On the Electrodynamics of Curved Spacetime

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0 A Note to the Reader

The contents of this paper are designed to be readily accessible to anyone with a solid understanding of vector calculus; topics covering Lagrangian Mechanics, Tensor Calculus, Differential Geometry, and of course, General Relativity, are covered, but they are done so in a way that builds from an assumed knowledge of not much. The majority of the math is in the appendix, meaning that which is in the actual report is pertaining only to 1) the most important equations, and 2) the actual calculations regarding the purpose of this report.

As for the actual purpose of this report, it delves into the topic of Electromagnetic radiation around large bodies in spacetime, taking into account the curvature of spacetime and *how* that changes as opposed to Euclidean space.

While this report *is* written to be comprehensible to those who have not studied any of the topics mentioned prior, it may be beneficial to know the following things:

1) What a Lagrangian is and how it is used, as well as how to utilize the Euler-Lagrange equation.

2)What a tensor is, and how tensor operations function.

3) A familiarity with the Einstein Summation Convention, also called the Subscript/Superscript Summation Notation.

Now that that's out of the way, the fun can begin.

1 There's a Problem

It is well known that the Electromagnetic wave equations can easily be derived from Maxwell's Equations in free space, namely:

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

From which we can utilize a vector calculus identity to derive the electric field wave equation:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{5}$$

Similarly, we can also derive the magnetic field wave equation:

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{6}$$

However, while the equations *do* describe the propagation of electromagnetic radiation in a flat, 3-Dimensional spacetime, they do *not* describe the propagation of EM-radiation in a curved, 4-Dimensional Minkowski spacetime. In order to accurately analyze the behavior of electric and magnetic fields around massive objects, we must utilize the formulation of Maxwell's Equations in curved spacetime via the usage of christoffel symbols, lagrangians, and covariant derivatives.

2 An Extension to General Relativity

For the following calculations, in which I shall analyze the behavior of the electric and magnetic fields near the Sun, a Neutron Star, and a Black Hole, we firstly must define the metric that shall be used that properly described the curvature of spacetime. For all cases, we shall use the Schwarzschild metric, given as follows:

$$ds^2 = -(1-\frac{r_s}{r})c^2dt^2 + (1-\frac{r_s}{r})^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2) \tag{7}$$

From this line element, and with coordinates given by $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$, we can write the metric in matrix form as:

$$\begin{bmatrix} -(1-\frac{r_s}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{r_s}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2sin\theta \end{bmatrix} \tag{8}$$

This matrix, called the metric tensor $g_{\mu\nu}$, allows us to define a suitable metric that can then be used to examine and analyze the behavior of objects near a gravitational field. In order for us to properly examine the behavior of electromagnetic radiation, we must formulate what is called the **Covariant Derivative**, which essentially tells us how a derivative is changing along a manifold with a changing basis vector, which in this case is the curvature of spacetime due to the mass-energy of the examined celestial object. However, in order to calculate the covariant derivative, we must first calculate something else called the **Christoffel Symbols**, of which the directions are as listed below:

1. Using the following equation for the Lagrangian: $L = \frac{1}{2}g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}$, where the components of $g_{\mu\nu}$ are the same as they are in the matrix formulation of the metric tensor, construct the Lagrangian for the Schwarzschild metric.
2. Use the Euler-Lagrange equation to solve for the lagrangians for each coordinate, (ct, r, θ, ϕ), for a total of four different equations of motion for all coordinate directions.
3. Solve the geodesic equation: $\ddot{x^\lambda} + \Gamma^\lambda_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = 0$, where λ runs over the dimensions (0,1,2,3), and both μ and ν sum over all possible combinations of (0,1,2,3), meaning there will be four expressions each consisting of sixteen individual terms.
4. After writing out the full expression for the geodesic equation, compare the geodesic equation and the Euler-Lagrange equation to identity the connection coefficients in front of the velocity terms of both equations.The terms that survive are the non-zero Christoffel symbols for the Schwarzschild metric.

For succinctness, I shall write down the four calculated Euler-Lagrange equations for the four spacetime coordinates of the Schwarzschild metric:

$$\boxed{\ddot{ct} - \dot{ct}\dot{r}\frac{r_s}{r(r-r_s)} = 0} \tag{9}$$

$$\boxed{\ddot{r} + \dot{ct}^2\frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3}) + \dot{r}^2\frac{1}{2}\frac{r_s}{r(r_s-r)} + \dot{\theta}^2(r_s - r) + \dot{\phi}^2(r_s-r)sin^2\theta = 0} \tag{10}$$

$$\boxed{\ddot{\theta} + \dot{\theta}\dot{r}\frac{2}{r} - \dot{\phi}^2 sin\theta cos\theta = 0} \tag{11}$$

$$\boxed{\ddot{\phi} + \dot{r}\dot{\phi}\frac{2}{r} + \dot{\theta}\dot{\phi}2cot\theta = 0} \tag{12}$$

3: Solving the Geodesic Equation

$$\ddot{x^\lambda} + \Gamma^\lambda_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = 0; \lambda = 0, 1, 2, 3 \tag{13}$$

$$\Rightarrow \boxed{\ddot{ct} + \Gamma^0_{00}\dot{ct}^2 + 2\Gamma^0_{01}\dot{ct}\dot{r} + 2\Gamma^0_{02}\dot{ct}\dot{\theta} + 2\Gamma^0_{03}\dot{ct}\dot{\phi} + \Gamma^0_{11}\dot{r}^2 + 2\Gamma^0_{12}\dot{r}\dot{\theta} + 2\Gamma^0_{13}\dot{r}\dot{\phi} + \Gamma^0_{22}\dot{\theta}^2 + 2\Gamma^0_{23}\dot{\theta}\dot{\phi} + \Gamma^0_{33}\dot{\phi}^2 = 0} \tag{14}$$

$$\Rightarrow \boxed{\ddot{r} + \Gamma^1_{00}\dot{ct}^2 + 2\Gamma^1_{01}\dot{ct}\dot{r} + 2\Gamma^1_{02}\dot{ct}\dot{\theta} + 2\Gamma^1_{03}\dot{ct}\dot{\phi} + \Gamma^1_{11}\dot{r}^2 + 2\Gamma^1_{12}\dot{r}\dot{\theta} + 2\Gamma^1_{13}\dot{r}\dot{\phi} + \Gamma^1_{22}\dot{\theta}^2 + 2\Gamma^1_{23}\dot{\theta}\dot{\phi} + \Gamma^1_{33}\dot{\phi}^2 = 0} \tag{15}$$

$$\Rightarrow \boxed{\ddot{\theta} + \Gamma^2_{00}\dot{ct}^2 + 2\Gamma^2_{01}\dot{ct}\dot{r} + 2\Gamma^2_{02}\dot{ct}\dot{\theta} + 2\Gamma^2_{03}\dot{ct}\dot{\phi} + \Gamma^2_{11}\dot{r}^2 + 2\Gamma^2_{12}\dot{r}\dot{\theta} + 2\Gamma^2_{13}\dot{r}\dot{\phi} + \Gamma^2_{22}\dot{\theta}^2 + 2\Gamma^2_{23}\dot{\theta}\dot{\phi} + \Gamma^2_{33}\dot{\phi}^2 = 0} \tag{16}$$

$$\Rightarrow \boxed{\ddot{\phi} + \Gamma^3_{00}\dot{ct}^2 + 2\Gamma^3_{01}\dot{ct}\dot{r} + 2\Gamma^3_{02}\dot{ct}\dot{\theta} + 2\Gamma^3_{03}\dot{ct}\dot{\phi} + \Gamma^3_{11}\dot{r}^2 + 2\Gamma^3_{12}\dot{r}\dot{\theta} + 2\Gamma^3_{13}\dot{r}\dot{\phi} + \Gamma^3_{22}\dot{\theta}^2 + 2\Gamma^3_{23}\dot{\theta}\dot{\phi} + \Gamma^3_{33}\dot{\phi}^2 = 0} \tag{17}$$

4: Comparing the Euler-Lagrange Equation and the Geodesic Equation. From direct observation, we can identify the following non-zero Christoffel symbols:

$$\Gamma^0_{01} = \frac{1}{2}\frac{r_s}{r^2 - rr_s}; \quad \Gamma^0_{10} = \frac{1}{2}\frac{r_s}{r^2 - rr_s} \tag{18}$$

$$\Gamma^1_{00} = \frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3}); \quad \Gamma^1_{11} = \frac{1}{2}\frac{r_s}{r(r_s - r)}; \quad \Gamma^1_{22} = r_s - r; \quad \Gamma^1_{33} = (r_s - r)sin^2\theta \tag{19}$$

$$\Gamma^2_{12} = \frac{1}{r}; \quad \Gamma^2_{21} = \frac{1}{r}; \quad \Gamma^2_{33} = -sin\theta cos\theta \tag{20}$$

$$\Gamma^3_{13} = \frac{1}{r}; \quad \Gamma^3_{23} = cot\theta; \quad \Gamma^3_{31} = \frac{1}{r}; \quad \Gamma^3_{32} = cot\theta; \tag{21}$$

3 A Little Bit More Math

After solving for the Christoffel symbols, we can finally construct the argument for the Covariant Derivative, for which the formula is given as follows:

$$\nabla_\mu F_{r\lambda} = \partial\mu F_{\nu\lambda} + \Gamma^{\rho}_{\mu\nu} F_{\rho\lambda} + \Gamma^{\rho}_{\mu\lambda} F_{\nu\rho}, \tag{22}$$

where I have chosen, for simplicity's sake, to only consider a) that there only be an electric field, and b) that the radial component of the electric field, that is, only E_r will be non-zero.

Or, written after being expanded out in terms of its christoffel symbols:

$$\Rightarrow \boxed{\nabla_\mu F_{r\lambda} = \frac{\partial E_r}{\partial r}\delta^\mu_r + B_z[\frac{r_s}{r^2} - \frac{r_s^2}{r^3} + \frac{r_s}{r(r_s - r)} + 2r_s - 2r + 2r_s sin^2\theta - 2r sin^2\theta]} \tag{23}$$

Which is the complete expression for the covariant derivative of a purely radial electric field.

4 An Application from the Universe

Finally, we can begin to consider what happens to electromagnetic radiation as it propagates around through the gravitational field of celestial bodies. I will consider three separate cases with three separate objects, with the polar angle θ being equal to $\frac{\pi}{2}$ in all cases:

$$1) The\ Sun: \nabla_\mu F_{r\lambda} = B_z(\tfrac{2.9685\cdot10^3}{4.84416\cdot10^{17}} - \tfrac{8.81199\cdot10^6}{3.3715\cdot10^{26}} + \tfrac{2.9685\cdot10^3}{-4.8441\cdot10^{17}} + 2(2.9685\cdot10^3) - 2(6.96\cdot10^8) + 2(2.9685\cdot10^3) - 2(6.96\cdot10^8) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$= B_z(6.128\cdot10^{15} - 2.614\cdot10^{-20} - 6.128\cdot10^{-15} + 5937 - 1338000000 + 5937 - 1338000000) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$= B_z(-2675988126) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$\Rightarrow \boxed{-2.675988\cdot10^9 B_z + \frac{\partial E_r}{\partial r}\delta^\mu_r}$$

$$2) A\ Neutron\ Star: \nabla_\mu F_{r\lambda} = B_z(\tfrac{4.156\cdot10^3}{1\cdot10^8} - \tfrac{1.727\cdot10^7}{1\cdot10^{12}} + \tfrac{4.156\cdot10^3}{-5.844\cdot10^7} + 4(4.156\cdot10^3) - 4(1\cdot10^4) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$= B_z(-2.3376\cdot10^4) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$\Rightarrow \boxed{-2.3376\cdot10^4 B_z + \frac{\partial E_r}{\partial r}\delta^\mu_r}$$

Lastly, I will consider the supermassive black hole at the center of the Milky Way galaxy, Saggitarius A*, which has a radius of $5.18\cdot10^{10}$m and a mass of $8.54\cdot10^{36}$kg:

$$3) Saggitarius\ A*: \nabla_\mu F_{r\lambda} = B_z(\tfrac{1.2676\cdot10^{10}}{2.68324\cdot10^{21}} - \tfrac{1.6068\cdot10^{20}}{1.38992\cdot10^{32}} + \tfrac{1.2676\cdot10^{10}}{-2.0266232\cdot10^{21}} + 4(1.2676\cdot10^{10}) - 4(5.18\cdot10^{10}) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$= B_z(-1.42096\cdot10^{11}) + \tfrac{\partial E_r}{\partial r}\delta^\mu_r$$

$$\Rightarrow \boxed{-1.42096\cdot10^{11} B_z + \frac{\partial E_r}{\partial r}\delta^\mu_r}$$

5 So what does this mean?

From this, even without giving actual values for the electric field, we can see a proof of the Ampère–Maxwell law: **A changing electric field produces a magnetic field**. As is seen in the three cases, as our electric field propagates through curved spacetime, even if it's magnitude isn't changing, the basis vector *is*, and so it is in and of itself a changing electric field, which then produces a magnetic field in addition to the change in the electric field brought about by the changing basis vectors.

Furthermore, while physicists of the past have experimentally proven the validity of their scientific theories via direct experimentation, there is unfortunately very few ways for aspiring students to test the credibility of such a theory like General Relativity, at least without access to advanced equipment and precise measurements to observe and verify.

Take, for example, two of the most common and widely accepted "tests" of General Relativity: Time Dilation and Gravitational Lensing, both of which have indeed proven to be a reliable indication that General Relativity works, but both of which are impossible for an ordinary student to feasibly conduct a study upon.

Thus, for a student, it seems as if the only option is to study purely theoretical applications, which as was seen over the course of this paper can be at times, for a lack of a better descriptive word, *scary*. However, in the event that it can be sucked up and dealt with, it truly becomes beautiful as you are able to see other laws of physics that are intricately intertwined with each other, my case in point being the pseudo-proof of the Ampère–Maxwell law.

Whereas Maxwell experimentally proved that a changing electric field produced a magnetic field, I have over the course of this paper proved theoretically that a changing electric field produces a magnetic field. Though of course Maxwell did his experiment on the Earth and my theoretical calculations assume movement along a Pseudo-Riemannian manifold with a changing basis vector, the heart of the idea is still the same: There is a way to prove these theories true even if you don't have access to the cutting edge equipment, for the idea of a changing basis vector for an electric field would not make sense if the General Theory of Relativity were not also true.

6 Appendix: The Math Behind it All

6.1 The Lagrangians

1: Constructing the Lagrangian

$$x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi \tag{24}$$

$$ds^2 = -(1-\frac{r_s}{r})c^2dt^2 + (1-\frac{r_s}{r})^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2) \tag{25}$$

$$L = \frac{1}{2}g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = \frac{1}{2}g_{00}\dot{x^0}\dot{x^0} + \frac{1}{2}g_{11}\dot{x^1}\dot{x^1} + \frac{1}{2}g_{22}\dot{x^2}\dot{x^2} + \frac{1}{2}g_{33}\dot{x^3}\dot{x^3} \tag{26}$$

$$\rightarrow \dot{x}^0 = \dot{ct}, \dot{x}^1 = \dot{r}, \dot{x}^2 = \dot{\theta}, \dot{x}^3 = \dot{\phi}; g_{00} = -(1-\frac{r_s}{r}), g_{11} = (1-\frac{r_s}{r})^{-1}, g_{22} = r^2, g_{33} = r^2sin^2\theta \tag{27}$$

$$\Rightarrow \boxed{L = -\frac{1}{2}(1-\frac{r_s}{r})\dot{ct}^2 + \frac{1}{2}(1-\frac{r_s}{r})^{-1}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2sin^2\theta\dot{\phi}^2} \tag{28}$$

2: Using the Euler-Lagrange Equation

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{ct}} - \frac{\partial L}{\partial ct} = 0 \rightarrow \frac{d}{d\tau}\frac{\partial}{\partial \dot{ct}}(L) - \frac{\partial}{\partial ct}(L) = 0 \tag{29}$$

$$i)\frac{\partial L}{\partial ct} = 0; \quad ii)\frac{\partial L}{\partial \dot{ct}} = -(1-\frac{r_s}{r})\dot{ct}$$

$$\rightarrow \frac{d}{d\tau}((\frac{r_s}{r}-1)\dot{ct}) = \ddot{ct}(\frac{r_s}{r}-1) + \dot{ct}(-\frac{r_s}{r^2}\dot{r}) \Rightarrow \boxed{\ddot{ct} - \dot{ct}\dot{r}\frac{r_s}{r(r-r_s)} = 0}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \rightarrow \frac{d}{d\tau}\frac{\partial}{\partial \dot{r}}(L) - \frac{\partial}{\partial r}(L) = 0 \tag{30}$$

$$i)\frac{\partial L}{\partial r} = -\frac{r_s}{2r^2}\dot{ct}^2 - \frac{2r_s}{(r_s-r)}^2\dot{r}^2 + r\dot{\theta}^2 + rsin^2\theta\dot{\phi}^2; \quad ii)\frac{\partial L}{\partial \dot{r}} = (1-\frac{r_s}{r})\dot{r}$$

$$\rightarrow \frac{d}{d\tau}((1-\frac{r_s}{r})\dot{r}) = \ddot{r}(1-\frac{r_s}{r}) + \dot{r}^2(\frac{r_s}{r^2})$$

$$\rightarrow \frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \ddot{r}(1-\frac{r_s}{r}) + \dot{r}^2(\frac{r_s}{r^2}) + \frac{r_s}{2r^2}\dot{ct}^2 + \frac{2r_s}{(r_s-r)}^2\dot{r}^2 - r\dot{\theta}^2 - rsin^2\theta\dot{\phi}^2$$

$$\Rightarrow \boxed{\ddot{r} + \dot{ct}^2\frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3}) + \dot{r}^2\frac{1}{2}\frac{r_s}{r(r_s-r)} + \dot{\theta}^2(r_s-r) + \dot{\phi}^2(r_s-r)sin^2\theta = 0}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \rightarrow \frac{d}{d\tau}\frac{\partial}{\partial \dot{\theta}}(L) - \frac{\partial}{\partial \theta}(L) = 0 \tag{31}$$

$$i)\frac{\partial L}{\partial \theta} = r^2sin\theta cos\theta\dot{\phi}^2; \quad ii)\frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$$

$$\rightarrow \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \ddot{\theta}r^2 + \dot{\theta}\dot{r}2r - r^2sin\theta cos\theta\dot{\phi}^2 \Rightarrow \boxed{\ddot{\theta} + \dot{\theta}\dot{r}\frac{2}{r} - \dot{\phi}^2sin\theta cos\theta = 0}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{d}{d\tau}\frac{\partial}{\partial \dot{\phi}}(L) - \frac{\partial}{\partial \phi}(L) = 0 \tag{32}$$

$$i)\frac{\partial L}{\partial \phi} = 0; \quad ii)\frac{\partial L}{\partial \dot{\phi}} = r^2 sin^2\theta\dot{\phi}$$

$$\rightarrow \frac{d}{d\tau}(r^2 sin^2\theta\dot{\phi}) = \ddot{\phi}r^2 sin^2\theta + 2rsin^2\theta\dot{r}\dot{\phi} + 2r^2 sin\theta cos\theta\dot{\theta}\dot{\phi} \Rightarrow \boxed{\ddot{\phi} + \dot{r}\dot{\phi}\frac{2}{r} + \dot{\theta}\dot{\phi}2cot\theta = 0}$$

6.2 The Geodesic Equation Expanded

3: Solving the Geodesic Equation

$$\ddot{x^\lambda} + \Gamma^\lambda_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = 0; \lambda = 0, 1, 2, 3 \tag{33}$$

1) $\ddot{x}^0 + \Gamma^0_{00}\dot{x}^0\dot{x}^0 + \Gamma^0_{01}\dot{x}^0\dot{x}^1 + \Gamma^0_{02}\dot{x}^0\dot{x}^2 + \Gamma^0_{03}\dot{x}^0\dot{x}^3 + \Gamma^0_{10}\dot{x}^1\dot{x}^0 + \Gamma^0_{11}\dot{x}^1\dot{x}^1 + \Gamma^0_{12}\dot{x}^1\dot{x}^2 + \Gamma^0_{13}\dot{x}^1\dot{x}^3 + \Gamma^0_{20}\dot{x}^2\dot{x}^0 + \Gamma^0_{21}\dot{x}^2\dot{x}^1 + \Gamma^0_{22}\dot{x}^2\dot{x}^2 + \Gamma^0_{23}\dot{x}^2\dot{x}^3 + \Gamma^0_{30}\dot{x}^3\dot{x}^0 + \Gamma^0_{31}\dot{x}^3\dot{x}^1 + \Gamma^0_{32}\dot{x}^3\dot{x}^2 + \Gamma^0_{33}\dot{x}^3\dot{x}^3 = 0$

2) $\ddot{x}^1 + \Gamma^1_{00}\dot{x}^0\dot{x}^0 + \Gamma^1_{01}\dot{x}^0\dot{x}^1 + \Gamma^1_{02}\dot{x}^0\dot{x}^2 + \Gamma^1_{03}\dot{x}^0\dot{x}^3 + \Gamma^1_{10}\dot{x}^1\dot{x}^0 + \Gamma^1_{11}\dot{x}^1\dot{x}^1 + \Gamma^1_{12}\dot{x}^1\dot{x}^2 + \Gamma^1_{13}\dot{x}^1\dot{x}^3 + \Gamma^1_{20}\dot{x}^2\dot{x}^0 + \Gamma^1_{21}\dot{x}^2\dot{x}^1 + \Gamma^1_{22}\dot{x}^2\dot{x}^2 + \Gamma^1_{23}\dot{x}^2\dot{x}^3 + \Gamma^1_{30}\dot{x}^3\dot{x}^0 + \Gamma^1_{31}\dot{x}^3\dot{x}^1 + \Gamma^1_{32}\dot{x}^3\dot{x}^2 + \Gamma^1_{33}\dot{x}^3\dot{x}^3 = 0$

3) $\ddot{x}^2 + \Gamma^2_{00}\dot{x}^0\dot{x}^0 + \Gamma^2_{01}\dot{x}^0\dot{x}^1 + \Gamma^2_{02}\dot{x}^0\dot{x}^2 + \Gamma^2_{03}\dot{x}^0\dot{x}^3 + \Gamma^2_{10}\dot{x}^1\dot{x}^0 + \Gamma^2_{11}\dot{x}^1\dot{x}^1 + \Gamma^2_{12}\dot{x}^1\dot{x}^2 + \Gamma^2_{13}\dot{x}^1\dot{x}^3 + \Gamma^2_{20}\dot{x}^2\dot{x}^0 + \Gamma^2_{21}\dot{x}^2\dot{x}^1 + \Gamma^2_{22}\dot{x}^2\dot{x}^2 + \Gamma^2_{23}\dot{x}^2\dot{x}^3 + \Gamma^2_{30}\dot{x}^3\dot{x}^0 + \Gamma^2_{31}\dot{x}^3\dot{x}^1 + \Gamma^2_{32}\dot{x}^3\dot{x}^2 + \Gamma^2_{33}\dot{x}^3\dot{x}^3 = 0$

4) $\ddot{x}^3 + \Gamma^3_{00}\dot{x}^0\dot{x}^0 + \Gamma^3_{01}\dot{x}^0\dot{x}^1 + \Gamma^3_{02}\dot{x}^0\dot{x}^2 + \Gamma^3_{03}\dot{x}^0\dot{x}^3 + \Gamma^3_{10}\dot{x}^1\dot{x}^0 + \Gamma^3_{11}\dot{x}^1\dot{x}^1 + \Gamma^3_{12}\dot{x}^1\dot{x}^2 + \Gamma^3_{13}\dot{x}^1\dot{x}^3 + \Gamma^3_{20}\dot{x}^2\dot{x}^0 + \Gamma^3_{21}\dot{x}^2\dot{x}^1 + \Gamma^3_{22}\dot{x}^2\dot{x}^2 + \Gamma^3_{23}\dot{x}^2\dot{x}^3 + \Gamma^3_{30}\dot{x}^3\dot{x}^0 + \Gamma^3_{31}\dot{x}^3\dot{x}^1 + \Gamma^3_{32}\dot{x}^3\dot{x}^2 + \Gamma^3_{33}\dot{x}^3\dot{x}^3 = 0$

6.3 The Covariant Derivative

We must expand the formula for the covariant derivative out in terms of its Christoffel symbols. Going term by term, we have:

$$1) \quad \partial\mu F_{\nu\lambda} = \partial\mu = \frac{\partial E_r}{\partial x^\mu} = \frac{\partial E_r}{\partial r}\delta^\mu_r$$

$$2) \quad \Gamma^\rho_{\mu\nu}F_{\rho\lambda} = \Gamma^r_{\mu\nu}F_{r\lambda} + \Gamma^t_{\mu\nu}F_{t\lambda}$$

$$3) \quad \Gamma^\rho_{\mu\lambda}F_{\nu\rho} = \Gamma^r_{\mu\lambda}F_{\nu r} + \Gamma^t_{\mu\lambda}F_{\nu t}$$

Which, when put together, gives us the complete expression for the covariant derivative:

$$\nabla_\mu F_{r\lambda} = \frac{\partial E_r}{\partial r}\delta^\mu_r + \Gamma^r_{\mu\nu}F_{r\lambda} + \Gamma^t_{\mu\nu}F_{t\lambda} + \Gamma^r_{\mu\lambda}F_{\nu r} + \Gamma^t_{\mu\lambda}F_{\nu t} \tag{34}$$

Moreover, there are four terms in this expression which are summed over themselves four times, with a total of four possible combinations, meaning there is a total of sixty-four terms per term containing a christoffel symbol, and since there are four terms with a christoffel symbol, this gives rise to a single equation containing 256 terms, plus the one term from the partial derivative.

$$i) \quad \Gamma^r_{\mu\nu}F_{r\lambda} \tag{35}$$

$$ii) \quad \Gamma^t_{\mu\nu}F_{t\lambda} \tag{36}$$

$$iii) \quad \Gamma^r_{\mu\nu} F_{\nu r} \tag{37}$$

$$iv) \quad \Gamma^t_{\mu\nu} F_{\nu t} \tag{38}$$

However, we can make this process easier by eliminating the christoffel symbols that have a value of zero, meaning that we will only have as many terms as we have non-zero christoffel symbols that were calculated earlier using the Euler-Lagrange equation and the Geodesic equation.

$$i) \quad \Gamma^r_{\mu\nu} F_{r\lambda} = \Gamma^r_{tt} F_{r\lambda} + \Gamma^r_{rr} F_{r\lambda} + \Gamma^r_{\theta\theta} F_{r\lambda} + \Gamma^r_{\phi\phi} F_{r\lambda}$$

$$\rightarrow \quad = \tfrac{1}{2}(\tfrac{r_s}{r} - \tfrac{r_s^2}{r^3}) F_{r\lambda} + \tfrac{1}{2}\tfrac{r_s}{r(r_s - r)} F_{r\lambda} + (r_s - r) F_{r\lambda} + ((r_s - r) sin^2\theta) F_{r\lambda}$$

Let us, for the sake of sanity, denote the christoffel symbols, $\Gamma^1_{00} = \frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3})$; $\Gamma^1_{11} = \frac{1}{2}\frac{r_s}{r(r_s-r)}$; $\Gamma^1_{22} = r_s - r$; $\Gamma^1_{33} = (r_s - r)sin^2\theta$, by A, B, C, and D respectively.

$$= AF_{r0} + AF_{r1} + AF_{r2} + AF_{r3} + BF_{r0} + BF_{r1} + BF_{r2} + BF_{r3} + AF_{r0} + CF_{r1} + CF_{r2} + CF_{r3} + DF_{r0} + DF_{r1} + DF_{r2} + DF_{r3},$$

Where the terms $F_{r0}, F_{r,1}$, etc. are the components of the **Electromagnetic Field Tensor**, which is given in its entirety at the end of the appendix; though for this particular scenario, we only care about a few components:

$$F_{r0} = F_{10} = \tfrac{E_x}{c}; F_{r1} = F_{11} = 0; F_{r2} = F_{12} = B_z; F_{r3} = F_{13} = -B_y$$

Thus, we have

$$\rightarrow \tfrac{AE_x}{c} + AB_z - AB_y + \tfrac{BE_x}{c} + BB_z - BB_y + \tfrac{CE_x}{c} + CB_z - CB_y + \tfrac{DE_x}{c} + DB_z - DB_y$$

$$= A(\tfrac{E_x}{c} + B_z - B_y) + B(\tfrac{E_x}{c} + B_z - B_y) + C(\tfrac{E_x}{c} + B_z - B_y) + D(\tfrac{E_x}{c} + B_z - B_y)$$

$$= [\tfrac{1}{2}(\tfrac{r_s}{r} - \tfrac{r_s^2}{r^3})](\tfrac{E_x}{c} + B_z - B_y) + [\tfrac{1}{2}\tfrac{r_s}{r(r_s-r)}](\tfrac{E_x}{c} + B_z - B_y) + [r_s - r](\tfrac{E_x}{c} + B_z - B_y) + [(r_s - r)sin^2\theta](\tfrac{E_x}{c} + B_z - B_y)$$

$$\boxed{\Gamma^r_{\mu\nu} F_{r\lambda} = [\frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3})](\frac{E_x}{c} + B_z - B_y) + [\frac{1}{2}\frac{r_s}{r(r_s-r)}](\frac{E_x}{c} + B_z - B_y) + [r_s - r](\frac{E_x}{c} + B_z - B_y) + [(r_s - r)sin^2\theta](\frac{E_x}{c} + B_z - B_y)}$$

$$ii) \quad \Gamma^t_{\mu\nu} F_{t\lambda} = \Gamma^t_{tr} F_{t\lambda} + \Gamma^t_{rt} F_{t\lambda} \rightarrow \quad = \tfrac{1}{2}\tfrac{r_s}{r^2 - rr_s} F_{t\lambda} + \tfrac{1}{2}\tfrac{r_s}{r^2 - rr_s} F_{t\lambda} = \tfrac{r_s}{r^2 - rr_s} F_{t\lambda}$$

Again, we will denote $\frac{r_s}{r^2 - rr_s}$ by E:

$$= EF_{t0} + EF_{t1} + EF_{t2} + EF_{t3}$$

Where,

$$F_{t0} = F_{00} = 0; F_{t1} = F_{01} = -\tfrac{E_x}{c}; F_{t2} = F_{02} = -\tfrac{E_y}{c}; F_{t3} = F_{03} = -\tfrac{E_z}{c}$$

Therefore,

$$\rightarrow A(-\tfrac{E_x}{c}-\tfrac{E_y}{c}-\tfrac{E_z}{c}) = \tfrac{r_s}{r^2-rr_s}(-\tfrac{E_x}{c}-\tfrac{E_y}{c}-\tfrac{E_z}{c})$$

$$\boxed{\Gamma^t_{\mu\nu}F_{t\lambda} = \frac{r_s}{r^2-rr_s}(-\frac{E_x}{c}-\frac{E_y}{c}-\frac{E_z}{c})}$$

$$iii)\quad \Gamma^r_{\mu\nu}F_{\nu r} = \Gamma^r_{\mu\nu}F_{tr} + \Gamma^r_{\mu\nu}F_{\theta r} + \Gamma^r_{\mu\nu}F_{\phi r}$$

$$a)\quad \Gamma^r_{\mu\nu}F_{tr} = \Gamma^1_{00}F_{tr} + \Gamma^1_{11}F_{tr} + \Gamma^1_{22}F_{tr} + \Gamma^1_{33}F_{tr} = [\tfrac{1}{2}(\tfrac{r_s}{r}-\tfrac{r_s^2}{r^3})]F_{tr} + [\tfrac{1}{2}\tfrac{r_s}{r(r_s-r)}]F_{tr} + [r_s-r]F_{tr} + [(r_s-r)sin^2\theta]F_{tr}$$

Where again as before, I will denote the christoffel symbols by A, B, C,and D accordingly. This time, however, there is only *one* tensor component that need be considered:

$$F_{tr} = F_{01} = -\tfrac{E_x}{c}$$

Therefore,

$$\Gamma^r_{\mu\nu}F_{tr} = -\tfrac{AE_x}{c} - \tfrac{BE_x}{c} - \tfrac{CE_x}{c} - \tfrac{DE_x}{c}$$

$$b)\quad \Gamma^r_{\mu\nu}F_{\theta r} = ...; F_{\theta r} = F_{21} = B_z \rightarrow = ... = AB_z + BB_z + CB_z + DB_z$$

$$c)\quad \Gamma^r_{\mu\nu}F_{\phi r} = ...; F_{\phi r} = F_{31} = B_y \rightarrow = ... = AB_y + BB_y + CB_y + DB_y$$

$$\rightarrow \quad \Gamma^r_{\mu\nu}F_{\nu r} = AB_y + BB_y + CB_y + DB_y + AB_z + BB_z + CB_z + DB_z - \tfrac{AE_x}{c} - \tfrac{BE_x}{c} - \tfrac{CE_x}{c} - \tfrac{DE_x}{c}$$

$$= A(B_y + B_z - \tfrac{E_x}{c}) + B(B_y + B_z - \tfrac{E_x}{c}) + C(B_y + B_z - \tfrac{E_x}{c}) + D(B_y + B_z - \tfrac{E_x}{c})$$

$$\boxed{\Gamma^r_{\mu\nu}F_{\nu r} = [\frac{1}{2}(\frac{r_s}{r}-\frac{r_s^2}{r^3})](B_y + B_z - \frac{E_x}{c}) + [\frac{1}{2}\frac{r_s}{r(r_s-r)}](B_y + B_z - \frac{E_x}{c}) + [r_s-r](B_y + B_z - \frac{E_x}{c}) + [(r_s-r)sin^2\theta](B_y + B_z - \frac{E_x}{c})}$$

$$iv)\quad \Gamma^t_{\mu\nu}F_{\nu t} = \Gamma^t_{\mu\nu}F_{rt} + \Gamma^t_{\mu\nu}F_{\theta t} + \Gamma^t_{\mu\nu}F_{\phi t}$$

$$a)\Gamma^t_{\mu\nu}F_{rt} = \Gamma^0_{01}F_{rt} + \Gamma^0_{10}F_{rt} \rightarrow \quad = \tfrac{1}{2}\tfrac{r_s}{r^2-rr_s}F_{rt} + \tfrac{1}{2}\tfrac{r_s}{r^2-rr_s}F_{rt} = \tfrac{r_s}{r^2-rr_s}F_{rt}$$

Using the following:

$$F_{rt} = F_{10} = \tfrac{E_x}{c}$$

We get,

$$\Gamma^t_{\mu\nu}F_{rt} = (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_x}{c}$$

$$b) \quad \Gamma^t_{\mu\nu}F_{\theta t} = ... =; F_{\theta t} = F_{2,0} = \tfrac{E_y}{c} \rightarrow = ... = (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_y}{c}$$

$$c) \quad \Gamma^t_{\mu\nu}F_{\phi t} = ... =; F_{\phi t} = F_{3,0} = \tfrac{E_z}{c} \rightarrow = ... = (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_z}{c}$$

$$\rightarrow \Gamma^t_{\mu\nu}F_{\nu t} = (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_x}{c} + (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_y}{c} + (\tfrac{r_s}{r^2 - rr_s})\tfrac{E_z}{c}$$

$$\boxed{\Gamma^t_{\mu\nu}F_{\nu t} = [\frac{r_s}{r^2 - rr_s}](\frac{E_x}{c} + \frac{E_y}{c} + \frac{E_z}{c})}$$

Compiling everything, we have the following 4 expressions for the christoffel symbols terms:

$$\boxed{\Gamma^r_{\mu\nu}F_{r\lambda} = [\frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3})](\frac{E_x}{c} + B_z - B_y) + [\frac{1}{2}\frac{r_s}{r(r_s - r)}](\frac{E_x}{c} + B_z - B_y) + [r_s - r](\frac{E_x}{c} + B_z - B_y) + [(r_s - r)sin^2\theta](\frac{E_x}{c} + B_z - B_y)}$$

$$\boxed{\Gamma^t_{\mu\nu}F_{t\lambda} = \frac{r_s}{r^2 - rr_s}(-\frac{E_x}{c} - \frac{E_y}{c} - \frac{E_z}{c})}$$

$$\boxed{\Gamma^r_{\mu\nu}F_{\nu r} = [\frac{1}{2}(\frac{r_s}{r} - \frac{r_s^2}{r^3})](B_y + B_z - \frac{E_x}{c}) + [\frac{1}{2}\frac{r_s}{r(r_s - r)}](B_y + B_z - \frac{E_x}{c}) + [r_s - r](B_y + B_z - \frac{E_x}{c}) + [(r_s - r)sin^2\theta](B_y + B_z - \frac{E_x}{c})}$$

$$\boxed{\Gamma^t_{\mu\nu}F_{\nu t} = [\frac{r_s}{r^2 - rr_s}](\frac{E_x}{c} + \frac{E_y}{c} + \frac{E_z}{c})}$$

And we can see that there is beautiful symmetry between the four expressions, leading to the complete cancelling of two of them and numerous terms from the other two. I will, once again, denote the christoffel symbols by A, B, C, and D. Thus, finally, we can put everything together to attain a complete expression for the covariant derivative:

$$i) + ii) + iii) + iv) = A(\tfrac{E_x}{c} + B_z - B_y) + B(\tfrac{E_x}{c} + B_z - B_y) + C(\tfrac{E_x}{c} + B_z - B_y) + D(\tfrac{E_x}{c} + B_z - B_y) + A(B_y + B_z - \tfrac{B_x}{c}) + B(B_y + B_z - \tfrac{B_x}{c}) + C(B_y + B_z - \tfrac{B_x}{c}) + D(B_y + B_z - \tfrac{B_x}{c})$$

$$= 2AB_z + 2BB_z + 2CB_z + 2DB_z$$

$$= (\tfrac{r_s}{r} - \tfrac{r_s^2}{r^3})B_z + (\tfrac{r_s}{r(r_s - r)})B_z + (2(r_s - r))B_z + (2(r_s - r)sin^2\theta)B_z$$

$$= B_z[\tfrac{r_s}{r^2} - \tfrac{r_s^2}{r^3} + \tfrac{r_s}{r(r_s - r)} + 2r_s - 2r + 2r_s sin^2\theta - 2r sin^2\theta]$$

6.4 Deriving the Electromagnetic Field Tensor

Firstly, we shall first write the electromagnetic field vectors **E** and **B** in terms of the vector potential as:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{39}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{40}$$

Moreover, using the Einstein summation convention, we can express these two equations as:

$$\mathbf{E} = -\frac{\partial \phi}{\partial x_\mu} - \frac{\partial A_\mu}{\partial t} \tag{41}$$

$$\mathbf{B} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{42}$$

If we let $A_\mu = (\phi, -\mathbf{A})$, by algebraic manipulation, we will find the following relations:

$$\begin{gathered}
F_{01} = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = -\frac{E_x}{c} \\
F_{02} = -\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y} = -\frac{E_y}{c} \\
F_{03} = -\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z} = -\frac{E_z}{c} \\
F_{12} = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = B_z \\
F_{12} = \frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} = -B_y \\
F_{12} = -\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial z} = B_x
\end{gathered} \tag{43}$$

Surprisingly, these component equations are in fact *not* four-vectors, but instead six components that together are used to define an antisymmetrical tensor called the **Electromagnetic Field Tensor**.

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{44}$$