ORF527 - Problem Set 2

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Q.1

Let $\varepsilon > 0, x \in [0, 1]$.

- $\sup |f^n - f| \to_n 0$, let $n \in \mathbb{N}$ $\forall t \in [0, 1]$ $|f^n(t) - f(t)| \leq \frac{\varepsilon}{3}$.
- f^n is continuous, let $\delta > 0$ such that $\forall y \in [0, 1], |x - y| < \delta \Rightarrow |f^n(x) - f^n(y)| < \frac{\varepsilon}{3}$.
- Let $y \in [0, 1]$ such that $|x - y| < \delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f^n(x)| + |f^n(x) - f^n(y)| + |f^n(y) - f(y)| \\ &\leq 3\frac{\varepsilon}{3} \\ &\leq \varepsilon \end{aligned}$$

Which conclude the proof.

Q.2

3.2

let $X \sim \mathcal{N}(0, 1)$, and $\varepsilon \sim \mathcal{B}(-1, 1, \frac{1}{2})$ be two independant rv. And Let $Y = \varepsilon X$

- By symmetry of the distribution of X: $F_Y(y) = P(Y \leq y) = P(\varepsilon X \leq y) = E[P(\varepsilon X \leq y|\varepsilon)] = \frac{1}{2}P(X \leq y) + \frac{1}{2}P(-X \leq y) = P(X \leq x)$ so $Y \sim \mathcal{N}(0, 1)$.
- $cov(X, Y) = E[X^2\varepsilon] = E[X^2]E[\varepsilon] = 0$
- X, Y are not independent. Indeed, Let $\alpha := \mathbb{P}(|X| > 0.5)$.
  - $P(|X| > 0.5, |Y| > 0.5) = P(|X| > 0.5) = \alpha$,
  - $P(|X| > 0.5)P(|Y| > 0.5) = P(|X| > 0.5)^2 = \alpha^2$

  But since $\alpha \notin \{0, 1\}$, $\alpha \neq \alpha^2$.

3.3

For a random variable X, let's call Φ_X its caracteristic function.

(a) For $t \in \mathbb{R}^n$, $\Phi_{AV}(t) = \mathbb{E}[e^{iV^TA^Tt}] = \Phi_V(A^Tt) = e^{(A\mu)^TV - \frac{1}{2}(t^TA\Sigma A^T)t}$, so $V \sim \mathcal{N}(A\mu, A\Sigma A^T)$

(b) By symmetry of the gaussisan distribution, $-Y$ has the same distribution as Y. So it suffices to show that the result holds for $X + Y$. By independence: $\Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) = \Phi_X(t)^2 = e^{i2\mu t - \frac{1}{2}2t^2}$, so $X + Y \sim \mathcal{N}(0, 2)$.

(c) If $cov(X,Y) = 0$, the covariance matrix of the guassian process (X,Y) is the identity, therefore

$$\forall t = (t_1,t_2) \in \mathbb{R}^2 \ \Phi_{(X,Y)}(t) = E[e^{i\mu_X t_1 + i\mu_Y t_2 - \frac{1}{2}\sigma_X^2 t_1^2 - -\frac{1}{2}\sigma_Y^2 t_2^2}] = \Phi_X(t_1)\Phi_Y(t_2)$$

Where $X \sim \mathcal{N}(\mu_X, \sigma_X^2), Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Therefore X, Y are independent.

(d) Let $\alpha := \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}$. $(Y - \alpha X, X)$ is gaussian and $\operatorname{cov}(Y - \alpha X, X) = 0$. So $Y - \alpha X \perp X$. Therefore:

$$\mu_{Y|X=x} = E[Y|X=x] = E[Y - \alpha X] + E[\alpha X | X = x] = \mu_Y - \alpha\mu_X + \alpha x = \mu_Y + \frac{\operatorname{cov}(X,Y)}{\operatorname{var} X}(x - \mu_X)$$

$$\sigma_Y^2 = \operatorname{var}(Y|X=x) = \operatorname{var}(Y - \alpha X) + \underbrace{\operatorname{var}(\alpha X | X = x)}_{=0} = \operatorname{var} Y + \alpha^2 \operatorname{var} X - 2\alpha \operatorname{cov}(X,Y) = \sigma_Y^2 - \frac{\operatorname{cov}(X,Y)^2}{\operatorname{var} X}$$

Q.3

(a) Since $\forall n > 0, \Delta_n(1) = 0$, $B_1 = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(1) = \lambda_0 Z_0 \Delta_0(1) = \lambda_0 Z_0$. But $B_t - U_t = \lambda_0 Z_0 \Delta_0(t) = t\lambda_0 Z_0 = tB_t$, which proves the result.

(b) If $s < t$, then $cov(B_s, B_t) = \operatorname{cov}(B_s - B_t, B_s) + \operatorname{var}(B_s) = s$.

$\operatorname{cov}(U_s, U_t) = \operatorname{cov}(B_s - sB_1, B_t - tB_1) = \operatorname{cov}(B_s, B_t) + ts\operatorname{cov}(B_1, B_1) - t\operatorname{cov}(B_s, B_1) - s\operatorname{cov}(B_1, B_t) = s + ts - ts - ts = s - ts = s(1-t)$.

(c) For $s < t$, $cov(X_t, X_s) = g(t)g(s)(h(s) \wedge h(t))$, should be equal to $s(1-t)$. We can take $g : t \to 1 - t$, $h : s \to \frac{s}{1-s}$ defined on $[0, 1)$

(d) Since the two guaissian processes U_t and X_t have the same mean (0) and covariance matrix, they have the same distribution. Therefore, Y_t has the same distribution as $(1+t)X_{\frac{t}{1+t}} = (1+t)g(1+t)B_{h(\frac{t}{1+t})} = B_t$. In addition, U_t is continuous as a composition of two continuous functions, it is a brownian motion.