$X = \underbrace{M}_{\text{local martingale}} + \underbrace{A}_{\text{bounded variation process}}$

Ito: $f \in \mathcal{C}^2, df(X_t) = f'(X_s)dX_s + \frac{1}{2}f''(X_s)d < M >_s$

1 Basic concepts of SPT

Starting point: semimartingale market models, ie:

$$dB(t) = r(t)B(t)dt \tag{1}$$

$$dX_i(t) = X_i(t)\left(b_i(t)dt + \sum_\nu \sigma_{i,\nu} dW_\mu(t)\right) \tag{2}$$

Here:

- $B(t)$ is the value of the bank accound if we start from 1 dollar today.
- $X_i(t)$ stands for the price of one share of stock of company i.
- $r(t)$ is the short rate.
- $b_i(t)$ rate of return of stock i.
- $\sigma_{i,\nu}(t)$ volatility of stock i with respect to W_ν.

Theorem 1 (Solutions)**.** *(1) and (2) admist solutions (as long as we know the ?)* $B(t) = e^{\int_0^t r_s ds}$ $X_i(t) = X_i(0)e^{\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)}$ *where* $\gamma_i(t) = b_i(t) - \frac{1}{2}a_{ii}(t) = b_i(t) - \frac{1}{2}\sum_{\mu=1}^d \sigma_{i\mu}(t)$

Proof.

- $e^{\int_0^t r(s)ds}$ is a process of bounded variations. ($\int_0^t r(s)ds = \int_0^t r(s)^+ds - r(s)^-ds$) By Ito's formula for the semi martingale $\int_0^t r(s)ds$ and $f = \exp$ $\mathrm{d}e^{\int_0^t r(s)ds} = e^{\int_0^t r(s)ds}d(\int_0^t r(s)ds) = e^{\int_0^t r(s)ds}r(t)dt$.
- $$X_i(t) = X_i(0)e^{\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)}$$
  $$d\,log(X_i(t)) = d(\int_0^t \gamma_i(s)ds + \int \Sigma_{\nu=1}^d \sigma_{i,\nu}(s)dW_\nu(s)) = \gamma_i(t)dt + \Sigma_{\nu=1}^d \sigma_{i,\nu}(t)dW_\nu(t)$$
  $$\begin{aligned} d\,log(X_i(t)) &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\frac{1}{X_i(t)^2}\underbrace{X_i(t)^2\sum \sigma_{i\mu}^2(t)dt}_{d<X_i>(t)} \\ &= \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\sum \sigma_{i\mu}^2(t)dt \end{aligned}$$

□

Definition 1 (Portfolios)**.** *Fix a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *such that* B, X_i, r, b, σ *are adapated to it. A portfolio* $\Pi(t) = (\Pi_1(t), \ldots, \Pi_n(t))$*is a bounded progressively measurable process with respect to* $(\mathcal{F}_t)_t$ *such that:*

$$\sum_i \Pi_i(t) = 1\ \forall t$$

We Π *call long-only portfolio if* $\pi_i(t) \geq 0 \forall i$

Definition 2 (Progessively measurable)**.** $\Pi(t)$ *measurable with respect to* $\cup_{s<t}\mathcal{F}_s$

Example 1.

- *Equal weigted portfolio:* $\Pi_1(t) = \ldots = \Pi_n(t) = \frac{1}{n}$.
- *Market portfolio: Suppose company* i *has* $N_i(t)$ *shares at time* t $\Pi_i(t) = \frac{X_i(t)V_i(t)}{\sum X_j(t)V_j(t)}$

Assumption: All portfolios Π are self financing($\iff$ we immediately re investing all gain from traind). Mathematically, the portfolio value $V^{(\pi)}(t) = \sum \Pi_i(t)X_i(t)$ satisfies the equation $\frac{dV^\pi(t)}{V_i^{pi}(t)} = \sum_i \pi_i(t)\frac{dX_i(t)}{X_i(t)}$.

Theorem 2. *Has an explicit solution*

$$V^{(\pi)}(t) = V^{(\pi)}(0)\exp\left(\int_0^t \gamma_\pi(u)du + \int_0^t \sum_\nu \sigma_{\pi\nu}(u)dW_\nu(u)\right)$$

$\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \gamma^*_\pi(t)$ $\gamma^*_\pi(t) = \frac{1}{2}(\sum \pi_i(t)a_{ii}(t) - \sum_{i,j} \pi_i(t)\pi_j(t)a_{i,j}(t))$

$$\sigma_{\pi\nu}(t) = \sum_i \pi_i(t)\sigma_{i\mu}(t)$$