

# An ALE Finite Element Method for Vorticity-Streamfunction Formulation with Species Transport Equation

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# Outline

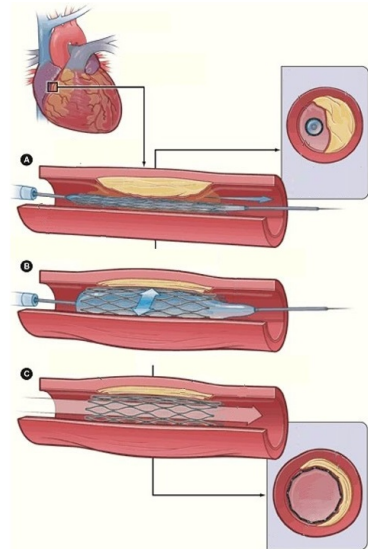
1. Introduction
2. Mathematical Model
3. Computational Code
4. Validation
5. Results
6. Conclusion

## Motivation:

- Ischaemic heart disease and stroke have remained the leading death causes globally in the last 15 years [1]

## Goals:

- To develop a ALE-FE code for the Vorticity-Streamfunction Formulation
- To create new drug-eluting stent design patent



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# Arbitrary Lagrangian-Eulerian (ALE)

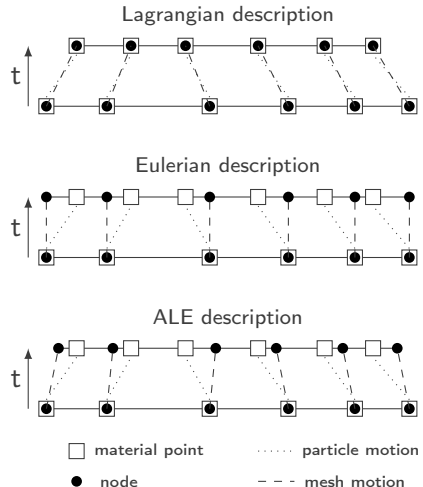
The Arbitrary Lagrangian-Eulerian combines the classical motion descriptions, while it provides [2]:

Advantages:

- Simulations in fluid-structure and moving boundary problems

Disadvantages:

- The computational mesh requires an extensive topological treatment



[2] Donea, J., Huerta, A., Ponthot, J.-P. and Rodríguez-Ferran, A. (2004). Arbitrary Lagrangian-Eulerian Methods. In Encyclopedia of Computational Mechanics *doi:10.1002/0470091355.ecm009*

# Governing Equations

Assumptions [3]:

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Constant Mass Difusivity
6. Single-phase Flow
7. Two-dimensional flow

$$\frac{\partial \omega_z}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla \omega_z = \frac{1}{Re} \nabla^2 \omega_z$$

$$\nabla^2 \psi = -\omega_z$$

$$\frac{\partial e}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla e = \frac{1}{ReSc} \nabla^2 e$$

- If the mesh velocity field  $\hat{\mathbf{v}} = \mathbf{v}$  (*Lagrangian*) or  $\hat{\mathbf{v}} = 0$  (*Eulerian*)
- The material velocity field  $\mathbf{v} = (v_x, v_y)$  is calculated by:  
 $v_x = \partial \psi / \partial y$  and  $v_y = -\partial \psi / \partial x$

# Semi-Lagrangian Method

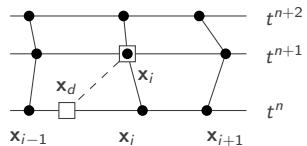
The convective term was replaced by material derivative in the direction of characteristic trajectory, that is:  $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$

$$\frac{D\omega_z}{Dt} = \frac{1}{Re} \nabla^2 \omega_z$$

$$\nabla^2 \psi = -\omega_z$$

$$\frac{De}{Dt} = \frac{1}{ReSc} \nabla^2 e$$

The departure node is calculated by  $x_d^n = x_i^{n+1} - (\mathbf{v} - \hat{\mathbf{v}}) \Delta t$ . Then, a searching procedure is required to find  $x_d^n$  using barycentric coordinates



The implicit semi-Lagrangian time discretization provides [4]:

Advantages:

- ▶ Symmetric linear systems
- ▶ Unconditionnal stability

$$\frac{\omega_i^{n+1} - \omega_d^n}{\Delta t} = \frac{1}{Re} \nabla^2 \omega^{n+1}$$

$$\nabla^2 \psi = -\omega$$

Disadvantages:

- ▶ Numerical Diffusion
- ▶ Searching procedure may lead to excessive computational cost if it is not well designed

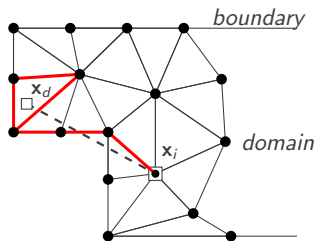
$$\frac{e_i^{n+1} - e_d^n}{\Delta t} = \frac{1}{ReSc} \nabla^2 e^{n+1}$$

[4] Pironneau, O. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numer. Math. 38, 309–332 (1982). <https://doi.org/10.1007/BF01396435>

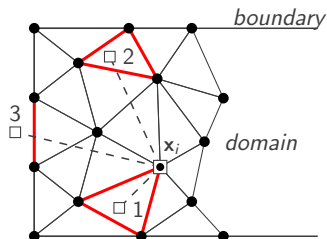


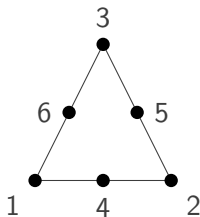
# Searching Procedure

The start searching procedure was used. This procedure has presented an acceptable computational cost, in addition to the concave domains can be used

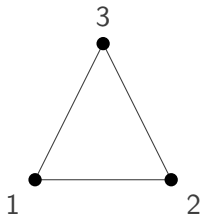


The searching procedure must be prepared for three situations that may occur. After the departure node is found, the interpolation is performed





Quad Element



Linear Element

$$\left[ \frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{Re} \right] \omega_i^{n+1} = \frac{\mathbf{M}}{\Delta t} \omega_d^n$$

$$\mathbf{K}\psi = \mathbf{M}\omega$$

$$\left[ \frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{ReSc} \right] e_i^{n+1} = \frac{\mathbf{M}}{\Delta t} e_d^n$$

- The material velocity field is calculated by:

$$\mathbf{M}v_x = \mathbf{G}_y\psi$$

$$\mathbf{M}v_y = -\mathbf{G}_x\psi$$

The computational mesh velocity  $\hat{\mathbf{v}}$  is defined as a linear combination of other velocities, such as:

$$\hat{\mathbf{v}} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2$$

where,  $\mathbf{v}_1$  is the Lagrangian velocity,  $\mathbf{v}_2$  is the Laplacian Smoothing velocity,  $\beta_1$  and  $\beta_2$  are the parameters control the Lagrangian and Laplacian motion respectively.

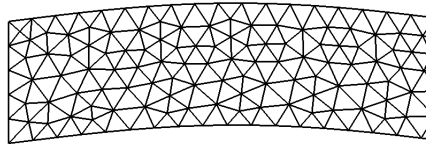
# Laplacian Smoothing

To avoid the fast degradation of the computational elements due to ALE description, it was used the Laplacian Smoothing Method [5]

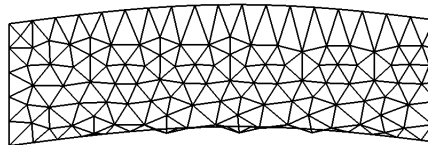
The new node position  $\hat{x}_i$  can be approximated by:

$$\hat{x}_i = w_{ij} \sum_j^{N_1} (x_j - x_i)$$

where,  $N_1$  is the 1-ring neighbors of the node,  $w_{ij}$  is the weight and was calculated by the inverse distance from neighbors vertices

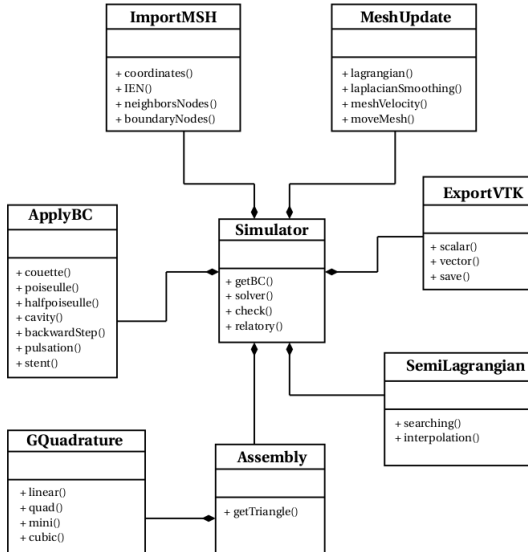


with Laplacian Smoothing



no Laplacian Smoothing

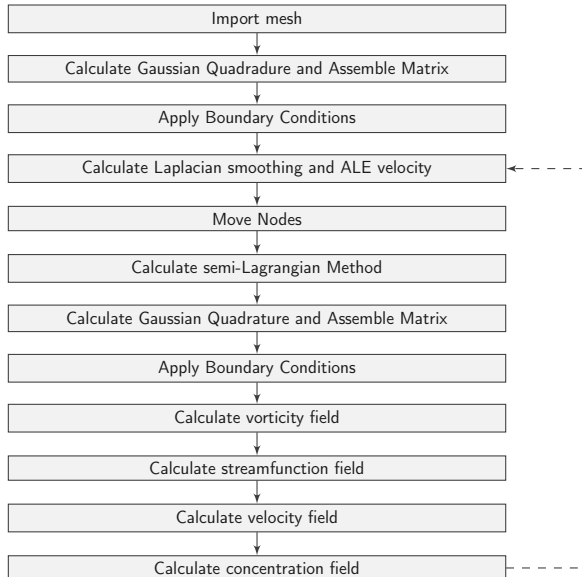
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2. Mathematical Model
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Process	AVG Computational Cost (%)
Mesh import	1.24
Assembly	73.87
BC Apply	6.70
Mesh update	10.05
Semi-Lagrangian	2.27
Vorticity Solver	5.51
VTK export	0.36

Average computational cost for the linear triangular element.

# Solution Algorithm



Repeat the procedure  
for the next time step  
until the steady state



1. Introduction
2. Mathematical Model
3. Computational Code
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# Validation - Poiseuille Flow

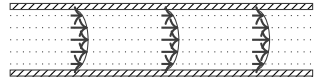
Inflow condition:  $v_x = u_{analytical}$ ,  $v_y = 0$

Top plate:  $v_x = 0$ ,  $v_y = 0$ ,  $\psi = 1$

Bottom plate:  $v_x = 0$ ,  $v_y = 0$ ,  $\psi = 0$

Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

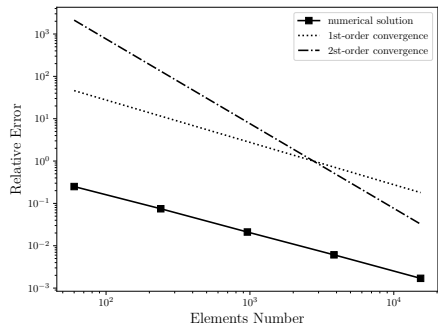
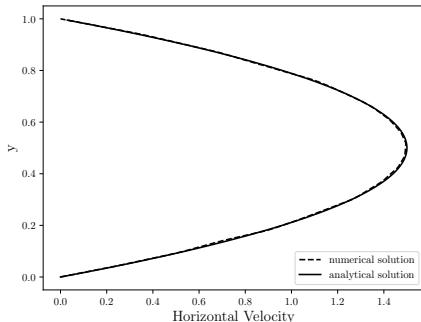
$\beta_1 = 0$        $\beta_2 = 1$



Nodes: 3835

Elements: 7299

Relative Error: 0.4%



# Validation - Symmetric Poiseuille Flow

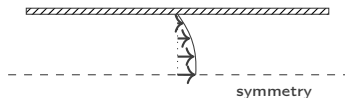
Inflow condition:  $v_x = u_{analytical}$ ,  $v_y = 0$

Top plate:  $v_x = 0$ ,  $v_y = 0$ ,  $\psi = 1$

Symmetric axis:  $v_y = 0$ ,  $\psi = 0$

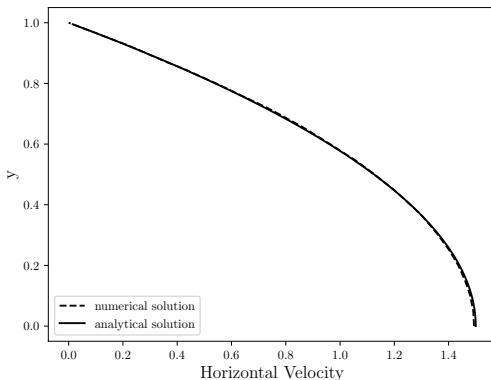
Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

$\beta_1 = 0$        $\beta_2 = 1$



Nodes: 3835

Elements: 7299



# Validation - Lid Driven Cavity Flow

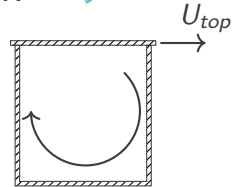
Bottom and side plates:  $v_x = 0$ ,  $v_y = 0$  e  $\psi = 0$

Top plate:  $v_x = 1$ ,  $v_y = 0$  e  $\psi = 0$

Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

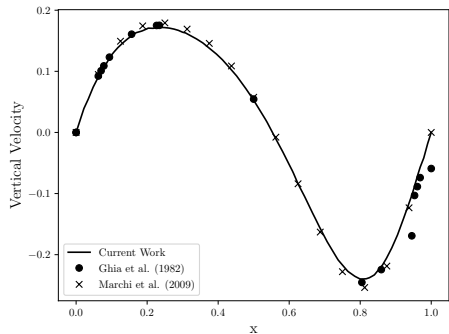
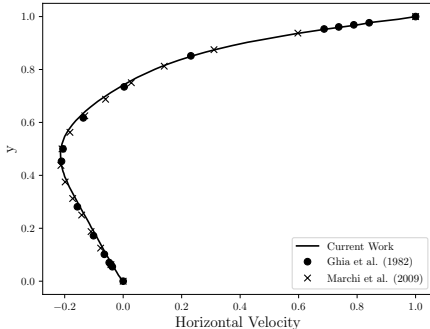
$\beta_1 = 0$        $\beta_2 = 1$

$Re = 100$



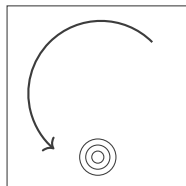
Nodes: 1563

Elements: 2988



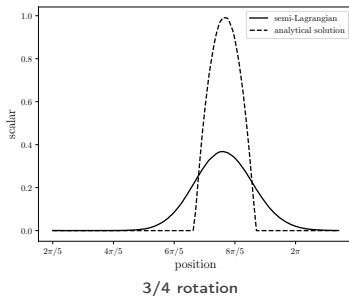
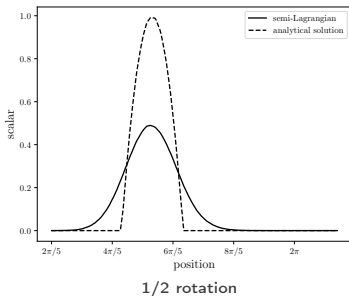
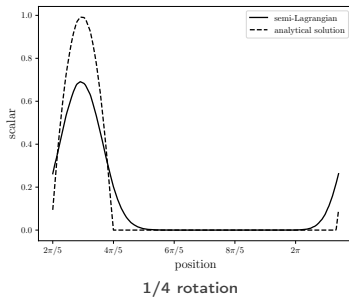
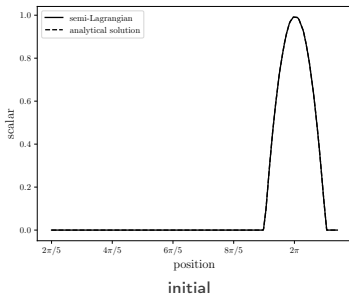
## Assumptions

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Two-dimensional Flow
6. Single-phase Flow
7. High Reynolds number ( $Re = \infty$ )

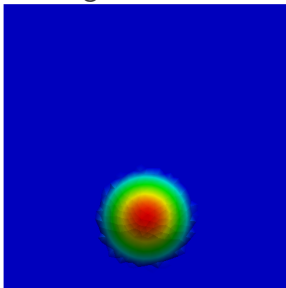


$$\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla \alpha = 0$$

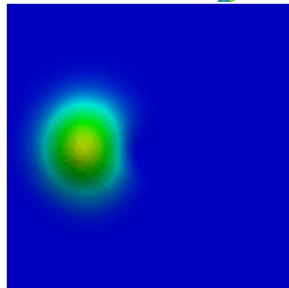
# Linear Triangular Element



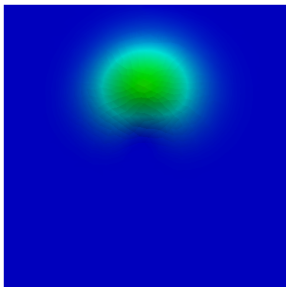
# Linear Triangular Element



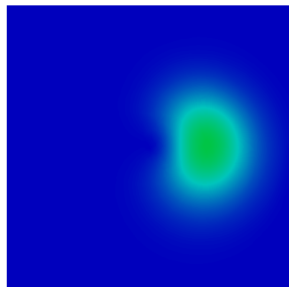
initial



1/4 rotation

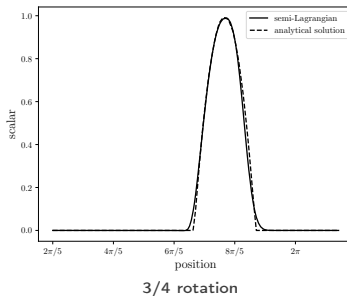
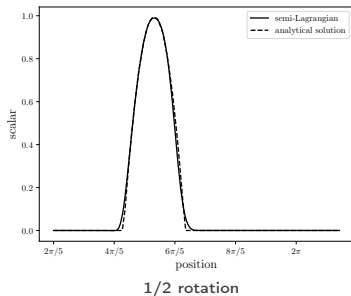
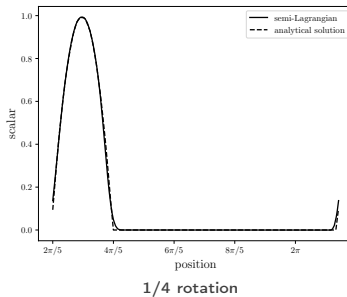
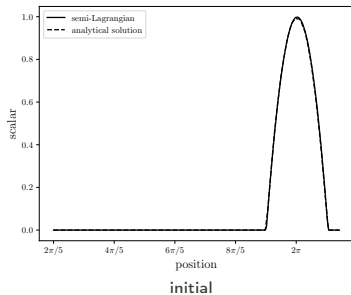


1/2 rotation



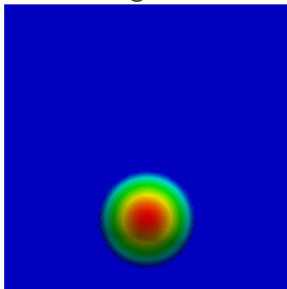
3/4 rotation

# Quadratic Triangular Element

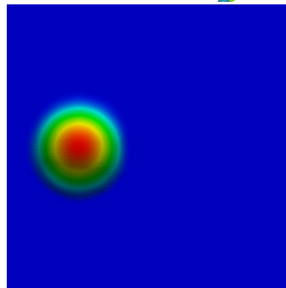




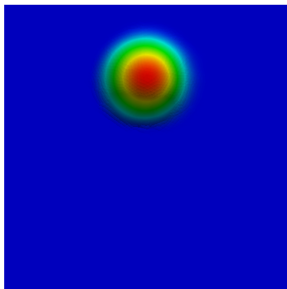
## Quadratic Triangular Element



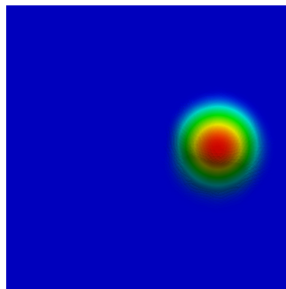
initial



1/4 rotation



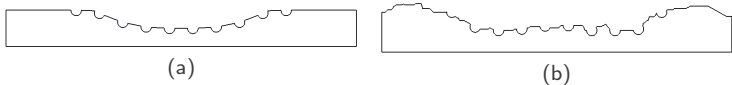
1/2 rotation



3/4 rotation

1. Introduction
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3. Computational Code
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# Results



Non-dimensional symmetric geometry for blood flow in coronary artery with drug-eluting stent placed by Wang et al. (2017): (a) Curved Channel with Stent (b) Real Channel with Stent.

Inflow condition:  $v_x = u_{analytical}$ ,  $v_y = 0$ ;

Top plate:  $v_x = 0$ ,  $v_y = 0$ ,  $\psi = 1$ ;

Symmetry condition:  $v_y = 0$ ,  $\psi = 0$ ;

Drug-eluting stent:  $v_x = 0$ ,  $v_y = 0$ ,  $\psi = 1$ ,  $e = 1$

Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

$\beta_1 = 0$        $\beta_2 = 1$

$R = 0.0015m$

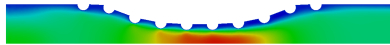
$\mu = 0.0035Pa.s$

$\rho = 1060kg/m^3$

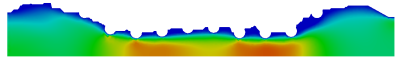
$u = 12cm/s$

$Re = 54.5$

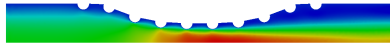
# Results - Velocity Field



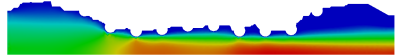
$t = 1.0s$



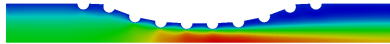
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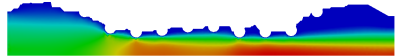
$t = 5.0s$



$t = 5.0s$

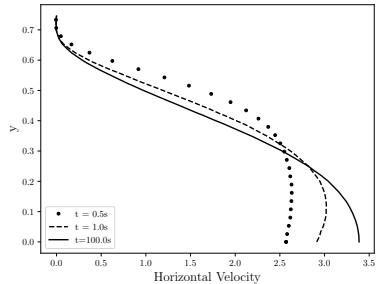
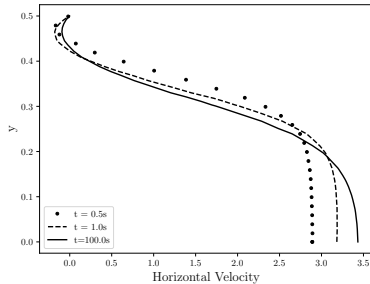


$t = 100.0s$



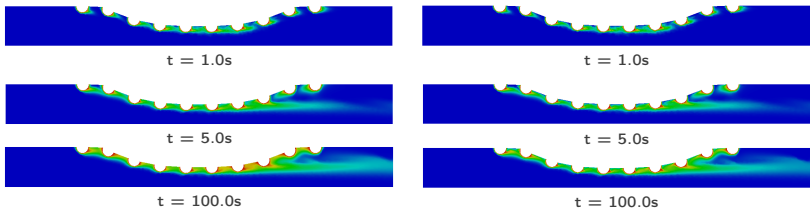
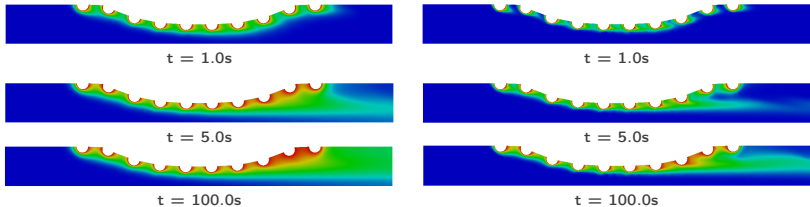
$t = 100.0s$

Evolution in time and space of velocity field:  
Curved Channel (left column) and Real Channel (right column)

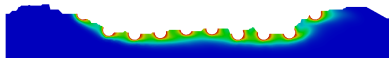


Evolution of velocity profile in centerline ( $x = 0.5L$ ):  
(a) Curved Channel and (b) Real Channel

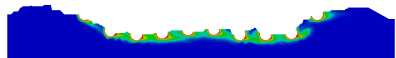
# Results - Concentration Field



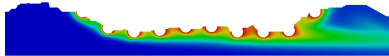
# Results - Concentration Field



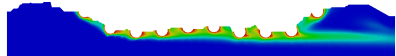
$t = 1.0s$



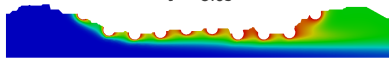
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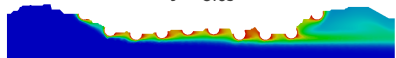
$t = 5.0s$



$t = 5.0s$



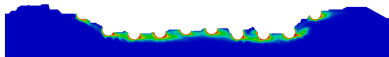
$t = 100.0s$



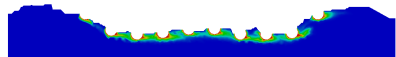
$t = 100.0s$

Evolution in time and space of concentration field in Real Channel:

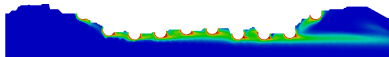
$Sc = 1$  (left column) and  $Sc = 10$  (right column)



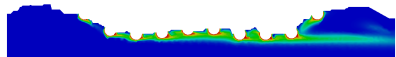
$t = 1.0s$



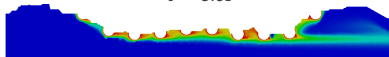
$t = 1.0s$



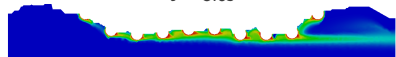
$t = 5.0s$



$t = 5.0s$



$t = 100.0s$



$t = 100.0s$

Evolution in time and space of concentration field in Real Channel:

$Sc = 100$  (left column) and  $Sc = 1000$  (right column)

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1. The ALE description allows to perform simulations in the fluid-structure and moving boundary problems, in addition it is possible to assign several mesh velocity values in specific regions of the problem in order to improve the accuracy of the numerical results
2. The searching procedure of the semi-Lagrangian Method used in this work was well designed since it corresponds to only 2% of average computational cost of the numerical simulation. Therefore, the method showed to be very useful due to the unconditional stability and symmetric linear system
3. As expected, the chemical species transport in blood flow is directly influenced by Schmidt number, where the diffusion is increased as the Schmidt number decreases. It is possible that the density and viscosity of the blood are affected by chemical species diffusion. However, this influence is not considered in this work



# Further developments:

1. Increase assembly performance
2. The use of primitive variables in the 3D Navier-Stokes equation
3. Blood flow model as a multiphase problem
4. Blood flow model as a non-Newtonian fluid

# Thank you!

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