Problem set 8, ORF527

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1 9.2 (Steele)

- $\mu(t,x) = tx$
- $\sigma(t,x) = e^{\frac{t^2}{2}}$

  For $t \leq T$, μ, σ, are trivially lipschiz in x uniformly in t.

  Let X_t be the unique solution, and define $U_t = X_t e^{-\frac{t^2}{2}}$, then:

$$dU_t = -tU_t dt + e^{-\frac{t^2}{2}} dX_t = -tU_t dt + tU_t dt + dB_t = dB_t$$

  so $U_t = U_0 + B_t = 1 + B_t$, and therefore $X_t = e^{\frac{t^2}{2}}(1 + B_t)$

2 9.4 (Steele)

Let f be a smooth function:

$$\begin{aligned} dY_t &= \frac{1}{2}\sigma^2(X_t) f''(X_t) dt + f'(X_t) dX_t \\ &= (\frac{1}{2}\sigma^2 f'' + af')(X_t) dt + (\sigma f')(X_t) dB_t \end{aligned}$$

Take

- $f(x) = \int_0^x \frac{1}{\sigma(s)} ds$, note that $f' = \frac{1}{\sigma} > 0$, so f is increasing. Since f continuous it is invertible.
- Define b as $b(f(x)) = (\frac{1}{2}\sigma^2 f'' + f'a)(x)$, or equivalently $b(y) = (-\frac{1}{2}\sigma' + \frac{a}{\sigma})(f^{-1}(y))$

  Note that:

- $\sigma(x) f'(x) = 1$
- $\frac{1}{2}\sigma^2(x) f''(x) + a(x) = b(f(x))$

so that $dY_t = b(f(X_t))dt + dB_t$, and $Y_t = Y_0 + \int_0^t b(Y_s) ds + B_t$

3 Q.2

a

Let $b, d = 0$
Consider:

$$\begin{cases} dZ_t = aZ_t dt + cZ_t dW_t \\ Z_0 = X_0 \end{cases}$$

$Z_t = X_0 e^{(a-\frac{1}{2}c^2)t+cW_t}$ We check easily that this is a solution to the SDE. It is also the unique solution because the coefficients are linear in x and don't depend on t.

Now consider the general equation:

$$\begin{cases} dX_t = (aX_t + b)dt + (cX_t + d)dW_t \\ X_0 = x_0 \end{cases}$$

$Z_t > 0$ a.s
$f(x,z) = \frac{x}{z}$
Let $U_t = \frac{X_t}{Z_t}$

$$\begin{aligned}
dU_t &= \frac{1}{Z_t}dX_t - \frac{X_t}{Z_t^2}dZ_t + \frac{1}{2}[-2\frac{(cX_t+d)cZ_t}{Z_t^2} + 2(cZ_t)^2\frac{X_t}{Z_t^3}]dt \\
&= \frac{1}{Z_t}dX_t - \frac{X_t}{Z_t}\frac{dZ_t}{Z_t} + [-\frac{(c^2X_t+cd)}{Z_t} + c^2\frac{X_t}{Z_t}]dt \\
&= \frac{1}{Z_t}dX_t - \frac{X_t}{Z_t^2}dZ_t - \frac{cd}{Z_t}dt \\
&= \frac{1}{Z_t}(dX_t - X_t\frac{dZ_t}{Z_t} - cd\,dt) \\
&= \frac{1}{Z_t}(X_t(adt + cdW_t) + bdt + ddW_t - X_t(adt + cdW_t) - cd\,dt) \\
&= \frac{1}{Z_t}((b-cd)dt + ddW_t)
\end{aligned}$$

Since $X_0 = Z_0$:

$$X_t = U_tZ_t = Z_t\left[1 + (b+cd)\int_0^t \frac{ds}{Z_s} + d\int_0^t \frac{dW_s}{Z_s}\right]$$

b

$dX_t = aX_t(b - X_t)dt + cX_tdW_t = X_t(a(b-X_t)dt + cdW_t)$
Define $U_t := \frac{1}{X_t}$
Ito:

$$\begin{aligned}
dU_t &= d\frac{1}{X_t} \\
&= -\frac{1}{X_t^2}dX_t + \frac{1}{X_t^3}(cX_t)^2dt \\
&= -\frac{1}{X_t}\frac{dX_t}{X_t} + c^2\frac{1}{X_t}dt \\
&= -U_t[abdt - aX_tdt + cdW_t] + c^2U_tdt \\
&= -abU_tdt + adt - cU_tdW_t + c^2U_tdt \\
&= \left[a + (-ab + c^2)U_t\right]dt - cU_tdW_t
\end{aligned}$$

Define V_t the solution to the homogenious SDE:

$$dV_t = (-ab + c^2)V_tdt - cV_tdW_t, V_0 = U_0$$

so that:

$$V_t = \frac{1}{X_0}\exp\left((\frac{1}{2}c^2 - ab)t - cW_t\right)$$

by part a): $U_t = V_t\left[1 + a\int_0^t \frac{ds}{V_s}\right]$

$$X_t = \frac{1}{U_t} = \frac{1}{V_t\left[1 + a\int_0^t \frac{ds}{V_s}\right]}$$

When $a > 0$, since $V_t > 0$, this solution is well definied, and we can check easily that it verifies the SDE.

If $a < 0$ and $c = -1$ and b is such that $c^2 = 2ab$, we prove that with positive probability, $\exists t > 0, \int_0^t \frac{ds}{V_s} \geq -\frac{1}{a}$, and as a result X_t is not well defined.

Define the stopping times:

- $\tau_1 = \inf\{t > 0, W_t = 1 + \log(-\frac{X_0}{a})\}$
- $\tau_2 = \inf\{t \geq \tau_1, W_t = 2 + \log(-\frac{X_0}{a})\}$
- $\tau_3 = \inf\{t \geq \tau_2, W_t = 1 + \log(-\frac{X_0}{a})\}$

All the τ_i are finite a.s.

Furthermore, if $\tau_3 \geq \tau_2 + 1$, $\int_0^t \frac{ds}{V_s} \geq \int_{\tau_2}^{\tau_3} \frac{ds}{V_s} \geq -\frac{1}{a}$

Now, by strong markov property: $\mathbb{P}(\tau_3 \geq \tau_2 + 1) = \mathbb{P}(\tau_1 \geq 1 | W_0 = 2 + \log(-\frac{X_0}{a}))$. This probability is positive because $E[\tau_1] = \infty$

Q.3

a

$$
\begin{aligned}
.|X_t^n - X_t| &= |\int_0^t [b(X_s) - b(X_s^n)]ds + \rho(W_t^n - W_t)| \\
&\leq L\int_0^t |X_s - X_s^n| ds + |\rho||W_t^n - W_t| \\
&\leq L\int_0^t \underbrace{|X_s - X_s^n|}_{f(s)} ds + \underbrace{|\rho| \sup_{s \leq s} |W_s^n - W_s|}_{g(t)}
\end{aligned}
$$

g is non-descreasing, Gronwall implies:

$$|X_t^n - X_t| \leq |\rho(W_t^n - W_t)| \exp(LT)$$

so that:

$$\sup_{[0,T]} |X_t^n - X_t| \leq |\rho| \sup |W_t^n - W_t| \exp(LT) \to_n 0$$

b

1. SDE of Z_t

   First assume that Z_t exists. Write:

   $Z_t = \mu(Z_t)dt + \alpha(Z_t)$

   If we definie f, c as in 9.4, $c(y) = (-\frac{1}{2}\alpha' + \frac{\mu}{\alpha})(f^{-1}(y))$

   we have that:

$$\begin{pmatrix} f(Z_t) & = f(Z_0) & + \int_0^t c(f(Z_s))ds & + B_t \\ \uparrow & \uparrow & \uparrow & \uparrow \\ f(Z_t^n) & f(Z_0^n) & \int_0^t c(f(Z_s^n))ds & B_t^n \end{pmatrix}$$

   Where the convergence holds a.s.

   To find a candidate for the SDE, let's assume: $\forall n \; f(Z_t^n) = f(Z_0^n) + \int_0^t c(f(Z_s^n))ds + B_t^n$

   Note that:

- $f' = \frac{1}{\alpha}$, $f'' = -\frac{\alpha'}{\alpha^2}$
- $f^{-1\prime}(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(z)} = \alpha(z)$
- $f^{-1\prime\prime}(y) = (\frac{1}{f'(f^{-1}(y))})' = -\frac{f''(f^{-1}(y))}{((f^{-1}(y))')^2} = \alpha'(z)\alpha(z)$

Note $Y_t^n = f(Z_n^t)$

$$\begin{aligned}
dZ_t^n &= df^{-1}(Y_n^t) \\
&= f^{-1\prime}(Y_t^n)dY_t^n \\
&= \alpha(Z_t^n)(c(Y_n^t)dt + dB_t^n) \\
&= \alpha(Z_n^t)c(Y_n^t)dt + \underbrace{\alpha}_{\sigma}(Z_t^n)dB_t^n \\
&= \underbrace{(\mu - \frac{1}{2}\alpha\alpha')}_{b}(Z_t^n)dt + \underbrace{\alpha}_{\sigma}(Z_t^n)dB_t^n
\end{aligned}$$

By identification, $\alpha = \sigma$, $\mu = b + \frac{1}{2}\alpha\alpha'$ In conclusion, Z_t verifies:

$dZ_t = (b + \frac{1}{2}\sigma\sigma')(Z_t)dt + \sigma(Z_t)dB_t$

In the next part we consider the solution to this SDE, and we prove that, indeed, Z_t^n converges to Z_t uniformly in t.

2. Existence of Z_t Define Z_t as the solution of the SDE:

$$dZ_t = \underbrace{b + \frac{1}{2}\sigma\sigma'}_{\text{Lipschiz}}(Z_t)dt + \underbrace{\sigma}_{\text{Lipschiz}}(Z_t)dB_t$$

Following the last part,

$$df(Z_t) = c \circ f(Z_t)dt + dB_t$$

We also have that:

$df(Z_t^n) = f'(Z_t^n)dZ_t^n = \frac{b}{\sigma}(Z_t^n)dt + dB_t^n$ b/σ is Lipschiz, by part **a**:

$\sup_{t\in[0,T]} |f(Z_t^n) - f(Z_t)| \to_n 0$

We know also that f^{-1} is Lipschiz in $[0, T]$. Indeed, $f^{-1\prime}(y) = \sigma(f^{-1}(y))$ is bounded in that interval.

So

$$\sup_{[0,T]} |Z_t - Z_t^n| = \sup_{[0,T]} |f^{-1} \circ f(Z_t) - f^{-1} \circ f(Z_t^n)| \leq ||f^{-1\prime}||_\infty \sup_{t\in[0,T]} |f(Z_t^n) - f(Z_t)| \to_n 0$$

Which proves the existence of Z_t.