1 measure theory

Definition 1 (Sigma Algebra) *$\mathcal{F}$ σ-algebra:*

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- $\cup_n A_n \in \mathcal{F}$

Definition 2 (Probability measure) *Probability measure*

- $\mathbb{P}(A) \in [0,1]$
- $\mathbb{P}(\Omega) = 1$
- $A \cap B = \emptyset \to \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Theorem 1 (Equivalence additive measure) *The following are equivalent fo μ finitely additive measure:*

- *$\mu\sigma-$ additive*
- *μ continuous from below / above/ at 0.*

Definition 3 (Monotone class theorem) *Monotne class $\mathcal{M} \subset \mathcal{P}(\Omega)$, and is closed under countable monotone unions and intersections.*

Theorem 2 (Monote class theorem) *G an algebra, $\sigma(G) = M(G)$*

Theorem 3 ($\lambda - \pi$) *D is a Dynkin system if:*

- $\Omega \in D$
- $A \in D \Rightarrow A^c \in D$
- *$A_1, \ldots \in D$ pariwise disjoint, $\cup A_i \in D$*

Equivalently

- $\Omega \in D$
- $A, B \in D; A \subset B \Rightarrow B \setminus A \in D$
- *$A_1, \ldots \in D$ increasing, $\cup A_i \in D$*

$P\pi$-system: closed under finite interesection.
$P \subset D \Rightarrow \sigma(P) \subset P$

Theorem 4 (Sigma in out)

$$\sigma(f^{-1}(A) : A \in \epsilon) = \{f^{-1}(A) : A \in \sigma(\epsilon)\}$$

Definition 4 (Semi-ring)

- $\emptyset \in S$
- $A \cap B \in S \forall A, B \in S$
- *For al $A, B \in S$ there exist pairwise disjoint subset $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i \leq n} C_i$*

Theorem 5 (Caratheodory's Extension Theorem)

- *A measure μ on a semi-ring S can be extendted to a measure on $\sigma(S)$.*
- *If μ is σ-finite, the extension is unique.*

Definition 5 (Consistence)

- $\mathbb{P}^{i_1,\ldots,i_n}[A_1 \times ... \times A_n] = \mathbb{P}^{\pi(i_1),\ldots,\pi(i_n)}[A_{\pi(1)} \times ... \times A_{\pi(n)}]$
- $\mathbb{P}^{i_1,\ldots,i_{n-1}}[A_1 \times ... \times A_{n-1}] = \mathbb{P}^{i_1,\ldots,i_n}[A_1 \times ... \times A_{n-1} \times \mathbb{R}]$

Theorem 6 (Kolmogorov's Extension Theorem) *I non empty. $(\mathbb{P}^{i_1,\ldots,i_n})_{i_1,\ldots,i_n \in I}$ consistent family. There exists a unique probability measure on $\mathbb{P}$ on $(\mathbb{R}^I, \mathbb{B}(\mathbb{R})^{\times I})$ such that*

$$\mathbb{P}[\{\omega \in R^I : (\omega_{i_1}, ..., \omega_{i_n}) \in B] = \mathbb{P}^{i_1,\ldots,i_n}[B]$$

2 Integrals

Theorem 7 (Monotone Convergencen) *$f_1, \dots$ be a pointwise non-decreasing sequence of non-negative valued measurable functions, set $\sup f_n = f$. Then f is measurable and $\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.*

Theorem 8 (Fatou) *Let $f1, f2, f3, \dots$ be a sequence of non-negative measurable functions. Define $f = \liminf_{n\to\infty} f_n$. Then f is measurable and $\int_S f \, d\mu \leq \liminf_n \int_S f_n \, d\mu$.*

Theorem 9 (Dominated Convergence) *$g, f_1, f_2, \dots$ measurable functions suchat $\int |g| < \infty$, $|f_n| \leq g \forall n$ a.s., $f_n \overset{a.s.}{\to} f$, then $\int |f| \leq \int |g| < \infty$ and $\lim \int |f_n - f| \to 0$, $\lim \int f_n \to \int f$*

Theorem 10 (Funbini) *μ_1, μ_2 are σ-finite.*

- $\int_{\Omega_1\times\Omega_2} |f| d(\mu_1 \times \mu_2) < \infty \Rightarrow \int_{\Omega_1\times\Omega_2} f d(\mu_1\times\mu_2) = \int_{\Omega_1} \int \Omega_2 f$
- $f \geq 0 \; a.s \Rightarrow \int_{\Omega_1\times\Omega_2} f d(\mu_1\times\mu_2) = \int_{\Omega_1}\int \Omega_2 f$

Theorem 11 (Inequalities)
- *Holder: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \int |fg| \leq (\int |f|^p)^{\frac{1}{p}} (\int |g|^q)^{\frac{1}{q}}$*
- *Minkowsky: $\forall p \leq 0 \; ||f+g||_p \leq ||f||_p + ||g||_p$*

Theorem 12 (Borel Cantelli)
- $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n\geq m} A_n] = 0$
- *(A_n), indepedent, $\sum \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n\geq m} A_n] = 1$*

3 Random Variables

Definition 6 (Uniform integrability) *(X_i) u.i iff $\lim_c \sup_i \int_{|X_i|>c} |X_i| d\mathbb{P} = 0$ iff $\lim_n \sup_i \mathbb{E}[1_{|X_i|>c}|X_i|] = 0$*

Theorem 13 (Caracterisation)
- *$\forall i |X_i| \leq X \in L_1 \Rightarrow (X_i)$ uc*
- *uc iff:*
  - $\sup E[|X_i|] < \infty$
  - $\forall \epsilon > 0, \exists \delta > 0 \forall A \mathbb{P}(A) < \delta \Rightarrow \forall i \int_A |X_i| < \epsilon$

Theorem 14 (L_1 Convergencen) *$X_i \overset{\mathbb{P}}{\to} X$, X_i uc. Then $X \in L_1$, $X_i \overset{L_1}{\to} X$*

Theorem 15 (De la Valle-Pousson) *X_i uc $\iff \exists \Phi : \mathbb{R}^+ \to \mathbb{R}^+, \frac{\Phi(x)}{x} \to \infty st \sup E[\Phi|X_i|] < \infty$. Φ can be assumed convex and non-decreasing.*

Theorem 16 (Week Law of large numbers) *$X_i \in L_2$ uncorrelated, $E[X_i] = m, \sup E[X_i^2] < \infty$, then $\frac{\sum_i X_i}{n} \to m$ in L_2.*

Theorem 17 (Caracteristic Function)
- $|\Phi_X(u)| < \Phi_X(0) = 1$
- $\Phi_X(-u) = \bar{\Phi_X}(y)$
- $\Phi_X \in \mathbb{R} \iff X \overset{\mathbb{D}}{=} -X$
- *Φ_x is unifromly continuous.*
- *$E[|X|^n] < \infty \Rightarrow \exists \Phi_X^k \forall k \leq n$, and $\Phi_X^k(u) = E[(iX)^k e^{iuX}]$, and $\Phi_X(u) = \sum_k^n \frac{(iu)^k}{k!} E[X^k] + \frac{(iu)^n}{n!}\mathcal{E}_n(u)$, with $\mathcal{E}_n \to_0 0$*
- $\exists \Phi_X^{2k}(0) \Rightarrow E[X^{2k}] < \infty$
- *Inversion Formula: $\frac{F_X(b)+F_X(b^-)}{2} - \frac{F_X(a)+F_X(a^-)}{2} = \lim \frac{}{2\pi} \int_{-c}^{c} \frac{e^{-iua} - e^{-iub}}{iu} \Phi_X(u) du$*
- $\int_R |\Phi_X| < \infty \Rightarrow f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \Phi_X(u) du$
- *$X = (X_1, \dots X_n)$ indepdendent $\iff \Phi_X = \prod \Phi_{X_i}$*

Theorem 18 (Continuity Theorem)
- $X_n \overset{D}{\to} X \iff \Phi_{X_n} \to \Phi_X$
- *$\Phi_{X_n} \to \Phi$ and Φ continuous at 0 then $\exists X \; X_n \overset{D}{\to} X$*
- $X_n \overset{D}{\to} X \iff F_n \overset{in\, C(F_X)}{\to} F_X$

Theorem 19 (LLN) *X_i iid in L_1, $\frac{\sum X_i}{n} \to E[X]$ as and in L_1*

Theorem 20 (CLT) *X_n iid $Var(X) = \sigma^2 < \infty$ then $\frac{1}{\sqrt{n}} \sum \frac{X_i - E[X]}{\sigma} \to \mathcal{N}(0,1)$*

4 Martingales

Theorem 21 (Radon-Nikodym) *$\mu_2 << \mu_1 \Rightarrow \exists f$ unique μ_1-a.s $f = \frac{d\mu_2}{d\mu_1}$*

Theorem 22 (Stopping times)
- *$X_n^\tau = X_0 + (V.X)_n$ is martingale because V is predictable.*
- *If τ bounded $E[X_0] = E[X_\tau]$*
- *$M \geq \tau \geq \sigma$ stopping times, then $E[X_\tau|F_\sigma] = X_\sigma$*

Theorem 23 (Upcrossing inequality) *X_n submartingale. $E[B_n(a,b)] \leq \frac{E[(X_n-a)^+]}{b-a}$*

Theorem 24 (Convergence)
- *X_n submartingale, L_1 bounded, then there exists X_∞ such that $X_n \overset{a.s}{\to} X_\infty$, and $E[|X_\infty|] < \sup E[|X_n|]$*
- *a submartingale that is bounded above converges a.s.*
- *X_n ui submartingale, then there exists $X_\infty \in L_1$ such that $X_n \to X_\infty$ in L_1. Moreover $E[X_\infty|F_n] \geq X_n$.*
- *(F_n) filtration, $E[X|F_n] \to E[X| \cup F_n]$ a.s and L_1 (because u.i.)*
- *X_i iid, $\mathcal{G} = \cup\sigma(X_n, \ldots)$, then $\forall A \in \mathcal{G}$ $P(A) \in \{0,1\}$ (because $1_A = E[1_A]$)*
- *(G_i) dec-filtration, $E[X|G_n] \to E[X| \cap G_n]$ as and in L_1.*

Theorem 25 (Doob Maximal inequality) *X_n non-negative submartingale.*
- *$\forall\lambda > 0$, then $\lambda^p \mathbb{P}[\max_{k\leq n} X_k \geq \lambda] \leq E[X_n^p]$*
- *$|\max_{k\leq n} X_k|_p \leq \frac{p}{p-1}|X_n|_p$*
- *$|\max_{k\leq n} X_k|_1 \leq \frac{e}{e-1}(1 + |X_n \log(X_n)|_1)$*

Theorem 26 (Random Walk)
- *Fair random walk, $S_0 = 0, \tau = \inf\{n : S_n \in \{A, -B\}\}$:*

$$P(X_\tau = A) = \frac{B}{A+B}$$
$$E[\tau] = AB$$
$$P(\tau < \infty) \geq P(\tau_A < \tau_{-B}) \to 1$$

- *$p \neq \frac{1}{2}$, $P(X_\tau = A) = \frac{(\frac{1-p}{p})^B - 1}{(\frac{1-p}{p})^{A+B} - 1}$,*

$$P(\tau_A < \infty) = \lim_B P(\tau_A < \tau_{-B}) = \begin{cases} 1 & \text{if } p > \frac{1}{2} \\ (\frac{p}{1-p})^A & \text{else} \end{cases}$$

$$E[\tau_A] = \lim E[\tau_A \wedge \tau_{-B}] = \begin{cases} \frac{A}{2p-1} & \text{if } p > \frac{1}{2} \\ \infty & \text{else} \end{cases}$$

5 Markov

Theorem 27 (Markov property)
- *(X_n) markov (λ, P). Conditional on $X_m = i$, X_{n+m} is markov (δ_i, P) independent of $X_0, ..., X_m$.*
- *(X_n) markov (λ, P). Conditional on $X_T = i$, X_{n+T} is markov (δ_i, P) independent of $X_0, ..., X_T$.*

Definition 7 (Some defs)
- *Communicating classes: $I/\sim$ where $i \sim j \iff i \leftrightarrow j$*
- *C Closed $\iff i \in C, i \to j \Rightarrow j \in C$*
- *P irreducible $\iff \forall i, j, i \to j \iff$ there is only one communicating class.*
- *$H_i = \inf\{n \geq 0; X_n = i\}, T_i = \inf\{n \geq 1; X_n = i\}, V_i := \sum_n 1_{\{X_n = i\}}, f_i = P_i(T_i < \infty), m_i = E[T_i]$*
- *i is reccurent if $P_i(V_i = \infty) = 1 \iff f_i = 1 \iff \sum p_{ii}^{(n)} = \infty$, otherwise transient.*
- *i is positibe recurrent $\iff m_i < \infty$*
- *$P_i(V_i \geq k+1) = f_i^k$*

- *In a communcating class all cstates are transient or all are reccurrent.*
- *recurrence* $\Rightarrow$ *closed*
- *finite + closed* $\Rightarrow$ *recurrent.*
- P *irreducible + recurrent* $\Rightarrow P(T_j < \infty)$
- i *aperiodic* $\iff p_{ii}^n > 0$ *for large* n
- $\lambda_i p_{ij} = \lambda_j p_{ji} \Rightarrow \lambda$ *is invariant.*

Theorem 28 (Invariant Distribution)
- *I finite, if for some* $i \in I$ $p_{ij}^{(n)} \to \pi_j \forall j \in I$ *then* π *is an invariant distribution.*
- *if* P *irreducible and* $\lambda \geq 0$ *invariant, then* $\lambda \in \{0, \infty, \mathbb{R}^n\}$
- $\gamma_i^k = E_k[\sum_{n=0}^{T_k-1} 1_{X_n=i}]$. *If* P *irreducible and recurrent, then*
  - $\gamma_k^k = 1$
  - γ^k *is invariant*
  - $0 < \gamma^k < \infty$
- *If* P *irreducible and* λ *invariant with* $\lambda_k = 1$ *then* $\lambda \geq \gamma^k$. *If* P *is reccurent,* $\lambda = \gamma^k$.
- *If* P *irreducible, every state is positive recurrent* $\iff$ *state* i *is pos rec* $\iff$ P *has invariant distribution* π. *Moreover* $\pi_i = 1/m_i$

Theorem 29 (Convergences) P *transition matrix of an ergodic Markov chain (irreducible, aperiodic and positive recurrent), with invariant measure* π, *then for any initial distribution,* $P(X_n = j) \to \pi_j$

Theorem 30 (Ergodic theorem)
- P *irreducible, then* $\frac{V_n(n)}{n} \to \frac{1}{m_i}$ *as. If* P *is irreducible and positive recurrent, for every bounded function:* $\frac{1}{n}\sum^{n-1} f(X_k) \to \sum_i \pi_i f(i)$ *a.s*

Theorem 31 (Time reversal) P *irreducible and have an invariant distribution* π. *if* $X_i \sim Markov(\pi, (P_{ij}))$ *then* $X_{N-i} \sim Markov(\pi, (P_{ji}))$

Theorem 32 (Coupling theorem) $X, Y \sim Markov(\lambda/\pi, P)$ *indenpendent,* $W_n = (X_n, Y_n) \sim Markov(\lambda \otimes \pi, P \otimes P)$

6 Complex Analysis

Theorem 33 (Cauchy) *Suppose* U *is an open subset of the complex plane* $\mathbb{C}$, $f : U \to \mathbb{C}$ *is a holomorphic function and the closed disk* $D = \{z : |z - z_0| \leq r\}$ *is completely contained in* U. *Let* γ *be the circle forming the boundary of* D. *Then for every* a *in the interior of* D:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - a}\, dz$$

where the contour integral is taken counter-clockwise.

7 Annexe

Theorem 34 (Tower)

$$\begin{aligned}
E[1_{Y\in B}F(y)] &= \int 1_{y\in B}F(y)f(y)dy \\
&= \int 1_{y\in B}(\int 1_{x\leq y}f(x)dx)f(y)dy \\
&= \int 1_{y\in B}1_{x\leq y}f(x)f(y)dxdy && \textit{Tonnelli, positive} \\
&= \int 1_{y\in B}1_{x\leq y}dP_{X,Y} \\
&= E[1_{Y\in B}1_{X\leq Y}]
\end{aligned}$$

$F(Y)$ *is* $\sigma(Y)$*-measurable. So* $P(X \leq Y|Y) = E[1_{X\leq Y}|Y] = F(Y)$

1. *Continuous distributions.*

	Name	Parameters	Density $f_X(x)$	Ch. function $\varphi_X(t)$
1	Uniform	$a < b$	$\frac{1}{b-a}\,\mathbf{1}_{[a,b]}(x)$	$\frac{e^{-ita}-e^{-itb}}{it(b-a)}$
2	Symmetric Uniform	$a > 0$	$\frac{1}{2a}\,\mathbf{1}_{[-a,a]}(x)$	$\frac{\sin(at)}{at}$
3	Normal	$\mu \in \mathbb{R},\ \sigma > 0$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$	$\exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$
4	Exponential	$\lambda > 0$	$\lambda\exp(-\lambda x)\mathbf{1}_{[0,\infty)}(x)$	$\frac{\lambda}{\lambda - it}$
5	Double Exponential	$\lambda > 0$	$\frac{1}{2}\,\lambda\exp(-\lambda\,\lvert x\rvert)$	$\frac{\lambda^2}{\lambda^2+t^2}$
6	Cauchy	$\mu \in \mathbb{R}, \gamma > 0$	$\frac{\gamma}{\pi(\gamma^2+(x-\mu)^2)}$	$\exp(i\mu t - \gamma\,\lvert t\rvert)$

2. *Discrete distributions.*

	Name	Parameters	Distribution μ_X,	Ch. function $\varphi_X(t)$
7	Dirac	$c \in \mathbb{R}$	δ_c	$\exp(itc)$
8	Biased Coin-toss	$p \in (0,1)$	$p\delta_1 + (1-p)\delta_{-1}$	$\cos(t) + (2p-1)i\sin(t)$
9	Geometric	$p \in (0,1)$	$\sum_{n\in\mathbb{N}_0} p^n(1-p)\delta_n$	$\frac{1-p}{1-e^{it}p}$
10	Poisson	$\lambda > 0$	$\sum_{n\in\mathbb{N}_0} e^{-\lambda}\,\frac{\lambda^n}{n!}\,\delta_n,\ n \in \mathbb{N}_0$	$\exp(\lambda(e^{it}-1))$

3. *A singular distribution.*

	Name	Ch. function $\varphi_X(t)$
11	Cantor	$e^{it/2}\prod_{k=1}^{\infty}\cos(\frac{t}{3^k})$

Gamma	$\frac{\lambda x^{r-1}e^{-\lambda x}}{\Gamma(r)}, 0 < x, 0 < r, 0 < \lambda$	r/λ	r/λ^2

Figure 1: Distributions(gamma mean var)

8 Common distribution

Order	Non-central moment	Central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$
5	$\mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	0
6	$\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$	$15\sigma^6$
7	$\mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$	0
8	$\mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\sigma^8$	$105\sigma^8$