An ALE Finite Element Method for Vorticity-Streamfunction Formulation with Species Transport Equation

Student Researcher: Leandro Marques
Advisors: Gustavo Anjos and Jose Pontes

State University of Rio de Janeiro
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GESAR

Outline

1. Introduction
2. Mathematical Model
3. Computational Code
4. Validation
5. Results
6. Conclusion

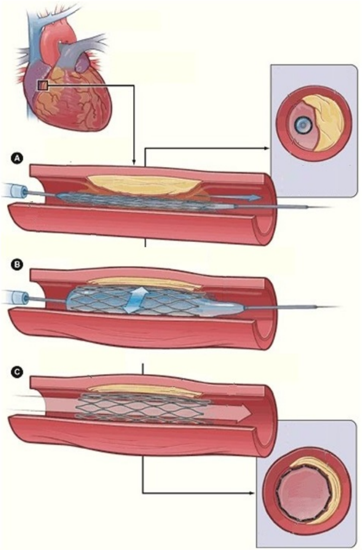
Introduction

Motivation:

- Ischaemic heart disease and stroke have remained the leading death causes globally in the last 15 years [1]

Goals:

- To develop a Finite Element code for the Vorticity-Streamfunction Formulation with species transport equation using the Arbitrary Lagrangian-Eulerian (ALE) approach and semi-Lagrangian Method
- To create new drug-eluting stent design patent



[1] World Health Organization, 2018, "The top 10 causes of death". *www.who.int/news-room/fact-sheets/detail/the-top-10-causes-of-death* [Online accessed 06-04-20 08:27pm]

Arbitrary Lagrangian-Eulerian (ALE)

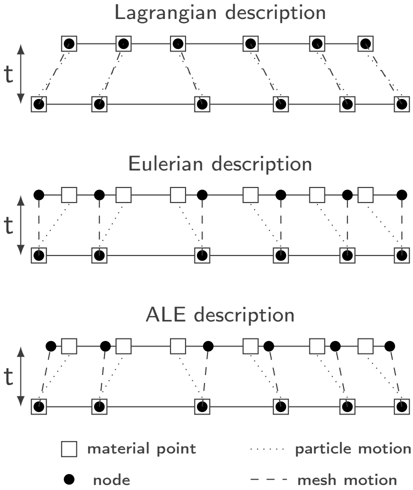
The Arbitrary Lagrangian-Eulerian combines the classical motion descriptions, while it provides [2]:

Advantages:

- Simulations in fluid-structure and moving boundary problems

Disadvantages:

- The computational mesh requires an extensive topological treatment



[2] Donea, J., Huerta, A., Ponthot, J.-P. and Rodríguez-Ferran, A. (2004). Arbitrary Lagrangian–Eulerian Methods. In Encyclopedia of Computational Mechanics *doi:10.1002/0470091355.ecm009*

Governing Equations

Assumptions [3]:

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Constant Mass Difusivity
6. Single-phase Flow
7. Two-dimensional flow

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla \omega = \frac{1}{Re} \nabla^2 \omega$$

$$\nabla^2 \psi = -\omega$$

$$\frac{\partial c}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla c = \frac{1}{ReSc} \nabla^2 c$$

- If the mesh velocity field $\hat{\mathbf{v}} = \mathbf{v}$ (*Lagrangian*) or $\hat{\mathbf{v}} = 0$ (*Eulerian*)
- The material velocity field $\mathbf{v} = (v_x, v_y)$ is calculated by: $v_x = \partial \psi / \partial y$ and $v_y = -\partial \psi / \partial x$

[3] Panton, R. (2013). Incompressible Flow John Wiley & Sons, Ltd

Semi-Lagrangian Method

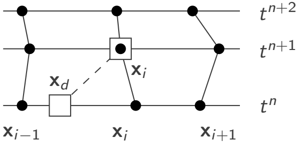
The convective term was replaced by material derivative in the direction of characteristic trajectory, that is: $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$

$$\frac{D\omega}{Dt} = \frac{1}{Re}\nabla^2\omega$$

$$\nabla^2\psi = -\omega$$

$$\frac{Dc}{Dt} = \frac{1}{ReSc}\nabla^2 c$$

The departure node is calculated by $x_d^n = x_i^{n+1} - (\mathbf{v} - \hat{\mathbf{v}})\,\Delta t$. Then, a searching procedure is required to find x_d^n using barycentric coordinates



Semi-Lagrangian Method

The implicit semi-Lagrangian time discretization provides [4]:

Advantages:

- Symmetric linear systems
- Unconditionnal stability

Disadvantages:

- Numerical Diffusion
- Searching procedure may lead to excessive computational cost if it is not well designed

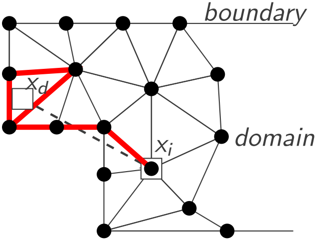
$$\frac{\omega_i^{n+1} - \omega_d^n}{\Delta t} = \frac{1}{Re}\nabla^2 \omega^{n+1}$$

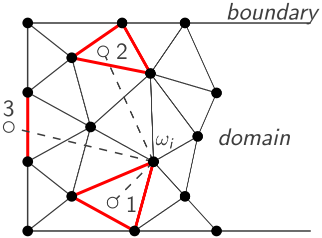
$$\nabla^2 \psi = -\omega$$

$$\frac{c_i^{n+1} - c_d^n}{\Delta t} = \frac{1}{ReSc}\nabla^2 c^{n+1}$$

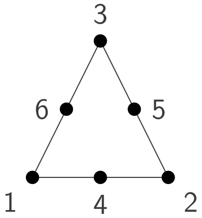
[4] Pironneau, O. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numer. Math. 38, 309–332 (1982). *https://doi.org/10.1007/BF01396435*

Searching Procedure

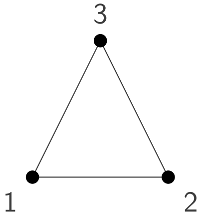




Galerkin FE Method



Quad Element



Linear Element

$$\left[\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{Re}\right] \omega_i^{n+1} = \frac{\mathbf{M}}{\Delta t} \omega_d^n$$

$$\mathbf{K}\psi = \mathbf{M}\omega$$

$$\left[\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{ReSc}\right] c_i^{n+1} = \frac{\mathbf{M}}{\Delta t} c_d^n$$

- The material velocity field is calculated by:
  $\mathbf{M} v_x = \mathbf{G_y} \psi$
  $\mathbf{M} v_y = -\mathbf{G_x} \psi$

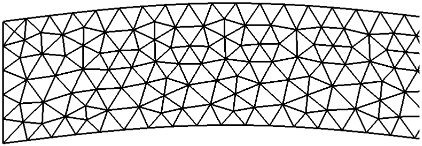
Laplacian Smoothing

To avoid the fast degradation of the computational elements due to ALE description, it was used the Laplacian Smoothing Method [5]

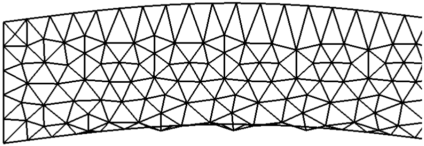
The new node position $\hat{\mathbf{x}}_\mathbf{i}$ can be approximated by:

$$\hat{\mathbf{x}}_\mathbf{i} = \sum_{i}^{np} \sum_{j}^{N_1} w_{ij}(\mathbf{x}_j - \mathbf{x}_i)$$

where, np is node number, N_1 is the 1-ring neighbors of a node, w_{ij} is the weight and was calculated by the inverse distance from neighbors vertices



with Laplacian Smoothing



no Laplacian Smoothing

[5] Desbrun, M. Meyer, P. Schröder, A. Barr, Implicit fairing of irregular meshes using diffusion and curvature flow, in: Proceedingsof Siggraph, 1999, pp. 317–324

Computational Cost

Process	AVG Computational Cost (%)
Mesh import	1.24
Assembly	73.87
BC Apply	6.70
Mesh update	10.05
Semi-Lagrangian	2.27
Vorticity Solver	5.51
VTK export	0.36

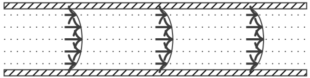
Tabela: Average computational cost for several linear triangular elements.

Solution Algorithm

1. Import mesh
2. Calculate Gaussian Quadradure and Assemble Matrix
3. Apply Boundary Conditions
4. Calculate Laplacian smoothing and ALE velocity
5. Move Nodes
6. Calculate semi-Lagrangian Method
7. Calculate Gaussian Quadrature and Assemble Matrix
8. Apply Boundary Conditions
9. Calculate vorticity field
10. Calculate streamfunction field
11. Calculate velocity field
12. Calculate concentration field

Repeat the procedure for the next time step until the steady state

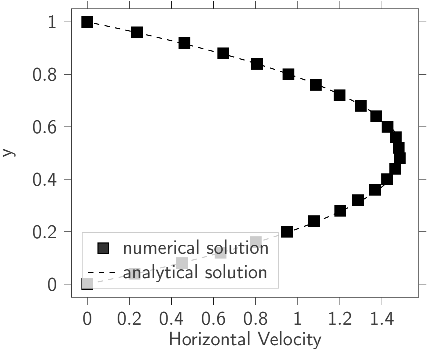
Validation - Poiseuille Flow

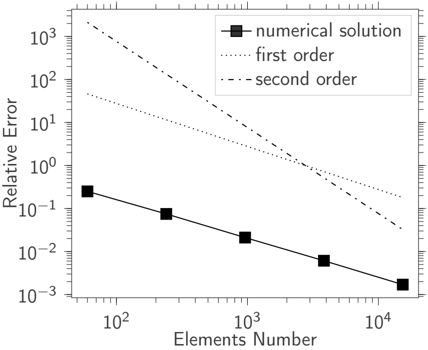


Boundaries Conditions:

Inflow condition: $u = u_{analytical}$, $v = 0$
Top plate: $u = 0$, $v = 0$, $\psi = 1$
Bottom plate: $u = 0$, $v = 0$, $\psi = 0$

Nodes: 1757
Elements: 3263
Relative Error: 1.2%





Validation - Lid Driven Cavity Flow

Boundaries Conditions:

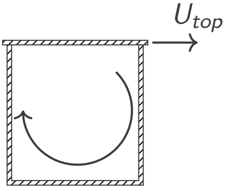
Bottom and side plates: $u = 0$, $v = 0$ e $\psi = 0$

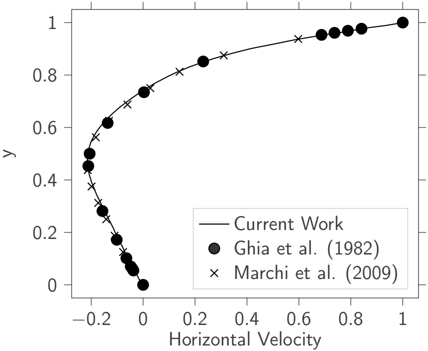
Top plate: $u = 1$, $v = 0$ e $\psi = 0$

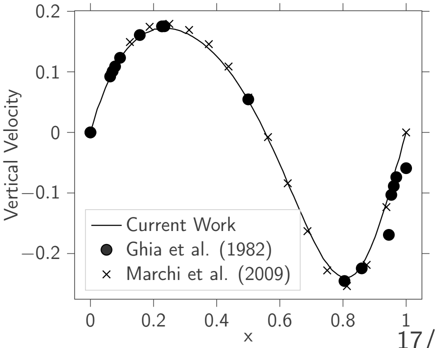
Nodes: 3798

Elements: 7382

Re: 100







Coming Soon

- Backward-Facing Step Validation
- Pulsation Flow Validation
- Drug-Eluting Stent Cases

Work Plan

	2018				2019												2020							
ACTIVITIES	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8
SUBJECTS	■	■	■	■	■	■	■	■	■	■	■	■												
LITERATURE REVIEW													■	■	■	■								
CODE IMPLEMENTATION															■	■	■	■	■	■	■			
CODE VALIDATION																				■	■	■		
RESULTS SIMULATION																							■	
PRESENTATION																								■

SUBJECTS:

- CONTINUUM MECHANICS I
- ADVANCED CALCULUS
- FLUID MECHANICS
- COMPUTATIONAL FLUID MECHANICS I
- COMPUTATIONAL FLUID MECHANICS II
- FINITE ELEMENT METHOD
- COMPUTATIONAL METHODS
- MULTIPHASE FLOWS

LITERATURE REVIEW:

- ARBITRARY LAGRANGIAN-EULERIAN (ALE)
- SEMI-LAGRANGIAN METHOD
- LAPLACIAN SMOOTHING

Final Presentation

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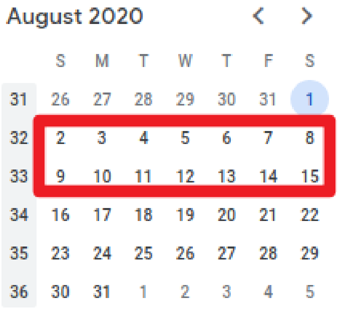
Expected Date for Final Presentation



Thank you!

marquesleandro67@gmail.com
gustavo.rabello@mecanica.coppe.ufrj.br
jose.pontes@uerj.br

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