ORF526 - Problem Set 9

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Question 1

Because it is predictable we have that $E[M_{n+1}|F_n] = M_{n+1}$, and because it is a martingale $E[M_{n+1}|F_n] = M_n$. Therefore M_n is constant equal to M_0.

Question 2

Let $M_n = X_0 + \sum_{i=0}^{n-1} X_{i+1} - E[X_{i+1}|F_i]$, $A_n = X_n - M_n = \sum_{i=0}^{n-1} E[X_{i+1}|F_i] - X_i$ and as a convention $A_0 = 0$.
Then

- (M_n) is an (F_n) martingale because
  - It is (F_n)-adapted: For all n, M_n for $i = 0...n-1$, X_{i+1} and $E[X_{i+1}|F_i]$ are F_n measurable.
  - $M_{n+1} - M_n = X_{n+1} - E[X_{n+1}|F_n]$, so

$$E[M_{n+1}|F_n] - M_n = E[M_{n+1} - M_n|F_n] = E[X_{n+1}|F_n] - E[E[X_{n+1}|F_n]|F_n] = 0$$

- (A_n) is a non-decreasing predictable process because:
  - (A_n) is predictable because for $i < n$, X_i and $E[X_{i+1}|F_i]$ are F_i (and F_{n-1}) measurable
  - (A_n) is non-decreasing: $A_{n+1} - A_n = E[X_{n+1}|F_n] - X_n \geq 0$ because X_n is a submartingale.

The decomposition is unique, because if there exist an other decomposition $X_n = M'_n + A'_n$ with the same properties then: $M_n - M'_n = A_n - A'_n$, which is a marintgale (as the difference of two martingales), and predictable (as the difference of two predictable processes). By question 1, this sequence is constant equal to $A_0 - A'_0 = 0$

Question 3

Let $n, p \in \mathbb{N}$,
By the itereated expectation:

$$E[M_{n+i+1}M_{n+i}] = E[E[M_{n+i+1}|F_n]M_{n+i}] = E[M_n^2]$$

So:

$$||M_{n+p} - M_n||_2^2 = E[|M_{n+p} - M_n|^2] = E[M_{n+p}^2] + E[M_n^2] - 2E[M_{n+p}M_n] = E[M_{n+p}^2] - E[M_n^2]$$

M_n^2 is a submartingale, so $E[M_n^2]$ is non-decreasing, and since it is bounded, it converges and therefore $E[M_{n+p}^2] - E[M_n^2] \to_{n,p} 0$ c/c: $||M_{n+p} - M_n||_2 \to_{n,p} 0$, and (M_n) is a cauchy sequence.

Question 4

1. M_n is L^p bounded and $p > 1$, so (M_n^p) is submartingale that is L_1 bounded, and therefore: $|M_n|^p$ converges to $S \in L_1$ a.s. Moreover, M_n is L_1 bounded so it has an as limit M_∞. Therefore $M_\infty^p = S \in L_1$, so $M_\infty \in L_p$

2. Let Ω the set of measure 1 where $S_n = \max_{k\leq n}|M_k|$ holds. Then for every $\omega \in \Omega$, $S_n(\omega)$ is pointwise non-decreasing as., so it has a limit $S(\omega)$, and by monotonuous convergence theorem $E[S_n^p] \to E[S^p]$ By Doobs inequality
$$||S_n||_p = ||\max_{k\leq n}|M_k|||_p \leq \frac{p}{p-1}||M_n||_p \leq \frac{p}{p-1}\sup_k ||M_k||_p$$
By taking the limit, and taking the p-th power:
$$E[S^p] = ||\max_{k\leq n}|M_k|||_p^p \leq \left(\frac{p}{p-1}\right)^p \sup ||M_k||_p^p$$

3. Since $|M_\infty - M_n| \leq |M_\infty| + |M_n|$, $|M_n| \leq S_n \leq S$ and $|M_\infty| = \lim_n |M_n| \leq \lim_n S_n \leq S$:
$$|M_\infty - M_n| \leq 2S$$
Therefore $|M_\infty - M_n|^p \leq 2^p S^p \in L_1$,and by Dominated Convergence theorem $E[|M_\infty - M_n|^p] \to 0$

Question 5

Let F_n be the filtration generated by B_n

1. B_n is F_n adapted, so is $B_n^2 - n$

2. B_n^2 is L_1, and
$$\begin{aligned}E[B_{n+1}^2 - (n+1)|B_n] &= E[(B_{n+1} - B_n + B_n)^2 - (n+1)]\\ &= E[(B_{n+1} - B_n)^2|B_n] + E[B_n^2|B_n] + 2E[(B_{n+1} - B_n)B_n|F_n] - (n+1)\\ &= E[\mathcal{N}(0,1)^2] + B_n^2 - (n+1)\\ &\text{(because } B_{n+1} - B_n \sim \mathcal{N}(0,1) \text{ and is independent from } B_n)\\ &= B_n^2 - n\end{aligned}$$
so $B_n^2 - n$ is a martingale.

3. $\exp(\sigma B_{n+1} - \frac{1}{2}\sigma(n+1)^2)$ is L_1 (using the moment generating function of the normal distribtion)
$Y_i = B_{i+1} - B_i \sim \mathcal{N}(0,1)$ and independent from F_i.
$E[\exp(\sigma B_{n+1} - \frac{1}{2}\sigma(n+1)^2)|F_n] = \exp(\sigma B_n - \frac{1}{2}\sigma(n+1)^2)E[\exp(\sigma Y_n)|F_n] = \exp(\sigma B_n - \frac{1}{2}\sigma(n+1)^2)\exp(\frac{1}{2}\sigma^2) = \exp(\sigma B_n - \frac{1}{2}\sigma n^2)$

4. By defintion of the borwnian motion (B_t), $B^{(m)}$ is a normal vector that has expectation 0 and covariance matrix:
$$cov(B_{i/m}^{(m)}, B_{j/m}^{(m}) = \frac{1}{m}cov(B_i, B_j) = \frac{1}{m}(i \wedge j) = (i/m \wedge j/m)$$
So $B^{(m)}$ has the same distribution as $(B_n)_{n=1..m}$
   - It's a martingale
   - The increments are independent
   - It is an discretisation of the brownian motion

Question 6

1.

$$\begin{aligned} a\log(b) \le alog(a) + \frac{b}{e} &\iff \log(b/a) \le \frac{b}{a}e^{-1} \\ &\iff \log(x) \le \frac{x}{e} && (x = \frac{b}{a}) \\ &\iff \log(x) \le \log'(e)(x-e) + \log(e) \\ &\text{(true because log is concave)} \end{aligned}$$

2.
- if $b \le 1$, then the inequality in equivalent to $0 \le alog^{+}(a) + b/e$ which is true because all the terms are non negative.
- if $b > 1$ and $a > 1$, the inequality in equivalent to the one of question 1.
- if $b > 1$ and $a \le 1$, by question 1, $a\log(b) \le 1\log(b) \le 1\log(1) + \frac{b}{e}$, or $a\log^{+}(n) \le \frac{b}{e}$ which is what we want to prove.

Question 7

1. It is clear that

   - $E[X|G]$ is $\sigma(G, H)$ measurable
   - $E[X|G]$ is L_1

   In addition, by linearity of the conditional probability, we only need to show that it holds for non negative rv. Let's assume $X \ge 0$ as.

   Let's consider the two measures on $\sigma(G, H)$: $\mu_1 : C \to E[1_C E[X|G]]$, and $\mu_2 : C \to E[1_C X]$, and consider D the set on which they agree.

   Also, the set D on which they agree is a Dynkin system:

   - $\Omega \in D$ because $\mu_1(\Omega) = \mu_1(\Omega) = E[X]$.
   - if $A \subset B$, so is $B \setminus A$ because$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$.
   - if A_i is an non-dreasing sequence of subsets in D, then $\cup A_i \in D$, because $\mu_1(\cup_i A_i) = \lim_i \mu_1(A_i) = \lim_i \mu_1(A_i) = \mu_2(\cup_i A_i)$

   Let's define $S := \{A \cap B | A \in G, B \in H\}$, $H \cup G \subset S \subset \sigma(H, G)$, so S is a generator of $\sigma(H, G)$. Furthermore, it is stable by intersection. It is then a π-system.

   Let $A \in G, B \in H$, and let's show that $E[1_{A\cap B}E[X|G]] = E[1_{A\cap B}X]$.

   Indeed by indepdence of $E[1_A X|G]$ and 1_B: $E[1_A 1_B E[X|G]] = E[1_B E[1_A X|G]] = E[1_B]E[E[1_A X|G]] = E[1_B]E[1_A X] = E[1_B 1_A X]$

   So μ_1, μ_2 agree on S, e.g. $S \subset D$, By $\pi - \lambda$ theorem, $\sigma(S) = \sigma(H, G) \subset D$, eg μ_1, μ_2 agree everywhere.

   As a result $E[X|G] = E[X|\sigma(G, H)]$

2. Let X, Y two independent bernouilli variable (taking values 1 and 0 with probability $\frac{1}{2}$). And let $Z = X + Y \bmod 2$. Then it is easy to verify (see problemset 1) that:

   - $\sigma(Y)$ and $\sigma(Z)$ are independent, because $P(Y = a, Z = b) = P(Y = a, X = b + a\, mod(2)) = P(Y = a)P(X = b + a\, mod(2)) = \frac{1}{4} = P(Y = a)P(Z = b)$, for $a, b \in \{0, 1\}$
   - $\sigma(X)$ and $\sigma(Z)$ are independent (same argument).

But

- $E[X|\sigma(Y,Z)] = Z - Y \bmod(2) = X$
- By independence $E[X|\sigma(Z)] = E[X] = \frac{1}{2} \neq X$

Question 8

By indepedence of the X_i, $E[X_1|S_n, S_{n+1}...] = E[X_1|\sigma(S_n, \sigma(X_{n+1}, ...))] = E[X_1|S_n]$ because $\sigma(X_1, S_n)$ and $\sigma(X_{n+1}, ...)$ are independent.

By symmetry, when $i \leq n$, $E[X_i|S_n] = E[X_1|S_n]$, and therefore:
$S_n = E[S_n|S_n] = \sum_{i=1}^n E[X_i|S_n] = nE[X_1|S_n]$,
ie $E[X_1|S_n, S_{n+1}...] = \frac{S_n}{n}$