Structures of General Relativity and Gauge Theory

General Relativity as a Lorentz gauge theory: gauge field from the Levi-Civita connection?

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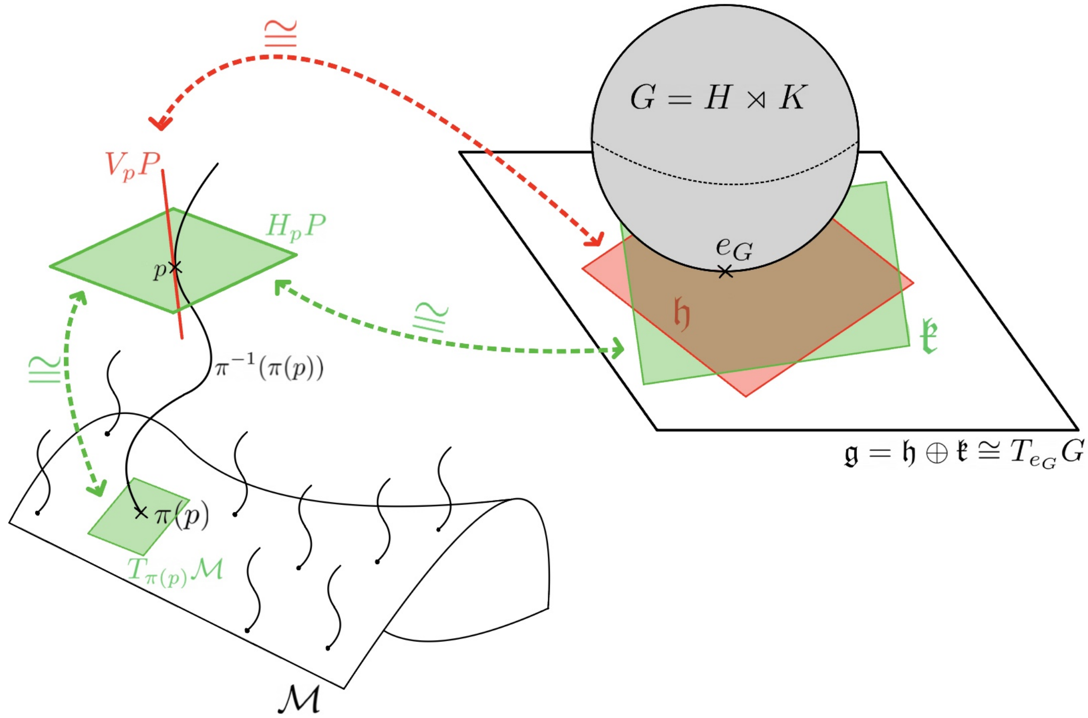

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0.1 Mathematical foundations

0.1.1 Manifolds and differential forms

From a geometrical perspective, most physical theories model our world by means of **manifolds**, a type of space with particular properties.

Definition 0.1 (Topological manifold) *A real n-dimensional **topological manifold**[1] is a topological space[2] $\mathcal{M}$ that is locally euclidean. More precisely, for each point $x \in \mathcal{M}$, there exists an open neighborhood U_x of x that is homeomorphic[3] to an open subspace V of $\mathbb{R}^n$:*

$$U_x \cong V \subseteq \mathbb{R}^n$$

From now on, we implicitly assume that the manifolds we will consider are topological and of dimension n unless specified otherwise.

Example 0.1 *The 2-sphere S^2 (Fig. 1) is a natural example of a manifold: it has a certain geometry, but from each point on the surface, it appears to be flat (which is the reason why some people believe the Earth to be flat).*
However, it is a known result in topology that a sphere with a hair (Fig. 2) is not a manifold: the junction point between the sphere and the hair is not locally euclidean.

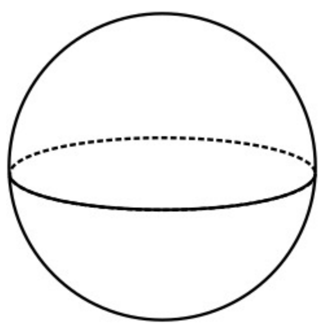

Figure 1: The 2-sphere is a manifold.

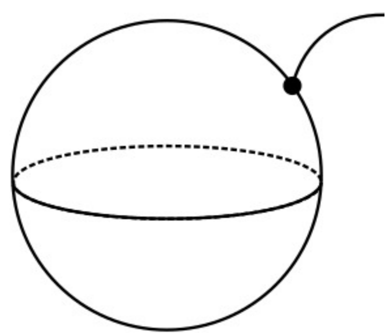

Figure 2: The 2-sphere with a hair is *not* a manifold.

Definition 0.2 (Coordinate charts) *A **coordinate chart** on $\mathcal{M}$ is a pair (U, ϕ) consisting of an open subset $U \subset \mathcal{M}$ and a homeomorphism $\phi : U \to V \subseteq \mathbb{R}^n$*

Remark 0.1 *It is a common abuse of notation to omit the map ϕ and denote by x both the point on the manifold and its coordinates.*

Definition 0.3 (Atlas) *An **atlas** on $\mathcal{M}$ is a family of coordinate charts (U_i, ϕ_i) such that*

i The subsets U_i cover the whole manifold: $\mathcal{M} = \cup_i U_i$

ii Each map[4] $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is C^∞ (Fig. 3).

[1] By speaking about *topological* manifolds, we assume the manifold is Hausdorff and second-countable, but this is not relevant for our work so we will not discuss it.

[2] A topological space is a set of elements called points, endowed with a topology, which can intuitively be thought of as a set of neighborhoods at each point satisfying a set of axioms.

[3] A map $f : X \to Y$ between topological spaces is called an *homeomorphism* if f is bijective, continuous and has a continuous inverse map f^{-1}.

[4] This map is called the *overlap function*.

An atlas defines[5] *a **differential structure** on $\mathcal{M}$.*

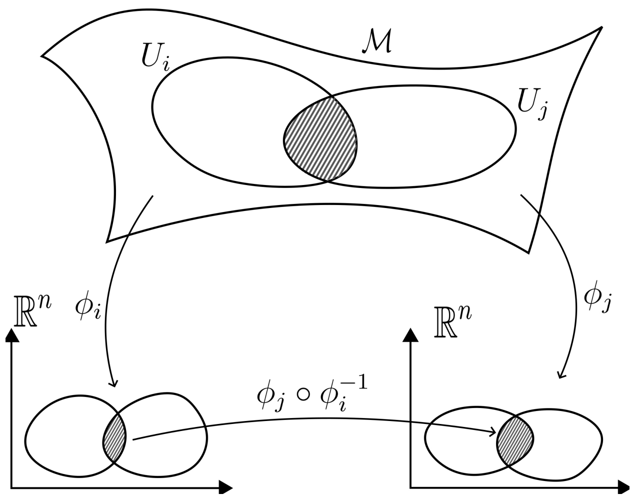

Figure 3: (Overlap) map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$

Definition 0.4 (Differentiable manifold) *A manifold $\mathcal{M}$ endowed with a differential structure is called a **differential manifold**.*

Given a starting point $x \in \mathcal{M}$, the smoothness of the manifold allows us to consider infinitesimal movements in various directions. The set of all these directions in which we can move is called the *tangent space*. Intuitively, infinitesimal movements correspond essentially to differentiation of smooth curves along $\mathcal{M}$ at x, and these derivatives yield vectors tangent to $\mathcal{M}$. More precisely,

Definition 0.5 (Tangent bundle) *The **tangent space** $T_x\mathcal{M}$ to $\mathcal{M}$ at $x \in \mathcal{M}$ is the set of all tangent vectors to $\mathcal{M}$ at x (Fig. 4)*[6].

*We define the **tangent bundle** as $T\mathcal{M} := \bigsqcup_{x \in \mathcal{M}} T_x\mathcal{M}$ and, as we will see, sections of this bundle will give rise to vector fields.*

Remark 0.2 *A tangent space carries a vector space structure.*

Definition 0.6 (Riemannian manifold) *A **Riemannian manifold** is a differentiable manifold equipped with a metric.*

Once the tangent bundle structure has been defined, we can decide to "choose" one vector from each tangent space and this procedure will give rise to a *vector field*. In other words,

[5] In reality, we also require that the atlas is *complete* (or maximal), in other words, that it is not contained in any other atlas.

[6] There are more formal definitions (using derivations, equivalence classes of curves, etc) but this geometrical definition will be enough for our work.

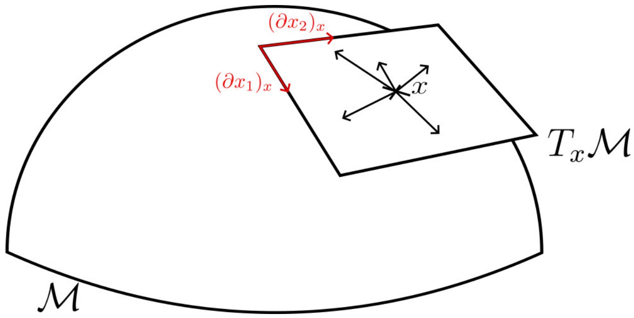

Figure 4: Tangent space $T_x\mathcal{M}$ to a 2-dimensional manifold $\mathcal{M}$ at x.

Definition 0.7 (Vector fields) *A* ***vector field*** *X on $\mathcal{M}$ is a smooth assignment of a tangent vector $X_x \in T_x\mathcal{M}$ to each point $x \in \mathcal{M}$.*

Equivalently, we can see a vector field X to be a smooth section[7] *$\xi \in \Gamma(T\mathcal{M})$ of the tangent bundle $T\mathcal{M}$ (Fig. 5). For each x, we have a corresponding vector $X_x := \xi(x) \in T_x\mathcal{M}$:*

$$\xi(x) = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$$

where ξ^i are smooth functions and $B_x = \left\{ \frac{\partial}{\partial x^i} \right\}$ is the coordinate basis of $T_x\mathcal{M}$.

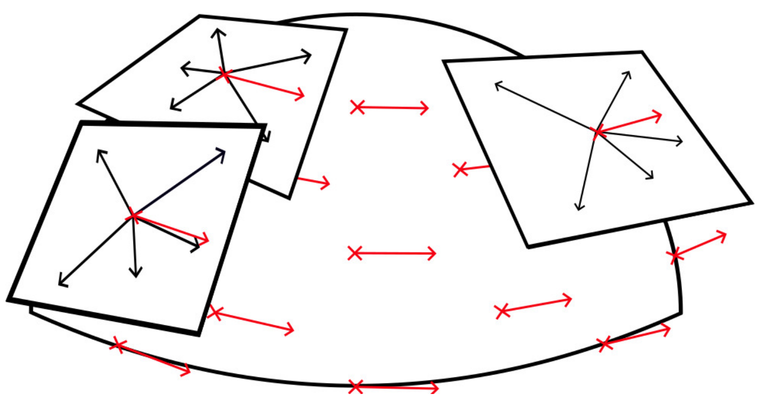

Figure 5: A vector field corresponds to "choosing" a vector from each tangent space.

So far, we have discussed tangent vectors and how they allow for infinitesimal movements on the manifold. We shall now ask ourselves about the objects acting on them which, in turn, will be of the utmost importance.

Definition 0.8 (Dual space) *The (algebraic*[8]*)* ***dual space*** *V^* of a (real) vector space V is the set of all linear maps $\phi : V \to \mathbb{R}$. The space V^* inherits a real vector space structure.*

[7] We will introduce this concept in the upcoming sections when we discuss fibre bundles. For now, the heuristic interpretation of"assigning a vector to each point" is enough.

[8] In opposition to the algebraic dual, the topological dual consists of all *continuous* linear maps. However, in finite-dimensional vector spaces, both duals are isomorphic and we do not bother with this distinction.

We claimed that the tangent spaces carry vector space structure, and therefore it is possible to define the dual space of a tangent space.

Definition 0.9 (Cotangent bundle) *The **cotangent space** $T^*_x\mathcal{M}$ is the dual space of $T_x\mathcal{M}$, i.e., it is the set of all linear maps $\phi : T_x\mathcal{M} \to \mathbb{R}$.*

*Again, we define the **cotangent bundle** as the set of all cotangent vectors at all points of the manifold: $T^*\mathcal{M} := \bigsqcup_{x\in\mathcal{M}} T^*_x\mathcal{M}$.*

Remark 0.3 (Musical isomorphism) *There exists an isomorphism (Fig. 6) between the tangent space $T_x\mathcal{M}$ and the cotangent space $T^*_x\mathcal{M}$*

$$\begin{aligned} \flat : T\mathcal{M} &\to T^*\mathcal{M} \\ v &\to \flat v := v^* \end{aligned} \qquad\qquad \begin{aligned} \sharp : T^*\mathcal{M} &\to T\mathcal{M} \\ v^* &\to \sharp v^* := v \end{aligned}$$

where $v^* \in T^*_x\mathcal{M}$ is the covector (or dual vector) associated to $v \in T_x\mathcal{M}$.

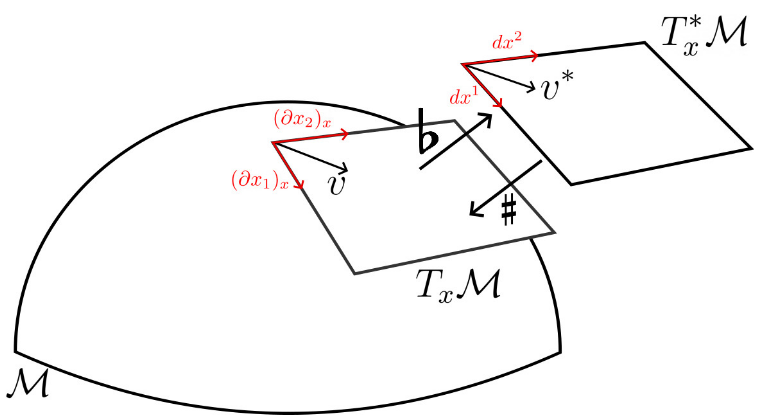

Figure 6: Musical isomorphism between the tangent and cotangent space.

Remark 0.4 *We have written the bases of the tangent and cotangent space as $\{\partial x_i\}$ and $\{dx^i\}$. However, they can be written in a more general way as $\{e_a\}$ (tetrads) and $\{e^a\}$ (co-tetrads), known as non-coordinate bases, such that*

$$e^\mu_a e^b_\nu = \delta^\nu_\mu \tag{1}$$

where $e_a = e^\mu_a \partial_\mu$ and $e^a = e^a_\mu dx^\mu$.

What we have implicitly done until now and will keep doing in the following work, is to select particular bases $\{e_a\} = \{\partial_\mu\}$ and $\{e^a\} = \{dx^\mu\}$, called coordinate bases, which are defined by the fact that they commute

$$[\partial_\mu, \partial_\nu] = 0, \qquad \partial_\mu(dx^\nu) = \delta^\mu_\nu \tag{2}$$

In the same way that a section of the tangent bundle gave us vector fields, the section of the cotangent bundle will yield a new object: a *one-form*. Further discussion will be done on differential forms, but for now let us give the following preliminary definition:

Definition 0.10 (One-form) *A* ***one-form*** ω *is a smooth assignment of a cotangent vector* ω_x *to each point* $x \in \mathcal{M}$

In order to generalize the discussion to general differential forms of order p, we need to introduce the concept of *tensors*.

Definition 0.11 (Tensor fields) *A general* V*-valued tensor* T_x *of type* (p,q) *at* $x \in \mathcal{M}$ *and of type* (r,s) *in* V *is an element of the tensor product space*

$$T_x \in T^p_{q,x}(\mathcal{M}) \otimes T^r_s(V) := \left(\bigotimes^p T_x\mathcal{M}\right) \otimes \left(\bigotimes^q T^*_x\mathcal{M}\right) \otimes \left(\bigotimes^r V\right) \otimes \left(\bigotimes^s V^*\right)$$

What we mean by this is an element of a field that assigns a (p,q)*-tensor to each point of the manifold, where the components of this tensor are themselves* (r,s)*-tensors that take values in* V*. As an example, consider a vector field with values in* $\mathbb{R}^n$*: to each point on the manifold, we assign a vector* $v \in T_x\mathcal{M}$*. However, this vector takes values in* $\mathbb{R}^n$*, i.e. its components are themselves* n*-dimensional vectors. In total, we get a "vector of vectors" in* $T_x\mathcal{M} \otimes \mathbb{R}^n$ *with* $n \times m$ *components.*

Such a tensor can be developed in the different basis as

$$T_x = (T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q,x})^{a_1\ldots a_r}{}_{b_1\ldots b_s}\left(\bigotimes_{i=1}^p \partial_{\mu_i}\right) \otimes \left(\bigotimes_{j=1}^q dx^{\nu_j}\right) \otimes \left(\bigotimes_{k=1}^r E_{a_k}\right) \otimes \left(\bigotimes_{l=1}^s E^{b_l}\right)$$

A ***tensor field*** T *of type (p,q) can be seen as a smooth assignment of a (p,q)-tensor to each point* x *on the manifold. Equivalently, it can be seen as a smooth section*

$$T \in \Gamma(\mathcal{T}^{(p,q;r,s)}(\mathcal{M}, V))$$

of the ***tensor bundle***

$$\mathcal{T}^{(p,q;r,s)}(\mathcal{M}, V) :=\bigsqcup_{x\in\mathcal{M}}\left[\left(\bigotimes^p T_x\mathcal{M}\right) \otimes \left(\bigotimes^q T^*_x\mathcal{M}\right) \otimes \left(\bigotimes^r V\right) \otimes \left(\bigotimes^s V^*\right)\right]$$

$$= \left(\bigotimes^p T\mathcal{M}\right) \otimes \left(\bigotimes^q T^*\mathcal{M}\right) \otimes \left(\bigotimes^r V\right) \otimes \left(\bigotimes^s V^*\right)$$

We can now speak about p-forms, and to do so let us consider a tensor field of type $(0,p)$:

$$T \in T^0_p = \left(\bigotimes^p T^*\mathcal{M}\right) \otimes V$$

which can be expressed as

$$T = (T_{\mu_1\ldots\mu_p})^a dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p} \otimes E_a$$

Definition 0.12 (p-forms) *A **p-form** ω $(0 \leq p \leq n)$ is an antisymmetric tensor field of type $(0,p)$.*
*After imposing this condition, we can replace the tensor products by **wedge products** (which we will define shortly)*

$$\omega = (\omega_{\mu_1 \dots \mu_p})^a dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \otimes E_a \tag{3}$$

and we can define the set of all p-forms as

$$\omega \in \Omega^p(\mathcal{M}, V) = \left(\bigwedge\nolimits^p T^*\mathcal{M}\right) \otimes V$$

Remark 0.5 *More generally, it is possible to define the set of all differential forms on $\mathcal{M}$:*

$$\bigwedge(\mathcal{M}) = \bigoplus_{p=0}^{n} \Omega^p(\mathcal{M})$$

We shall now finish our preliminary discussion on differential geometry by introducing operations on the differential forms.

Definition 0.13 (Exterior product) *Consider two differential forms ω_1 and ω_2 of rank n_1 and n_2 respectively. The **wedge product**, also known as exterior product, of ω_1 and ω_2 is the $(n_1 + n_2)$-form defined by*

$$\omega_1 \wedge \omega_2 = \frac{1}{n_1! n_2!} \sum_{PermP} (-1)^{degP} (\omega_1 \otimes \omega_2)^P$$

where $PermP$ gives us the different permutations of ω_1 and ω_2.

Example 0.2 *Let us consider two 1-forms dx and dy. Then,*

$$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$$

Definition 0.14 (Interior product) *The **interior product** ι_ξ is a graded derivation operator of degree -1*

$$\iota_\xi : \Omega^p(\mathcal{M}, V) \to \Omega^{p-1}(\mathcal{M}, V)$$

It acts on differential p-forms ω and lowers their degree by one.

Indeed, given a p-form ω, the interior product $\iota_\xi \omega$ is the $(p-1)$-form obtained by the contraction of ω with the vector field ξ:

$$(\iota_\xi \omega)(X_1, \dots, X_{p-1}) = \omega(\xi, X_1, \dots, X_{p-1})$$

for some vector fields $X_1, \dots, X_{p-1}$. This is telling us that that applying ι_ξ to ω gives a form that behaves exactly like ω, with the difference that the first component is "fixed" to be ξ.
In index notation,

$$\iota_\xi(\omega_{\mu_1 \dots \mu_p}) = (\iota_\xi \omega)_{\mu_1 \dots \mu_{p-1}}$$

$$= \xi^\nu \omega_{\nu\ldots\mu_p}$$

We can also define the complementary operator j_ξ:

$$\begin{aligned} j_\xi : \Omega^p(\mathcal{M}, V) &\to \Omega^{p+1}(\mathcal{M}, V) \\ \omega &\to j_\xi\omega = \flat\xi \wedge \omega \end{aligned}$$

such that

$$\langle \iota_\xi \alpha, \beta \rangle = \langle \alpha, j_\xi \beta \rangle \tag{4}$$

Proposition 0.1 *The interior product has the following properties:*

i By convention, if ω is a 0*-form (scalar field), then $\iota_\xi \omega = 0$.*

ii Let α and β be two p-forms. Then,

$$\iota_\xi(\alpha \wedge \beta) = (\iota_\xi \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_\xi \beta) \tag{5}$$

Definition 0.15 (Exterior derivative) *The **exterior derivative** d is a graded derivation operator of degree* +1

$$d : \Omega^p(\mathcal{M}, V) \to \Omega^{p+1}(\mathcal{M}, V)$$

acting on a p-form and giving a $(p+1)$-form.

Proposition 0.2 *The exterior derivative has the following properties:*

i It is closed, i.e. $d(d\omega) = 0$ for any p-form ω

ii Since forms on an n-dimensional manifold can be of, at most, rank n, then the exterior derivative on an n-form is null.

iii Let α and β be two p-forms. Then,

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) \tag{6}$$

Using (3), we can act on a general p-form with the exterior derivative $d : \Omega^p(\mathcal{M}, V) \to \Omega^{p+1}(\mathcal{M}, V)$:

$$d\phi = \partial_{\mu_0}(\phi_{\mu_1\ldots\mu_p})^a dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \otimes E_a \tag{7}$$

Definition 0.16 (Push-forward and pullback) *Let h be a smooth endormorphism $h : \mathcal{M} \to \mathcal{M}$* [9]*.*

We can define

*i The **push-forward**:*

$$\begin{aligned} h_* : T_x\mathcal{M} &\to T_{h(x)}\mathcal{M} \\ v &\to h_* v \end{aligned}$$

[9] In the general case, $h : \mathcal{M} \to \mathcal{N}$ is only a smooth map, where $\mathcal{M}$ and $\mathcal{N}$ are two distinct manifolds.

*ii The **pullback**:*

$$
\begin{aligned}
h^* : T^*_{h(x)}\mathcal{M} &\to T^*_x\mathcal{M} \\
\omega &\to h^*\omega
\end{aligned}
$$

such that

$$(h^*\omega)_x(v) = \omega_{h(x)}(h_* v) \tag{8}$$

as shown in Fig. 7.

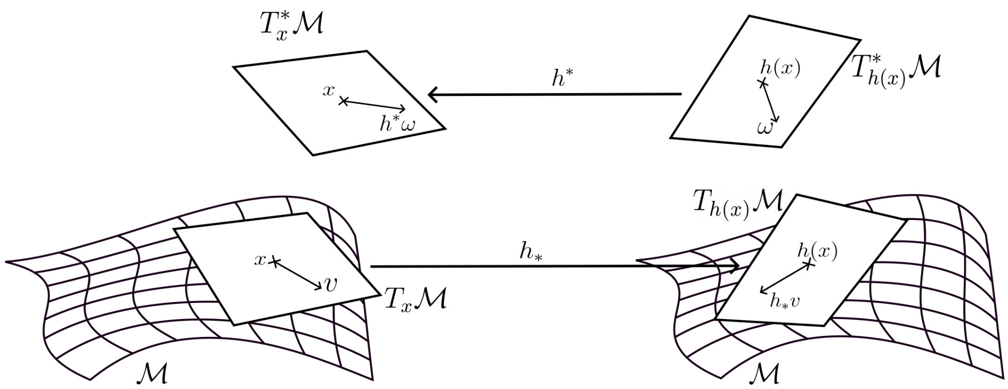

Figure 7: Pullback and push-forward

Furthermore, the push-forward and pullback commute with the musical isomorphism (Fig. 8).

$$
\begin{tikzcd}
T_x^*\mathcal{M} \arrow[d, shift left, "\flat"] & T^*_{h(x)}\mathcal{M} \arrow[l, "h^*"'] \arrow[d, shift left, "\flat"] \\
T_x\mathcal{M} \arrow[u, shift left, "\sharp"] \arrow[r, "h_*"'] & T_{h(x)}\mathcal{M} \arrow[u, shift left, "\sharp"]
\end{tikzcd}
$$

Figure 8: Commutative diagram of the musical isomorphism

Definition 0.17 (Lie derivative) *Let us consider a form ω and a vector field ξ. The **Lie derivative** $\mathcal{L}_\xi \omega$ of the form ω tells us how it changes along the flow of ξ:*

$$\mathcal{L}_\xi \omega = \frac{d}{dt}\bigg|_{t=0} (\phi_t^* \omega)$$

where ϕ_t^* is the pullback of the flow ϕ_t generated by ξ.

The flow of ξ is a map $\phi_t : \mathcal{M} \to \mathcal{M}$ that "moves" the points of the manifold along ξ. For example, $p' := \phi_t(p)$ is the point where $p \in \mathcal{M}$ ends up after flowing along ξ for a time t. Since the Lie derivatives tells us about the flow of a form (and not a point), it makes use of the pullback of the flow.
The Lie derivative can also be expressed by "Cartan's magic formula"

$$\mathcal{L}_\xi \omega = \iota_\xi (d\omega) + d(\iota_\xi \omega) \tag{9}$$

Definition 0.18 (Hodge operator) *Consider a Riemannian manifold*[10] *endowed with a metric g.*
The ***Hodge star operator*** *is a linear operator that maps p-forms to $(n-p)$-forms.*
It is defined by the following property

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_g \tag{10}$$

where $\alpha, \beta \in \Omega^p$, $\langle \cdot, \cdot \rangle$ is the scalar product induced by the metric g and $\omega_g = \star 1 \in \Omega^n$ is the unit volume form given by

$$\omega_g = \sqrt{|det(g)|} dx^1 \wedge \cdots \wedge dx^n$$

In index notation,

$$\star\alpha = \frac{\sqrt{|det(g)|}}{(n-p)!} \alpha_{\mu_1 \dots \mu_p} \varepsilon^{\mu_1 \dots \mu_p}{}_{\nu_{p+1} \dots \nu_n} dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_n}$$

Example 0.3 *The Hodge dual of a 1-form in 4-dimensional spacetime is given by*

$$\star dx^\mu = \sqrt{|det(g)|} \frac{1}{3!} \varepsilon^\mu{}_{\nu_1 \nu_2 \nu_3} dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\nu_3} \tag{11}$$

0.1.2 Lie theory and representations

Arguably, one of the most important concepts in this work, and perhaps in all of physics, is that of *symmetries*: transformations that leave a system unchanged. As straightforward as this topic might seem at first glance, there is a lot to say about symmetries, and we will give a formal definition in the upcoming sections. For now, let our focus be on *continuous* symmetries: transformations that vary smoothly and infinitesimally.
This concept is the main motivation for this whole section.

When we speak of transformations in physics, the next idea that should come to mind is that of *groups*[11]. Indeed, this is the main mathematical tool with which we describe transformations (rotations, flipping with respect to a symmetry axis, etc).

[10] For the following construction, it is enough to consider a pseudo-Riemannian manifold, but we will not make the distinction here.

[11] By group, we understand the usual linear algebra definition: a set equipped with a binary operation satisfying associativity, which has an identity element and which has an inverse element for every element of the group.

However, when we impose the condition that our transformation be continuous, i.e. that it can be done infinitesimally, we require this property to be reflected by the group describing the transformation, and this is achieved by the group having a differentiable structure.

This idea of *smooth groups* is the idea behind *Lie groups.*

Definition 0.19 (Lie group) *A **Lie group** G is a group that has a differentiable manifold structure.*

A Lie group is said to be abelian if the group operation is commutative: $g_1g_2 = g_2g_1$ $\forall g_1, g_2 \in G$.

Indeed, what makes Lie groups special is that we can go from one element to another in a smooth way, thanks to its differentiable manifold structure, whereas this is not necessarily the case in general groups, where the elements can be "discrete".
A good example to showcase this contrast is the comparison between the transformation that consists of rotating a circle by an angle α (continuous) and the one that consists of flipping an image with respect to an axis (discrete).
In the case of the circle, it is easy to see that we can do a rotation by a certain angle, and then slowly keep doing small rotations (infinitesimally). However, this does not happen for the flipping: either we flip an image or we do not, with no in-between states.

When we say that we model a transformation by a group G, what we are really saying is that G describes *all* transformations T of that type (for example, rotations) and that, to each particular transformation $T_i \in T$ of this type (for example, a rotation by 180°), there is an associated element of the group $g_i \in G$.

The interpretation of Lie groups as describing "transformations" can be formalized by introducing the general concept of the *action* of a group.

Definition 0.20 (Group action) *The (left[12]) **group action** of a group G on a set X is a map*

$$\begin{aligned} L : G \times X &\to X \\ (g, x) &\to gx \end{aligned}$$

that satisfies the following properties

i Identity: $L(e, x) = x$

ii Compatibility: $L(g_1, L(g_2, x)) = L(g_1g_2, x)$

Instead of having a map $L : G \times X \to x$*, it will come in handy to define a set of maps* $\{L_g\}_{g\in G}$*:*

$$L_g : X \to X$$

[12] The same construction can be done to define a right action. This is standard for the usual case, however it will be of interest for us to define the action on the lesser used left side.

$$x \to L_g x = gx$$

satisfying

i $L_e x = x$

ii $L_{g_1} \circ L_{g_2} x = L_{g_1 g_2} x$

Remark 0.6 *It is important to note that sometimes we will use the abusive notation mentioned in Rem. 0.1 and speak of the action on $x \in \mathcal{M}$ or on $x^\mu \in \mathbb{R}^n$ interchangeably (Fig. 9). While one is reached from the other by use of a map $p : \mathcal{M} \to \mathbb{R}^n$, $p(x) = x^\mu$, we make the abuse of notation of calling everything x.*

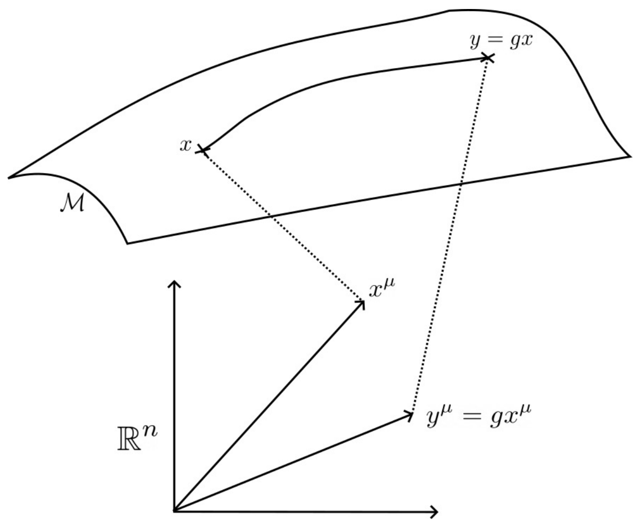

Figure 9: Sometimes we refer as x to both the point in the manifold and its coordinates in $\mathbb{R}^n$, that are related by a map $p : \mathcal{M} \to \mathbb{R}^n$, $p(x) = x^\mu$.

Proposition 0.3 *The action of a group can be of different types*[13]*:*

i The action of a group G on $\mathcal{M}$ is ***free*** *if, for $x \in \mathcal{M}$, $gx = x \implies g = e_G$, i.e. it is only the identity element that acts trivially.*

ii The action of a group G on $\mathcal{M}$ is ***transitive*** *if any two points on the manifold can be connected by an element of the group, i.e. if for all pairs on the manifold, it is possible to reach the second point just by acting with the group on the first one: $\forall (x_1, x_2) \in \mathcal{M}, \exists g \in G : x_2 = gx_1$.*

We can act with a Lie group on vector fields and some of them will have the property of being invariant.

[13]We only list here the types of action that will be useful for us.

Definition 0.21 (Left-invariant vector fields) *A vector field X is invariant under the (left) action of G if:*

$$(L_g)_* X(h) = X(gh) \tag{12}$$

where $g, h \in G$ and $(L_g)_*$ is the push-forward of L_g (0.16) . We can define the set of left-invariant vector fields under the action of G as $\mathfrak{X}(G)$.

In particular, if we consider the vector field at the identity:

$$(L_g)_* X(e_G) = X(g)$$

*This tells us that the value of X at any point $g \in G$ can be obtained by pushing forward $X(e_G)$ with L_g or, in other words, that **a left-invariant vector field is uniquely determined by its value at identity** $X(e_G)$.*

The left-invariant vectors under the action of G form an extremely important set.

Definition 0.22 (Lie algebra of a Lie group) *The Lie algebra $\mathfrak{g}$ of a Lie group G is the set of left-invariant vector fields on G:*

$$\mathfrak{g} = \{X \in \mathfrak{X}(G) | (L_g)_* X = X, g \in G\} \tag{13}$$

Let us go back to our heuristic discussion of transformations and let us consider doing a rotation of 0 degrees to a circle or, in other words, doing nothing, which we formally call the identity transformation. This transformation, like any other, has a corresponding element in the Lie group, which turns out to be the identity element $e_G \in G$.
Just as we did in the case of general differentiable manifolds when studying tangent spaces, we can place ourselves at the point e_G and consider the different directions in which we can move (in the example of the circle, this corresponds to doing small rotations parametrized by a small angle ε).
Again, these infinitesimal transformations are modeled by differentiation at the point, which yields a tangent space at the identity element.
However, as we noted in (0.21), the identity element also determines left-invariant vector fields, which in turn form the Lie algebra. We can already hint at a connection between these two ideas.

Indeed, on the one hand, infinitesimal transformations around the identity element yield a tangent space. On the other hand, applying transformations to the identity element gives rise to the Lie algebra through left-invariant vector fields.
This link gives rise to a fundamental result in Lie theory:

Theorem 0.1 *The tangent space $T_{e_G}G$ to a Lie group G at the identity element e_G is isomorphic to the **Lie algebra** $\mathfrak{g}$ of G [14]:*

$$T_{e_G}G \cong \mathfrak{g}$$

as shown in Fig. 10.

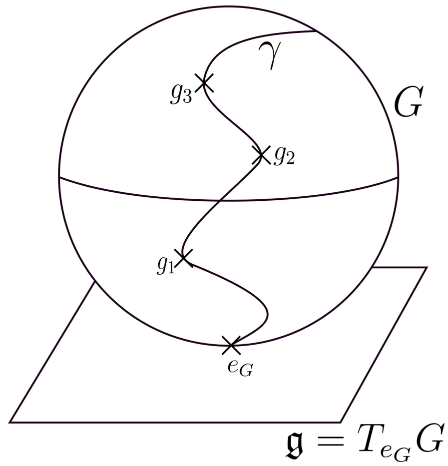

Figure 10: Lie algebra $\mathfrak{g}$ of a (simply connected) Lie group G. All elements g_i of the group are connected to the identity element e_G by a smooth path γ.

The path γ shown in Fig. 10 is formally known as a *one-parameter subgroup.*

Definition 0.23 (One-parameter subgroup) *A one-parameter subgroup of G is a smooth group homomorphism*

$$\gamma : \mathbb{R} \to G$$

such that

i It starts at the identity: $\gamma(0) = e_G$

ii It respects the group structure: $\gamma(r + s) = \gamma(r)\gamma(s)$

It is important to note that, while we have introduced the concept of Lie algebras as emerging from Lie groups, they exist as standalone entities: all Lie groups have a corresponding Lie algebra, generating the infinitesimal elements of the group, but not all Lie algebras stem from a Lie group.
We shall now give a formal definition of abstract Lie algebras as entities independent of Lie groups.

Definition 0.24 (Lie algebra) *A real* ***Lie algebra*** $\mathfrak{g}$ *is an $\mathbb{R}$-vector space equipped with a map $\{\cdot, \cdot\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the* ***Lie bracket****, that satisfies the following properties:*

i Bilinearity:

$$\{X, aY + bZ\} = a\{X, Y\} + b\{X, Z\}, \quad \{aX + bY, Z\} = a\{X, Z\} + b\{Y, Z\}$$

where $X, Y, Z \in \mathfrak{g}$ and $a, b \in K$.

ii Antisymmetry:

$$\{X, Y\} = -\{Y, X\}$$

[14]Lie's third theorem tells us that every finite-dimensional real Lie algebra corresponds to the Lie algebra of some simply connected Lie group.

iii The Jacobi identity

$$\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0$$

Remark 0.7 *Thanks to the Lie bracket, the Lie algebra not only gives us information about the elements of the corresponding Lie group, but it also describes their interactions. For example, rotations in 3-dimensional space do not commute: doing a rotation around the z-axis and then around the y-axis does not give the same result as doing the rotations in reverse order (Fig. 11).*

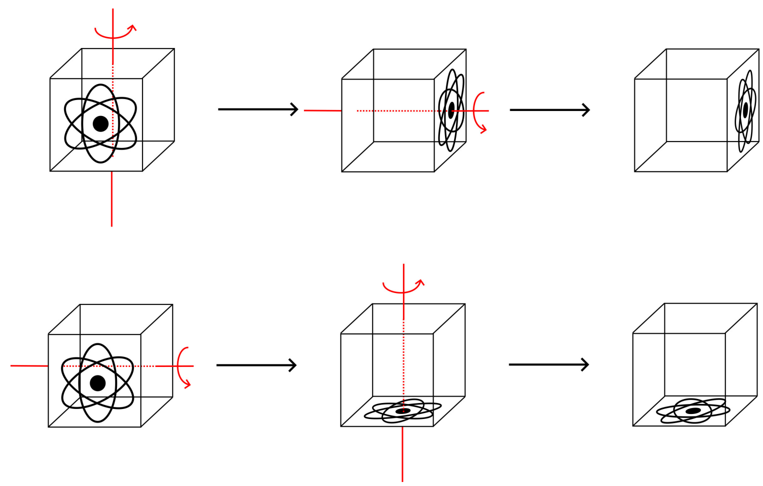

Figure 11: Rotations in 3D space do not commute

This distinction plays a crucial role in gauge theories: the main difference between theories like quantum electrodynamics and quantum chromodynamics is that the first is an abelian (commutative) gauge theory while the second is not. This has very deep implications as it results in self-interactions and complex dynamics such a confinement or asymptotic freedom.

Remark 0.8 *In the case of the Lie algebra of a Lie group G, the Lie bracket simply corresponds to the usual bracket of vector fields:*

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

where $X_1, X_2 \in \mathfrak{X}(G)$

For the rest of our work, we will consider the case where each Lie algebra $\mathfrak{g}$ corresponds to a Lie group G.
In this context, just as we defined the Lie algebra as the space of elements tangent to the identity element of the Lie group, we can do the inverse work and obtain elements of the Lie group[15] in terms of elements of the Lie algebra by means of the *exponential map*:

[15]In reality, this is only true for elements that are *connected* to the identity element. However, since our focus will be made on infinitesimal elements around the identity, we do not need to make this distinction.

Definition 0.25 (Exponential map) *The **exponential map** is a map*

$$\begin{aligned} exp : \mathfrak{g} &\to G \\ X &\to exp(X) = g(1) \end{aligned}$$

where $g(t)$ is the unique one-parameter subgroup of G such that

$$\left.\frac{d}{dt}g(t)\right|_{t=0} = X \tag{14}$$

or, in other words, the smooth path in G whose tangent vector at e_G is X, as shown in Fig. 12. This tells us that we can construct an element $g \in G$ by taking the exponential map of $X \in \mathfrak{g}$ and it allows us to move from infinitesimal transformations (Lie algebra) to finite transformations (Lie group).

Remark 0.9 *In the case of a matrix Lie group, the one-parameter subgroup satisfying (14) is given by the matrix exponential*

$$g(t) = e^{tX}$$

and we can use the usual properties of the exponential function, such as writing the infinitesimal element as

$$g(t) = e^{tX} \simeq 1 + tX$$

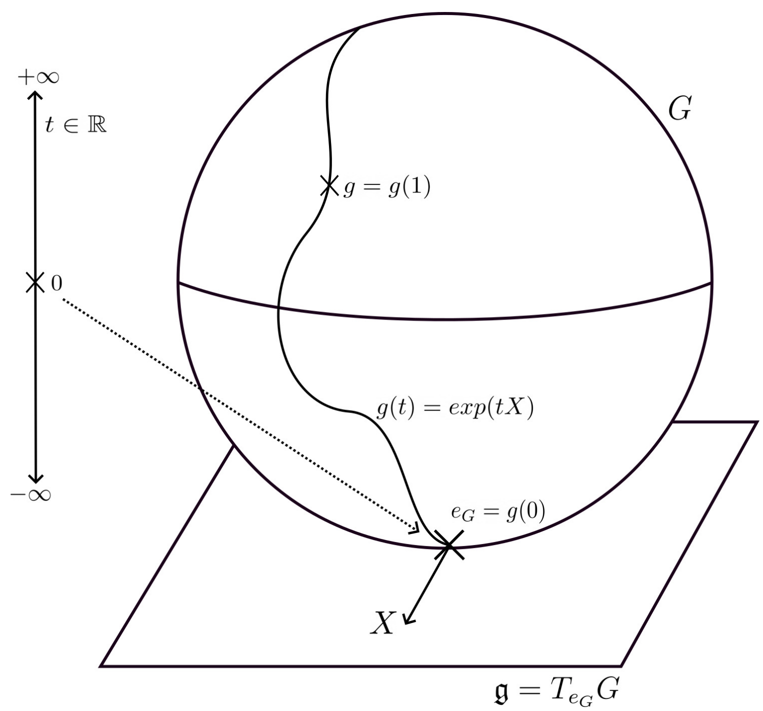

Figure 12: Exponential map of a Lie group

We have hereby presented the main actors of our play, and we shall now discuss their roles. In the rest of this work, we will consider the case of Lie groups acting on manifolds.

As stated before, Lie groups themselves have a manifold structure (all Lie groups are manifolds but not all manifolds are Lie groups) and we can therefore consider the case where the manifold being acted upon by G is the G itself.

In this particular case, the product between group elements is already defined by the group operation and the action can be applied directly:

$$L_{g_1} g_2 = g_1 g_2$$

However, this is generally not the case and the group needs a *representation* to act on the manifold.

Definition 0.26 (Representations of Lie groups) *A* ***representation*** *of a Lie group G on a vector space V is a smooth group homomorphism* [16]

$$\begin{aligned} \rho : G &\to Aut(V) \\ g &\to \rho(g) \end{aligned}$$

where $Aut(V)$ is the automorphism group on V (the group of all the invertible linear transformations from V to itself). In other words, $\rho(g) : V \to V$.

We can now act with a Lie group G on a general manifold $\mathcal{M}$:

$$L_g x = \rho(g) x$$

Remark 0.10 *Two important properties of the action should be noted:*

i If $\rho(g)$ represents a left action then, by construction, its pullback $\rho(g)^*$ represents a right action (as presented in theorem 13.4.1 of [1]).

ii If $\rho(g)$ represents a left action, then $\rho(g^{-1})$ represents a right action (as stated in theorem 13.1.1 of [1]).

Just as we wrote elements of a Lie group in terms of the elements of the Lie algebra, it is possible to do the same construction for representations using the exponential map. For that, let us introduce the concept of *Lie algebra homomorphisms*:

Definition 0.27 (Lie algebra homomorphism) *A* ***Lie algebra homomorphism*** *is a map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie bracket*

$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

where $X, Y \in \mathfrak{g}$.

[16] That is, a map h between two groups $(G, *)$, $(H, \star)$ such that $h(g_1 * g_2) = h(g_1) \star h(g_2)$, $\forall g_1, g_2 \in G$.

Definition 0.28 (Representations of Lie algebras) *A* ***representation*** *of a Lie algebra* $\mathfrak{g}$ *on* $\mathcal{M}$ *is a Lie algebra homomorphism*

$$\begin{aligned} \Pi : \mathfrak{g} &\to \mathfrak{aut}(V) \\ X &\to \rho'(X) \end{aligned}$$

where $\mathfrak{aut}(V)$ *is the Lie algebra of the automorphism group on* V*.*
We can express the representation $\rho(g)$ *of* G *in terms of the representation* $\rho'(X)$ *of* $\mathfrak{g}$*:*

$$\rho(g) = e^{g\rho'(X)}$$

and, in this case, we say that ρ' *is the derivate representation.*

Remark 0.11 *In particular, for a transformation* $g \in G$ *connected to the identity, we can write its infinitesimal left action on* $\mathcal{M}$*:*

$$\begin{aligned} L_{g(\varepsilon)}x &= \rho(g(\varepsilon))x \\ &= e^{\varepsilon\rho'(X)}x \\ &\simeq (id_G + \varepsilon\rho'(X))x \\ &= x + \varepsilon\rho'(X)x \end{aligned} \tag{15}$$

Remark 0.12 *The derivate representation* ρ' *commutes with* ρ *following Fig. 13:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho'"] \arrow[d, "\exp"'] & \mathfrak{aut}(V) \arrow[d, "\exp"] \\ G \arrow[r, "\rho"] & \mathrm{Aut}(V) \end{tikzcd}$$

Figure 13: Commutative diagram between ρ and ρ'

Remark 0.13 *It is a well known result in modern differential geometry that the Lie algebra of the diffeomorphism group on* $\mathcal{M}$ *can be identified to the space of smooth vector fields in* $\mathcal{M}$*, denoted* $\mathfrak{X}(\mathcal{M})$ *.*
In this context, we can interpret the derived representation ρ' *as a map giving a vector field (0.7)*

$$\rho'(X) =: \xi_X$$

which we call the ***fundamental field****, and so the infinitesimal action of* G *on* $x \in \mathcal{M}$ *becomes*

$$L_{g(\varepsilon)}x = x + \varepsilon\xi_X x$$

This result can be extended to the action on forms by use of the Lie derivative:

$$L_{g(\varepsilon)}\omega = \omega + \varepsilon\mathcal{L}_{\xi_X}\omega$$

0.1.3 Fibre bundles and connections

Definition 0.29 (Bundle) *A **bundle** is a triple $(P, \pi, \mathcal{M})$, where P and $\mathcal{M}$ are topological spaces and $\pi : P \to \mathcal{M}$ is a continuous map.*

*The space P is called the **total space**, $\mathcal{M}$ is the **base space** and the map π is the **projection**.*

*The image of the inverse map $\pi^{-1} : \mathcal{M} \to P$ at x, i.e. all the points in P that are projected onto the same point $x \in \mathcal{M}$, is called the **fibre** over x.*

In most of the cases that are relevant to us physicists, the fibres at every point of the manifold are equal to a common space. In this case, we say that the common space is the fibre of the bundle and the bundle is said to be a **fibre bundle**.

Definition 0.30 (Fibre bundle) *A **fibre bundle**, often denoted $F \to P \xrightarrow{\pi} \mathcal{M}$, is a bundle $(P, \pi, \mathcal{M})$ together with a topological space F (footnote 2) such that*

i Each fiber $\pi^{-1}(x)$ is diffeomorphic to F for all $x \in \mathcal{M}$.

ii There exists a local trivialization: as shown in Fig. 14, there exists an open neighborhood $U_x \subset \mathcal{M}$ around each x and an homeomorphism $\phi : \pi^{-1}(U_x) \to U_x \times F$ such that π corresponds to projecting onto the first factor[17], i.e. $\pi = proj_1 \circ \phi$.
This condition can be thought of as saying that, locally, the total space P looks like the product space $F \times U_x$.

$$\begin{tikzcd}
\pi^{-1}(U_x) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U_x \times F \arrow[dl, "\mathrm{proj}_1"] \\
U_x &
\end{tikzcd}$$

Figure 14: Local trivialization

Definition 0.31 (Global section) *A **global section** of $(P, \pi, \mathcal{M})$ is a map $\zeta : \mathcal{M} \to P$ such that $\zeta(x) \in \pi^{-1}(x)$, $\forall x \in \mathcal{M}$, i.e. $\zeta \circ \pi = id_{\mathcal{M}}$.*
In other words, the section "chooses" a point from the fibre over each point on the manifold (Fig. 15).
Quite often, it is local sections that are considered. The term "local" refers to the fact that we only consider an open $U \in \mathcal{M}$ and not the whole manifold.

[17] We call $\mathrm{proj}_1 : U_x \times F \to U_x$, $\mathrm{proj}_1(u, f) = u$ the *natural projection*

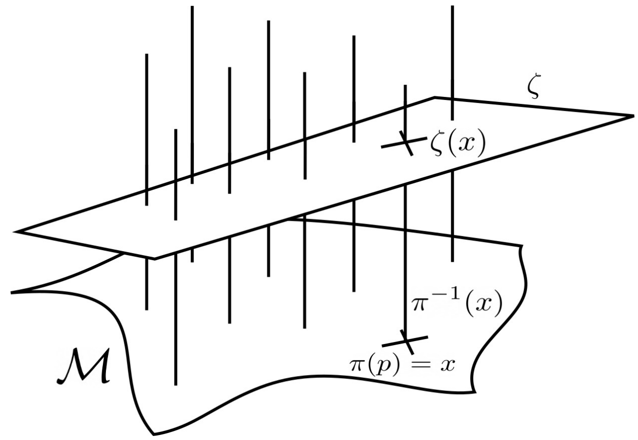

Figure 15: A section ζ "chooses" a point $\zeta(x) \in \pi^{-1}$ on the fibre at each point (the section is here represented by a flat plane for simplicity but it can be any smooth curved surface).

Definition 0.32 (Principal fibre bundle) *A **principal G-bundle** is a fibre bundle $(P, \pi, \mathcal{M}, G)$, where the fibre G is a Lie group that acts freely and transitively on the left*[18] *on P:*

$$\begin{aligned} L_g : P &\to P \\ p &\to gp \end{aligned}$$

Definition 0.33 (Associated bundle) *Let us consider a G-bundle $G \to P \xrightarrow{\tilde{\pi}} \mathcal{M}$ and a representation $\rho : G \to Aut(V)$ of G on a vector space V. We can construct the **associated bundle** $E = P \times_\rho V$*[19]*, a vector bundle where each fibre $E_x := \tilde{\pi}^{-1}(x)$ is diffeomorphic to V.*

Example 0.4 (The frame bundle) *The frame bundle is a principal $GL(n, \mathbb{R})$-bundle with associated vector bundle $E = P \times_\rho V$, where the fibre over each $x \in \mathcal{M}$ corresponds to the set of basis of E_x, which are acted upon naturally by elements of the general linear group (whose elements correspond to change of basis matrices).*
Moreover, we can note that we fix a certain basis, we can associate each new basis to the transformation matrix between the starting basis and the new one. This isomorphism allows us to identify the frame bundle with a principal $GL(n, \mathbb{R})$-bundle.

Let us place ourselves in a principal bundle $G \to P \xrightarrow{\pi} \mathcal{M}$.
Tangent spaces are a very special type of fibres that will be key to our work, and just as we defined the tangent space $T_x\mathcal{M}$ to a point on the manifold, we can also define a more abstract version of it: the tangent space T_pP to a point p in the total space P. This tangent space can be decomposed in a vertical and an horizontal subspaces (Fig. 16).

Definition 0.34 (Vertical and horizontal subspaces) *The **vertical subspace** $V_pP \subset T_pP$ is the set of all vertical vectors of T_pP:*

$$V_pP := ker(\pi_*) = \{v \in T_pP | \pi_* v = 0\}$$

[18] Principal fibre bundles are usually defined by means of the right action. However, this definition is equivalent and it will simplify our work to use the left action instead.

[19] The formal construction is a bit more complicated: the bundle is defined as a quotient space and we need to introduce an equivalence class. However, for our work, this level of detail is not necessary.

where $\pi_* : T_pP \to T_{\pi(p)}P$ *is the push-forward of* π.
We can then define, in a complementary way, the ***horizontal subspace*** $H_pP \subset T_pP$ *such that*

$$T_pP = H_pP \oplus V_pP$$

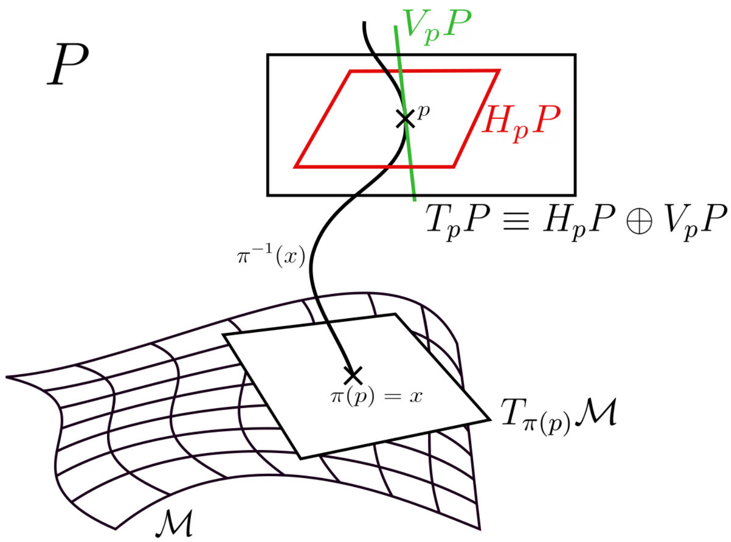

Figure 16: Decomposition of the tangent space T_pP to the total space P at p into horizonatal and vertical subspaces.

Let us now imagine that we wanted to perform the parallel transport of a vector (a key procedure in general relativity). Symbolically, this is done by taking the vector at a point $x_2 \in \mathcal{M}$ and comparing it to itself at the previous point $x_1 \in \mathcal{M}$ to make sure it stays parallel. However, sooner than later we will realize that this comparison cannot be done so easily: the vectors at each point belongs to a different vector space (the tangent spaces at each point) and are therefore written in terms of different basis, so there is no straightforward way to compare them.
A *connection* is a tool that allows us to perform these type of comparisons: it can be thought of as connecting fibres over different points in $\mathcal{M}$.

Definition 0.35 (Connection) *A* ***connection*** *on a principal G-bundle* $G \to P \to \mathcal{M}$ *is a* $\mathfrak{g}$*-valued 1-form* $\omega \in \Omega^1(P, \mathfrak{g})$ *that performs a smooth assignment of horizontal subspaces* $H_pP \subset T_pP$ *to each* $p \in P$ *defined by*

$$ker(\omega) = H_pP$$

Each horizontal subspace must satisfy the following properties:

i $T_pP \cong V_pP \oplus H_pP$

ii $g(H_pP) = H_{gp}P$

The second condition says that this assignment should respect the action of the group, but is not of particular importance for our considerations.
Moreover, ω must satisfy some conditions: it must transform in a way compatible by the action by satisfying the following:

i Under the left action by g, the connection transforms in the adjoint representation: $Ad_{g^{-1}}(L_g) = \omega$, where $Ad_{g^{-1}}(\xi) = g^{-1}\xi g$

ii It acts trivially on vertical vectors: $\omega(X_\xi) = \xi$

Remark 0.14 *Just by looking at Fig. 16, the idea of "assigning" a horizontal subspace might seem trivial: we just take the subspace "orthogonal" to V_pP. However, this picture is misleading.*
In fact, the only condition we have is that an element of T_pP must be decomposed as in terms of vectors of H_pP and V_pP. This implies that these vectors must be linearly independent, but that does not imply orthogonality. Indeed, as shown in Fig. 17, there are several (infinitely many) horizontal subspaces verifying this condition and the operation performed by the connection is not trivial at all.

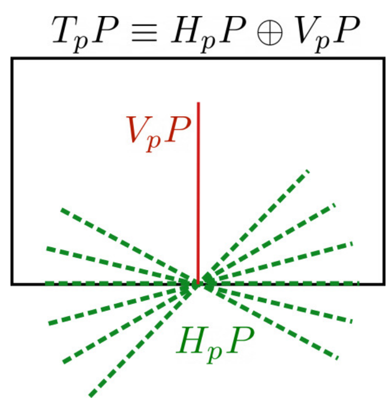

Figure 17: Several horizontal subspaces H_pP are linearly independent of V_pP.

These horizontal subspaces allow us to define *parallel transport* in the language of bundles. Indeed, if we consider a principal G-bundle and a connection ω, horizontal vectors are defined as the kernel of ω: $v \in H_pP$ if $\omega(v) = 0$.

Doing the parallel transport of a vector means that, as we move long a path $\gamma(t) \in \mathcal{M}$, we want to lift vectors to a path $\tilde{\gamma}(t) \in P$ such that the tangent vectors $\dot{\tilde{\gamma}}(t)$ lie in the horizontal subspaces $H_{\tilde{\gamma}(t)}P$ (Fig. 18).

However, when we consider a closed path (loop) on $\mathcal{M}$, the final tangent vector might not lie in the same horizontal subspace as the first one because of the curvature of $\mathcal{M}$. This can be illustrated, as shown in Fig. 19, with the case of a closed loop on a sphere: the starting and ending vectors will not be parallel to each other due to the curvature of the sphere.
This "failure" to stay horizontal is encoded in the *curvature* form of the connection.

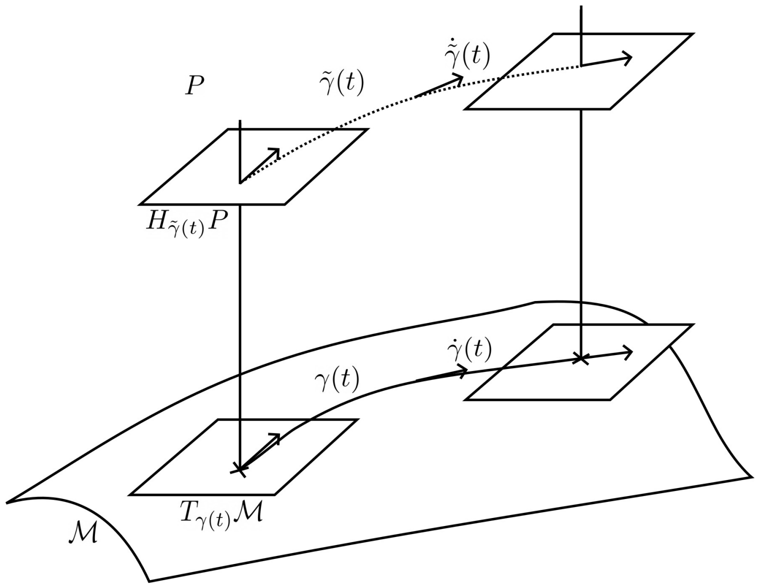

Figure 18: Parallel transports along a path in the manifold and its lifted path on the total space.

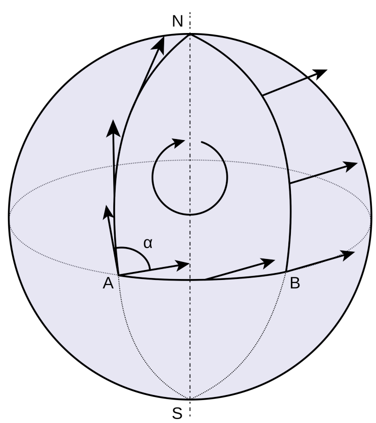

Figure 19: Parallel transport of a vector around a closed loop on the sphere, by Fred the Oyster, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=35124171

Definition 0.36 (Curvature form) *Let ω be a connection on $\mathcal{M}$. The* ***curvature form*** *of ω is the $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ defined by*

$$\Omega = d\omega + \omega \wedge \omega \tag{16}$$

0.2 Physical background

0.2.1 Symmetries and conservation laws

One of the starting points for most (if not all) theories in modern physics is **symmetries**.

The concept of symmetry is one of the few mathematical concepts that are intuitive to

most of us: we are aware of it even if we don't fully understand it.
One can easily realize that the images of objects reflected on a mirror do not correspond to what we perceive with our eyes or that one can rotate a golf ball and it will still look the same, without a need for awareness of the mathematical description of symmetries. Indeed, we can find symmetrical patterns everywhere, from nature, like the wings of a butterfly, to man-made, like the rose window of Notre-Dame. From an early age, we are exposed to these geometrical patterns and our not fully developed brains are already capable of recognizing that there is something particular about them [2].

However, when one studies this phenomenon more thoroughly, one is rewarded with much deeper implications that have become a cornerstone of modern physics.
In 1918, Emmy Nother published her *Invariante Variationsprobleme* where she drew a correspondence between symmetries of physical systems and conserved quantities. This revolutionary result has become one of the foundations of the standard model of particle physics.

The concept of symmetry is also the keystone of this work, so let us give a more formal definition.

Definition 0.37 (Symmetry) *A **symmetry** of a system is a transformation that leaves a property of the system unchanged, i.e. the system is invariant under said transformation.*

Mathematically, and as discussed in (0.1.2), a continuous transformation is described by a (Lie) group which, in this context, is called the symmetry group, and the action of said group on the system is precisely the transformation that leaves it invariant.

We can distinguish symmetries depending on how and on what they act .
*If G acts in the same way for all points $x \in \mathcal{M}$, the symmetry is said to be **global**, i.e. $g =$ constant. If it acts differently depending on the point, i.e. $g = g(x)$, then it is **local**.*
*On the other hand, depending on whether the symmetry acts on $\mathcal{M}$ or not, it is said to be **internal** or **external**.*

To summarize,

$$\begin{cases} \textit{Type of action} \begin{cases} \textit{Global: same action everywhere} \\ \\ \textit{Local: different action at each point} \end{cases} \\ \\ \textit{Aim of action} \begin{cases} \textit{Internal: action only affects the fields, is independent of spacetime} \\ \\ \textit{External: action affects both the fields } \textbf{\textit{and}} \textit{ the spacetime} \end{cases} \end{cases}$$

Remark 0.15 *The definition of a group being internal is equivalent to the condition that the fundamental fields (Rem. 0.13) need to be zero:*

$$\rho(g) = id_{\mathcal{M}}$$

$$\begin{aligned}
&\iff e^{\varepsilon\rho'(X)} = id_{\mathcal{M}} \\
&\iff \rho'(X) = 0 \\
&\iff \xi_X = 0
\end{aligned} \tag{17}$$

Theorem 0.2 (Noether's theorem) *Consider an action*

$$S[\phi] = \int d^4x \mathscr{L}(\phi, \partial_\mu \phi, x^\mu)$$

where $\phi = \phi(x^\mu)$ are the fields described by the Lagrangian. Now consider this Lagrangian to be invariant under a continuous infinitesimal transformation

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \varepsilon \delta x^\mu \\ \\ \phi \to \phi' = \phi + \varepsilon \delta \phi \end{cases}$$

Noether's (first [20]) theorem states that there exists a current

$$J^\mu = \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) - \mathscr{L} \right) \delta x^\mu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta \phi \tag{18}$$

that is conserved on-shell[21]:

$$\partial_\mu J^\mu = 0 \tag{19}$$

Example 0.5 (Internal symmetry) *Let us consider a massless complex scalar field ϕ in flat space described by the Klein-Gordon Lagrangian $\mathscr{L}(\partial_\mu \phi, \partial_\mu \phi^*) = \eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi$, and hence satisfying the Klein-Gordon equations of motion*

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$

This field is invariant under global $U(1)$ phase transformations:

$$\begin{cases} x^\mu \to x'^\mu = x^\mu \\ \\ \phi \to \phi' = e^{-iq\alpha} \phi \sim (1 - iq\alpha\phi) \\ \\ \phi^* \to \phi^{*\prime} = e^{iq\alpha} \phi^* \sim (1 + iq\alpha\phi^*) \end{cases} \iff \begin{cases} \delta x^\mu = 0 \\ \\ \delta \phi = -iq\alpha\phi \\ \\ \delta \phi^* = iq\alpha\phi^* \end{cases}$$

The Noether currents become

$$\begin{aligned}
J^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^* + \mathscr{L} \delta x \\
&= (\partial^\mu \phi^*)(-iq\alpha\phi) + (\partial^\mu \phi)(iq\alpha\phi^*) + 0
\end{aligned}$$

[20] This theorem, colloquially known as Noether's theorem, is actually one of two theorems. The second one, which will not be of interest to us, regarding the case of gauge theories.

[21] That is, if the equations of motion are satisfied by the field.

$$= iq\alpha(\phi^* \, \partial^\mu \phi - \phi \, \partial^\mu \phi^*) \tag{20}$$

and we can easily check that they are indeed conserved

$$\partial_\mu J^\mu = (\partial_\mu \phi^*)(\partial^\mu \phi) - \phi^*(\partial_\mu \partial^\mu \phi) - (\partial_\mu \phi)(\partial^\mu \phi^*) + \phi(\partial_\mu \partial^\mu \phi^*) = 0$$

under the condition that $\partial_\mu \partial^\mu \phi = 0$ (on-shell).

Example 0.6 (External symmetry) *Let us consider the same setting as 0.5, this time invariant under global $\mathbb{R}^4$ translations*

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu \\ \\ \phi \to \phi' = \phi \\ \\ \phi^* \to \phi^{*\prime} = \phi^* \end{cases}$$

We should note that the total variation of the field $\delta\phi$ has two parts: the intrinsic variation of the field itself and the functional variation of the field $\bar{\delta}\phi$, which accounts for the variation of the field due to the variation of the coordinates. This is captured by the Lie derivative $\mathcal{L}_{\delta x^\mu}\phi$ which tells us how the field transforms under the coordinate shift δx^μ. For translations, the Lie derivative of ϕ reduces to $\varepsilon^\mu \partial_\mu \phi$

We can then compute the infinitesimal transformations

$$\begin{cases} \delta x^\mu = \varepsilon^\mu \\ \\ \delta\phi = 0 \implies \bar{\delta}\phi = -\mathcal{L}_{\delta x^\mu}\phi = -\delta x^\mu \partial_\mu \phi = -\varepsilon^\mu \partial_\mu \phi \\ \\ \bar{\delta}\phi^* = -\mathcal{L}_{\delta x^\mu}\phi^* = -\varepsilon^\mu \partial_\mu \phi^* \end{cases}$$

This transformation is ***external*** *since it transforms the fields as well as the spacetime coordinates themselves.*

We can write an expression for the conserved currents

$$\begin{aligned} J^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \bar{\delta}\phi + \mathscr{L} \, \delta x^\mu \\ &= (\partial^\mu \phi)(-\varepsilon^\nu(x^\nu)\partial_\nu \phi) + \mathcal{L} \, \varepsilon^\mu(x^\mu) \\ &= (\mathscr{L} \, \delta^\mu{}_\nu - \partial^\mu \phi \, \partial_\nu \phi)\varepsilon^\nu(x^\nu) \\ &=: T^\mu{}_\nu \, \varepsilon^\nu(x^\nu) \end{aligned} \tag{21}$$

where we have introduced the canonical energy-momentum tensor $T^{\mu}{}_{\nu}$

$$T^{\mu}{}_{\nu} := \mathscr{L}\delta^{\mu}{}_{\nu} - \partial^{\mu}\phi^{*}\,\partial_{\nu}\phi - \partial^{\mu}\phi\,\partial_{\nu}\phi^{*} \tag{22}$$

The conservation of this current allows us to write the conservation of the energy-momentum tensor

$$\begin{aligned}
\partial_{\mu}J^{\mu} &= (\partial_{\mu}T^{\mu}{}_{\nu})\varepsilon^{\nu} + T^{\mu}{}_{\nu}(\partial_{\mu}\varepsilon^{\nu}) \\
&= (\partial_{\mu}T^{\mu}{}_{\nu})\varepsilon^{\nu} \\
&= (\partial_{\mu}T^{\mu}{}_{\nu})\varepsilon^{\nu} \\
\implies \partial_{\mu}J^{\mu} &= 0 \\
\iff \partial_{\mu}T^{\mu}{}_{\nu} &= 0
\end{aligned}$$

0.2.2 Gauge Theories

0.2.2.1 Physical motivation for Gauge Theories Let us consider the action of a complex scalar field satisfying the Klein-Gordon equation in flat space

$$S[\phi] = \int d^{4}x\,\eta^{\mu\nu}\partial_{\mu}\phi^{*}\partial_{\nu}\phi \tag{23}$$

where we have omitted the mass term for simplicity.

As we saw in 0.5, this action remains invariant under global $U(1)$ phase transformations

$$\begin{cases} \phi \to \phi' = e^{iq\alpha}\phi \\ \\ \phi^{*} \to \phi^{*\prime} = e^{-iq\alpha}\phi^{*} \end{cases}$$

This symmetry can be promoted to local, i.e. $\alpha = \alpha(x)$, and it is in this the framework that we discuss *gauge theories*.

A gauge theory is a field theory where the action remains invariant under the *local* action of an (internal [22]) Lie group, referred to in this context as the *gauge group*. The transformation encoded by this group is known as *gauge transformation* and the property of invariance of the action, as *gauge invariance*.

If we promote the $U(1)$ symmetry to local, we can observe that the action (23) is no longer invariant under this transformation

$$\partial_{\mu}\phi \to \partial_{\mu}\phi' = \partial_{\mu}(e^{iq\alpha(x)}\phi)$$

[22] As already discussed, a Lie group can encode both internal and external symmetries. However, it will be the goal of this work to showcase that only internal symmetries can be interpreted as giving rise to gauge theories.

$$= e^{iq\alpha(x)}(\partial_\mu \phi + iq\phi\partial_\mu \alpha)$$

$$\neq e^{iq\alpha(x)}(\partial_\mu \phi)$$

Indeed,

$$\partial_\mu \phi^* \partial^\mu \phi \to \partial_\mu (e^{-iq\alpha(x)}\phi^*)\partial^\mu (e^{iq\alpha(x)}\phi)$$

$$= e^{-iq\alpha(x)}e^{iq\alpha(x)}(\partial_\mu \phi^* - iq\phi^* \partial_\mu \alpha)(\partial^\mu \phi + iq\phi\partial^\mu \alpha)$$

$$= \partial_\mu \phi^* \partial^\mu \phi + iq(\phi\partial_\mu \phi^* \partial^\mu \alpha - \phi^* \partial^\mu \phi \partial_\mu \alpha) + q^2 \partial_\mu \alpha \partial^\mu \alpha$$

$$= \partial_\mu \phi^* \partial^\mu \phi + iq(\phi\partial_\mu \phi^* - \phi^* \partial_\mu \phi)\partial^\mu \alpha + q^2 \partial_\mu \alpha \partial^\mu \alpha$$

$$= \partial_\mu \phi^* \partial^\mu \phi + J_\mu \partial^\mu \alpha + q^2 \partial_\mu \alpha \partial^\mu \alpha \tag{24}$$

In order to restore the invariance, we must introduce a new field to counter the extra terms $J_\mu \partial^\mu \alpha + q^2 \partial_\mu \alpha \partial^\mu \alpha$. The first term $J_\mu \partial^\mu \alpha$ tells us that the field we introduce should couple to J^μ and, in order to do so, we conclude it must be a vector field.
Hence, we introduce the field A_μ, appearing in the Lagrangian by means of replacing the usual partial derivative ∂_μ with a *covariant derivative* $D_\mu \phi$:

$$D_\mu \phi = \partial_\mu \phi + iqA_\mu \phi \tag{25}$$

In order for our new action to be locally invariant, we need to postulate the transformation law for D_μ:

$$D'_\mu \phi' = e^{iq\alpha} D_\mu \phi$$

which gives in turn a transformation for A_μ:

$$D'_\mu \phi' = e^{iq\alpha} D_\mu \phi$$

$$\iff (\partial_\mu + iqA'_\mu)(e^{iq\alpha}\phi) = e^{iq\alpha}(\partial_\mu + iqA_\mu)\phi$$

$$\iff \partial_\mu (e^{iq\alpha}\phi) + iqe^{iq\alpha}A'_\mu \phi = e^{iq\alpha}\partial_\mu \phi + iqe^{iq\alpha}A_\mu \phi$$

$$\iff e^{iq\alpha}\partial_\mu \phi + iqe^{iq\alpha}\phi\partial_\mu \alpha + iqe^{iq\alpha}A'_\mu \phi = e^{iq\alpha}\partial_\mu \phi + iqe^{iq\alpha}A_\mu \phi$$

$$\iff iqe^{iq\alpha}A'_\mu \phi = -iqe^{iq\alpha}\phi\partial_\mu \alpha + iqe^{iq\alpha}A_\mu \phi$$

$$\iff A'_\mu = A_\mu - \partial_\mu \alpha \tag{26}$$

We should note that this transformation law is specific for the abelian case. In the non-abelian case, it becomes more complicated:

$$A_\mu \to gA_\mu g^{-1} + \frac{1}{q} g dg^{-1}$$

Replacing (25) in (23), the Lagrangian becomes

$$\begin{aligned}
\mathscr{L} &= D_\mu \phi^* D^\mu \phi \\
&= (\partial_\mu \phi^* + iqA_\mu \phi^*)(\partial^\mu \phi + iqA^\mu \phi) \\
&= \partial_\mu \phi^* \, \partial^\mu \phi + iq(A_\mu \phi^* \, \partial^\mu \phi + A^\mu \phi \, \partial_\mu \phi^*) - q^2 A_\mu A^\mu \phi^* \phi \\
&= \partial_\mu \phi^* \, \partial^\mu \phi + iqA_\mu (\phi^* \, \partial^\mu \phi + \phi \, \partial_\mu \phi^*) - q^2 A_\mu A^\mu \phi^* \phi \\
&= \partial_\mu \phi^* \, \partial^\mu \phi + iqA_\mu J^\mu - q^2 A_\mu A^\mu \phi^* \phi
\end{aligned} \tag{27}$$

where we have recognized the Noether currents arising from $U(1)$ symmetry (20).

It is of utmost importance for this work to note that, when we impose local invariance through A_μ, we couple the gauge field to the Noether current, which physically represents the interaction between the gauge and matter field. This is the case for both abelian and non-abelian gauge theories.

Moreover, the gauge field A_μ should be a dynamical field that satisfies its own equations of motion, and this can be encoded in the Lagrangian by manually adding a term (which must obviously be gauge invariant as well):

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the *field strength*. It can be shown that this is the unique lowest order term that satisfies Lorentz and gauge invariance and parity.
The full *locally* invariant action becomes

$$S[\phi] = \int d^4x (\eta^{\mu\nu} D_\mu \phi^* D_\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu})$$

It can be shown that the variation of the last term with respect to A_μ yields

$$\partial_\mu F^{\mu\nu} = 0$$

which we recognize as the source-free Maxwell equations in vacuum and lets us identify the gauge field A_μ as the electromagnetic potential.

Indeed, gauge fields are of the utmost importance in physics: in quantum field theories, excitations of the quantum version of the gauge fields are known as *gauge bosons*, and they are responsible for mediating the interaction between other fields (for example, in our case, the excitations of the quantized version of A_μ are photons). This formulation works very well in flat spacetime. However, to get a deeper understanding of how these gauge theories work, we will need a more general geometric description.

0.2.2.2 Differential geometry description of Gauge Theories Let us consider a gauge theory on flat Minkowski space, which is how we describe the standard model of particle physics, and let us consider again the example of $U(1)$ symmetry.
When the symmetry is global, it is very simple: we assign the same phase $e^{i\alpha}$ to every point x in the space. However, when the symmetry is promoted to local, we need to choose a specific phase $e^{i\alpha(x)}$ for each point (*gauge fixing*): this means that for each point on the manifold, we have a whole collection of phases that we can choose from. This collection of phases is precisely the group $U(1)$.
This copy of the group at each point x is the fibre $\pi^{-1}(x)$ of a principal $U(1)$-bundle $U(1) \to P \xrightarrow{\pi} \mathcal{M}$ over Minkowski space (the base manifold $\mathcal{M}$), and it describes the degrees of freedom of our field: observables are invariant under movement *within* the fibre. Indeed, different elements of the fibre correspond to different states that are physically indistinguishable from each other (since $|e^{i\alpha(x)}|= 1\ \forall \alpha(x) \in \mathbb{R}$).
In this context, a gauge transformation corresponds to moving within the fibre.

Formally, a specific choice of gauge (gauge fixing) corresponds to a smooth section (0.31) of the bundle, and gauge transformations correspond to changing the section.

As discussed in earlier sections, the tangent space to a point p in the bundle P represents all the directions in which we can move:

i Vertically: moving within the fibre at the same point on base space, which corresponds to pure gauge transformations.

ii Horizontally: moving along the base space, through different fibres in the way defined by the connection, with a fixed gauge.

A principal connection ω on the bundle (0.35), mathematically described by a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, is the tool allows us to "connect" different fibres by defining what moving horizontally means.

Let us consider a local section $\zeta : U \subset \mathcal{M} \to P$ (0.31). We can perform the pullback of ω by ζ

$$(\zeta^*\omega)_x(v) = \omega_\zeta(x)(d\zeta_x(v))$$

which gives the $\mathfrak{g}$-valued 1-form $A := \zeta^*\omega$ on $\mathcal{M}$.

Since ω takes its values in $\mathfrak{g}$, we can write it as $\omega = \omega^i E_i$, where E_i are the elements of the basis of $\mathfrak{g}$, and the pullback becomes

$$\zeta^*\omega = (\zeta^*\omega^i)E_i$$

$$=: A^i E_i$$

where $A^i = A^i{}_\mu dx^\mu$ is a $\mathfrak{g}$-valued 1-form.
This connection $A := \zeta^*\omega \in \Omega^1(\mathcal{M}, \mathfrak{g})$ is precisely the gauge field, and it transforms under gauge transformations in the adjoint representation[23]

$$A_\mu \to g A_\mu g^{-1} + g^{-1} dg \tag{28}$$

So far, we have given a geometrical description of what the gauge groups are abstractly and what they represent - internal degrees of freedom of matter fields - but we still have not discussed what the matter fields are.

Let us consider a representation $\rho : G \to V$ of G on some vector space V. It is possible to construct the associated bundle $E = P \times_\rho V$ (0.33). We can define a *matter field* as a smooth section of E:

$$\phi : \mathcal{M} \to E, \quad \tilde{\pi} \circ \phi = id_{\mathcal{M}} \tag{29}$$

and $\phi(x) \in E_x$ corresponds to the value of ϕ at $x \in \mathcal{M}$, which transform under local gauge transformations $g(x) \in G$ as

$$\phi(x) \to \rho(g(x))\phi(x)$$

Summary The take-away from this section is to note that, in gauge theories,

i We place ourselves in a **principal G-bundle**, where the Lie group G is the gauge group.

ii We ensure *local* invariance under this symmetry group by means of a **covariant derivative D_μ**.

iii The covariant derivative introduces a new field, the gauge field, represented by a **connection 1-form** on $\mathcal{M}$. More specifically, it introduces the **coefficients** A_μ of the connection in the differential basis: $A = A_\mu dx^\mu$.

iv The connection appears in the Lagrangian by **coupling to the Noether currents**, mediating the interaction between the matter and gauge field.

0.2.3 General Relativity

0.2.3.1 Physical motivation for General Relativity In standard gauge theories, all forces except gravity are described by fields on flat Minkowski space, characterized by a metric $\eta_{\mu\nu}$, where the fields vary dynamically and couple to each other. This metric is *fixed* and allows us to define the dynamics of the fields: it allows us to construct time intervals, lightcones and to describe how particles move.
Some of these particles are free, meaning they do not couple to other forces and should therefore not be affected by them: in the absence of other interactions, the motion of these

[23]The transformation law for ω can be written in a more general way as $\omega \to Ad_{g^{-1}}\omega + g^*\Xi$, where Ξ is the *Maurer-Cartan form*, which reduces to our equation in the case of a matrix Lie group.

particles should be solely determined by the metric. However, this is not what we observe.

Take the example of light. We know that light should always follow an extremal path (a geodesic) which, in flat space, is a straight line. However, observations show that light actually follows *curved* lines, as seen in gravitational lensing. This apparent paradox can be resolved if we note that curved paths are still geodesics, but geodesics in *curved space*.

This hints at the fact that the background itself, the metric, must be dynamical, varying under the influence of some "force".
This universal "force" that couples to and therefore affects all fields is precisely what we refer to as gravity.

Moreover, Einstein's interpretation of the equivalence principle tells us that, locally, accelerating frames are indistinguishable from gravitational fields: that is, this geometry should reduce to flat space locally.

Thus, we request for gravity to not only be encoded by a dynamical geometry, but that this geometry must locally resemble the one found in special relativity. These conditions lead naturally to the description of spacetime as a smooth manifold endowed with a generalized metric $g_{\mu\nu}$.

In this generalized setting, because of the presence of curvature, we lose several of the key properties that we had in special relativity. For example, we no longer have Poincaré invariance: the new symmetry of our theory is that of *diffeomorphism invariance*. This symmetry tells us that the laws of physics should be the same under coordinate transformations[24], and this poses a problem because partial derivatives do not transform covariantly.

Indeed, let us consider a contravariant vector field $V^\mu(x)$. Its transformation law under coordinate change $x^\mu \to x^{\mu'}$ is given by

$$V^\mu(x) \to V^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu(x) \tag{30}$$

We can check the transformation for its partial derivative:

$$\partial_\nu V^\mu \to \partial_{\nu'} V^{\mu'} = \frac{\partial}{\partial x^{\nu'}} \left(\frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \right)$$

$$= \frac{\partial x^\rho}{\partial x^{\nu'}} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \right)$$

[24] Actually, this is just a consequence of *general covariance*, a requirement for diffeomorphism invariance. The fundamental symmetry in General Relativity is deeper than a mere *passive* change in coordinates, and there exists an ongoing discussion about this topic. However, making this distinction is not necessary for us.

$$= \frac{\partial x^\rho}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\mu'}}{\partial x^\rho \partial x^\mu} V^\mu\right)$$

$$= \left(\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\rho}\right)\left(\frac{\partial x^\rho}{\partial x^{\nu'}} V^\mu\right) + \left(\frac{\partial x^\rho}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\rho}\right) V^\mu$$

$$= \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\rho}{\partial x^{\nu'}}\partial_\rho V^\mu + \left(\frac{\partial x^\rho}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\rho}\right) V^\mu \tag{31}$$

The first term corresponds indeed to the kind of transformation we would expect. However, we get an extra second term that breaks covariance.
In order to fix this and ensure invariance under coordinate change, we need to introduce a new derivative: a covariant derivative.

We define this new derivative as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\rho} V^\rho \tag{32}$$

where $\Gamma^\mu{}_{\nu\rho}$, known as the *Christoffel symbols*, are some coefficients that transform as

$$\Gamma^{\mu'}{}_{\nu'\rho'} = \frac{\partial x^{\mu'}}{\partial x^\gamma}\frac{\partial x^\eta}{\partial x^{\nu'}}\frac{\partial x^\sigma}{\partial x^{\rho'}}\Gamma^\gamma{}_{\eta\sigma} + \frac{\partial^2 x^\gamma}{\partial x^{\nu'}\partial x^{\rho'}}\frac{\partial x^{\mu'}}{\partial x^\gamma} \tag{33}$$

Using (33), it can be shown that

$$\nabla_{\nu'} V^{\mu'} = \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\nabla_\nu V^\mu \tag{34}$$

which, unlike (31), preserves general covariance.

We can think of it as the fact that, when we take the (regular) derivative of a field along a direction, we are moving infinitesimally in that direction. In curved spacetime, this is not as straightforward. We need a derivative that keeps track of the curvature and this is precisely what the covariant derivative does. It performs the usual partial derivative and then corrects it with an extra term, the Christoffel symbols, which encodes the information of how space curves (in the direction we are derivating along).

0.2.3.2 Differential geometry description of General Relativity As we showed in the previous section, the objects considered in General Relativity have a very geometrical interpretation, and so differential geometry gives us a great language in which to describe this theory.

As motivated, spacetime is modeled by a 4-dimensional smooth manifold endowed with a Lorentzian[25] metric $g_{\mu\nu} = g_{\mu\nu}(x)$

[25] This term refers to the fact that the metric has signature $(1, n-1)$, in our case, (1,3).

On this manifold, it is possible construct the tangent bundle $T\mathcal{M}$ (0.5), which has a vector bundle structure: it is a fibre bundle, where each fibre (a vector space) corresponds to the tangent space $\pi^{-1}(x) = T_x\mathcal{M}$.
As we mentioned in (0.33), we can recover a vector bundle from a principal fibre bundle: in this case, our (associated) tangent bundle is define through the *frame bundle*

$$T\mathcal{M} = F\mathcal{M} \times_{GL(4,\mathbb{R})} \mathbb{R}^4 \tag{35}$$

Each tangent space can be thought of as a collection of basis (different basis of the same vector space). If we start at a certain basis, which we can symbolically name B_0 and, to each other basis B_i, we associate the matrix $M_i \in GL(4,\mathbb{R})$ that transforms the basis B_0 into B_i, then this collection of basis becomes a collection of matrices, which is precisely $GL(4,\mathbb{R})$, the group of 4x4 (since we are in 4 dimensions) invertible matrices (since we want to be able to go from B_0 to B_i back and forth). In other words, each tangent space (fibre) is isomorphic to $GL(4,\mathbb{R})$.

The frame bundle is then a principal $GL(4,\mathbb{R})$-bundle:

i Each fibre corresponds to the set of all basis of the associated vector space at x, i.e. $\pi^{-1}(x) = \{\text{set of all basis } \{e_i\} \text{ of } T_x\mathcal{M}\ \}$.

ii The action $GL(4,\mathbb{R})$ on a fibre $\pi^{-1}(x)$ corresponds to the change of basis $\{e_a\} \to \{e_b\}$ of $T_x\mathcal{M}$.

This bundle has even deeper structures that will be of interest to us. In particular, since the metric allows us to define distances and angles which, in turn, allow us to define orthonormality, we can select an orthonormal frame (or basis) among all the possible frames of the tangent space at each point:

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} \tag{36}$$

This reduces $GL(4,\mathbb{R})$ to $SO(1,3) \subset GL(4,\mathbb{R})$, and we will then work on a principal $SO(1,3)$-(sub)bundle.

Let us place ourselves in the principal $SO(1,3)$-bundle. The covariant derivative we defined in (32) introduced the Christoffel symbols ${\Gamma^\mu}_{\nu\rho}$ (33). These objects are the coefficients of a connection

$$\omega = {\Gamma^\mu}_{\nu\rho} dx^\mu {E^\nu}_\rho \in \Omega^1(\mathcal{M}, \mathfrak{so}(1,3))$$

in the differential basis.

This connection, known in physics as the Levi-Civita, is the pullback via section of a corresponding connection on the total space.

Moreover, we should note that, while we are working on the (coordinate) differential basis $\{dx^\mu\}$, it is possible, as per (0.4), to do the same work in a general non-coordinate basis $\{e^a\}$ satisfying (36).
This yields a different approach to General Relativity known as the *tetrad formalism*.

Summary In this section we have shown that, in General Relativity,

i We place ourselves in a **principal** $SO(1,3)$**-bundle**.

ii We ensure *local* invariance under diffeomorphisms by means of a **covariant derivative** ∇_μ.

iii The covariant derivative introduces introduces the coefficients $\Gamma^\mu{}_{\nu\rho}$ of a **connection 1-form** on $\mathcal{M}$ in the differential basis.

0.2.4 Cartan formalism

We took the time to introduce the language of differential geometry because, as announced, we will make use of it in order to draw our comparisons between the two theories.
For this reason, we will use *Cartan's formalism*[26] in the upcoming work.
Cartan's formalism is a much more geometrical way of describing physical theories in differential geometry. It makes use of differential forms, a very natural way of discussing connections, the key object of our study, and it makes the parallelism between General Relativity and Gauge Theory more transparent by encoding information in a coordinate-independent way, which allows us to avoid using coordinate-heavy notation, such as the tensorial formalism of General Relativity.
We already gave a small taste of this formalism in the previous sections, representing connections as $\mathfrak{g}$-valued differential forms, but we will extend this further to other objects.

In Cartan's formalism (in 4-dimensions),

1. Fields are represented by differential forms:

   i Scalar fields are 0-forms: $\phi = \phi^i E_i$

   ii Gauge fields are 1-forms: $A = (A^i_\mu E_i) dx^\mu$

   iii Field strengths are 2-forms $F = dA + A \wedge A$

2. Connections are described as $\mathfrak{g}$-valued 1-forms

3. The Lagrangian is a top-degree form (in our case, a 4-form) that is integrated over the manifold
$$S = \int_{\mathcal{M}} \mathscr{L}, \quad \mathscr{L} \in \Omega^4(\mathcal{M})$$

4. i The Noether currents $\star J$ are 3-forms

   ii The conservation law for the Noether currents is written as
$$d(\star J) = 0 \tag{37}$$

[26] In reality, we will not use pure Cartan formalism because we will stay in coordinate basis, instead of veilbeins, and we will not make use of Cartan's structure equations. However, we will emphasize on a geometrical, mostly coordinate-free, description built on differential forms.

5. The dynamics of the fields are encoded using wedge products and Hodge duals

An example of (5.), which we will be using throughout the upcoming sections, is that we can re-write the Klein-Gordon Lagrangian for a massless complex scalar field as

$$\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi \to d\phi^* \wedge \star d\phi \tag{38}$$

and the action becomes

$$S[\phi] = \int_{\mathcal{M}} d\phi^* \wedge \star d\phi \tag{39}$$

Indeed, injecting (7) and (11) and using (10) in the previous equation,

$$\begin{aligned}
\int_{\mathcal{M}} d\phi^* \wedge \star d\phi &= \int_{\mathcal{M}} \langle d\phi^*, d\phi\rangle \omega_g \\
&= \int_{\mathcal{M}} (\partial_\mu\phi^*\partial_\nu\phi)\langle dx^\mu, dx^\nu\rangle \omega_g \\
&= \int_{\mathcal{M}} \partial_\mu\phi^*\partial_\nu\phi g^{\mu\nu}\omega_g \\
&= \int_{\mathcal{M}} \quad g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi\sqrt{|\det(g)|}dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\nu_3} \wedge dx^{\nu_4} \\
&= \int d^4x \, \sqrt{|\det(g)|} \, g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi
\end{aligned}$$

which, in flat space, reduces to the usual Klein-Gordon action

$$S[\phi] = \int d^4x \, \eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi$$

The replacement $\partial_\mu\phi \to d\phi$ can be done for other expressions. For example, the expression (20) for the $U(1)$ Noether currents becomes

$$\star J_{U(1)} = iq\alpha(\phi \star d\phi^* - \phi^* \star d\phi) \tag{40}$$

As we claimed, this formalism overly simplifies calculations. For example, the Noether currents (18) require writing explicitly two contributions to the variation of the field:

i A term for the intrinsic variation of the field $\delta\phi$:

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi$$

ii A term for the variation of the field induced by the variation of the coordinates δx:

$$\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)\right)\delta x$$

In this new formalism, it is sufficient to write the single term

$$\frac{\partial \mathscr{L}}{\partial(d\phi)}\delta(d\phi)$$

because the differential forms $d\phi = \partial_\mu \phi dx^\mu$ are coordinate-free since they carry the spacetime dependence implicitly, which simplifies all the variational calculations. Moreover, the Klein-Gordon equation can be written as

$$(\delta d + m^2)\phi = 0 \tag{41}$$

where δd is the *codifferential* operator given by

$$\delta d = - \star d \star d \tag{42}$$

0.2.5 Geometrical comparison

To summarize the previous sections, as shown in Tab. 0.2.5, in both theories we place ourselves in principal fibre bundles and in both cases we seek to ensure local invariance under a symmetry group. This invariance compels us to introduce a covariant derivative which, in turn, introduces the coefficients of a connection 1-form (0.2.5). In gauge theories, these coefficients, the gauge fields, appear in the Lagrangian as coupling to the Noether currents (27) arising from the symmetry whose Lie algebra gives values to the connection.

Theory	General Relativity	(Abelian) Gauge theory
Setting	Principal $SO(1,3)$-bundle	Principal G-bundle
Symmetry group	$SO(1,3) \subset GL(4,\mathbb{R})$	G
Covariant derivative (Cartan)	$\nabla V = dV + \omega \wedge V$	$D\psi = d\psi + A \wedge \psi$
Connection (Cartan)	$\omega = \omega^a{}_b E^b{}_a$	$A = A^i E_i$

Table 1: Geometrical comparison between General Relativity and abelian Gauge theories