ORF527 - Problem Set 3

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Problem 4.4

- Let's prove that $(Z_t = X_{t+\varepsilon} = Y_{\varepsilon+t} - Y_\varepsilon)_{t\geq 0}$ is a BM. Indeed:
  - Z_t is clearly a continuous guassian process.
  - for $t \geq s$, $cov(Z_t, Z_s) = cov(Y_{\varepsilon+t}, Y_{\varepsilon+s}) - cov(Y_{\varepsilon+t}, Y_\varepsilon) + cov(Y_{\varepsilon+s}, Y_\varepsilon) + cov(Y_\varepsilon, Y_\varepsilon) = \varepsilon + s - \varepsilon - \varepsilon + \varepsilon = s$

  It follows that $(X_t)_{t\geq\varepsilon}$ is martingale (with respect to its natural filtration $\mathcal{F}_t$).

- For $0 < s \leq t$, let $\varepsilon = \frac{s}{2}$, then $t - s = \mathbb{E}(Y_t - Y_\varepsilon - (Y_s - Y_\varepsilon))^2 = \mathbb{E}(Y_t - Y_s)^2$

- $\mathbb{E}[Y_t^2] = t^2\mathbb{E}[B_{\frac{1}{t}}^2] = t$, so $Y_t \to_{L_2} 0$, and therefore in probability, so there exist a subsequence $t_n > 0$, $(Y_{t_n})_n$ such that $t_n \to 0$ and $Y_{t_n} \to 0$ a.s.

  Let $n \in \mathbb{N}^*$, $(Y_{t_n+t} - Y_{t_n})_{t\geq 0}$ is a martingale, and $\mathbb{E}[Y_t^2] = t$, by doobs maximal inequality

$$\mathbb{P}(\sup_{t\in[t_m,t_n]} |Y_t - Y_{t_m}| > \alpha) \leq \frac{t_n}{\alpha^2}$$

  so:

$$\mathbb{P}(\sup_{t\in[t_m,t_n]} |Y_t| - |Y_{t_m}| > \alpha) \leq \frac{t_n}{\alpha^2}$$

  so:

$$\mathbb{P}(\sup_{t\in[t_m,t_n]} |Y_t| > \alpha + |Y_{t_m}|) \leq \frac{t_n}{\alpha^2}$$

  $\sup_{t\in[t_m,t_n]} |Y_t| \to_m \sup_{t\in(0,t_n]} |Y_t|$ a.s, because:

  - $(\sup_{t\in[t_m,t_n]} |Y_t|)_m$ is non-decreasing, so it has a limit L.
  - $(0, t_n] = \cup_m [t_m, t_n]$, so $\sup_{t\in(0,t_n]} |Y_t| = \sup_m \sup_{t\in[t_m,t_n]} |Y_T| = L$

  So $\sup_{t\in[t_m,t_n]} |Y_t| - |Y_{t_m}| \to_m \sup_{t\in(0,t_n]} |Y_t|$ a.s. and in distribution. Therefore $\mathbb{P}(\sup_{t\in(0,t_n]} |Y_t| \geq \alpha) \leq \frac{t_n}{\alpha^2}$

$$\mathbb{P}(\limsup_0 |Y_t| \geq \frac{\alpha}{2}) = \lim_n \mathbb{P}(\sup_{t\leq t_n} |Y_t| \geq \frac{\alpha}{2}) \leq \frac{4t_n}{\alpha^2} \to 0$$

  ie $P(\limsup_0 |Y_t| \neq 0) = P(\exists n \in \mathbb{N}^+ \ \limsup_0 |Y_t| \leq \frac{1}{n}) \leq \sum_{n>0} P(\limsup_0 |Y_t| \leq \frac{1}{n}) = 0.$

Problem 4.6

- $X_{t\wedge\tau}$ is a bounded martingale
- τ is finite a.s
- $X_{t\wedge\tau} \to X_\tau$ a.s
- Dominated convergence theorem :

  $1 = \mathbb{E}[X_\tau] = \mathbb{P}(B_\tau = A)\exp(\alpha A)E[\exp(-\alpha^2\tau/2)|B_\tau = A] + \mathbb{P}(B_\tau = -A)\exp(-\alpha A)E[\exp(-\alpha^2\tau/2)|B_\tau = -A]$
- By symmetry of the brownian motion B_t, is it easy that $P(\tau \le t|B_\tau = A) = P(\tau \le t|B_\tau = -A)$, so τ is independent of $\{B_\tau = A\}$, and the last equation becomes:

  $$1 = \mathbb{E}[X_\tau] = \mathbb{P}(B_\tau = A)\exp(\alpha A)E[\exp(-\alpha^2\tau/2)] + \mathbb{P}(B_\tau = -A)\exp(-\alpha A)E[\exp(-\alpha^2\tau/2)]$$

  By symmetry: $\mathbb{P}(B_\tau = A) = \mathbb{P}(B_\tau = -A) = \frac{1}{2}$

so $1 = \frac{1}{2}(\exp(\alpha A) + \exp(-\alpha A))\Phi(-\alpha^2/2)$ If $\lambda > 0$, set $\alpha = \sqrt{2\lambda}$, $\Phi(\lambda) = \frac{2}{\exp(\sqrt{2\lambda}A)+\exp(-\sqrt{2\lambda}A)}$

- $e^{\lambda x} = 1 - \int_0^\lambda xe^{-\sigma x} d\sigma$
- $xe^{\lambda x} = x - \int_0^\lambda x^2 e^{-\sigma x} d\sigma$

By fubini

- $\mathbb{E}[\tau e^{\lambda\tau}] = 1 - \int_0^\lambda \mathbb{E}[\tau e^{-\sigma\tau}]d\sigma$
- $\mathbb{E}[\tau^2 e^{\lambda\tau}] = 1 - \int_0^\lambda \mathbb{E}[\tau^2 e^{-\sigma\tau}]d\sigma$

e.g.

- $\Phi'(\lambda) = \mathbb{E}[-\tau e^{-\lambda\tau}] = -1 + \int_0^\lambda \mathbb{E}[\tau^2 e^{-\sigma\tau}]d\sigma$
- $\Phi''(\lambda) = \mathbb{E}[\tau^2 e^{-\sigma\tau}]$
- $\Phi''(0) = E[\tau^2]$

  Let $x = \lambda 2A^2$, then:

$$\begin{aligned}
\Phi(x) &= \frac{1}{\cosh(\sqrt{x})} \\
&= \frac{1}{1 + \frac{x}{2} + \frac{x^2}{4!} + o(x^2)} \\
&= 1 - (\frac{x}{2} + \frac{x^2}{4!}) + (\frac{x}{2} + \frac{x^2}{4!})^2 + o(x^2) \\
&= 1 - \frac{x}{2} + (\frac{1}{4} - \frac{1}{4!})x^2 + o(x^2) \\
&= 1 - A^2 x + A^4 \frac{5}{6} x^2 + o(x^2)
\end{aligned}$$

so $\Phi''(0) = \frac{5A^4}{3}$

$$E[\tau^2] = \frac{5A^4}{3}$$

When the boundary is not symmetric, we face the difficulty of calculating figuring out the dependence between τ and B_τ

Problem 2

1. $\tau_A = \inf\{t \geq 0 : W_t = A\} = \inf\{t \geq 0 : \frac{W_t}{A} = 1\} = \inf\{A^2 t : t \geq 0, \frac{W_{tA^2}}{A} = 1\} = A^2 \underbrace{\inf\{t \geq 0, \tilde{W}_t = 1\}}_{\tilde{\tau}_1}$

   Where $\tilde{W}_t = \frac{W_{tA^2}}{A}$ is a brownian motion, so $\tilde{\tau}_1 \stackrel{d}{=} \tau_1$, indeed:

   - By continuity of the brownian motion:
     $\{\tau \leq t\} = \cap_{\varepsilon > 0} \cup_{s \leq t, s \in \mathbb{Q}} \{d(W_s, A) < \varepsilon\}$
   - The measure of this quantity depends only on the distribution of $(W_s)_{s \in Q}$, which is the same for all brownian motions. That is true because they all have the same finite distribution, and Kolmogorov extension theorem garantees the uniqueness when we extend that to countably many variable.

   Therefore $\tau_A \stackrel{d}{=} A^2 \tau_1$.

2. $$\begin{aligned} \mathbb{P}(W_t \leq 0 \; \forall t \leq T) &\leq \mathbb{P}(W_t \leq \frac{A}{2} \; \forall t \leq T) \\ &\leq \mathbb{P}(T \leq \tau_A) \\ &= \mathbb{P}(\frac{T}{A^2} \leq \tau_1) \rightarrow_{A0} \mathbb{P}(\infty \leq \tau_1) \\ &= 0 \end{aligned}$$

   (because $\frac{T}{A^2} \rightarrow \infty$ as and $\tau_1 < \infty$)

3. Beucase of the continuity of (W_t), $W_t = 0$ has a finite number of solutions on $[0, T) \implies$ there exists $n \in \mathbb{N}*$, $n \geq \frac{1}{T}$ such that W_t dones't change sign on $[0, \frac{1}{n}]$.

   $$\begin{aligned} &P(W_t = 0 \text{ for finitely many } t \in [0, T]) \\ &\leq P((\exists n \geq \frac{1}{T}) \; (\forall t \; \leq \frac{1}{n} W_t \geq 0) \vee (\forall t \; \leq \frac{1}{n} W_t \leq 0)) \\ &\leq \sum_{n \in \mathbb{N}^*} P(\forall t \leq \frac{1}{n} W_t \leq 0) + P(\forall t \leq \frac{1}{n} W_t \geq 0) && \text{(Union bound)} \\ &= \sum_{n \in \mathbb{N}^*} 2P(\forall t \leq \frac{1}{n} W_t \leq 0) && \text{(By considering } -W_t) \\ &= 0 \end{aligned}$$

Problem 3

a) $\leq$ follows from the fact that $\sup(A + B) \leq \sup A + \sup B$ for any $A, B \subset \mathbb{R}$

   $\geq$ Let $0 = t_0 \leq \ldots \leq t_n = t$ be partition included in $0 = s_0 \leq \ldots \leq s_m = t$, then $\sum_k (g(t_k) - g(t_{k-1}))^+ = \sum_k (\sum_{t_{k-1} \leq s_i \leq t_k} g(s_i) - g(s_{i-1}))^+ \leq \sum_k \sum_{t_{k-1} \leq s_i \leq t_k} (g(s_i) - g(s_{i-1}))^+ \leq \sum_i (g(s_i) - g(t_{i-1}))^+$

   Let's call $\pi_n = \{t_1 \leq \ldots \leq t_n\}$ a partition of n elements, and call $|\pi_n| = \max_i |t_{i+1} - t_i|$, and for two parition π^1, π^2, let $\pi^1 \vee \pi^2$ be the smallest partition including π^1 and π^2. We have proved the sup over partitions can only increase when n increases or when taking the union of two partitions. Therefore

$$\begin{aligned}
g^+(t) + g^-(t) &= \lim_n \sup_{\pi_n} \sum_{t_k \in \pi_n^1} (g(t_k) - g(t_{k-1}))^+ + \lim_n \sup_{\pi_n^2} \sum_{t_k \in \pi_n} (g(t_k) - g(t_{k-1}))^+ \\
&\leq \lim_n \sup_{\pi = \pi_n^1 \vee \pi^2{}_n} \sum_{t_k \in \pi} (g(t_k) - g(t_{k-1}))^+ + \sum_{t_k \in \pi} (g(t_k) - g(t_{k-1}))^- \\
&\leq \lim_n \sup_{\pi = \pi_n^1 \vee \pi^2{}_n} \sum_{t_k \in \pi} |g(t_k) - g(t_{k-1})| \\
&\leq TV[g, t]
\end{aligned}$$

b) Let π_n^1, π_n^2 two sequences of partitions that converge to the sup, eg: $g^+(t) = \lim_n \sum_{\pi_n^1} (g(t_k) - g(t_{k+1}))^+$ $g^-(t) = \lim_n \sum_{\pi_n^2} (g(t_k) - g(t_{k+1}))^-$

   But $\lim_n \sum_{\pi_n^1} (g(t_k) - g(t_{k+1}))^+ \leq \lim_n \sum_{\pi_n^1 \vee \pi_n^2} (g(t_k) - g(t_{k+1}))^+ \leq g^+(t)$ so $g^+(t) = \lim_n \sum_{\pi_n^1 \vee \pi_n^2} (g(t_k) - g(t_{k+1}))^+$ Similarly: $g^-(t) = \lim_n \sum_{\pi_n^1 \vee \pi_n^2} (g(t_k) - g(t_{k+1}))^-$

   so $g^+(t) - g^-(t) = \lim_n \sum_{\pi_n^1 \vee \pi_n^2} (g(t_k) - g(t_{k+1}))^+ - (g(t_k) - g(t_{k+1}))^- = g(t) - g(0)$

c) g^+, g^- are both non-decreasing functions. Indeed, for $s \leq t$, for any partition of $[0, t]$, we can always include the point s, and we can see that $g^+(t) - g^+(s)$ is a limit of a non-negative quantity.

d) if g is non-decreasing, $\sup_{n, \pi_n} \sum_k |g(t_k) - g(t_{k+1})| = \sup_{n, \pi_n} \sum_k (g(t_k) - g(t_{k+1})) = g(t) - g(0) < \infty$

e) part b) proved that if g has finite variation, it can be written as the difference of two nondecreasing functions. By the triangular inequality we can prove that fow two functions g, f, $TV[g + f, t] \leq TV[g] + TV[f]$, so the variation of the difference of two nondecreasing functions is smaller than the sum of their respective variation, so its finite.

f) The difference of two right continuous functions is right continuous, if they both have finite variation, then their difference has finite variation and is rightconinuous.

   Let f be a right continuous function with finite variation, and let's prove that g^+ is right continuous. Fix t, we know that f^+ is non-decreasing, so f^+ has a right limit at t. Let's prove that this limit is equal to $g^+(t)$.

   Notation: $\Pi_n[a, b]$ the set of partitions of $[a, b]$. $TV(f)[a, b] := \sup_{\pi \in \Pi_n[a,b]} \sum_{t_k \in \pi} |f(t_{k+1}) - f(t_k)|$.

   If $c \in (a, b)$, by adding c to any parition of $[a, b]$ we can see that $TV(f)[a, b] := TV(f)[a, c] + TV(f)[c, b]$.

   Let $\delta > 0$

   - f is rightcontinuous at t. For ε smaller than some threshold ε_0: $f(t + \varepsilon) - f(t) \leq \delta$.
   - f has finite variation, let π be a parition of $[t, t + \varepsilon_0]$ such that:

$$\sum_{t_k \in \pi} (f(t_{k+1}) - f(t_k))^+ \geq TV(f)[t, t + \varepsilon_0] - \delta$$

   - Let t_δ the smallest element in π bigger than t. Again, without loss of generality, we can assume $t_\delta - t < \varepsilon_0$.
     Therefore:

$$\sum_{t_k \in \pi} (f(t_{k+1}) - f(t_k))^+ = (f(t_{k*}) - f(t))^+ + \sum_{t_k \in \pi, t_k \neq t_\delta} (f(t_{k+1}) - f(t_k))^+ \leq \delta + TV[f](t + t_\delta, t + \varepsilon_0)$$

- We have proved that: $TV(f)[t, t+\varepsilon_0] - \delta \leq \delta + TV(f)[t+t_\delta, t+\varepsilon_0]$
  e.g $TV(f)[t, t+t_\delta] \leq \delta + \delta$
  eg $f(t+t_\delta) - f(t) \leq 2\delta$

By taking $\delta = \frac{1}{2n}$, we get the existence of subsequence t_n^* such that: $t_n^* \to 0$, such that $0 \leq g^+(t_n^*) - g^+(t) \leq \frac{1}{n} \to 0$. So the g^+ is right continuous, and $g^- = g^+(t) - g(t) + g(0)$ also is right continuous and $g = g^+ - g^- + g(0)$

g) Let's first prove the equality of the two integrals when f is simple and $h = h_+$

Let R denote the Riemann-Stieltjes integral and L define the Lebesgue ingtegral.

$$R(f) = \sum_{i=1}^{n} f(t_i)(h(t_i) - h(t_{i-1}))$$

$$L(f) = \sum_{i=1}^{n} f(t_i)\mu_h([t_i, t_{i-1}]) = \sum_{i=1}^{n} f(t_i)(h(t_{i-1}) - h(t_{i-1}))$$

This equality holds for $h = h_+ - h_-$ by linearity of the two integrals.

Now let f be Riemann-Stieltjes integrable as a uniform limit of simple functions $f_n(t) = \sum_i f_n(t_i^n) 1_{[t_{i-1}^n - t_{i+1}^n]}$, then we consider $g_n(t) = \sum_i \max(f_n(t_i^n), f(t_i^n)) 1_{[t_{i-1}^n - t_{i+1}^n]}$, and $f_n^{\uparrow} = \max_{k \leq n} g_n$, it is easy to see that $f_n^{\uparrow}$ converge uniformly to to f, and $f_n^{\uparrow} \uparrow f$.

- $R(f_n^{\uparrow}) \to R(f)$ by uniform convergence.
- $L(f_n^{\uparrow}) \to L(f)$ by monotone convergence theorem.