

$$1, a) P(I_{i=1} | \pi, \mu_1, \mu_2, \sigma_1, \sigma_2, x, I_i) \propto P(x | I_i, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot P(I_{i=1})$$

$$= \prod_{i=1}^n P(x_i | I_i, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot \pi$$

$$\propto P(x_i | I_i, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot \pi$$

$$= \phi(x_i; \mu_1, \sigma_1^2) \cdot \pi$$

p.s.s.

$$P(I_i=2 | \pi, \mu_1, \mu_2, \sigma_1, \sigma_2, I_{-i}) \propto \phi(x_i; \mu_2, \sigma_2^2) (1-\pi)$$

$S \in$

$$P(I_{i=1} | \dots) = \frac{\pi \cdot \phi(x_i; \mu_1, \sigma_1^2)}{\pi \phi(x_i; \mu_1, \sigma_1^2) + (1-\pi) \phi(x_i; \mu_2, \sigma_2^2)}$$

$$1b) P(\lambda | \beta, \sigma^2, y, x) = \frac{P(\beta, \sigma^2, \lambda | y, x)}{P(\beta, \sigma^2 | y, x)} \propto P(\beta, \sigma^2, \lambda | y, x)$$

$$P(\beta, \sigma^2, \lambda | y) \propto P(y | \beta, \sigma^2, \lambda) P(\beta, \sigma^2, \lambda)$$

$$= P(y | \beta, \sigma^2, \lambda) P(\beta | \sigma^2, \lambda) P(\sigma^2 | \lambda) P(\lambda)$$

$$= \prod_{i=1}^n P(y_i | \beta, \sigma^2, \lambda) \cdot P(\beta | \sigma^2, \lambda) P(\sigma^2 | \lambda) P(\lambda)$$

$$= \prod_{i=1}^n \underbrace{\frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{1}{2\sigma^2} (y_i - x_i^\top \beta)^\top (y_i - x_i^\top \beta)\right)}_{\text{constant with respect to } \lambda} \cdot P(\beta | \sigma^2, \lambda) P(\sigma^2 | \lambda)$$

$$S_0, \quad P(\lambda | \beta, \sigma^2, y, x) \propto P(\beta | \sigma^2, \lambda) \underbrace{P(\sigma^2 | \lambda)}_{\text{const.}} P(\lambda)$$

$$\text{Now } \beta | \sigma^2, \lambda \sim N(0, \lambda^{-1} I_p)$$

$$\sigma^2 | \lambda \sim \text{Inv-}\chi^2 \quad \text{Does not depend on } \lambda$$

$$\lambda \sim \text{Inv-}\chi^2(n_0, \lambda_0)$$

$$\begin{aligned} P(\lambda | \beta, \sigma^2, y, x) &\propto \prod_{i=1}^p \frac{1}{\sqrt{2\pi/\lambda}} \exp\left(-\frac{1}{2\lambda} \beta_i^2\right) \cdot \frac{\exp\left(-\frac{n_0 \lambda_0}{2\lambda}\right)}{\lambda^{1+n_0/2}} \\ &\propto \lambda^{p/2} \exp\left(-\frac{1}{2} \sum_{i=1}^p \beta_i^2\right) \exp\left(-\frac{n_0 \lambda_0}{2\lambda}\right) \cdot \lambda^{(n_0/2 - 1)} \\ &= \lambda^{\frac{p}{2} - n_0/2 - 1} \exp\left(-\frac{\lambda}{2} \sum_{i=1}^p \beta_i^2 - \frac{n_0 \lambda_0}{2\lambda}\right) \end{aligned}$$

Not so good... Better if prior was
of the form

$$\lambda^a \exp(-\lambda b).$$

This the Gamma distribution.

The way to go.

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2(n_0, \lambda_0) \Rightarrow$$

$$\frac{1}{\lambda} \sim \text{Inv Gamma}\left(\frac{n_0}{2}, \frac{n_0 \lambda_0}{2}\right) \Rightarrow$$

$$\lambda \sim \text{Gamma}\left(\alpha = \frac{n_0}{2}, \beta = \frac{n_0 \lambda_0}{2}\right)$$

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2(n_0, \lambda_0) \Rightarrow \frac{1}{\lambda} \sim \text{Inv-Gamma} \left(\frac{n_0}{2}, \frac{n_0 \lambda_0}{2} \right)$$

$p(\lambda) \propto \exp\left(-\frac{\nu \lambda^2}{2\lambda}\right) \lambda^{-n_0}$

$$\Rightarrow \lambda \sim \text{Gamma} \left(\frac{n_0}{2}, \frac{n_0 \lambda_0}{2} \right)$$

$$p(\lambda | \beta, \delta^2, y) \propto \lambda^{p/2} \exp\left(-\lambda \frac{\sum \beta_i^2}{2}\right) \lambda^{\frac{n_0 k - 1}{2}} \exp\left(-\lambda \frac{n_0 \lambda_0}{2}\right)$$

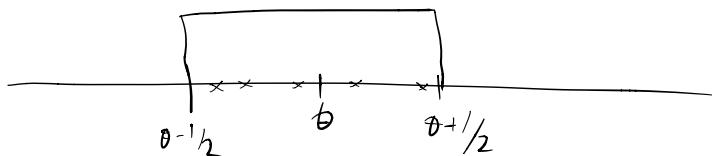
$$= \lambda^{(p+n_0)/2 - 1} \exp\left(-\lambda \frac{\sum \beta_i^2 + n_0 \lambda_0}{2}\right)$$

$$\lambda | \beta, \delta^2, y \sim \text{Ga} \left(\frac{p+n_0}{2}, \frac{\sum \beta_i^2 + n_0 \lambda_0}{2} \right)$$

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2 \left(p+n_0, \frac{\sum \beta_i^2 + n_0 \lambda_0}{p+n_0} \right)$$

$$E\left(\frac{1}{\lambda}\right) \approx \frac{\sum \beta_i^2 + n_0 \lambda_0}{p+n_0} \approx \frac{\sum \beta_i^2}{p} = \text{Var}(\beta_i) \quad \text{when } p > n_0$$

2a)



$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n} \quad \sigma^2 = \text{Var}(X_i) \quad X_i \sim U(\theta - 1/2, \theta + 1/2)$$

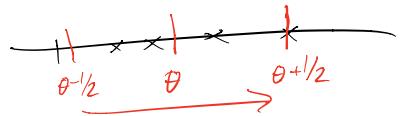
$$X \sim U(0,1) \quad \text{Var}(X) = \frac{1}{12}$$

$$\text{So } \text{Var}(\bar{X}) = \frac{1}{12n}$$

$$2b) P(\theta | x_1, \dots, x_n) \propto P(x_1, \dots, x_n | \theta) P(\theta)$$

$$= \prod_{i=1}^n P(x_i | \theta) P(\theta)$$

$$= \prod_{i=1}^n I\left(\theta - \frac{1}{2} \leq x_i \leq \theta + \frac{1}{2}\right) \cdot 1$$



$$\begin{aligned} \theta + \frac{1}{2} &\geq x_{\max} \\ \theta - \frac{1}{2} &\leq x_{\min} \end{aligned} \Rightarrow \theta \in [x_{\max} - \frac{1}{2}, x_{\min} + \frac{1}{2}]$$



$$P(\theta | x_1, \dots, x_n) \propto 1 \quad \text{for } \theta \in [x_{\max} - \frac{1}{2}, x_{\min} + \frac{1}{2}]$$

= Otherwise.

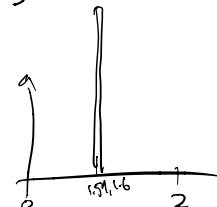
$$\hat{\theta} | x_1, \dots, x_n \sim U(x_{\max} - \frac{1}{2}, x_{\min} + \frac{1}{2})$$

$$2c) \text{ Frequentist: } \hat{\theta} = \bar{x} = 1.53$$

$$\text{Var}(\hat{\theta}) = \frac{1}{12n} = \frac{1}{12 \cdot 3} = 0.027777$$

$$SD(\hat{\theta}) = 0.1666$$

$$\text{Bayesian: } \theta | x_1, x_2, x_3 \sim U(1.53, 1.6)$$



$$3a) \quad \theta | x_1, \dots, x_n \sim \text{Beta}(\alpha+s, \beta+f)$$

$$p(\theta|x) \propto \theta^{\alpha+s-1} (1-\theta)^{\beta+f-1} \Rightarrow \ln p(\theta|x) \propto (\alpha+s-1) \ln \theta + (\beta+f-1) \ln (1-\theta)$$

$$\frac{\partial \ln p(\theta|x)}{\partial \theta} = \frac{\alpha+s-1}{\theta} + \frac{\beta+f-1}{1-\theta} (-1)$$

$$\begin{aligned} \frac{\partial \ln p(\theta|x)}{\partial \theta} &= 0 \Rightarrow \frac{\alpha+s-1}{\theta} = \frac{\beta+f-1}{1-\theta} \\ &\Rightarrow \hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2} \end{aligned}$$

$$3b) \quad \theta | x_1, \dots, x_n \stackrel{\text{optimal}}{\sim} N(\hat{\theta}, -\mathbb{E}_{\theta|x}^{-1}) \quad 1-\hat{\theta} = \frac{\beta+f-1}{\alpha+\beta+n-2}$$

$$\frac{\partial^2 \ln p(\theta|x)}{\partial \theta^2} = -\frac{\alpha+s-1}{\theta^2} + \frac{\beta+f-1}{(1-\theta)^2} (-1)$$

$$\begin{aligned} \frac{\partial^2 \ln p(\theta|x)}{\partial \theta^2} \Big|_{\theta=\hat{\theta}} &= -\left(\frac{\alpha+s-1}{\left(\frac{\alpha+s-1}{\alpha+\beta+n-2} \right)^2} + \frac{\beta+f-1}{\left(\frac{\beta+f-1}{\alpha+\beta+n-2} \right)^2} \right) \\ &= -(\alpha+\beta+n-2)^2 \left(\frac{1}{\alpha+s-1} + \frac{1}{\beta+f-1} \right) \\ &= -(\alpha+\beta+n-2)^2 \left(\frac{\alpha+\beta+n-2}{(\alpha+s-1)(\beta+f-1)} \right) \\ &= -\frac{(\alpha+\beta+n-2)^3}{(\alpha+s-1)(\beta+f-1)} \end{aligned}$$

$$\theta | x_1, \dots, x_n \stackrel{\text{approx}}{\sim} N\left(\hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2}, -J_{\hat{\theta}, x}^{-1} = \frac{(\alpha+s-1)(\beta+f-1)}{(\alpha+\beta+n-2)^3}\right)$$

$$\text{Check: } \begin{aligned} \text{Var}(\hat{\theta}) &= \frac{(\alpha+s)(\beta+f)}{(\alpha+s+\beta+f)^2(\alpha+s+\beta+f+1)} \\ &= \frac{(\alpha+s)(\beta+f)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)} \end{aligned}$$

3 c) Sc R-code
 3 d) —— //

$$4 a) n_A = 5 \quad n_B = 5 \quad n_C = 12$$

$$\begin{aligned} P(A | T_1, T_2) &\propto P(T_1, T_2 | A) P(A) \\ &= P(T_1 | A) \cdot P(T_2 | A) P(A) \end{aligned}$$

$$\begin{matrix} P(A) & P(B) & P(C) & \text{from} & \text{Dirichlet} \\ \theta_1 & \theta_2 & \theta_3 & & \backslash \end{matrix}$$

$$(\theta_1, \theta_2, \theta_3) | n_A, n_B, n_C \sim \text{Dirichlet}_{\text{Dirichlet}}(5+1, 5+1, 10+1) / \text{Dirichlet}(6, 6, 11)$$

$$E(\theta_1 | n_A, n_B, n_C) = \frac{6}{23} \approx 0.26$$

$$E(\theta_2 | \cdot) = \frac{6}{23} = 0.26$$

$$E(\theta_3 | \cdot) = \frac{11}{23} \approx 0.48$$

$$P(T_i | A) \quad N(\mu_{IA}, 1)$$

$\mu_{IA} \bar{x}_1 = 1.2$	$\sim N(1.2, \frac{1}{5})$	$\boxed{\mu \sim N(\bar{x}, \frac{\sigma^2}{n})}$
$\mu_{IB} \bar{x}_1 = 1.4$	$\sim N(1.4, \frac{1}{5})$	
$\mu_{IC} \bar{x}_1 = 0.7$	$\sim N(0.7, \frac{1}{10})$	
$\mu_{IA} \bar{x}_2 = 2.1$	$\sim N(2.1, \frac{1}{5})$	
$\mu_{IB} \bar{x}_2 = 3.5$	$\sim N(3.5, \frac{1}{5})$	
$\mu_{IC} \bar{x}_2 = 4.2$	$\sim N(4.2, \frac{1}{10})$	

$$P(A | T_1 = 1.3, T_2 = 4.2) \propto P(T_1 = 1.3 | A) \cdot P(T_2 = 4.2 | A) \cdot P(A)$$

$$= \phi(1.3, \mu = 1.2, \sigma^2 = \frac{1}{5}) \cdot \phi(4.2, \mu = 2.1, \sigma^2 = \frac{1}{5}) \cdot \frac{6}{23} = 3e-6$$

$$P(B | T_1 = 1.3, T_2 = 4.2) \propto \phi(1.3 | 1.4, \frac{1}{5}) \phi(4.2 | 3.5, \frac{1}{5}) \cdot \frac{6}{23} = 0.060$$

$$P(C | T_1 = 1.3, T_2 = 4.2) \propto \phi(1.3 | 0.7, \frac{1}{10}) \cdot \phi(4.2 | 4.7, \frac{1}{10}) \cdot \frac{11}{23} = 0.036$$

Normalized posterior probabilities: (0.00, 0.62, 0.38)