1 Prelude to stochastic integration

$(M_n)_n$ mtg, $(A_n)_n$ adapted, $(A.M)_n = \sum_k A_k(M_{k+1} - M_k)$ martingale transformation nof M by A.

In continuous time $(M_t)_t$ mtg, $(A_t)_t$ progressively measurable. $0 = t_0 < \ldots < t_n = t$

$$\sum_{k=1}^{n} A_{t_{k-1}}(M_{t_k} - M_{t_k}) \underset{|t_k - t_{k-1}| \to 0}{\to} "\int_0^t A_s dM_s"$$

Lessons:

- Key ingredients: estimate of form $|\int f df| \leq ||f||$ for f simple. The norm determines the class of f that is integrable: $||.||$-limits of simple functions.
- Counterexample when $TV[g, T] = \infty$ required $f(t_{k-1})$ to anticipate sign of $g(t_k) - g(t_{k-1})$

2 Ito integral

Definition 1. *Square integrable A process (X_t) is said to be square-integrable $\in \mathcal{H}[0, T]$ if*

- *X is progressively measurable.*
- $E[\int_0^t X_s^2 ds] < \infty$

Definition 2. *Simple functions $(X_t) \in \mathcal{H}^2[0, T]$ is called simple if $X_t = \sum_k 1_{]t_{k-1}, t_k]}(t) Y_{k-1}$ where:*

- *$t_0, \ldots, t_n$ deterministic times.*
- *Y_k is $\mathcal{F}_k$-measurable.*

For $X \in \mathcal{H}_0^2[0, T]$ define:

$$\int_0^t X_s dB_s := \sum_k Y_{k-1}(B_{t_k \wedge t} - B_{t_{k-1} \wedge t})$$

Lemma 1. *Ito Isometry For $X \in \mathcal{H}_0^2[0, T]$, $\mathbb{E}(\int_0^t X_s dB_s)^2 = \mathbb{E} \int_0^T X_s^2 ds$ $||\int_0^T X_s dB_s||_{L^2(\Omega)} = ||X||_{L^2(\Omega \times [0, T])}$*