

# An ALE Finite Element Method for Vorticity-Streamfunction Formulation with Species Transport Equation

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# Outline

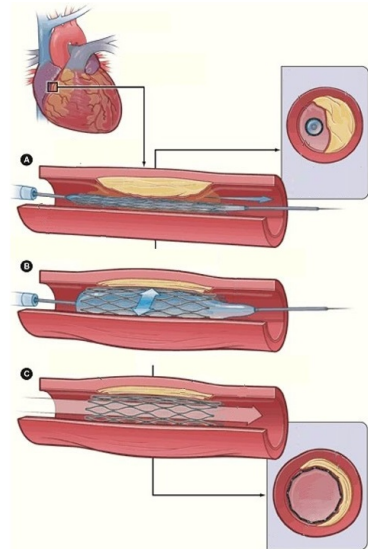
1. Introduction
2. Mathematical Model
3. Computational Code
4. Validation
5. Results
6. Conclusion

## Motivation:

- Ischaemic heart disease and stroke have remained the leading death causes globally in the last 15 years [1]

## Goals:

- To develop a ALE-FE code for the Vorticity-Streamfunction Formulation
- To create new drug-eluting stent design patent



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# Arbitrary Lagrangian-Eulerian (ALE)

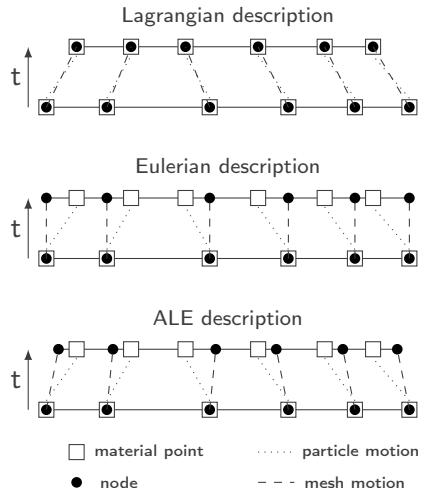
The Arbitrary Lagrangian-Eulerian combines the classical motion descriptions, while it provides [2]:

Advantages:

- Simulations in fluid-structure and moving boundary problems

Disadvantages:

- The computational mesh requires an extensive topological treatment



[2] Donea, J., Huerta, A., Ponthot, J.-P. and Rodríguez-Ferran, A. (2004). Arbitrary Lagrangian-Eulerian Methods. In Encyclopedia of Computational Mechanics *doi:10.1002/0470091355.ecm009*

# Governing Equations

Assumptions [3]:

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Constant Mass Difusivity
6. Single-phase Flow
7. Two-dimensional flow

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla \omega = \frac{1}{Re} \nabla^2 \omega$$

$$\nabla^2 \psi = -\omega$$

$$\frac{\partial e}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla e = \frac{1}{ReSc} \nabla^2 e$$

- If the mesh velocity field  $\hat{\mathbf{v}} = \mathbf{v}$  (*Lagrangian*) or  $\hat{\mathbf{v}} = 0$  (*Eulerian*)
- The material velocity field  $\mathbf{v} = (v_x, v_y)$  is calculated by:  
 $v_x = \partial \psi / \partial y$  and  $v_y = -\partial \psi / \partial x$

# Semi-Lagrangian Method

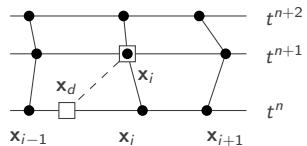
The convective term was replaced by material derivative in the direction of characteristic trajectory, that is:  $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$

$$\frac{D\omega}{Dt} = \frac{1}{Re} \nabla^2 \omega$$

$$\nabla^2 \psi = -\omega$$

$$\frac{De}{Dt} = \frac{1}{ReSc} \nabla^2 e$$

The departure node is calculated by  $x_d^n = x_i^{n+1} - (\mathbf{v} - \hat{\mathbf{v}}) \Delta t$ . Then, a searching procedure is required to find  $x_d^n$  using barycentric coordinates



The implicit semi-Lagrangian time discretization provides [4]:

Advantages:

- ▶ Symmetric linear systems
- ▶ Unconditionnal stability

$$\frac{\omega_i^{n+1} - \omega_d^n}{\Delta t} = \frac{1}{Re} \nabla^2 \omega^{n+1}$$

$$\nabla^2 \psi = -\omega$$

Disadvantages:

- ▶ Numerical Diffusion
- ▶ Searching procedure may lead to excessive computational cost if it is not well designed

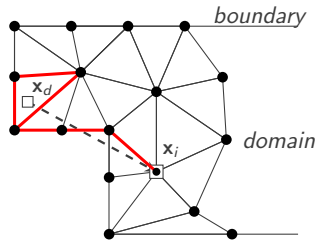
$$\frac{e_i^{n+1} - e_d^n}{\Delta t} = \frac{1}{ReSc} \nabla^2 e^{n+1}$$

[4] Pironneau, O. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numer. Math. 38, 309–332 (1982). <https://doi.org/10.1007/BF01396435>

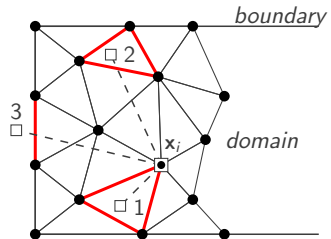


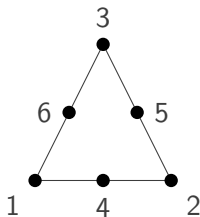
# Searching Procedure

The convective term was replaced by material derivative in the direction of characteristic trajectory, that is:  $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$

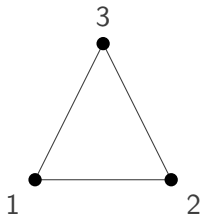


The departure node is calculated by  $x_d^n = x_i^{n+1} - (\mathbf{v} - \hat{\mathbf{v}}) \Delta t$ . Then, a searching procedure is required to find  $x_d^n$  using barycentric coordinates





Quad Element



Linear Element

$$\left[ \frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{Re} \right] \omega_i^{n+1} = \frac{\mathbf{M}}{\Delta t} \omega_d^n$$

$$\mathbf{K}\psi = \mathbf{M}\omega$$

$$\left[ \frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{ReSc} \right] e_i^{n+1} = \frac{\mathbf{M}}{\Delta t} e_d^n$$

- The material velocity field is calculated by:

$$\mathbf{M}v_x = \mathbf{G}_y\psi$$

$$\mathbf{M}v_y = -\mathbf{G}_x\psi$$

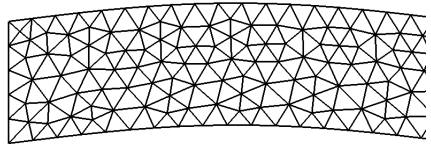
# Laplacian Smoothing

To avoid the fast degradation of the computational elements due to ALE description, it was used the Laplacian Smoothing Method [5]

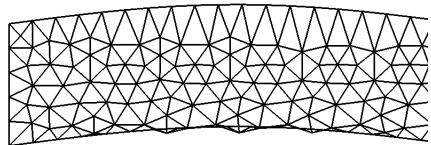
The new node position  $\hat{x}_i$  can be approximated by:

$$\hat{x}_i = w_{ij} \sum_j^{N_1} (x_j - x_i)$$

where,  $N_1$  is the 1-ring neighbors of the node,  $w_{ij}$  is the weight and was calculated by the inverse distance from neighbors vertices

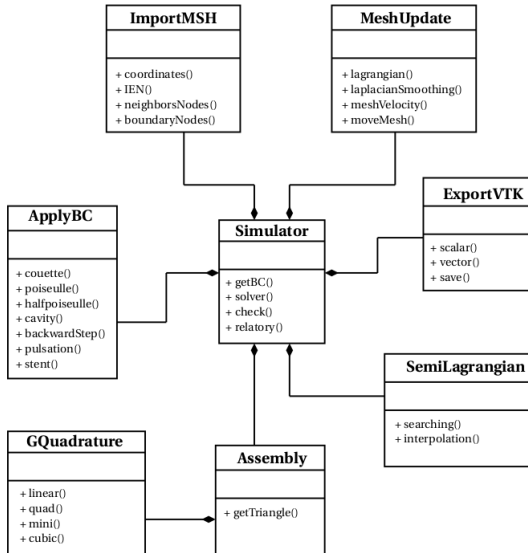


with Laplacian Smoothing



no Laplacian Smoothing

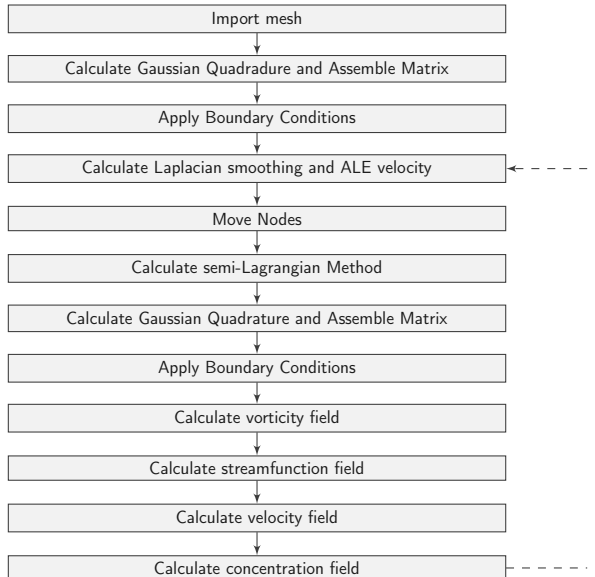
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Process	AVG Computational Cost (%)
Mesh import	1.24
Assembly	73.87
BC Apply	6.70
Mesh update	10.05
Semi-Lagrangian	2.27
Vorticity Solver	5.51
VTK export	0.36

Average computational cost for the linear triangular element.

# Solution Algorithm



Repeat the procedure  
for the next time step  
until the steady state

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# Validation - Poiseuille Flow

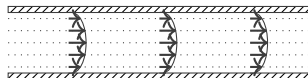
Inflow condition:  $u = u_{analytical}$ ,  $v = 0$

Top plate:  $u = 0$ ,  $v = 0$ ,  $\psi = 1$

Bottom plate:  $u = 0$ ,  $v = 0$ ,  $\psi = 0$

Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

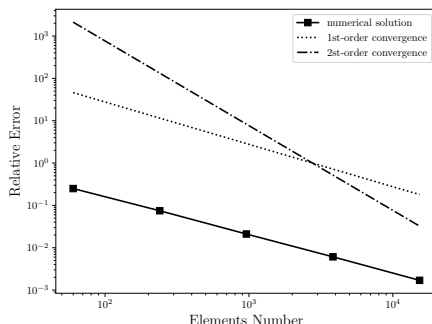
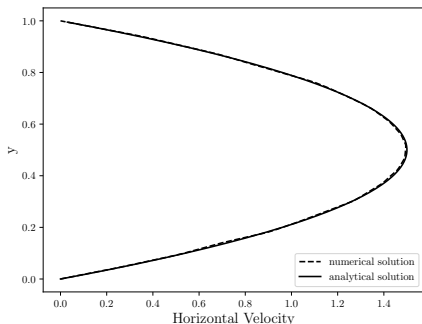
$\beta_1 = 0$        $\beta_2 = 1$



Nodes: 3835

Elements: 7299

Relative Error: 0.4%



# Validation - Half Poiseuille Flow

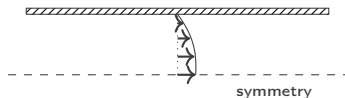
Inflow condition:  $u = u_{analytical}$ ,  $v = 0$

Top plate:  $u = 0$ ,  $v = 0$ ,  $\psi = 1$

Symmetric axis:  $v = 0$ ,  $\psi = 0$

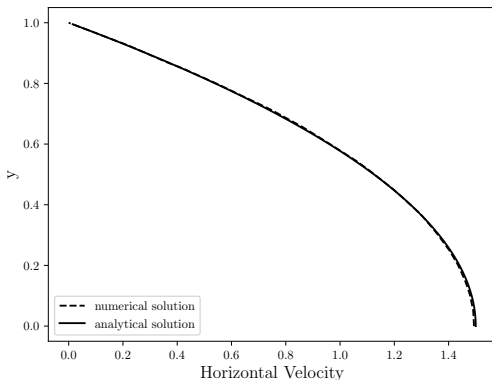
Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

$\beta_1 = 0$        $\beta_2 = 1$



Nodes: 3835

Elements: 7299



# Validation - Lid Driven Cavity Flow

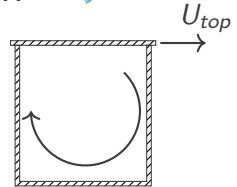
Bottom and side plates:  $u = 0$ ,  $v = 0$  e  $\psi = 0$

Top plate:  $u = 1$ ,  $v = 0$  e  $\psi = 0$

Vorticity condition:  $\omega = \nabla \times \mathbf{v}$

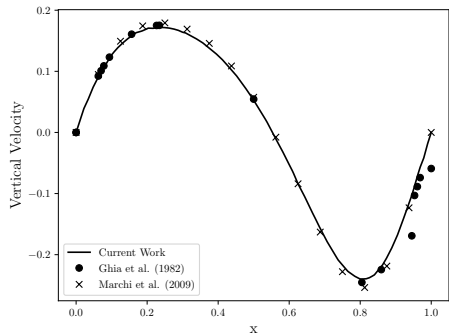
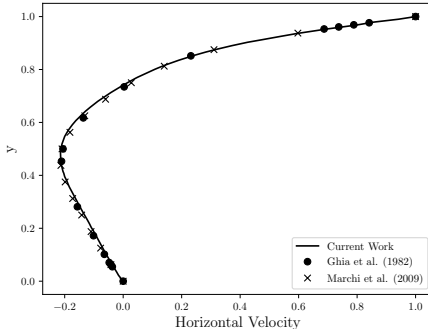
$\beta_1 = 0$        $\beta_2 = 1$

$Re = 100$



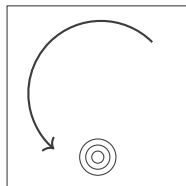
Nodes: 1563

Elements: 2988



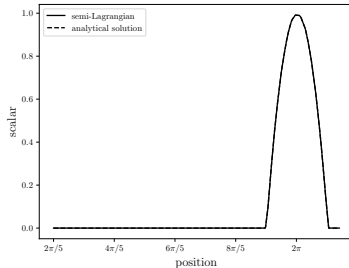
## Assumptions

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Two-dimensional Flow
6. Single-phase Flow
7. High Reynolds number ( $Re = \infty$ )

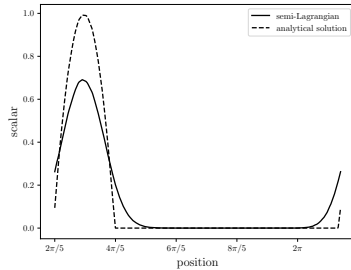


$$\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla \alpha = 0$$

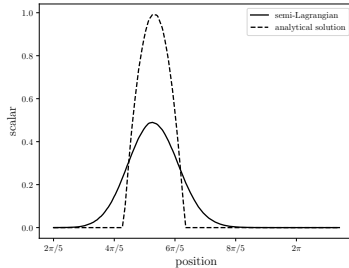
# Linear Triangular Element



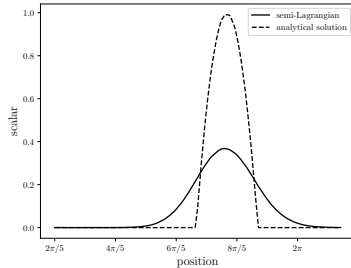
initial



1/4 rotation

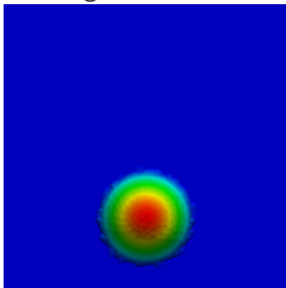


1/2 rotation

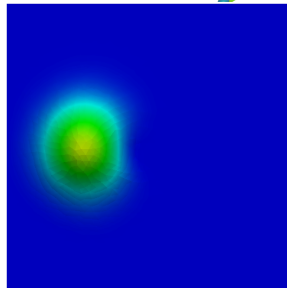


3/4 rotation

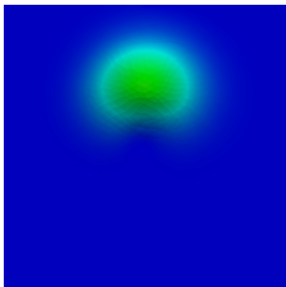
# Linear Triangular Element



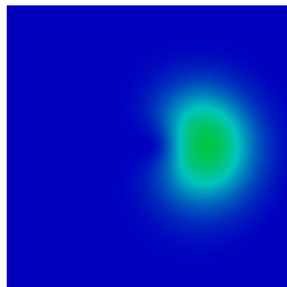
initial



1/4 rotation

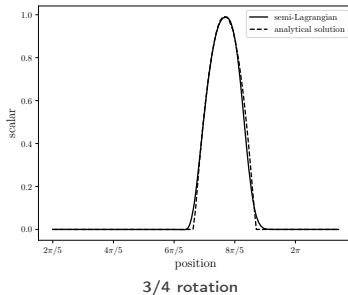
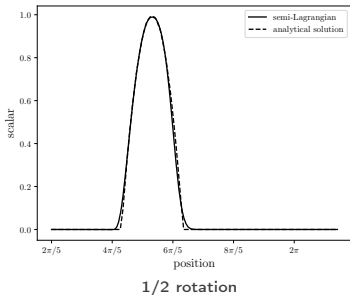
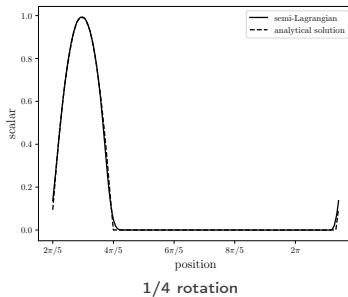
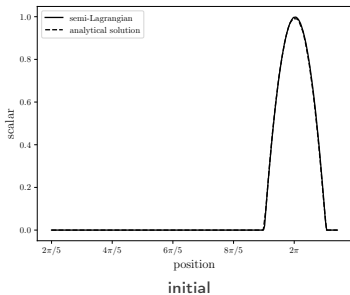


1/2 rotation

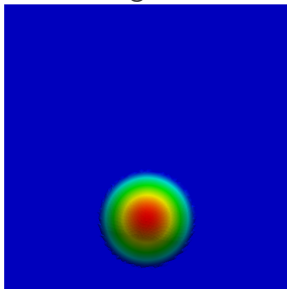


3/4 rotation

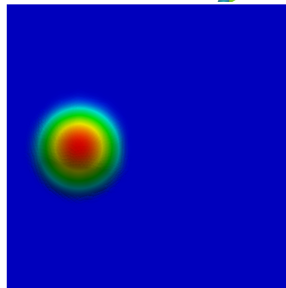
# Quadratic Triangular Element



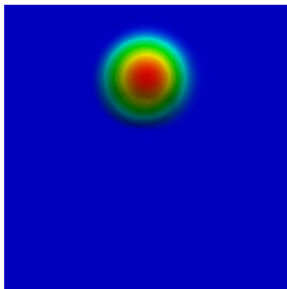
# Quadratic Triangular Element



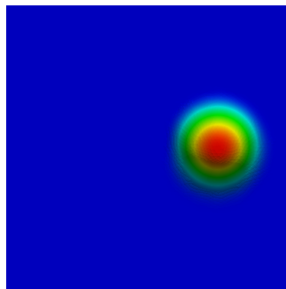
initial



1/4 rotation



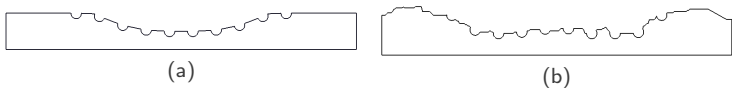
1/2 rotation



3/4 rotation



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Non-dimensional symmetric geometry for blood flow in coronary artery with drug-eluting stent placed by Wang et al. (2017): (a) Curved Channel with Stent (b) Real Channel with Stent.

Boundaries Conditions:

Inflow condition:  $u = 1$ ,  $v = 0$  e  $\psi = y$ ;

Outflow condition:  $\psi = y$ ;

Top plate:  $u = 0$ ,  $v = 0$ ,  $\psi = 1$ ;

Symmetry condition:  $v = 0$ ,  $\psi = 0$ ;

Drug-eluting stent:  $u = 0$ ,  $v = 0$ ,  $\psi = 1$  e  $c = 1$

$$R = 0.0015m$$

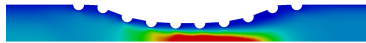
$$\mu = 0.0035Pa.s$$

$$\rho = 1060kg/m^3$$

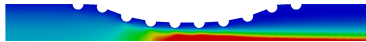
$$u = 12cm/s$$

$$Re = 54.5$$

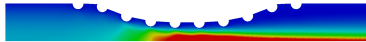
# Results - Velocity Field



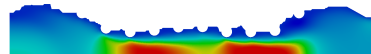
$t = 1.0$



$t = 5.0$



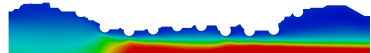
$t = 10.0$



$t = 1.0$

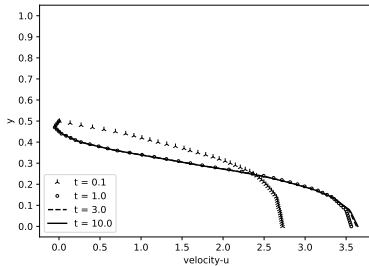


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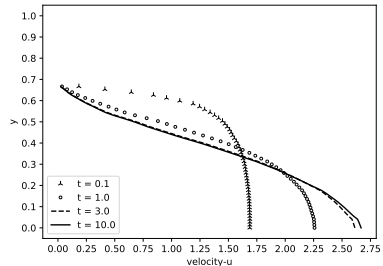


$t = 10.0$

Evolution in time and space of velocity field:  
Curved Channel (left column) and Real Channel (right column)



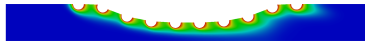
(a)



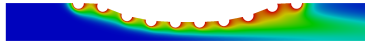
(b)

Evolution of velocity profile in centerline ( $x = 0.5L$ ):  
(a) Curved Channel and (b) Real Channel

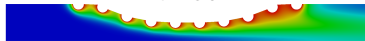
# Results - Concentration Field



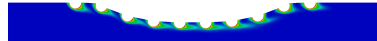
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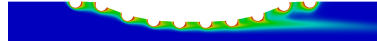
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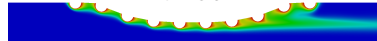
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$t = 1.0$

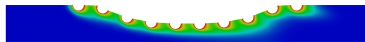


$t = 5.0$

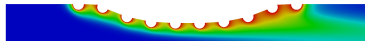


$t = 10.0$

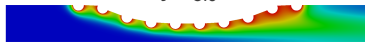
Evolution in time and space of concentration field in Curved Channel:  
 $Sc = 1$  (left column) and  $Sc = 10$  (right column)



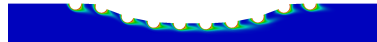
$t = 1.0$



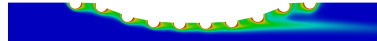
$t = 5.0$



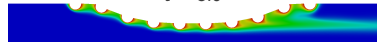
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$t = 1.0$



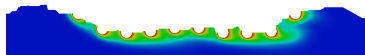
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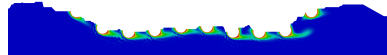
$t = 10.0$

Evolution in time and space of concentration field in Real Channel:  
 $Sc = 1$  (left column) and  $Sc = 10$  (right column)

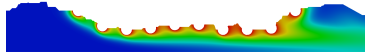
# Results - Concentration Field



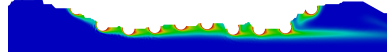
$t = 1.0$



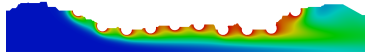
$t = 1.0$



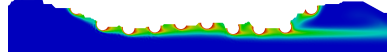
$t = 5.0$



$t = 5.0$

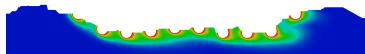


$t = 10.0$

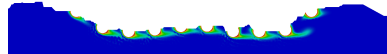


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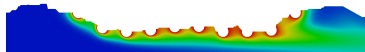
Evolution in time and space of concentration field in Curved Channel:  
 $Sc = 1$  (left column) and  $Sc = 10$  (right column)



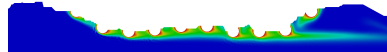
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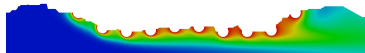
$t = 1.0$



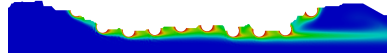
$t = 5.0$



$t = 5.0$



$t = 10.0$



$t = 10.0$

Evolution in time and space of concentration field in Real Channel:  
 $Sc = 1$  (left column) and  $Sc = 10$  (right column)

1. Introduction
2. Mathematical Model
3. Computational Code
4. Validation
5. Results
6. Conclusion

1. The streamfunction-vorticity formulation showed an useful approach for to calculate the velocity and concentration fields since the variables are scalars allowing a smooth implementation
2. Due to generalized construction of the code, the simulator is able to describe drug-eluting stent problem in coronary artery as well as flows of Newtonian fluids with scalar transport (concentration or temperature)
3. The ALE description allows moving boundary problems to be simulated

# Thank you!

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