$$X = \underbrace{M}_{\text{local martingale}} + \underbrace{A}_{\text{bounded variation process}}$$

Ito: $f \in \mathcal{C}^2, df(X_t) = f'(X_s)dX_s + \frac{1}{2}f''(X_s)d < M >_s$

1 Basic concepts of SPT

Starting point: semimartingale market models, ie:

$$dB(t) = r(t)B(t)dt \tag{1}$$

$$dX_i(t) = X_i(t)\left(b_i(t)dt + \sum_\nu \sigma_{i,\nu} dW_\mu(t)\right) \tag{2}$$

Here:

- $B(t)$ is the value of the bank accound if we start from 1 dollar today.
- $X_i(t)$ stands for the price of one share of stock of company i.
- $r(t)$ is the short rate.
- $b_i(t)$ rate of return of stock i.
- $\sigma_{i,\nu}(t)$ volatility of stock i with respect to W_ν.

Theorem 1 (Solutions)**.** *(1) and (2) admist solutions (as long as we know the ?)* $B(t) = e^{\int_0^t r_s ds}$

$$X_i(t) = X_i(0)\exp(\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s))$$

where

$$\gamma_i(t) = b_i(t) - \frac{1}{2}a_{ii}(t) = b_i(t) - \frac{1}{2}\sum_{\mu=1}^d \sigma_{i\mu}(t)$$

is called the growth rate.

Proof.

- $e^{\int_0^t r(s)ds}$ is a process of bounded variations. ($\int_0^t r(s)ds = \int_0^t r(s)^+ds - r(s)^-ds$) By Ito's formula for the semi martingale $\int_0^t r(s)ds$ and $f = \exp$ $\mathrm{d}e^{\int_0^t r(s)ds} = e^{\int_0^t r(s)ds}\mathrm{d}(\int_0^t r(s)ds) = e^{\int_0^t r(s)ds}r(t)dt$.
-
$$X_i(t) = X_i(0)e^{\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)}$$
$$d\,log(X_i(t)) = d(\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)) = \gamma_i(t)dt + \Sigma_{\nu=1}^d \sigma_{i,\nu}(t)dW_\nu(t)$$
$$\begin{aligned} d\,log(X_i(t)) &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\frac{1}{X_i(t)^2}\underbrace{X_i(t)^2\sum \sigma_{i\mu}^2(t)dt}_{d<X_i>(t)} \\ &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\sum \sigma_{i\mu}^2(t)dt \end{aligned}$$

□

Remak 1 (growth rate)**.**

$$\frac{1}{T}\log X_i(t) - \frac{1}{T}\int_0^T \gamma_i(t)dt \to 0$$

Whenever σ does not grow too fast in T.

Proof.

$$\frac{1}{T}\log X_i(t) - \frac{1}{T}\int_0^T \gamma_i(t)dt = \frac{1}{T}\int_0^T \sum_\nu \gamma_{i\nu}(t)dW_\nu(t)$$

□

Theorem 2 (Time change martingale)**.** *Every stochastic integral* $I_t = \sum \int h_\nu dW_\nu(s)$ *can be written as a time change of a brownian motion* β *where*

$$\beta(s) = I_{\tau_s}$$

$$\tau_s = \inf\{t : \int_0^t \sum h_\nu(s)^2 ds\}$$

$I_t = \beta(< I >_t)$

2 Class Portfolios old theory

Definition 1 (Portfolios)**.** *Fix a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *such that* B, X_i, r, b, σ *are adapated to it. A portfolio* $\Pi(t) = (\Pi_1(t), \ldots, \Pi_n(t))$*is a bounded progressively measurable process with respect to* $(\mathcal{F}_t)_t$ *such that:*

$$\sum_i \Pi_i(t) = 1 \ \forall t$$

We Π *call long-only portfolio if* $\pi_i(t) \geq 0 \forall i$

Definition 2 (Progessively measurable)**.** $\Pi(t)$ *measurable with respect to* $\cup_{s<t}\mathcal{F}_s$

Example 1.

- *Equal weigted portfolio:* $\Pi_1(t) = \ldots = \Pi_n(t) = \frac{1}{n}$.
- *Market portfolio: Suppose company i has* $N_i(t)$ *shares at time t* $\Pi_i(t) = \frac{X_i(t)V_i(t)}{\sum X_j(t)V_j(t)}$

Assumption: All portfolios Π are self financing($\iff$ we immediately re investing all gain from traind). Mathematically, the portfolio value $V^{(\pi)}(t) = \sum \Pi_i(t)X_i(t)$ satisfies the equation $\frac{dV^\pi(t)}{V_i^{pi}(t)} = \sum_i \pi_i(t)\frac{dX_i(t)}{X_i(t)}$.

Theorem 3. *Has an explicit solution*

$$V^{(\pi)}(t) = V^{(\pi)}(0)\exp(\int_0^t \gamma_\pi(u)du + \int_0^t \sum_\nu \sigma_{\pi\nu}(u)dW_\nu(u))$$

$\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t)$ $\gamma_\pi^*(t) = \frac{1}{2}(\sum \pi_i(t)a_{ii}(t) - \sum_{i,j}\pi_i(t)\pi_j(t)a_{i,j}(t))$

$$\sigma_{\pi\nu}(t) = \sum_i \pi_i(t)\sigma_{i\mu}(t)$$

Definition 3 (Portfolio)**.**

- *Classical portfolios:*

$$\zeta(t) = (\underbrace{\zeta_i(t)}_{(\#\ of\ share)})_i$$

- *Self financing condition: portfolio value* $V(t) = \zeta.X$ *satisfies* $dV = \zeta.dX$
- *in SPT, we wwwwant to think about weights.* $\Pi_i(t) = \frac{\zeta_i(t)X_i(t)}{\zeta.X}$
- *It only make sens to think of V in relative terms:*

$$\frac{dV^{(\pi)}(t)}{V^\pi(t)} = \sum_i \pi_i(t)\frac{dX_i(t)}{X_i(t)}$$

Theorem 4. *Has an explicit solution*

$$V^{(\pi)}(t) = V^{(\pi)}(0)\exp(\int_0^t \gamma_\pi(u)du + \int_0^t \sum_\nu \sigma_{\pi\nu}(u)dW_\nu(u))$$

$$\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \underbrace{\gamma_\pi^*(t)}_{excess\ growth\ rate}$$

$$\gamma_\pi^*(t) = \frac{1}{2}(\sum \pi_i(t)a_{ii}(t) - \sum_{i,j}\pi_i(t)\pi_j(t)a_{i,j}(t))$$

$$\sigma_{\pi\nu}(t) = \sum_i \pi_i(t)\sigma_{i\mu}(t)$$

We can prove that $\frac{1}{T}\log(V^\pi(t)) - \frac{1}{T}\int_0^T \gamma^\pi(u)du \to 0$

Remak 2 (Market portfolios and market weights)**.** ***Disclaimer:*** *Fron now on, think of* $X_i(t)$ *as the market capitalization of company i (# shares . price per share).*

2.0.1 The market portfolio

Recall: the market portfolio has weights $\pi_i(t) = \frac{X_i(t)}{\sum X_j} = \mu_i(t)$. For the market portfolio:

$$\frac{1}{T}\int_0^T \gamma^\mu du = \frac{1}{T}\int_0^T \sum \gamma_i(u)\mu_i(u)du + \frac{1}{T}\int_0^T \gamma_\mu^*(u)du$$

If in the original model for X_i the coefficients only depend on the μ_is: $b_i(t) = \bar{b}_i.\mu, \sigma_{i\nu}(t) = \bar{\sigma}_{i\nu}.\mu$ then we are taking the average of a function on μ:

$$\frac{1}{T}\int_0^T f(\mu_1(t), \ldots, \mu_n(t))dt$$

$\mu \to \int_0^T f(\mu(t))dt$ is aclled an additive functional. To understand market portfolio:

- Need to understand how μ begaves in the real world.
- Select a class of models compatible with that.
- Study the assymptotics of the additive functional, which will give us the asymptotic growth of market portfolio.

Main observation (Fernholz): rank the market weights: $\mu_{(1)} \geq \ldots \geq \mu_{(n)}$

- the curve $\log k \to \log \mu_{(k)}(t)$ is very stable over time.
- shape is close to linear (weights decay poly)
- $\implies$ look for models of $(\mu_1(t), \ldots, \mu_n(t))$ so that $(\mu_{(1)}(t), \ldots, \mu_{(n)}(t))$ is stochastically stable. e.g. there exist an initial distribution of $(\mu_{(1)}(0), \ldots, \mu_{(n)}(0)) \stackrel{d}{=} (\mu_{(1)}(t), \ldots, \mu_{(n)}(t))$ Such a distribution is a called a stationary / invariant distribution of the process.