ORF527: Problem Set 4

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Q1

(6.1 in the book)

- We calculate the the $L_2([0,t] \times \Omega)$ -norm of the variables.

$$\begin{aligned}\int_0^t E[|B_s|]ds &= \int_0^t E[|B_s|]ds \\ &= \sqrt{\frac{2}{\pi}} \int_0^t \sqrt{s}ds \\ &= \frac{2}{3}\sqrt{\frac{2}{\pi}} t^{\frac{3}{2}} < \infty\end{aligned}$$

Since $|B_s|^{\frac{1}{2}}$ is progressively measurable, it is $\mathcal{H}^2$, and $Var([|B_s|^{\frac{1}{2}}) = \frac{2}{3}\sqrt{\frac{2}{\pi}} t^{\frac{3}{2}}$

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$$\begin{aligned}\int_0^t E[(B_s + s)^4] &= \int_0^t s^4 + 6s^3 + 3s^2 ds \\ &= \frac{t^5}{5} + \frac{3}{2}t^4 + t^3\end{aligned}$$

Since $(B_s + s)^2$ is progressively measurable, it is $\mathcal{H}^2$, and

$$Var(\int_0^t (B_s + s)^2 dB_s)) = \frac{t^5}{5} + \frac{3}{2}t^4 + t^3$$

(6.2 in the book)

1. - By Tonneli: $E[\int_0^t |B_s|] = \int_0^t \sqrt{s}dsE[|\mathcal{N}(0,1)|] < \infty$

   By Fubini:

   $$E[\int_0^t B_s] = \int_0^t \sqrt{s}dsE[\mathcal{N}(0,1)] = 0$$

   - $E[\int_0^t \int_0^t |B_u||B_s|dsdu] = (\int_0^t \sqrt{s}ds)^2 E[|\mathcal{N}(0,1)|]^2 < \infty$

By Fubini:

$$Var(I_1) = E[(\int_0^t B_s ds)^2] = \int_{[0,t]^2} E[B_s B_u] duds = \int_{[0,t]^2} (u \wedge s) duds = 2\int_0^t (\int_0^s udu) ds = \frac{t^3}{3}$$

2. - By Tonneli: $E[I_2] = \int_0^t E[B_s^2] ds = \int_0^t sds = \frac{t^2}{2}$
   - $Var(I_2) = E[I_2^2] - E[I_2]^2$

   By Toneli: $E[I_2^2] = \int_{[0,t]^2} E[B_s^2 B_u^2] dsdu$.

   If $u < s$, using the fact that $B_s^2 - s$ is martingale: $E[B_s^2 B_u^2] = E[B_u^2 E[B_s^2 | B_u]] = E[B_u^2(B_u^2 - u + s)] = 3u^2 - u(u-s) = 2u^2 + us$, so:

$$E[I_2^2] = 2\int_0^t \int_0^s (2u^2 + us) duds = \frac{7}{3}\int_0^t s^3 = \frac{7}{12}t^4$$

   As a conclusion:

$$Var(I_2) = \frac{7t^4}{12} - \frac{t^4}{4} = \frac{1}{3}t^4$$

(6.3 in the book)

1. f is bounded so $\int_0^t E[f(X_s)^2] ds < \infty$.

Fubini: $E[V_t] = \int_0^t E[f(X_s)] ds = \int_0^t E[f(\sqrt{s}Z)] ds$
Since $E[f(\sqrt{s}Z)] = E[f(\sqrt{s}\mathcal{N}(0,1))] = E[f(B_s)]$,
$\int_0^t E[f(\sqrt{s}Z)] ds = \int_0^t E[f(B_s)] ds$
By fubini again by a similar argument:
$\int_0^t E[f(B_s)] ds = E[\int_0^t f(B_s) ds]$
So $E[V_t] = E[U_t]$

1.

Let $f(s) = s$
$Var(\int_0^t f(B_s) ds) = Var(I_1) = \frac{5}{6}$
$Var(\int_0^t f(X_s) ds) = E(Z^2 (\int_0^1 \sqrt{s} ds)^2) = \frac{4}{9}$
Let $f_n(s) = s1_{|s|<n} + sign(s)n1_{|s|\geq n}$

- f_n are contiuous, bounded functions, such that $f_n(s) \to f(s)$
- $f_n(B_s)f_n(B_u)$, $f_n(X_s)f_n(X_u)$ are bounded respectively by $|B_s B_u|$ and $|X_s X_u|$ which are in $L_1([0,1] \times [0,1] \times \Omega)$
- By Fubini, $Var(\int_0^t f_n(B_s) ds) = E\int_{[0,t]^2} f_n(B_s) f_n(B_u) dsdu$, the same holds for X_s

By dominated convergence theorem:

- $Var(\int_0^1 f_n(B_s) ds) \to Var(\int_0^1 f(B_s) ds)$
- $Var(\int_0^1 f_n(X_s) ds) \to Var(\int_0^1 f(X_s) ds)$

Since the limits are different, for n large enough, $Var(\int_0^1 f_n(B_s) ds) \neq Var(\int_0^1 f_n(X_s) ds)$.

Q2

$s \to X_s(\omega)$ can be apporiximated by the sequence of simple process $X_s^n(\omega) = \sum_{i=1}^{2^n} X_{t_i^n}(\omega) 1_{(t_i^n, t_{i+1}^n]}(s)$ where $t_i^n = \frac{it}{2^{-n}}$.

Indeed, for (s, ω) fixed and $\varepsilon > 0$, by continuity of $u \to X_u(\omega)$ there exist a $\delta > 0$ such that $|s - u| \leq \delta \implies X_s(\omega) - X_{s'}(\omega)| < \varepsilon$.

Let n be large enough so that $t2^{-n} < \delta$, and let $i \leq 2^n$ be such that $t_i^n < s \leq t_{i+1}^n$, then $X_s^n(\omega) = X_{t_i^n}(\omega)$, and $|X_s^n(\omega) - X_s(\omega)| \leq |X_{t_i^n}(\omega) - X_s(\omega)| \leq \varepsilon$

- Since X is adapated, each $X_{t_i^n}$ is measurable with respect to $\mathcal{F}_{t_i^n} \subset \mathcal{F}_t$, and since it dones't depend on t, it is progressively measurable.
- $1_{(t_i^n, t_{i+1}^n]}$ is measurable with respect to t, and constant with respect to ω so it is progressively measurable.

  As a result, X^n is progressively measurable, so is the pointwise limit X.

Q3

a) $\omega \in A_n$ infinitely often $\iff \omega \in \cap_m \cup_{n \geq m} A_n$

By continuity of P from below:

$$1 - P(A_n \text{ infinitely often }) = P(\cup_m \cap_{n \geq m} A_n^c) = \lim_m P(\cap_{n \geq m} A_n^c)$$

By independence, this is equal to

$$\lim_m \prod_n P(A_n^c) = \lim_m \prod_{n \geq m} (1 - P(A_n)) \leq \lim_m e^{-\underbrace{\sum_{n \geq m} P(A_n)}_{\infty}} = 0$$

b) Choose $t_n = \frac{1}{n}$, $\alpha_n = \sqrt{\frac{n(n+1)}{\log\log(n)}}$ so that $\alpha_n^2(t_n - t_{n+1}) = \frac{1}{\log\log n}$

$\int_0^T f_n(t)^2 dt = \alpha_n^2(t_n - t_{n+1}) = \frac{1}{\log\log(n)} \to 0$

$$\int_0^T f_n(t) dW_t = \alpha_n(W_{t_n} - W_{t_{n+1}})$$

$$P(\int_0^T f_n(t) dW_t > 1) = P(\underbrace{\alpha_n(W_{t_n} - W_{t_{n+1}}) > 1}_{A_n}) = P(\mathcal{N}(0,1) > \frac{1}{\alpha_n \sqrt{t_n - t_{n+1}}})$$

It is clear that the A_n are independent.

Lemma: $\int_x^\infty e^{-\frac{t^2}{2}} \geq \int_x^{x+1} e^{-\frac{t^2}{2}} \geq e^{-\frac{(x+1)^2}{2}}$

Using this lemma: $P(\int_0^T f_n(t) dW_t > 1) \geq e^{-(\frac{1}{\alpha_n \sqrt{t_n - t_{n_1}}} + 1)^2} = e^{-(\sqrt{\log\log n} + 1)^2}$

Let n be large enough so that $\log^4(n) \leq n$ and $\sqrt{\log\log(n)} \geq 1$, then $(\sqrt{\log\log(n)} + 1)^2 \leq 4\log\log(n)$, and:

$$P(\int_0^T f_n(t) dW_t > 1) \geq e^{-4\log\log n} = \frac{1}{\log^4(n)} \geq \frac{1}{n}$$

Since the harmonic Series divergese: $\sum_n P(\int_0^T f_n(t)dW_t > 1) = \infty$ and we have the answer using the previous question.

c) Let X^n be a sequence of simple functions that converges to X in $L_2([0,T] \times \Omega)$, so that $\int_0^t X_t^n dW_t \overset{L_2}{\to} \int_0^T X_t dW_t$. The sequence converges in $L_2(\Omega) \implies$ converges in probability $\implies$ convergence a.s. along a subsequence.