# Time-Varying LPs and SDPs

Joint work with Amirali Ahmadi

Bachir El Khadir

Princeton University

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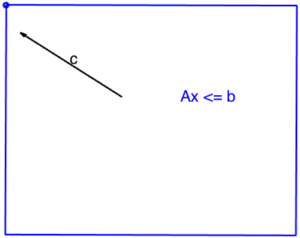
# Outline

# Topic

# TV-LP

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \langle c, x \rangle \\ \text{subject to} \quad & Ax \leq b \end{aligned} \tag{LP}$$



# TV-LP

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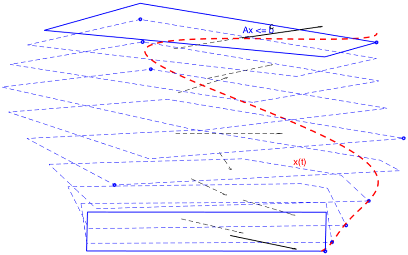
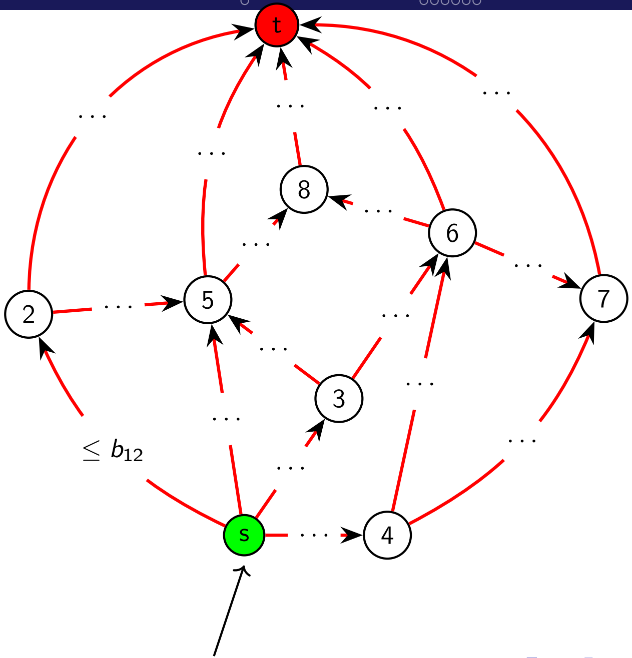



- $A, b, c$ polynomials.
- Polynomials are general enough.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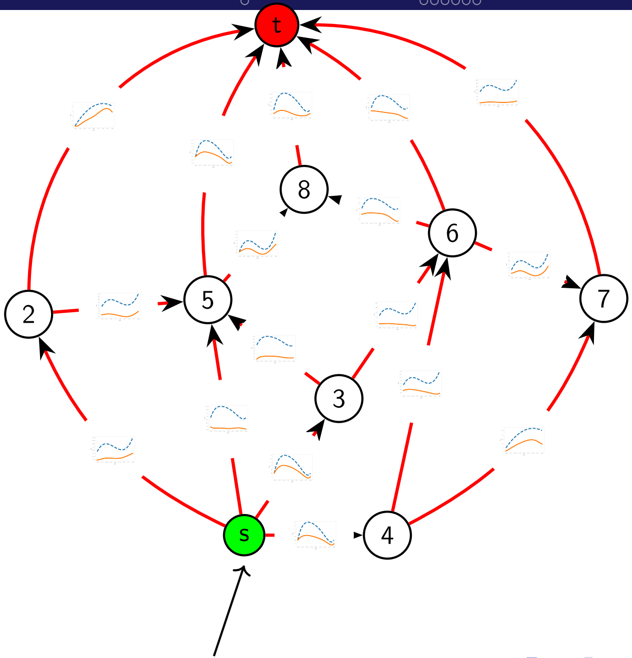
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- Study existence and optimality of polynomial solutions.
- Find the best polynomial solution of a given degree to a TV-LP / TV-SDP using a (non time-varying) SDP.

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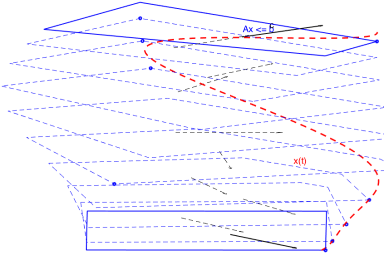
- Generally no!



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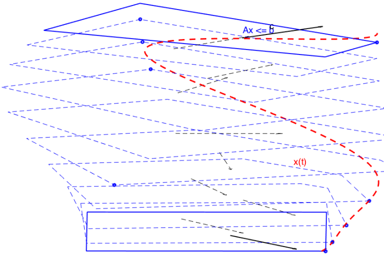
## Near Optimality

$\forall \varepsilon > 0$, $\exists x(t) \in \mathbb{R}^n[t]$ such that:

- $A(t)x(t) \leq b(t)$
- $opt - \int_{-1}^{1} \langle c(t), x(t) \rangle dt \leq \varepsilon$.



# Topic

# We are interested in continuous solutions

## Problems in practice:

- Deciding the transmission power of a cell tower during the day.
- Chosing the optimal control of a robotic arm.
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We want smooth solutions!

## Positivstellensatz for TV-LPs (Polya-Szego, 1976)

Every nonnegative univariate polynomial $p(t)$ on $[-1,1]$ can be written as

$$p = \sigma_0 + (1-t)\sigma_1 + (1+t)\sigma_2 + (1-t^2)\sigma_3,$$

where $\sigma_i \in SOS$, $i = 0, \ldots, 3$, with degree bounded by $deg(p)$.

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## In our case

- Constraint $A_i(t)x(t) \leq b_i(t) \quad \forall t \in [-1, 1]$
- Becomes
  $b_i(t) - A_i(t)x(t) = \sigma_0(t) + (1 - t)\sigma_1(t) + (1 + t)\sigma_2(t) + (1 - t^2)\sigma_3(t)$

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## That's a (non time-varying) SDP

$$\sigma(t) \in SOS \iff \exists Q \succeq 0, \sigma(t) = \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{\frac{n}{2}} \end{pmatrix}^T Q \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{\frac{n}{2}} \end{pmatrix}$$

# Topic

# Geometry of a TV-LP

$$\begin{aligned} \underset{x(t)}{\text{maximize}} \quad & \int_{-1}^{1} \langle c(t), x(t) \rangle dt \\ \text{subject to} \quad & A(t)x(t) \leq b(t) \;\; \forall t \in [-1, 1] \end{aligned} \tag{TV-LP}$$

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## Assumptions

The feasible set $\mathcal{P}_t$ at time $t \in [-1, 1]$ is:

- nonempty
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## Theorem (Geometry of the Feasible Set)

- *There exist:*
  - *$N$ break points $-1 = t_1 < \cdots < t_N = 1$,*
  - *$N - 1$ finite sets of rational functions $\mathcal{V}_1, \ldots, \mathcal{V}_{N-1} \subset \mathbb{R}^n(t)$,*

*such that:*

$$\mathcal{P}_t = conv\{v(t),\ v \in \mathcal{V}_i\}$$

*for every $i \in [N - 1]$, $t \in (t_i, t_{i+1})$.*

- *Every $v \in \mathcal{V}_i$ has the form $v(t) = A_{B_v}(t)^{-1} b_{B_v}(t)$.*

# Topic

# Continuous Solutions: What could go wrong?

## Good news

Continuous Feasibility $\implies$ Continuous Optimality.

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## Example (What could go wrong?)

A "discontinuous" TV-LP $\mathcal{P}_t := \{x \in \mathbb{R}, tx \geq 0, t(x-1) \geq 0\}$.

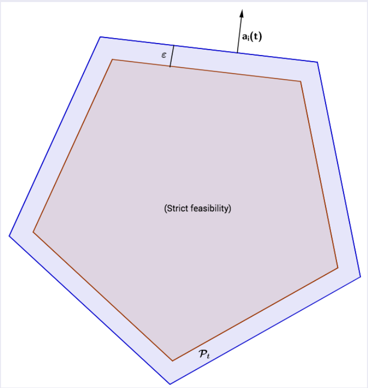
- $\mathcal{P}_t = [1, \infty)$ when $t > 0$.
- $\mathcal{P}_t = (-\infty, 0]$ when $t < 0$. No continuous solution!

# Strict Feasibility

## Definition (Strict Feasibility)

A TV-LP is *strictly feasible* if there exists a (not necessarily continuous) function $x^s : [-1,1] \to \mathbb{R}^n$ and a scalar $\varepsilon > 0$ such that

$$A(t)x^s(t) \leq b(t) - \varepsilon \mathbf{1},\ \forall t \in [-1,1].$$



**Theorem (Strict feasibility $\implies$ Continuous solutions)**

*If a TV-LP is strictly feasible, then it has a continuous near optimal solution. Futhermore, the continuous solution can be chosen to be strictly feasible.*

# Polynomials: What could go wrong?

Optimality of continuous functions $\implies$ Optimality of polynomials?

**Example (No! A "Tight" TV-LP)**

- $(1 + t^2)x(t) = 1$
- Only one solution $x(t) = \frac{1}{1+t^2}$. Not polynomial.

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## Definition (Continuous Full-Dimensionality)

TV-LP is *continuously full-dimensional* if there exists a constant $\delta > 0$ and a continuous function $x^c : [-1, 1] \to \mathbb{R}^n$ such that $B(x^c(t), \delta) \subset \mathcal{P}_t, \ \forall t \in [-1, 1]$.

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**Full-Dimensionality $\Longrightarrow$ Optimality of Polynomials**

- Approximate $x^c(t)$ by a polynomial.

# Strict Feasibility vs Continuous Full-dimensionality

- Strict Feasibility provides slackness in the space of the constraints.
- Continuous full-dimensionality provides slackness in the space of the variables.

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**Theorem (Strict feasibility $\implies$ Optimality of Polynomial Solutions)**

*Strict feasibility $\implies$ Full dimensionality $\implies$ Optimality of Polynomial Solutions.*

# Application: MinCut

## Maxflow (Primal)

$$\max_{f_{ij}} \sum_{j\sim 1} f_{1j}$$
$$\sum_{j\sim i} f_{ij} - f_{ji} = 0, \quad i \in V$$
$$0 \leq f_{ij} \leq b_{ij}, \quad i \sim j$$

Live simulation.

## Mincut (Dual)

$$\min_{d_{ij}, p_i} \sum_{i\sim j} b_{ij} d_{ij}$$
$$d_{ij} - p_i + p_j \geq 0, \quad i \sim j$$
$$p_1 - p_n \geq 1$$
$$p_i \geq 0, \quad i \in V$$
$$d_{ij} \geq 0, \quad i \sim j$$

# Application: MinCut

## Maxflow (Primal)

$$
\begin{aligned}
&\max_{f_{ij}} \int_{-1}^{1} \sum_{j\sim 1} f_{1j}(t)dt \\
&\sum_{j\sim i} f_{ij}(t) - f_{ji}(t) = 0, \quad i \in V \\
&0 \leq f_{ij}(t) \leq b_{ij}(t), \quad i \sim j
\end{aligned}
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## Simulation

- Mincut is strictly feasible.
- Find best polynomial solution to both of degree 9.
- $85.42 \leq opt \leq 85.52$.

# Topic

# TV-SDPs

$$\begin{aligned} \underset{X}{\text{maximize}} \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_i, X \rangle \leq b_i \qquad \forall i \in [m], \\ & X \succeq 0 \end{aligned} \tag{TV-SDP}$$

- Generalisation of TV-LPs where we allow psd constraints

$$X \succeq 0.$$

# TV-SDPs

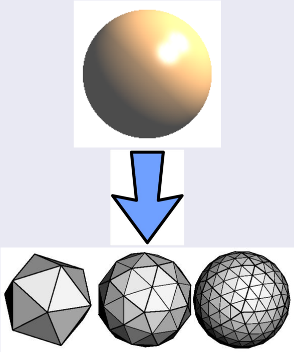
$$\begin{array}{lll} \underset{X(t)}{\text{maximize}} & \int_{-1}^{1} \langle C(t), X(t) \rangle dt & \\ \text{subject to} & \langle A_i(t), X(t) \rangle \leq b_i(t) & \forall i \in [m],\ \forall t \in [-1,1] \\ & X(t) \succeq 0 & \forall t \in [-1,1] \end{array} \qquad \text{(TV-SDP)}$$

- Generalisation of TV-LPs where we allow psd constraints

$$X(t) \succeq 0 \quad \forall t \in [-1,1].$$

# Approximating a spectrahedron by a polyhedron

- $N(\varepsilon)$ a $\varepsilon$ -covering of $\{X \succeq 0, ||X|| = 1\}$.
- Replace $X(t) \succeq 0$ by $\sum_{Y \in N(\varepsilon)} \underbrace{\alpha_Y(t)}_{\geq 0} Y$.



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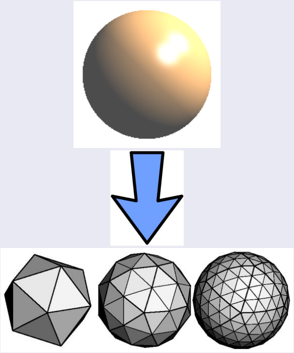
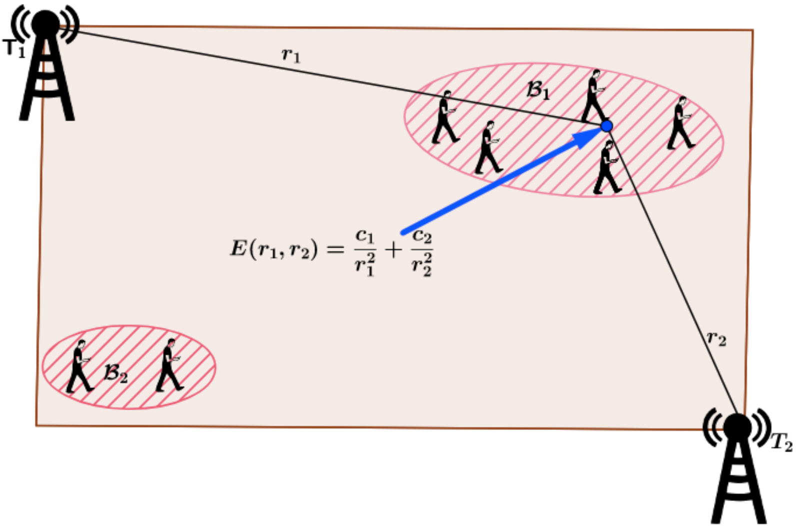


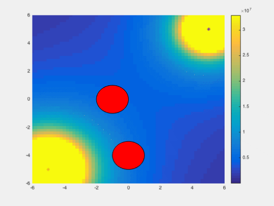
If a TV-SDP is strictly feasible, then polynomials are near optimal.

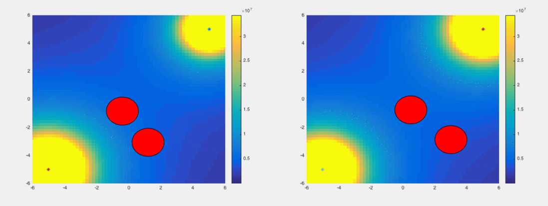



$$r_i^2 = (x - \bar{x}_i)^2 + (y - \bar{y}_i)^2, i = 1, 2$$

# Results

| $d$ | $c_1(t)$ | $c_2(t)$ | $\int_{-1}^{1}(c_1(t)+c_2(t))dt$ |
|---|---|---|---|
| 0 | 31.96 | 21.63 | 107.19 |
| 1 | $28.97 + 4.07t$ | $24.23 - 3.7t$ | 106.38 |
| 2 | $26.67 + 6.1t + 0.47t^2$ | $25.78 - 5.82t + 0.44t^2$ | 105.49 |
| 7 | $26.21 + 7.49t + 0.43t^2$<br>$-3.27t^3 + 2.95t^4 - 0.15t^5$<br>$-0.63t^6$ | $26.18 + 7.16t + 0.81t^2$<br>$3.02t^3 - 3.38t^4 + 0.44t^5$<br>$0.63t^6$ | 105.42 |





# Conclusion and Future Work

- Algorithms to optimize over polynomial solutions to TV-LPs / TV-SDPs using SOS optimization.
- Sufficient conditions under which polynomial solutions are optimal.

## Possible improvements

- Strict feasibility excludes equality constraints.
- Except for TV-LPs, SOS optimization scales poorly. What about SOCP? QCQP?
- Add new dimension.