Problem set 5, ORF527

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1 Q1 (7.2 in Steele)

Let $\tau_n = inf\{t, |X_t| \geq n\}$ be a localizing sequence of (X_t), so that $X_{t\wedge\tau_n}$ is bounded. τ_n is non-decreasing and diverges to because the continuous function $t \to X_t$ is bounded on every compact set $[0,T]$, and if n is larger than this bounde then $\tau_n \geq T$.

Since ϕ is continuous, $Y_{t\wedge\tau_n} = \phi(X_{t\wedge n})$ is bounded.

So for $s < t$, $E[Y_{t\wedge\tau_n}|F_s] = E[\phi(X_{t\wedge\tau_n})|F_s] \underbrace{\leq}_{\text{Jensen}} \phi(E[X_{t\wedge\tau_n}|F_s]) = \phi(Y_{s\wedge n})$ Counter example:

- $\Omega = (0,1)$, P the uniform measure.
- $\phi(x) = x^2$
- $X(t,\omega) = \frac{1}{\omega^{\frac{1}{2}}}$ is a integrable constant in t, so it is a martingale, but $\phi(X) = \frac{1}{\omega}$ is not integrable.

2 Q1 (7.3)

Let X_t be a continuous local submartingale martingale verifying (7.35), and τ_n a localizing sequence, and $s < t < T$, then:

- $E[|X_t|] \leq E[\sup_{[0,T]}|X_s|] < \infty$
- $E[X_{t\wedge\tau_n}|F_s] \geq X_{s\wedge\tau_n}$

Using the fact that $X_{s\wedge\tau_n}$ is uniformly bounded by the L_1 function $\sup_{[0,T]}|X_s|$, we use dominated convergence theorem to prove that $E[X_t|F_s] \geq X_s$, so X_s is a submartingale.

A bounded local martingale trivially verifies (7.35)

3 Q2

a. By Ito isometry and linearity:

$$E[(\int_0^T X_s^n dW_s - \int_0^T X_s dW_s)^2] = E[\int_0^T (X_s^n - X_s)^2 ds] \to 0$$

b. $\tau_n = \inf\{t, \int_0^t X_s^2 ds \geq n\} \wedge T$ is a localizing sequence. By markov inequality:

$$P(|\int_0^T X_{t\wedge\tau_n} dW_t| \geq \varepsilon) \leq \frac{E[|\int_0^T X_{t\wedge\tau_n} dW_t|^2]}{\varepsilon^2}$$

By Ito:

$$P(|\int_0^T X_{t\wedge\tau_n} dW_t| \geq \varepsilon) \leq \frac{E[\int_0^T X_{t\wedge\tau_n}^2 dt]}{\varepsilon^2}$$

- For $0 < \delta < \varepsilon$

$$P(|\int_0^T X_t dW_t| \geq \varepsilon) \leq P(|\int_0^{\tau_n} X_t dW_t| \geq \varepsilon - \delta) + P(|\int_{\tau_n}^T X_t dW_t| \geq \delta) \tag{1}$$
$$\leq \frac{E[|\int_0^{\tau_n} X_t dW_t|^2]}{(\varepsilon-\delta)^2} + P(|\int_{\tau_n}^T X_t dW_t| \geq \delta) \tag{2}$$
$$\leq \frac{E[\int_0^{\tau_n} X_t^2 dt]}{(\varepsilon-\delta)^2} + P(|\int_{\tau_n}^T X_t dW_t| \geq \delta) \tag{3}$$
$$\leq \frac{N}{(\varepsilon-\delta)^2} + P(\tau_n < T) \tag{4}$$
$$\leq \frac{N}{(\varepsilon-\delta)^2} + P(\int_0^T X_s^2 \geq N) \tag{5}$$

We get the result by taking δ to 0.

c. By b.

$$P(|\int_0^T (X_t - X_t^n) dW_t|^2 > \varepsilon) \leq P(\int_0^T (X_t - X_t^n)^2 dt \geq \varepsilon) + \frac{N}{\varepsilon^2}$$

Taking the lim sup with respect to n:

$$\limsup_n P(|\int_0^T (X_t - X_t^n) dW_t|^2 > \varepsilon) \leq \frac{N}{\varepsilon^2}$$

And thus for all $N > 0$. We conclude by taking the $N \to 0$.

d. Let $X \in \mathcal{H}^{loc}[0,T]$, let τ_n be a localizing sequence, so that $X1_{[0,\tau_n]} \in \mathcal{H}[0,T]$.

Since $\mathcal{H}_0[0,T]$ is dense in $\mathcal{H}[0,T]$ with respect to the $L_2(\Omega \times [0,T])$ norm, there exist a sequence $X_n \in \mathcal{H}_0$ such that: $E[\int_0^T (X1_{[0,\tau_n]}(s) - X_n(s))^2 ds] \to_n 0$.

$P(\int_0^T (X(s)1_{[0,\tau_n]}(s) - X_n(s))^2 ds > \varepsilon) \leq \frac{E[\int_0^T (X1_{[0,\tau_n]}(s) - X_n(s))^2 ds]}{\varepsilon} \to_n 0$

So $\int_0^T (X(s)1_{[0,\tau_n]}(s) - X_n(s))^2$ converges to 0 in probability.

We also know that $\int_0^T X^2(s)1_{[0,\tau_n]}(s) ds \to \int_0^T X^2(s) ds$ alsmost surely, and thus in probability.

Now, $\int_0^T (X(s) - X_n(s))^2 ds \leq \int_0^T (X(s) - X(s)1_{[0,\tau_n]}(s))^2 + \int_0^T (X1_{[0,\tau_n]}(s) - X_n(s))^2 \xrightarrow{P} 0$, which gives the result.

Using c., We can definie the integral of $X \in \mathcal{H}^{loc}[0,T]$ as the limit in probability of a sequence of simple function that converge to X in the sense of c.

4 Q3

All functions considered here are $\mathcal{H}^{loc}$ as continuous function / integrals of brownian motions. a. $d(e^t W_t) = e^t W_t dt + e^t dW_t$

b. $fx \to \frac{1}{1+x^2}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f''(x) = \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} = \frac{-2+6x^2}{(1+x^2)^3}$

$$d(1+W_t^2)^{-1} = \frac{-2W_t}{(1+W_t^2)^2} dW_t + \frac{-1+3W_t^2}{(1+W_t^2)^3} dt$$

and at 0 the value is 1.

c.

- $Y_t = \int_0^t \sqrt{|W_s|} dW_s$, $dY_t = \sqrt{|W_s|} dW_s$
- $d\cos(Y_t) = -\sin(Y_t) dY_t - \frac{1}{2}\cos(Y_t) d < Y >_t = -\sin(Y_t) dY_t - \frac{1}{2}\cos(Y_t)|W_s| ds$
- $Z_t = e^{\alpha W_t + \sigma t}$

- $dZ_t = \sigma Z_t dt + \alpha Z_t dW_t + \frac{1}{2}\alpha^2 Z_t dt = Z_t((\sigma + \frac{1}{2}\alpha^2)dt + \alpha dW_t)$
- $U_t = e^{\alpha W_t + \sigma t}\cos(\int_0^t \sqrt{|W_s|}dW_s)$
- $V_t = e^{\alpha W_t + \sigma t}\sin(\int_0^t \sqrt{|W_s|}dW_s)$
- $d\cos(Y_t)dZ_t = -\alpha sin(Y_t)\sqrt{|W_t|}Z_t dt = -\alpha\sqrt{|W_t|}V_t$

$$\begin{aligned}
d(e^{\alpha W_t + \sigma t}\cos(\int_0^t \sqrt{|W_s|}dW_s)) &= \cos(Y_t)dZ_t + Z_t d\cos(Y_t) + dZ_t dcos(Y_t) \\
&= \cos(Y_t)Z_t((\sigma + \frac{1}{2}\alpha^2)dt + \alpha dW_t) - Z_t(\sin(Y_t)dY_t + \frac{1}{2}\cos(Y_t)|W_t|dt) - \alpha sin(Y_t)\sqrt{|W_t|}Z_t d \\
&= U_t(\sigma + \frac{1}{2}\alpha^2)dt + U_t\alpha dW_t - Z_t\sin(Y_t)dY_t - \frac{1}{2}U_t|W_t|dt - \alpha sin(Y_t)\sqrt{|W_t|}Z_t dt \\
&= U_t(\sigma + \frac{1}{2}\alpha^2 - \frac{|W_t|}{2})dt + (\alpha U_t - \sin(Y_t)\sqrt{|W_t|}Z_t)dW_t - \alpha sin(Y_t)\sqrt{|W_t|}Z_t dt \\
&= \left(U_t(\sigma + \frac{1}{2}\alpha^2 - \frac{|W_t|}{2}) - \alpha\sqrt{|W_t|}V_t\right)dt + (\alpha U_t - \sqrt{|W_t|}V_t)dW_t
\end{aligned}$$

at 0 the value is 1. d.

- $U_t = \int_0^t W_s d\tilde{W}_s$
- $dU_t = W_s d\tilde{W}_s$
- $V_t = W_t U_t$
- $dV_t = W_t dU_t + U_t dW_t = W_t^2 d\tilde{W}_t + U_t dW_t$
- $d\exp(V_t) = \exp(V_t)(dV_t + \frac{1}{2}(W_t^4 + U_t^2)dt)$

$$\begin{aligned}
d(\exp(W_t\int_0^t W_s d\tilde{W}_s)W_t) &= d(\exp(V_t)W_t) \\
&= W_t d(\exp(V_t)) + \exp(V_t)dW_t + d(\exp(V_t))dW_t \\
&= W_t\exp(V_t)(dV_t + \frac{1}{2}(W_t^4 + U_t^2)dt) + \exp(V_t)dW_t + \exp(V_t)U_t dt \\
&= W_t\exp(V_t)(W_t^2 d\tilde{W}_t + U_t dW_t + \frac{1}{2}(W_t^4 + U_t^2)dt) + \exp(V_t)dW_t + \exp(V_t)U_t dt \\
&= \exp(V_t)W_t^3 d\tilde{W}_t + \exp(V_t)\left(W_t U_t + 1\right)dW_t + \exp(V_t)\left(\frac{1}{2}W_t^5 + \frac{1}{2}W_t U_t^2 + U_t\right)dt
\end{aligned}$$

at 0 the value is 0
e. Ito Forumla:

$$cos(W_t) = cos(0) - \underbrace{\int_0^t \sin(W_s)dW_s}_{\text{martingale}} - \frac{1}{2}\int_0^t cos(W_s)ds$$

Taking the expectation on both sides, and swapping E and $\int$ because cos is bounded:

$$E[cos(W_t)] = 1 - \frac{1}{2}\int_0^t E[cos(W_s)]ds$$

So $t \to E[cos(W_t)]$ is solution the differential equation: $f = 1 - \frac{1}{2}\int_0^t f$ Since the solution is unique($e^{-\frac{s}{2}}$):

$$\log(E[cos(W_t)]) = -\frac{t}{2}$$

$$\frac{\partial}{\partial t}[cos(W_t)] = -\frac{1}{2}$$