An ALE Finite Element Method for Vorticity-Streamfunction Formulation with Species Transport Equation

Student Researcher: Leandro Marques
Advisors: Gustavo Anjos and Jose Pontes

State University of Rio de Janeiro
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GESAR

Outline

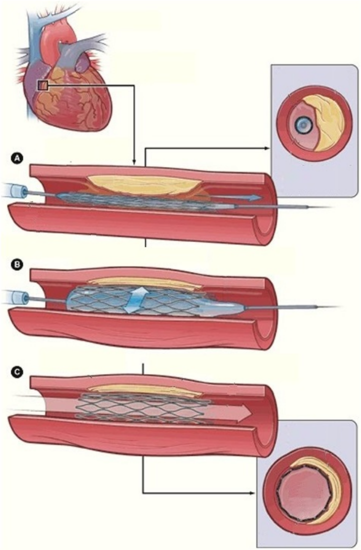
Introduction

Motivation:

- Ischaemic heart disease and stroke have remained the leading death causes globally in the last 15 years [1]

Goals:

- To develop a ALE-FE code for the Vorticity-Streamfunction Formulation
- To create new drug-eluting stent design patent



[1] World Health Organization, 2018, "The top 10 causes of death". *www.who.int/news-room/fact-sheets/detail/the-top-10-causes-of-death* [Online accessed 06-04-20 08:27pm]

1. Introduction
2. Mathematical Model
3. Computational Code
4. Validation
5. Results
6. Conclusion

Arbitrary Lagrangian-Eulerian (ALE)

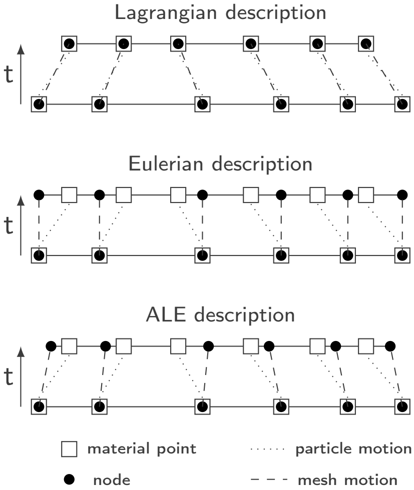
The Arbitrary Lagrangian-Eulerian combines the classical motion descriptions, while it provides [2]:

Advantages:

- Simulations in fluid-structure and moving boundary problems

Disadvantages:

- The computational mesh requires an extensive topological treatment



[2] Donea, J., Huerta, A., Ponthot, J.-P. and Rodríguez-Ferran, A. (2004). Arbitrary Lagrangian–Eulerian Methods. In Encyclopedia of Computational Mechanics *doi:10.1002/0470091355.ecm009*

Governing Equations

Assumptions [3]:

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Constant Mass Difusivity
6. Single-phase Flow
7. Two-dimensional flow

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla \omega = \frac{1}{Re} \nabla^2 \omega$$

$$\nabla^2 \psi = -\omega$$

$$\frac{\partial e}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla e = \frac{1}{ReSc} \nabla^2 e$$

- If the mesh velocity field $\hat{\mathbf{v}} = \mathbf{v}$ (*Lagrangian*) or $\hat{\mathbf{v}} = 0$ (*Eulerian*)
- The material velocity field $\mathbf{v} = (v_x, v_y)$ is calculated by:
  $v_x = \partial\psi/\partial y$ and $v_y = -\partial\psi/\partial x$

[3] Panton, R. (2013). Incompressible Flow John Wiley & Sons, Ltd

Semi-Lagrangian Method

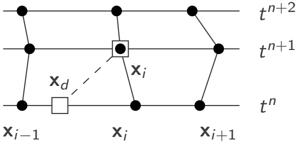
The convective term was replaced by material derivative in the direction of characteristic trajectory, that is: $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$

$$\frac{D\omega}{Dt} = \frac{1}{Re}\nabla^2\omega$$

$$\nabla^2\psi = -\omega$$

$$\frac{De}{Dt} = \frac{1}{ReSc}\nabla^2 e$$

The departure node is calculated by $x_d^n = x_i^{n+1} - (\mathbf{v} - \hat{\mathbf{v}})\,\Delta t$. Then, a searching procedure is required to find x_d^n using barycentric coordinates



Semi-Lagrangian Method

The implicit semi-Lagrangian time discretization provides [4]:

Advantages:

- Symmetric linear systems
- Unconditionnal stability

Disadvantages:

- Numerical Diffusion
- Searching procedure may lead to excessive computational cost if it is not well designed

$$\frac{\omega_i^{n+1} - \omega_d^n}{\Delta t} = \frac{1}{Re}\nabla^2 \omega^{n+1}$$

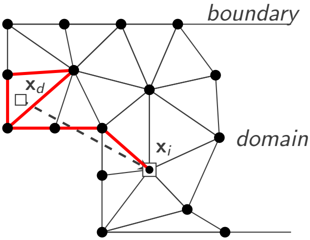
$$\nabla^2 \psi = -\omega$$

$$\frac{e_i^{n+1} - e_d^n}{\Delta t} = \frac{1}{ReSc}\nabla^2 e^{n+1}$$

[4] Pironneau, O. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numer. Math. 38, 309–332 (1982). *https://doi.org/10.1007/BF01396435*

Searching Procedure

The convective term was replaced by material derivative in the direction of characteristic trajectory, that is: $\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v} - \hat{\mathbf{v}}) \cdot \nabla(\cdot)$



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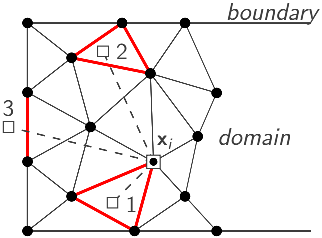
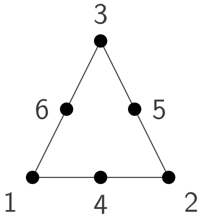
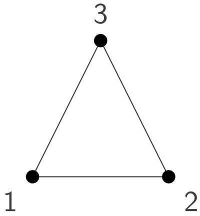


Galerkin FE Method



Quad Element



Linear Element

$$\left[\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{Re}\right] \omega_i^{n+1} = \frac{\mathbf{M}}{\Delta t}\omega_d^n$$

$$\mathbf{K}\psi = \mathbf{M}\omega$$

$$\left[\frac{\mathbf{M}}{\Delta t} + \frac{\mathbf{K}}{ReSc}\right] e_i^{n+1} = \frac{\mathbf{M}}{\Delta t}e_d^n$$

- The material velocity field is calculated by:
  $\mathbf{M}v_x = \mathbf{G_y}\psi$
  $\mathbf{M}v_y = -\mathbf{G_x}\psi$

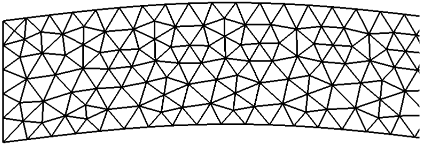
Laplacian Smoothing

To avoid the fast degradation of the computational elements due to ALE description, it was used the Laplacian Smoothing Method [5]

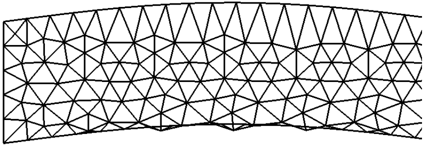
The new node position $\hat{\mathbf{x}}_\mathbf{i}$ can be approximated by:

$$\hat{\mathbf{x}}_\mathbf{i} = w_{ij} \sum_{j}^{N_1} (\mathbf{x}_j - \mathbf{x}_i)$$

where, N_1 is the 1-ring neighbors of the node, w_{ij} is the weight and was calculated by the inverse distance from neighbors vertices



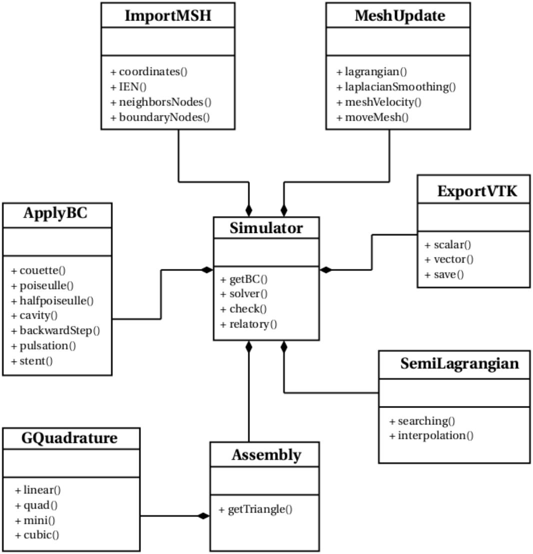
with Laplacian Smoothing



no Laplacian Smoothing

[5] Zheng, Y. Lewis, R. Gethin, D., Three-dimensional unstructured mesh generation: Part 2. surface meshes. Computer Methods in Applied Mechanics and Engineering, Amsterdam: North-Holland Pub. Co., c1972-, v. 134, n. 3, p. 269–284, 1996.

Code Framework



Simplified Class Diagram

Computational Cost

Process	AVG Computational Cost (%)
Mesh import	1.24
Assembly	73.87
BC Apply	6.70
Mesh update	10.05
Semi-Lagrangian	2.27
Vorticity Solver	5.51
VTK export	0.36

Average computational cost for the linear triangular element.

Solution Algorithm

Import mesh

Calculate Gaussian Quadradure and Assemble Matrix

Apply Boundary Conditions

Calculate Laplacian smoothing and ALE velocity

Move Nodes

Calculate semi-Lagrangian Method

Calculate Gaussian Quadrature and Assemble Matrix

Apply Boundary Conditions

Calculate vorticity field

Calculate streamfunction field

Calculate velocity field

Calculate concentration field

Repeat the procedure for the next time step until the steady state

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Validation - Poiseuille Flow

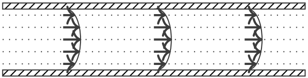
Inflow condition: $u = u_{analytical}$, $v = 0$

Top plate: $u = 0$, $v = 0$, $\psi = 1$

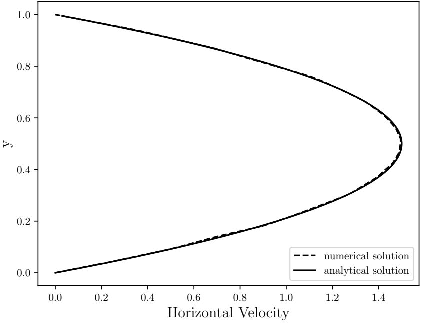
Bottom plate: $u = 0$, $v = 0$, $\psi = 0$

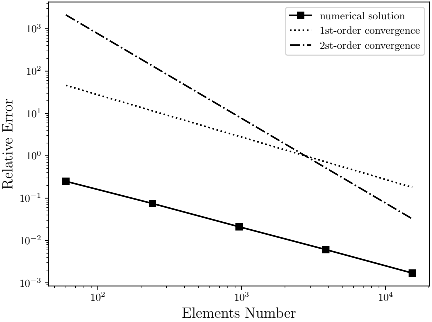
Vorticity condition: $\omega = \nabla \times \mathbf{v}$

$\beta_1 = 0 \qquad \beta_2 = 1$



Nodes: 3835
Elements: 7299
Relative Error: 0.4%





Validation - Half Poiseuille Flow

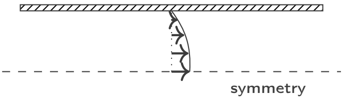
Inflow condition: $u = u_{analytical}$, $v = 0$

Top plate: $u = 0$, $v = 0$, $\psi = 1$

Symmetric axis: $v = 0$, $\psi = 0$

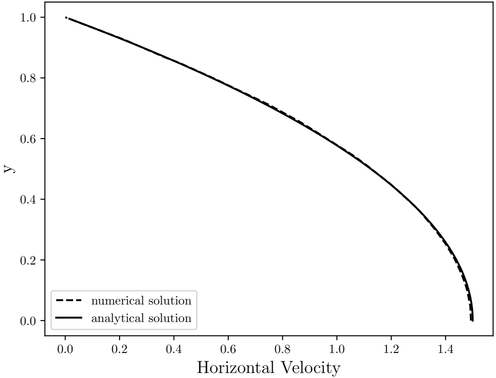
Vorticity condition: $\omega = \nabla \times \mathbf{v}$

$\beta_1 = 0$ $\beta_2 = 1$



Nodes: 3835

Elements: 7299



Validation - Lid Driven Cavity Flow

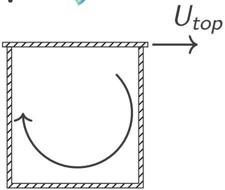
Bottom and side plates: $u = 0$, $v = 0$ e $\psi = 0$

Top plate: $u = 1$, $v = 0$ e $\psi = 0$

Vorticity condition: $\omega = \nabla \times \mathbf{v}$

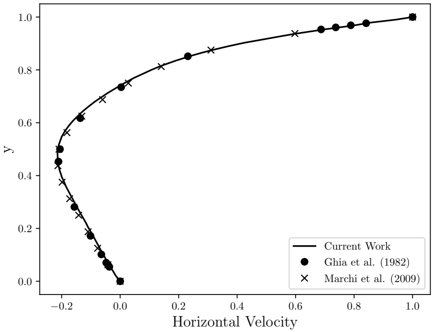
$\beta_1 = 0 \qquad \beta_2 = 1$

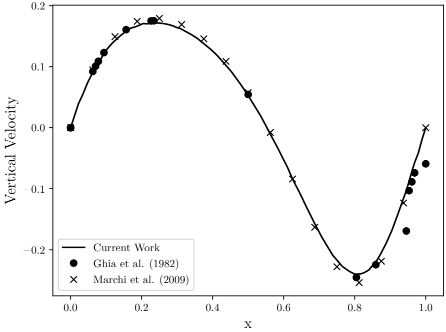
$Re = 100$



Nodes: 1563

Elements: 2988

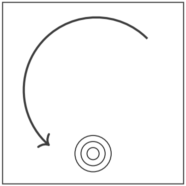




Validation - Pure Advection Flow

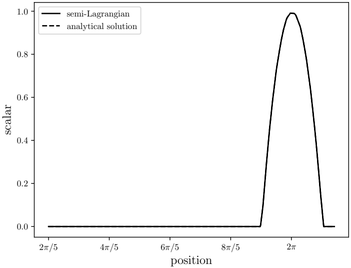
Assumptions

1. Continuum hypothesis
2. Homogeneous and Isotropic
3. Incompressible
4. Newtonian
5. Two-dimensional Flow
6. Single-phase Flow
7. High Reynolds number ($Re = \infty$)

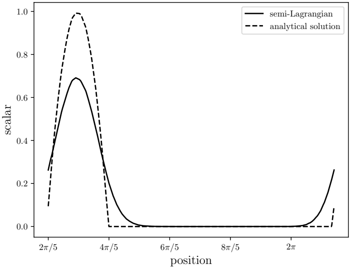


$$\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla \alpha = 0$$

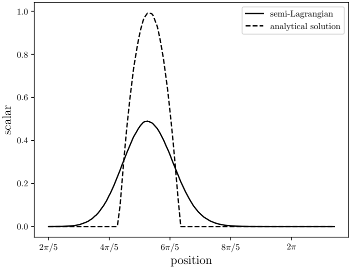
Linear Triangular Element



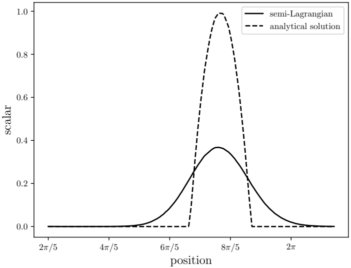
initial



1/4 rotation

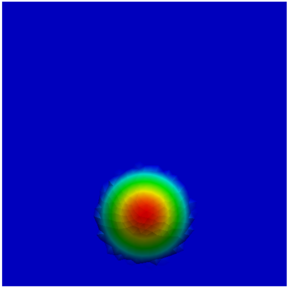


1/2 rotation

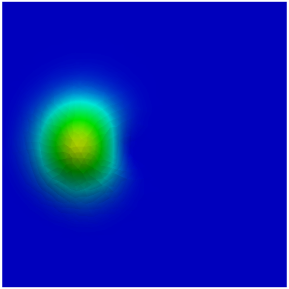


3/4 rotation

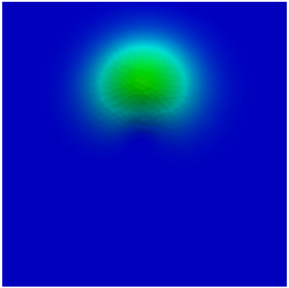
Linear Triangular Element



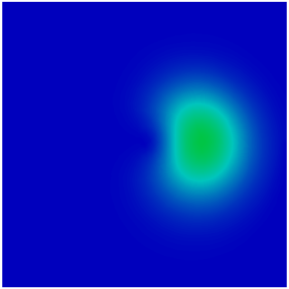
initial



1/4 rotation

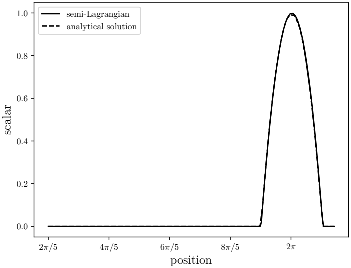


1/2 rotation

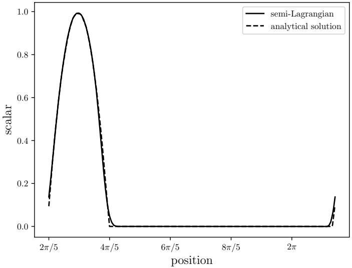


3/4 rotation

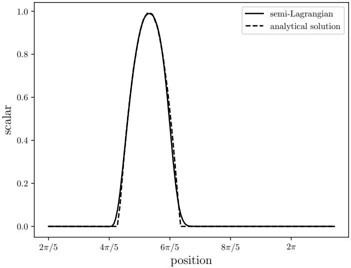
Quadratic Triangular Element



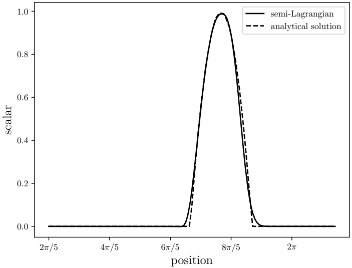
initial



1/4 rotation

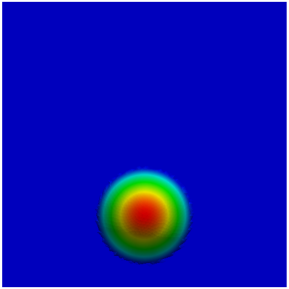


1/2 rotation

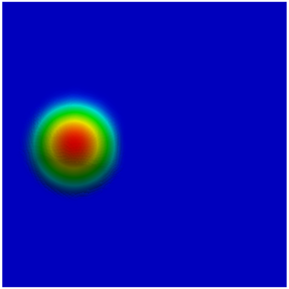


3/4 rotation

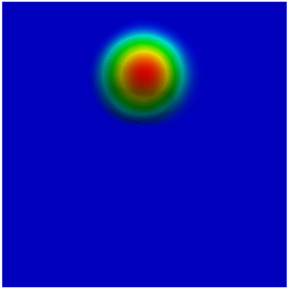
Quadratic Triangular Element



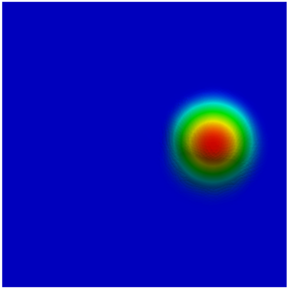
initial



1/4 rotation

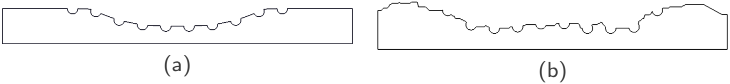


1/2 rotation



3/4 rotation

Results



Non-dimensional symmetric geometry for blood flow in coronary artery with drug-eluting stent placed by Wang et al. (2017): (a) Curved Channel with Stent (b) Real Channel with Stent.

Inflow condition: $u = u_{analytical}$, $v = 0$;

Top plate: $u = 0$, $v = 0$, $\psi = 1$;

Symmetry condition: $v = 0$, $\psi = 0$;

Drug-eluting stent: $u = 0$, $v = 0$, $\psi = 1$ e $c = 1$

$\beta_1 = 0$ $\quad \beta_2 = 1$

$R = 0.0015m$

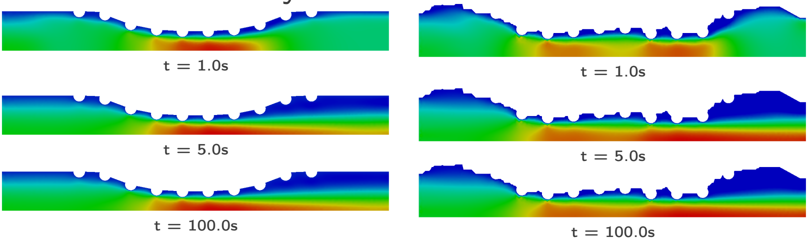
$\mu = 0.0035Pa.s$

$\rho = 1060kg/m^3$

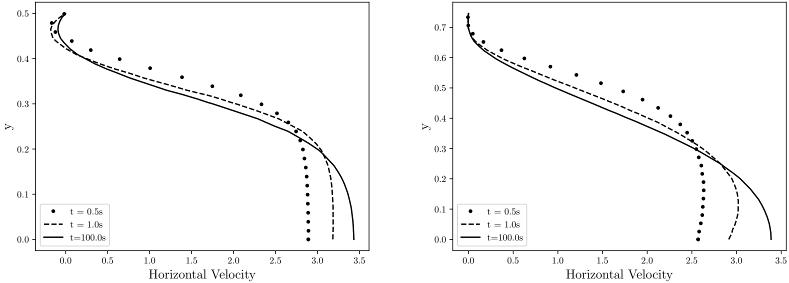
$u = 12cm/s$

$Re = 54.5$

Results - Velocity Field

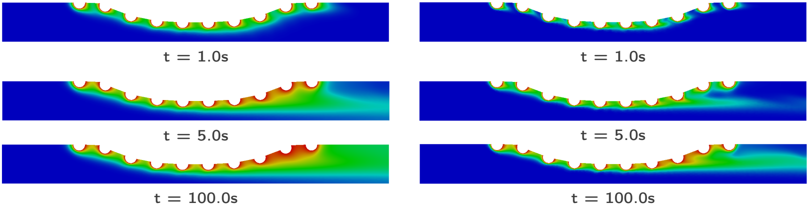


Evolution in time and space of velocity field:
Curved Channel (left column) and Real Channel (right column)

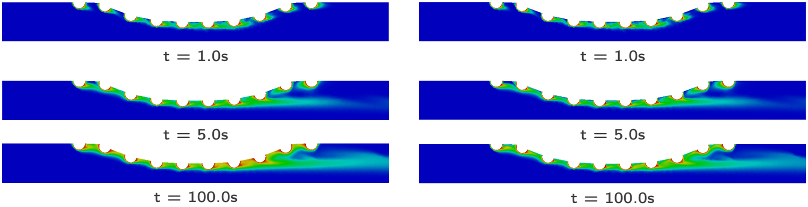


Evolution of velocity profile in centerline ($x = 0.5L$):
(a) Curved Channel and (b) Real Channel

Results - Concentration Field

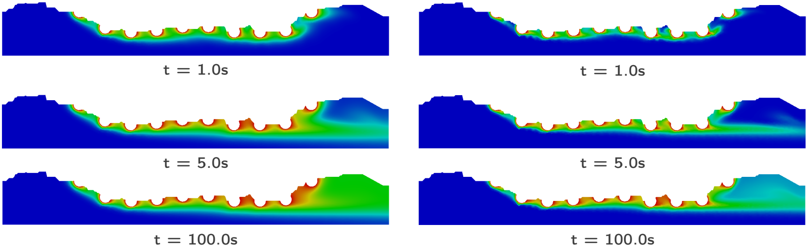


Evolution in time and space of concentration field in Curved Channel:
$Sc = 1$ (left column) and $Sc = 10$ (right column)

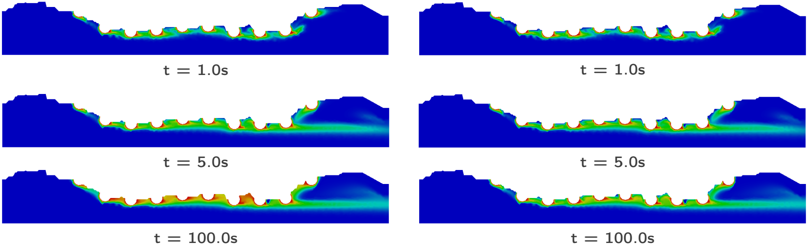


Evolution in time and space of concentration field in Curved Channel:
$Sc = 100$ (left column) and $Sc = 1000$ (right column)

Results - Concentration Field



Evolution in time and space of concentration field in Real Channel:
$Sc = 1$ (left column) and $Sc = 10$ (right column)



Evolution in time and space of concentration field in Real Channel:
$Sc = 100$ (left column) and $Sc = 1000$ (right column)

Conclusion

1. The streamfunction-vorticity formulation showed an useful approach for to calculate the velocity and concentration fields since the variables are scalars allowing a smooth implementation
2. Due to generalized construction of the code, the simulator is able to describe drug-eluting stent problem in coronary artery as well as flows of Newtonian fluids with scalar transport (concentration or temperature)
3. The ALE description allows moving boundary problems to be simulated

Thank you!

marquesleandro67@gmail.com
gustavo.rabello@mecanica.coppe.ufrj.br
jose.pontes@uerj.br

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