Problem set 2, ORF550

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October 19, 2016

1 Problem 2.11

a. By polarization identity

$$\mathcal{E}(P_t f, P_t g) = \frac{1}{4}[\mathcal{E}(P_t(f+g), P_t(f+g)) - E(P_t(f-g), P_t(f-g))]$$

Since

$$var_\mu(P_t f) = 2\int_0^\infty \mathcal{E}(P_t f, P_t f)dt$$

Taking the integral on both sides:

$$\int_0^\infty \mathcal{E}(P_t f, P_t g)dt = \frac{1}{8}[var_\mu(f+g) - var_\mu(f-g)$$

e.g:

$$\int_0^\infty \mathcal{E}(P_t f, P_t g)dt = \frac{1}{2}cov_\mu(f,g)$$

b. $cov_\mu(f,g) = \int_0^\infty$
c. $dY_t = -Y_t dt + dB_t$
Y_t is a reversible ergodic Markov process, with:

- Stationary measure $\mu = N(0, I)$
- Semi-group $P_t f(Y) = E[f(e^{-t}Y + \sqrt{1-e^{-2t}}\epsilon)]$, $\epsilon \sim \mathcal{N}(0, I)$. $P_t f \geq 0$ when $f \geq 0$
- Generator $\mathcal{E}(f,g) = \mu < \nabla f, \nabla g >$

Let $f_x(Y) = f(\Sigma^{\frac{1}{2}} Y)$, so that :

- $\nabla f_x(Y) = \Sigma^{\frac{1}{2}} \nabla f(X)$
- $\nabla P_t f_x(Y) = e^{-t}\Sigma^{\frac{1}{2}} P_t \nabla f(X)$
- f is coordinate-wise non-decreasing, so $\nabla f(X) \geq 0$, and therefore $P_t \nabla f(X) \geq 0$
- The same applies to g.
- $\Sigma \geq 0$, so $u'\Sigma u \geq 0$ whenever $u \geq 0$

$$\begin{aligned}
cov(f(X), g(X)) &= cov(f_x(Y), g_x(Y)) \\
&= \int_0^\infty \mathcal{E}(P_t f_x, P_t g_x)dt \\
&= \int_0^\infty E[\nabla P_t f_x(Y).\nabla P_t g_x(Y)]dt \\
&= \int_0^\infty e^{-2t} E[P_t \nabla f(X)'\Sigma P_t \nabla g(X)]dt \\
&\geq 0
\end{aligned}$$

2 Problem 3.1

a. X σ^2 sub-gaussian, so $\psi(\lambda) = \log E[e^{\lambda(X-E[X])}] \leq \frac{\lambda^2\sigma^2}{2}$

Because $e^x \geq 1 + \frac{1}{2}x^2$ when $x \geq 0$, we have that: $e^{\frac{\lambda^2\sigma^2}{2}} \geq E[e^{|\lambda(X-E[X])|}] \geq 1 + \frac{1}{2}E[(\lambda(X-E[X]))^2] \geq 1 + \frac{\lambda^2}{2}var(X)$

Which proves that $var(X) \leq \frac{e^{\lambda^2/2\sigma^2}-1}{\lambda^2/2} \rightarrow_\lambda \sigma^2$

b.

$$\Phi(|X|) = \Phi(0) + \int_0^{|X|} \Phi'(t)dt = \Phi(0) + \int_0^\infty \Phi'(t)1_{t\leq|X|}dt$$

We conclude by taking the expectation on both sides. The swapping of E and $\int$ is justified by Fubini for non-negative functions.

c. X σ^2 sub-gaussian, so: $P(X \geq t) \wedge P(X \geq -t) \leq e^{-t^2/2\sigma^2}$

$P(|X| \geq t) \leq P(X \geq t) + P(X \geq -t) \leq 2e^{-t^2/2\sigma^2}$

d.

$$\begin{aligned}
E[e^{X^2/6\sigma^2}] &= 1 + \frac{1}{3}\sigma^2 \int_0^\infty te^{t^2/6\sigma^2} P(|X| \geq t)dt \\
&\leq 1 + \frac{2}{3}\sigma^2 \int_0^\infty te^{t^2/6\sigma^2 - t^2/2\sigma^2} dt \\
&\leq 1 + \int_0^\infty (2t)\frac{1}{3}\sigma^2 e^{-\frac{t^2}{3\sigma^2}} dt \\
&\leq 1 + \int_0^\infty \frac{d}{dt} - e^{-\frac{t^2}{3\sigma^2}} dt \\
&\leq 2
\end{aligned}$$

e. Let's prove the two inequalities in the hint

$$e^u \leq 1 + \frac{1}{2}u^2 e^{|u|} \tag{1}$$

- When $u \leq 0$, it is trivially verified.
- When $u \geq 0$, this is can be proven using the following:

$e^u = 1 + u + \frac{1}{2}u^2 \sum_{k=0}^\infty u^k \underbrace{\frac{2}{(k+2)!}}_{\leq \frac{1}{k!}} \leq 1 + \frac{u^2}{2}e^u$

$$|\lambda x| \leq \frac{a\lambda^2}{2} + \frac{x^2}{2a} \tag{2}$$

This follows from:

$$0 \leq (\sqrt{a}\lambda + \frac{1}{\sqrt{a}}x)^2 = \frac{a\lambda^2}{2} + \frac{x^2}{2a} - \lambda x$$

$$0 \leq (\sqrt{a}\lambda - \frac{1}{\sqrt{a}}x)^2 = \frac{a\lambda^2}{2} + \frac{x^2}{2a} + \lambda x$$

We now show the X is subgaussian:

$$\begin{aligned}
E[e^{\lambda X}] &\leq 1 + \frac{\lambda^2}{2} E[X^2 e^{|\lambda X|}] && \text{(by(1))} \\
&\leq 1 + \frac{\lambda^2}{2} e^{a\frac{\lambda^2}{2}} E[X^2 e^{\frac{X^2}{2a}}] && \text{(by(2))} \\
&\leq 1 + \frac{\lambda^2}{2b} e^{\frac{a}{2}\lambda^2} E[e^{(\frac{1}{2a}+b)X^2}] && (X^2 \leq \frac{1}{b} e^{bX^2}) \\
&\leq 1 + \frac{2\lambda^2}{1/a - 1/(3\sigma^2)} e^{a\frac{\lambda^2}{2}} && (\frac{1}{2a} + b = \frac{1}{6}\sigma^2) \\
&\leq 1 + 12\sigma^2\lambda^2 e^{\sigma^2\lambda^2} && (a = 2\sigma^2) \\
&\leq 1 + (e^{12\sigma^2\lambda^2} - 1) e^{\sigma^2\lambda^2} \\
&\leq \underbrace{1 - e^{\sigma^2\lambda^2}}_{\leq 0} + e^{13\sigma^2\lambda^2} \\
&\leq e^{13\sigma^2\lambda^2}
\end{aligned}$$

So X is 26-subguaussian. f.

$$E[X^{2q}] = 2q \int_0^\infty t^{2q-1} P(|X| \geq t) dt \leq 4q \int_0^\infty t^{2q-1} e^{-t^2/2\sigma^2} dt = (4\sigma^2)^q q!$$

g.

Fubini for non negative functions:

$$\begin{aligned}
E[e^{X^2/8\sigma^2}] &= \sum_q \frac{1}{((8\sigma^2)^q q!} E[X^{2q}] \\
&\leq \sum_q \frac{1}{((8\sigma^2)^q q!} (4\sigma^2)^q q! \\
&\leq \sum_q \frac{1}{2^q} \\
&= 2
\end{aligned}$$

3 Problem 3.7

$$f(\varepsilon_1, \ldots, \varepsilon_1) = \sup_{t \in T} \sum_k t_k \varepsilon_k$$

McDiarmid's inequality gives the following variance proxy for the subguaussian property:

$$\sigma^2 = \frac{1}{4} \sum_k ||D_k f||_\infty^2$$

$$D_k f = \sup_{\varepsilon_k', \varepsilon_k} \sup_{t', t \in T} \sum_{i \neq k} \varepsilon_i (t_i - t_i') + \varepsilon_k t_k - \varepsilon_k' t_k' \geq \sup_{\varepsilon_k', \varepsilon_k} \sup_{t', t \in T} (\varepsilon_k - \varepsilon_k') t_k \geq 2 \sup t_k$$

So at best $||D_k f||_\infty = 4 \sup t_k^2$, and

$$\sigma^2 = \sum_k \sup_{t \in T} t_k^2$$

Now, pick T the set of vectors that have one component equal to 1, and the rest equal to 0, then

$$\sum_{k=1}^n \sup_T t_k^2 = n$$

while

$$\sup_T \sum_{k=1}^n t_k^2 = 1$$

4 Problem 3.8

$$f(X_1,\dots,X_n) = \sup_{C\in\mathcal{C}} |\frac{\#\{X_k \in C\}}{n} - \mu(C)| = \sup_{C\in\mathcal{C}} |\frac{\sum_i (1_{X_i\in C} - \mu(C))}{n}| = |\frac{\sum_{i\neq k}(1_{X_i\in C} - \mu(C))}{n} + \frac{1_{X_k\in C} - \mu(C)}{n}|$$

$$\begin{aligned}
D_k f &= \sup_{X_k, X_k'} \left\{ \sup_C |\frac{\sum_{i\neq k}(1_{X_i\in C} - \mu(C))}{n} + \frac{1_{X_k\in C} - \mu(C)}{n}| - \sup_{C'} |\frac{\sum_{i\neq k}(1_{X_i\in C'} - \mu(C'))}{n} + \frac{1_{X_k'\in C'} - \mu(C')}{n}| \right\} \\
&\leq \sup_{X_k, X_k'} \sup_C |\frac{\sum_{i\neq k}(1_{X_i\in C} - \mu(C))}{n}| + \sup_C |\frac{1_{X_k\in C} - \mu(C)}{n}| - \sup_{C'} |\frac{\sum_{i\neq k}(1_{X_i\in C'} - \mu(C'))}{n}| + \sup_{C'} |\frac{1_{X_k'\in C'} - \mu(C')}{n}| \\
&\leq \frac{2}{n}
\end{aligned}$$

$||D_k f||_\infty = \frac{2}{n}$
as a result Z_n is subgaussian with proxy variance $\sigma^2 \leq \frac{1}{n}$, therefore the result.

5 Problem 3.10

- Let X_i be the set of nodes $j > i$ that are connected to i. Then χ is a function of the X_i. Let's write $\chi = f(X_1,\dots,X_n)$.
- Notice that the X_i are independent, because they each depend on a disjoint subset of edges.
- χ can vary by at most 1 when one of the X_i changes its value, because at worst, we add a new color for the node i, or we delete the color of the node i.
- $|D_i f| \leq 1$, so that $\sum_k ||D_k f||_\infty^2 \leq n$.
- By McDiarmid inequality:
  $$P[\chi - E\chi \geq \sqrt{n}t] \leq e^{-2(\sqrt{n})t^2/n} = e^{-2t^2}$$
  and similarly
  $$P[E\chi - \chi \leq -\sqrt{n}t] \leq e^{-2t^2}$$
- By Union Bound:
  $$P[|\chi - E\chi| \geq \sqrt{n}t] \leq 2e^{-2t^2}$$

6 Problem 3.14

Notations:

- $$f(\varepsilon) = \sup_{t\in T}\langle \varepsilon, t\rangle$$
- Let $(t^{(n)}(\varepsilon))_n$ be a sequence in T that verifies $\langle t^{(n)}(\varepsilon), \varepsilon\rangle \to f(\varepsilon)$
- Let $\varepsilon^- = \arg\min_{\varepsilon_j' = \varepsilon_j \forall j\neq i} f(\varepsilon')$

We have that:

$$
\begin{aligned}
D_i^- f(\varepsilon) &= f(\varepsilon) - \inf_{\varepsilon_i} \sup_{t\in T} \langle \varepsilon, t\rangle \\
&= \sup_{t\in T} \langle \varepsilon, t\rangle - \sup_{t\in T} \langle \varepsilon^-, t\rangle \\
&= \lim_n \langle \varepsilon, t^{(n)}\rangle - \sup_{t\in T} \langle \varepsilon^-, t\rangle \\
&\le \lim_n \langle \varepsilon - \varepsilon^-, t^{(n)}\rangle \\
&\le \lim_n 2|t_i^{(n)}(\varepsilon)|
\end{aligned}
$$

So:

$$
|| \sum_i D_i^- f(\varepsilon))^2||_\infty \le || \sum_i \lim_n 4|t_i^{(n)}(\varepsilon)|^2||_\infty \le 4 \sup \sum_{k=1}^n t_k^2 := \sigma^2
$$

We conclude by Bounded difference inequality.