Problem 1

1. Let's consider $g : u \to \log(1+e^u)$. g is non-decreasing and convexe because $g'(u) = \frac{e^u}{1+e^u} = \frac{1}{1+e^{-u}}$ is increasing.

   We notice that $f(x_1, x_2) = g(x_1 - x_2) + x_2$.

   - $x_2 \to x_2$ is linear
   - $x_2 \to x_1 - x_2$ is linear, g convexe and non-decreasing, so $g(x_1 - x_2)$ is convexe

   c/c: f is convexe.

2. The following transformation is a bijection from $(2,3) \times (0,\infty) \times (0,\infty)$ to $(\frac{\log 2}{2}, \frac{\log 3}{2}) \times \mathbb{R} \times \mathbb{R}$

$$
\begin{aligned}
x_1 &= 2\log x \\
x_2 &= \log y - \log z \\
x_3 &= \log y
\end{aligned}
$$

   $\frac{x}{y} = z^2 = e^{2\log z} = e^{2\log y - 2x_2} = e^{2x_3 - 2x_2}$ Minimizing $\frac{x}{y}$ is the same as minimizing $a(x_1, x_2, x_3) := e^{2x_3 - 2x_2}$ which is convexe as the composition of a linear function and a convexe and increasing one exp.

   - $\frac{x}{y} = z \iff \log x - \log y = \log z \iff \frac{1}{2}x_1 - x_3 = x_3 - x_2 \iff \frac{1}{2}x_1 + x_2 - 2x_3 = 0$ and $b(x_1, x_2, x_3) := \frac{1}{2}x_1 + x_2 - 2x_3$ is linear.
   - $x^2 + \frac{y}{z} \leq \sqrt{y} \iff e^{x_1} + e^{x_2} \leq \sqrt{e^{x_3}} \iff \log(e^{x_1} + e^{x_2}) \leq \frac{1}{2}x_3 \iff f(x_1, x_2) - \frac{1}{2}x_3 \leq 0$ and $c(x_1, x_2, x_2) := f(x_1, x_2) - \frac{1}{2}x_3$ is convexe as the sum of two convexe functions

   c/c: the optimization problem is equivalent to:

$$
\max a(x_1, x_2, x_2) \text{ s.t. } b(x_1, x_2, x_2) = 0, c(x_1, x_2, x_2) \leq 0, (x_1, x_2, x_2) \in (\frac{\log 2}{2}, \frac{\log 3}{2}) \times \mathbb{R} \times \mathbb{R}
$$

   which is a convexe problem.

Problem 2

$\Rightarrow$) Let's suppose f convexe.

$$
\begin{aligned}
\nabla f^T(x)(y-x) &= \lim_{\alpha 0} \frac{f(x + \alpha(y-x)) - f(x)}{\alpha} \\
&= \lim_{\alpha 0} \frac{f((1-\alpha)x + \alpha y) - f(x)}{\alpha} \\
&\leq \lim_{\alpha 0} \frac{(1-\alpha)f(x) + \alpha f(y) - f(x)}{\alpha} && \text{(because } f \text{ convexe)} \\
&\leq f(x) - f(y)
\end{aligned}
$$

$\Leftarrow$) Let's suppose $\forall x, y\, \nabla f^T(y-x) \leq f(y) - f(x)$

Let $\alpha \in (0,1)$, and $u = (1-\alpha)x + \alpha y$

$$
\begin{aligned}
f(x) - f(u) &\geq \nabla f(u)(x-u) \\
f(y) - f(u) &\geq \nabla f(u)(y-u)
\end{aligned}
$$

By multiplying the first ineqaulity by $1-\alpha$ and the second one by α and summing, we get: $(1-\alpha)f(x) + \alpha f(y) - f(u) \geq 0$

Which proves that f convexe.

Problem 3

Problem 4

1. For $y \in \mathbb{R}^n$ $\sup_y L(x,y) \geq L(x,u)$

   By taking the $\inf_x$: $\inf_x \sup_y L(x,y) \geq \inf_x L(x,u)$

   By taking the $\sup_u$: $\inf_x \sup_y L(x,y) \geq \sup_u \inf_x L(x,u)$

2. Let $f(x) := \max_y L(x,y)$ We know that $x^* \in \arg\min f$ and f is convexe, so $\partial f(x^*) = 0$.

   L is continuous the Danskin's theorem, we have that: $0 \in \{\nabla_x L(x^*, y) | y \in \arg\min L(x^*, y^*)\} = \{\nabla_x L(x^*, y^*)\}$, wich means that $\nabla_x L(x^*, y^*) = 0$, and symmetrically, $\nabla_y L(x^*, y^*) = 0$.

   c/c:

$$
\begin{aligned}
x^* &= x^* - \alpha \nabla_x L(x^*, y^*) \\
y^* &= y^* - \alpha \nabla_y L(x^*, y^*)
\end{aligned}
$$

Problem 4

1. By taylor equality, there exist $y \in X$ such that:

$$f(x^k) - f(x^*) = \nabla f(x^*)(x^k - x^*) + \nabla^2 f(x^*)$$

Problem 5

1. Let's call S_t the price of the stock at time t, and $C_t(S_t)$ the price of the corresponding american action (with strike K)

   - State $x_t = S_t$, $t = 1..T$
   - Action:

$$u_t = \begin{cases} 1 & \text{meaning we exercise the option} \\ 0 & \text{meaning we don't} \end{cases}$$

   - Randomness: The change in the stock price $w_t = \frac{S_{t+1}}{S_t}$ s.t $x_{t+1} = w_t x_t$, $P(w_t = u) = 1 - P(w_t = d) = p$
   - Transitional cost:

     $g(x_k, u_k = 0, w_t) = 0$ $g(x_k, u_k = 1, w_t) = (x_t - K)^+ - C_{t+1}(x_t + w_t)$ Explication: If we exercise the option, we gain $(x_t - K)^+$ but we lose the right to the option $(-C_{t+1})$. $g(x_T) = (x_T - K)^+$: We are forced to exericse the option at time T

   The price problem:

$$C_k(x) = \max_{\mu} E[g(x_T) + \sum_{t=k}^{T-1} g(x_k, \mu_k(x_k), w_t) | x_k = x]$$

2. Bellman equation:

$$C_k(x) = \max_{\mu_k} E[g_k(x, \mu_k(x), w_k) + C_{k+1}(w_k x_{k+1}) | x_k = x] = \max\left\{(x-K)^+, pC_{k+1}(ux) + (1-p)C_{k+1}(dx)\right\}$$

$$C_T(x) = (x - K)^+$$

3. LP: Let $J(t, S)$ be the price of the option at time t is $S_t = S$, and we decide to adopt the strategy J verifies: $J(t, S) = \max\{E[J(t+1, S_{t+1})|S_t = S], S - K\} = [\max_\mu(P_\mu J + g_\mu)](t, S)$ where $\mu(t, S) \in \{\text{HOLD}, \text{EXEC}\}$,

$$(P_\mu J)(t, S) = \left\{ \begin{array}{cc} pJ(t+1, uS) + (1-p)J(t+1, dS) & \text{if } \mu(t, S) = \text{HOLD} \\ 0 & \text{otherwise} \end{array} \right\}$$

$$g_\mu(t, S) = \left\{ \begin{array}{cc} 0 & \text{if } \mu(t, S) = \text{HOLD} \\ S - K & \text{otherwise} \end{array} \right\}$$