ORF523 - Problem Set 1

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Q.1

1. Let λ be an eigen value of A^TA corresponding to an eigen vector $u \neq 0$, then $0 \leq ||Au||^2 = u^TA^TAu = \lambda||u||^2$, therefore $\lambda > 0$.
2. Let λ be an eigen value of A corresponding to an eigen vector u, the $A^TAu = A(Au) = \lambda^2 u$, so λ^2 is an eigen value of A^TA. Since A has n eigen values (accounting for multplicity), the eigen values of A^TA are exactly the squares of the eigen values of A, and therefore the singular values of A are the absolute values of the eigen values of A.
3. $$u_i^Tu_j = u_i^T\frac{A^TAu_j}{\lambda_j} = \frac{u_i^TA^TA}{\lambda_j}u_j$$

   Since $\lambda_i \neq \lambda_j$, $u_i^Tu_j = 0$

Q.2

- The L_2 norm for vectors is unitarly invariant: Let O unitary matrix and X a vector, then $||OX||^2 = X^TO^TOX = X^TX = ||X||^2$.
- Since O is invertible, the application $S \to S, X \to OX$, where S is the L_2 sphere, is a bijection. So $\{x, ||x||_2 = 1\} = \{Ox, ||x||_2 = 1\}$
- The L_2 norm for matrices is unitarly invariant. If A a matrix, then $||AO|| = \max_{||x||_2=1}||AOx|| = \max_{||Ox||_2=1}||AOx|| = \max_{||y||_2=1}||Ay|| = ||A||$ and $||OA|| = \max_{||x||_2=1}||OAx|| = \max_{||x||_2=1}||Ax|| = ||A||$.
- Let B be a matrix of rank at most k. $||A-B|| = ||U(\Sigma - U^TBV)V^T|| = ||\Sigma - U^TBV||$ Let $U'\Sigma'V'$ be the SVD of B, by a similar argument: $||A-B|| = ||\Sigma - \Sigma'||$.

  $rank(B) = rank(\Sigma') \leq k$, so Σ' can be written as $\Sigma'^{(k)} = diag(\sigma'_1, \dots, \sigma'_k, ldots)$. $||A-B|| = \sqrt{\sum_{i=1\dots k}(\sigma_i - \sigma'_i)^2 + \sum_{i=k+1\dots n}\sigma_i^2} = \sqrt{\sum_{i=1\dots k}(\sigma_i - \sigma'_i)^2 + ||A - A^{(k)}||^2} \geq ||A - A^{(k)}||$, Since $rank(A^{(k)}) = k$ we have proved the result.