Problem set 6, ORF527

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1 Q1 (8.2 in Steele)

$f(t,x) = h(t)x \in C^{1,2}(R^+ \times R)$, $f_t = h'(x), f_x = h(t), f_{xx} = 0$

So by Ito formula:

$$d(h(t)B_t) = h'(t)B_t dt + h(t)dB_t$$

e.g

$$\int_0^t h(s)dB_s = h(t)B_t - \int_0^t h'(s)B_s ds$$

2 Q2

a.

$f \in C^\infty$, $f_t = (\mu - \sigma^2/2)f$, $f_x = \sigma f$, $f_x x = \sigma^2 f$, By ito formula:
$dX_t = df(t, W_t) = (f_t + \frac{1}{2}f_x x)dt + f_x dW_t = \mu X_t dt + \sigma X_t dW_t$
$dX_t dX_t = \sigma^2 X_t^2 dt$
$\log' = \frac{1}{x}, \log'' = -\frac{1}{x^2}$, so:
$d\log(X_t) = \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t}\sigma^2 X_t^2 dt = (\mu - \sigma^2/2)dt + \sigma dW_t$

b.

$\tau_n = \inf\{t \geq 0, X_t \leq \frac{1}{n}\}$
τ_n is non-decreasing, and since $X_t > 0$, $\tau_n \to \infty$ (otherwise $X_{\tau_n} \to X_{\lim \tau_n} = 0$)
Let $\phi \in C^\infty(\mathbb{R}^+)$ such that:

$$\phi(x) = \begin{matrix} 1 & \text{when } x \geq 1 \\ 0 & \text{when } x \leq \frac{1}{2} \end{matrix}$$

(The construction has been done in class)

Let $f^n(t,x) = f(t,x)\phi(nx) \in C^{1,2}(\mathbb{R}^+ \times R^+)$. f^n is equal to f when $x \in [\frac{1}{n}, \infty)$.
Then $\forall t \geq 0$, $f(t \wedge \tau_n, X_{t\wedge\tau_n}) = f^n(t \wedge \tau_n, X_{t\wedge\tau_n})$
Let n be large enough so that $\frac{1}{n} < X_0$. Ito formula applied to f^n:
$f^n(t, X_t) = f^n(0, X_0) + \int_0^t (f_t^n + \frac{[X]_s}{2} f_{xx}^n)(s, X_s)ds + \int_0^t (f_t^n + \frac{[X]_s}{2} f_{xx}^n)(s, X_s)ds$
So:

$$\begin{aligned} f(t \wedge \tau_n, X_{t\wedge\tau_n}) &= f^n(t \wedge \tau_n, X_{t\wedge\tau_n}) \\ &= f(0, X_0) + \int_0^{t\wedge\tau_n} (f_t^n + \frac{[X]_s}{2} f_{xx}^n)(s, X_s)ds + \int_0^{t\wedge\tau_n} f_x^n(s, X_s)dX_s \\ &= f(0, X_0) + \int_0^{t\wedge\tau_n} (f_t + \frac{[X]_s}{2} f_{xx})(s, X_s)ds + \int_0^{t\wedge\tau_n} f_x(s, X_s)dX_s \end{aligned}$$

$|f(s) - f^n(s)1_{s\leq\tau_n}| \leq |f(s) - f^n(s)| + |1_{s\leq\tau_n} - 1||f^n(s)| \leq |f - f^n|_\infty + 2|1_{s\leq\tau_n} - 1||f|_\infty$
As a conclusion

$$f(t, X_t) = f(0, X_0) + \int_0^t (f_t + \frac{[X]_s}{2} f_{xx})(s, X_s)ds + \int_0^t f_x(s, X_s)dX_s$$

3 Q3

a.

$$QV[B,t] = \lim_{n\infty} \frac{t}{2^n} \sum_{k=1}^{2^n} |N_k|^2$$

Where $N_k \overset{iid}{\sim} \mathcal{N}(0,1)$
By law of large number $QV[B,t] \to tE[N_1^2] = t$
b.

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s$$

Using $(b-a)^2 = b^2 - a^2 - 2a(b-a)$

$$\begin{aligned}
QV[X,T] &= \lim_{n\infty} \sum_{k=1}^{2^n} (X_{k2^{-n}t} - X_{(k-1)2^{-n}t})^2 \\
&= \lim_{n\infty} \sum_{k=1}^{2^n} (X_{k2^{-n}t}^2 - X_{(k-1)2^{-n}t}^2) + \sum_{k=1}^{2^n} 2X_{(k-1)2^{-n}t}(X_{k2^{-n}t} - X_{(k-1)2^{-n}t}) \\
&= X_t^2 - X_0^2 + 2\lim_{n\infty} \sum_{k=1}^{2^n} X_{(k-1)2^{-n}t}(X_{k2^{-n}t} - X_{(k-1)2^{-n}t})
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{2^n} X_{(k-1)2^{-n}t}(X_{k2^{-n}t} - X_{(k-1)2^{-n}t}) &= \sum_{k=1}^{2^n} X_{(k-1)2^{-n}t} \int_{t_{k-1}}^{t_k} F_s ds + X_{(k-1)2^{-n}t} \int_{t_{k-1}}^{t_k} G_s dB_s \\
&= \int_0^t \tilde{X}_s^n F_s ds + \int_0^t \tilde{X}_s^n G_s dB_s
\end{aligned}$$

Where $\tilde{X}_t^n = \sum_{k=1}^{2^n} X_{t_k} 1_{t_k \le t \le t_{k+1}}$
$|\int_0^t \tilde{X}_s^n F_s ds - \int_0^t \tilde{X}_s F_s ds| \le \int_0^t |\tilde{X}_s^n - X_s||F_s| ds \le \underbrace{\sup_{|u-v|\le 2^{-n}} |X_u - X_v|}_{\to_n 0 (\text{uniform continuity})} |F|_{H_1}$

Let $\tau_m = \inf\{t, |G_t| \ge m \vee |X_t| \ge m\}$ be a localising sequence. Note that since G_t, X_t are finite a.s, $\tau_m \to \infty$ too.

$$\begin{aligned}
P(|\int_0^t \tilde{X}_s^n G_s dB_s - \int_0^t X_s G_s dB_s| > \varepsilon) &\le P(|\int_0^t \tilde{X}_s^n G_s dB_s - \int_0^t X_s G_s dB_s| > \varepsilon, t \le \tau_m) + P(t \ge \tau_m) \\
&\le P(|\int_0^{t\wedge\tau_m} \tilde{X}_s^n G_s dB_s - \int_0^{t\wedge\tau_m} X_s G_s dB_s| > \varepsilon) + P(t \ge \tau_m) \\
&\le \frac{1}{\varepsilon^2} E|\int_0^{t\wedge\tau_m} (\tilde{X}_s^n - X_s) G_s dB_s|^2 + P(t \ge \tau_m) \qquad \text{(markov)} \\
&\le \frac{m^2}{\varepsilon^2} \int_0^t E|1_{t\le\tau_m}(\tilde{X}_s^n - X_s)^2| ds + P(t \ge \tau_m) \\
&\le \frac{m^2 t}{\varepsilon^2} \int_0^t E 1_{t\le\tau_m}(\tilde{X}_s^n - X_s)^2 ds + P(t \ge \tau_m)
\end{aligned}$$

By continuity: $\forall s\ \tilde{X}_s^n \to X_s$, and $|1_{t\le\tau_m}(\tilde{X}_s^n - X_s)^2| \le m$, by dominated convergence theorem:

$$\int_0^t E 1_{t\le\tau_m}(\tilde{X}_s^n - X_s)^2 ds \to_n 0$$

so $\limsup_n P(|\int_0^t \tilde{X}_s^n G_s dB_s - \int_0^t X_s G_s dB_s| > \varepsilon) \leq P(t \geq \tau_m) \to_m 0$
We have just proved that

$$QV(X,t) \xrightarrow{\mathbb{P}} X_t^2 - X_0^2 - 2\int_0^2 X_s dX_s$$

c
By Ito formula
$d(X_t^2) = 2X_t dX_t + G_t^2 dt$
So $X_t^2 - X_0^2 - \int_0^t X_s dX_s = \int_0^t G_s^2 ds$
e.g $QV(X,t) = \int_0^t G_s^2 ds$.

4 Q4

Let's assume $X_0 + \int_0^t F_s ds + \sum_{k=1}^m \int_0^t G_s^k dW_s^k = \tilde{X}_0 + \int_0^t \tilde{F}_s ds + \sum_{k=1}^m \int_0^t \tilde{G}_s^k dW_s^k$
By setting t to 0, $X_0 = \tilde{X}_0$
By linearity: $\int_0^t (F_s - \tilde{F}_s) ds = \sum_{k=1}^m \int_0^t (\tilde{G}_s^k - G_s^k) dW_s^k$,
Taking the quadratic variation of both sides, we find that:

$$0 = \sum_{k=1}^m \int_0^t (\tilde{G}_s^k - G_s)^2 ds$$

Which proves $\forall T \geq t \geq 0$ $\int_0^t ||\tilde{G}_s - \tilde{G}_s||^2 ds = 0$, e.g $G_s \overset{H^2}{=} \tilde{G}_s$ and that $\sum_{k=1}^m \int_0^t (\tilde{G}_s^k - G_s^k) dW_s^k = 0$.
This leads to $\forall T \geq t \geq 0$ $\int_0^t F_s - \tilde{F}_s = 0$, by taking the derivative, $F_s = \tilde{F}_s$, therefore $\int_0^t |F_t - \tilde{F}_t| dt = 0$