1 measure theory

Definition 1 (Sigma Algebra) *$\mathcal{F}$ σ-algebra:*

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- $\cup_n A_n \in \mathcal{F}$

Definition 2 (Probability measure) *Probability measure*

- $\mathbb{P}(A) \in [0,1]$
- $\mathbb{P}(\Omega) = 1$
- $A \cap B = \emptyset \rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Theorem 1 (Equivalence additive measure) *The following are equivalent fo μ finitely additive measure:*

- $\mu\sigma-$ *additive*
- μ *continuous from below / above/ at 0.*

Definition 3 (Monotone class theorem) *Monotne class $\mathcal{M} \subset \mathcal{P}(\Omega)$, and is closed under countable monotone unions and intersections.*

Theorem 2 (Monote class theorem) *G an algebra, $\sigma(G) = M(G)$*

Theorem 3 (Sigma in out)

$$\sigma(f^{-1}(A) : A \in \epsilon) = \{f^{-1}(A) : A \in \sigma(\epsilon)\}$$

Definition 4 (Semi-ring)

- $\emptyset \in S$
- $A \cap B \in S \forall A, B \in S$
- *For al $A, B \in S$ there exist pairwise disjoint subset $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i \leq n} C_i$*

Theorem 4 (Caratheodory's Extension Theorem)

- *A measure μ on a semi-ring S can be extendted to a measure on $\sigma(S)$.*
- *If μ is σ-finite, the extension is unique.*

Definition 5 (Consistence)

- $\mathbb{P}^{i_1,...,i_n}[A_1 \times ... \times A_n] = \mathbb{P}^{\pi(i_1),...,\pi(i_n)}[A_{\pi(1)} \times ... \times A_{\pi(n)}]$
- $\mathbb{P}^{i_1,...,i_{n-1}}[A_1 \times ... \times A_{n-1}] = \mathbb{P}^{i_1,...,i_n}[A_1 \times ... \times A_{n-1} \times \mathbb{R}]$

Theorem 5 (Kolmogorov's Extension Theorem) *I non empty. $(\mathbb{P}^{i_1,...,i_n})_{i_1,...,i_n \in I}$ consistent family. There exists a unique probability measure on $\mathbb{P}$ on $(\mathbb{R}^I, \mathbb{B}(\mathbb{R})^{\times I})$ such that*

$$\mathbb{P}[\{\omega \in R^I : (\omega_{i_1}, ..., \omega_{i_n}) \in B] = \mathbb{P}^{i_1,...,i_n}[B]$$

2 Integrals

Theorem 6 (Monotone Convergencen) *$f_1, \ldots$ be a pointwise non-decreasing sequence of non-negative valued measurable functions, set $\sup f_n = f$. Then f is measurable and $\lim_{k\to\infty} \int f_k \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$.*

Theorem 7 (Fatou) *Let $f1, f2, f3, \ldots$ be a sequence of non-negative measurable functions. Define $f = \liminf_{n\to\infty} f_n$. Then f is measurable and $\int_S f \, d\mu \leq \liminf_n \int_S f_n \, d\mu$.*

Theorem 8 (Dominated Convergence) *$g, f_1, f_2, \ldots$ measurable functions suchat $\int |g| < \infty$, $|f_n| \leq g \forall n$ a.s., $f_n \overset{a.s.}{\rightarrow} f$, then $\int |f| \leq \int |g| < \infty$ and $\lim |f_n - f| \rightarrow 0$, $\lim \int f_n \rightarrow \int f$*

Theorem 9 (Funbini) *μ_1, μ_2 are σ-finite.*

- $\int_{\Omega_1 \times \Omega_2} |f| d(\mu_1 \times \mu_2) < \infty \Rightarrow \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int \Omega_2 f$
- $f \geq 0 \; a.s \Rightarrow \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int \Omega_2 f$

Theorem 10 (Inequalities)

- *Holder:* $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \int |fg| \leq (\int |f|^p)^{\frac{1}{p}} (\int |g|^q)^{\frac{1}{q}}$
- *Minkowsky:* $\forall p \leq 0 \; ||f + g||_p \leq ||f||_p + ||g||_p$

Theorem 11 (Borel Cantelli)

- $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n \geq m} A_n] = 0$
- (A_n), *indepedent,* $\sum \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}[\cap_m \cup_{n \geq m} A_n] = 1$

3 Random Variables

Definition 6 (Uniform integrability) *(X_i) u.i iff $\lim_n \sup_i \int_{|X_i|>c} |X_i| d\mathbb{P} = 0$ iff $\lim_n \sup_i \mathbb{E}[1_{|X_i|>c}|X_i|] = 0$*

Theorem 12 (Caracterisation)
- $\forall i |X_i| \leq X \in L_1 \Rightarrow (X_i)$ *uc*
- *uc iff:*
  - $\sup E[|X_i|] < \infty$
  - $\forall \epsilon > 0, \exists \delta > 0 \forall A \mathbb{P}(A) < \delta \Rightarrow \forall i \int_A |X_i| < \epsilon$

Theorem 13 (L_1 Convergencen) *$X_i \xrightarrow{\mathbb{P}} X$, X_i uc. Then $X \in L_1$, $X_i \xrightarrow{L_1} X$*

Theorem 14 (De la Valle-Pousson) *X_i uc $\iff \exists \Phi : \mathbb{R}^+ \to \mathbb{R}^+, \frac{\Phi(x)}{x} \to \infty st \sup \Phi|X_i| < \infty$. Φ can be assumed convex and non-decreasing.*

Theorem 15 (Week Law of large numbers) *$X_i \in L_2$ uncorrelated, $E[X_i] = m, \sup E[X_i^2] < \infty$, then $\frac{\sum_i X_i}{n} \to m$ in L_2.*

Theorem 16 (Caracteristic Function)
- $|\Phi_X(u)| < \Phi_X(0) = 1$
- $\Phi_X(-u) = \bar{\Phi_X}(y)$
- $\Phi_X \in \mathbb{R} \iff X \overset{\mathbb{D}}{=} -X$
- *Φ_x is unifromly continuous.*
- *$E[|X|^n] < \infty \Rightarrow \exists \Phi_X^k \forall k \leq n$, and $\Phi_X^k(u) = E[(iX)^k e^{iuX}]$, and $\Phi_X(u) = \sum_k^n \frac{(iu)^k}{k!} E[X^k] + \frac{(iu)^n}{n!} \mathcal{E}_n(u)$, with $\mathcal{E}_n \to_0 0$*
- $\exists \Phi_X^{2k}(0) \Rightarrow E[X^{2k}] < \infty$
- *Inversion Formula: $\frac{F_X(b)+F_X(b^-)}{2} - \frac{F_X(a)+F_X(a^-)}{2} = \lim \frac{1}{2\pi} \int_{-c}^{c} \frac{e^{-iua}-e^{-iub}}{iu} \Phi_X(u) du$*
- $\int_R |\Phi_X| < \infty \Rightarrow f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \Phi_X(u) du$
- *$X = (X_1, \dots X_n)$ indepdendent $\iff \Phi_X = \prod \Phi_{X_i}$*

Theorem 17 (Continuity Theorem)
- $X_n \xrightarrow{D} X \iff \Phi_{X_n} \to \Phi_X$
- *$\Phi_{X_n} \to \Phi$ and Φ continuous at 0 then $\exists X$ $X_n \xrightarrow{D} X$*
- $X_n \xrightarrow{D} X \iff F_n \overset{in\, C(F_X)}{\to} F_X$

Theorem 18 (LLN) *X_i iid in L_1, $\frac{\sum X_i}{n} \to E[X]$ as and in L_1*

Theorem 19 (CLT) *X_n iid $Var(X) = \sigma^2 < \infty$ then $\frac{1}{\sqrt{n}} \sum \frac{X_i - E[X]}{\sigma} \to \mathcal{N}(0,1)$*

4 Martingales

Theorem 20 (Radon-Nikodym) *$\mu_2 << \mu_1 \Rightarrow \exists f$ unique μ_1-a.s $f = \frac{d\mu_2}{d\mu_1}$*

Theorem 21 (Stopping times)
- *$X_n^\tau = X_0 + (V.X)_n$ is martingale because V is predictable.*
- *If τ bounded $E[X_0] = E[X_\tau]$*
- *$M \geq \tau \geq \sigma$ stopping times, then $E[X_\tau | F_\sigma] = X_\sigma$*

Theorem 22 (Upcrossing inequality) *X_n submartingale. $E[B_n(a,b)] \leq \frac{E[(X_n-a)^+]}{b-a}$*

Theorem 23 (Convergence)
- *X_n submartingale, L_1 bounded, then there exists X_∞ such that $X_n \xrightarrow{a.s} X_\infty$, and $E[|X_\infty|] < \sup E[|X_n|]$*
- *a submartingale that is bounded above converges a.s.*
- *X_n ui submartingale, then there exists $X_\infty \in L_1$ such that $X_n \to X_\infty$ in L_1. Moreover $E[X_\infty | F_n] \geq X_n$.*
- *(F_n) filtration, $E[X|F_n] \to E[X| \cup F_n]$ a.s and L_1 (because u.i.)*
- *X_i iid, $\mathcal{G} = \cup \sigma(X_n, \dots)$, then $\forall A \in \mathcal{G}$ $P(A) \in \{0,1\}$ (because $1_A = E[1_A]$)*
- *(G_i) dec-filtration, $E[X|G_n] \to E[X| \cap G_n]$ as and in L_1.*

Theorem 24 (Doob Maximal inequality) *X_n non-negative submartingale.*
- *$\forall \lambda > 0$, then $\lambda^p \mathbb{P}[\max k \leq n X_k \geq \lambda] \leq E[X_n^p]$*
- $|\max_{k \leq n} X_k|_p \leq \frac{p}{p-1} |X_n|_p$
- $|\max_{k \leq n} X_k|_1 \leq \frac{e}{e-1}(1 + |X_n \log(X_n)|_1)$

5 Markov

Theorem 25 (Markov property)

- *(X_n) markov (λ, P). Conditional on $X_m = i$, X_{n+m} is markov (δ_i, P) independent of $X_0, ..., X_m$.*
- *(X_n) markov (λ, P). Conditional on $X_T = i$, X_{n+T} is markov (δ_i, P) independent of $X_0, ..., X_T$.*

Definition 7 (Some defs)

- *Communicating classes: $I/\sim$ where $i \sim j \iff i \leftrightarrow j$*
- *C Closed $\iff i \in C, i \to j \Rightarrow j \in C$*
- *P irreducible $\iff \forall i, j, i \to j \iff$ there is only one communicating class.*
- *$H_i = \inf\{n \geq 0; X_n = i\}, T_i = \inf\{n \geq 1; X_n = i\}, V_i := \sum_n 1_{\{X_n = i\}}, f_i = P_i(T_i < \infty), m_i = E[T_i]$*
- *i is reccurent if $P_i(V_i = \infty) = 1 \iff f_i = 1 \iff \sum p_{ii}^{(n)} = \infty$, otherwise transient.*
- *$P_i(V_i \geq k+1) = f_i^k$*
- *In a communcating class all cstates are transient or all are reccurrent.*
- *recurrence $\Rightarrow$ closed*
- *finite + closed $\Rightarrow$ recurrent.*
- *P irreducible + recurrent $\Rightarrow P(T_j < \infty)$*

Theorem 26 (Invariant Distribution)

- *I finite, if for some $i \in I$ $p_{ij}^{(n)} \to \pi_j \forall j \in I$ then π is an invariant distribution.*