Time-Varying LPs and SDPs

Bachir El Khadir

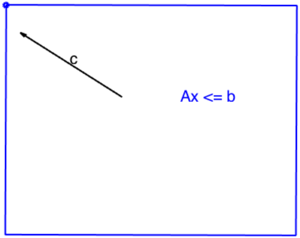
2017-05-17

Outline

Topic

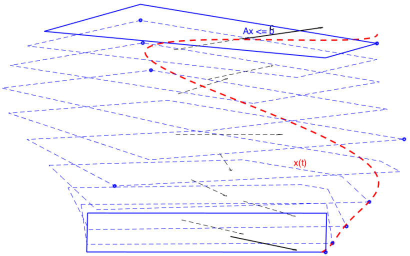
TV-LP

$$\begin{array}{ll} \underset{x}{\text{maximize}} & \langle c, x \rangle \\ \text{subject to} & Ax \leq b \end{array} \qquad \text{(LP)}$$

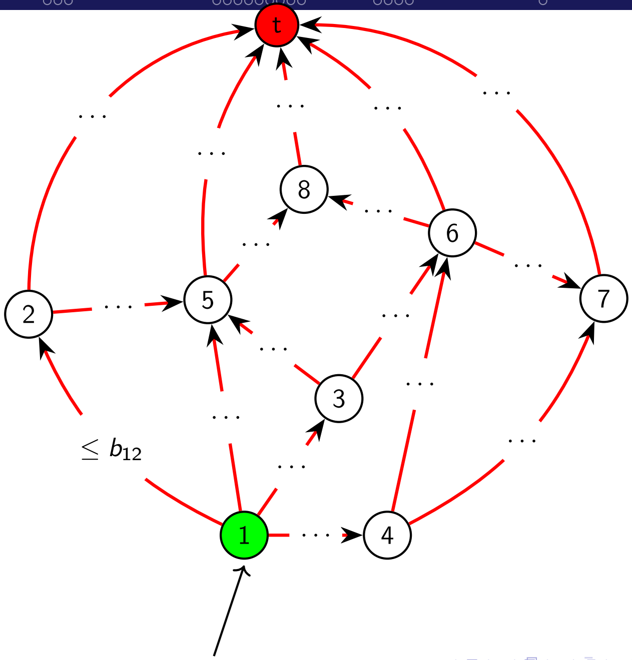


TV-LP

$$\begin{aligned} \underset{x(t)}{\text{maximize}} \quad & \int_{-1}^{1} \langle c(t), x(t) \rangle dt \\ \text{subject to} \quad & A(t)x(t) \leq b(t) \qquad \forall t \in [-1, 1] \end{aligned} \qquad \text{(TV-LP)}$$



- A, b, c polynomials.
- Polynomials are dense.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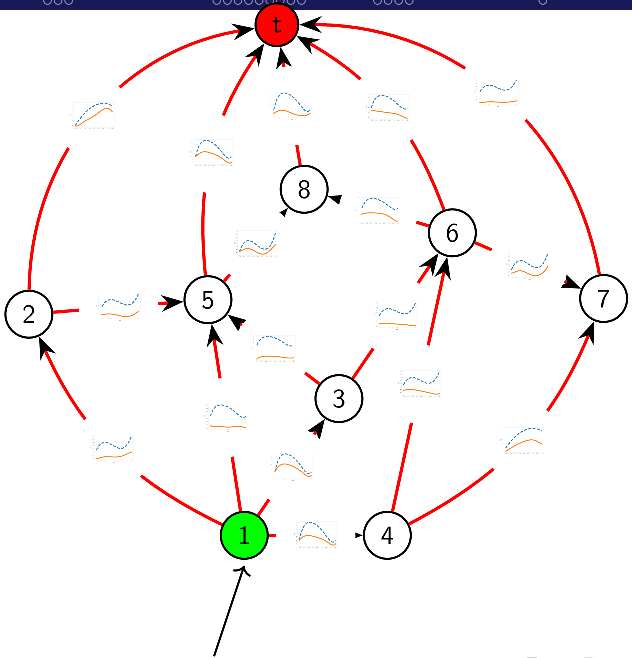
t
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TV-SDP

$$\begin{aligned} \underset{X \in \mathcal{S}_n}{\text{maximize}} \quad & \langle C, X \rangle \\ \text{subject to} \quad & A_i X \leq b_i \qquad \forall i \in [m], \end{aligned} \tag{TV-SDP}$$

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Contributions

- Find the best polynomial solution of a given degree to a TV-LP / TV-SDP using a (non varying) SDP.
- Study when polynomial solutions are optimal in several level of details.

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- Find the best polynomial solution of a given degree to a TV-LP / TV-SDP using a (non varying) SDP.
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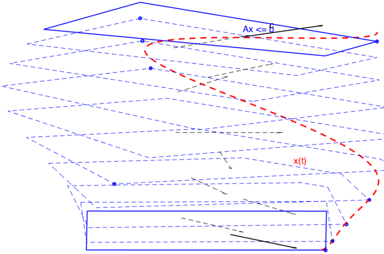
- When are polynomials near optimal to a TV-LP or TV-SDP?
- Almost never!



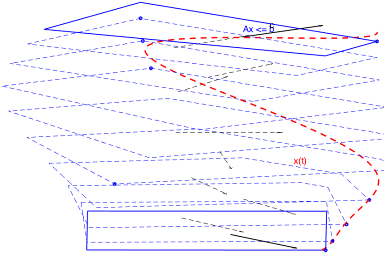
Contributions

- Find the best polynomial solution of a given degree to a TV-LP / TV-SDP using a (non varying) SDP.
- Study when polynomial solutions are optimal in several level of details.

- When are polynomials near optimal to a TV-LP or TV-SDP?
- Almost never!

Near optimal $\iff \forall \varepsilon > 0$, $\exists x(t) \in \mathbb{R}^n[t]$ such that:

- $A(t)x(t) \leq b(t)$
- $opt - \int_{-1}^{1} \langle c(t), x(t) \rangle dt \leq \varepsilon$.



Topic

We are interested in continuous solutions

Problems in practice:

- Deciding the power of transmission of a cell tower during the day.
- Chosing the optimal control of a robot hand.
- ...

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We want smooth solutions!

Positivestellnaz (TV-LPs)

Positivestellnaz for TV-LPs

Every non negative univariate polynomials $p(t)$ on $[-1, 1]$ can be written as

$$p = \sigma_0 + (1 - t)\sigma_1 + (1 + t)\sigma_2 + (1 - t^2)\sigma_3,$$

where $\sigma_i \in SOS$, $i = 0, \ldots, 3$, with degree bounded by $deg(p)$.

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In our case

- Constraint $A(t)x(t) \leq b(t) \quad \forall t \in [-1, 1]$
- Becomes $b(t) - A(t)x(t) = \sigma_0 + (1-t)\sigma_1 + (1+t)\sigma_2 + \sigma_3(1-t^2)$
- Efficient search!

Positivestellnaz (TV-SDPs)

Positivestellnaz for TV-SDPs, (H. Dette and W. J. Studden, 2002)

A polynomial matrix $X(t)$ is psd (i.e. $X(t) \succeq 0$) for all $t \in [-1, 1]$ iff it can be written as

$$X = \Sigma_0 + (1 - t)\Sigma_1 + (1 + t)\Sigma_2 + (1 - t^2)\Sigma_3,$$

where $\Sigma_i(t) = A_i(t)^T A_i(t)$, $i = 0, \ldots, 3$ with $deg(A_i) \leq deg(X)$.

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Topic

Geometry of a TV-LP

$$\begin{array}{ll} \underset{x(t)}{\text{maximize}} & \int_{-1}^{1} \langle c(t), x(t) \rangle dt \\ \text{subject to} & A(t)x(t) \leq b(t) \ \ \forall t \in [-1, 1] \end{array} \tag{TV-LP}$$

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Assumptions

The feasible set $\mathcal{P}_t$ at time $t \in [-1,1]$ is:

- not empty
- bounded.

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Theorem

- There exist :
  - N break points $-1 = t_1 < \cdots < t_N = 1$,
  - $N-1$ finite sets of rational functions $\mathcal{V}_1, \ldots, \mathcal{V}_{N-1} \subset \mathbb{R}^n(X)$.

such that:

$$\mathcal{P}_t = conv\{v(t), \ v \in \mathcal{V}_i\}$$

for every $i \in [N-1], t \in (t_i, t_{i+1})$.

- Every $v \in \mathcal{V}_i$ has the form $v(t) = A_{B_v}(t)^{-1} b_{B_v}(t)$.

Algorithms

Computing the t_i

A vertex $v(t) = A_B(t)b_B(t)$ disappears if:

Find $\mathcal{V}_i$

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- Pick a time $t \in (t_i, t_{i+1})$, and find the vertices of $\mathcal{P}_t$.
- We need a lower bound on $\min_i t_{i+1} - t_i$.
- Result of Mahler: If α, β two distinct roots of $P = \sum_i^n a_i X_i$, then

$$|\alpha - \beta| \geq C_n \frac{1}{\max |a_i|^{n-1}}.$$

Algorithms

Finding the optimal solution:

A vertex $v(t) \in \mathcal{V}_i$ is no longer optimal if:

- $t \to \langle c(t), v(t) \rangle - \langle c(t), w(t) \rangle$ changes sign for some other vertex $w \in \mathcal{V}_i$.

Topic

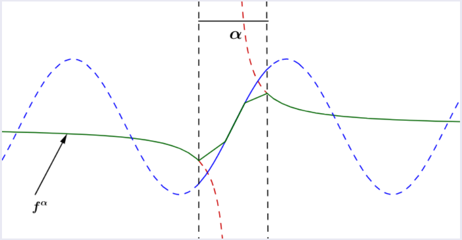
Feasibility and Near-Optimality are equivalent

Theorem

If the TV-LP admits a feasible continuous solution f_0, then, continuous solutions are near optimal.

Proof.

Construct a near optimal solution f^{α} that lives on the optimal vertex, travels to the continuous solution f_0 to get through the possibly problematic time t_i.



Continuity with respect to perturbations

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \langle c, x \rangle \\ \text{subject to} \quad & Ax \leq b \end{aligned} \tag{LP}$$

- $\Omega := \{(A, b) \mid \quad \{x \in \mathbb{R}^n, Ax \leq b\}$ is non empty and bounded$\}$.
- $opt(A, b, c) := \max_{Ax \leq b} \langle c, x \rangle$ defined for $(A, b, c) \in \Omega \times \mathbb{R}^n$.

Theorem (D. H. Martin, 1975)

$opt(A, b, c)$ is

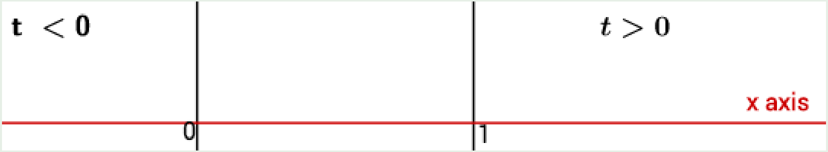
- *Continuous with respect to the variables b and c.*
- *Upper semi-continuous with respect to the variable A.*

What could go wrong?

Example (A "discontinuous" TV-LP)

$\mathcal{P}_t := \{x \in \mathbb{R}, tx \geq 0, t(x-1) \geq 0\}$ for $t \in [-1, 1]$.

- $\mathcal{P}_t = [1, \infty)$ when $t > 0$.
- $\mathcal{P}_t = (-\infty, 0]$ when $t < 0$. No continuous solution!



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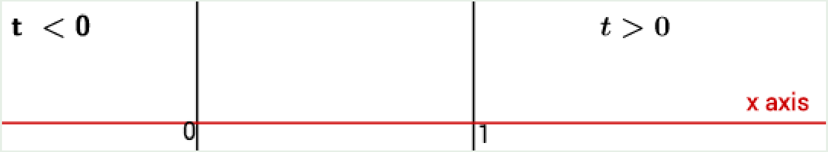

Equivalent condition

Theire exists a continuous feasible solution if and only if

$$conv\{v(t_i), v \in \mathcal{V}_i\} \cap conv\{v(t_i), v \in \mathcal{V}_{i+1}\} \neq \emptyset$$

for $i = 1, \ldots, N-1$.

An easy case

$A(t)$ is fixed

If $A(t) =$ is constant, their always exists a continuous feasible solution.

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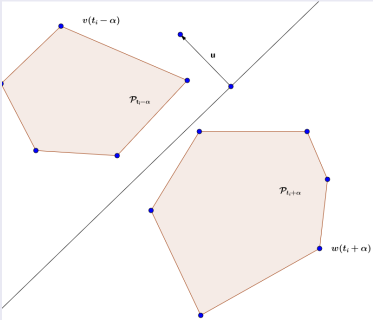
  $\underset{v \in \mathcal{V}_i}{conv}\, v(t_i) \cap \underset{w \in \mathcal{V}_{i+1}}{conv}\ v(t_i) = \emptyset$.
- $\exists u \in \mathbb{R}^n$ and $\delta > 0$:
  - $\langle v(t_i), u \rangle > \delta$ for $v \in \mathcal{V}_i$.
  - $\langle w(t_i), u \rangle < -\delta$ for $w \in \mathcal{V}_{i+1}$.



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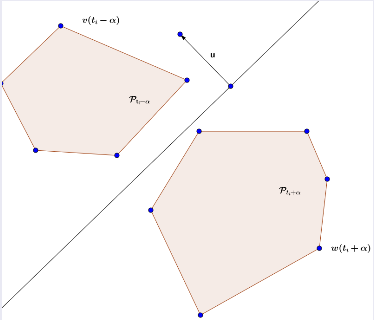
- $\exists u \in \mathbb{R}^n$ and $\delta > 0$:
  - $\langle v(t_i), u\rangle > \delta$ for $v \in \mathcal{V}_i$.
  - $\langle w(t_i), u\rangle < -\delta$ for $w \in \mathcal{V}_{i+1}$.
- Contradicts the continuity of the optimal value of

$$\underset{x \in P_{t_i+\alpha}}{\text{minimize}} \langle x, u\rangle.$$

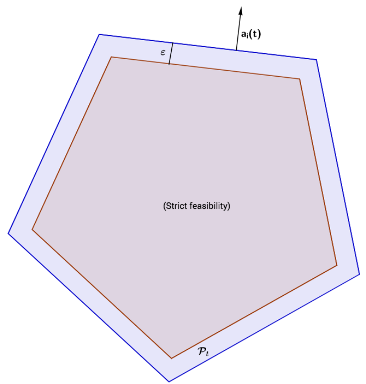


Strict Feasibility

Definition (Strict Feasibility)

A TV-LP is *strictly feasible* if there exists a (not necessarily continuous) function $x^s : [-1, 1] \to \mathbb{R}^n$ and a scalar $\varepsilon > 0$ such that

$$A(t)x^s(t) \leq b(t) - \varepsilon \mathbf{1}, \ \forall t \in [-1, 1].$$



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Proof.

- It is enough to prove the existence of a continuous feasible solution $x^c(t)$.

We construction $x^c(t)$ in two steps:

- near the problematic points t_i.
- away from the t_i.

□

Near the problematic points t_i:

- Choose an arbitrary vertex $w := A_b(t)^{-1}(b(t) - \varepsilon\mathbf{1})$ of the non-empty polytope $\{x \in \mathbb{R}^n | A(t_i)x \leq b(t_i) - \varepsilon\mathbf{1}\}$.
- Define $w_i^{near}(t) := A_B(t)^{-1}(b_B(t) - \varepsilon 1)$.
- By continuity, $\exists$ a neighborhood $[t_i - \alpha, t_i + \alpha]$, such that $w_i^{near}(t)$ is a well defined continuous function and $w_i^{near}(t)$ is strictly feasible.
- Furthermore, since the number of breakpoints t_i s is finite, we can make the same choice of α for all $i = 1, \cdots, N$.

Far away from the t_i:

- For $t \in (t_i, t_{i+1})$, let $w_i^{far}(t) := \frac{\sum_{u \in \mathcal{V}_i} u(t)}{|\mathcal{V}_i|} \in \mathcal{P}_t$.
- $\delta_i := \min_{t \in J_i, j=1,\ldots,m} (b(t) - A(t) w_i^{far}(t))_j$.
- Observe that $\delta_i > 0$. O.w., by continuity, there exist $\hat{j}$ and $\hat{t} \in J_i$ such that $(b(\hat{t}) - A(\hat{t}) w^{far}(\hat{t}))_{\hat{j}} = 0$.
    - This means that $\mathcal{P}_{\hat{t}} \subseteq \{x \in \mathbb{R}^n |\ A_{\hat{j}}(\hat{t})^T x = b_{\hat{j}}(\hat{t})\}$

Connecting the patches:

Recap

We have constructed w_i^{near}, w_i^{far} that are continuous and strictly feasible on (t_i, t_{i+1}) and $(t_{i+1} - \alpha, t_{i+1} + \alpha)$ resp.

We get a continuous feasible solution on $[-1, 1]$ simply by "connecting" the solutions w_i^{far}, w_i^{near} by interpolating from one to the other linearly.

Topic

What could go wrong?

Optimality of continuous functions $\implies$ Optimality of polynomials?

Example (No! A "Tight" TV-LP)

- $(1 + t^2)x(t) \leq 1$
- $-(1 + t^2)x(t) \leq -1$

Only one solution $x(t) = \frac{1}{1+t^2}$. Not polynomial.

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Definition (Continuous Full-Dimensionality)

TV-LP is *continuously full-dimensional* if there exists a constant $\delta > 0$ and a continuous function $x^c : [-1,1] \to \mathbb{R}^n$ such that $B(x^c(t), \delta) \subset \mathcal{P}_t,\ \forall t \in [-1,1]$.

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Full-Dimensionality $\implies$ Optimality of Polynomials

- Approximate $x^c(t)$ by a polynomial.

Strict feasibility vs Continuous Full dimensionality

- Strict Feasibility provides slackness in the space of the constraints.
- Continuous Full dimensionality provides slackness in the space of the variables.

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Yes, if the rows of $A(t)$ don't cancel!

- Let $\varepsilon := \min_{i=1,\ldots,n} \min_{t\in[-1,1]} (b(t) - A(t)x^c(t))_i$.
- $\varepsilon > 0$.Otherwise, $\exists (t_m, i_m)$ for which $b_{i_m}(t_m) - A_{i_m}(t_m)x^c(t_m) = 0$.
- If $u \in \mathbb{R}^n$ has norm smaller than δ, then $b_{i_m}(t) - A_{i_m}(t_m)(x^c(t_m) + u) \geq 0$, which leads to $A_i(t_m)^T u \geq 0$, and to $A_i(t_m) = 0$.

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Yes

- Strict feasibility $\implies$ existence of continuous strict feasible solution $x^c(t)$, $A(t)x^c(t) \leq b(t) - \mathbf{1}\varepsilon$.
- If $||y|| = 1$, $A(t)(x^c(t) + \delta y) \leq b(t) - \mathbf{1}\varepsilon + \delta A(t)y \leq b(t) - (\varepsilon - \delta||A||)\mathbf{1}$.

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Theorem (Strict feasibility $\implies$ Optimality of Polynomial solutions)

If a TV-LP is strictly feasible, then polynomials as near optimal.

Application: MinCut

Maxflow(Primal)

$$\max_{f_{ij}} \sum_{j\sim 1} f_{1,j}$$
$$\sum_{j\sim i} f_{i,j} - f_{j,i} = 0, \quad i \in V$$
$$0 \leq f_{i,j} \leq b_{ij}, \quad i \sim j$$

Mincut (Dual)

$$\min_{d_{ij}, p_i} \sum_{i\sim j} b_{ij} d_{ij}$$
$$d_{ij} - p_i + p_j \geq 0, \quad i \sim j$$
$$p_1 - p_n \geq 1$$
$$p_i \geq 0, \quad i \in V$$
$$d_{ij} \geq 0, \quad i \sim j$$

Application: MinCut

Maxflow(Primal)

$$\max_{f_{ij}} \int_{-1}^{1} \sum_{j\sim 1} f_{1,j}(t)dt$$
$$\sum_{j\sim i} f_{i,j}(t) - f_{j,i}(t) = 0, \quad i \in V$$
$$0 \leq f_{i,j}(t) \leq b_{ij}(t), \quad i \sim j$$

Mincut (Dual)

$$\min_{d_{ij}, p_i} \int_{-1}^{1} \sum_{i\sim j} b_{ij}(t) d_{ij}(t)dt$$
$$d_{ij}(t) - p_i(t) + p_j(t) \geq 0, \quad i \sim j$$
$$p_1(t) - p_n(t) \geq 1$$
$$p_i(t) \geq 0, \quad i \in V$$
$$d_{ij}(t) \geq 0, \quad i \sim j$$

Application: MinCut

Maxflow(Primal)

$$
\begin{aligned}
&\max_{f_{ij}} \int_{-1}^{1} \sum_{j\sim 1} f_{1,j}(t)dt \\
&\sum_{j\sim i} f_{i,j}(t) - f_{j,i}(t) = 0, \quad i \in V \\
&0 \le f_{i,j}(t) \le b_{ij}(t), \qquad i \sim j
\end{aligned}
$$

Mincut (Dual)

$$
\begin{aligned}
&\min_{d_{ij}, p_i} \int_{-1}^{1} \sum_{i\sim j} b_{ij}(t) d_{ij}(t)dt \\
&d_{ij}(t) - p_i(t) + p_j(t) \ge 0, \quad i \sim j \\
&p_1(t) - p_n(t) \ge 1 \\
&p_i(t) \ge 0, \qquad i \in V \\
&d_{ij}(t) \ge 0, \qquad i \sim j
\end{aligned}
$$

Simulation

- Mincut is strictly feasible.
- Find best polynomial solution to both of degree 9.
- $85.42 \le opt \le 85.52$.

Topic

Numerical Stability

- Choice of points $t_0 < \cdots < t_{2k}$.
- Choice of basis of $\mathbb{R}_k[t]$, $p_0, \ldots, p_k$.
- $A^{(l)} = (p_i(t_l)p_j(t_l))_{ij}$.
- $q(t)$ is in SOS if and only if there exists $X \succeq 0$ such that $q(t_l) = \langle X, A^{(l)} \rangle$, $\forall l \in [2k]$

Choice of breakpoints and basis

- $t_i = \cos((i + \frac{1}{2})\frac{\pi}{2k+1})$ for $i \in [2k]$
- $(p_j(t))_{j\in[k]}$ scaled Chebyshev polynomial.

$$p_0(t) = \frac{1}{2k+1}, p_1(t) = \sqrt{\frac{2t}{2k+1}}, p_i(t) = 2tp_{i-1}(t) - p_{i-2}(t) \text{ for } i = 2, 3 \ldots$$

- This makes the columns of the matrix $A^{(l)}$ orthonormal.

Topic

$$\begin{aligned} \underset{X \in \mathcal{S}_n}{\text{maximize}} \quad & \langle C, X \rangle \\ \text{subject to} \quad & A_i X \leq b_i \qquad \forall i \in [m], \end{aligned} \tag{TV-SDP}$$

$$\begin{aligned}
\underset{X(t)\in\mathcal{S}_n[t]}{\text{maximize}} \quad & \int_{-1}^{1}\langle C(t), X(t)\rangle dt \\
\text{subject to} \quad & A_i(t)X(t) \leq b_i(t) \qquad \forall i \in [m],\ \forall t \in [-1,1]
\end{aligned} \tag{TV-SDP}$$

$$\max_{X(t)\in\mathcal{S}_n[t]} \quad \int_{-1}^{1}\langle C(t), X(t)\rangle dt \qquad \text{(TV-SDP)}$$
$$\text{subject to} \quad A_i(t)X(t) \leq b_i(t) \qquad \forall i \in [m],\ \forall t \in [-1,1]$$

Definition (Strict Feasibility for TV-SDPs)

A TV-SDP is *strictly feasible* if there exists a (not necessarily continuous) function $X^s : [-1,1] \to \mathcal{S}_n$ and a scalar $\varepsilon > 0$ such that

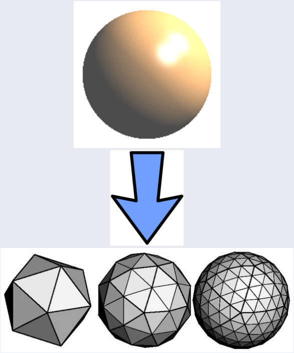
- $X^s(t) \succeq \varepsilon I,\ \forall t \in [-1,1]$.
- $\langle A_i(t), X^s(t)\rangle \leq b_i(t) - \varepsilon,\ \forall t \in [-1,1]$.

If a TV-SDLP is strictly feasible, then polynomials are near optimal.

Approximating a spectrahedron by a polyhedron

- $N(\varepsilon)$ a ε -covering of $\{X \succeq 0, ||X|| = 1\}$.
- Replace $X(t) \succeq 0$ by $X(t) \in \sum \mathbb{R}^+[t]N(\varepsilon)$.

$$\begin{aligned}
&\max_{X(t)} \int_{-1}^{1} \langle X(t), C(t)\rangle dt \\
&\text{s.t.} \\
&X(t) = \sum_{Y \in N(\varepsilon)} \alpha_Y(t) Y \\
&\langle A_i(t), X(t)\rangle \leq b_i(t), \\
&i \in [m], t \in [-1, 1] \\
&\qquad\qquad\qquad (APPROX - LP_\varepsilon)
\end{aligned}$$



$APPROX - LP_\varepsilon$

$$\max_{X(t)} \int_{-1}^{1} \langle C(t), X(t)\rangle dt$$
s.t.
$$X(t) = \sum_{Y \in N(\varepsilon)} \alpha_Y(t) Y$$
$$\langle A_i(t), X(t)\rangle \leq b_i(t)$$

TV-SDP

$$\max_{X(t) \in \mathcal{S}_n[t]} \int_{-1}^{1} \langle C(t), X(t)\rangle dt$$
s.t.
$$A_i(t)X(t) \leq b_i(t)$$

Lemma

As $\varepsilon \to 0$, the optimal value of $APPROX - LP_\varepsilon$ converges to the optimal value of the TV-SDP.

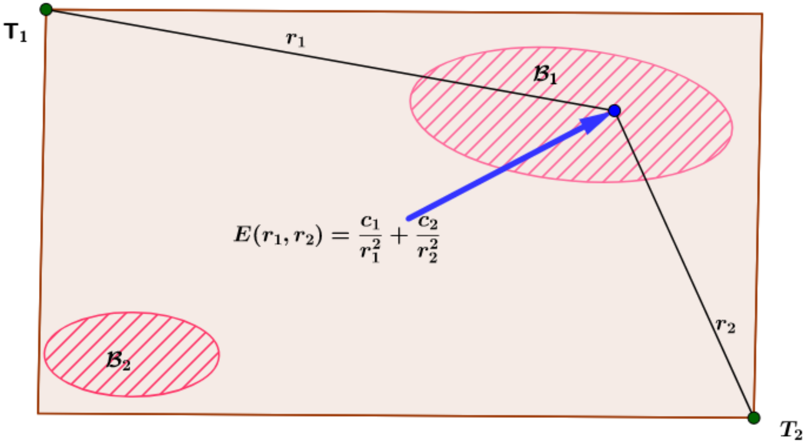
Feasible set of $APPROX - LP_\varepsilon$ $\to$ feasible set of the TV-SDP.

Lemma

Polynomial solutions are near optimal for $APPROX - LP_\varepsilon$.

TV-SDP strictly feasible $\implies$ $APPROX - LP_\varepsilon$ strictly feasible

Wireless Coverage Problem



$$\mathcal{B}_j = \{(x, y), \quad || \begin{pmatrix} x \\ y \end{pmatrix} - z_j|| \leq 1\}, j = 1, 2.$$

Minimize $c_1 + c_2$

$$E(x, y) \geq C \quad \forall (x, y) \in \mathcal{B}_1 \cup \mathcal{B}_2.$$

Equivalently:

$$p(x, y) := -C \prod_{i=1}^{2} [(x - \bar{x}_i)^2 + (y - \bar{y}_i)^2] + \sum_{i=1}^{2} [(x - \bar{x}_i)^2 + (y - \bar{y}_i)^2] c_i \geq 0$$

SOS relaxation:

$$p(x, y) = \sigma^{(j)} + \mu^{(j)}(1 - (x - \bar{x}_j)^2 - (y - \bar{y}_j)^2) \quad j = 1, 2$$

$$\sigma^{(j)} = z' P^{(j)} z, \mu^{(j)} = z' Q^{(j)} z, , z \text{ vector of monomials in x and y.}$$

$$P^{(j)}, Q^{(j)} \succeq 0$$

$$\mathcal{B}_j = \{(x, y, t), \quad || \begin{pmatrix} x \\ y \end{pmatrix} - z_j(t) || \leq 1\}, j = 1, 2.$$

Minimize $\int_{-1}^{1} c_1(t) + c_2(t) dt$

$$E(x, y, t) \geq C \quad \forall (x, y, t) \in \mathcal{B}_1 \cup \mathcal{B}_2.$$

Equivalently:

$$p(x, y, t) := -C \prod_{i=1}^{2} [(x - \bar{x}_i)^2 + (y - \bar{y}_i)^2] + \sum_{i=1}^{2} [(x - \bar{x}_i)^2 + (y - \bar{y}_i)^2] c_i(t) \geq 0$$

SOS relaxation:

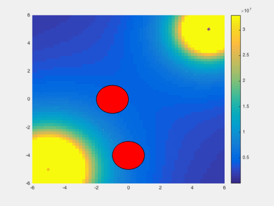
$$p(x, y, t) = \sigma^{(j)}(t) + \mu^{(j)}(t)(1 - (x - \bar{x}_j)^2 - (y - \bar{y}_j)^2) \quad j = 1, 2$$

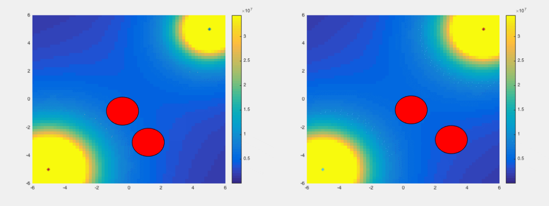
$$\sigma^{(j)}(t) = z' P^{(j)}(t) z, \mu^{(j)} = z' Q^{(j)}(t) z, , z \text{ vector of monomials in x and y.}$$

$$P^{(j)}(t), Q^{(j)}(t) \succeq 0 \quad , t \in [-1, 1]$$

Results

d	$c1(t)$	$c2(t)$	$\int_{-1}^{1}(c_1(t)+c_2(t))dt$
0	31.96	21.63	107.19
1	$28.97 + 4.07t$	$24.23 - 3.7t$	106.38
2	$26.67 + 6.1t + 0.47t^2$	$25.78 - 5.82t + 0.44t^2$	105.49
7	$26.21 + 7.49t + 0.43t^2$ $-3.27t^3 + 2.95t^4 - 0.15t^5$ $-0.63t^6$	$26.18 + 7.16t + 0.81t^2$ $3.02t^3 - 3.38t^4 + 0.44t^5$ $0.63t^6$	105.42





Conclusion and Future Work

- Natural method to optimize over polynomial solutions to TV convex program using SOS.
- Sufficient conditions under which polynomial solutions are optimal.
- Strict feasibility exclude equality constraints.
- Except for LPs, SOS optimization scales poorly.