Problem 1

1. Let's consider $g : u \to \log(1 + e^u)$. g is non-decreasing and convexe because $g'(u) = \frac{e^u}{1+e^u} = \frac{1}{1+e^{-u}}$ is increasing.

   We notice that $f(x_1, x_2) = g(x_1 - x_2) + x_2$.

   - $x_2 \to x_2$ is linear
   - $x_2 \to x_1 - x_2$ is linear, g convexe and non-decreasing, so $g(x_1 - x_2)$ is convexe

   c/c: f is convexe.

2. The following transformation is a bijection from $(2, 3) \times (0, \infty) \times (0, \infty)$ to $(\frac{\log 2}{2}, \frac{\log 3}{2}) \times \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} x_1 &= 2 \log x \\ x_2 &= \log y - \log z \\ x_3 &= \log y \end{aligned}$$

   $\frac{x}{y} = z^2 = e^{2 \log z} = e^{2 \log y - 2x_2} = e^{2x_3 - 2x_2}$ Minimizing $\frac{x}{y}$ is the same as minimizing $a(x_1, x_2, x_3) := e^{2x_3 - 2x_2}$ which is convexe as the composition of a linear function and a convexe and increasing one exp.

   - $\frac{x}{y} = z \iff \log x - \log y = \log z \iff \frac{1}{2}x_1 - x_3 = x_3 - x_2 \iff \frac{1}{2}x_1 + x_2 - 2x_3 = 0$ and $b(x_1, x_2, x_3) := \frac{1}{2}x_1 + x_2 - 2x_3$ is linear.
   - $x^2 + \frac{y}{z} \leq \sqrt{y} \iff e^{x_1} + e^{x_2} \leq \sqrt{e^{x_3}} \iff \log(e^{x_1} + e^{x_2}) \leq \frac{1}{2}x_3 \iff f(x_1, x_2) - \frac{1}{2}x_3 \leq 0$ and $c(x_1, x_2, x_2) := f(x_1, x_2) - \frac{1}{2}x_3$ is convexe as the sum of two convexe functions

   c/c: the optimization problem is equivalent to:

$$\max a(x_1, x_2, x_2) \text{ s.t. } b(x_1, x_2, x_2) = 0, c(x_1, x_2, x_2) \leq 0, (x_1, x_2, x_2) \in (\frac{\log 2}{2}, \frac{\log 3}{2}) \times \mathbb{R} \times \mathbb{R}$$

   which is a convexe problem.

Problem 2

$\Rightarrow$) Let's suppose f convexe.

$$\begin{aligned} \nabla f^T(x)(y - x) &= \lim_{\alpha 0} \frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \\ &= \lim_{\alpha 0} \frac{f((1 - \alpha)x + \alpha y) - f(x)}{\alpha} \\ &\leq \lim_{\alpha 0} \frac{(1 - \alpha)f(x) + \alpha f(y) - f(x)}{\alpha} && \text{(because } f \text{ convexe)} \\ &\leq f(x) - f(y) \end{aligned}$$

$\Leftarrow$) Let's suppose $\forall x, y\, \nabla f^T(y - x) \leq f(y) - f(x)$

Let $\alpha \in (0, 1)$, and $u = (1 - \alpha)x + \alpha y$

$$\begin{aligned} f(x) - f(u) &\geq \nabla f(u)(x - u) \\ f(y) - f(u) &\geq \nabla f(u)(y - u) \end{aligned}$$

By multiplying the first ineqaulity by $1-\alpha$ and the second one by α and summing, we get: $(1-\alpha)f(x) + \alpha f(y) - f(u) \geq 0$

Which proves that f convexe.

Problem 3

1. D being definite positive, It can be written as a diagonal matrix in an orthonormal basis. Since rotations are isometries, witout loss of generality we can assume that $D = \text{diag}(d_1, ...d_n)$ is diagonal in the canonical basis. Let's call λ the biggest of the eigen values of D, and β the smallest.

   let's define the norm $||u||_D^2 := u^T D^{-1} u = ||\sqrt{D^{-1}}u||_2^2$ and the associated scalar product $< .,. >_D$ We have that: $\frac{||u||_2^2}{\lambda} \leq ||u||_D^2 \leq \frac{||u||_2^2}{\beta}$

   We assume that the projection $[.]^+$ is done with respect to the norm $|| \, ||_D$ rather than the euclidian norm. Let $y_{k+1} := x_k - \alpha D \nabla f(x_k)$ such that $x_k = [y_k]^+$

$$
\begin{aligned}
f(x_k) - f(x^*) &\leq \nabla f(x_k)(x_k - x^*) \\
&= \frac{1}{\alpha}(D^{-1}(x_k - y_{k+1}))^T(x_k - x^*) \\
&= \frac{1}{2\alpha}(||x_k - y_{k+1}||_D^2 + ||x_k - x^*||_D^2 - ||y_{k+1} - x^*||_D^2)
\end{aligned}
$$

   Since $\nabla^2 f$ is uniformly bounded by $||H||$, we have that ∇f is L-Lipschiz. We assume that

$$\nabla f$$

   is uniformly bounded by a constant L.

   And By non expansiveness of the projection $[.]^+$: $||y_{k+1} - x^*||_D \geq ||x_{k+1} - x^*||$ So:

$$f(x_k) - f(x^*) \leq \frac{1}{2\alpha}(\alpha^2 L + ||x_k - x^*||_D^2 - ||x_{k+1} - x^*||_D^2)$$

   By summing over k:

$$\sum_{i \leq k}(f(x_i) - f(x^*)) \leq \frac{k}{2}\alpha L + \frac{1}{2\alpha}||x_0 - x^*||_D^2$$

   Let's show that $f(x_i)$ is non-increasing. Indeed, We have that

   - By non expansiveness of the projection:

$$\nabla f(x_k)(x_{k+1} - x_k) = -\frac{1}{\alpha}(y_{k+1} - x_k)' D^{-1}(x_{k+1} - x_k) = -\frac{1}{\alpha} < y_{k+1} - x_k, x_{k+1} - x_k >_D \leq -\frac{1}{\alpha}||x_{k+1} - x_k||_D^2$$

   - By Cauchy Schwartz and the non expansiveness of the projection:

$$
\begin{aligned}
||x_{k+1} - x_k||_2^2 &\leq \lambda ||x_{k+1} - x_k||_D^2 \\
&\leq \lambda ||y_{k+1} - x_k||_D^2 \\
&\leq \alpha^2 \lambda \nabla f(x_k)' D \nabla f(x_k) \\
&\leq \alpha^2 \lambda^2 ||\nabla f(x_k)||_2^2 = \alpha^2 \lambda^2 L^2
\end{aligned}
$$

   - Taylor approximation:

$$
\begin{aligned}
f(x_{k+1}) - f(x_k) &\leq \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}||H||^2 ||x_{k+1} - x_k||_2^2 \\
&\leq -\frac{1}{\alpha}||x_{k+1} - x_k||_D^2 + \alpha^2 \frac{\lambda^2 L^2 ||H||^2}{2}
\end{aligned}
$$

   Which is smaller than 0 when α is small enough. Therefore: $f(x_k) - f(x^*) \leq \frac{1}{k}\sum_{i \leq k}(f(x_i) - f(x^*)) \leq \frac{\alpha L}{2} + \frac{1}{2\alpha k}||x_0 - x^*||_D^2$

As a result:

$$f(x_k) - f(x^*) = \min_{i \leq k} f(x_i) - f(x^*) \leq \frac{1}{k} \sum_{i \leq k} (f(x_i) - f(x^*)) \leq \frac{\alpha L}{2} + \frac{||x_0 - x^*||_D^2}{2\alpha k}$$

2. Let $g : \alpha \to \frac{\alpha L}{2} + \frac{||x_0 - x^*||_D^2}{2\alpha k} := a\alpha + \frac{b}{\alpha}$ so that $g'(\alpha) = a - \frac{b}{\alpha^2}$, $g''(\alpha) = \frac{2b}{\alpha^3} > 0$
   g is convexe, so it is minimal when $g'(\alpha) = 0$, ie
   $\alpha = \sqrt{\frac{b}{a}} = \sqrt{\frac{||x_0 - x^*||_D^2}{Lk}}$, $\min g = g(\alpha) = 2\sqrt{\frac{b}{a}} = 2\sqrt{\frac{||x_0 - x^*||_D^2}{Lk}}$
   Therefore the optimal bound is:

$$f(x_k) - f(x^*) \leq \sqrt{\frac{||x_0 - x^*||_D^2}{Lk}} = O(k^{-\frac{1}{2}})$$

3. Since ∇f is Lipshiz: $||\nabla f(x_k) - \nabla f(x^*)||_2 \leq ||H||||x_k - x^*||_2^2$ We will also need the inequality: $||Du||_D \leq \frac{||Du||_2}{\beta} \leq \frac{\lambda}{\beta}||u||_2$ And the fact that $x^* = [x^* - \alpha D\nabla f(x^*)]^+$

$$\begin{aligned}
||x^{k+1} - x^*||_D^2 &\leq ||[x_k - \alpha D\nabla f(x_k)]^+ - [x^* - \alpha D\nabla f(x^*)]^+||_D^2 \\
&\leq ||x_k - x^* - \alpha D(\nabla f(x_k) - \nabla f(x^*))||_D^2 \\
&\leq ||x_k - x^*||_D^2 - 2\alpha < D(\nabla f(x_k) - \nabla f(x^*)), (x_k - x^*) >_D + \alpha^2||D(\nabla f(x_k) - \nabla f(x^*))||_D^2 \\
&\leq ||x_k - x^*||_D^2 - 2\alpha(\nabla f(x_k) - \nabla f(x^*))'(x_k - x^*) + \alpha^2\lambda^2||\nabla f(x_k) - \nabla f(x^*)||_2^2 \\
&\leq ||x_k - x^*||_D^2 - 2\alpha\sigma||x_k - x^*||_2^2 + \alpha^2\frac{\lambda^2}{\beta^2}||H||||x_k - x^*||_2^2 \quad \text{By s} \\
&\leq ||x_k - x^*||_D^2 - 2\alpha\frac{\sigma}{\beta^2}||x_k - x^*||_D^2 + \alpha^2\frac{\lambda^2}{\beta^4}||H||||x_k - x^*||_D^2 \quad \text{By s} \\
&\leq (1 - 2\frac{\sigma}{\beta^2}\alpha + \frac{||H||\lambda^2}{\beta^4}\alpha^2)||x_k - x^*||_D^2 \\
&\leq \rho||x_k - x^*||_D^2 \\
&(\rho = 1 - 2\frac{\sigma}{\beta^2}\alpha + \frac{||H||\lambda^2}{\beta^4}\alpha^2) \\
&\leq \rho^{k+1}||x_0 - x^*||_D^2 \\
&\text{By reccurence}
\end{aligned}$$

4. ρ is quadratic in α, so it is minimal when $\frac{\partial\rho}{\partial\alpha} = 0$, ie $\alpha = \frac{\beta^2\sigma}{||H||\lambda^2}$, $\min\rho = 1 - \frac{\sigma^2}{||H||\lambda^2}$

Problem 4

1. For $y \in \mathbb{R}^n$ $\sup_y L(x, y) \geq L(x, u)$
   By taking the $\inf_x$: $\inf_x \sup_y L(x, y) \geq \inf_x L(x, u)$
   By taking the $\sup_u$: $\inf_x \sup_y L(x, y) \geq \sup_u \inf_x L(x, u)$
2. Let $f(x) := \max_y L(x, y)$ We know that $x^* \in \arg\min f$ and f is convexe, so $\partial f(x^*) = 0$.
   L is continuous the Danskin's theorem, we have that: $0 \in \{\nabla_x L(x^*, y) | y \in \arg\min L(x^*, y^*)\} = \{\nabla_x L(x^*, y^*)\}$, wich means that $\nabla_x L(x^*, y^*) = 0$, and symmetrically, $\nabla_y L(x^*, y^*) = 0$.
   c/c:

$$\begin{aligned}
x^* &= x^* - \alpha\nabla_x L(x^*, y^*) \\
y^* &= y^* - \alpha\nabla_y L(x^*, y^*)
\end{aligned}$$

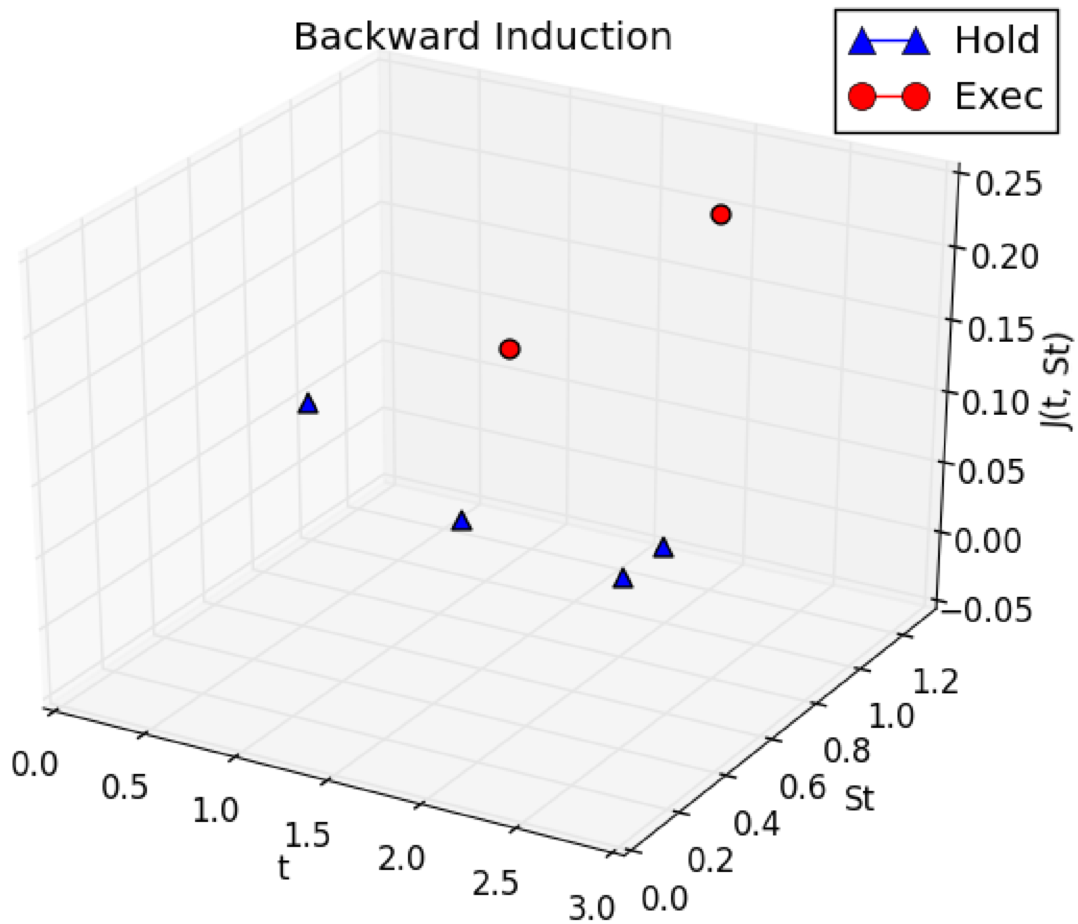
Problem 5

1. Let's call S_t the price of the stock at time t, and $V_t(S_t)$ the price of the corresponding american action (with strike K)

   - State $x_t = S_t$, $t = 1..T$
   - Action:
   $$u_t = \begin{cases} \text{EXEC} & \text{meaning we exercise the option} \\ \text{HOLD} & \text{meaning we don't} \end{cases}$$
   - Randomness: The change in the stock price $w_t = \frac{S_{t+1}}{S_t}$ s.t $x_{t+1} = w_t x_t$, $P(w_t = u) = 1 - P(w_t = d) = p$
   - Transitional cost:
   $g(x_k, u_k = \text{HOLD}, w_t) = 0$ $g(x_k, u_k = \text{EXEC}, w_t) = x_t - K$ $g(x_T) = (x_T - K)^+$: We exericse the option at time T only if $x_T > K$

   The price problem:
   $$V_k(x) = \max_{\mu} E[g(x_T) + \sum_{t=k}^{T-1} g(x_k, \mu_k(x_k), w_t) | x_k = x]$$

2. Bellman equation:
   $$V_k(x) = \max \{x - K, pV_{k+1}(ux) + (1-p)V_{k+1}(dx)\}$$
   $$V_T(x) = (x - K)^+$$
   Result for $K = S_0 = 1., P(w_t = \text{down}) = 0.45, \text{up} = 1.1, \text{down} = 0.9, T = 100 \text{ days}, \text{time step} = 10$:



3. LP: Let $J(t,S)$ be the price of the option at time t is $S_t = S$, and we decide to adopt the strategy J verifies: $J(t,S) = \max\{E[J(t+1,S_{t+1})|S_t = S], S-K\} = [\max_\mu(P_\mu J + g_\mu)](t,S)$ where:

$$\mu(t,S) \in \{\text{HOLD}, \text{EXEC}\}$$

$$(P_\mu J)(t,S) = \begin{cases} pJ(t+1,uS) + (1-p)J(t+1,dS) & \text{if } \mu(t,S) = \text{HOLD} \\ 0 & \text{otherwise} \end{cases}$$

$$g_\mu(t,S) = \begin{cases} 0 & \text{if } \mu(t,S) = \text{HOLD} \\ S-K & \text{otherwise} \end{cases}$$

The LP problem is:

$$\min e^T J \text{ s.t } \forall \mu\; J \geq P_\mu J + g_\mu$$

At time t, S can take the following values $\{u^k d^{t-k} S_0,\, k \leq t\}$. Let's denote by $\tilde{J}(t,k) := J(t, u^k d^{t-k} S_0)$

when $k \leq t$ and L otherewise where $L >> S_0$ is a very big constant. The problem can be written as:

$$\begin{aligned} &\min \sum_{t,k} \tilde{J}(t,k) \\ &\quad \text{s.t } \forall t, k \in \{1...T-1\} \\ &\quad \tilde{J}(t,k) \geq p\tilde{J}(t+1,k) + (1-p)\tilde{J}(t+1,k+1) \\ &\quad \tilde{J}(t,k) \geq u^k d^{t-k} S_0 - K \\ &\quad \tilde{J}(T,k) = u^k d^{t-k} S_0 - K \\ &\quad \tilde{J}(t,k) = L \text{ when } k > t \end{aligned}$$

Let $x_{t*T+k} = \tilde{J}(t,k)$, and define $A \in \mathcal{M}_{T^2,T^2}$, $B \in \mathcal{M}_{\frac{T(T-1)}{2},T^2}$ $U \in \mathbb{R}^{T^2}$ such that:

$$A := \begin{array}{r} \text{T(T-1)} \left\{ \vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}} \right. \\ \text{T} \left\{ \vphantom{\begin{matrix} 0 \\ \cdots \end{matrix}} \right. \end{array} \begin{bmatrix} 0 & p & (1-p) & 0 & \cdots \\ \vdots & & \ddots & \ddots & \cdots \\ 0 & & \cdots & p & (1-p) \\ 0 & 0 & 0 & 0 & 0 \\ & \cdots & \cdots & & \end{bmatrix}$$

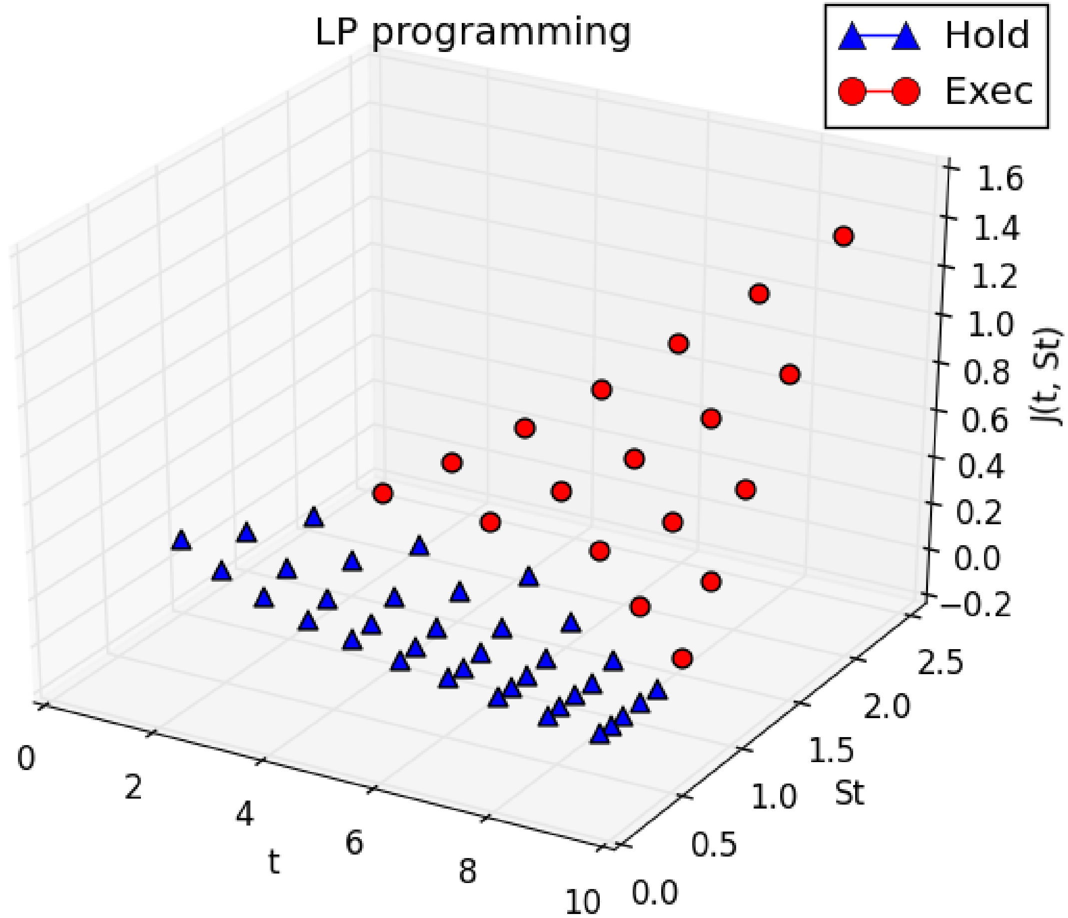
$$B := \begin{array}{r} \text{T-1} \left\{ \vphantom{\begin{matrix} 0 \\ \ddots \\ 1 \end{matrix}} \right. \\ \text{T-2} \left\{ \vphantom{\begin{matrix} 0 \\ \ddots \\ 1 \end{matrix}} \right. \\ \vdots \left\{ \vphantom{\ddots} \right. \\ \text{T - (T-1)} \left\{ \vphantom{0} \right. \end{array} \left[\begin{array}{c|c|c|c} \begin{matrix} 0 & 1 & & \cdots \\ & & \ddots & \\ & & & 1 \end{matrix} & & & \\ \hline & \begin{matrix} 0 & 0 & 1 & \cdots \\ & & \ddots & \\ & & & 1 \end{matrix} & & \\ \hline & & \ddots & \\ \hline & & & \begin{matrix} 0 & \cdots & 1 \end{matrix} \end{array} \right]$$

$U \in R^{T^2}$ such that : $U_{tT+k} := u^k d^{t-k} S_0 - K$

The LP problem is equivalent to:

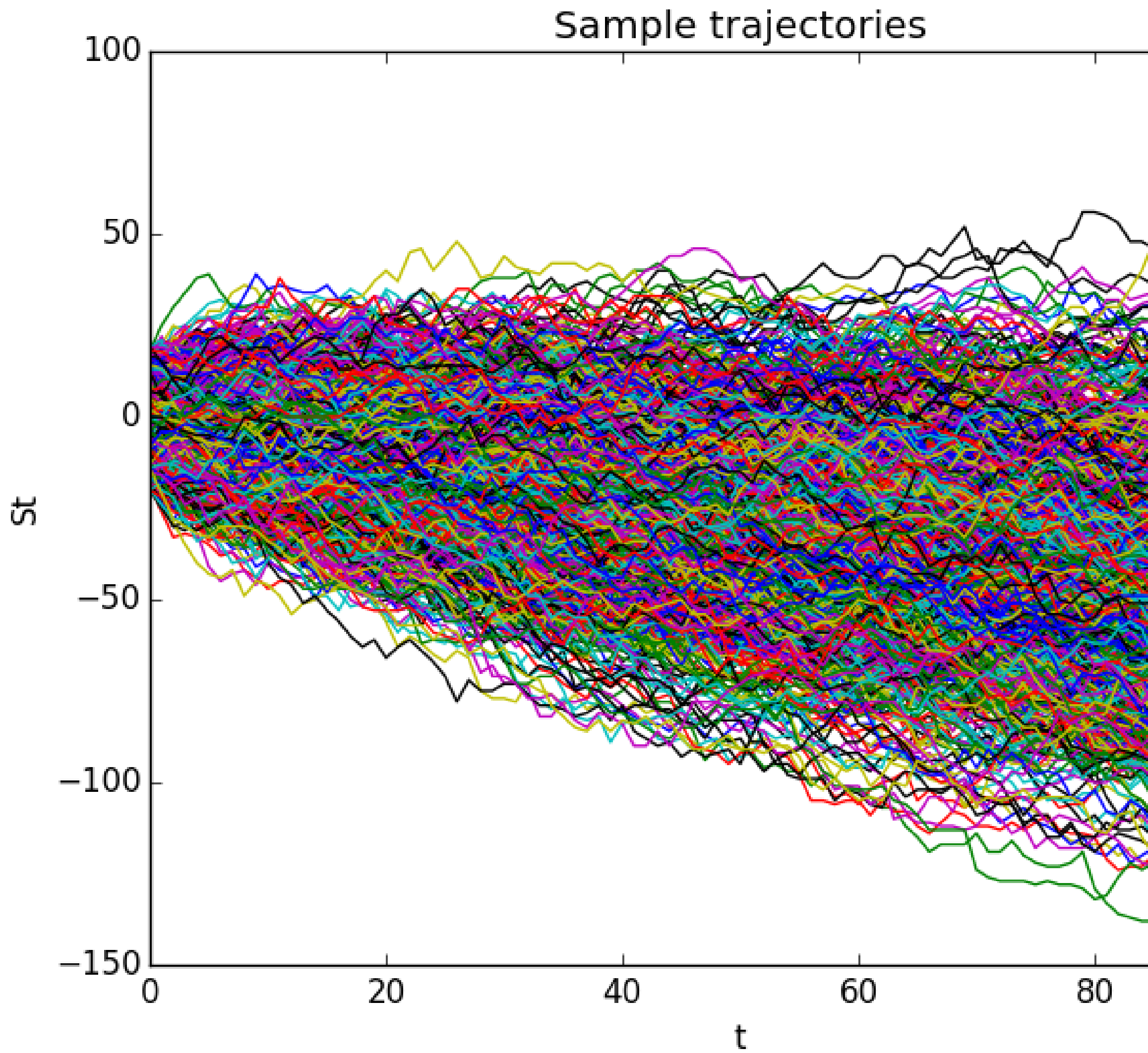
$$\begin{aligned} &\min e^T x \\ &\quad \text{s.t} \\ &\quad x \geq Ax \\ &\quad x \geq U \\ &\quad Bx = L1_{\frac{T(T-1)}{2}} \end{aligned}$$

Result for the same parameters as before:



Problem 6

- We choose a discrete random walk model for generating S_t: $S_t = Y + \sum_{i \le t} X_i$ where the X_i are iid $\mathcal{N}(\mu, sigma)$ and Y uniform on $\{-10, ..., 10\}$ and independent from the X_i



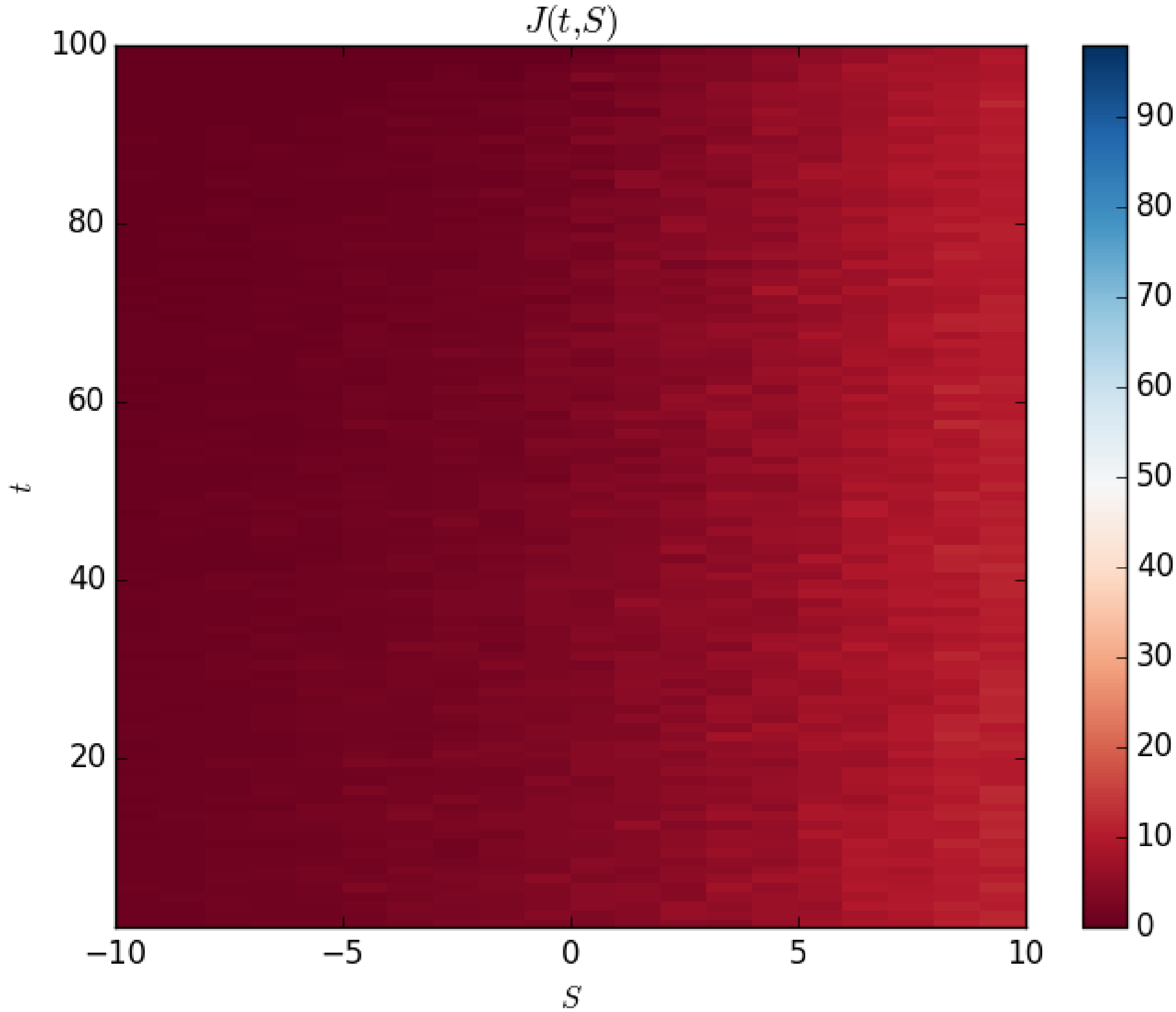
Sample trajectories:

- By generating the paths $(S_t^{(i)})_i$, we simulate the transitions $(t, S_t) \to (t+1, S_{t+1})$

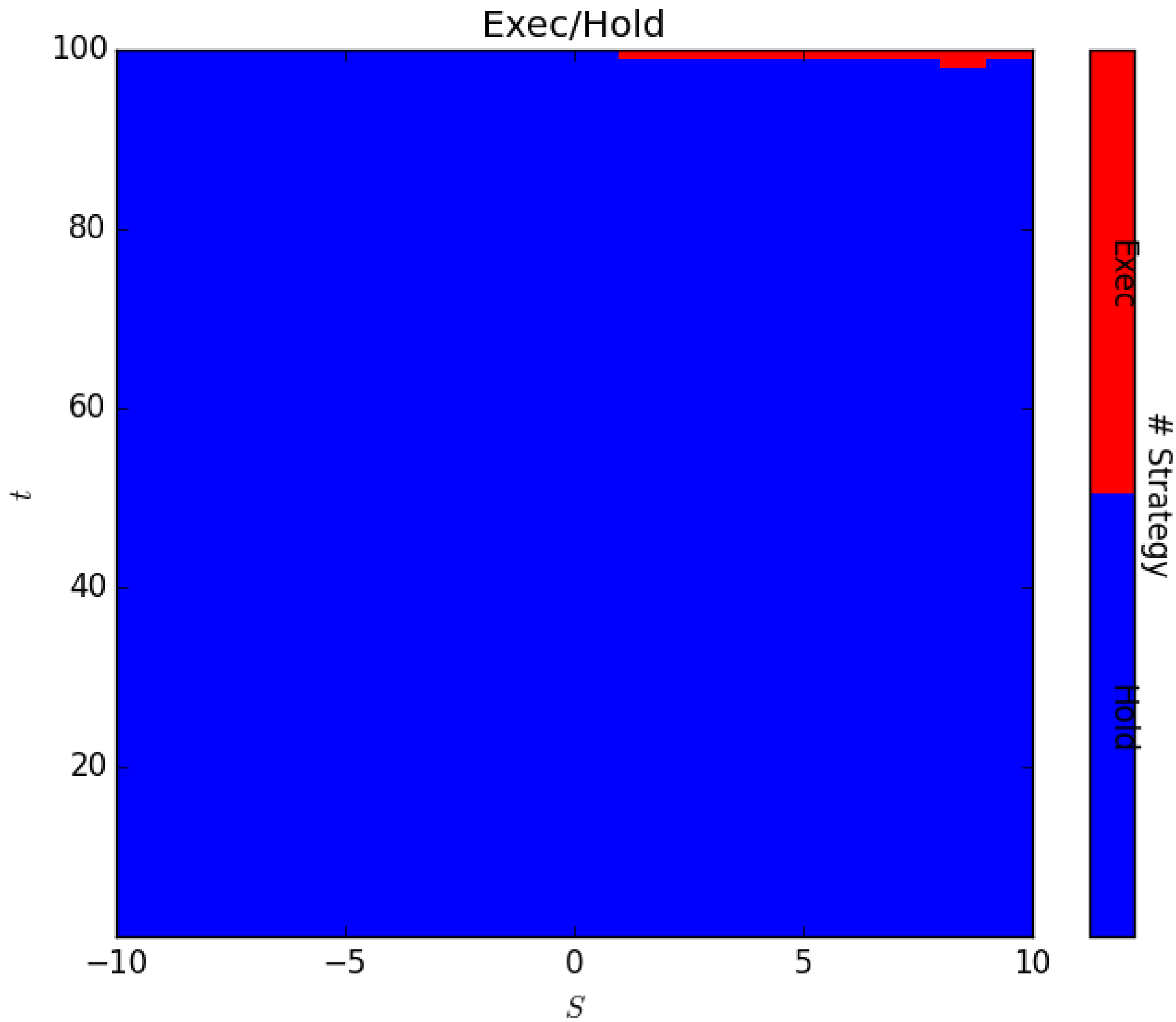
  We adopt the same notation as in lecture 23 slide 36. At step $i \in \{0, ..., M-1\}$:

$$Q_{i+1}((t, S_t)) = (1-\gamma)Q_i((t, S_t)) + \gamma \max(S_{t+1} - K, Q_i((t+1, S_{t+1}))$$

$$J((t, S_t)) = \max\{S_t - K, Q_M((t, S_t))\}$$



$J(t,S)$
100
80
60
40
20
t
−10
−5
0
5
10
S
90
80
70
60
50
40
30
20
10
0


Exec/Hold
100
80
60
40
20
t
−10
−5
0
5
10
S
Exec
Hold
Strategy