Lecture notes

Masters of QFT

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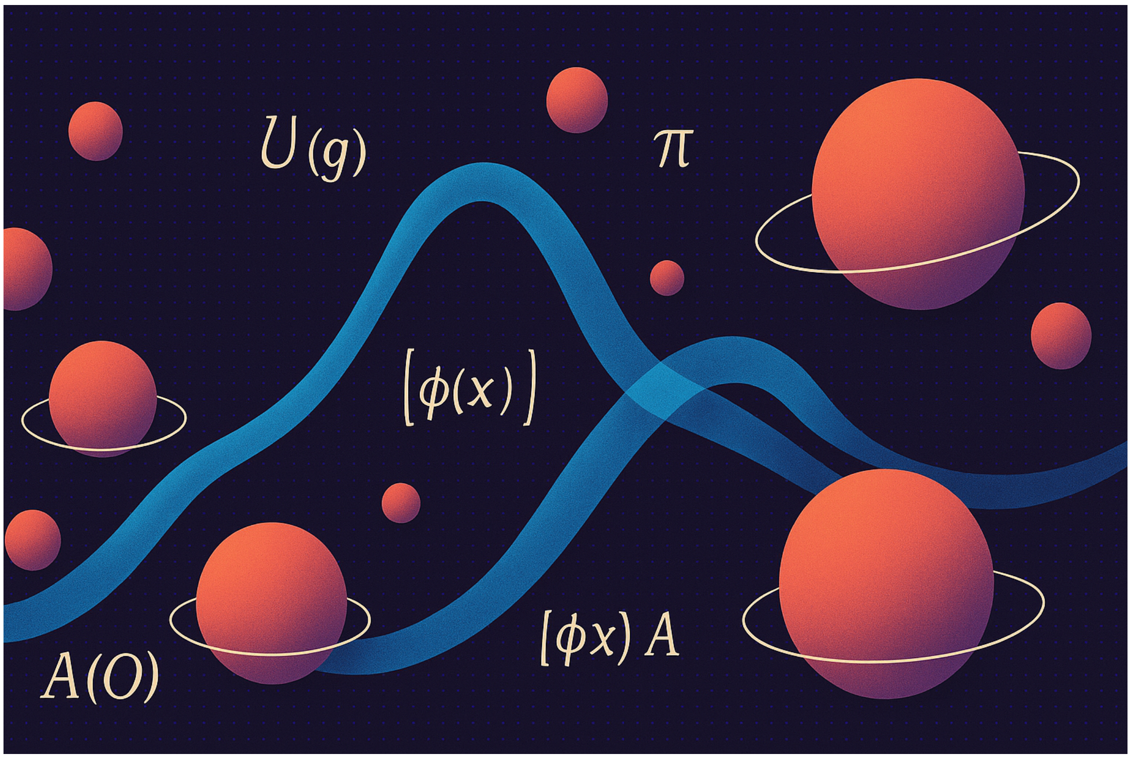

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1 One Quantum System

1.1 Postulates of Quantum Mechanics

Postulate 1 The dynamical degrees of freedom of a quantum system are associated to a **Hilbert space** $\mathcal{H}$.

Postulate 2 A **pure state** is a vector $|\psi\rangle \in \mathcal{H}$ such that

$$\langle\psi|\psi\rangle = 1 \tag{1}$$

Definition 1.1. *An* ***operator*** $\hat{A}$ *is a linear map*

$$\begin{aligned} \hat{A} : \mathcal{H} &\to \mathcal{H} \\ |\psi\rangle &\mapsto \hat{A}\,|\psi\rangle \end{aligned}$$

Postulate 3 An **observable** $\hat{A}$ is an operator that is **self-adjoint**:

$$\hat{A} = \hat{A}^\dagger \tag{2}$$

Its expected value in a state $|\psi\rangle$ is given by

$$\langle\hat{A}\rangle_{|\psi\rangle} = \langle\psi|\,\hat{A}\,|\psi\rangle \tag{3}$$

From Eq. (3) we see that two states that differ by a global phase yield the same expected vales for all observables. Thus, we have the identification

$$|\psi\rangle \sim \mathrm{e}^{\mathrm{i}\phi}\,|\psi\rangle \tag{4}$$

with $\phi \in [0, 2\pi)$.

Definition 1.2 (Time-evolution operator)**.** *A* ***time-evolution operator*** $\hat{U}$ *is an operator that evolves the state* $|\psi\rangle$ *from time* t_0 *to* t*:*

$$|\psi(t_0)\rangle \mapsto |\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle \tag{5}$$

and that preserves the norm:

$$\langle\psi(t_0)|\psi(t_0)\rangle = 1 \implies \langle\psi(t)|\psi(t)\rangle = 1 \tag{6}$$

Corollary 1.1. *Property* (6) *can be written as:*

$$\langle\psi|\,\hat{U}^\dagger\hat{U}\,|\psi\rangle \overset{!}{=} 1 \quad \forall\,|\psi\rangle \in \mathcal{H} \tag{7}$$

and, therefore,

$$\hat{U}^\dagger\hat{U} = \mathbb{I}, \quad \hat{U}\hat{U}^\dagger = \mathbb{I} \tag{8}$$

This means that time-evolution operators are ***unitary*** *operators.*

Speaking of time evolution, the idea of time derivatives comes naturally. Indeed, we can take the temporal derivative of Eq. (5):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = \frac{\mathrm{d}\hat{U}}{\mathrm{d}t}\left|\psi(t_0)\right\rangle \tag{9}$$

Moreover, Eq. (5) can be "flipped" and re-written as

$$\left|\psi(t_0)\right\rangle = \hat{U}^\dagger\left|\psi(t)\right\rangle \tag{10}$$

Injecting Eq. (10) in Eq. (9), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = \frac{\mathrm{d}\hat{U}}{\mathrm{d}t}\hat{U}^\dagger\left|\psi(t)\right\rangle \tag{11}$$

Moreover, the Schrödinger equation relates the time evolution of the state with the so-called Hamiltonian operator $\hat{H}(t)$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle \tag{12}$$

identifying Eq. (11) and Eq. (12), we find that

$$\hat{H}(t) = \mathrm{i}\frac{\mathrm{d}\hat{U}}{\mathrm{d}t}\hat{U}^\dagger(t) \tag{13}$$

and

$$\hat{H}^\dagger(t) = -\mathrm{i}\frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t}\hat{U}(t) \tag{14}$$

The solutions to the Schrödinger equation are of the form

$$\hat{U}(t) = \mathcal{T}\exp\left(-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,\hat{H}(t')\right) \tag{15}$$

where $\mathcal{T}\exp$ is the *time-ordered* exponential.

Definition 1.3 (Strongly continuous unitary group)**.** *It can easily be shown that*

$$\hat{U}(t+s) = \hat{U}(t)\hat{U}(s) \tag{16}$$

*This property is precisely a group operation that defines an operator group, known as the "**strongly continuous unitary group**".*

Homework 1 Show that $\hat{H}(t) = \hat{H}^\dagger(t)$ from Eq. (8). Injecting Eq. (15) into Eq. (8),

$$\hat{U}^\dagger\hat{U} = \mathcal{T}\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,\hat{H}^\dagger(t')\right)\mathcal{T}\exp\left(-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t''\,\hat{H}(t'')\right) \tag{17}$$

Expanding the time-ordered exponentials as Dyson series,

$$\mathcal{T}\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,\hat{H}^\dagger(t')\right)\mathcal{T}\exp\left(-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t''\,\hat{H}(t'')\right)$$

$$\sim \left[\mathbb{I} + \mathrm{i}\varepsilon \int_{t_0}^{t} \mathrm{d}t_1' \hat{H}^\dagger(t_1') - \varepsilon^2(-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t_1' \int_{t_0}^{t_1'} \mathrm{d}t_2' \hat{H}^\dagger(t_1')\hat{H}^\dagger(t_2') + \dots\right]$$
$$\left[\mathbb{I} - \mathrm{i}\varepsilon \int_{t_0}^{t} \mathrm{d}t_1'' \hat{H}(t_1'') + \varepsilon^2(-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t_1'' \int_{t_0}^{t_1''} \mathrm{d}t_2'' \hat{H}(t_1'')\hat{H}(t_2'') + \dots\right]$$

$$\sim \mathbb{I} + \mathrm{i}\varepsilon \int_{t_0}^{t} \mathrm{d}t_1' \hat{H}^\dagger(t_1') - \mathrm{i}\varepsilon \int_{t_0}^{t} \mathrm{d}t_1'' \hat{H}(t_1'') + \mathcal{O}(\varepsilon^2)$$

$$= \mathbb{I} + \mathrm{i}\varepsilon \int_{t_0}^{t} \mathrm{d}t_1 \left(\hat{H}^\dagger(t_1) - \hat{H}(t_1)\right) + \mathcal{O}(\varepsilon^2) \tag{18}$$

We can then see that Eq. (8) is verified if $\hat{H}^\dagger(t) = \hat{H}(t)$. $\square$

1.2 Pictures of Quantum Mechanics

Schrödinger picture In the Schrödinger picture, observables are fixed and the time dependence is encoded solely on the states

$$|\psi(t)\rangle = \hat{U}(t, t_0)\, |\psi(t_0)\rangle \tag{19}$$

which evolve following the Schrödinger equation.

The fact that the evolution operator is time-dependent does not contradict the assumptions for this picture, at this operator is a time-evolution operator and not an observable.

The expectation value of an observable $\hat{A}_S$ in this picture is simply

$$\langle \hat{A}_S \rangle(t) = \langle \psi(t)|\, \hat{A}_S\, |\psi(t)\rangle$$

$$= \langle \psi(t_0)|\, \hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0)\, |\psi(t_0)\rangle \tag{20}$$

However, it is totally acceptable to define an observable $\hat{A}_H(t, t_0) = \hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0)$ so that the time-dependence is carried by observables and not states. This leads to the Heisenberg picture.

Heisenberg picture In the Heisenberg picture, the states are fixed and it is the observables that evolve in time:

$$\hat{A}_H(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t) \tag{21}$$

In this picture, we can compute the evolution of the observables:

$$\frac{\mathrm{d}\hat{A}_H}{\mathrm{d}t} = \frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t}\hat{A}\hat{U}(t) + \hat{U}^\dagger(t)\hat{A}\frac{\mathrm{d}\hat{U}}{\mathrm{d}t}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t}\underbrace{\hat{U}\hat{U}^\dagger}_{\mathbb{I}}\hat{A}\hat{U}(t) + \hat{U}^\dagger(t)\hat{A}\underbrace{\hat{U}^\dagger\hat{U}}_{\mathbb{I}}\frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t}\hat{U}(t)\hat{A}(t) + \hat{A}(t)\hat{U}^\dagger(t)\frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \qquad (22)
\end{aligned}
$$

Homework 2 Show that

$$
\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \mathrm{i}[\hat{A}(t), \hat{H}(t)] \qquad (23)
$$

Using Eqs. (13)-(14) and the commutation properties of unitary operators (Eq. (8)),

$$
\begin{aligned}
\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} =& \frac{d\hat{U}^\dagger}{dt}\hat{U}(t)\hat{A}(t) + \hat{A}(t)\hat{U}^\dagger(t)\frac{d\hat{U}}{dt} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{U}^\dagger(t)\hat{A}\hat{U}(t)\right) \\
&= \frac{\mathrm{d}\hat{U}}{\mathrm{d}t}\hat{U}^\dagger(t)\hat{A}(t) + \hat{A}(t)\hat{U}^\dagger(t)\frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \\
&= \mathrm{i}\hat{H}(t)\hat{A}(t) + \mathrm{i}\hat{A}(t)\hat{H}(t) \\
&= \mathrm{i}[\hat{A}(t), \hat{H}(t)] \qquad (24)
\end{aligned}
$$

□

The redefinition introduced in Eq. (21) is not unique and we can keep doing so:

$$
|\psi(t)\rangle = \langle\psi(t_0)|\,(\hat{V}^\dagger\hat{V})\hat{U}^\dagger(t,t_0)\hat{A}_S\hat{U}(t,t_0)(\hat{V}^\dagger\hat{V})\,|\psi(t_0)\rangle \qquad (25)
$$

In this case, we have introduced a new time-evolution operator and, while the states $|\psi\rangle$ remain fixed, the observable now evolves as

$$
\hat{A}(t) \equiv \hat{V}\hat{U}^\dagger(t)\hat{A}\hat{U}(t)\hat{V}^\dagger \qquad (26)
$$

If we now give a time dependence to $\hat{V}$, we arrive to the interaction picture.

Interaction picture In the interaction picture, we split the Hamiltonian from the Schrödinger picture into two parts:

$$\hat{H}_S(t) = \hat{H}_{S,0}(t) + \hat{H}_{S,1}(t) \tag{27}$$

where, typically, $\hat{H}_{S,0}$ is an exactly solvable Hamiltonian and $\hat{H}_{S,1}$ is a harder-to-solve time-dependent Hamiltonian, encoding perturbations of the system. Analogously to Eqs. (13)-(14),

$$\hat{H}_{S,0}(t) = \mathrm{i}\frac{\mathrm{d}\hat{U}_{S,0}}{\mathrm{d}t}\hat{U}^\dagger_{S,0}(t), \quad \hat{H}_S(t) = \mathrm{i}\frac{\mathrm{d}\hat{U}_S}{\mathrm{d}t}\hat{U}^\dagger_S(t) \tag{28}$$

and

$$\hat{H}^\dagger_{S,0}(t) = -\mathrm{i}\frac{\mathrm{d}\hat{U}^\dagger_{S,0}}{\mathrm{d}t}\hat{U}_{S,0}(t), \quad \hat{H}^\dagger_S(t) = -\mathrm{i}\frac{\mathrm{d}\hat{U}^\dagger_S}{\mathrm{d}t}\hat{U}_S(t) \tag{29}$$

A state in the interaction picture can be found by applying a time-dependent unitary transformation to a state in the Schrödinger picture:

$$\begin{aligned}
|\psi_I(t)\rangle &= \hat{U}^\dagger_{S,0}(t)\,|\psi_S(t)\rangle \\
&= \hat{U}^\dagger_{S,0}(t)\hat{U}_S(t)\,|\psi_S(0)\rangle \\
&=: \hat{U}_I(t)\,|\psi_S(0)\rangle
\end{aligned} \tag{30}$$

Following previous constructions,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\psi_I(t)\rangle &= \frac{d\hat{U}_I}{dt}|\psi_S(0)\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{U}^\dagger_{S,0}(t)\hat{U}_S(t)\right)|\psi_S(0)\rangle \\
&= \mathrm{i}\left(\frac{\mathrm{d}\hat{U}^\dagger_{S,0}}{\mathrm{d}t}\hat{U}_S(t) + \hat{U}^\dagger_{S,0}(t)\frac{\mathrm{d}\hat{U}_S}{\mathrm{d}t}\right)|\psi_S(0)\rangle \\
&= \mathrm{i}\left(\mathrm{i}\hat{U}^\dagger_{S,0}(t)\hat{H}_{S,0}\hat{U}_S(t) - \mathrm{i}\hat{U}^\dagger_{S,0}(t)H_S\hat{U}_S(t)\right)|\psi_S(0)\rangle \\
&= \mathrm{i}\left(\mathrm{i}\hat{U}^\dagger_{S,0}(t)\hat{H}_{S,0}\hat{U}_{S,0}(t)\hat{U}^\dagger_{S,0}(t)\hat{U}_S(t) - \mathrm{i}\hat{U}^\dagger_{S,0}(t)\hat{H}_S\hat{U}_{S,0}(t)\hat{U}^\dagger_{S,0}(t)\hat{U}_S(t)\right)|\psi_S(0)\rangle \\
&= \mathrm{i}\left(\mathrm{i}\hat{U}^\dagger_{S,0}(t)\hat{H}_{S,0}\hat{U}_{S,0}(t)\hat{U}_I(t) - \mathrm{i}\hat{U}^\dagger_{S,0}(t)\hat{H}_S\hat{U}_{S,0}(t)\hat{U}_I(t)\right)|\psi_S(0)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left(\hat{U}^{\dagger}_{S,0}(t)(\hat{H}_S - \hat{H}_{S,0})\hat{U}_{S,0}(t)\hat{U}_I(t) \right) |\psi_S(0)\rangle \\
&= \left(\hat{U}^{\dagger}_{S,0}(t)\hat{H}_{S,1}\hat{U}_{S,0}(t)\hat{U}_I(t) \right) |\psi_S(0)\rangle \\
&=: \left(\hat{H}_I\hat{U}_I(t) \right) |\psi_S(0)\rangle
\end{aligned} \tag{31}
$$

where we have introduced the **interaction Hamiltonian**

$$
\hat{H}_I(t) = \hat{U}^{\dagger}_{S,0}(t)\hat{H}_{S,1}\hat{U}_{S,0}(t) \tag{32}
$$

as the time evolved second part of the Hamiltonian in the Schrödinger picture.

The expected value of an observable becomes

$$
\begin{aligned}
\langle\hat{A}\rangle(t) &= \langle\psi(0)|\, \hat{U}^{\dagger}_S(t)\hat{A}_S(t)\hat{U}_S(t)\, |\psi(0)\rangle \\
&= \langle\psi(0)|\, \hat{U}^{\dagger}_S(t)\hat{U}_{S,0}(t)\hat{U}^{\dagger}_{S,0}(t)\hat{A}_S(t)\hat{U}_{S,0}(t)\hat{U}^{\dagger}_{S,0}(t)\hat{U}_S(t)\, |\psi(0)\rangle \\
&= \langle\psi(0)|\, \hat{U}^{\dagger}_I(t)\hat{U}^{\dagger}_{S,0}(t)\hat{A}_S(t)\hat{U}_{S,0}(t)\hat{U}_I(t)\, |\psi(0)\rangle \\
&=: \langle\psi(0)|\, \hat{U}^{\dagger}_I(t)\hat{A}_I(t)\hat{U}_I(t)\, |\psi(0)\rangle
\end{aligned} \tag{33}
$$

Example 1.1. *Let us consider a full Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_I$, constant in time. The evolution operators are:*

$$
\hat{U}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t}, \quad \hat{U}_I(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_I t} \tag{34}
$$

then, using the Baker-Campbell-Hausdorff formula,

$$
\begin{aligned}
\hat{U}(t)\hat{U}_I(t) &= \mathrm{e}^{-\mathrm{i}\hat{H}t}\mathrm{e}^{-\mathrm{i}\hat{H}_I t} \\
&= exp\left(-\mathrm{i}(\hat{H} + \hat{H}_I + \frac{1}{2}[\hat{H}, \hat{H}_I] + ...)t \right) \\
&= \mathrm{e}^{-\mathrm{i}(\hat{H}+\hat{H}_I)t} \\
&=: \mathrm{e}^{-\mathrm{i}\hat{H}_0 t}
\end{aligned} \tag{35}
$$

if $\hat{H}$ and $\hat{H}_I$ commute. We have introduce the Hamiltonian operator $\hat{H}_0$ and we choose its associated evolution operator $\hat{U}_0(t) \equiv \hat{U}(t)\hat{U}^{\dagger}_I(t)$ such that

$$
\mathrm{i}\frac{\mathrm{d}\hat{U}_0}{\mathrm{d}t}\hat{U}_0 = \hat{H}_0 \implies \hat{U}(t) = \hat{U}_0(t)\hat{U}_I(t) \tag{36}
$$

1.3 Measurements

Definition 1.4 (Measurement)**.** *A* ***measurement*** *is a decomposition*

$$\mathbb{I} = \sum_i \hat{P}_i \tag{37}$$

where $\hat{P}_i$ are orthogonal projectors satisfying

$$\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i \tag{38}$$

Given a state $|\psi\rangle \in \mathcal{H}$, the probability p_i of obtaining the i^{th} outcome is

$$p_i = \langle\psi| \hat{P}_i |\psi\rangle \tag{39}$$

Moreover, since the probabilities must sum up to 1, we have:

$$\sum_i p_i = \langle\psi| \sum_i \hat{P}_i |\psi\rangle = 1 \tag{40}$$

Observation 1.1. *If $\hat{A}$ is a self-adjoint operator, then*

$$\hat{A} = \sum_i^n \lambda_i \hat{P}_i \tag{41}$$

Moreover, if $\hat{P}_0$ is the projector into Ker$(\hat{A})$ and $\lambda_i \neq 0$,

$$\hat{\mathbb{I}} = \hat{P}_0 + \sum_i^n \hat{P}_i \tag{42}$$

and this measurement is defined by $\hat{A}$.

Definition 1.5 (Compatible measurements)**.** *Let $\mathbb{I} = \sum_i \hat{P}_i = \sum_j \hat{Q}_j$ be measurements. We say that they are* ***compatible*** *if*

$$[\hat{P}_i, \hat{Q}_j] = 0 \quad \forall i, j \tag{43}$$

Theorem 1.1. *After an outcome i is obtained, the state $|\psi\rangle$ is updated to*

$$|\psi_i\rangle = \frac{\hat{P}_i |\psi\rangle}{||\hat{P}_i |\psi\rangle ||} \tag{44}$$

This "collapse" is non-linear!

Observation 1.2. *If $\hat{P}_i$ and $\hat{Q}_j$ are compatible, the order in which they are measured is not important.*

1.4 Density operators

Definition 1.6 (Density operator)**.** *A* ***density operator*** $\hat{\rho}$ *is a a linear operator* $\hat{\rho} : \mathcal{H} \to \mathcal{H}$*, with* $\hat{\rho} \geq 0$ *(*$\langle\psi| \hat{\rho} |\psi\rangle \geq 0 \; \forall \, |\psi\rangle$*) and* $\mathrm{Tr}(\hat{\rho}) = 1$*.*
We can define the space of density operators in $\mathcal{H}$ *as* $\mathcal{L}(\mathcal{H})$*.*

Theorem 1.2. *Let* $\hat{\rho} \in \mathcal{L}(\mathcal{H})$*, then*

$$\hat{\rho} = \sum_i p_i \, |\psi_i\rangle \langle\psi_i| \tag{45}$$

where $\sum_i p_i = 1$*,* $p_i \geq 0 \quad \forall i$*.*

Observation 1.3. *Given a state* $|\psi\rangle$*, we can define a corresponding density operator* $\hat{\rho} := |\psi\rangle\langle\psi|$*. This tells us that there exists an isomorphism between states and density operators!*

Theorem 1.3. *Given an ensemble* $\{p_i, |\psi_i\rangle\}$*, the expected value of* $\hat{A}$ *in* $\{p_i, |\psi_i\rangle\}$ *is*

$$\begin{aligned}
\langle \hat{A} \rangle_{\{p_i, |\psi_i\rangle\}} &:= \sum_i p_i \, \langle\psi_i| \hat{A} |\psi_i\rangle \\
&= \sum_i p_i \, \mathrm{Tr}\Big(|\psi_i\rangle\langle\psi_i| \, \hat{A} \Big) \\
&= \mathrm{Tr}\left[\left(\sum_i p_i \, |\psi_i\rangle \langle\psi_i| \right) \hat{A} \right] \\
&= \mathrm{Tr}\Big(\hat{\rho} \hat{A} \Big)
\end{aligned} \tag{46}$$

Theorem 1.4. *A density operator* $\hat{\rho}$ *can be evolved in time by:*

$$\begin{aligned}
\hat{\rho}(t) &= \sum_i p_i \hat{U}(t) \, |\psi_i\rangle\langle\psi_i| \, \hat{U}^\dagger(t) \\
&= \hat{U}(t) \hat{\rho} \hat{U}^\dagger(t)
\end{aligned} \tag{47}$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \hat{\rho} \hat{U}^\dagger(t) + \hat{U} \hat{\rho} \frac{\mathrm{d}\hat{U}^\dagger(t)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \hat{U}^\dagger \hat{U} \hat{\rho} \hat{U}^\dagger + \hat{U} \hat{\rho} \hat{U}^\dagger \hat{U} \frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\hat{U}}{\mathrm{d}t} \hat{U}^\dagger \hat{\rho}(t) + \hat{\rho}(t) \hat{U} \frac{\mathrm{d}\hat{U}^\dagger}{\mathrm{d}t}
\end{aligned} \tag{48}$$

Homework 5 Show that

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H}(t), \hat{\rho}(t)] \tag{49}$$

By using

$$-\mathrm{i}\hat{H}(t) = \frac{\mathrm{d}\hat{U}}{\mathrm{d}t}\hat{U}^\dagger \tag{50}$$

Eq. (48) becomes:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}\hat{H}(t)\hat{\rho}(t) + i\hat{\rho}(t)\hat{H}(t)$$

$$= -\mathrm{i}[\hat{H}(t), \hat{\rho}(t)] \tag{51}$$

□

Definition 1.7 (CPTP operations)**.** *Let $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, satisfying*

$$\begin{cases} \mathcal{E}(\hat{\rho}) \geq 0 \;\; \forall \hat{\rho} \in \mathcal{L}(\mathcal{H}) \\ \\ \mathrm{Tr}(\mathcal{E}(\hat{\rho})) = 1 \\ \\ \mathcal{E}(\hat{\rho} + \alpha\hat{\sigma}) = \mathcal{E}(\hat{\rho}) + \alpha\mathcal{E}(\hat{\sigma}) \end{cases}$$

This set of operations is called the set of ***CPTP operations****, where CP stands for complete positivity and TP for trace preserving.*

Theorem 1.5. *An operator $\mathcal{E}$ being CPTP is equivalent to the condition that there exists a set $\{\hat{K}_i\}$ such that*

$$\sum_i \hat{K}_i^\dagger \hat{K}_i = \mathbb{I} \quad \text{and} \quad \mathcal{E}(\hat{\rho}) = \sum_i \hat{K}_i \hat{\rho} \hat{K}_i^\dagger \tag{52}$$

This decomposition of $\mathcal{E}$ is known as the ***Krauss decomposition****.*

Example 1.2. *Let us consider the case $\mathcal{H} = \mathbb{C}^n$ and an orthonormal basis $\{|i\rangle\}$. Then,*

$$\hat{\rho} = \frac{1}{n}\sum_i^n |i\rangle\langle i| = \frac{1}{n}\hat{\mathbb{I}} \tag{53}$$

and the expectation value is:

$$\langle \hat{A} \rangle_{\hat{\rho}} = Tr(\hat{A}\hat{\rho}) = \frac{1}{n}Tr(\hat{A}) \tag{54}$$

Let $\hat{P}$ be a 1-dimensional projector,

$$\langle \hat{P} \rangle_{\hat{\rho}} = \frac{1}{n}Tr(\hat{P}) = \frac{1}{n} \quad \forall \hat{P} \tag{55}$$

This corresponds to a maximally mixed state: it has equal probability of being in any state!

Homework 6 Let $\mathcal{H} = \mathbb{C}^2$ and σ_x, σ_y, σ_z the Pauli matrices.
Compute $\hat{\rho}$, $\langle\sigma_x|\sigma_x\rangle_{\hat{\rho}}$, $\langle\sigma_y|\sigma_y\rangle_{\hat{\rho}}$ and $\langle\sigma_z|\sigma_z\rangle_{\hat{\rho}}$ when $\hat{\rho}$ represents

$$\begin{cases} P_\psi = 1, |\psi\rangle = |0\rangle \\ \\ P_\psi = \frac{1}{2}, |\psi\rangle = |0\rangle \ \ \& \ P_\phi = \frac{1}{2}, |\phi\rangle = |1\rangle \\ \\ P_\psi = \cos^2\theta, P_\phi = \sin^2\theta, |\psi\rangle = |0\rangle, |\phi\rangle = |1\rangle \\ \\ P_\psi = \frac{1}{2}, |\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \ \& \ P_\phi = \frac{1}{2}, |\phi\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ \\ P_\psi = \frac{1}{2}, |\psi\rangle = |0\rangle, P_\phi = \frac{1}{2}, |\phi\rangle = |+\rangle \end{cases}$$

□

How do we determine if $\hat{\rho}$ is a pure state? According to Eq. (45), in the general case the density operator is

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{56}$$

For it to be a pure state, we need it to be of the form

$$\hat{\rho} = |\psi\rangle\langle\psi| \tag{57}$$

or, in other words, we need it to be a projector.
Moreover the defining property of projectors is

$$\hat{\rho}^2 = \hat{\rho} \tag{58}$$

and therefore,

$$\mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}(\hat{\rho}) = 1 \tag{59}$$

Homework 7 Prove that, if $\hat{\rho}$ is not a projector, then

$$\hat{\rho}^2 = \sum_i \sum_j p_i p_j \Big(|\psi_i\rangle \langle\psi_i| \Big)\Big(|\psi_j\rangle \langle\psi_j| \Big) \Rightarrow \mathrm{Tr}(\hat{\rho}^2) < 1 \tag{60}$$

Using Eq. (46),

$$\sum_i \sum_j p_i p_j \Big(|\psi_i\rangle \langle\psi_i| \Big)\Big(|\psi_j\rangle \langle\psi_j| \Big) = \Big(\sum_i p_i |\psi_i\rangle \langle\psi_i| \Big)\Big(\sum_j p_j |\psi_j\rangle \langle\psi_j| \Big) \tag{61}$$

□

Definition 1.8 (Purity)**.** *The purity of $\hat{\rho}$ is encoded in $\mathrm{Tr}(\hat{\rho}^2)$. We say that a density operator is pure if and only if*

$$\mathrm{Tr}\big(\hat{\rho}^2\big) = 1 \tag{62}$$

Theorem 1.6. *If $\hat{\rho}$ is pure, then it cannot be decomposed as a convex combination of $\hat{\rho}_i$:*

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i \tag{63}$$

where $\sum_i p_i = 1$ and $p_i > 0$.

Before, we defined a measurement as a decomposition

$$\hat{\mathbb{I}} = \sum_i \hat{P}_i \tag{64}$$

Each outcome has then a probability

$$p_i = \mathrm{Tr}(\hat{\rho}\hat{P}_i) \tag{65}$$

If $\hat{\rho}$ is pure,

$$\mathrm{Tr}(\hat{\rho}\hat{P}_i) = \mathrm{Tr}\Big(|\psi\rangle\langle\psi| \, \hat{P}_i \Big) = \langle\psi| \, \hat{P}_i \, |\psi\rangle \tag{66}$$

Using Eq. (44), the evolved operator $\hat{\rho}_i$ becomes:

$$\begin{aligned} \hat{\rho}_i &= |\psi\rangle\langle\psi| \\ &= \frac{\hat{P}_i \, |\psi\rangle\langle\psi| \, \hat{P}_i}{\langle\psi| \, \hat{P}_i \, |\psi\rangle} \\ &= \frac{\hat{P}_i \hat{\rho} \hat{P}_i}{\mathrm{Tr}(\hat{\rho}\hat{P}_i)} \end{aligned} \tag{67}$$

Definition 1.9 (Generalized measurement)**.** *We can now define a* ***generalized notion of measurement****. Let $\{\hat{M}_i\}$ be a basis such that*

$$\sum_i \hat{M}_i^\dagger \hat{M}_i = \mathbb{I} \tag{68}$$

If the outcome $\hat{M}_i$ is obtained, then the density operator evolves

$$\hat{\rho} \mapsto \hat{\rho}_i = \frac{\hat{M}_i \hat{\rho} \hat{M}_i^\dagger}{Tr(\hat{\rho}\hat{M}_i^\dagger \hat{M}_i)} \tag{69}$$

with probability

$$p_i = Tr(\hat{\rho}\hat{M}_i^\dagger \hat{M}_i) \tag{70}$$

Definition 1.10 (Non-selective measurement)**.** *In a* ***non-selective measurement****, the result is ignored and the state is evolved to a statistical mixture of all possible outcomes:*

$$\begin{aligned} \hat{\rho} \mapsto \sum_i p_i \hat{\rho}_i &= \sum_i Tr(\hat{\rho}\hat{P}_i) \frac{\hat{P}_i \hat{\rho} \hat{P}_i}{Tr(\hat{\rho}\hat{P}_i)} \\ &= \sum_i \hat{P}_i \hat{\rho} \hat{P}_i \end{aligned} \tag{71}$$

Observation 1.4. *The operation $\mathcal{E}(\hat{\rho}) = \sum_i \hat{P}_i \hat{\rho} \hat{P}_i$ is a CPTP operation.*

1.5 Quantum Information

Definition 1.11 (Von Neumann entropy)**.** *The **Von Neumann entropy** is a quantity defined as*

$$S(\hat{\rho}) = -\operatorname{Tr}(\hat{\rho}\ \log\hat{\rho}) \tag{72}$$

where $\hat{\rho} = \sum_i p_i \left|i\right\rangle\left\langle i\right|$ and $\{\left|i\right\rangle\}$ is an orthonormal basis.

Observation 1.5. *If $\hat{A} = \sum_i \lambda_i \hat{P}_i$, with $\hat{A} = \hat{A}^\dagger$, then*

$$\begin{aligned} f(\hat{A}) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\hat{A}^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\left(\sum_i \lambda_i \hat{P}_i\right)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\sum_i \lambda_i^n \hat{P}_i \\ &= \sum_i \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\lambda_i^n\right)\hat{P}_i \\ &= \sum_i f(\lambda_i)\hat{P}_i \end{aligned} \tag{73}$$

Corollary 1.2. *Applying this result to Eq.* (72),

$$\begin{aligned} S(\hat{\rho}) &= -\sum_i p_i\ \log p_i \operatorname{Tr}\Big(\ |i\rangle\langle i|\ \Big) \\ &= -\sum_i p_i\ \log p_i \end{aligned} \tag{74}$$

Example 1.3. *If $\hat{\rho}$ is a pure state, its entropy is null:*

$$S\Big(\ |\psi\rangle\langle\psi|\ \Big) = 0 \tag{75}$$

Example 1.4. *If $\hat{\rho}$ is a maximally mixed state, its entropy is maximal:*

$$S\left(\frac{1}{n}\hat{\mathbb{I}}\right) = -\sum_i \frac{1}{n}\ \log\frac{1}{n}\ \operatorname{Tr}\left(\hat{\mathbb{I}}\right) = \log\ n \tag{76}$$

Theorem 1.7. *The Von Neumann entropy is bounded in $\mathbb{C}^n$:*

$$0 \le S \le \log\ n \tag{77}$$

2 Two Quantum Systems

2.1 Multipartite systems

Postulate If $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ represent two quantum systems, then $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ represents the composite system.
This tensor space has basis of the form

$$\mathcal{B} = \{|i_{\mathrm{A}}\rangle \otimes |i_{\mathrm{B}}\rangle\}_{i_{\mathrm{A}}=1, i_{\mathrm{B}}=1}^{n_{\mathrm{A}}, n_{\mathrm{B}}} \tag{78}$$

Observation 2.1. *Let $\hat{T}$ be an operator $\hat{T} : \mathcal{H}_{\mathrm{A}} \oplus \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{A}} \oplus \mathcal{H}_{\mathrm{B}}$. Then $\hat{T}$ can simply be written as*

$$\hat{T} = \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{B} \end{pmatrix} \tag{79}$$

with

$$\begin{cases} \hat{A} : \mathcal{H}_{\mathrm{A}} \to \mathcal{H}_{\mathrm{A}} \\ \\ \hat{B} : \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{B}} \end{cases}$$

Example 2.1. *If $\mathcal{H}_{\mathrm{A}} \simeq \mathcal{H}_{\mathrm{B}} \simeq \mathbb{C}^2$, then $\hat{T} : \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ can be written as*

$$\hat{T} = \begin{pmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{pmatrix} \tag{80}$$

where neither operator acts individually on $\mathcal{H}_{\mathrm{A}}$ or $\mathcal{H}_{\mathrm{B}}$.

2.1.1 Independent states and observables

Definition 2.1. *If $\hat{A} \in \mathcal{L}(\mathcal{H}_{\mathrm{A}})$ and $\hat{B} \in \mathcal{L}(\mathcal{H}_{\mathrm{B}})$, then we define*

$$\Big(\hat{A} \otimes \hat{B}\Big)\Big(\,|\psi_{\mathrm{A}}\rangle \otimes |\psi_{\mathrm{B}}\rangle\,\Big) = \hat{A}\,|\psi_{\mathrm{A}}\rangle \otimes \hat{B}\,|\psi_{\mathrm{B}}\rangle \tag{81}$$

Example 2.2. *Let $\mathcal{H}_{\mathrm{A}} \simeq \mathcal{H}_{\mathrm{B}} \simeq \mathbb{C}^2$,*

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \implies \hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & a_{12}\hat{B} \\ a_{21}\hat{B} & a_{22}\hat{B} \end{pmatrix} \tag{82}$$

Theorem 2.1. *An operator $\hat{A} \in \mathcal{L}(\mathcal{H}_{\mathrm{A}})$ can be extended to $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$:*

$$\hat{A} \to \hat{A} \otimes \hat{\mathbb{I}}_{\mathrm{B}} \tag{83}$$

Observation 2.2. *This can be generalized to n-systems.*

Definition 2.2 (Combined state)**.** *If $|\psi_{\mathrm{A}}\rangle \in \mathcal{H}_{\mathrm{A}}, |\psi_{\mathrm{B}}\rangle \in \mathcal{H}_{\mathrm{B}}$ (and these are completely independent), then there exists a state $|\psi_{\mathrm{AB}}\rangle$:*

$$|\psi_{\mathrm{AB}}\rangle := |\psi_{\mathrm{A}}\rangle \otimes |\psi_{\mathrm{B}}\rangle \in \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}} \tag{84}$$

*which represents the **combined state**.*
In this case, if $\hat{A} \in \mathcal{L}(\mathcal{H}_{\mathrm{A}})$, then:

$$\begin{aligned}
\langle \psi_{\mathrm{AB}} | \hat{A} \otimes \hat{\mathbb{I}}_{\mathrm{B}} | \psi_{\mathrm{AB}} \rangle &= \langle \psi_{\mathrm{A}} | \otimes \langle \psi_{\mathrm{B}} | \left(\hat{A} | \psi_{\mathrm{A}} \rangle \otimes \hat{\mathbb{I}}_{\mathrm{B}} | \psi_{\mathrm{B}} \rangle \right) \\
&= \langle \psi_{\mathrm{A}} | \hat{A} | \psi_{\mathrm{A}} \rangle \otimes \langle \psi_{\mathrm{B}} | \psi_{\mathrm{B}} \rangle \\
&= \langle \hat{A} \rangle_{|\psi_{\mathrm{A}}\rangle}
\end{aligned} \tag{85}$$

Definition 2.3 (Separable state)**.** *We say that a state of the form Eq. (84) is **separable**.*

Theorem 2.2. *If $\hat{\rho}_{\mathrm{A}} \in \mathcal{H}_{\mathrm{A}}$ and $\hat{\rho}_{\mathrm{B}} \in \mathcal{H}_{\mathrm{B}}$, then there exists a **density operator***

$$\hat{\rho}_{\mathrm{AB}} = \hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}} \tag{86}$$

that describes the combined state.

Definition 2.4. *We say that a density operator is **separable** if it can be written in the form*

$$\hat{\rho}_{\mathrm{AB}} = \sum_i p_i \hat{\rho}_{A,i} \otimes \hat{\rho}_{B,i} \tag{87}$$

Definition 2.5 (Partial trace)**.** *Let $\mathcal{H}_C$ be any Hilbert space. We can define the **partial trace** Tr_{B} as a map*

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{B}} : \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_{\mathrm{B}}) &\to \mathcal{L}(\mathcal{H}_C) \\
\hat{C} \otimes \hat{B} &\to \mathrm{Tr}\left(\hat{B}\right)\hat{C}
\end{aligned} \tag{88}$$

such that

$$\begin{aligned}
\mathrm{Tr}\left(\hat{A} \otimes \hat{B}\right) &= \mathrm{Tr}\left(\mathrm{Tr}_{\mathrm{B}}(\hat{B})\hat{A}\right) \\
&= \sum_{i_{\mathrm{A}}} \sum_{i_{\mathrm{B}}} \langle i_{\mathrm{A}} i_{\mathrm{B}} | \hat{A} \otimes \hat{B} | i_{\mathrm{A}} i_{\mathrm{B}} \rangle \\
&= \left(\sum_{i_{\mathrm{A}}} \langle i_{\mathrm{A}} | \hat{A} | i_{\mathrm{A}} \rangle \right) \left(\sum_{i_{\mathrm{B}}} \langle i_{\mathrm{B}} | \hat{B} | i_{\mathrm{B}} \rangle \right) \\
\implies \mathrm{Tr}\left(\hat{A} \otimes \hat{B}\right) &= \mathrm{Tr}\left(\hat{A}\right) \mathrm{Tr}\left(\hat{B}\right)
\end{aligned} \tag{89}$$

This also implies

$$\mathrm{Tr}(\hat{\rho}_{\mathrm{AB}}) = \mathrm{Tr}(\hat{\rho}_{\mathrm{A}}) \ \mathrm{Tr}(\hat{\rho}_{\mathrm{B}}) = 1 \tag{90}$$

The positivity condition becomes:

$$\langle i_{\mathrm{A}} i_{\mathrm{B}} | \hat{\rho}_{\mathrm{AB}} | i_{\mathrm{A}} i_{\mathrm{B}} \rangle = \langle i_{\mathrm{A}} | \hat{\rho}_{\mathrm{A}} | i_{\mathrm{A}} \rangle \langle i_{\mathrm{B}} | \hat{\rho}_{\mathrm{B}} | i_{\mathrm{B}} \rangle \tag{91}$$

Observation 2.3. *If $\hat{\rho}_{\text{AB}} = \hat{\rho}_{\text{A}} \otimes \hat{\rho}_{\text{B}}$ and $\hat{A} \in \mathcal{L}(\mathcal{H}_{\text{A}})$, then*

$$\begin{aligned}
\langle \hat{A} \otimes \hat{\mathbb{I}}_{\text{B}} \rangle_{\hat{\rho}_{\text{AB}}} &= \mathrm{Tr}\Big(\hat{\rho}_{\text{AB}} \hat{A} \otimes \hat{\mathbb{I}}_{\text{B}}\Big) \\
&= \mathrm{Tr}\Big(\hat{\rho}_{\text{A}} \hat{A} \otimes \hat{\rho}_{\text{B}}\Big) \\
&= \mathrm{Tr}\Big(\hat{\rho}_{\text{A}} \hat{A}\Big)\, \mathrm{Tr}(\hat{\rho}_{\text{B}}) \\
&= \mathrm{Tr}\Big(\hat{\rho}_{\text{A}} \hat{A}\Big) \\
&= \langle \hat{A} \rangle_{\hat{\rho}_{\text{A}}}
\end{aligned} \tag{92}$$

2.1.2 General states and observables

Not all states in $\mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$ are separable (Eq. (84)).

Definition 2.6 (Partial state)**.** *We can define* ***partial states*** *as*

$$\hat{\rho}_{\text{A}} = \mathrm{Tr}_{\text{B}}(\hat{\rho}_{\text{AB}}), \quad \hat{\rho}_{\text{B}} = \mathrm{Tr}_{\text{A}}(\hat{\rho}_{\text{AB}}) \tag{93}$$

Example 2.3. *Let $|\psi_{\text{AB}}\rangle = |\psi_{\text{A}}\rangle \otimes |\psi_{\text{B}}\rangle$. Then,*

$$\begin{aligned}
\hat{\rho}_{\text{A}} &= \mathrm{Tr}_{\text{B}}\left(|\psi_{\text{AB}}\rangle \langle \psi_{\text{AB}}| \right) \\
&= \mathrm{Tr}_{\text{B}}\left(|\psi_A\rangle \langle \psi_A| \otimes |\psi_B\rangle \langle \psi_B| \right) \\
&= |\psi_A\rangle \langle \psi_A|
\end{aligned} \tag{94}$$

Let $\hat{\rho}_{\text{AB}} = \sum_i p_i \hat{\rho}_{A,i} \otimes \hat{\rho}_{B,i}$, then

$$\begin{aligned}
\hat{\rho}_{\text{A}} &= \mathrm{Tr}_{\text{B}}\left(\sum_i p_i \hat{\rho}_{A,i} \otimes \hat{\rho}_{B,i} \right) \\
&= \sum_i p_i \hat{\rho}_{A,i}
\end{aligned} \tag{95}$$

Theorem 2.3. *If $\hat{A} \in \mathcal{L}(\mathcal{H}_{\text{A}})$, then*

$$\langle \hat{A} \otimes \hat{\mathbb{I}}_{\text{B}} \rangle_{\hat{\rho}_{\text{AB}}} = \langle \hat{A} \rangle_{\hat{\rho}_{\text{A}}} \tag{96}$$

This is true even if $\hat{\rho}$ is <u>not separable</u>.

Proof

$$
\begin{aligned}
\langle \hat{A} \otimes \hat{\mathbb{I}}_{\mathrm{B}} \rangle_{\hat{\rho}_{\mathrm{AB}}} &= \mathrm{Tr}\Big(\hat{\rho}_{\mathrm{AB}} \hat{A} \otimes \hat{\mathbb{I}}_{\mathrm{B}}\Big) \\
&= \sum_i \sum_j \rho_{ij}\, \mathrm{Tr}\Big(\hat{A}_i \hat{A}\Big)\, \mathrm{Tr}\Big(\hat{B}_j\Big) \\
&= \langle \hat{A} \rangle_{\hat{\rho}_{\mathrm{A}}} \qquad (97)
\end{aligned}
$$

→ "The partial state contains all the necessary information about the local measurements."

Definition 2.7 (Local operator)**.** *If $\hat{T} \in \mathcal{L}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}})$ such that $\hat{T} = \hat{A} \otimes \hat{\mathbb{I}}$, we say that $\hat{T}$ is a* ***local operator*** *in $\hat{A}$.*

Observation 2.4. *If $\hat{\rho}_{\mathrm{AB}} = \hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}}$, then all the relevant information about $\hat{\rho}_{\mathrm{AB}}$ is encoded in local measurements.*

Observation 2.5. *If $\hat{\rho}_{\mathrm{AB}} = \sum_i p_i \hat{\rho}_{Ai} \otimes \hat{\rho}_{Bi}$, then*

$$
\hat{\rho}_{\mathrm{A}} = \sum_i p_i \hat{\rho}_{Ai}, \quad \hat{\rho}_{\mathrm{B}} = \sum_i p_i \hat{\rho}_{Bi}
$$

which, in turn, implies

$$
\hat{\rho}_{\mathrm{AB}} \neq \hat{\rho}_{\mathrm{A}} \otimes \hat{\rho}_{\mathrm{B}}
$$

Example 2.4. *Let $\mathcal{H}_{\mathrm{A}} \simeq \mathcal{H}_{\mathrm{B}} \simeq \mathbb{C}^2$ and*

$$
\begin{aligned}
\hat{\rho} &= \frac{1}{2} |0\rangle \langle 0| \otimes |1\rangle \langle 1| + \frac{1}{2} |1\rangle \langle 1| \otimes |0\rangle \langle 0| \\
&= \frac{1}{2} |01\rangle \langle 01| \otimes |10\rangle \langle 10| \\
&= \frac{1}{2} \hat{P}_{01} + \frac{1}{2} \hat{P}_{10}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\hat{\rho}_{\mathrm{A}} &= \mathrm{Tr}_{\mathrm{B}}(\hat{\rho}_{\mathrm{AB}}) \\
&= \frac{1}{2}\Big(|0\rangle \langle 0| \otimes |1\rangle \langle 1| \Big) \\
&= \frac{1}{2} \hat{\mathbb{I}}
\end{aligned}
$$

so

$$\hat{\rho}_{\mathrm{AB}} \neq \frac{1}{4}\hat{\mathbb{I}} \;\; \textit{maximally mixed state}$$

$$\to \hat{\rho}_{\mathrm{AB}} = \langle \hat{P}_{01} \rangle_{\hat{\rho}_{\mathrm{AB}}} \hat{P}_{01} + \langle \hat{P}_{10} \rangle_{\hat{\rho}_{\mathrm{AB}}} \hat{P}_{10}, \quad \hat{P}_{01} \neq \hat{A} \otimes \hat{\mathbb{I}}_{\mathrm{B}} \neq \hat{P}_{10}$$

is recoverable by measuring non-local operators $\hat{A} \otimes \hat{B}$, which encode correlations.

2.2 Mutual information

Definition 2.8 (Mutual information)**.** *We define the **mutual information** $I(\hat{\rho}_{\mathrm{AB}})$ as :*

$$I(\hat{\rho}_{\mathrm{AB}}) := S(\hat{\rho}_{\mathrm{A}}) + S(\hat{\rho}_{\mathrm{B}}) - S(\hat{\rho}_{\mathrm{AB}}) \tag{98}$$

Observation 2.6. *We can see that, if $\hat{\rho}_{\mathrm{AB}}$ is separable, then*

$$I(\hat{\rho}_{\mathrm{AB}}) = 0 \tag{99}$$

If $\hat{\rho}_{\mathrm{AB}}$ is pure, but not $\hat{\rho}_A$ and $\hat{\rho}_B$, then we have full information about AB combined but only partial information about A or B individually.

2.3 Entanglement

2.3.1 Entanglement for pure states

Definition 2.9 (Entangled state)**.** *A pure state which cannot be written as*

$$|\psi\rangle = |\psi_{\mathrm{A}}\rangle \otimes |\psi_{\mathrm{B}}\rangle$$

*(is not separable), is said to be an **entangled state**.*

Example 2.5. *Let $\mathcal{H}_{\mathrm{A}} \simeq \mathcal{H}_{\mathrm{B}} \simeq \mathbb{C}^2$. Then the state*

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle \Big)$$

is entangled. Moreover,

$$\langle \hat{P}_{01} \rangle = \langle \hat{P}_{10} \rangle = \frac{1}{2}$$

and

$$\begin{aligned}
\hat{\rho}_{\mathrm{A}} &= \mathrm{Tr}\Big(|\psi\rangle \langle\psi| \Big) \\
&= \frac{1}{2}\,\mathrm{Tr}_{\mathrm{B}}\Big[\Big(|01\rangle + |10\rangle \Big)\Big(\langle 01| + \langle 10| \Big) \Big] \\
&= \frac{1}{2}\,\mathrm{Tr}_{\mathrm{B}}\Big[|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10| \Big]
\end{aligned}$$

$$= \frac{1}{2}\bigg(|0\rangle \langle 0| + |1\rangle \langle 1| \bigg)$$

$$= \frac{1}{2}\hat{\mathbb{I}}$$

$$\hat{\rho}_{\mathrm{B}} = \frac{1}{2}\hat{\mathbb{I}}$$

are maximally mixed states.
We have information about the full state but not the partial states.

Definition 2.10 (Entanglement entropy)**.** *If $|\psi\rangle \in \mathcal{H}_{\mathrm{A}} \otimes H_{\mathrm{B}}$ is a state, with $\hat{\rho} = |\psi\rangle \langle\psi|$, then the **entanglement entropy** of $\hat{\rho}$ is defined as*

$$S_E(\hat{\rho}) = S(\hat{\rho}_{\mathrm{A}}) = S(\hat{\rho}_{\mathrm{B}}) \tag{100}$$

This quantifies the entanglement for pure states.

Definition 2.11 (Maximal entanglement)**.** *If $|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ and $S_E(|\psi\rangle \langle\psi|) = \log n$, then we say that $|\psi\rangle$ is a **maximally entangled state**.*

Example 2.6. *Let*

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\bigg(|00\rangle \pm |11\rangle \bigg), \quad |\psi_\pm\rangle = \frac{1}{\sqrt{2}}\bigg(|01\rangle \pm |10\rangle \bigg)$$

The partial states are $\frac{1}{2}\hat{\mathbb{I}}$ for all of these. Hence, for each state, $S_E = \log 2$.

Theorem 2.4. *If the entanglement entropy is null, then the state is separable:*

$$S_E\bigg(|\psi\rangle \langle\psi| \bigg) = 0 \iff |\psi\rangle = |\psi_{\mathrm{A}}\rangle \otimes |\psi_{\mathrm{B}}\rangle \tag{101}$$

2.3.2 LOOCs (Local Operations and Classical Communication)

Theorem 2.5. *The entanglement does not increase under local operations, even with classical communication.*

Example 2.7. $\hat{U}_{\mathrm{A}} = \hat{U} \otimes \hat{\mathbb{I}}$, $\hat{\rho} \to \hat{U}_{\mathrm{A}}\hat{\rho}\hat{U}_{\mathrm{A}}^\dagger$, $\mathcal{E}(\hat{\rho}) = \sum_i (\hat{K}_i \otimes \hat{\mathbb{I}})\hat{\rho}(\hat{K}_i \otimes \hat{\mathbb{I}})^\dagger$

Definition 2.12 (Classical communication)**.** ***Classical communication** refers to using classical (non-quantum) information from A to perform operations in B and vice versa.*

Example 2.8.

$$S_E\bigg((\hat{U} \otimes \hat{\mathbb{I}}) |\psi\rangle \langle\psi| (\hat{U} \otimes \hat{\mathbb{I}})^\dagger \bigg) = S(\hat{U}\hat{\rho}_{\mathrm{A}}\hat{U}^\dagger)$$

$$= S(\hat{\rho}_{\mathrm{A}})$$

2.3.3 Entanglement quantifiers

Theorem 2.6. *If $\hat{\rho}$ is not separable, then it is entangled.*

Definition 2.13 (Entanglement of Formation)**.** *Let $\hat{\rho} = \sum_i p_i \hat{\rho}_i$, with $\hat{\rho}_i \in \mathcal{L}(\mathcal{H})$. The* ***entanglement of Formation*** *of $\hat{\rho}$ is defined as*

$$E_F(\hat{\rho}) = \inf_{\{p_i, \hat{\rho}_i\}} \sum_i p_i S_E(\hat{\rho}_i) \tag{102}$$

Observation 2.7. *$E_F(\hat{\rho})$ is hard to compute even in $\mathbb{C}^2 \otimes \mathbb{C}^2$!*

Definition 2.14 (Distillable entanglement)**.** *The* ***distillable entanglement*** *of $\hat{\rho}$ is defined as*

$$E_D(\hat{\rho}) = \lim_{n \to \infty} \frac{1}{n} b_n(\hat{\rho}) \tag{103}$$

where $b_n(\hat{\rho})$ is the number of maximally entangled states that can be obtained from n copies of $\hat{\rho}$ by using only LOOCs.

Observation 2.8. *The entanglement of formation quantifies <u>all</u> possible entanglement, while the distillable entropy quantifies only the distillable entanglement (as there is entanglement that cannot be converted to maximally entangled states).*

Definition 2.15 (Bound entanglement)**.** *A non-distillable entanglement is called* ***bound entanglement****.*

2.3.4 Negativity

Reminder $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a valid operation if it is CPTP:

$$\mathrm{Tr}\Big(\mathcal{E}(\hat{A})\Big) = \mathrm{Tr}\Big(\hat{A}\Big)$$

$$(\mathcal{E} \otimes \hat{\mathbb{I}}_C)(\hat{B}) \geq 0 \;\; \forall \hat{B} \geq 0, \hat{B} \in \mathcal{L}(\mathcal{H}_{\text{A}} \otimes \mathcal{H}_C)$$

Definition 2.16 (Partial transpose)**.** *We define the* ***partial transpose*** *as*

$$(\hat{A} \otimes \hat{B})^{\top_{\text{B}}} = \hat{A} \otimes \hat{B}^{\top} \tag{104}$$

Example 2.9. *Let $\{|i_{\text{A}}\rangle\}$ and $\{|j_{\text{B}}\rangle\}$ be bases for $\mathcal{H}_{\text{A}}$ and $\mathcal{H}_{\text{B}}$, then the partial transpose acts in the basis of operators as*

$$(|i_{\text{A}}\rangle \langle i'_{\text{A}}| \otimes |j_{\text{B}}\rangle \langle j'_{\text{B}}|)^{\top_{\text{B}}} = |i_{\text{A}}\rangle \langle i'_{\text{A}}| \otimes |j'_{\text{B}}\rangle \langle j_{\text{B}}|)$$

If $\hat{\rho}_{\text{AB}} = \sum_i p_i \hat{\rho}_{Ai} \otimes \hat{\rho}_{Bi}$, then $\hat{\rho}_{\text{AB}}^{\top}$ is a density operator because the transpose is positive and trace preserving.
However, this is not always the case if $\hat{\rho}_{\text{AB}}$ is not separable. Indeed, it can be that

$$\mathrm{Tr}\big(\hat{\rho}_{\text{AB}}^{\top_{\text{B}}}\big) = 1 \;\; \text{but} \;\; \hat{\rho}_{\text{AB}} < 0$$

Theorem 2.7. *The transpose is a positive operation but not CP.*

Example 2.10. *Let* $|\Phi_+\rangle = \frac{1}{2}\Bigg(|00\rangle + |11\rangle \Bigg)$*, then*

$$\hat{\rho} = \frac{1}{2}\Bigg(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \Bigg)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\implies \hat{\rho}^{\top_{\mathrm{B}}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Recognizing the first Pauli matrix, we find that the spectrum of this operator is $\sigma_\lambda = \{-1, 1, 1, 1\}$*, and we can see that one of the eigenvalues is negative.*

Theorem 2.8 (Peres' criterion)**.** *If* $\hat{\rho}^{\top_{\mathrm{B}}}$ *is not positive, then* $\hat{\rho}$ *is entangled.*

Definition 2.17 (Negativity)**.** *The* ***negativity*** *of* $\hat{\rho}$ *is defined as*

$$\mathcal{N}(\hat{\rho}) = \sum_{\lambda_i \in \sigma(\hat{\rho}^{\top_{\mathrm{B}}}),\ \lambda_i < 0} |\lambda_i| \tag{105}$$

It is a quantifier of distillable entanglement.

Observation 2.9. *There is non bound entanglement in* $\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^3$*. Therefore, if* $\mathcal{N}(\hat{\rho}) > 0$*, then* $\hat{\rho}$ *is entangled in these spaces.*

3 Calculus

3.1 Manifold/Spacetime

3.1.1 Metric

Definition 3.1 (Spacetime)**.** *A manifold* $\mathcal{M}$ *with a symmetric metric* $g \in \Gamma(T_{(0,2)}(\mathcal{M}))$ *and signature (-1,1,1,1) is called a* ***spacetime***.

Theorem 3.1. *If $g = g_{\mu\nu} dx^\mu dx^\nu$, then there exists $e_{\nu'} = O^\mu{}_{\nu'} \partial_\mu$ such that*

$$g(e_{\mu'}, e_{\nu'}) = \eta_{\mu'\nu'} = O^\mu{}_{\mu'} O^\nu{}_{\nu'} g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{106}$$

$\{e_\mu\}$ are called an orthonormal frame.

Definition 3.2. *(Vector types) Let $v \in \Gamma(T\mathcal{M})$. If*

$$\begin{cases} g(v,v) < 0 \rightarrow v \text{ is a } \textbf{\textit{timelike}} \text{ vector} \\ g(v,v) = 0 \rightarrow v \text{ is a } \textbf{\textit{lightlike}} \text{ vector} \\ g(v,v) > 0 \rightarrow v \text{ is a } \textbf{\textit{spacelike}} \text{ vector} \end{cases}$$

Definition 3.3 (Curve types)**.** *Let $\gamma : I \subseteq \mathbb{R} \rightarrow \mathcal{M}$. If $\dot{\gamma}$ is time/light/spacelike, then γ is said to be* ***time/light/spacelike****.*

Definition 3.4 (Proper time)**.** *If γ is timelike, we can parametrize it as $\gamma = \gamma(u)$ and we define the* ***proper time*** *$\tau(u)$ as*

$$\tau(u) = \int_{u_0}^{u} \mathrm{d}u' \sqrt{-g_{\mu\nu}(\gamma(u)) \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}u} \frac{\mathrm{d}\gamma^\nu}{\mathrm{d}u}} \tag{107}$$

3.1.2 Covariant derivative

Definition 3.5 (Levi-Civita connection)**.** *The* ***Levi-Civita connection*** *is the unique torsionless connection ∇_ν such that*

$$\nabla_\mu g = 0 \tag{108}$$

Definition 3.6 (Acceleration)**.** *Let γ be a timelike curve. Its 4-acceleration is*

$$\begin{aligned} a^\mu(\tau) &= \frac{D\dot{\gamma}^\mu}{\mathrm{d}\tau} \\ &= \nabla_{\dot{\gamma}} \dot{\gamma}^\mu \end{aligned} \tag{109}$$

where

$$\dot{\gamma}^\mu = \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}$$

and the ***proper acceleration*** *can be computed as*

$$a(\tau) = \sqrt{g_{\mu\nu} a^\mu a^\nu} \tag{110}$$

Observation 3.1. *If* $u \equiv \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\tau}$ *is the 4-velocity, then*

$$g_{\mu\nu}u^\mu u^\nu = -1 \quad \forall \tau \tag{111}$$

and therefore

$$\frac{D}{\mathrm{d}t}(g_{\mu\nu}u^\mu u^\nu) = 2a_\mu u^\mu \stackrel{!}{=} 0 \tag{112}$$

which implies that a_μ *and* u^μ *are orthogonal.*

Definition 3.7 (Geodesic)**.** *Let* γ *be a curve. We say that* γ *is a* ***geodesic*** *if*

$$\dot{\gamma}^\mu \nabla_\mu \dot{\gamma}^\nu = 0 \tag{113}$$

Definition 3.8 (Laplacian)**.** *We define the* ***Laplacian*** *as*

$$\Box = \nabla_\mu \nabla^\mu \tag{114}$$

Observation 3.2. *In inertial coordinates in Minkowski spacetime, the Laplacian takes the form*

$$\Box = \partial_t^2 - \vec{\nabla}^2 \tag{115}$$

3.1.3 Separations

Definition 3.9. *Let* $x \in \mathcal{M}$*. We define*

$$\begin{cases} I(\mathsf{x}) = \{\mathsf{x}' \in \mathcal{M} | \exists\, \gamma : [u_0, u_1] \to \mathcal{M} \ \ \textit{timelike} : \gamma(u_0) = \mathsf{x} \ \ \textit{and} \ \ \gamma(u_1) = \mathsf{x}'\} \\ \\ J(\mathsf{x}) = \{\mathsf{x}' \in \mathcal{M} | \exists\, \gamma : [u_0, u_1] \to \mathcal{M} \ \ \textit{timelike or lightlike} : \gamma(u_0) = \mathsf{x} \ \ \textit{and} \ \ \gamma(u_1) = \mathsf{x}'\} \end{cases}$$

Observation 3.3. *This construction can also be made for neighborhoods. Let* $U \subseteq \mathcal{M}$*, then*

$$\begin{cases} \mathcal{J}(U) = \bigcup_{\mathsf{x} \in \mathcal{M}} \mathcal{J}(\mathsf{x}) \\ \\ \mathcal{I}(U) = \bigcup_{\mathsf{x} \in \mathcal{M}} \mathcal{I}(\mathsf{x}) \end{cases}$$

Definition 3.10 (Spacelike separation)**.** *Let* $U_1, U_2 \subseteq \mathcal{M}$ *such that* $\mathcal{J}(U_1) \bigcap U_2 = \emptyset$*, then we say that* U_1 *and* U_2 *are* ***spacelike separated****, or causally disconnected.*

Definition 3.11 (Time-oriented spacetime)**.** *Let* t *be a timelike coordinate. If* $t : \mathcal{M} \to \mathbb{R}$ *is globally defined, then we call it a* ***global time parameter*** *and* $(\mathcal{M}, g, t)$ *is a* ***time-oriented spacetime****.*

Definition 3.12 (Orientation)**.** *If* v *is a time/lightlike vector field, then*

$$\begin{cases} g(v, \partial_t) < 0 \to v \ \textit{is} \ \textbf{\textit{future-oriented}} \\ \\ g(v, \partial_t) > 0 \to v \ \textit{is} \ \textbf{\textit{past-oriented}} \end{cases}$$

Definition 3.13. *If spacetime is future/past-oriented, then we define*

$$\begin{cases} J^{\pm} \to \textit{\textbf{causal future/past}} \\ \\ I^{\pm} \to \textit{\textbf{chronological future/past}} \end{cases}$$

Definition 3.14 (Domain of dependence)**.** *Let $U \subseteq \mathcal{M}$, we define the* ***past/future domaine of dependence*** *$\mathcal{D}^{\pm}(U)$ of U as the set of all past/future directed causal curves starting at x, $\forall x \in \mathcal{M}$, that intersect U if maximally extended.*

Definition 3.15 (Cauchy surface)**.** *Let $\Sigma \subseteq \mathcal{M}$ be a 3-dimensional Riemannian manifold with the metric induced by g. We say that Σ is a spacelike hypersurface.*
Moreover, if $\mathcal{D}(\Sigma) = \mathcal{M}$, we say that Σ is a ***Cauchy surface****.*

Definition 3.16 (Foliation)**.** *If (t, x^i) are coordinates in $\mathcal{M}$ such that $\Sigma_t = \{\mathsf{x} \in \mathcal{M} : \mathsf{x} = (t, x^i)\}$ is a Cauchy surface for each $t \in \mathbb{R}$, then $\{\Sigma_t\}$ is a* ***foliation by Cauchy surfaces****.*

More generally, if

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t \quad \textit{and} \quad \{\Sigma_t\} \textit{ are disjoint}$$

then $\{\Sigma_t\}$ is a foliation by Cauchy surfaces.

Definition 3.17 (Convex causality)**.** *Let $U \subseteq \mathcal{M}$ such that x, x' are causally connected $\forall x, x' \in U$. If every causal curve that connects x and x' is fully contained in U, we say that U is* ***causally convex****.*

Definition 3.18 (Causal hull)**.** *We define the* ***causal hull*** *$\overline{U}$ of U as*

$$\overline{U} = \bigcap_{U \subseteq \overline{U} : U' \, \textit{causally convex}} U' \tag{116}$$

3.1.4 Integration

Definition 3.19 (Volume form)**.** *We define the* ***volume form*** *as*

$$\begin{aligned} \mathrm{d}V &= \sqrt{-g}\mathrm{d}\mathsf{x}^0 \wedge \mathrm{d}\mathsf{x}^1 \wedge \mathrm{d}\mathsf{x}^2 \wedge \mathrm{d}\mathsf{x}^3 \\ &\equiv \sqrt{-g}\ \mathrm{d}^4\mathsf{x} \end{aligned} \tag{117}$$

Theorem 3.2 (Stoke's theorem)**.**

$$\int_{\Omega} \nabla_{\mu} v^{\mu} \mathrm{d}V = \int_{\partial\Omega = \Sigma} \mathrm{d}\Sigma_{\mu} v^{\mu} \tag{118}$$

In particular,

$$\int_{\Omega} \nabla_{\mu} v^{\mu} dV = \int_{\Sigma_{t_1}} \mathrm{d}\Sigma_{\mu} v^{\mu} - \int_{\Sigma_{t_0}} \mathrm{d}\Sigma_{\mu} v^{\mu} \tag{119}$$

where

$$\begin{cases} v|_{\partial\Omega} = 0 \\ \mathrm{d}\Sigma_\mu = n_\mu \mathrm{d}\Sigma \\ \mathrm{d}\Sigma = \sqrt{g_\Sigma}\mathrm{d}^3x \end{cases}$$

3.2 Distributions

When dealing with differential equations, sometimes ordinary functions are not enough, and we are required to introduce a new class of function-like objects, called *distributions*, which are defined by their action on functions: to each function, we associate a complex number.

Definition 3.20 (Distribution)**.** *A* ***distribution*** ϕ *is a linear functional:*

$$\begin{aligned} \phi : \mathcal{D} = C_0^\infty(\mathcal{M}) &\to \mathbb{C} \\ f &\to \phi(f) \end{aligned}$$

The set of distributions corresponds to the (topological) dual of $C_0^\infty(\mathcal{M})$ and can be denoted as $\mathcal{D}'(\mathcal{M}) \subseteq \mathcal{D}^*(\mathcal{M})$.

Observation 3.4. *Distributions can be thought of as generalized functions. For example, as we will see, it is not possible to define a field operator as a function $\phi : \mathcal{M} \to \mathbb{C}$, evaluated at a specific point $\mathsf{x} \in \mathcal{M}$. Instead, we use a test function f to smear this field over a region of spacetime and the distribution $\phi(f)$ becomes a defined operator.*

Example 3.1. *Given $\phi(\mathsf{x})$, we define the "smearing" of f: $\phi(f) = \int \mathrm{d}V \phi(\mathsf{x}) f(\mathsf{x})$.*

Example 3.2. *Possibly the most important distribution, the Dirac delta, is defined as: $\phi(f) \equiv f(\mathsf{x}_0), \mathsf{x}_0 \in \mathcal{M}$, where $\phi(f)$ is usually denoted δ_{x_0}:*

$$\delta_{\mathsf{x}_0}(f) = f(\mathsf{x}_0)$$

As with regular functions, we can differentiate a distribution. The idea is to define a

Definition 3.21 (Distributional derivative)**.** *Let ϕ be a distribution. We define the* ***distributional derivative*** ∇_v *as*

$$\begin{aligned} \nabla_v : \mathcal{D}'(\mathcal{M}) &\to \mathcal{D}'(\mathcal{M}) \\ \phi &\mapsto \nabla_v \phi \end{aligned}$$

such that

$$\nabla_v \phi(f) = -\phi(\nabla_\mu (v^\mu f)) \tag{120}$$

where ∇_μ is the usual covariant derivative acting on functions.

Observation 3.5. *This construction is done by integrating by parts*

$$\nabla_v \phi(f) = \int \mathrm{d}V f(\nabla_v \phi)$$

$$= \int \mathrm{d}V \nabla_\mu (v^\mu f \phi) - \int \mathrm{d}V \nabla_\mu (v^\mu f)\phi$$

$$= -\int \mathrm{d}V \nabla_\mu (v^\mu f)\phi$$

$$= -\phi(\nabla_\mu (v^\mu f))$$

Example 3.3. $\nabla_v \delta_{\mathsf{x}_0}[f] = \delta_{\mathsf{x}_0}(-\nabla_\mu (v^\mu f)) = -\nabla_\mu (v^\mu f)\Big|_{\mathsf{x}_0}$

As introduced before, distributions are functionals that act on functions themselves, not points. Hence, if we want to evaluate a distribution at a point, we need to define explicitly what we mean by this.

Definition 3.22. *Let* $\{f_n^{\mathsf{x}_0}\}$ *be a sequence of test functions approximating the delta function:*

$$\lim_{n\to\infty} \int \mathrm{d}V g(\mathsf{x}) f_n^{\mathsf{x}_0}(x) = g(\mathsf{x}_0) \ \ \forall g \in C_0^\infty(\mathcal{M}) \tag{121}$$

If $\lim_{n\to\infty} \phi(f_n)$ *converges, we define*

$$\phi_{dist}(\mathsf{x}_0) := \lim_{n\to\infty} \phi(f_n^{\mathsf{x}_0}) \tag{122}$$

Observation 3.6. *If* $\phi(\mathsf{x}) \in C_0^\infty(\mathcal{M})$ *(or even* $\in L^2(\mathcal{M})$*), then*

$$\phi(\mathsf{x}_0) = \phi_{dist}(\mathsf{x}_0) \tag{123}$$

Example 3.4.

$$\lim_{n\to\infty} \delta_{\mathsf{x}_1}(f_n^{\mathsf{x}_0}) = \begin{cases} 0 & ,\mathsf{x}_0 \neq x_1 \\ \\ \infty & ,\mathsf{x}_0 = \mathsf{x}_1 \end{cases}$$

Definition 3.23 (Integral of a distribution)**.** *Let* ϕ *be a distribution. We define its* ***integral*** $I[\phi]$ *as its action on the constant function* $f(\mathsf{x}) = 1 \ \forall \mathsf{x} \in \mathcal{M}$*:*

$$I : \mathcal{D}'(\mathcal{M}) \to \mathbb{C}$$
$$\phi \to \phi(1) = \int \mathrm{d}V \phi(\mathsf{x}).$$

3.3 Bidistributions

Definition 3.24 (Bidistribution). *A* ***bidistribution*** *A is a bilinear functional:*

$$\mathcal{D}(\mathcal{M}) \times \mathcal{D}(\mathcal{M}) \to \mathbb{C}$$
$$(f,g) \to A(f,g)$$

Example 3.5. *Given $A(\mathsf{x},\mathsf{x}')$, we define $A(f,g) = \int \mathrm{d}V \mathrm{d}V' A(\mathsf{x},\mathsf{x}') f(\mathsf{x}) g(\mathsf{x}')$*

Observation 3.7. *Just like with normal distributions, we can differentiate $A(\mathsf{x},\mathsf{x}')$, only this time with respect to either x or x' ($\nabla_\mu A(\mathsf{x},\mathsf{x}')$, $\nabla_{\mu'} A(\mathsf{x},\mathsf{x}')$, $\nabla_\mu \nabla_{\mu'} A(\mathsf{x},\mathsf{x}')$).*

Example 3.6. *In Minkowski spacetime,*

$$A(f,g) = \lim_{\varepsilon \to 0^+} \int \mathrm{d}^4\mathsf{x}\, \mathrm{d}^4\mathsf{x}' \frac{1}{-(t-t'-i\varepsilon)^2 + (\boldsymbol{x}-\boldsymbol{x}')^2} f(\mathsf{x}) g(\mathsf{x}')$$

$$= \lim_{\varepsilon \to 0^+} \int \mathrm{d}^4\mathsf{x}\, \mathrm{d}^4\mathsf{x}' A(\mathsf{x},\mathsf{x}') f(\mathsf{x}) g(\mathsf{x}')$$

Definition 3.25 (Symmetry). *If $A(f,g) = A(g,f) \ \ \forall f,g$, we say that A is* ***symmetric****. Moreover, if $A(f,g)^* = A(g^*,f^*)$, we say that A is* ***conjugate symmetric****.*

Observation 3.8.

$$\int \mathrm{d}V \mathrm{d}V' A^*(\mathsf{x},\mathsf{x}') f^*(\mathsf{x}) g^*(\mathsf{x}') = \int \mathrm{d}V \mathrm{d}V' A(\mathsf{x},\mathsf{x}') g^*(\mathsf{x}') f^*(\mathsf{x})$$

$$= \int \mathrm{d}V \mathrm{d}V' A(\mathsf{x}',\mathsf{x}) f^*(\mathsf{x}) g^*(\mathsf{x}')$$

$$\implies A(\mathsf{x}',\mathsf{x}) = A^*(\mathsf{x},\mathsf{x}')$$

If $A(f,g)$ is symmetric/conjugate symmetric, then so is $A(\mathsf{x},\mathsf{x}')$.

Definition 3.26. *Let A be a bidistribution and $g \in D$. We define*

$$Ag(\mathsf{x}) = \int \mathrm{d}V' A(\mathsf{x},\mathsf{x}') g(\mathsf{x}') \tag{124}$$

and hence

$$f(Ag) = Ag(f) = \int \mathrm{d}V\, Ag(\mathsf{x}) f(\mathsf{x}) = A(f,g) \tag{125}$$

Observation 3.9. *We are using the same name for different objects*

$$\begin{cases} A(\mathsf{x},\mathsf{x}'), \quad A : \mathcal{M} \times \mathcal{M} \to \mathbb{C} \\ A(f,g), \quad A : D \times D \to \mathbb{C} \\ Ag, \quad A : D \to C^\infty(\mathcal{M}) \end{cases}$$

Observation 3.10. *Let $P : D \to C^\infty(\mathcal{M})$ linear. Then,*

$$Pg(\mathsf{x}) = \int \mathrm{d}V' P(\mathsf{x},\mathsf{x}') g(\mathsf{x}') \tag{126}$$

3.4 Functional calculus

Definition 3.27 (Functional)**.** *Let $\phi : D \to \mathbb{C}$ (not necessarily linear). We say that ϕ is a* ***functional****.*

Example 3.7. $\phi(f) = \int \mathrm{d}V f^2(\mathsf{x})$

Definition 3.28 (Derivative of a functional)**.** *Let $\phi : V \to W$ be a function, where V, W are normed vector spaces. The* ***derivative*** *of ϕ at $v \in V$ is a linear operator*

$$D\phi(v) : V \to W \tag{127}$$

such that

$$\phi(v + \varepsilon) = \phi(v) + D\phi(v)\varepsilon + R_v(\varepsilon) \tag{128}$$

with $\lim_{||\varepsilon|| \to 0} \frac{R_v(\varepsilon)}{||\varepsilon||} = 0$.

The function $D\phi : V \times V \to W$ such that $D\phi(v) : V \to W$ is the derivative of ϕ.

Example 3.8. *If $\phi(f) = \int \mathrm{d}V f^2(x)$, then*

$$\begin{aligned}
\phi(f + \varepsilon) &= \int \mathrm{d}V (f + \varepsilon)^2(x) \\
&= \int \mathrm{d}V (f^2 + 2f\varepsilon + \varepsilon^2)(x) \\
&= \int \mathrm{d}V f^2 + 2\int \mathrm{d}V f\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \phi(f) + 2f(\varepsilon) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

So we find

$$D\phi(f) = 2\int \mathrm{d}V f \tag{129}$$

Observation 3.11. *Many people write $D\phi = \dfrac{\delta\phi}{\delta f}$ and then*

$$\phi(f + \delta f) = \phi(f) + \frac{\delta\phi}{\delta f}\delta f + \mathcal{O}(\delta f^2) \tag{130}$$

Observation 3.12. *If $\phi : D \to \mathbb{C}$, then $\dfrac{\delta\phi}{\delta f}$ is linear, and so it is a distribution. We can then interpret $\frac{\delta\phi}{\delta f(\mathsf{x})}$ as the kernel of a distribution.*

Example 3.9. *Let $F : \mathbb{C} \to \mathbb{C}$ and $S(f) = \int \mathrm{d}V F(f(\mathsf{x}))$. Then,*

$$
\begin{aligned}
S(f + \delta f) &= \int \mathrm{d}V F(f + \delta f) \\
&= \int \mathrm{d}V \Big(F(f) + F'(f)\delta f + \mathcal{O}(\delta f^2) \Big) \\
&= S(f) + \int \mathrm{d}V F'(f(\mathsf{x}))\delta f(\mathsf{x}) \\
&=: S(f) + \int \mathrm{d}V \frac{\delta S}{\delta f(\mathsf{x})} \delta f(\mathsf{x})
\end{aligned}
$$

4 Day 5: Classical Field Theory

4.1 Fields

Definition 4.1 (Vector bundle)**.** *Let $\mathcal{M}$ and $\mathcal{E}$ be manifolds, V a vector space and $\pi : \mathcal{E} \to \mathcal{M}$ a map such that $\pi^{-1}(\{p\}) \equiv \mathcal{E}_p \cong V$ for each $p \in \mathcal{M}$.*

If, for each $p \in \mathcal{M}$, there exists an open neighbourhood $U \subseteq \mathcal{M}$ where one can define a diffeomorphism $\varphi : U \times V \to \mathcal{E}_U$ such that $\pi(\varphi(p, v)) = p$ and $\varphi(p, .) : V \to \mathcal{E}_p$ is an isomorphism $\forall p \in \mathcal{M}$, then we say that

$$
\begin{cases}
(\mathcal{M}, \mathcal{E}, \pi) \ \textit{is a } \textbf{\textit{vector bundle}} \textit{ over } \mathcal{M}. \\[1em]
\mathcal{E}_p \ \textit{is a } \textbf{\textit{fiber}} \textit{ for each } p \in \mathcal{M}. \\[1em]
\pi \ \textit{is the } \textbf{\textit{projection map}}. \\[1em]
\varphi \ \textit{is the } \textbf{\textit{local trivialization}} \textit{ of } \mathcal{E}\ .
\end{cases}
$$

Definition 4.2 (Vector field)**.** *Let $v : \mathcal{M} \to \mathcal{E},\ p \to v_p$ be a function such that $\pi(v_p) = p$. Then v is called a* ***vector field****.*
Geometrically, it can be defined as a ***section*** *of $\mathcal{E}$. The set of all sections of $\mathcal{E}$ is denoted $\Gamma(\mathcal{E})$.*

Example 4.1. *The tangent bundle $T\mathcal{M}$ is a bundle over $\mathcal{M}$ ($V = \mathbb{R}^n, n = \dim(\mathcal{M})$). Elements of the section of this bundle are vector fields, and so $\Gamma(T\mathcal{M}) \cong \mathfrak{X}(\mathcal{M})$.*

Example 4.2. *$\mathcal{M} \times \mathbb{R}$ is a bundle over $\mathcal{M}$ with $V = \mathbb{R}$, and therefore $\Gamma(\mathcal{M} \times \mathbb{R}) = C^\infty(\mathcal{M})$: sections of this bundle corresponds to scalar fields.*

4.2 Equations of motion

A **physical field** $\phi \in \Gamma(\mathcal{E})$ an equation of motion

$$P(\phi) = 0 \tag{131}$$

where P is a tensorial differential operator.

Example 4.3. *Consider a scalar field $\phi \in C^\infty(\mathcal{M})$, then P could be*

$$P(\phi) = \Box\phi + \phi^2 - 2\phi^3$$

Observation 4.1. *From now on, we will work under the assumption that P is linear, and then we can write the equations of motion as:*

$$P\phi = 0 \tag{132}$$

Definition 4.3 (Globally hyperbolic space)**.** *If $\mathcal{M}$ admits a Cauchy surface, then it is called a **globally hyperbolic space**.*

Observation 4.2. *If $\mathcal{M}$ is globally hyperbolic, then it admits a global time function t and so it admits a foliation by Cauchy surfaces.*

Definition 4.4. *Let P be a linear differentiable operator $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that there exists two functions $G_R, G_{\mathrm{A}} : \Gamma_0(\mathcal{E}) \to \Gamma(\mathcal{E})$ verifying*

$$\begin{cases} PG_R f = f, PG_{\mathrm{A}} f = f, \;\; \forall f \in \Gamma_0(\mathcal{E}) \\ G_R P f = f, G_{\mathrm{A}} P f = f, \;\; \forall f \in \Gamma_0(\mathcal{E}) \\ supp(G_R f) \subseteq J^+(supp f) \\ supp(G_{\mathrm{A}} f) \subseteq J^-(supp f) \\ G_R, G_{\mathrm{A}} \;\; \text{are unique given the conditions above} \end{cases}$$

We call G_R and G_{A} the retarded and advanced Green functions, and they live in the future/past causal cones of the source f.
Intuitively, we can think of G_R as a solution, i.e. a field, propagating forward in time after the source is "turned on".

Example 4.4. *Consider ϕ with a source such that $P\phi = f$, then $\phi = G_R f$ if $supp\phi \subseteq J^+(supp f)$ (respectively $supp\ \phi \subseteq J^-(supp f)$ for G_{A}).*

Observation 4.3. *These functions can be written in an integral form*

$$G_R f = \int \mathrm{d}V G_R(\mathsf{x}, \mathsf{x}') f(\mathsf{x}')$$

$$G_{\mathrm{A}} f = \int \mathrm{d}V G_{\mathrm{A}}(\mathsf{x}, \mathsf{x}') f(\mathsf{x}')$$

Definition 4.5 (Causal propagator)**.** *We can use these functions to define the* ***causal propagator*** $E : \Gamma_0(\mathcal{E}) \to \Gamma(\mathcal{E})$*:*

$$E = G_R - G_{\mathrm{A}} \tag{133}$$

Observation 4.4. *The causal propagator can be now used to recover solutions to the equation:*

$$PEf = P(G_R f - G_{\mathrm{A}} f) = f - f = 0 \tag{134}$$

Theorem 4.1. *Any solution of $P\phi = 0$ can be written as $\phi = Ef$ for some $f \in \Gamma_0(\mathcal{E})$.*

Observation 4.5. *Therefore, a physical field can be represented as if it came from a compactly supported sourced, propagated via the causal propagator.*

Observation 4.6. *We can see that $E(f+Ph) = Ef + EPh = Ef$, and hence the association $f \mapsto Ef$ is not unique.*

Let $S = \ker P = \{\phi \in \Gamma_0(\mathcal{E}) : P\phi = 0\}$. Then, the previous observations tells us that a map $\varphi : \Gamma_0(\mathcal{E}) \to S, \;\; f \mapsto Ef$, is surjective but not injective, and hence not bijective.

Let us then define another map $\varphi : \Gamma_0(\mathcal{E})/P\Gamma_0(\mathcal{E}) \to S, \;\; f+Ph \to Ef$, where $\Gamma_0(\mathcal{E})/P\Gamma_0(\mathcal{E}) = \{f + Ph | f, h \in \Gamma_0(\mathcal{E})\}$. This map is now bijective.

Definition 4.6 (Observable)**.** *We can now define an* ***observable*** *as a map $f : S \to \mathbb{R}$. In other words, an observable associates a real number to a physical field satisfying an equation of motion.*

Example 4.5. *Let $f(\phi) = \int \mathrm{d}V f(\mathsf{x})\phi(\mathsf{x})$ be an observable. If $\phi = Eg$, then*

$$f(\phi) = \int \mathrm{d}V f(\mathsf{x}) \int \mathrm{d}V' E(\mathsf{x}, \mathsf{x}') g(\mathsf{x}')$$

$$\int \mathrm{d}V \mathrm{d}V' E(\mathsf{x}, \mathsf{x}') f(\mathsf{x}) g(\mathsf{x}')$$

$$= E(f, g)$$

Let us go back to the equations of motion. We will now study different ways of solving them.

Method 1 The simplest method is to find a function $f \in \Gamma_0(\mathcal{E})$ such that $\phi = Ef$. Then, according to Eq. (134), this ϕ is a solution.

Let us now introduce a few more definitions in order to describe further methods of finding solutions.

Definition 4.7. *An* ***initial value problem*** *consists of differential equation and a set of initial conditions*

$$\begin{cases} P\phi = 0 \\ \\ \phi\Big|_{\Sigma} = \Phi_0 \\ \\ n^{\mu}\nabla_{\mu}\phi\Big|_{\Sigma} = \Pi_0 \end{cases}$$

where Σ is a Cauchy surface, n^{μ} the future directed unit normal and $\Phi_0, \Pi_0 : \Sigma \to \mathcal{E}$. Each pair of initial values Φ_0, Π_0 corresponds to a function f such that $\phi = Ef$.

Method 2 A second method of solving the equation of motion is to solve the corresponding initial value problem.

Observation 4.7. *It might be possible that S has more structure (like an inner product). In such cases, we can split its basis as $\{u_k, v_k\}$, where $\{u_k\}$ is a basis for S^+ and $\{v_k\}$ for S^- and S^+, S^- are the subspaces with positive and respectively negative inner product ($\langle u_k, u_{k'}\rangle = \delta_{kk'}$ $\langle v_k, v_{k'}\rangle = -\delta_{kk'}$, $\langle u_k, v_k\rangle = 0$), with $S = S^+ \oplus S^-$.*

In such case, we can write the solutions as

$$\phi = \sum\!\!\!\!\!\!\int_k \alpha_k u_k + \beta_k v_k \tag{135}$$

where

$$\alpha_k := \langle u_k, \phi\rangle, \quad \beta_k := -\langle v_k, \phi\rangle$$

To each pair Φ_0, Π_0 corresponds a pair α_k, β_k.

Observation 4.8. *If we decompose $S = S^+ \oplus S^-$ and $\{u_k\}$ is a basis for S^+, then*

$$(u_k, u_{k'}) = \delta_{kk'} \tag{136}$$

and so

$$(u_k^*, u_{k'}^*) = -\delta_{kk'} \tag{137}$$

which means that $\{u_k^*\}$ is a basis for S^-.

Example 4.6. *In Minkowski spacetime, the elements of the basis are*

$$u_{\boldsymbol{k}}(\mathsf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{\sqrt{2|\boldsymbol{k}|}} \tag{138}$$

Method 3 We can find a solution ϕ by finding the coefficients and expanding the solution in the basis.

Observation 4.9. *The split $S = S^+ \oplus S^-$ is not unique and this has deep implications. Since we require this split in order to define the creation and annihilation operators (and so to define particles), this means there is not a canonical definition of particles: different observers can disagree on their operators.*

4.3 The Klein-Gordon field

Let $P = \nabla_\mu \nabla^\mu - V(\mathsf{x})$, $\phi \in C^\infty(\mathcal{M})$. P is a global hyperbolic operator with

$$G_R(\mathsf{x}, \mathsf{x}') = G_{\mathrm{A}}(\mathsf{x}', \mathsf{x}) \iff E(\mathsf{x}, \mathsf{x}') = -E(\mathsf{x}', \mathsf{x}) \iff E(f, g) = -E(g, f)$$

Example 4.7. *In Minwkowski spacetime, the propagators are written as*

$$\begin{cases} G_R(\mathsf{x}, \mathsf{x}') = \dfrac{1}{4\pi}\delta(-(t-t')^2 + (\boldsymbol{x} - \boldsymbol{x}')^2)\theta(t - t') \\ \\ G_{\mathrm{A}}(\mathsf{x}, \mathsf{x}') = \dfrac{1}{4\pi}\delta(-(t-t')^2 + (\boldsymbol{x} - \boldsymbol{x}')^2)\theta(t' - t) \end{cases}$$

Definition 4.8 (Klein-Gordon inner product)**.** *The **Klein-Gordon inner product** is defined by*

$$(\phi_1, \phi_2) = -\mathrm{i}\int \mathrm{d}\Sigma_\mu (\phi_1^* \nabla^\mu \phi_2 - \phi_2 \nabla^\mu \phi_1^*) \tag{139}$$

$$=: -\mathrm{i}\int \mathrm{d}\Sigma_\mu j^\mu \tag{140}$$

Observation 4.10. *The current j^μ is conserved:*

$$\begin{aligned} \nabla_\mu j^\mu &= \nabla_\mu \phi_1^* \nabla^\mu \phi_2 + \phi_1^* \nabla_\mu \nabla^\mu \phi_2 - \nabla_\mu \phi_2 \nabla^\mu \phi_1^* - \phi_2 \nabla_\mu \nabla^\mu \phi_1^* \\ &= \nabla_\mu \phi_1^* \nabla^\mu \phi_2 + \phi_1^* V \phi_2 - \nabla_\mu \phi_2 \nabla^\mu \phi_1^* - \phi_2 V^* \phi_1^* \\ &= \nabla_\mu \phi_1^* \nabla^\mu \phi_2 + \phi_1^* V \phi_2 - \nabla_\mu \phi_2 \nabla^\mu \phi_1^* - \phi_2 V \phi_1^* \\ &= 0 \end{aligned} \tag{141}$$

Moreover, if we integrate this current over a volume Ω,

$$\int_\Omega \mathrm{d}V \nabla_\mu j^\mu = 0 \iff \int_{\Sigma_1} \mathrm{d}\Sigma_\mu j^\mu - \int_{\Sigma_2} \mathrm{d}\Sigma_\mu j^\mu = 0 \tag{142}$$

which means that the inner product (ϕ_1, ϕ_2) is independent of the choice of Σ.

Example 4.8. *As previously mentioned, in Minkowski space we have*

$$u_{\boldsymbol{k}}(\mathsf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{\sqrt{2|\boldsymbol{k}|}}$$

Injecting this in the inner product,

$$(\phi_1, \phi_2) = -\mathrm{i} \int \mathrm{d}^3\boldsymbol{x}(u^*_{\boldsymbol{k}}(\partial_t u_{\boldsymbol{k}'}) - u_{\boldsymbol{k}'}(\partial_t u^*_{\boldsymbol{k}}))$$

$$= -\mathrm{i} \int \mathrm{d}^3\boldsymbol{x}(u^*_{\boldsymbol{k}}(-\mathrm{i}\boldsymbol{k}' u_{\boldsymbol{k}'}) - u_{\boldsymbol{k}'}(\mathrm{i}\boldsymbol{k} u^*_{\boldsymbol{k}}))$$

Since the conservation of the current is independent of the choice of Cauchy surface, we can in particular choose $t = 0$ and so we get

$$-\mathrm{i} \int \mathrm{d}^3\boldsymbol{x}(u^*_k(-\mathrm{i}\boldsymbol{k}' u_{\boldsymbol{k}'}) - u_{\boldsymbol{k}'}(\mathrm{i}\boldsymbol{k}\, u^*_{\boldsymbol{k}})) = \frac{1}{2\pi^3} \int \mathrm{d}^3\boldsymbol{x} e^{-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{x}} \frac{\boldsymbol{k}+\boldsymbol{k}'}{2\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}'}}$$

$$= \int \mathrm{d}^3\boldsymbol{x}\, \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{k}) \frac{\boldsymbol{k}+\boldsymbol{k}'}{2\sqrt{\boldsymbol{k}\boldsymbol{k}'}}$$

$$= 1$$

with $n^\mu = (1, \mathbf{0})$ and so $n^\mu \nabla_\mu = \partial_t$.

Observation 4.11. *If $V(\mathsf{x}) = m^2$, then $|\boldsymbol{k}|$ becomes $\omega_{\boldsymbol{k}} = \sqrt{|\boldsymbol{k}|^2 + m^2}$*

Overall, we can write a real solution as

$$\phi = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} a_{\boldsymbol{k}} u_{\boldsymbol{k}} + a^*_{\boldsymbol{k}} u^*_{\boldsymbol{k}} \tag{143}$$

where

$$a_{\boldsymbol{k}} = (u_{\boldsymbol{k}}, \phi)$$

$$= \mathrm{i} \int \mathrm{d}\Sigma_\mu (u^*_{\boldsymbol{k}} \nabla^\mu \phi - \phi \nabla^\mu u_{\boldsymbol{k}})$$

$$= \mathrm{i} \int \mathrm{d}\Sigma_\mu (u^*_{\boldsymbol{k}} n_\mu \nabla^\mu \phi - \phi n_\mu \nabla^\mu u_{\boldsymbol{k}})$$

$$= \mathrm{i} \int \mathrm{d}\Sigma_\mu (u^*_{\boldsymbol{k}} \Pi_0 - \Phi_0 n_\mu \nabla^\mu u_{\boldsymbol{k}}) \tag{144}$$

Therefore, given initial conditions, one computes the $\{a_{\boldsymbol{k}}\}$ which give the solution ϕ.

Observation 4.12. *$E(f,g) = -\mathrm{i}((Ef)^*, Eg) = f(Eg) = -g(Ef)$ and so, for any $\psi \in S$,*

$$(\psi^*, Eg) = \mathrm{i}g(\psi) \tag{145}$$

Moreover,

$$\begin{aligned} Eg(\mathsf{x}) &= \int \mathrm{d}V' \frac{1}{\mathrm{i}} \left(\sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(\mathsf{x}') g(\mathsf{x}') u_{\boldsymbol{k}}(\mathsf{x}) - u_{\boldsymbol{k}}(\mathsf{x}') g(\mathsf{x}') u^*_{\boldsymbol{k}}(\mathsf{x}) \right) \\ &= \int \mathrm{d}V' \frac{1}{\mathrm{i}} \left(\sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(\mathsf{x}') u_{\boldsymbol{k}}(\mathsf{x}) - u_{\boldsymbol{k}}(\mathsf{x}') u^*_{\boldsymbol{k}}(\mathsf{x}) \right) g(\mathsf{x}') \\ &:= \int \mathrm{d}V' E(\mathsf{x}, \mathsf{x}') g(\mathsf{x}') \end{aligned} \tag{146}$$

Definition 4.9 (Smeared field observable)**.** *We define the* ***smeared field observable*** *as*

$$\phi(f) = \int \mathrm{d}V \phi(\mathsf{x}) f(\mathsf{x}) \tag{147}$$

Observation 4.13. *In addition, we should note that*

$$\begin{aligned} \phi(f + Ph) &= \int \mathrm{d}V (\phi(\mathsf{x}) f(\mathsf{x}) + \phi(\mathsf{x}) Ph(\mathsf{x})) \\ &= \int \mathrm{d}V (\phi(\mathsf{x}) f(\mathsf{x}) + P\phi(\mathsf{x}) h(\mathsf{x})) \\ &= \int \mathrm{d}V \phi(\mathsf{x}) f(\mathsf{x}) \end{aligned}$$

and so we can interpret $\phi(f)$ as the distributional form of the solution.
At the level of distribution, the equation of motion can be written as

$$\phi(Ph) = 0 \tag{148}$$

4.4 Lagrangian formulation

If the equation $P(\phi) = 0$ can be obtained from an action $S(\phi) = \int \mathrm{d}V \mathscr{L}(\phi, \nabla_\mu \phi)$, then by extremizing it,

$$P(\phi) = \frac{\partial f}{\partial \phi} - \nabla_\mu \left(\frac{\partial f}{\partial (\nabla_\mu \phi)} \right) = \frac{\delta S}{\delta \phi} \tag{149}$$

We call $\mathscr{L}$ the Lagrangian (density) of the field.

Definition 4.10 (Stress-energy tensor)**.** *The **stress-energy tensor** $T_{\mu\nu}$ is defined by*

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \tag{150}$$

Example 4.9. *For a Klein-Gordon field,*

$$\begin{aligned}
T_{\mu\nu} &= -2\bigg(\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu}\mathscr{L}\bigg) \\
&= -2\bigg(- \tfrac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2} g_{\mu\nu}(\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + V(\mathsf{x})\phi^2)\bigg) \\
&= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2} g_{\mu\nu}\bigg(\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2}V(\mathsf{x})\phi^2\bigg)
\end{aligned} \tag{151}$$

Observation 4.14. *Given a timelike trajectory $z(\tau)$ with 4-velocity u^μ, we can define vectors $e_i^\mu(\tau)$ orthogonal to $u^\mu(\tau)$ such that $\{u(\tau), e_i(\tau)\}$ is an orthonormal basis at $z(\tau)$. Then, we can use the tensor $T_{\mu\nu}$ to define observables*

$$\begin{cases}
\rho \equiv u^\mu u^\nu T_{\mu\nu} \quad \textit{(energy density)} \\
p_i \equiv -u^\mu e_i^\nu T_{\mu\nu} \quad \textit{(momentum density in direction } e_i\textit{)} \\
P_i \equiv e_i^\mu e_i^\nu T_{\mu\nu} \quad \textit{(pressure in direction } e_i\textit{)} \\
\tau_{ij} \equiv e_i^\mu e_j^\nu T_{\mu\nu} \quad \textit{(shear plane } e_i, e_j\textit{)}
\end{cases}$$

5 Algebraic Quantum Theory

5.1 Algebras

Definition 5.1 (Algebra)**.** *An **algebra** $\mathcal{A}$ is a vector space with a product operation such that*

$$\begin{cases}
(\alpha A + B)C = \alpha AC + BC \ \ \forall A, B, C \in \mathcal{A}, \forall \alpha \in \mathbb{C} \\
A(\alpha B + C) = \alpha AB + AC \\
(AB)C = A(BC)
\end{cases}$$

Example 5.1. *Lie algebras are an example of algebras.*

Definition 5.2 (Unital algebra)**.** *We say that an algebra $\mathcal{A}$ is **unital** if there exists an identity operator $\hat{\mathbb{I}} \in \mathcal{A}$ such that*

$$\mathbb{I}A = A\mathbb{I} = A \ \ \forall A \in \mathcal{A} \tag{152}$$

Definition 5.3 (Normed algebra)**.** *We say that an algebra $\mathcal{A}$ is **normed** if there exists a norm $||.|| : \mathcal{A} \to \mathbb{R}$ such that*

$$||AB|| \leq ||A||\ ||B|| \tag{153}$$

Definition 5.4 ($*$-algebra)**.** *We say an algebra $\mathcal{A}$ is a **$*$-algebra** if $* : \mathcal{A} \to \mathcal{A}$ is an operation such that*

$$\begin{cases} (\alpha A + B)^* = \alpha^* A^* + B^* \\ \\ (AB)^* = B^* A^* \end{cases}$$

and we will denote $A^\dagger \equiv A^*$.

Definition 5.5 (C^*-algebra)**.** *We say an algebra $\mathcal{A}$ is a C^***-algebra** if it's a $*$-algebra and normed with*

$$||A^*|| = ||A|| \quad \forall A \in \mathcal{A} \tag{154}$$

Definition 5.6 (Set of generators)**.** *A set $\{\hat{A}_i\} \subseteq \mathcal{A}$ is called a **set of generators** if $G(\{\hat{A}_i\})$ (the set of all possible products of $\{\hat{A}_i\}$) contains a basis of $\mathcal{A}$. In other words, if*

$$\mathcal{A} = span(G(\{\hat{A}_i\})) \tag{155}$$

Example 5.2. *If $\{\hat{A}_i\} = \{\hat{A}_1, \hat{A}_2\}$, then $G(\{\hat{A}_i\}) = \{\hat{A}_1, \hat{A}_2, \hat{A}_1\hat{A}_2, \hat{A}_2\hat{A}_1, \hat{A}_1^2, \hat{A}_2^2, \hat{A}_1^2\hat{A}_2, \dots\}$.*

Example 5.3. *Let $\{\mathbb{I}, X, Y\}$ be generators for an algebra $\mathcal{A}$ such that $XY = -YX$, $X^2 = Y^2 = \mathbb{I}$. Then,*

$$G(\{\mathbb{I}, X, Y\}) = \{\mathbb{I}, X, Y, XY, YX, X^2Y, Y^2X, \dots\}$$

$$= \{\mathbb{I}, X, Y, XY, -XY, Y, X, \dots\}$$

so

$$spanG = span(\{\mathbb{I}, X, Y, XY\})$$

If we define $XY \equiv \mathrm{i}Z$, then we find the commutation relations

$$\begin{cases} [X, Y] = 2\mathrm{i}Z \\ [Y, Z] = \mathrm{i}(YXY - XYY) = 2\mathrm{i}X[Z, X] = 2\mathrm{i}Y. \end{cases}$$

and, for $A \in \mathcal{A}$, $A = A_0\mathbb{I} + a_1X + a_2Y + a_3Z$. If we identity these elements to the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \ Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then the commutation relations are satisfied and match $\mathcal{A} = \mathcal{L}(\mathbb{C}^2)$.

Theorem 5.1. *An algebra is entirely determined by the commutator between its elements.*

Definition 5.7 (Subalgebra)**.** *Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be an algebra. Then $\tilde{\mathcal{A}}$ is called a **subalgebra**.*

Observation 5.1. *A subalgebra $\tilde{\mathcal{A}}$ inherits an algebra structure: $\tilde{A}\tilde{B} \in \tilde{\mathcal{A}}\ \forall \tilde{A}, \tilde{B} \in \tilde{\mathcal{A}}$.*

Example 5.4. *The only non-trivial subalgebra of $\mathcal{L}(\mathbb{C}^2)$ is $\{\alpha\mathbb{I} + \beta A : A \neq a\mathbb{I}\}$*

Definition 5.8 (Commutant)**.** *Let $\tilde{A}$ be a subalgebra. We define the* ***commutant*** *$\tilde{A}'$ as*

$$\tilde{A}' = \{B \in \mathcal{A} : [A, B] = 0\ \ \forall A \in \mathcal{A}\} \tag{156}$$

Definition 5.9 (Tensor product algebra)**.** *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two algebras, we define their tensor product as the algebra generated by*

$$\{A_1 \otimes \mathbb{I}_2, \mathbb{I}_1 \otimes A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\} \tag{157}$$

and we define

$$(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1B_1 \otimes A_2B_2,\ \ A_1, B_1 \in \mathcal{A}_1, A_2, B_2 \in \mathcal{A}_2 \tag{158}$$

Observation 5.2. *The tensor product is linear:*

$$(\alpha A_1 + B_1) \otimes B_2 = \alpha A_1 \otimes B_2 + B_1 \otimes B_2 \tag{159}$$

Definition 5.10 (Self-adjointness)**.** *An element $A \in \mathcal{A}$ such that $A = A^\dagger$ is called* ***self-adjoint****.*

Definition 5.11 (Unitarity)**.** *If $U \in \mathcal{A}$ such that $U^\dagger U = UU^\dagger = \mathbb{I}$, then U is called* ***unitary****.*

Definition 5.12 (Algebra morphisms)**.** *Let $\mathcal{E} : \mathcal{A}_1 \to \mathcal{A}_2$ be linear such that*

$$\mathcal{E}(A_1, B_1) = \mathcal{E}(A_1)\mathcal{E}(B_1) \tag{160}$$

then $\mathcal{E}$ is called an ***algebra homomorphism****.*

In addition, if the inverse $\mathcal{E}^{-1}$ exists and is also a homomorphism, then $\mathcal{E}$ is an ***algebra isomorphism****.*

Observation 5.3. *If $\mathcal{E}(A^*) = A^*$, $\forall A \in \mathcal{A}$, then $\mathcal{E}$ is a* ***$*$-isomorphism****.*

Moreover, if $||\mathcal{E}(A)|| = ||A||$, then $\mathcal{E}$ is a ***normed algebra isomorphism****.*

Definition 5.13 (Faithful representation)**.** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ a $*$-algebra and $\mathcal{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ a $*$-algebra isomorphism. Then, we say that $(\mathcal{A}, \mathcal{H}, \mathcal{E})$ defines a* ***faithful representation****.*

Example 5.5. *$\mathcal{E}(A) = 0\ \ \forall A \in \mathcal{A}$ is called the trivial representation.*

5.2 Algebraic Quantum Theory

5.2.1 Postulates

Postulate 1 A quantum system is associated to a $*$-algebra $\mathcal{A}$.

Postulate 2 A state is a linear function $\omega : \mathcal{A} \to \mathbb{C}$ such that

$$\begin{cases} \omega(\mathbb{I}) = 1 \\ \\ \omega(\hat{A}^\dagger \hat{A}) \geq 0 \end{cases}$$

The expected value of $\hat{A}$ is $\omega(\hat{A})$.

Observation 5.4. *This corresponds exactly to what we had in the "normal" formulation, and a state was a map* $|\psi\rangle : \hat{A} \to \langle\psi| \hat{A} |\psi\rangle$ *with*

$$\begin{cases} \mathbb{I} \to 1 \\ \\ \hat{A}^\dagger \hat{A} \to \langle\psi| \hat{A}^\dagger \hat{A} |\psi\rangle \geq 0 \end{cases}$$

Postulate 3 Observables are associated with self-adjoint operators.

Observation 5.5 (Density operator)**.** *A* ***density operator*** *is now a map* $\hat{\rho} : \hat{A} \to \mathrm{Tr}\left(\hat{\rho}\hat{A}\right)$ *and we can see it verifies the previous properties*

$$\begin{cases} \mathbb{I} \to \mathrm{Tr}(\hat{\rho}) = 1 \\ \\ \hat{A}^\dagger \hat{A} \to \mathrm{Tr}\left(\hat{\rho}\hat{A}^\dagger \hat{A}\right) \geq 0 \end{cases}$$

Observation 5.6. *If we decompose* $\mathbb{I} = \sum_i \hat{P}_i$*, then* $\omega(\mathbb{I})) = \sum_i \omega(\hat{P}_i) = 1$ *and* $\omega(\hat{P}_i)$ *gives the probability of each outcome.*

Example 5.6. *Let* $U \in \mathcal{A}$ *be unitary. Then* $\mathcal{E}(A) = U^\dagger A U$ *is a* $*$*-isomorphism.*

Consider the state $\omega(.) \to \omega(U^\dagger.U)$. In the Schrödinger picture, this state evolves as

$$\omega_t(\hat{A}) = \omega_0(U^\dagger(t)\hat{A}U(t)) = \omega_0(\hat{A}(t)) \tag{161}$$

but, in general, it is hard to talk about ω_t as it requirse to talk about its action on all operators. Therefore, it is more convenient to directly update the operators and just use the ω_0 which we already have.

If $\omega_0(\hat{A}) = \mathrm{Tr}\left(\hat{\rho}_0\hat{A}\right)$, then

$$\begin{aligned} \omega_t(\hat{A}) &= \omega_0(\hat{U}^\dagger(t)\hat{A}\hat{U}(t)) \\ \\ &= \mathrm{Tr}\left(\hat{\rho}_0\hat{U}^\dagger(t)\hat{A}\hat{U}(t)\right) \\ \\ &= \mathrm{Tr}\left(\hat{U}(t)\hat{\rho}_0\hat{U}^\dagger(t)\hat{A}\right) \end{aligned}$$

$$= \mathrm{Tr}\Big(\hat{\rho}(t)\hat{A}\Big)$$

In the algebraic formulation, instead of updating the expected value (state in the Schrödinger picture, ω_t here), we update the observables ($\hat{A}$) of which we take the expectation value. **[[I am not sure what you mean by this sentence. In all cases the only meaningful quantity to be updated is the expected value, whether you decide to update the states or the algebras is a choice]]**.

Definition 5.14 (Mixed and pure states)**.** *Let ω be a state. If there exists states ω_1 and ω_2 such that $\omega = p_1\omega_1 + p_2\omega_2$ (where p_1, p_2 are probabilities), then ω is said to be a **mixed state**. Otherwise, it is **pure**.*

6 Algebraic Quantum Field Theory

6.1 Introduction

Definition 6.1 (Quantum field theory)**.** *A **quantum field theory** is an association $\mathcal{A} : \mathcal{O} \to \mathcal{A}(\mathcal{O})$ from causally convex subsets $\mathcal{O} \subseteq \mathcal{M}$ to $*$-algebras of (not necessarily self-adjoint) observables, satisfying*

- *$\mathcal{A}(\mathcal{O})$ is a unital $*$-algebra for each $\mathcal{O}$ and $\mathcal{A}(\mathcal{M})$ is generated by the combination of all the $\mathcal{A}(\mathcal{O})$*

$$\mathcal{O} \xrightarrow{\quad\quad\mathcal{A}\quad\quad} \mathcal{A}(\mathcal{O}) = \{\mathbb{I}, \dots\}$$

- *If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ ($\mathcal{A}_1 \subseteq \mathcal{A}_2$).*

- *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then*

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0 \tag{162}$$

This condition is known as microcausality.

- *If $\mathcal{O}_1$ contains a Cauchy surface of $\mathcal{O}_2$ ($\mathcal{O}_2 \subseteq D(\Sigma)$), then $\mathcal{A}(\mathcal{O}_1) = \mathcal{A}(\mathcal{O}_2)$. Therefore, we only need to study the causal diamonds.*

Observation 6.1. *Given any $\mathcal{O} \subseteq \mathcal{M}$, we can define*

$$\mathcal{A} : \mathcal{O} \to \mathcal{A}(D(\mathcal{O})) = \mathcal{A}(\bar{\mathcal{O}}) \tag{163}$$

Observation 6.2. *A state is a linear functional $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ such that $\omega(\hat{A}^\dagger\hat{A}) \geq 0$ and $\omega(\mathbb{I}) = 1$.*

Observation 6.3. *There are, in principle, no local states in QFT.*

Observation 6.4. *The above formulation naturally incorporates time evolution to the observables.*

Example 6.1. *Say that someone interacts with a field in a region $\mathcal{O}$ with a unitary operator $\hat{U} \in \mathcal{A}(\mathcal{O})$ such that a state is updated following*

$$\omega(.) \to \omega(\hat{U}^\dagger.\hat{U})$$

Equivalently, one could update all observables

$$\hat{A} \to \hat{U}^\dagger \hat{A}\hat{U}$$

If the operation is localized in $\mathcal{O}$, then the state should not change outside of $J^+(\mathcal{O})$. However, the states are global. If $\hat{A}' \in \mathcal{A}(\mathcal{O}')$, where $\mathcal{O}' \cap J(\mathcal{O}) = \emptyset$, then, following microcausality, $\omega(\hat{A}') = \omega(\hat{U}^\dagger \hat{A}'\hat{U})$

Observation 6.5. *Operations in $\mathcal{O}$ do not change the expected values of observables spacelike to $\mathcal{O}$.*

Example 6.2. *Consider a classical Klein-Gordon theory, where observables are functions $O \in C^\infty(S)$. Notice that:*

- *$C^\infty(S)$ is a $*$-algebra:*

$$\begin{cases} (O_1O_2)\phi \equiv O_1(\phi)O_2(\phi) \implies [O_1, O_2] = 0 \\ \\ O_1^\dagger \equiv (O_1(\phi))^* \\ \\ \mathbb{I}(\phi) = 1 \end{cases}$$

  for $O_1, O_2 : S \to \mathbb{C}$.

- *$C^\infty(S)$ is generated by linear functionals*

$$f(\phi) = \int \mathrm{d}V \phi(\mathsf{x})f(\mathsf{x}) \equiv \phi(f)$$

Therefore, a set of generators is $\{1, \phi(f) : f \in C_0^\infty(\mathcal{M})\}$.

Then, the association $\mathcal{O} \to \mathcal{A}(\mathcal{O})$, where $\mathcal{A}(\mathcal{O})$ is generated by $\{1, \phi(f) : f \in C_0^\infty(\mathcal{M})\}$, defines an AQFT.

[[You didn't mention anything about representations here]]

6.2 AQFT for a real scalar field

6.2.1 Basics

Consider a Klein-Gordon field ϕ with equation of motion $P\phi = 0$ and causal propagator E. Let the association $f \to \hat{\phi}(f)$ for $f \in C_0^\infty(\mathcal{M})$, satisfying

$$\begin{cases} \hat{\phi}(\alpha f + g) = \alpha\hat{\phi}(f) + \hat{\phi}(g) \\ (\hat{\phi}(g))^\dagger = \hat{\phi}(f^*) \\ \hat{\phi}(f + Ph) = \hat{\phi}(f) \;\; f, h \in C_0^\infty(\mathcal{M}) \\ [\hat{\phi}(f), \hat{\phi}(g)] = \mathrm{i}E(f, g) \end{cases}$$

The association $\mathcal{O} \to \mathcal{A}(\mathcal{O})$, where $\mathcal{A}(\mathcal{O})$ is generated by $\{1, \phi(f) : f \in C_0^\infty(\mathcal{M})\}$, matches the AQFT defiition for a real scalar quantum field.

Observation 6.6. *We can think of $\hat{\phi}(f)$ as*

$$\hat{\phi}(f) = \int \mathrm{d}V \hat{\phi}(\mathsf{x}) f(\mathsf{x}) \tag{164}$$

where $\hat{\phi}(\mathsf{x})$ is the field used in "classical" QFT, which is an ill-defined operator.

In such cases, $\hat{\phi}$ can be thought of as an "operator-valued distribution" $\hat{\phi} : C_0^\infty(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$, satisfying $\hat{\phi}^\dagger(\mathsf{x}) = \hat{\phi}(\mathsf{x})$ and the equations of motion $P\hat{\phi}(\mathsf{x}) = 0$.

Observation 6.7. *We can recover the* ***equal times commutation relations****: given Σ,*

$$[\hat{\phi}(\mathsf{x})\Big|_\Sigma, n^\mu \nabla_\mu \hat{\phi}(\mathsf{x}')\Big|_\Sigma] = \mathrm{i}\delta_\Sigma(\mathsf{x}, \mathsf{x}') \tag{165}$$

Observation 6.8. *The function f defines the region where one has access to the field, and they can measure observables generated by linear combinations of $\hat{\phi}(f)$.*

Observation 6.9. *If we define $\hat{\pi}(f) \equiv n^\mu \nabla_\mu \hat{\phi}(f)$, then Eq. (120) tells us that, if we know $\hat{\phi}(f)$, then we know $\hat{\pi}(f)$.*

6.3 States

We defined a state as a linear map $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$.
Let us now consider the map $f \to \hat{\phi}(f)$. Then, the map $f \to \omega(\hat{\phi}(f))$ maps $C_0^\infty(\mathcal{M}) \to \mathbb{C}$ and is linear: it is then a **distribution**:

$$\omega(\hat{\phi}(f)) = \int \mathrm{d}V W_1(\mathsf{x}) f(\mathsf{x}) \tag{166}$$

At the same time, $f, g \to \omega(\hat{\phi}(f)\hat{\phi}(g))$ is bilinear, and so it is a **bidistribution**:

$$W_2(f, g) \equiv \omega(\hat{\phi}(f)\hat{\phi}(g)) = \int \mathrm{d}V \mathrm{d}V' W_2(\mathsf{x}, \mathsf{x}') f(\mathsf{x}) g(\mathsf{x}') \tag{167}$$

Observation 6.10. *We can continue iterating the same procedure and define an n-distribution*

$$\omega(\hat{\phi}(f_1) \dots \hat{\phi}(f_n)) = \int \mathrm{d}V_1 \dots \mathrm{d}V_n W_n(\mathsf{x}_1, \dots, \mathsf{x}_n) f_1(\mathsf{x}_1) \dots f_n(\mathsf{x}_n) \tag{168}$$

Definition 6.2 (Quasi-free state)**.** *Let ω be a state such that*

$$\omega(\hat{\phi}(f_1) \dots \hat{\phi}(f_{2n+1})) = 0 \;\; \forall n \in \mathbb{N} \tag{169}$$

and that satisfies Wick contractions for even products of $\hat{\phi}(f)$. Then ω is called a ***quasi-free state****.*

Example 6.3. *Consider a quasi-free state $\omega(\hat{\phi}(f_1)\hat{\phi}(f_2)\hat{\phi}(f_3)\hat{\phi}(f_4))$. Then, Wick's contraction rule gives*

$$\begin{aligned}\omega(\hat{\phi}(f_1)\hat{\phi}(f_2)\hat{\phi}(f_3)\hat{\phi}(f_4)) &= \omega(\hat{\phi}(f_1)\hat{\phi}(f_2))\omega(\hat{\phi}(f_3)\hat{\phi}(f_4)) \\ &\quad + \omega(\hat{\phi}(f_1)\hat{\phi}(f_3))\omega(\hat{\phi}(f_2)\hat{\phi}(f_4)) + \omega(\hat{\phi}(f_1)\hat{\phi}(f_4))\omega(\hat{\phi}(f_2)\hat{\phi}(f_3))\end{aligned}$$

Observation 6.11. *Quasi-free states are* ***fully*** *defined by W_2.*

Observation 6.12. *The standard vacua, thermal states and particles states are all quasifree particles.*

Convention The distribution W_2 is denoted simply by W and is called the **Wightman function**.

Observation 6.13. *The Wightman function of two test functions $W(f,g)$ encodes the correlations between the field operator smeared through these functions $\hat{\phi}(f), \hat{\phi}(g)$.*

Observation 6.14. *Consider the Wightman function. It can be written as*

$$\begin{aligned}W(f,g) &= \omega(\hat{\phi}(f)\hat{\phi}(g)) \\ &= \omega\left(\frac{1}{2}\{\hat{\phi}(f),\hat{\phi}(g)\} + \frac{1}{2}[\hat{\phi}(f),\hat{\phi}(g)]\right) \\ &= \frac{1}{2}\omega\left(\{\hat{\phi}(f),\hat{\phi}(g)\}\right) + \frac{1}{2}\omega\left([\hat{\phi}(f),\hat{\phi}(g)]\right) \\ &= \frac{1}{2}\omega\left(\{\hat{\phi}(f),\hat{\phi}(g)\}\right) + \frac{1}{2}\mathrm{i}E(f,g)\omega(\mathbb{I}) \\ &\equiv \frac{1}{2}H_\omega(f,g) + \frac{1}{2}\mathrm{i}E(f,g) \qquad (170)\end{aligned}$$

where we have introduced the ***Hadamard bidistribution*** *$H_\omega(f,g)$.*

Definition 6.3. *Given that $E(f,g)$ is state independent, the Hadamard bidistribution defines entirely a quasi-free state.*

Observation 6.15. *Since the Hadamard function is defined by means of the anticommutator, it is then symmetric and so*

$$\begin{aligned}W(g,f) &= \frac{1}{2}H(g,f) + \frac{1}{2}\mathrm{i}E(g,f) \\ &= \frac{1}{2}H(f,g) - \frac{1}{2}\mathrm{i}E(f,g)\end{aligned}$$

$$\implies W(g^*, f^*) = \frac{1}{2}H(f^*, g^*) - \frac{1}{2}\mathrm{i}E(f^*, g^*)$$

$$= (W(f, g))^* \tag{171}$$

Observation 6.16. *H and W cannot just be any functions. Indeed, we require $\hat{\phi}(Ph) = 0 \ \forall h \in C_0^\infty(\mathcal{M})$, which translates to*

$$W(Ph_1, g) = W(f, Ph_2) \stackrel{!}{=} 0$$

$$\iff \begin{cases} P_{\mathsf{x}} W(\mathsf{x}, \mathsf{x}') = 0 \\ P_{\mathsf{x}'} W(\mathsf{x}, \mathsf{x}') = 0 \end{cases}$$

*Hence, $H(\mathsf{x}, \mathsf{x}')$ and $W(\mathsf{x}, \mathsf{x}')$ must be **bi-solutions** of the equations of motion.*

Observation 6.17. *If $\phi_0(\mathsf{x})$ is a classical solution, then $H(\mathsf{x}, \mathsf{x}') = \phi_0(\mathsf{x})\phi_0(\mathsf{x}')$ is also a solution.*

More generally, given any solutions $\phi_n(\mathsf{x})$,

$$H(\mathsf{x}, \mathsf{x}') = \sum_{n=0}^{\infty} h_n \mathfrak{Re}\{\phi_n(\mathsf{x})\phi_n^*(\mathsf{x}')\} \tag{172}$$

is an acceptable H if the coefficients $\{h_n\}$ are real.

Observation 6.18. *The support of $H(\mathsf{x}, \mathsf{x}')$ is not restricted by causality as the compatibility between measurements is determined by the commutator (and not the anticommutator).*

Observation 6.19. *If f and g are causally disconnected, then $E(f, g) = 0$ and $H(f, g)$ encodes the correlations between $\hat{\phi}(f)$ and $\hat{\phi}(g)$.*

Observation 6.20. *$\hat{\phi}(f)$ is always an unbounded operator such that its spectrum is $\mathbb{R}$.*

Definition 6.4 (Feynman propagator)**.** *The **Feynman propagator** is defined as*

$$G_F(\mathsf{x}, \mathsf{x}') \equiv W(\mathsf{x}, \mathsf{x}')\theta(t - t') + W(\mathsf{x}', \mathsf{x})\theta(t' - t) \tag{173}$$

$$\tag{174}$$

$$= \omega(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\theta(t - t') + \hat{\phi}(\mathsf{x}')\hat{\phi}(\mathsf{x})\theta(t' - t)) \tag{175}$$

where t is any positive oriented time coordinate.

Observation 6.21. *We can see that $G_F(f, g) \neq W(f, g)\theta(t - t')$, so the time ordering needs to happen inside the integral.*

Observation 6.22. *We can write the Feynman propagator in terms of the causal propagator:*

$$G_F(\mathsf{x},\mathsf{x}') = \frac{1}{2}H(\mathsf{x},\mathsf{x}') + \frac{1}{2}\mathrm{i}[E(\mathsf{x},\mathsf{x}')\theta(t-t') + E(\mathsf{x}',\mathsf{x})\theta(t'-t)]$$

$$= \frac{1}{2}H(\mathsf{x},\mathsf{x}') + \frac{1}{2}\mathrm{i}[G_R(\mathsf{x},\mathsf{x}') - E(\mathsf{x},\mathsf{x}')\theta(t'-t)]$$

$$= \frac{1}{2}H(\mathsf{x},\mathsf{x}') + \frac{1}{2}\mathrm{i}[G_R(\mathsf{x},\mathsf{x}') - (-G_{\mathrm{A}}(\mathsf{x},\mathsf{x}'))]$$

$$\Longrightarrow \; G_F(f,g) = \frac{1}{2}H(f,g) + \frac{1}{2}\mathrm{i}\Delta(f,g) \tag{176}$$

where $\Delta(\mathsf{x},\mathsf{x}') = G_R(\mathsf{x},\mathsf{x}') + G_{\mathrm{A}}(\mathsf{x},\mathsf{x}')$ *is the* ***symmetric propagator****.*

Observation 6.23. *Notice that*

$$G_{\mathrm{A}}(\mathsf{x},\mathsf{x}') = G_R(\mathsf{x},\mathsf{x}') \implies \Delta(\mathsf{x},\mathsf{x}') = \Delta(\mathsf{x}',\mathsf{x})$$

and, therefore,

$$G_F(\mathsf{x},\mathsf{x}') = G_F(\mathsf{x},\mathsf{x}) \tag{177}$$

6.4 Coherent states

Let ω be a quasi-free state. Then, $\omega(\hat{\phi}(t)) = 0$.
Let $f_0 \in C_0^\infty(\mathcal{M})$, with $\phi_0 = E\, f_0$. Then, we define the unitary operation:

$$\omega(.) \mapsto \omega_{f_0} \equiv \omega(\mathrm{e}^{-\mathrm{i}\hat{\phi}(f_0)}.\mathrm{e}^{\mathrm{i}\hat{\phi}(f_0)}) \tag{178}$$

Observation 6.24. *We can compute* $\mathrm{e}^{-\mathrm{i}\hat{\phi}(f_0)}\hat{\phi}(f)\mathrm{e}^{\mathrm{i}\hat{\phi}(f_0)}$ *for a certain* $\hat{\phi}(f)$*:*

$$\mathrm{e}^{-\mathrm{i}\hat{\phi}(f_0)}\hat{\phi}(f)\mathrm{e}^{\mathrm{i}\hat{\phi}(f_0)} = \hat{\phi}(f) - \mathrm{i}[\hat{\phi}(f_0),\hat{\phi}(f)] + \dots$$

$$= \hat{\phi}(f) - \mathrm{i}E(f_0,f)\hat{\mathbb{I}}$$

$$= \hat{\phi}(f) + E(f_0,f)\hat{\mathbb{I}}$$

$$= \hat{\phi}(f) - E(f,f_0)\hat{\mathbb{I}}$$

$$= \hat{\phi}(f) - Ef_0(f)\hat{\mathbb{I}}$$

$$= \hat{\phi}(f) - \phi_0(f)\hat{\mathbb{I}}$$

$$\Longrightarrow \; w_{f_0}(\hat{\phi}(f)) = \phi_0(f) \tag{179}$$

6.5 Basis of solutions

6.5.1 Basis

We will build a quasi-free state ω. In order to do so, let us consider a basis $\{u_{\boldsymbol{k}}, u^*_{\boldsymbol{k}}\}$ of S and define

$$W(\mathsf{x}, \mathsf{x}') = \sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(\mathsf{x}) u^*_{\boldsymbol{k}}(\mathsf{x}') \tag{180}$$

Observation 6.25. *This sum (typically) diverges when* $\mathsf{x} = \mathsf{x}'$, *but* $W(f, g) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(f) u^*_{\boldsymbol{k}}(g)$ *converges for all* $f, g \in C_0^\infty(\mathcal{M})$

Observation 6.26. *Because* $\{u_{\boldsymbol{k}}, u^*_{\boldsymbol{k}}\}$ *is a basis of* S*, then* $P_{\mathsf{x}} u_{\boldsymbol{k}}(\mathsf{x}) = P_{\mathsf{x}'} W(\mathsf{x}, \mathsf{x}') = 0$ *and therefore*

$$P_{\mathsf{x}} W(\mathsf{x}, \mathsf{x}') = P_{\mathsf{x}'} W(\mathsf{x}, \mathsf{x}') = 0 \tag{181}$$

Observation 6.27. *We can retrieve Eq.* (146)

$$\begin{aligned} W(\mathsf{x}, \mathsf{x}') - W(\mathsf{x}', \mathsf{x}) &= \mathrm{i}\left(\frac{1}{\mathrm{i}} \sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(\mathsf{x}) u^*_{\boldsymbol{k}}(\mathsf{x}') - u_{\boldsymbol{k}}(\mathsf{x}') u^*_{\boldsymbol{k}}(\mathsf{x})\right) \\ &= \mathrm{i} E(\mathsf{x}, \mathsf{x}') \end{aligned} \tag{182}$$

All previous three observations imply that this $W(\mathsf{x}, \mathsf{x}')$ is a valid Wightman function and defines a state ω.

Observation 6.28. *It is important to note that we are making a* ***choice*** *by selecting a basis! In Minkowski spacetime, there is a preferred choice, but this is not the case in curved spacetimes.*

Theorem 6.1 (GNS representation)**.** *If* $\mathcal{A}$ *is a* $*$*-algebra and* ω *is a state, then there exists*

$$\begin{cases} \textit{a Hilbert space } \mathcal{F}(\mathcal{H}) \\ \textit{a unit vector } |\Omega\rangle \in \mathcal{F}(\mathcal{H}) \\ \textit{a faithful representation } \pi_\omega : \mathcal{A} :\to \mathcal{L}(\mathcal{F}(\mathcal{H})) \end{cases}$$

such that

$$\begin{cases} \pi_\omega(\mathcal{A}) \, |\Omega\rangle \ \textit{ is dense in } \mathcal{F}(\mathcal{H}) \\ \omega(\hat{A}) = \langle\Omega| \, \pi_\omega(\hat{A}) \, |\Omega\rangle \quad \forall \hat{A} \in \mathcal{A} \end{cases}$$

The state $|\Omega\rangle$ *is interpreted as the ground state or vacuum.*

Observation 6.29. *For the state* ω *defined by* $\{u_{\boldsymbol{k}}\}$*, it turns out there is a faithful representation.*

Remember that $S = S^+ \oplus S^-$ and that, therefore, we can decompose a solution $\phi \in S$ as

$$\phi(\mathsf{x}) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\mathsf{x}) + a^*_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(\mathsf{x}) \tag{183}$$

The portion of $\phi(\mathsf{x})$ in S^+ is simply

$$\phi^+(\mathsf{x}) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\mathsf{x}) \tag{184}$$

If we define a map $K : S \to S^+$, $\phi = Ef \mapsto \phi^+$, for any given $f \in C_0^\infty(\mathcal{M})/PC_0^\infty(\mathcal{M})$, then we can write an element of S^+ as

$$\phi^+ = KEf = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(f) u_{\boldsymbol{k}} \tag{185}$$

In S^+,

$$\begin{aligned}
(KEf, KEg) &= \left(\sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(f) u_{\boldsymbol{k}}, \sum\!\!\!\!\!\!\int_{\boldsymbol{k}'} u^*_{\boldsymbol{k}'}(g) u_{\boldsymbol{k}'} \right) \\
&= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \sum\!\!\!\!\!\!\int_{\boldsymbol{k}'} u_{\boldsymbol{k}}(f^*) u^*_{\boldsymbol{k}'}(g) (u_{\boldsymbol{k}}, u_{\boldsymbol{k}'}) \\
&= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \sum\!\!\!\!\!\!\int_{\boldsymbol{k}'} u_{\boldsymbol{k}}(f^*) u^*_{\boldsymbol{k}'}(g) \delta_{\boldsymbol{k}\boldsymbol{k}'} \\
&= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(f^*) u^*_{\boldsymbol{k}}(g) \\
&= W(f^*, g)
\end{aligned} \tag{186}$$

Observation 6.30. *This gives an inner product for a Hilbert space $\mathcal{H} = C_0^\infty(\mathcal{M})/PC_0^\infty(\mathcal{M})$:*

$$\langle f|g \rangle \coloneqq W(f^*, g) \tag{187}$$

Definition 6.5 (Fock space)**.** *Given a Hilbert space $\mathcal{H}$ as defined in Obs. 6.30, we can define the space consisting of n symmetric tensor products of $\mathcal{H}$*

$$\mathcal{H}^{(n)} = \mathcal{H} \odot \cdots \odot \mathcal{H} \tag{188}$$

where we set $\mathcal{H}^{(0)} \equiv \mathbb{C}$.

*We can then define the **Fock space** $\mathcal{F}(\mathcal{H})$ as*

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \tag{189}$$

and a vector $|\psi\rangle \in \mathcal{F}(\mathcal{H})$ can be written as a list

$$|\psi\rangle = (\alpha, f, f_1 \odot f_2, \dots) \tag{190}$$

where $\alpha \in \mathcal{H}^{(0)} = \mathbb{C}$, $f \in \mathcal{H}^{(1)} = C_0^\infty(\mathcal{M})/PC_0^\infty(\mathcal{M})$, ...

Observation 6.31. *The Fock space is endowed with the inner product defined in Obs. 6.30:*

$$\langle\psi|\psi\rangle = ||\alpha||^2 + W(f^*, f) + W(f_1^*, f_1)W(f_2^*, f_2) + \dots \tag{191}$$

Definition 6.6 (Creation/annihilation operators)**.** *We can define a couple of operators $\hat{a}, \hat{a}^\dagger$: $C_0^\infty(\mathcal{M})/PC_0^\infty(\mathcal{M}) \to \mathcal{L}(\mathcal{F}(\mathcal{H}))$, with $(\hat{a}(g))^\dagger = \hat{a}^\dagger(g^*)$, such that*

$$\begin{aligned} \hat{a}^\dagger(g)\,|\psi\rangle &= |g\rangle \odot |\psi\rangle \\ &= (0, \alpha g, g \odot f, \dots) \end{aligned} \tag{192}$$

satisfying

$$\begin{cases} [\hat{a}(f), \hat{a}^\dagger(g)] = W(f, g)\ \hat{\mathbb{I}}_{\mathcal{F}(\mathcal{H})} \\ [\hat{a}^\dagger(f), \hat{a}^\dagger(g)] = [\hat{a}(f), \hat{a}(g)] = 0. \end{cases}$$

Observation 6.32. *These operators are defined over the whole manifold ($\hat{a}, \hat{a}^\dagger \in \mathcal{A}(\mathcal{M})$) and therefore they are **global**.*

Observation 6.33. *There exists a representation π_ω such that*

$$\pi_\omega(\hat{\phi}(f)) = \hat{a}^\dagger(f) + \hat{a}(f) \tag{193}$$

Observation 6.34. *Any state in $\mathcal{F}(\mathcal{H})$ can be built by successively applying $\hat{a}^\dagger(f)$ to $|\Omega\rangle$.*

Observation 6.35. *Using the commutation relations of the creation and annihilation operators, we can compute the commutator of the field operator with itself:*

$$\begin{aligned} [\hat{\phi}(f), \hat{\phi}(g)] &= [\hat{a}^\dagger(f), \hat{a}(g)] + [\hat{a}(f), \hat{a}^\dagger(g)] \\ &= \hat{\mathbb{I}}\ (W(f, g)W(g, f)) \\ &= \mathrm{i}E(f, g)\ \hat{\mathbb{I}} \end{aligned} \tag{194}$$

Observation 6.36. *One can think of these operators as "operator-valued distributions":*

$$\hat{a}(f) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(f)\hat{a}_{\boldsymbol{k}} \tag{195}$$

and so we can define

$$|f\rangle = \hat{a}^\dagger(f)\,|\Omega\rangle$$

$$
= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}^*(f)\hat{a}_{\boldsymbol{k}}^\dagger |\Omega\rangle
$$

$$
=: \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \tilde{f}(\boldsymbol{k}) |\boldsymbol{k}\rangle \tag{196}
$$

where we have introduced

$$
\tilde{f}(\boldsymbol{k}) \equiv \int \mathrm{d}V \, u_{\boldsymbol{k}}(\mathsf{x}) f(\mathsf{x}), \quad |\boldsymbol{k}\rangle = \hat{a}_{\boldsymbol{k}}^\dagger |\Omega\rangle \tag{197}
$$

However, $|\boldsymbol{k}\rangle$ *is ill-defined, as* $\langle \boldsymbol{k}|\boldsymbol{k}\rangle = \delta^{(3)}(\mathbf{0}) = \infty$.

In order for these functions to be valid, we need them to be normalized:

$$
(f, f) = \langle\Omega| \hat{a}(f^*)\hat{a}^\dagger(f) |\Omega\rangle
$$

$$
= \langle\Omega| [\hat{a}(f^*), \hat{a}^\dagger(f)] |\Omega\rangle
$$

$$
= W(f^*, f)
$$

$$
= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}(f^*) u_{\boldsymbol{k}}^*(f)
$$

$$
= \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} |\tilde{f}(\boldsymbol{k})|^2 \stackrel{!}{=} 1 \tag{198}
$$

Observation 6.37.

$$
\hat{\phi}(f) |\Omega\rangle = \hat{a}^\dagger(f) |\Omega\rangle = |f\rangle \tag{199}
$$

Observation 6.38.

$$
\pi_\omega(\hat{\phi}(f)) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}^*(f)\hat{a}_{\boldsymbol{k}}^\dagger + u_{\boldsymbol{k}}(f)\hat{a}_{\boldsymbol{k}}
$$

$$
= \int \mathrm{d}V \left(\sum\!\!\!\!\!\!\int_{\boldsymbol{k}} u_{\boldsymbol{k}}^*(\mathsf{x})\hat{a}_{\boldsymbol{k}}^\dagger + u_{\boldsymbol{k}}(\mathsf{x})\hat{a}_{\boldsymbol{k}} \right) f(\mathsf{x})
$$

$$
=: \int \mathrm{d}V \, \text{``}\hat{\phi(\mathsf{x})}\text{''} f(\mathsf{x}) \tag{200}
$$

Definition 6.7 (Number operator)**.** *We define the* ***number operator*** *(within* $\mathcal{H}^{(n)}$*) as*

$$
\hat{N} = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} \tag{201}
$$

and it returns

$$\hat{N}\,|\psi\rangle = n\,|\psi\rangle\,,\ \ |\psi\rangle \in \mathcal{H}^{(n)} \tag{202}$$

Observation 6.39. *The number operator does not exist as a standalone entity. Indeed, it depends completely on the choice of basis for the Hilbert space.*

Observation 6.40. *The state ω is quasi-free and pure.*

6.5.2 Change of basis

Consider a new basis $\{v_{\boldsymbol{k}}, v^*_{\boldsymbol{k}}\}$ and a state $\tilde{\omega}$ defined by

$$\tilde{W}(\mathsf{x},\mathsf{x}') = \sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} v_{\boldsymbol{k}}(\mathsf{x})v^*_{\boldsymbol{k}}(\mathsf{x}') \tag{203}$$

We can do the same construction as before and find some operators $\hat{b}(f), \hat{b}^\dagger(f)$, defined analogously to $\hat{a}$ and $\hat{a}^\dagger$ through the commutation relations giving the Wightman function $\tilde{W}$ and its actions on vectors in $\mathcal{F}(\tilde{\mathcal{H}})$. They can be written as

$$\hat{b}(f) = \sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} v_{\boldsymbol{k}}(f)\hat{b}_{\boldsymbol{k}} \tag{204}$$

such that

$$\begin{aligned} \pi_{\tilde{\omega}}(\hat{\phi}(f)) &= \hat{b}(f) + \hat{b}^\dagger(f) \\ &= \int \mathrm{d}V \left(\sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} v^*_{\boldsymbol{k}}(\mathsf{x})\hat{b}^\dagger_{\boldsymbol{k}} + v_{\boldsymbol{k}}(\mathsf{x})\hat{b}_{\boldsymbol{k}} \right) f(\mathsf{x}) \end{aligned} \tag{205}$$

Observation 6.41. *Both π_ω and $\pi_{\tilde{\omega}}$ are faithful, and so the operators $\hat{a}_{\boldsymbol{k}}(f), \hat{a}^\dagger(f)$ can be written in terms of $\hat{b}_{\boldsymbol{k}}(f), \hat{b}^\dagger(f)$.*

$$\begin{tikzcd} & \mathcal{A}(\mathcal{M}) \arrow[dl, "\pi_\omega"'] \arrow[dr, "\pi_{\tilde{\omega}}"] & \\ \mathcal{L}(\mathcal{F}(\mathcal{H})) \arrow[rr, leftrightarrow, "\textit{because of faithfulness}"'] & & \mathcal{L}(\mathcal{F}(\widetilde{\mathcal{H}})) \end{tikzcd}$$

However, $\mathcal{H} \neq \widetilde{\mathcal{H}}$.

Observation 6.42. *We can define another number operator*

$$\tilde{\hat{N}} = \sum\!\!\!\!\!\!\!\int_{\boldsymbol{k}} \hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}} \tag{206}$$

Definition 6.8. *If there exists a unitary operator $\hat{U} : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\widetilde{H})$ such that*

$$\hat{U}\pi_\omega(\hat{A})\hat{U}^\dagger = \pi_{\tilde{\omega}}(\hat{A}) \ \forall \hat{A} \in \mathcal{A}(\mathcal{M}) \tag{207}$$

Then we say that π_ω and $\pi_{\tilde{\omega}}$ are ***unitarily equivalent*** *and all states in $\mathcal{H}$ correspond to states in $\widetilde{\mathcal{H}}$.*

Observation 6.43. *This is usually not the case, as not all states in $\mathcal{A}(\mathcal{M})$ can be represented in a given $\mathcal{F}(\mathcal{M})$). Indeed, the Fock space can only describe so many states: there is an infinite amount of states that are encoded in the algebra only, and that cannot be written as vectors in the Fock space.*

If $v_{\boldsymbol{k}'} = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \alpha_{\boldsymbol{k}\boldsymbol{k}'} u_{\boldsymbol{k}} + \beta_{\boldsymbol{k}\boldsymbol{k}'} u^*_{\boldsymbol{k}}$, then the operators can be written as

$$\hat{a}_{\boldsymbol{k}} = \sum\!\!\!\!\!\!\!\!\int_{\boldsymbol{k}'} \alpha_{\boldsymbol{k}\boldsymbol{k}'}\hat{b}_{\boldsymbol{k}'} + \beta^*_{\boldsymbol{k}\boldsymbol{k}'}\hat{b}^\dagger_{\boldsymbol{k}'} \tag{208}$$

In that case, $\widetilde{\omega}(\hat{N}) = \sum\!\!\!\!\!\!\int_{\boldsymbol{k}} \sum\!\!\!\!\!\!\int_{\boldsymbol{k}'} |\beta_{\boldsymbol{k}\boldsymbol{k}'}|^2$, although $\widetilde{\omega}(\hat{\widetilde{N}}) = 0$.

Observation 6.44. *If there exists a state $\hat{\rho}_{\tilde{w}} \in \mathcal{L}(\mathcal{F}(\mathcal{H}))$ such that*

$$\widetilde{\omega}(\hat{A}) = \mathrm{Tr}\Big(\hat{\rho}_{\tilde{w}}\pi_\omega(\hat{A})\Big) \ \forall \hat{A} \in \mathcal{A}(\mathcal{M}) \tag{209}$$

then we say that we can represent a state $\widetilde{\omega}$ as a state in $\mathcal{F}(\mathcal{H})$.

In this case, it can be shown that there exists no state $\hat{\rho}$ such that the expectation value of the number operator $\langle\hat{N}\rangle_{\hat{\rho}}$ of the Fock space $\mathcal{F}(\mathcal{H})$ is also infinite. Therefore, we cannot represent all states and there is no unitary equivalence between π_ω and $\pi_{\tilde{\omega}}$.

Observation 6.45. *$\hat{N}$ is not a well defined element of $\mathcal{A}(\mathcal{M})$.*

Observation 6.46. *This implies that there is not a canonical definition of particle/vacuum! Different observers will disagree on their definitions. For example, an accelerating observer will see an observer at rest's vacuum as a thermal bath. This is known as the Unruh effect.*

6.6 Minkowski spacetime

Typically, for a massless scalar field in Minkowski spacetime, one chooses the basis

$$u_{\boldsymbol{k}}(\mathsf{x}) = \frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{(2\pi)^{\frac{3}{2}}\sqrt{2|\boldsymbol{k}|}} \tag{210}$$

Definition 6.9 (Minkowski vacuum)**.** *The state ω_0 defined by this choice of basis is called* ***the Minkowski vacuum****. It is the unique state in this theory that is pure and invariant under spacetime translations.*
In the GNS representation, we denote it $|0\rangle$.

Let $\mathsf{x} = (t, \boldsymbol{x})$ be inertial coordinates and $f_i(\mathsf{x}) := \chi_i(t)F_i(\boldsymbol{x})$. The Wightman function be expressed as

$$W(\mathsf{x}, \mathsf{x}') = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|} \mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t'} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'} \tag{211}$$

or, in terms of these functions,

$$W(f_1, f_2) = \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^4\mathsf{x}\mathrm{d}^4\mathsf{x}' \frac{\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{(2\pi)^{\frac{2}{3}}\sqrt{2|\boldsymbol{k}|}} \chi_1(t)F_1(\boldsymbol{x}) \frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t'}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}}{(2\pi)^{\frac{2}{3}}\sqrt{2|\boldsymbol{k}|}} \chi_2(t')F_2(\boldsymbol{x}')$$

$$= \frac{1}{(2\pi)^{\frac{2}{3}}} \int \mathrm{d}^3\boldsymbol{k} \frac{1}{2|\boldsymbol{k}|} \widetilde{\chi}_1(|\boldsymbol{k}|)\widetilde{F}_1(-\boldsymbol{k})\widetilde{\chi}_2(-|\boldsymbol{k}|)\widetilde{F}_2(\boldsymbol{k})$$

where we have have introduced

$$\widetilde{\chi}(\omega) = \int \mathrm{d}t\, \chi(t)\mathrm{e}^{-\mathrm{i}\omega t}, \;\; \widetilde{F}(\boldsymbol{k}) = \int \mathrm{d}t\, \chi(t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{212}$$

Moreover, we can also write the kernel of this distribution as

$$W(\mathsf{x}, \mathsf{x}') = \lim_{\varepsilon \to 0^+} \frac{1}{4\pi^2(-(t - t' - \mathrm{i}\varepsilon)^2 + (\boldsymbol{x} - \boldsymbol{x}')^2)} \tag{213}$$

Example 6.4. *A 1-particle state would be defined by*

$$|f\rangle = \int \mathrm{d}^3\boldsymbol{k} \tilde{f}(\boldsymbol{k})\hat{a}^\dagger_{\boldsymbol{k}} |0\rangle = \hat{a}^\dagger_{\boldsymbol{k}} |0\rangle \tag{214}$$

with $\int \mathrm{d}^3\boldsymbol{k}|f(\boldsymbol{k})|^2 = 1$ *and*

$$\tilde{f}(\boldsymbol{k}) = u^*_{\boldsymbol{k}}(f)$$

$$= \int \mathrm{d}^4\mathsf{x} \frac{\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|t}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{(2\pi)^{\frac{2}{3}}\sqrt{2|\boldsymbol{k}|}} f(x)$$

$$= \frac{1}{(2\pi)^{\frac{2}{3}}\sqrt{2|\boldsymbol{k}|}} \widetilde{\chi}(-|\boldsymbol{k}|)\widetilde{F}(\boldsymbol{k})$$

6.7 The Hadamard condition

At this stage, we still don't have a way of computing finite expectation values of operators of the form $\hat{\phi}^2(f)$ (not to confuse with $\hat{\phi}(f)^2$).

Let ω_0 be a reference state such that, for a given state ω,

$$W(\mathsf{x}, \mathsf{x}') - W_0(\mathsf{x}, \mathsf{x}') \equiv h(\mathsf{x}, \mathsf{x}') \tag{215}$$

where h is regular in the $\mathsf{x} \mapsto \mathsf{x}'$.

Then, we define

$$\omega(:\hat{\phi}^2(f):) \equiv \omega(\hat{\phi}^2(f)) - \omega_0(\hat{\phi}^2(f))$$

$$= \int \mathrm{d}V \mathrm{d}V' f(\mathsf{x}) \Big(W(\mathsf{x}, \mathsf{x}') - W_0(\mathsf{x}, \mathsf{x}') \Big) \delta(\mathsf{x}, \mathsf{x}')$$

$$= \int \mathrm{d}V f(\mathsf{x}) h(\mathsf{x}, \mathsf{x}) < \infty$$

Observation 6.47. *For a state* $|f\rangle$,

$$\langle f| \hat{\phi}(h)\hat{\phi}(g) |f\rangle = W_{|f\rangle}(h, g)$$

$$= W_{|\Omega\rangle}(h, g) + \langle f| \hat{\phi}(h) |f\rangle \langle f| \hat{\phi}(g) |f\rangle$$

The Hadamard condition Let ω be a state and $W(\mathsf{x}, \mathsf{x}')$ its Wightman function. We say that ω satisfies the **Hadamard condition** if, $\forall \mathsf{x} \in \mathcal{M}$, there exists a neighborhood $N_{\mathsf{x}} \subseteq \mathcal{M}$ of x where

$$W(\mathsf{x}, \mathsf{x}') = \frac{\Delta^{\frac{1}{2}}(\mathsf{x}, \mathsf{x}')}{8\pi^2 \sigma_\varepsilon(\mathsf{x}, \mathsf{x}')} + v(\mathsf{x}, \mathsf{x}') \log\left(\frac{\sigma_\varepsilon(\mathsf{x}, \mathsf{x}')}{l^2}\right) + u(\mathsf{x}, \mathsf{x}') \tag{216}$$

where

$$\begin{cases} \Delta(\mathsf{x}, \mathsf{x}') \text{ is the Van-Vleck determinant} \\ \\ \sigma_\varepsilon(\mathsf{x}, \mathsf{x}') \text{ is } \frac{1}{2} \text{ the geodesic spacetime separation squared between } \mathsf{x}, \mathsf{x}' \\ \text{with a regulator i}\varepsilon\text{, analogous to Minkowski spacetime} \\ \\ v(\mathsf{x}, \mathsf{x}'), u(\mathsf{x}, \mathsf{x}') \text{ are regular in the limit } \mathsf{x} \mapsto \mathsf{x}' \\ \\ l^2 \text{ is a parameter with units of length} \end{cases}$$

Theorem 6.2. *If two states have a Wightman function of the form Eq. (216), then h is regular and $\omega(:\hat{\phi}^2(f):)$* ***is finite****.*

Definition 6.10 (Hadamard state)**.** *A* ***Hadamard state*** *is a state that verifies the Hadamard condition.*

Example 6.5. *The Wightman function of the Minkowski vacuum is precisely of the Hadamard form and so the Minkowski vacuum is Hadamard state (it is actually the prototipical Hadamard state).*

Observation 6.48. *For a Hadamard state, the stress-energy tensor is well defined, even without being smeared with a test function.*

7 Local probes of QFT

7.1 Motivation

Consider an atom $\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$, $\psi_n(\boldsymbol{x}) = \langle \boldsymbol{x}|\psi_n\rangle$. The interaction of an external electric field with the atom (in some regimes) can be modeled by a dipole interaction

$$\hat{H}_I = q\hat{\boldsymbol{x}}\cdot\boldsymbol{E}(\hat{\boldsymbol{x}},t) \tag{217}$$

Observation 7.1. *If $\boldsymbol{E}$ is classical, then $\boldsymbol{E}(\hat{\boldsymbol{x}},t)$ is not an association of spacetime to operators:*

$$\boldsymbol{E}:\mathcal{M}\to\mathbb{R}^3 \ \text{ but } \ \boldsymbol{E}(\hat{\boldsymbol{x}},t):\mathcal{M}\not\to\mathcal{L}(\mathcal{H}) \tag{218}$$

[[Maybe say what E is then]]

If $\boldsymbol{E}$ is treated as quantum field,

$$\hat{\boldsymbol{E}}(\mathsf{x}) = \sum_{s=1}^{2}\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2|\boldsymbol{k}|}}\left(\mathrm{e}^{\mathrm{i}\mathsf{k}\cdot\mathsf{x}}\hat{a}_{\boldsymbol{k},s}+\mathrm{e}^{-\mathrm{i}\mathsf{k}\cdot\mathsf{x}}\hat{a}^\dagger_{\boldsymbol{k},s}\right)\boldsymbol{\varepsilon}_s(\boldsymbol{k}) \tag{219}$$

in a given GNS representation and gauge, where $\mathsf{k}\cdot\mathsf{x} = -|\boldsymbol{k}|t+\boldsymbol{k}\cdot\boldsymbol{x}$.

Then, the Hamiltonian becomes

$$\hat{H}_I = q\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t):\mathcal{H}_{\text{atom}}\otimes\mathcal{F}(\mathcal{H}_E)\to\mathcal{H}_{\text{atom}}\otimes\mathcal{F}(\mathcal{H}_E) \tag{220}$$

Example 7.1. *Let us compute the expectation value of $\boldsymbol{E}(\hat{\boldsymbol{x}},t)$:*

$$\begin{aligned}\langle\psi_n|\,\boldsymbol{E}(\hat{\boldsymbol{x}},t)\,|\psi_n\rangle &= \langle\psi_n|\,\mathbb{I}\,\boldsymbol{E}(\hat{\boldsymbol{x}},t)\,|\psi_n\rangle\\ &= \int\mathrm{d}^3\boldsymbol{x}\,\langle\psi_n|\boldsymbol{x}\rangle\,\boldsymbol{E}(\boldsymbol{x},t)\,\langle\boldsymbol{x}|\psi_n\rangle\\ &= \int\mathrm{d}^3\boldsymbol{x}\,\boldsymbol{E}(\boldsymbol{x},t)|\psi_n(\boldsymbol{x})|^2\end{aligned}$$

In $H_I(t)$, the expectation value of $\boldsymbol{E}(\hat{\boldsymbol{x}},t)$ depends on a state in $\mathcal{H}_{atom}\otimes\mathcal{F}(\mathcal{H}_E)$.

Consider operators $\hat{\rho}_\omega\in\mathcal{L}(\mathcal{F}(\mathcal{H}_E))$, $|\psi_n\rangle\!\langle\psi_n|\in\mathcal{L}(\mathcal{H}_{atom})$, then

$$\begin{aligned}\langle\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t)\rangle_{|\psi_n\rangle\!\langle\psi_n|\otimes\hat{\rho}_\omega} &= \mathrm{Tr}\left(\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t)\,|\psi_n\rangle\!\langle\psi_n|\otimes\hat{\rho}_\omega\right)\\ &= \mathrm{Tr}\left(\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t)\mathbb{I}\,|\psi_n\rangle\!\langle\psi_n|\otimes\hat{\rho}_\omega\right)\end{aligned}$$

$$= \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{Tr}\left(\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t)\hat{\rho}_\omega\right)|\psi_n(\boldsymbol{x})|^2$$

$$= \int \mathrm{d}^3\boldsymbol{x} \langle \hat{\boldsymbol{E}}(\boldsymbol{x},t)\rangle_{\hat{\rho}_\omega} |\psi_n(\boldsymbol{x})|^2$$

We can see that the expected value of the field operator evaluated at the position operator of the atom gives the expected value of the field operator smeared by the probability distribution associated to the atom.

We can write

$$\begin{aligned}
\hat{H}_I(t) &= q\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t) \\
&= q\int \mathrm{d}^3\boldsymbol{x}\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t) \\
&= q\int \mathrm{d}^3\boldsymbol{x}\, \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{E}}(\hat{\boldsymbol{x}},t)\, |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \\
&= q\int \mathrm{d}^3\boldsymbol{x} \left(\sum_n |\psi_n\rangle\langle\psi_n|\right)\left(\sum_m |\psi_m\rangle\langle\psi_m|\right) |\boldsymbol{x}\rangle\langle\boldsymbol{x}|\, \boldsymbol{x}\cdot\hat{\boldsymbol{E}}(\boldsymbol{x},t) \\
&= q\int \mathrm{d}^3\boldsymbol{x} \sum_n\sum_m |\psi_n\rangle\langle\psi_n|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\psi_m\rangle\langle\psi_m|\, \boldsymbol{x}\cdot\hat{\boldsymbol{E}}(\boldsymbol{x},t) \\
&= q\int \mathrm{d}^3\boldsymbol{x} \sum_n\sum_m \psi_n^*(\boldsymbol{x})\psi_m(\boldsymbol{x})\, |\psi_n\rangle\langle\psi_m|\, \boldsymbol{x}\cdot\hat{\boldsymbol{E}}(\boldsymbol{x},t) \\
&= q\int \mathrm{d}^3\boldsymbol{x} \sum_n\sum_m \psi_n^*(\boldsymbol{x})x^i\psi_m(\boldsymbol{x})\, |\psi_n\rangle\langle\psi_m|\, \hat{E}_i(\boldsymbol{x},t) \\
&=: \int \mathrm{d}^3\boldsymbol{x}\, \hat{h}_I(\mathsf{x})
\end{aligned} \tag{221}$$

where we have defined a Hamiltonian density

$$\hat{h}_I(\mathsf{x}) = q\sum_n\sum_m \psi_n^*(\boldsymbol{x})x^i\psi_m(\boldsymbol{x})\, |\psi_n\rangle\langle\psi_m|\, \hat{E}_i(\boldsymbol{x},t) \tag{222}$$

If we define

$$F^i_{nm}(\boldsymbol{x}) \equiv \psi_n^*(\boldsymbol{x})x^i\psi_m(\boldsymbol{x}) \tag{223}$$

then the Hamiltonian density becomes

$$\hat{h}_I(\mathsf{x}) = q\sum_n\sum_m F^i_{nm}\hat{E}_i(\boldsymbol{x},t)(\boldsymbol{x})\,|\psi_n\rangle\!\langle\psi_m| \tag{224}$$

We also want to incorporate the evolution of this Hamiltonian with respect to $\hat{H}_{\text{atom}} = \sum_n E_n\,|\psi_n\rangle\,\langle\psi_n|$:

$$\hat{h}_I(\mathsf{x}) \mapsto \mathrm{e}^{\mathrm{i}\hat{H}_{\text{atom}}t}\hat{h}_I\mathrm{e}^{-\mathrm{i}\hat{H}_{\text{atom}}t}$$

$$=q\sum_n\sum_m \mathrm{e}^{\mathrm{i}(E_n-E_m)t}F^i_{nm}\hat{E}_i(\boldsymbol{x},t)(\boldsymbol{x})\,|\psi_n\rangle\!\langle\psi_m|$$

We can restrict this interaction to two levels only, $|g\rangle$, $|e\rangle$ (ground and excited states), such that $E_g < E_e$ and we define the **energy gap** $\Omega = E_e - E_g$.

Then, the Hamiltonian density becomes

$$\hat{h}_I(\mathsf{x}) = q\Bigg(|\psi_g(\boldsymbol{x})|^2\,|g\rangle\!\langle g|+|\psi_e(\boldsymbol{x})|^2\,|e\rangle\!\langle e|+\psi_e^*(\boldsymbol{x})\psi_g(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\Omega t}\,|e\rangle\!\langle g|+\psi_g^*(\boldsymbol{x})\psi_e(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\Omega t}\,|g\rangle\!\langle e|\Bigg)x^i\hat{E}_i(\boldsymbol{x},t) \tag{225}$$

Neglecting the diagonal terms,

$$\hat{h}_I(\mathsf{x}) = q\Bigg(f^*(\boldsymbol{x})\hat{\sigma}^+(t) + f(\boldsymbol{x})\hat{\sigma}^-(t)\Bigg)\boldsymbol{x}\cdot\hat{\boldsymbol{E}}_i(\boldsymbol{x},t) \tag{226}$$

where we have introduced $\hat{\sigma}^- = |g\rangle\,\langle e|\,$, $\hat{\sigma}^+ = |e\rangle\,\langle g|\,$, $\sigma^\pm(t) = \mathrm{e}^{\pm\mathrm{i}\Omega t}\hat{\sigma}^\pm$ and $f(\boldsymbol{x}) = \psi_g^*(\boldsymbol{x})\psi_e(\boldsymbol{x})$. Implementing finite time interactions and assuming $f^* = f$, we can re-write yet again

$$\hat{h}_I(\mathsf{x}) = q\,\chi(t)f(\boldsymbol{x})(\hat{\sigma}^+(t)+\hat{\sigma}^-(t))\boldsymbol{x}\cdot\hat{\boldsymbol{E}}_i(\boldsymbol{x},t)$$

$$=:q\Lambda(\mathsf{x})(\hat{\sigma}^+(t)+\hat{\sigma}^-(t))\boldsymbol{x}\cdot\hat{\boldsymbol{E}}_i(\boldsymbol{x},t) \tag{227}$$

Observation 7.2. *The operators $\hat{\sigma}^\pm$ satisfy some properties that will be of use later:*

$$\begin{cases}(\hat{\sigma}^+)^2 = (\hat{\sigma}^-)^2 = 0\\ \hat{\sigma}^+\hat{\sigma}^-\hat{\sigma}^+ = \hat{\sigma}^+\\ \hat{\sigma}^-\hat{\sigma}^+\hat{\sigma}^- = \hat{\sigma}^-\end{cases}$$

7.2 The Unruh-De Witt (UDW) model

7.2.1 Two-level UDW

Let $\mathcal{M}$ be a spacetime with a scalar QFT for a field ϕ, $\mathsf{z}(\tau)$ a time-like trajectory and $\tau(\mathsf{x})$ an extension of τ locally around the curve.

Consider a qubit travelling along z such that its Hamiltonian is $\hat{H}_D = \Omega\,\hat{\sigma}^+\hat{\sigma}^- = \Omega\,|e\rangle\,\langle e|$ and define the interaction Hamiltonian density

$$\hat{h}_I(\mathsf{x}) = \lambda\Lambda(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega\tau}\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega\tau}\hat{\sigma}^-)\hat{\phi}(\mathsf{x}) \tag{228}$$

where λ is a coupling constant and $\Lambda(\mathsf{x})$ is a function localized around $z(\tau)$.

If one can find coordinates (τ, ξ^i) such that ξ^i parametrizes the rest surfaces $\tau = \mathrm{cst}$, it is common to assume that

$$\Lambda(\mathsf{x}) = \chi(\tau)\,f(\boldsymbol{\xi}) \tag{229}$$

In these coordinates,

$$\hat{h}_I(\tau, \boldsymbol{\xi}) = \lambda\,\chi(\tau) f(\boldsymbol{\xi})(\mathrm{e}^{\mathrm{i}\Omega\tau}\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega\tau}\hat{\sigma}^-)\hat{\phi}(\mathsf{x}) \tag{230}$$

We call $\chi(\tau)$ the *switching function*, $f(\boldsymbol{\xi})$ the *smearing function* and $\Lambda(\mathsf{x})$ the *spacetime smearing function.*

Let us assume the initial state of the detector is $\hat{\rho}_{\mathrm{D},0} = \hat{\sigma}^-\hat{\sigma}^+ = |g\rangle\,\langle g|$ and that the initial state of the field is ω, which can be represented as $\hat{\rho}_\omega$ in a given GNS representation. The initial combined state is then given by

$$\hat{\rho}_0 = \hat{\rho}_{\mathrm{D},0} \otimes \hat{\rho}_\omega = \hat{\sigma}^-\hat{\sigma}^+ \otimes \hat{\rho}_\omega \tag{231}$$

and it evolves to a final state $\hat{\rho}$ as

$$\hat{\rho}_0 \mapsto \hat{\rho} = \hat{U}_I\,\hat{\rho}_0\,\hat{U}_I^\dagger \tag{232}$$

where $\hat{U}_I$ is the operator defined as

$$\begin{aligned}\hat{U}_I &= \mathcal{T}_\tau\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}V \hat{h}_I(\mathsf{x})} \\ &\sim \hat{\mathbb{I}} + \hat{U}_I^{(1)} + \hat{U}_I^{(2)} + \dots\end{aligned} \tag{233}$$

with

$$\begin{cases}\hat{U}_I^{(1)} = -\mathrm{i}\int \mathrm{d}V \hat{h}_I(\mathsf{x}) \\ \hat{U}_I^{(2)} = -\int \mathrm{d}V\mathrm{d}V'\theta(\tau - \tau')\hat{h}_I(\mathsf{x})\hat{h}_I(\mathsf{x}')\end{cases}$$

Therefore, the final state can be written as

$$\begin{aligned}\hat{\rho} &= \hat{U}_I\,\hat{\rho}_0\,\hat{U}_I^\dagger \\ &\sim (\hat{\mathbb{I}} + \hat{U}_I^{(1)} + \hat{U}_I^{(2)} + \dots)\,\hat{\rho}_0\,(\hat{\mathbb{I}} + (\hat{U}_I^{(1)})^\dagger + (\hat{U}_I^{(2)})^\dagger + \dots)\end{aligned}$$

$$= \hat{\rho}_0 + \hat{U}_I^{(1)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(1)})^\dagger + \hat{U}_I^{(1)}\hat{\rho}_0(\hat{U}_I^{(1)})^\dagger + \hat{U}_I^{(2)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(2)})^\dagger + \mathcal{O}(\lambda^3) \tag{234}$$

It is now a matter of evaluating these terms.

Notice that

$$\hat{U}_I^{(1)} = -\mathrm{i}\int \mathrm{d}V \lambda\, \chi(\tau) f(\boldsymbol{\xi})(\mathrm{e}^{\mathrm{i}\Omega\tau}\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega\tau}\hat{\sigma}^-)\hat{\phi}(\mathsf{x})$$

$$= -\mathrm{i}\lambda\hat{\sigma}^+ \int \mathrm{d}V \Lambda^+(\mathsf{x})\hat{\phi}(\mathsf{x}) - \mathrm{i}\lambda\hat{\sigma}^- \int \mathrm{d}V \Lambda^-(\mathsf{x})\hat{\phi}(\mathsf{x})$$

$$= -\mathrm{i}\lambda\Big(\hat{\phi}(\Lambda^+)\hat{\sigma^+} + \hat{\phi}(\Lambda^-)\hat{\sigma^-}\Big) \tag{235}$$

where we have introduced $\Lambda^\pm(\mathsf{x}) = \mathrm{e}^{\pm\mathrm{i}\Omega\tau}\Lambda(\mathsf{x})$.

We can use this expression to compute the different terms:

- $\hat{U}_I^{(1)}\hat{\rho}_0 = -\mathrm{i}\lambda\Big(\hat{\phi}(\Lambda^+)\hat{\sigma^+} + \hat{\phi}(\Lambda^-)\hat{\sigma^-}\Big)\hat{\sigma}^-\hat{\sigma}^+ \otimes \hat{\rho}_\omega$

$= -\mathrm{i}\lambda\hat{\sigma}^+ \otimes \hat{\phi}(\Lambda^+)\hat{\rho}_\omega$

- $\hat{\rho}_0(\hat{U}_I^{(1)})^\dagger = (\hat{U}_I^{(1)}\hat{\rho}_0)^\dagger$

$= \mathrm{i}\lambda\hat{\sigma}^- \otimes \hat{\rho}_\omega\, \hat{\phi}(\Lambda^-)$

- $\hat{U}_I^{(1)}\hat{\rho}_0(\hat{U}_I^{(1)})^\dagger = \Big(-\mathrm{i}\lambda\hat{\sigma}^+ \otimes \hat{\phi}(\Lambda^+)\hat{\rho}_\omega\Big)\mathrm{i}\lambda\Big(\hat{\phi}(\Lambda^-)\hat{\sigma^-} + \hat{\phi}(\Lambda^+)\hat{\sigma^+}\Big)$

$= \lambda^2\hat{\sigma}^+\hat{\sigma}^- \otimes \hat{\phi}(\Lambda^+)\,\hat{\rho}_\omega\,\hat{\phi}(\Lambda^-)$

- $\hat{U}_I^{(2)}\hat{\rho}_0 = \Big(-\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\hat{h}_I(\mathsf{x})\hat{h}_I(\mathsf{x}')\Big)\hat{\rho}_0$

$$= -\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big(\lambda\Lambda(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega\tau}\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega\tau}\hat{\sigma}^-)\hat{\phi}(\mathsf{x})\Big)\Big(\lambda\Lambda(\mathsf{x}')(\mathrm{e}^{\mathrm{i}\Omega\tau'}\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega\tau'}\hat{\sigma}^-)\hat{\phi}(\mathsf{x}')\Big)\hat{\rho}_0$$

$$= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Lambda(\mathsf{x})\Lambda(\mathsf{x}')\Big(\mathrm{e}^{\mathrm{i}\Omega(\tau+\tau')}\hat{\sigma}^+\hat{\sigma}^+ + \mathrm{e}^{\mathrm{i}\Omega(\tau-\tau')}\hat{\sigma}^+\hat{\sigma}^- + \mathrm{e}^{-\mathrm{i}\Omega(\tau-\tau')}\hat{\sigma}^-\hat{\sigma}^+ + \mathrm{e}^{-\mathrm{i}\Omega(\tau+\tau')}\hat{\sigma}^-\hat{\sigma}^-\Big)\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_0$$

$$= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Lambda(\mathsf{x})\Lambda(\mathsf{x}')\Big(\mathrm{e}^{\mathrm{i}\Omega(\tau-\tau')}\hat{\sigma}^+\hat{\sigma}^- + \mathrm{e}^{-\mathrm{i}\Omega(\tau-\tau')}\hat{\sigma}^-\hat{\sigma}^+\Big)\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\sigma}^-\hat{\sigma}^+\otimes\hat{\rho}_\omega$$

$$= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Lambda(\mathsf{x})\Lambda(\mathsf{x}') + \mathrm{e}^{-\mathrm{i}\Omega(\tau-\tau')}\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\hat{\sigma}^-\hat{\sigma}^+$$

- $\hat{\rho}_0(\hat{U}_I^{(2)})^\dagger = (\hat{U}_I^{(2)}\hat{\rho}_0)^\dagger$

$$= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Lambda(\mathsf{x}')\Lambda(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega(\tau-\tau')}\hat{\rho}_\omega\hat{\phi}(\mathsf{x}')\hat{\phi}(\mathsf{x})\hat{\sigma}^-\hat{\sigma}^+$$

$$= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau'-\tau)\Lambda(\mathsf{x})\Lambda(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega(\tau-\tau')}\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\sigma}^-\hat{\sigma}^+ \tag{236}$$

We only have access to observables of the form $\hat{A}\otimes\hat{\mathbb{I}}_{\mathcal{F}(\mathcal{H})}$, with $\hat{A}\in\mathcal{L}(\mathbb{C}^2)$. These are entirely determined by $\hat{\rho}_D = \mathrm{Tr}_{\mathcal{F}(\mathcal{H})}(\hat{\rho})$.

Therefore, the final state is

$$\hat{\rho}_D = \mathrm{Tr}_\omega(\hat{\rho})$$

$$= \mathrm{Tr}_\omega(\hat{U}_I\,\hat{\rho}_0\,\hat{U}_I^\dagger)$$

$$= \mathrm{Tr}_\omega\left(\hat{\rho}_0 + \hat{U}_I^{(1)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(1)})^\dagger + \hat{U}_I^{(1)}\hat{\rho}_0(\hat{U}_I^{(1)})^\dagger + \hat{U}_I^{(2)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(2)})^\dagger\right) + \mathcal{O}(\lambda^3)$$

$$= \mathrm{Tr}_\omega(\hat{\rho}_0) + \mathrm{Tr}_\omega\left(\hat{U}_I^{(1)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(1)})^\dagger\right) + \mathrm{Tr}_\omega\left(\hat{U}_I^{(1)}\hat{\rho}_0(\hat{U}_I^{(1)})^\dagger\right) + \mathrm{Tr}_\omega\left(\hat{U}_I^{(2)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(2)})^\dagger\right) + \mathcal{O}(\lambda^3)$$

$$
\begin{aligned}
&= \mathrm{Tr}_\omega(\hat\sigma^-\hat\sigma^+\otimes\hat\rho_\omega) + \mathrm{Tr}_\omega\Bigg(-\mathrm{i}\lambda\hat\sigma^+\otimes\hat\phi(\Lambda^+)\hat\rho_\omega + \mathrm{i}\lambda\hat\sigma^-\otimes\hat\rho_\omega\,\hat\phi(\Lambda^-)\Bigg) + \mathrm{Tr}_\omega\left(\lambda^2\hat\sigma^+\hat\sigma^-\otimes\hat\phi(\Lambda^+)\,\hat\rho_\omega\,\hat\phi(\Lambda^-)\right)\\
&+ \mathrm{Tr}_\omega\left[-\lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')\Big(\theta(\tau-\tau')\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\hat\rho_\omega + \theta(\tau'-\tau)\hat\rho_\omega\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\Big)\hat\sigma^-\hat\sigma^+\right] + \mathcal{O}(\lambda^3)\\
&= \mathrm{Tr}_\omega(\hat\sigma^-\hat\sigma^+\otimes\hat\rho_\omega) - \mathrm{i}\lambda\hat\sigma^+\,\mathrm{Tr}_\omega\left(\hat\phi(\Lambda^+)\hat\rho_\omega\right) + \mathrm{i}\lambda\hat\sigma^-\,\mathrm{Tr}_\omega\left(\hat\rho_\omega\,\hat\phi(\Lambda^-)\right) + \lambda^2\hat\sigma^+\hat\sigma^-\,\mathrm{Tr}_\omega\left(\hat\phi(\Lambda^+)\,\hat\rho_\omega\,\hat\phi(\Lambda^-)\right)\\
&- \lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')\Big[\theta(\tau-\tau')\,\mathrm{Tr}_\omega\left(\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\hat\rho_\omega\right) + \theta(\tau'-\tau)\,\mathrm{Tr}_\omega\left(\hat\rho_\omega\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\right)\Big]\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)\\
&= \mathrm{Tr}_\omega(\hat\sigma^-\hat\sigma^+\otimes\hat\rho_\omega) - \mathrm{i}\lambda\hat\sigma^+\,\mathrm{Tr}_\omega\left(\hat\phi(\Lambda^+)\hat\rho_\omega\right) + \mathrm{i}\lambda\hat\sigma^-\,\mathrm{Tr}_\omega\left(\hat\phi(\Lambda^-)\,\hat\rho_\omega\right) + \lambda^2\hat\sigma^+\hat\sigma^-\,\mathrm{Tr}_\omega\left(\hat\phi(\Lambda^-)\hat\phi(\Lambda^+)\,\hat\rho_\omega\right)\\
&- \lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')\Big[\theta(\tau-\tau')\,\mathrm{Tr}_\omega\left(\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\hat\rho_\omega\right) + \theta(\tau'-\tau)\,\mathrm{Tr}_\omega\left(\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\hat\rho_\omega\right)\Big]\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)\\
&= \hat\sigma^-\hat\sigma^+ - \mathrm{i}\lambda\hat\sigma^+\langle\hat\phi(\Lambda^+)\rangle + \mathrm{i}\lambda\hat\sigma^-\langle\hat\phi(\Lambda^-)\rangle + \lambda^2\hat\sigma^+\hat\sigma^-\langle\hat\phi(\Lambda^-)\hat\phi(\Lambda^+)\rangle\\
&\quad - \lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')\Big(\theta(\tau-\tau') + \theta(\tau'-\tau)\Big)\langle\hat\phi(\mathsf{x})\hat\phi(\mathsf{x}')\rangle\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)\\
&= \hat\sigma^-\hat\sigma^+ - \mathrm{i}\lambda\hat\sigma^+\langle\hat\phi(\Lambda^+)\rangle + \mathrm{i}\lambda\hat\sigma^-\langle\hat\phi(\Lambda^-)\rangle + \lambda^2\hat\sigma^+\hat\sigma^- W(\Lambda^-,\Lambda^+)\\
&\quad - \lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')\Big(\theta(\tau-\tau') + \theta(\tau'-\tau)\Big)W(\mathsf{x},\mathsf{x}')\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)\\
&= \hat\sigma^-\hat\sigma^+ - \mathrm{i}\lambda\hat\sigma^+\langle\hat\phi(\Lambda^+)\rangle + \mathrm{i}\lambda\hat\sigma^-\langle\hat\phi(\Lambda^-)\rangle + \lambda^2\hat\sigma^+\hat\sigma^- W(\Lambda^-,\Lambda^+)\\
&\quad - \lambda^2\int \mathrm{d}V\mathrm{d}V'\Lambda^-(\mathsf{x})\Lambda^+(\mathsf{x}')W(\mathsf{x},\mathsf{x}')\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)\\
&= \hat\sigma^-\hat\sigma^+ - \mathrm{i}\lambda\hat\sigma^+\langle\hat\phi(\Lambda^+)\rangle + \mathrm{i}\lambda\hat\sigma^-\langle\hat\phi(\Lambda^-)\rangle + \lambda^2\hat\sigma^+\hat\sigma^- W(\Lambda^-,\Lambda^+) - \lambda^2 W(\Lambda^-,\Lambda^+)\hat\sigma^-\hat\sigma^+ + \mathcal{O}(\lambda^3)
\end{aligned}
\tag{237}
$$

Observation 7.3. *We can write this result in matrix form. If we define*

$$
|g\rangle = \begin{pmatrix}1\\0\end{pmatrix},\, |e\rangle = \begin{pmatrix}0\\1\end{pmatrix}
\tag{238}
$$

Then, the definitions of $\hat\sigma^\pm$ give:

$$
\hat\sigma^+|g\rangle = |e\rangle \iff \hat\sigma^+\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \iff \hat\sigma^+ = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}
$$

$$\hat{\sigma}^- |e\rangle = |g\rangle \iff \hat{\sigma}^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \hat{\sigma}^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and hence

$$\hat{\sigma}^- \hat{\sigma}^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}^+ \hat{\sigma}^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, the final state can be written as

$$\hat{\rho}_{\mathrm{D}} = \begin{pmatrix} 1 - \lambda^2 W(\Lambda^-, \Lambda^+) & \mathrm{i}\lambda \langle \hat{\phi}(\Lambda^-) \rangle \\ & \\ -\mathrm{i}\lambda \langle \hat{\phi}(\Lambda^+) \rangle & \lambda^2 W(\Lambda^-, \Lambda^+) \end{pmatrix} + \mathcal{O}(\lambda^3) \tag{239}$$

Observation 7.4. *The probability P_e of the detector becoming excited is given by*

$$P_e = \mathrm{Tr}(\hat{\rho}_{\mathrm{D}} |e\rangle \langle e|) = \lambda^2 W(\Lambda^-, \Lambda^+) \tag{240}$$

Example 7.2. *If we consider a gaussian spacetime smearing function and a positive energy gap, then the excitation probability at finite times is non-zero. This gives rise to the phenomenon of vacuum fluctuations!*

8 Two (or more) UDW

Consider two Unruh-De Witt detectors A, B with Hamiltonians $\hat{H}_{\mathrm{A}} = \Omega_{\mathrm{A}} \hat{\sigma}^+_{\mathrm{A}} \hat{\sigma}^-_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}} = \Omega_{\mathrm{B}} \hat{\sigma}^+_{\mathrm{B}} \hat{\sigma}^-_{\mathrm{B}}$ respectively and spacetime smearing functions $\Lambda_{\mathrm{A}}(\mathsf{x})$ and $\Lambda_{\mathrm{B}}(\mathsf{x})$.

These two qubits interact with a scalar field $\hat{\phi}(\mathsf{x})$ according to the interaction Hamiltonian densities

$$\begin{cases} \hat{h}_{I,\mathrm{A}} = \lambda \, \Lambda_{\mathrm{A}}(\mathsf{x}) (\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}} \tau_{\mathrm{A}}(\mathsf{x})} \hat{\sigma}^+_{\mathrm{A}} + \mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}} \tau_{\mathrm{A}}(\mathsf{x})} \hat{\sigma}^-_{\mathrm{A}}) \hat{\phi}(\mathsf{x}) \\ \\ \hat{h}_{I,\mathrm{B}} = \lambda \, \Lambda_{\mathrm{B}}(\mathsf{x}) (\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}} \tau_{\mathrm{B}}(\mathsf{x})} \hat{\sigma}^+_{\mathrm{B}} + \mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}} \tau_{\mathrm{B}}(\mathsf{x})} \hat{\sigma}^-_{\mathrm{B}}) \hat{\phi}(\mathsf{x}) \end{cases} \tag{241}$$

and we define the total interaction Hamiltonian density

$$\hat{h}_I(\mathsf{x}) = \hat{h}_{I,\mathrm{A}}(\mathsf{x}) + \hat{h}_{I,\mathrm{B}}(\mathsf{x}) \tag{242}$$

Given an initial state for the detectors and field, say in the ground states

$$\hat{\rho}_0 = \hat{\rho}_{\mathrm{D},0} \otimes \hat{\rho}_\omega, \quad \hat{\rho}_{\mathrm{D},0} = \hat{\rho}_{\mathrm{A},0} \otimes \hat{\rho}_{\mathrm{B},0} \tag{243}$$

where

$$\begin{cases} \hat{\rho}_{\mathrm{A},0} = |g_{\mathrm{A}}\rangle \langle g_{\mathrm{A}}| = \hat{\sigma}^-_{\mathrm{A}} \hat{\sigma}^+_{\mathrm{A}} \\ \\ \hat{\rho}_{\mathrm{B},0} = |g_{\mathrm{B}}\rangle \langle g_{\mathrm{B}}| = \hat{\sigma}^-_{\mathrm{B}} \hat{\sigma}^+_{\mathrm{B}} \end{cases} \tag{244}$$

we can compute the final state of the detectors:

$$\hat{\rho}_{\text{D}} = \text{Tr}_{\omega}(\hat{U}_I \hat{\rho}_0 \hat{U}_I^{\dagger}) \tag{245}$$

Again, we can write infinitesimally

$$\hat{U}_I\,\hat{\rho}_0\,\hat{U}_I^{\dagger} \sim (\hat{\mathbb{I}} + \hat{U}_I^{(1)} + \hat{U}_I^{(2)} + \dots)\,\hat{\rho}_0\,(\hat{\mathbb{I}} + (\hat{U}_I^{(1)})^{\dagger} + (\hat{U}_I^{(2)})^{\dagger} + \dots)$$

This time, the terms are harder to compute, but it can be done in the same way. Noticing that

$$\hat{U}_I^{(1)} = -\mathrm{i}\lambda\Big(\hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^- \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{A}}^+)\hat{\sigma}_{\text{A}}^+ \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\mathbb{I}}_{\text{A}} \otimes \hat{\sigma}_{\text{B}}^- + \hat{\phi}(\Lambda_{\text{B}}^+)\hat{\mathbb{I}}_{\text{A}} \otimes \hat{\sigma}_{\text{B}}^+\Big) \tag{246}$$

we find

- $$\begin{aligned}\hat{U}_I^{(1)}\hat{\rho}_0 = -\mathrm{i}\lambda\Big(&\hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^- \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{A}}^+)\hat{\sigma}_{\text{A}}^+ \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\mathbb{I}}_{\text{A}} \otimes \hat{\sigma}_{\text{B}}^- \\ &+ \hat{\phi}(\Lambda_{\text{B}}^+)\hat{\mathbb{I}}_{\text{A}} \otimes \hat{\sigma}_{\text{B}}^+\Big)\Big(\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\rho}_{\omega}\Big) \\ = -\mathrm{i}\lambda\Big(&\hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\mathbb{I}}_{\text{B}}\hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\rho}_{\omega} + \hat{\phi}(\Lambda_{\text{A}}^+)\hat{\sigma}_{\text{A}}^+\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\mathbb{I}}_{\text{B}}\hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\rho}_{\omega} \\ &+ \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\mathbb{I}}_{\text{A}}\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\rho}_{\omega} + \hat{\phi}(\Lambda_{\text{B}}^+)\hat{\sigma}_{\text{B}}^+\hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\mathbb{I}}_{\text{A}}\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\rho}_{\omega}\Big) \\ = -\mathrm{i}\lambda\Big(&\hat{\phi}(\Lambda_{\text{A}}^+)\hat{\sigma}_{\text{A}}^+ \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ \otimes \hat{\rho}_{\omega} + \hat{\phi}(\Lambda_{\text{B}}^+)\hat{\sigma}_{\text{B}}^+ \otimes \hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\rho}_{\omega}\Big)\end{aligned} \tag{247}$$

- $$\begin{aligned}\hat{\rho}_0(\hat{U}_I^{(1)})^{\dagger} &= (\hat{U}_I^{(1)}\hat{\rho}_0)^{\dagger} \\ &= \mathrm{i}\lambda\Big(\hat{\rho}_{\omega} \otimes \hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^- \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ + \hat{\rho}_{\omega} \otimes \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\sigma}_{\text{B}}^- \otimes \hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+\Big)\end{aligned} \tag{248}$$

- $$\begin{aligned}\hat{U}_I^{(1)}\hat{\rho}_0(\hat{U}_I^{(1)})^{\dagger} = \lambda^2\Big(&\hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^- \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{A}}^+)\hat{\sigma}_{\text{A}}^+ \otimes \hat{\mathbb{I}}_{\text{B}} + \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\sigma}_{\text{B}}^- \otimes \hat{\mathbb{I}}_{\text{A}} + \hat{\phi}(\Lambda_{\text{B}}^+)\hat{\sigma}_{\text{B}}^+ \otimes \hat{\mathbb{I}}_{\text{A}}\Big) \\ &\Big(\hat{\rho}_{\omega} \otimes \hat{\phi}(\Lambda_{\text{A}}^-)\hat{\sigma}_{\text{A}}^- \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ + \hat{\rho}_{\omega} \otimes \hat{\phi}(\Lambda_{\text{B}}^-)\hat{\sigma}_{\text{B}}^- \otimes \hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+\Big) \\ = \lambda^2\Big(&\hat{\phi}(\Lambda_{\text{A}}^+)\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\text{A}}^-) \otimes \hat{\sigma}_{\text{A}}^+\hat{\sigma}_{\text{A}}^- \otimes \hat{\sigma}_{\text{B}}^-\hat{\sigma}_{\text{B}}^+ + \hat{\phi}(\Lambda_{\text{A}}^+)\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\text{B}}^-)\hat{\sigma}_{\text{A}}^+\hat{\sigma}_{\text{A}}^-\hat{\sigma}_{\text{A}}^+ \otimes \hat{\sigma}_{\text{B}}^-\end{aligned}$$

$$+\hat{\phi}(\Lambda_{\mathrm{B}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{A}}^{-})\otimes\hat{\sigma}_{\mathrm{B}}^{+}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\sigma}_{\mathrm{B}}^{+}\otimes\hat{\sigma}_{\mathrm{A}}^{-}+\hat{\phi}(\Lambda_{\mathrm{B}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{B}}^{-})\otimes\hat{\sigma}_{\mathrm{B}}^{+}\hat{\sigma}_{\mathrm{B}}^{-}\otimes\hat{\sigma}_{\mathrm{A}}^{-}\hat{\sigma}_{\mathrm{A}}^{+}\Big)$$

$$=\lambda^2\Big(\hat{\phi}(\Lambda_{\mathrm{A}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{A}}^{-})\otimes\hat{\sigma}_{\mathrm{A}}^{+}\hat{\sigma}_{\mathrm{A}}^{-}\otimes\hat{\sigma}_{\mathrm{B}}^{-}\hat{\sigma}_{\mathrm{B}}^{+}+\hat{\phi}(\Lambda_{\mathrm{A}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{B}}^{-})\hat{\sigma}_{\mathrm{A}}^{+}\otimes\hat{\sigma}_{\mathrm{B}}^{-}$$
$$+\hat{\phi}(\Lambda_{\mathrm{B}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{A}}^{-})\otimes\hat{\sigma}_{\mathrm{B}}^{+}\otimes\hat{\sigma}_{\mathrm{A}}^{-}+\hat{\phi}(\Lambda_{\mathrm{B}}^{+})\hat{\rho}_{\omega}\hat{\phi}(\Lambda_{\mathrm{B}}^{-})\otimes\hat{\sigma}_{\mathrm{B}}^{+}\hat{\sigma}_{\mathrm{B}}^{-}\otimes\hat{\sigma}_{\mathrm{A}}^{-}\hat{\sigma}_{\mathrm{A}}^{+}\Big) \tag{249}$$

- $\hat{U}_I^{(2)}\hat{\rho}_0=\Big(-\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\hat{h}_I(\mathsf{x})\hat{h}_I(\mathsf{x}')\Big)\hat{\rho}_0$

$$=-\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big(\hat{h}_{I,A}(\mathsf{x})+\hat{h}_{I,B}(\mathsf{x})\Big)\Big(\hat{h}_{I,A}(\mathsf{x}')+\hat{h}_{I,B}(\mathsf{x}')\Big)\hat{\rho}_0$$

$$=-\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big(\hat{h}_{I,A}(\mathsf{x})\hat{h}_{I,A}(\mathsf{x}')+\hat{h}_{I,A}(\mathsf{x})\hat{h}_{I,B}(\mathsf{x}')+\hat{h}_{I,B}(\mathsf{x})\hat{h}_{I,A}(\mathsf{x}')+\hat{h}_{I,B}(\mathsf{x})\hat{h}_{I,B}(\mathsf{x}')\Big)\hat{\rho}_0$$

$$=-\lambda^2\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[$$
$$\Lambda_{\mathrm{A}}(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-})\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{-})\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{A}}(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-})\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{-})\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{B}}(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{-})\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{-})\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{B}}(\mathsf{x})(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{-})\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')(\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}+\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{-})\hat{\phi}(\mathsf{x}')\Big]\hat{\rho}_0$$

$$=-\lambda^2\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[$$
$$\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x}')\Big]\hat{\rho}_0$$

$$=-\lambda^2\int\mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x}')$$
$$+\Lambda_{\mathrm{A}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}(\mathsf{x})}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{B}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}'}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x}')+\Lambda_{\mathrm{B}}(\mathsf{x})\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{B}}\tau_{\mathrm{B}}(\mathsf{x})}\hat{\sigma}_{\mathrm{B}}^{+}\hat{\phi}(\mathsf{x})\Lambda_{\mathrm{A}}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{A}}\tau_{\mathrm{A}}'}\hat{\sigma}_{\mathrm{A}}^{+}\hat{\phi}(\mathsf{x}')$$

$$+ \Lambda_{\rm B}(\mathsf{x})\mathrm{e}^{-\mathrm{i}\Omega_{\rm B}\tau_{\rm B}(\mathsf{x})}\hat{\sigma}_{\rm B}^{-}\hat{\phi}(\mathsf{x})\Lambda_{\rm B}(\mathsf{x}')\mathrm{e}^{\mathrm{i}\Omega_{\rm B}\tau'_{\rm B}}\hat{\sigma}_{\rm B}^{+}\hat{\phi}(\mathsf{x}')\Big]\Big[\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\otimes\hat{\rho}_\omega\Big]$$

$$\begin{aligned}
&= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[\\
&\Big(\Lambda_{\rm A}^{-}(\mathsf{x})\hat{\phi}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}')\hat{\phi}(\mathsf{x}') + \Lambda_{\rm B}^{-}(\mathsf{x})\hat{\phi}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}')\hat{\phi}(\mathsf{x}')\Big)\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\otimes\hat{\rho}_\omega\\
&+\Big(\Lambda_{\rm A}^{+}(\mathsf{x})\hat{\phi}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}')\hat{\phi}(\mathsf{x}') + \Lambda_{\rm B}^{+}(\mathsf{x})\hat{\phi}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}')\hat{\phi}(\mathsf{x}')\Big)\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{+}\otimes\hat{\rho}_\omega\Big]
\end{aligned}$$

$$\begin{aligned}
&= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[\Big(\Lambda_{\rm A}^{-}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}') + \Lambda_{\rm B}^{-}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}')\Big)\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\otimes\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\\
&+\Big(\Lambda_{\rm A}^{+}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}') + \Lambda_{\rm B}^{+}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}')\Big)\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{+}\otimes\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\Big]
\end{aligned} \tag{250}$$

- $\hat{\rho}_0(\hat{U}_I^{(2)})^\dagger = (\hat{U}_I^{(2)}\hat{\rho}_0)^\dagger$

$$\begin{aligned}
&= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau-\tau')\Big[\Big(\Lambda_{\rm A}^{+}(\mathsf{x})\Lambda_{\rm A}^{-}(\mathsf{x}') + \Lambda_{\rm B}^{+}(\mathsf{x})\Lambda_{\rm B}^{-}(\mathsf{x}')\Big)\hat{\rho}_\omega\hat{\phi}(\mathsf{x}')\hat{\phi}(\mathsf{x})\otimes\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\\
&+\Big(\Lambda_{\rm A}^{-}(\mathsf{x})\Lambda_{\rm B}^{-}(\mathsf{x}') + \Lambda_{\rm B}^{-}(\mathsf{x})\Lambda_{\rm A}^{-}(\mathsf{x}')\hat{\phi}(\mathsf{x}')\Big)\hat{\rho}_\omega\hat{\phi}(\mathsf{x}')\hat{\phi}(\mathsf{x})\hat{\sigma}_{\rm A}^{-}\otimes\hat{\sigma}_{\rm B}^{-}\Big]\\
&= -\lambda^2\int \mathrm{d}V\mathrm{d}V'\theta(\tau'-\tau)\Big[\Big(\Lambda_{\rm A}^{+}(\mathsf{x}')\Lambda_{\rm A}^{-}(\mathsf{x}) + \Lambda_{\rm B}^{+}(\mathsf{x}')\Lambda_{\rm B}^{-}(\mathsf{x})\Big)\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\otimes\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\\
&+\Big(\Lambda_{\rm A}^{-}(\mathsf{x}')\Lambda_{\rm B}^{-}(\mathsf{x}) + \Lambda_{\rm B}^{-}(\mathsf{x}')\Lambda_{\rm A}^{-}(\mathsf{x})\Big)\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\sigma}_{\rm A}^{-}\otimes\hat{\sigma}_{\rm B}^{-}\Big]
\end{aligned} \tag{251}$$

- $\hat{U}_I^{(2)}\hat{\rho}_0 + \hat{\rho}_0(\hat{U}_I^{(2)})^\dagger = -\lambda^2\int \mathrm{d}V\mathrm{d}V'\Big[$

$$\begin{aligned}
&\Big(\Lambda_{\rm A}^{-}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}') + \Lambda_{\rm B}^{-}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}')\Big)\Big(\theta(\tau-\tau')\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega + \theta(\tau'-\tau)\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\Big)\otimes\hat{\sigma}_{\rm A}^{-}\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{-}\hat{\sigma}_{\rm B}^{+}\\
&+\theta(\tau-\tau')\Big(\Lambda_{\rm A}^{+}(\mathsf{x})\Lambda_{\rm B}^{+}(\mathsf{x}') + \Lambda_{\rm B}^{+}(\mathsf{x})\Lambda_{\rm A}^{+}(\mathsf{x}')\Big)\hat{\sigma}_{\rm A}^{+}\otimes\hat{\sigma}_{\rm B}^{+}\otimes\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\\
&+\theta(\tau'-\tau)\Big(\Lambda_{\rm A}^{-}(\mathsf{x}')\Lambda_{\rm B}^{-}(\mathsf{x}) + \Lambda_{\rm B}^{-}(\mathsf{x}')\Lambda_{\rm A}^{-}(\mathsf{x})\Big)\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\sigma}_{\rm A}^{-}\otimes\hat{\sigma}_{\rm B}^{-}\Big]
\end{aligned} \tag{252}$$

Therefore, the final composite state of the detectors is

$$
\begin{aligned}
\hat{\rho}_D = {} & \mathrm{Tr}_\omega\left(\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\rho}_\omega\right)\\
& - \mathrm{i}\lambda\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\rho}_\omega\right) - \mathrm{i}\lambda\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\rho}_\omega\right)\\
& + \mathrm{i}\lambda\,\mathrm{Tr}_\omega\left(\hat{\rho}_\omega\otimes\hat{\phi}(\Lambda^-_{\mathrm{A}})\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\right) + \mathrm{i}\lambda\,\mathrm{Tr}_\omega\left(\hat{\rho}_\omega\otimes\hat{\phi}(\Lambda^-_{\mathrm{B}})\hat{\sigma}^-_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\right)\\
& + \lambda^2\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\rho}_\omega\hat{\phi}(\Lambda^-_{\mathrm{A}})\otimes\hat{\sigma}^+_{\mathrm{A}}\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\right) + \lambda^2\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\rho}_\omega\hat{\phi}(\Lambda^-_{\mathrm{B}})\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\right)\\
& + \lambda^2\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\rho}_\omega\hat{\phi}(\Lambda^-_{\mathrm{A}})\otimes\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\right) + \lambda^2\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\rho}_\omega\hat{\phi}(\Lambda^-_{\mathrm{B}})\otimes\hat{\sigma}^+_{\mathrm{B}}\hat{\sigma}^-_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\right)\\
& - \lambda^2\int \mathrm{d}V\mathrm{d}V'\bigg\{\left[\Lambda^-_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\right]\\
& \left[\theta(\tau-\tau')\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right) + \theta(\tau'-\tau)\,\mathrm{Tr}_\omega\left(\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\right)\right]\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\\
& + \theta(\tau-\tau')\Big(\Lambda^+_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}') + \Lambda^+_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}')\Big)\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^+_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right)\\
& + \theta(\tau'-\tau)\Big(\Lambda^-_{\mathrm{A}}(\mathsf{x}')\Lambda^-_{\mathrm{B}}(\mathsf{x}) + \Lambda^-_{\mathrm{B}}(\mathsf{x}')\Lambda^-_{\mathrm{A}}(\mathsf{x})\Big)\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\rho}_\omega\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\right)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& = \hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\rho}_\omega\right)\\
& - \mathrm{i}\lambda\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\rho}_\omega\right) - \mathrm{i}\lambda\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\rho}_\omega\right)\\
& + \mathrm{i}\lambda\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}s\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{A}})\hat{\rho}_\omega\right) + \mathrm{i}\lambda\hat{\sigma}^-_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{B}})\ \hat{\rho}_\omega\right)\\
& + \lambda^2\hat{\sigma}^+_{\mathrm{A}}\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{A}})\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\rho}_\omega\right) + \lambda^2\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{B}})\hat{\phi}(\Lambda^+_{\mathrm{A}})\hat{\rho}_\omega\right)\\
& + \lambda^2\hat{\sigma}^+_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{A}})\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\rho}_\omega\ \right) + \lambda^2\hat{\sigma}^+_{\mathrm{B}}\hat{\sigma}^-_{\mathrm{B}}\otimes\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\Lambda^-_{\mathrm{B}})\hat{\phi}(\Lambda^+_{\mathrm{B}})\hat{\rho}_\omega\right)\\
& - \lambda^2\int \mathrm{d}V\mathrm{d}V'\bigg\{\left[\Lambda^-_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\right]\\
& \left[\theta(\tau-\tau')\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right) + \theta(\tau'-\tau)\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right)\right]\hat{\sigma}^-_{\mathrm{A}}\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\hat{\sigma}^+_{\mathrm{B}}-\\
& + \theta(\tau-\tau')\Big(\Lambda^+_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}') + \Lambda^+_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}')\Big)\hat{\sigma}^+_{\mathrm{A}}\otimes\hat{\sigma}^+_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right)\\
& + \theta(\tau'-\tau)\Big(\Lambda^-_{\mathrm{A}}(\mathsf{x}')\Lambda^-_{\mathrm{B}}(\mathsf{x}) + \Lambda^-_{\mathrm{B}}(\mathsf{x}')\Lambda^-_{\mathrm{A}}(\mathsf{x})\Big)\hat{\sigma}^-_{\mathrm{A}}\otimes\hat{\sigma}^-_{\mathrm{B}}\,\mathrm{Tr}_\omega\left(\hat{\phi}(\mathsf{x})\hat{\phi}(\mathsf{x}')\hat{\rho}_\omega\right)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \mathrm{i}\lambda\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^+_{\mathrm{A}})\rangle - \mathrm{i}\lambda\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^+_{\mathrm{B}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} s\langle\hat\phi(\Lambda^-_{\mathrm{A}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^-_{\mathrm{B}})\rangle \\
&+ \lambda^2\hat\sigma^+_{\mathrm{A}}\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{A}}) + \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{A}}) \\
&+ \lambda^2\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{B}}) + \lambda^2\hat\sigma^+_{\mathrm{B}}\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{B}}) \\
&- \lambda^2 \int \mathrm{d}V\mathrm{d}V' \Bigg\{ \Big[\Lambda^-_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\Big] \\
&\Big[\theta(\tau-\tau') + \theta(\tau'-\tau)\Big] W(\mathsf{x},\mathsf{x}')\hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&+ \theta(\tau-\tau')\Big(\Lambda^+_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}') + \Lambda^+_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}')\Big)\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^+_{\mathrm{B}} W(\mathsf{x},\mathsf{x}') \\
&+ \theta(\tau'-\tau)\Big(\Lambda^-_{\mathrm{A}}(\mathsf{x}')\Lambda^-_{\mathrm{B}}(\mathsf{x}) + \Lambda^-_{\mathrm{B}}(\mathsf{x}')\Lambda^-_{\mathrm{A}}(\mathsf{x})\Big)\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} W(\mathsf{x},\mathsf{x}') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \mathrm{i}\lambda\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^+_{\mathrm{A}})\rangle - \mathrm{i}\lambda\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^+_{\mathrm{B}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^-_{\mathrm{A}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^-_{\mathrm{B}})\rangle \\
&+ \lambda^2\hat\sigma^+_{\mathrm{A}}\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{A}}) + \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{A}}) \\
&+ \lambda^2\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{B}}) + \lambda^2\hat\sigma^+_{\mathrm{B}}\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{B}}) \\
&- \lambda^2 \int \mathrm{d}V\mathrm{d}V' \Big[\Lambda^-_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\Big] W(\mathsf{x},\mathsf{x}')\hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^+_{\mathrm{B}} \int \mathrm{d}V\mathrm{d}V' \Big(\Lambda^+_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\theta(\tau-\tau')W(\mathsf{x},\mathsf{x}') + \Lambda^+_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}')\theta(\tau-\tau')W(\mathsf{x},\mathsf{x}')\Big) \\
&- \lambda^2\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} \int \mathrm{d}V\mathrm{d}V' \Big(\Lambda^-_{\mathrm{A}}(\mathsf{x}')\Lambda^-_{\mathrm{B}}(\mathsf{x})\theta(\tau'-\tau)W(\mathsf{x},\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x}')\Lambda^-_{\mathrm{A}}(\mathsf{x})\theta(\tau'-\tau)W(\mathsf{x},\mathsf{x}')\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \mathrm{i}\lambda\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^+_{\mathrm{A}})\rangle - \mathrm{i}\lambda\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^+_{\mathrm{B}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^-_{\mathrm{A}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^-_{\mathrm{B}})\rangle \\
&+ \lambda^2\hat\sigma^+_{\mathrm{A}}\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{A}}) + \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{A}}) \\
&+ \lambda^2\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{B}}) + \lambda^2\hat\sigma^+_{\mathrm{B}}\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{B}}) \\
&- \lambda^2 \int \mathrm{d}V\mathrm{d}V' \Big[\Lambda^-_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{A}}(\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\Big] W(\mathsf{x},\mathsf{x}')\hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^+_{\mathrm{B}} \int \mathrm{d}V\mathrm{d}V' \Big(\Lambda^+_{\mathrm{A}}(\mathsf{x})\Lambda^+_{\mathrm{B}}(\mathsf{x}')\theta(\tau-\tau')W(\mathsf{x}',\mathsf{x}) + \Lambda^+_{\mathrm{B}}(\mathsf{x}')\Lambda^+_{\mathrm{A}}(\mathsf{x})\theta(\tau'-\tau)W(\mathsf{x}',\mathsf{x})\Big) \\
&- \lambda^2\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} \int \mathrm{d}V\mathrm{d}V' \Big(\Lambda^-_{\mathrm{A}}(\mathsf{x}')\Lambda^-_{\mathrm{B}}(\mathsf{x})\theta(\tau'-\tau)W(\mathsf{x},\mathsf{x}') + \Lambda^-_{\mathrm{B}}(\mathsf{x})\Lambda^-_{\mathrm{A}}(\mathsf{x}')\theta(\tau-\tau')W(\mathsf{x}',\mathsf{x})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} \\
&- \mathrm{i}\lambda\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^+_{\mathrm{A}})\rangle - \mathrm{i}\lambda\hat\sigma^+_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^+_{\mathrm{B}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}}\langle\hat\phi(\Lambda^-_{\mathrm{A}})\rangle + \mathrm{i}\lambda\hat\sigma^-_{\mathrm{B}} \otimes \hat\sigma^-_{\mathrm{A}}\hat\sigma^+_{\mathrm{A}}\langle\hat\phi(\Lambda^-_{\mathrm{B}})\rangle \\
&+ \lambda^2\hat\sigma^+_{\mathrm{A}}\hat\sigma^-_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}}\hat\sigma^+_{\mathrm{B}} W(\Lambda^-_{\mathrm{A}},\Lambda^+_{\mathrm{A}}) + \lambda^2\hat\sigma^+_{\mathrm{A}} \otimes \hat\sigma^-_{\mathrm{B}} W(\Lambda^-_{\mathrm{B}},\Lambda^+_{\mathrm{A}})
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda^2\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}W(\Lambda^-_{\text{A}},\Lambda^+_{\text{B}})+\lambda^2\hat{\sigma}^+_{\text{B}}\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}W(\Lambda^-_{\text{B}},\Lambda^+_{\text{B}})\\
&-\lambda^2\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}W(\Lambda^-_{\text{A}},\Lambda^+_{\text{A}})-\lambda^2\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}W(\Lambda^-_{\text{B}}\Lambda^+_{\text{B}})+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}\mathcal{M}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{M}^*\\
\\
&=\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\\
&-\mathrm{i}\lambda\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^+_{\text{A}})\rangle-\mathrm{i}\lambda\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^+_{\text{B}})\rangle+\mathrm{i}\lambda\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^-_{\text{A}})\rangle+\mathrm{i}\lambda\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^-_{\text{B}})\rangle\\
&+\hat{\sigma}^+_{\text{A}}\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{AA}+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{L}_{\text{BA}}+\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\mathcal{L}_{\text{AB}}+\hat{\sigma}^+_{\text{B}}\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\mathcal{L}_{BB}\\
&-\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{AA}-\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{BB}+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}\mathcal{M}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{M}^*\\
\\
&=\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}(1-\mathcal{L}_{AA}-\mathcal{L}_{BB})\\
&+\mathrm{i}\lambda\Big(-\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^+_{\text{A}})\rangle-\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^+_{\text{B}})\rangle+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^-_{\text{A}})\rangle+\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^-_{\text{B}})\rangle\Big)\\
&+\hat{\sigma}^+_{\text{A}}\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{AA}+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{L}_{\text{BA}}+\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\mathcal{L}_{\text{AB}}+\hat{\sigma}^+_{\text{B}}\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\mathcal{L}_{BB}\\
&+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}\mathcal{M}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{M}^*\\
\\
&=\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}(1-\mathcal{L}_{AA}-\mathcal{L}_{BB})\\
&+\mathrm{i}\lambda\Big(\color{red}{\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^-_{\text{A}})\rangle+\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^-_{\text{B}})\rangle}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\langle\hat{\phi}(\Lambda^-_{\text{A}})\rangle+\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\langle\hat{\phi}(\Lambda^-_{\text{B}})\rangle\Big)\\
&+\hat{\sigma}^+_{\text{A}}\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{AA}+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{L}_{\text{BA}}+\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\mathcal{L}_{\text{AB}}+\hat{\sigma}^+_{\text{B}}\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\mathcal{L}_{BB}\\
&+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}\mathcal{M}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{M}^*\\
\\
&=\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}(1-\mathcal{L}_{AA}-\mathcal{L}_{BB})\\
&+\hat{\sigma}^+_{\text{A}}\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\hat{\sigma}^+_{\text{B}}\mathcal{L}_{AA}+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{L}_{\text{BA}}+\hat{\sigma}^+_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\mathcal{L}_{\text{AB}}+\hat{\sigma}^+_{\text{B}}\hat{\sigma}^-_{\text{B}}\otimes\hat{\sigma}^-_{\text{A}}\hat{\sigma}^+_{\text{A}}\mathcal{L}_{BB}\\
&+\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}\mathcal{M}+\hat{\sigma}^-_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}\mathcal{M}^*
\end{aligned}
\tag{253}
$$

Where we have introduced

$$
\begin{cases}
\mathcal{L}_{ij}=\lambda^2W(\Lambda^-_i,\Lambda^+_j): \text{ the local noise, describing the local excitation probabilities of the detectors.}\\
\\
\mathcal{M}=-\lambda^2G_F(\Lambda^+_{\text{A}},\Lambda^+_{\text{B}}): \text{ nonlocal correlation amplitudes between detectors.}
\end{cases}
\tag{254}
$$

Observation 8.1. *We can also write the tensor products of operators $\hat{\sigma}^{\pm}$ in matrix form:*

$$
\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^+_{\text{B}}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\ \begin{matrix}0&0\\1&0\end{matrix}&0\end{pmatrix}
$$

$$
\hat{\sigma}^+_{\text{A}}\otimes\hat{\sigma}^-_{\text{B}}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\ \begin{matrix}0&1\\0&0\end{matrix}&0\end{pmatrix}
$$

$$\hat{\sigma}_{\text{A}}^{-} \otimes \hat{\sigma}_{\text{B}}^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \\ & \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_{\text{A}}^{-} \otimes \hat{\sigma}_{\text{B}}^{-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \\ & \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_{\text{A}}^{-} \hat{\sigma}_{\text{A}}^{+} \otimes \hat{\sigma}_{\text{B}}^{-} \hat{\sigma}_{\text{B}}^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \\ & \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_{\text{A}}^{-} \hat{\sigma}_{\text{A}}^{+} \otimes \hat{\sigma}_{\text{B}}^{+} \hat{\sigma}_{\text{B}}^{-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & 0 \\ & \\ 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_{\text{A}}^{+} \hat{\sigma}_{\text{A}}^{-} \otimes \hat{\sigma}_{\text{B}}^{-} \hat{\sigma}_{\text{B}}^{+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ & \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

Using this, we can write the final state in the matrix form

$$\hat{\rho}_D = \begin{pmatrix} 1 - \mathcal{L}_{AA} - \mathcal{L}_{BB} & 0 & 0 & \mathcal{M}^* \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{\text{AB}} & 0 \\ 0 & \mathcal{L}_{\text{BA}} & \mathcal{L}_{AA} & 0 \\ \mathcal{M} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{255}$$

It will be important to us to compute the partial transpose of this matrix:

$$\hat{\rho}_D^{\top_{\text{B}}} = \begin{pmatrix} 1-\mathcal{L}_{AA}-\mathcal{L}_{BB} & 0 & 0 & \mathcal{L}_{\text{AB}} \\ 0 & \mathcal{L}_{BB} & \mathcal{M} & 0 \\ 0 & \mathcal{M}^* & \mathcal{L}_{AA} & 0 \\ \mathcal{L}_{\text{BA}} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{256}$$

Observation 8.2. *This matrix has eigenvalues*

$$\sigma_{\hat{\rho}_D^{\top_{\text{B}}}} = \Big\{0, 1-\mathcal{L}_{AA}-\mathcal{L}_{BB}, \frac{1}{2}\Big(\mathcal{L}_{AA}+\mathcal{L}_{BB}-\sqrt{\mathcal{L}_{AA}^2-2\mathcal{L}_{AA}\mathcal{L}_{BB}+\mathcal{L}_{BB}^2+4\mathcal{M}\mathcal{M}^*}\Big), \\ \frac{1}{2}\Big(\mathcal{L}_{AA}+\mathcal{L}_{BB}+\sqrt{\mathcal{L}_{AA}^2-2\mathcal{L}_{AA}\mathcal{L}_{BB}+\mathcal{L}_{BB}^2+4\mathcal{M}\mathcal{M}^*}\Big)\Big\}$$

The first eigenvalue is $\sim 1-\lambda^2$, so it is positive for small values of λ (where our approximation is valid), and the third eigenvalue is a sum of positive terms. Therefore, the only potentially negative eigenvalue is

$$\lambda = \frac{1}{2}\Big(\mathcal{L}_{AA}+\mathcal{L}_{BB}-\sqrt{\mathcal{L}_{AA}^2-2\mathcal{L}_{AA}\mathcal{L}_{BB}+\mathcal{L}_{BB}^2+4\mathcal{M}\mathcal{M}^*}\Big) \tag{257}$$

and we can use it to compute the ***negativity****:*

$$\mathcal{N}(\hat{\rho}_D) = max\Bigg(0, \sqrt{|\mathcal{M}|^2-\left(\frac{\mathcal{L}_{AA}-\mathcal{L}_{BB}}{2}\right)^2}-\frac{\mathcal{L}_{AA}+\mathcal{L}_{BB}}{2}\Bigg) \tag{258}$$

Example 8.1. *In Minkowski spacetime,*

$$W(\mathsf{x},\mathsf{x}') = \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|}\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t-t')}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}$$

If we consider gaussian smearing functions

$$\Lambda_{\text{A}}(\mathsf{x}) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}}\,\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}, \quad \Lambda_{\text{B}}(\mathsf{x}) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}}\,\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{L}|^2}{2\sigma^2}} \tag{259}$$

Then we can find an explicit expression for $\mathcal{M}$:

$$\begin{aligned}\mathcal{M} &= -\lambda^2 G_F(\Lambda_{\text{A}}^+,\Lambda_{\text{B}}^+) \\ &= -\lambda^2\int\mathrm{d}V\mathrm{d}V'\Big(\Lambda_{\text{A}}(\mathsf{x})\Lambda_{\text{B}}(\mathsf{x}')\mathrm{e}^{i\Omega(t+t')}W(\mathsf{x},\mathsf{x}')\theta(t-t')+\Lambda_{\text{A}}(\mathsf{x}')\Lambda_{\text{B}}(\mathsf{x})\mathrm{e}^{i\Omega(t'+t)}W(\mathsf{x}',\mathsf{x})\theta(t'-t)\Big)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\lambda^2}{(2\pi\sigma^2)^3}\int \mathrm{d}V\mathrm{d}V'\Bigg(\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}}\mathrm{e}^{i\Omega(t+t')}W(\mathsf{x},\mathsf{x}')\theta(t-t')\\
&\qquad\qquad + \mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'|^2}{2\sigma^2}}\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{L}|^2}{2\sigma^2}}\mathrm{e}^{i\Omega(t'+t)}W(\mathsf{x}',\mathsf{x})\theta(t'-t)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|}\\
&\int \mathrm{d}t\,\mathrm{d}t'\Bigg(\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t-t')}\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{i\Omega(t+t')}\theta(t-t') + \mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t'-t)}\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{i\Omega(t'+t)}\theta(t'-t)\Bigg)\\
&\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}'\Bigg(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}} + \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\mathrm{e}^{-\frac{|\boldsymbol{x}'|^2}{2\sigma^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{L}|^2}{2\sigma^2}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|}\\
&\int \mathrm{d}t\,\mathrm{d}t'\Bigg(\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t-t')}\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{i\Omega(t+t')}\theta(t-t') + \mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t-t')}\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{i\Omega(t+t')}\theta(t-t')\Bigg)\\
&\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}'\Bigg(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}} + \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{4}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|}\int \mathrm{d}t\,\mathrm{d}t'\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|(t-t')}\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\mathrm{e}^{i\Omega(t+t')}\theta(t-t')\\
&\qquad\qquad\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}'\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{4}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{2|\boldsymbol{k}|}\int \mathrm{d}t\,\mathrm{d}t'\mathrm{e}^{\mathrm{i}(\Omega-|\boldsymbol{k}|)t}\mathrm{e}^{\mathrm{i}(\Omega+|\boldsymbol{k}|)t'}\,\mathrm{e}^{-\frac{t^2}{2T^2}}\,\mathrm{e}^{-\frac{t'^2}{2T^2}}\,\theta(t-t')\\
&\qquad\qquad\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\int \mathrm{d}^3\boldsymbol{x}'\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}}
\end{aligned}
$$

$$
= -\frac{1}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\,2\,Q(|\boldsymbol{k}|,\Omega)\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2}}\int \mathrm{d}^3\boldsymbol{x}'\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}\,\mathrm{e}^{-\frac{|\boldsymbol{x}'-\boldsymbol{L}|^2}{2\sigma^2}}
$$

$$
= -\frac{1}{(2\pi)^3}\frac{\lambda^2}{(2\pi\sigma^2)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\,2\,Q(|\boldsymbol{k}|,\Omega)(2\pi)^3\left(\sigma^3\mathrm{e}^{-\frac{1}{2}|\boldsymbol{k}|^2\sigma^2}\right)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{L}}\left(\sigma^3\mathrm{e}^{-\frac{1}{2}|\boldsymbol{k}|^2\sigma^2}\right)
$$

$$
= -\frac{\lambda^2}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\,2\,Q(|\boldsymbol{k}|,\Omega)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{L}}\,\mathrm{e}^{-|\boldsymbol{k}|^2\sigma^2}
$$

$$
\begin{aligned}
&= -\frac{\lambda^2}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^{\infty}\frac{\mathrm{d}|\boldsymbol{k}|}{|\boldsymbol{k}|}\,\sin(\theta)|\boldsymbol{k}|^2 2\,Q(|\boldsymbol{k}|,\Omega)\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}||\boldsymbol{L}|\cos(\theta)}\,\mathrm{e}^{-|\boldsymbol{k}|^2\sigma^2}\\
&= -\frac{4\pi\lambda^2}{(2\pi)^3}\int_0^{\infty}\frac{\mathrm{d}|\boldsymbol{k}|}{|\boldsymbol{k}|}\mathrm{e}^{-|\boldsymbol{k}|^2\sigma^2}\,2\,Q(|\boldsymbol{k}|,\Omega)\,2|\boldsymbol{k}|\frac{\sin(|\boldsymbol{k}||\boldsymbol{L}|)}{|\boldsymbol{L}|}\\
&= -\frac{4\pi\lambda^2}{(2\pi)^3}\int_0^{\infty}\mathrm{d}|\boldsymbol{k}|\;\mathrm{e}^{-|\boldsymbol{k}|^2\sigma^2}\,2\,Q(|\boldsymbol{k}|,\Omega)\,2|\boldsymbol{k}|\,\mathrm{sinc}(|\boldsymbol{k}||\boldsymbol{L}|)
\end{aligned}
\tag{260}
$$

Observation 8.3.

$$
\hat{\rho}_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}(\hat{\rho}_D) = \begin{pmatrix} 1-\mathcal{L}_{AA} & 0 \\ 0 & \mathcal{L}_{AA}\end{pmatrix}, \quad \hat{\rho}_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}(\hat{\rho}_D) = \begin{pmatrix} 1-\mathcal{L}_{BB} & 0 \\ 0 & \mathcal{L}_{BB}\end{pmatrix} \tag{261}
$$

Observation 8.4. *If the detectors start in the ground or excited state, to leading order, we cannot see the effect of exchange of energy between them. Indeed, emitting (or absorbing) a "particle" when starting in the excited (or ground) state is a second order effect, that therefore cannot be seen to leading order.*

9 Entanglement in Quantum Field Theory

Let us consider two regions A and B of spacetime. In Quantum Field Theory, states are global and there are no Hilbert spaces associated to different spacetime regions A or B:

$$
\mathcal{F}(\mathcal{H}) \neq \mathcal{F}(\mathcal{H}_{\mathrm{A}}) \otimes \mathcal{F}(\mathcal{H}_{\mathrm{B}}) \tag{262}
$$

Both $\mathcal{F}(\mathcal{H}_{\mathrm{A}})$ and $\mathcal{F}(\mathcal{H}_{\mathrm{B}})$ are not even properly defined. Therefore, it is very complicated to talk about entanglement in QFT. This is due to vacuum effects.

9.1 Entanglement harvesting

Let us consider two detectors smeared with $\Lambda_{\mathrm{A}}(\mathsf{x})$ and $\Lambda_{\mathrm{B}}(\mathsf{x})$ over two regions $\mathcal{O}_{\mathrm{A}}, \mathcal{O}_{\mathrm{B}}$ of spacetime, sampling the degrees of freedom of the algebra

$$
\mathcal{A}(\mathrm{supp}\,\Lambda_i) = \mathcal{A}(D(\mathrm{supp}\,\Lambda_i)) \tag{263}
$$

Moreover, in order to witness entanglement, we require these regions to be causally disconnected so that we can be sure that the entanglement of the detectors comes solely from the entanglement between the regions and not from classical communication.

There are two ways for the detectors to become entangled:

1. They communicate **through** the field: they sample non-independent degrees of freedom of the field. In this case, there is causal connection.

2. They extract previously existing entanglement in the field: they sample independent degrees of freedom.

This entanglement can be quantified by the *negativity*:

$$\mathcal{N}(\hat{\rho}_D) = \max(0, |\mathcal{M}| - \mathcal{L}) \tag{264}$$

if we assume $\mathcal{L}_{AA} = \mathcal{L}_{BB}$.

Observation 9.1. *The negativity is only a quantifier of the entanglement and it doesn't allow us to distinguish where the entanglement is coming from. This is the reason we require the causal disconnection.*

Indeed, for **causally disconnected regions of spacetime**,

1. If $\mathcal{L} > |\mathcal{M}|$, then $\mathcal{N}(\hat{\rho}_D) = 0$: the local noise overshadows the nonlocal correlations, so no entanglement can be extracted.

2. If $\mathcal{L} < |\mathcal{M}|$, then $\mathcal{N}(\hat{\rho}_D) > 0$: the nonlocal correlations are strong enough to overcome the local noise, and the entanglement can be extracted.

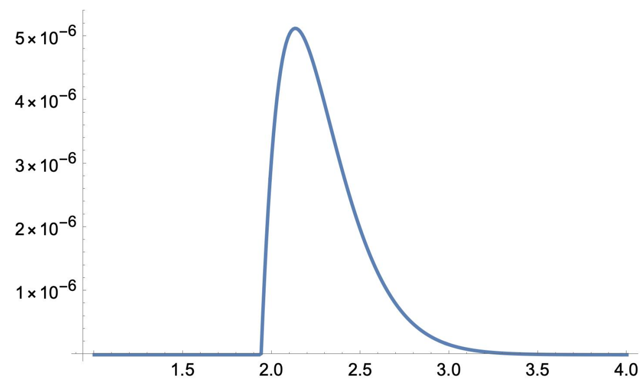

Figure 1: Negativity as a function of ΩT in Minkowski spacetime for $L = 5T, \sigma = 0.1T$

Observation 9.2. *Negativity is asymptotically zero, hence the reason we consider finite times.*

Observation 9.3. *As introduced earlier, $\mathcal{M} = -\lambda^2 G_F(\Lambda_{\mathrm{A}}^+, \Lambda_{\mathrm{B}}^+)$. Therefore,*

$$\mathcal{M} = -\frac{1}{2}\lambda^2\Big(H(\Lambda_{\mathrm{A}}^+, \Lambda_{\mathrm{B}}^+) + \mathrm{i}\Delta(\Lambda_{\mathrm{A}}^+, \Lambda_{\mathrm{B}}^+)\Big) \tag{265}$$

- *The delta term Δ:*

$$\begin{cases} \text{is only non-zero when } A \text{ and } B \text{ are causally connected.} \\ \\ \text{is state independent.} \end{cases} \tag{266}$$

$\rightarrow$ *It encodes classical communication.*

- *The Hadamard term H:*

$$\begin{cases} \textit{is always non-zero, in particular when } A \textit{ and } B \textit{ are causally disconnected.} \\ \\ \textit{is state dependent.} \end{cases} \tag{267}$$

$\rightarrow$ *It encodes previously existing entanglement.*

Observation 9.4. *Gaussian regions of spacetime are never truly causally disconnected because of the tails.*
In this case, we define

$$\mathcal{N}^+ \equiv |\mathcal{M}^+| - \mathcal{L} \tag{268}$$

where

$$\mathcal{M}^+ = -\frac{1}{2}\lambda^2 H(\Lambda^+_{\text{A}}, \Lambda^+_{\text{B}})$$

$$\Delta^+ = -\frac{1}{2}\lambda^2 \Delta(\Lambda^+_{\text{A}}, \Lambda^+_{\text{B}})$$

If $\frac{\Delta^+}{\mathcal{M}^+}$ is small enough, then we can conclude that entanglement between the probes is still mostly extracted from the field and the effect of signaling is negligible. That is, we can consider the regions to be causally disconnected.