Definition 1 (Brownian motion)**.** *X_t is a borwnian motion if:*

- $X_0 = 0$
- $X_t - X_s \sim \mathcal{N}(0, t-s)$
- $X_t - X_s \perp \sigma\{X_r, r \leq s\}$
- $t \to X_t$ *continuous*

Question: Does the process exist? Yes (Wiener, 1923) Motivation $X_t^{(N)} = \frac{1}{\sqrt{N}} \sum_{k=1}^{\lfloor Nt \rfloor} \epsilon_k$, ϵ_k iid, with expectation 0 and variance 1. Formally: "$X_t = \lim_N X_t^{(N)}$" (Convergence only in distribuion)

Innovations

1. Design random walks where as N increases, new pts are added **between** existing points. This makes the convergenece a.s.
2. Interpolate linearly between grid points. (To make the descrete paths continuous)
3. Work on a compact set: $t \in [0,1]$, then extend for all $t \geq 0$ at the end.

Lemma 1 (Interpolation)**.** *Take a grid: $0 \leq t_0 < t_1 \ldots < t_n$ Suppose given r.v $X_{t_0} = 0, X_{t_1}, \ldots, X_{t_n}$ st $X_{t_i} - X_{t_{i-1}} \sum \mathcal{N}(0, t_i - t_{i-1})$ and $\{X_{t_i} - X_{t_{i-1}}\}$ are independent. Let*

- $\epsilon \sim \mathcal{N}(0,1) \perp X_{t_0}, \ldots, X_{t_n}$,
- $s = \frac{t_i - t_{i-1}}{2}$
- $X_s := \frac{X_{t_i} - X_{t_{i-1}}}{2} + \frac{1}{2}\sqrt{t_i - t_{i-1}}\epsilon$

then $(X_{t_0}, \ldots, X_{t_n})$ satisfy the properties of the brownian motion.

Theorem 2 (Taking the limit)**.**

Proof. $X_t^{(N)} - X_t^{(N-1)} = \sum_{k=1}^{2^{N-1}} \underbrace{\epsilon_{N,k}}_{\mathcal{N}(0,1)} S_{N,k}(t)$, $S_{N,k}$ is the Schauder functions.

$$\begin{aligned}
\sum_N \sup |X_t^{(N)} - X_t^{(N-1)}| &= \sum_N \sup |\sum_{k=1}^{2^{N-1}} \epsilon_{N,k} S_{N,k}(t)| \\
&\leq \sum_N \max\{|\epsilon_{N,k}| 2^{-\frac{n+1}{2}}, k = 1 \ldots 2^{n-1} - 1\}
\end{aligned}$$

$$\mathbb{P}(\max_{k=1\ldots 2^{n-1}-1} |\epsilon_k| > \frac{1}{n^2}) \leq 2^{n-1}\mathbb{P}(|\epsilon| > \frac{2^{\frac{n-1}{2}}}{n^2}$$

$$\underbrace{\leq}_{\text{markov}} \underbrace{O^*}_{\text{up to a polynomial}} (2^{-\frac{n}{2}})$$

Borel-Cantelli: $\mathbb{P}(\sup |X^n - X^{n-1}| \leq \frac{1}{n^2}\text{eventually}) = 1$

$\Rightarrow \sum \sup |X^n - X^{n-1}| < \infty$

Using the lemma $X_t^{(N)}$ converges uniformly to a continuous process.

In addition, $X_t^{(N)}$ statisfies the properties of BM on the grod $\{k.2^{-N}\}$. Since X_t^N is constant eventually on dyadic rationals g^*, this properties are verified on g^*. □

Lemma 3 (Uniform convergence). *Let f^i a sequence of continuous functions on $[0,1]$. Suppose that*

$$\sum_{n\geq 1} \sup |f^n - f^{n-1}| < \infty$$

Then $\lim_n f^n(t) =: f(t)$ exists and f is continuous.

Proof. $f^n = f^0 + \underbrace{\sum f^k - f^{k-1}}_{\text{absolute convergence}}$, so f exists.

$$\sup |f - f^n| \leq \sum_{k>n} \sup |f^k - f^{k-1}| \to 0$$

. We have uniform convergence of continuous functions, therefore f is continuous. □