ORF526 - Problem Set 8

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Question 1

X_n is a martingale because

- It is adapted to $\mathcal{F}_n$, and L_1 by definition of conditional expectation.
- $E[X_{n+1}|\mathcal{F}_n] = E[E[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = E[X|F_n] = X_n$ because $F_n \subseteq F_{n+1}$

X_n are ui because:

- By Jensen inequality: $E[|X_n|] = E[|E[X|\mathcal{F}_n]|] \leq E[E[|X||\mathcal{F}_n]] \leq E[|X|] < \infty$
- For $\epsilon > 0$, Since $X \in L_1$, there exist $\delta > 0$ st $\forall A \in \mathcal{F}, P(A) < \delta \Rightarrow E[|X|1_A] < \epsilon$.
  Let c be large enough so that $\frac{E[|X|]}{c} \leq \delta$.

$$\begin{aligned}
E[|X_n|1_{|X_n|>c]}] &\leq E[E[|X||F_n]1_{|X_n|>c]}] \\
&\leq E[E[E[|X|1_{|X_n|>c]}|F_n]] && \text{because } 1_{|X_n|>c]} \text{ is } F_n \text{ measurable} \\
&\leq E[|X|1_{|X_n|>c]}] \\
&\leq \epsilon && \text{because } P(|X_n|>c) \leq \frac{E[|X_n|]}{c} \leq \frac{E[|X|]}{c} \leq \delta
\end{aligned}$$

Question 2

F_τ is a σ-algebra because: (n denotes a natural number)

- $\{\tau = n\} \cap \emptyset = \emptyset \in F_n$ so $\emptyset \in F_\tau$, and $F_\tau \neq \emptyset$
- Let $(A_i)_{i\in\mathbb{N}} \in F_\tau^{\mathbb{N}}$, $(\cup_{i\in\mathbb{N}} A_i) \cap \{\tau = n\} = \cup_{i\in\mathbb{N}}(\overbrace{A_i \cap \{\tau = n\}}^{\in F_n}) \in F_n$, so $\cup_{i\in\mathbb{N}} A_i \in F_\tau$
- Let $A \in F_\tau$, then $(A^c \cap \{\tau = n\})^c = A \cup \{\tau \neq n\} = (\overbrace{A \cap \{\tau = n\}}^{\in F_n}) \cup \overbrace{\{\tau \neq n\}}^{\in F_n} \in \mathcal{F}_n$, so $A^c \cap \{\tau = n\} \in F_n$ so $A^c \in F_\tau$

Question 3

Lemma 1. *if τ a stopping time adapted to (F_n), then for $k \leq n$, $\{\tau = k\} \in F_n$ and $\{\tau \leq n\} \in F_n$.*

proof: F_n is increasing and $\{\tau \leq n\} = \cup_{k=0..n} \overbrace{\{\tau = k\}}^{\in F_n} \in F_n$

Lemma 2. *if $A \cap \{\tau \leq n\} \in F_n \forall n$ then $A \in F_\tau$*

proof: $A \cap \{\tau = n\} = A \cap \{\tau \leq n\} \cap \{\tau \leq n-1\}^c \in F_n$ for all n, so $A \in F_\tau$
n is an arbitrary natural number:

a) $\{\tau+\sigma=n\} = \cup_{k=0..n}(\overbrace{\{\tau=k\}}^{\in F_n}\cap\overbrace{\{\sigma=n-k\}}^{\in F_n}) \in F_n$, so $\tau+\sigma$ is a stopping time.

b) $\{\tau\vee\sigma\leq n\} = \{\tau\leq n\}\cup\{\sigma\leq n\}\in F_n$, by lemma 2 $\tau\vee\sigma$ is a stopping time.

c) $\{\tau>k\}\in F_k$ because $\{\tau>k\}=\{\tau\leq k\}^c\in F_k$. Same for $\{\sigma>k\}$
$\{\tau\wedge\sigma>k\}=\{\tau>k\}\cap\{\sigma>k\}\in F_k$
But $\{\tau\wedge\sigma\leq n\}=\{\tau\wedge\sigma>n\}^c\in F_n$, so by lemma 2 $\tau\wedge\sigma$ is a stopping time.

d) $\Rightarrow$ Let $A\in F_\tau\cap F_\sigma$, then $A\cap\{\tau\leq n\}$ and $A\cap\{\sigma\leq n\}$ are in F_n, so is their intersection $A\cap\{\tau\wedge\sigma\leq n\}$.
c/c: $F_\tau\cap F_\sigma\subset F_{\tau\wedge\sigma}$

$\Leftarrow$ Let $A\in F_{\tau\wedge\sigma}$, then $A\cap\{\tau\wedge\sigma=n\}\in F_n$

$$\begin{aligned}
A\cap\{\tau=n\} &= (\cup_{k\leq n}A\cap\{\tau=n,\sigma=k\})\bigcup(\cup_{k>n}A\cap\{\tau=n,\sigma=k\})\\
&= (\cup_{k\leq n}A\cap\{\tau=n,\tau\wedge\sigma=k\})\bigcup(\cup_{k>n}A\cap\{\sigma=k,\tau\wedge\sigma=n\})\\
&= \left(\overbrace{A\cap\{\tau=n\}}^{\in F_n}\cap\overbrace{\{\tau\wedge\sigma\leq n\}}^{\in F_n}\right)\bigcup\left(\overbrace{A\cap\{\tau\wedge\sigma=n\}}^{\in F_n}\cap\overbrace{\{\sigma\leq n\}^c}^{\in F_n}\right)\\
&\in F_n
\end{aligned}$$

c/c: $F_{\tau\wedge\sigma}\subset F_\tau\cap F_\sigma$

As a conclusion $F_{\tau\wedge\sigma}=F_\tau\cap F_\sigma$

Question 4

Let M_n be such a martingale, then

- $\forall n$, $M_n\in L_1$ and is G_n adapated trivially.
- Since $E[M_{n+1}|F_n]=M_n$, we have that $E[E[M_{n+1}|F_n]|G_n]=E[M_n|G_n]=M_n$, so that $E[M_{n+1}|G_n]=M_n$ because $G_n\subset F_n$.

Question 5

a) Let $N\in mathbbN^*$, we have that

$$\cup_{n=1..N}\{Y_{(A+B)n+k}=1,k=1..(A+B)\}\subset\{\tau<\infty\}$$

$$\begin{aligned}
P(\tau<\infty) &\geq P(\cup_{n=1..N}\{Y_{(A+B)n+k}=1,k=1..(A+B)\})\\
&= 1-P(\cap_n\{Y_{(A+B)n+k}=1,k=1..(A+B)\})^c)\\
&= 1-\prod_{n=1..N}P(\{Y_{(A+B)n+k}=1,k=1..(A+B)\})^c) && \text{independence}\\
&= 1-\prod_{n=1..N}(1-P(\{Y_{(A+B)n+k}=1,k=1..(A+B)\})) && \text{independence}\\
&= 1-\prod_{n=1..N}(1-p^{A+B})\\
&= 1-(1-p^{A+B})^N\to 0 && \text{because } 0<p<1
\end{aligned}$$

so $\tau<\infty$ a.s.

b)

$$E[X_N|G_{N-1}] - X_{N-1} = X_N - X_{N-1} = Y_N \neq 0$$

So (X_n) is not martingale with respect to (G_n).