ORF526 - Problem Set 1

Bachir EL KHADIR

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Question 1

Let X and Y be the result of two independent coin tosses, and let

$$A_1 = \{X = H\}$$
$$A_2 = \{Y = H\}$$
$$A_3 = \{X = Y\}$$

Question 2

$$\begin{aligned}
\mathbb{E}[X] &:= \sum_{n=1}^{N} X(\omega_n) p_n \\
&= \sum_{n=1}^{N} \left[\mathrm{Re}(X)(\omega_n) + i\mathrm{Im}(X)(\omega_n)\right] p_n \\
&= \sum_{n=1}^{N} \mathrm{Re}(X)(\omega_n) p_n + i \sum_{n=1}^{N} \mathrm{Im}(X)(\omega_n) p_n \\
&= \mathbb{E}[\mathrm{Re}(X)] + i\,\mathbb{E}[\mathrm{Im}(X)]
\end{aligned}$$

Question 3

$$(i) \Rightarrow (ii)$$

$$\begin{aligned}
\mathbb{E}[f_1(X_1)...f_M(X_M)] &= \sum_{x_1,...,x_M} f_1(x_1)...f_M(x_M)\mathbb{P}(X_1 = x_1, ..., X_M = x_m) \\
&= \sum_{x_1,...,x_M} f_1(x_1)...f_M(x_M)\mathbb{P}(X_1 = x_1)...\mathbb{P}(X_M = x_M) \qquad \text{(because of } (i)\text{)} \\
&= \sum_{x_1} f_1(x_1)\mathbb{P}(X_1 = x_1)... \sum_{x_M} f_M(x_M)\mathbb{P}(X_M = x_M) \\
&= \mathbb{E}[f_1(X_1)]...\mathbb{E}[f_M(X_M)]
\end{aligned}$$

$$(ii) \Rightarrow (iii)$$

Take $f_i(x) = e^{iu_i x}$

$$(iii) \Rightarrow (i)$$

By linearity we prove that the equality holds for polynomials of complex exponentials of the random variables too.

Let $\{x_1, ..., x_n\}$ be the elements of Ω, and

$$\begin{aligned} f\ : \mathbb{C}^{n-1}[X] &\longrightarrow \mathbb{C}^n \\ P &\longrightarrow (P(e^{i\frac{x_i}{n}}))_i \end{aligned}$$

where n is large enough so that $e^{i\frac{x_i}{n}}$ are all different.

f is linear and injective (two polynomials of degree $< n$ who agree on n points are equal), it is then a bijection (because $dim(\mathbb{C}^{n-1}[X]) = dim(\mathbb{C}^n)$) As a consequence, for each indicator function of the form 1_{x_i} there exists a polynomial $P_i(e^{i\frac{u}{n}}) = 1_{u=x_i}$, ie (i) is verified.

Question 4

Immediate using definition (ii)

Question 5

a) if X and Y are independent, then

$cov(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}\left[f(X)g(Y)\right] = \mathbb{E}\left[f(X)\right]\mathbb{E}\left(Y\right)\right] = \mathbb{E}[X - \mathbb{E}[X]]\mathbb{E}[Y - \mathbb{E}[Y]] = 0$

b) Let X and ϵ be two independent uniform variables on $\{-1, 0, 1\}$ and $\{-1, 1\}$ respectively, then $cov(X, \epsilon X) = \mathbb{E}[\epsilon X^2] = \mathbb{E}[\epsilon]\mathbb{E}[X^2] = 0$, but $\mathbb{P}(X = 1, \epsilon X = 0) = 0 \neq \mathbb{P}(X = 1)\mathbb{P}(\epsilon X = 1) = \frac{1}{9}$

Question 6

A vector space $(V, +, ., \mathbb{K})$ over a field $\mathbb{K}$ verifies

For all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{K}$, then $u + v \in V$, $\lambda u \in V$ and

- $(V, +)$ is an Abelian group
- $\lambda(\mu u) = (\lambda\mu)u$.
- $(\lambda + \mu)u = \lambda u + \mu u$.
- $\lambda(u + v) = \lambda u + \lambda v$.
- $1u = u$.

examples: R^n, space of continuous functions from R to R, space of square matrices of dimension n^2 ...

Question 7

a) By using symetry, bilinearity and then symetry

b) When $y = 0$ it is trivial. When $y \neq 0$ and $\lambda = \frac{<x,y>}{||y||^2}$, $0 \leq < x - \lambda y, x - \lambda y > = \frac{||x||^2||y||^2 - <x,y>^2}{||y||^2}$

c)
 - Positive homogeneity is a result of Bilinearity.
 - Triangle inequality can be obtained by squaring both sides of the inequality and applying Cauchy-Shwartz.
 - Positive definiteness of the norm is a direct consequence of the Positive definiteness of the scalar product.

d) We can assume that X and Y are centred (adding a constant doesn't change the cov or var). Since the mapping $(X, Y) \longrightarrow cov(X, Y)$ is scalar product in the space of centered random variables on the finite probability space Ω, we then apply Cauchy-Shwarz.