Gauge Invariance as Applied to Maxwell's Equations

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To prove that the equations of Electromagnetism, namely Maxwell's Equations, are invariant under a gauge transformation, we can consider adding to the Lagrangian for Electromagnetism, $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$ an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$. The resulting proof is as follows.

The equations of motion for a flat-spacetime field theory can be derived from the Euler-Lagrange equations for a field theory in flat spacetime:

$$\boxed{\frac{\delta S}{\delta \Phi^i} = \frac{\partial \mathcal{L}_{\mathcal{C}}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\mathcal{C}}}{\partial(\partial_\mu \Phi^i)} \right) = 0} \tag{1}$$

Using this, we can apply it to our Lagrangian for Electromagnetism to receive the manifestly covariant non-homogeneous Maxwell Equation, which incorporates both Gauss' Law for Electricity and Ampère's Law: $\partial_\mu F^{\nu\mu} = J^\nu$. *This* is how we will demonstrate the properties of gauge invariance, by showing that an additional term to the Lagrangian will not change the end result.

Starting with our Lagrangian: $\mathcal{L}_{\mathcal{C}C} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + \tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, the first term in the Euler-Lagrange equation is straightforward:

$$\frac{\partial \mathcal{L}_{\mathcal{C}}}{\partial A_\nu} = \delta^\nu_\mu J^\mu = J^\nu \tag{2}$$

The second and third terms, however, are much trickier, and firstly require the rewriting the first and third terms of the Lagrangian:

$$\begin{aligned} &a) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\frac{1}{4}\eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} \\ &b)\tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = \tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}F_{\alpha\beta}F_{\gamma\lambda} \end{aligned} \tag{3}$$

Thus, we now have the following to evaluate:

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}(\mathcal{L}_{\mathcal{C}}) = \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(-\frac{1}{4}\eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} + A_\mu J^\nu + \tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}F_{\alpha\beta}F_{\gamma\lambda} \right) \tag{4}$$

Utilizing the Sum Rule for Differentiation, we can split up the calculation as follows:

$$= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(-\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} \right) + \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} F_{\alpha\beta} F_{\gamma\lambda} \right) + \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(A_\mu J^\nu \right) \tag{5}$$

The last term in (5) goes to zero as per the rules for partial differentiation. Continuing on, we can pull out the metric tensors and apply the Leibniz Rule for differentiation of products:

$$\begin{aligned} &= -\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \left[\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (F_{\alpha\beta}) \right) F_{\rho\sigma} + \left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (F_{\rho\sigma}) \right) F_{\alpha\beta} \right] \\ &+ \tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \left[\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (F_{\alpha\beta}) \right) F_{\gamma\lambda} + \left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (F_{\gamma\lambda}) \right) F_{\alpha\beta} \right] \end{aligned} \tag{6}$$

Now, we will utilize the following definition of the Electromagnetic Field Strength Tensor:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{7}$$

This definition comes from the fact that both the Electric and Magnetic fields can be represented by the vector potential A, and the fact that the 6 components of the Electric and Magnetic fields can be packaged together into a rank-two antisymmetric tensor, that being, of course, the Electromagnetic Field Strength Tensor, with the components given as follows:

$$F_{\mu\nu} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{8}$$

Returning to the problem at hand, we can express all the instances of the field strength tensor in this notation:

$$\begin{aligned} &= -\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \left[\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right) F_{\rho\sigma} + \left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \right) F_{\alpha\beta} \right] \\ &+ \tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \left[\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right) F_{\gamma\lambda} + \left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\gamma A_\lambda - \partial_\lambda A_\gamma) \right) F_{\alpha\beta} \right] \end{aligned} \tag{9}$$

Next, we will take advantage of the following identity:

$$\boxed{\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \right) = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha} \tag{10}$$

Now, we can utilize *this* notation to further simplify (9):

$$\begin{aligned} &= -\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \left[\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right) F_{\rho\sigma} + \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) F_{\alpha\beta} \right] \\ &+ \tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \left[\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right) F_{\gamma\lambda} + \left(\delta^\mu_\gamma \delta^\nu_\lambda - \delta^\mu_\lambda \delta^\nu_\gamma \right) F_{\alpha\beta} \right] \end{aligned} \tag{11}$$

Factoring in the constants outside of the brackets,

$$\begin{aligned} &= \left[-\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \delta^\mu_\alpha \delta^\nu_\beta + \frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \delta^\mu_\beta \delta^\nu_\alpha \right] F_{\rho\sigma} + \left[-\frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \delta^\mu_\rho \delta^\nu_\sigma + \frac{1}{4} \eta^{\alpha\rho}\eta^{\beta\sigma} \delta^\mu_\sigma \delta^\nu_\rho \right] F_{\alpha\beta} \\ &+ \left[\tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \delta^\mu_\alpha \delta^\nu_\beta - \tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \delta^\mu_\beta \delta^\nu_\alpha \right] F_{\gamma\lambda} \\ &\qquad + \left[\tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \delta^\mu_\gamma \delta^\nu_\lambda - \tilde{\epsilon}_{\mu\nu\rho\sigma} \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda} \delta^\mu_\lambda \delta^\nu_\gamma \right] F_{\alpha\beta} \end{aligned} \tag{12}$$

Now, we can begin contracting the Metric Tensors with the the Kronecker Delta's:

$$
\begin{aligned}
&= \left[-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} + \frac{1}{4}\eta^{\nu\rho}\eta^{\mu\sigma}\right] F_{\rho\sigma} + \left[-\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{1}{4}\eta^{\alpha\nu}\eta^{\beta\mu}\right] F_{\alpha\beta} \\
&+ \left[\tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}\delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right] F_{\gamma\lambda} \\
&\qquad + \left[\tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}\delta^{\mu}_{\gamma}\delta^{\nu}_{\lambda} - \tilde{\epsilon}_{\mu\nu\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\gamma}\eta^{\sigma\lambda}\delta^{\mu}_{\lambda}\delta^{\nu}_{\gamma}\right] F_{\alpha\beta}
\end{aligned}
\tag{13}
$$

Next, instead of contracting the Metric Tensors with the Kronecker Deltas, we will contract them with the rank-four anti-symmetric Levi-Civita Symbol:

$$
\begin{aligned}
&= \left[-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} + \frac{1}{4}\eta^{\nu\rho}\eta^{\mu\sigma}\right] F_{\rho\sigma} + \left[-\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{1}{4}\eta^{\alpha\nu}\eta^{\beta\mu}\right] F_{\alpha\beta} \\
&+ \left[\epsilon^{\alpha\beta\gamma\lambda}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \epsilon^{\alpha\beta\gamma\lambda}\delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right] F_{\gamma\lambda} + \left[\epsilon^{\alpha\beta\gamma\lambda}\delta^{\mu}_{\gamma}\delta^{\nu}_{\lambda} - \epsilon^{\alpha\beta\gamma\lambda}\delta^{\mu}_{\lambda}\delta^{\nu}_{\gamma}\right] F_{\alpha\beta}
\end{aligned}
\tag{14}
$$

Now, we can contract the Kronecker Deltas with the Levi-Civita Symbols:

$$
\begin{aligned}
&= \left[-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} + \frac{1}{4}\eta^{\nu\rho}\eta^{\mu\sigma}\right] F_{\rho\sigma} + \left[-\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu} + \frac{1}{4}\eta^{\alpha\nu}\eta^{\beta\mu}\right] F_{\alpha\beta} \\
&+ \left[\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right] F_{\gamma\lambda} + \left[\epsilon^{\alpha\beta\mu\nu} - \epsilon^{\alpha\beta\nu\mu} F_{\alpha\beta}\right]
\end{aligned}
\tag{15}
$$

Now, we will factor in the Field Strength Tensors:

$$
\begin{aligned}
&= \left(-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} + \frac{1}{4}\eta^{\nu\rho}\eta^{\mu\sigma}F_{\rho\sigma} + -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta} + \frac{1}{4}\eta^{\alpha\nu}\eta^{\beta\mu}F_{\alpha\beta}\right) \\
&+ \left[\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right] F_{\gamma\lambda} + \left[\epsilon^{\alpha\beta\mu\nu} - \epsilon^{\alpha\beta\nu\mu} F_{\alpha\beta}\right]
\end{aligned}
\tag{16}
$$

Next, we can contract the remaining Metric Tensors with the Field Strength Tensors:

$$
= -\frac{1}{4}F^{\mu\nu} + \frac{1}{4}F^{\nu\mu} - \frac{1}{4}F^{\mu\nu} + \frac{1}{4}F^{\nu\mu} + F_{\gamma\lambda}\left(\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right) + F_{\alpha\beta}\left(\epsilon^{\alpha\beta\mu\nu} - \epsilon^{\alpha\beta\nu\mu}\right) \tag{17}
$$

Furthermore, we can employ the anti-symmetric properties of the Field Strength Tensor, namely $F^{\alpha\beta} = -F^{\beta\alpha}$ to rewrite (17) as:

$$
= \frac{1}{4}F^{\nu\mu} + \frac{1}{4}F^{\nu\mu} + \frac{1}{4}F^{\nu\mu} + \frac{1}{4}F^{\nu\mu} + F_{\gamma\lambda}\left(\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right) + F_{\alpha\beta}\left(\epsilon^{\alpha\beta\mu\nu} - \epsilon^{\alpha\beta\nu\mu}\right) \tag{18}
$$

Or, of course,

$$
= F^{\nu\mu} + F_{\gamma\lambda}\left(\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right) + F_{\alpha\beta}\left(\epsilon^{\alpha\beta\mu\nu} - \epsilon^{\alpha\beta\nu\mu}\right) \tag{19}
$$

Moreover, we are free to relabel our "dummy indices" in the last two expressions of (19) any way we so choose to do so, as long as we are consistent; doing so allows us to write the following:

$$
= F^{\nu\mu} + F_{\gamma\lambda}\left(\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right) + F_{\gamma\lambda}\left(\epsilon^{\gamma\lambda\mu\nu} - \epsilon^{\gamma\lambda\nu\mu}\right) \tag{20}
$$

Next, we are going to utilize the cyclical permutation identity for the Levi-Civita Symbol:

$$
\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases} \tag{21}
$$

From this, it is easy to see that the permutation of the indices of the Levi-Civita Symbol inside the parentheses permute properly, allowing us to write the following:

$$
= F^{\nu\mu} + 2F_{\gamma\lambda}\left(\epsilon^{\mu\nu\gamma\lambda} - \epsilon^{\nu\mu\gamma\lambda}\right) \tag{22}
$$

Now, instead of trying to simplify this even more, let's try plugging this into our Euler-Lagrange Equations:

$$\frac{\partial \mathcal{L}_{\mathcal{C}}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\mathcal{C}}}{\partial(\partial_\mu A_\nu)} \right) = 0 \Rightarrow J^\nu - \partial_\mu \left(F^{\nu\mu} + 2F_{\gamma\lambda}\epsilon^{\mu\nu\gamma\lambda} - 2F_{\gamma\lambda}\epsilon^{\nu\mu\gamma\lambda} \right) = 0 \tag{23}$$

Again, we can employ the cyclical permutation property to obviously write the following:

$$J^\nu - \partial_\mu \left(F^{\nu\mu} + 2F_{\gamma\lambda}\epsilon^{\mu\nu\gamma\lambda} + 2F_{\gamma\lambda}\epsilon^{\mu\nu\gamma\lambda} \right) = 0 \tag{24}$$

Or simply,

$$J^\nu - \partial_\mu \left(F^{\nu\mu} + 4F_{\gamma\lambda}\epsilon^{\mu\nu\gamma\lambda} \right) = 0 \tag{25}$$

Finally, since the Electromagnetic Field Strength Tensor is constant in a special relativistic flat-spacetime theory, which is the assumption under which we are working under, the partial derivative in the μ coordinate direction, while it is able to act upon the first term $F^{\nu\mu}$, when it acts upon the second term, $4F_{\gamma\lambda}\epsilon^{\mu\nu\gamma\lambda}$, it goes to zero. This is the case because in general spacetimes, especially those where there is curvature present, we must generalize the partial derivative ∂_μ to a **Covariant Derivative** ∇_μ, which takes into account the presence of curvature via the usage of **Christoffel Symbols**. And so, (25) reduces to the following:

$$J^\nu - \partial_\mu F^{\mu\nu} = 0 \tag{26}$$

Or, as is more typically written,

$$\boxed{\partial_\mu F^{\nu\mu} = J^\nu} \tag{27}$$

Which of course, is the non-homogeneous Maxwell Equation described in the introduction.