 1_{\{\max=X\}}dX + 1_{\{\max=Y\}}dY - \frac{1}{2}dL_0^{\max(X,Y)-\min(X,Y)}(t) \end{cases}$$

Proof. Key identity: $max(X,Y) =: X \vee Y = \frac{X+Y}{2} + \frac{|X-Y|}{2}$

Ito Tanaka:

$$\begin{aligned} dX \vee Y &= \frac{dX + dY}{2} + \frac{1}{2}\left(sign(X-Y)d(X-Y) + dL_0^{|X-Y|}(t)\right) \\ &= \underbrace{\frac{1}{2}(1 + sign(X-Y))}_{1_{X\geq Y}} dX + \frac{1}{2}(1 - sign(X-Y))dY + \frac{1}{2}dL_0^{|\max - \min|} \end{aligned}$$

∎

Theorem 6 (Back to the first order model). *Consider a first order model with 2 stocks:*

$$dX_1(t) = 1_{\{X_1(t)=X_{(1)}(t)\}}(b_1dt + \sigma_1 dW_1(t)) + 1_{\{X_1(t)=X_{(2)}(t)\}}(b_2dt + \sigma_2 dW_2(t))$$
$$dX_2(t) = 1_{\{X_2(t)=X_{(1)}(t)\}}(b_2dt + \sigma_2 dW_2(t)) + 1_{\{X_2(t)=X_{(2)}(t)\}}(b_2dt + \sigma_2 dW_2(t))$$

There exist independent standard Brownian Motions β_1, β_2 such that:

$$dX_{(1)}(t) = b_1dt + \sigma_1 d\beta_1(t) - \frac{1}{2}dL_0^{X_{(1)}-X_{(2)}}$$

$$dX_{(2)}(t) = b_2dt + \sigma_2 d\beta_2(t) - \frac{1}{2}dL_0^{X_{(1)}-X_{(2)}}$$

($X_{(1)} = \min$)

Lemma 1 (Levy's caracterization of BM). *If $M_1, \dots, M_n$ are continuous local martingales and $< M_i, M_j > (t) = t1_{i=j}$, then: $(M_1, \dots, M_n)$ is a standard n-dimensional BM.*

Proof.

$$\begin{aligned}
dX_{(1)} &= dX_1 \vee X_2 \\
&= 1_{X_1=X_{(1)}}dX_1 + 1_{X_2=X_{(1)}}dX_2 - \frac{1}{2}dL_0^{X_{(2)}-X_{(1)}} \\
&= 1_{X_1=X_{(1)}}(b_1dt + \sigma_1 dW_1) + 1_{X_1=X_{(1)}=X_2}(b_1dt + \sigma_1 dW_2) \\
&+ 1_{X_2=X_{(1)}}(b_1dt + \sigma_2 dW_2) + 1_{X_2=X_{(1)}=X_1}(b_1dt + \sigma_1 dW_1) \\
&- \frac{1}{2}dL_0^{X_{(2)}-X_{(1)}} \\
&= 1_{X_1=X_{(1)}}(b_1dt + \sigma_1 dW_1) + 1_{X_2=X_{(1)}}(b_1dt + \sigma_1 dW_2) - \frac{1}{2}dL_0^{X_{(2)}-X_{(1)}} \qquad (\{t, 1_{X_1=X_2}\}\text{has measure 0}) \\
&= b_1dt + \sigma_1 1_{X_1=X_{(1)}}dW_1 + \sigma_2 1_{X_2=X_{(1)}}dW_2
\end{aligned}$$

$$\begin{aligned}
dX_{(2)} &= b_2dt + \sigma_1 1_{X_1=X_{(2)}}dW_1 + \sigma_2 1_{X_2=X_{(2)}}dW_2 \\
d\beta_{(1)} &= 1_{X_{(1)}=X_1}dW_1 + 1_{X_{(1)}=X_2}dW_2 \\
d\beta_{(2)} &= 1_{X_{(2)}=X_1}dW_1 + 1_{X_{(2)}=X_2}dW_2
\end{aligned}$$

Claim: β_1, β_2 are independent standard BM. By the lemma.

- a stochastic integral is continuous and a local martingale
- Ito isometry

□

Theorem 7 (Banner, Fernholz, Karatzan '05). *Start with the first order model with n companies:*

$$dX_i(t) = \sum_{k=1}^n 1_{\{X_i(t)=X_{(k)}(t)\}}b_k + \sum_{k=1}^n 1_{\{X_i(t)=X_{(k)}(t)\}}\sigma_k dW_i(t)$$

Then there exist independent standard BM $\beta_1, \dots, \beta_n$ such that $dX_{(k)} = b_kdt + \sigma_k d\beta_k(t) - \frac{1}{2}dL_0^{X_{(k+1)}-X_{(k)}}(t) + \frac{1}{2}dL_0^{X_{(k)}-X_{(k-1)}}(t)$

Proof. Difficuties

- Why are there no loca times of the form $L^{X_{(l)}-X_{(k)}}$ for $l \geq k+2$?
- Why is local time coefficient $\frac{1}{2}$?

From induction Hypothesis:

$$dX_{(k)}(t) = \sum_i^n 1_{X(k)=X_i(t)}\frac{1}{N_k(t)}dX_i(t) + \sum_l^{k-1}\frac{1}{N_k(t)}dL_0^{X_{(k)}-X_{(l)}}(t) - \sum_{l=k+1}^n \frac{1}{N_k(t)}dL_0^{X_{(l)}-X_{(k)}}(t)$$

Idea $X_{(1)} = \min(X_1, \dots, X_n) = \min(X_1, \min(X_2, \dots, X_n))$
Tasks at this point

- $N_k(t) = 1$ for all k and lebesgue a.e.t with probability 1.
- $L_0^{X_{(k)}-X(k)} = 0$ for all $|l-k| \geq 1$ with probability 1.
- $N_k(t) = 2$ under $dL_0^{X_{(k+1)}-X_{(k)}}$ with probability 1, ie $\mathbb{P}(\int_0^\infty 1_{\{N_k(t)\neq 2\}} dL_0^{X_{(k+1)}-X_{(k)}} = 0)$

$\square$

2.0.2 Skrodhod problems and reflected Brownian motions

Definition 5 (Skorokhod problem in 1D)**.** *Given a continuous path $\phi : [0,\infty) \to \mathcal{R}$ with $\phi(0) > 0$, want to find a non-decreasing path $\eta : [0,\infty) \to \mathcal{R}^+$ s.t*

- $\psi(t) = \phi(t) + \eta(t) \geq 0$ *for all* $t \geq 0$.
- $\int_0^\infty 1_{\psi(t)\neq 0} d\eta(t) = 0$

Theorem 8 (Skorokhold)**.** *There exists a unique solution of the skorokhod problem for any continuous ϕ s.t $\phi(0) > 0$.*

Proof.
- $\eta(t) = \sup_{0\leq s\leq t} \phi^-(s)$ $\int_0^\infty 1_{\psi(t)\neq 0} d\eta(t) = 0$?

  Note that a point of increase t of η($\iff$ the support of the corresponding measure $d\eta$) have the property $\phi(t)^- = \sup_{s\leq t} \phi(s)^- = \eta(t)$.

  We need the show that $\psi(t) = 0$ for such a point t.
- Uniquenss: $(\eta, \psi), (\hat{\eta}, \hat{\psi})$ two solutions.

  $\hat{\psi} - \psi = \hat{\eta} - \hat{\eta}$ BV process.

  Ito: $\frac{1}{2}(\hat{\psi} - \psi)^2 = \int_0^t (\hat{\psi} - \psi) d(\hat{\eta} - \eta) = -\int_0^t \hat{\psi} d\eta - \int_0^t \psi d\hat{\eta}$

$\square$

Definition 6 (Reflected Brownian motion in 1D)**.**

$$\begin{aligned} \Phi : C([0,\infty), \mathbb{R}) &\longrightarrow C([0,\infty), \mathbb{R}) \\ : \phi &\longrightarrow \psi \end{aligned}$$

A reflected Brownian motion with drift μ and diffusion coefficient σ is the process $\Phi(\mu t + \sigma B(t))$

Remak 3. *Consider a first order model with 2 companies: $dX_i = 1_{X_i = X_(1)} b_1 dt + 1_{X_i = X_(2)} b_2 dt + 1_{X_i = X_(1)} \sigma_1 dW_1 + 1_{X_i = X_(2)} \sigma_2 dW_2$*

Claim: $|X_1 - X_2| = X_{(2)} - X_{(1)}$ is a RBM with drift $b_2 - b_1$ and drift coefficient $\sqrt{\sigma_1^2 + \sigma_2}$. $\underbrace{X_{(2)}(t) - X_{(1)}(t)}_{\psi(t)} =$

$\underbrace{X_{(2)}(0) - X_{(1)}(0) + (b_2 - b_1)t + \sigma_2\beta_2(t) - \sigma_1\beta_1(t)}_{\phi(t)} + L_0^{X_{(2)}-X_{(1)}}(t)$

Definition 7 (Skorokhod problem in R_+^m)**.** *Consider a continuous path $\phi : \mathbb{R}_+ \to \mathbb{R}^m$ st $\phi(0) > 0$, and a matrix $R \in \mathbb{R}^{m\times n}$. We want to find a continuous path $\eta \mathbb{R}_+ \to \mathbb{R}^m$ such that:*

- *all componoents of η are non-decreasing.*
- $\psi(t) = \phi(t) + R\eta(t)$
- $\int_0^\infty 1_{t,\psi_k(t)\neq 0} d\eta_k(t) = 0$

Theorem 9 (Existence and Uniqueness of the Solution)**.** *$R = I - Q$ and Q has non negative entries, zero diagonal entries and spectral radius < 1, then the skorokhod problem has unique solution.*