AA 609 : Computational Methods for Astronomy and Space Sciences

Assignment 1

Due date : January 13, 2022

1. **ODE Algorithms and Error Estimates**
   Consider the equation
   $$\frac{dy}{dx} = -xy \tag{1}$$
   with initial condition $y(0) = 1$, which has the exact solution $y = \exp(-x^2/2)$. Study the numerical integration of this using the following methods a) Euler Method b) Runge-Kutta 2nd Order and c) Runge-Kutta 4th Order. In particular, verify that the errors (difference between numerical and exact solutions) decrease according to the expected power of the algorithm and discuss the difference between local error and global error.

2. **Stellar Orbits in Kepler Potential**
   The equation that governs the motion of a star in r-θ plane under a Kepler potential is given by
   $$\frac{d^2u}{d\theta^2} + u = \frac{1}{a(1-e^2)} \tag{2}$$
   where $u = 1/r$, a and e are constants.

   Solve the above equation using values $a = 5.0$ and $e = 0.8$, plot the trajectory of the star with a) Euler Method and b) Runge Kutta 2nd Order Method. Is the motion of the star what you expect i.e., a closed ellipse with eccentricity e and semi-major axis a ?. Justify the choices of initial conditions taken and any deviation from the expected orbit.

   Hint: The analytic solution between radius r and θ for the above equation can be given as
   $$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)} \tag{3}$$
   where $\theta - \theta_0$ is called the *true anamoly.*

3. **Chandrashekar Mass Limit**
   Write the equations that govern the distribution of stellar mass $M(r)$ and $P(r)$ as a function of density $\rho(r)$ following the argument of hydrostatic balance. Using the following Equation of state, express the equations in terms of variation of $M(r)$ and $\rho(r)$, where r is the spherical radius.
   $$P = \frac{1}{4}(3\pi^2)^{1/3}\hbar c \left(\frac{\alpha}{m_N}\right)^{4/3} \rho^{4/3} \tag{4}$$
   where, the parameter α is the number of electrons per nucleon in the star.

   For numerical work it is useful to rescale the variables involved to that their actual numerical values are neither too large nor too small. We therefore introduce : $\hat{r}$, $\hat{\rho}$ and $\hat{M}$ such that $r = R_0\hat{r}$, $\rho = \rho_0\hat{\rho}$, $M = M_0\hat{M}$. where,
   $$\rho_0 = \frac{n_0 m_N}{\alpha} \tag{5}$$

with $n_0 = \frac{m_e^3 c^3}{3\pi^2 \hbar^3}$ and $M_0 = 4\pi \rho_0 R_0^3$. Using the above scaling relations, find the expression for R_0 such that the scaled equations become

$$\frac{d\hat{M}}{d\hat{r}} = \hat{r}^2 \hat{\rho} \tag{6}$$

$$\frac{d\hat{\rho}}{d\hat{r}} = -\frac{\hat{M}\hat{\rho}^{2/3}}{\hat{r}^2} \tag{7}$$

Write a program to solve the above scaled equations using any algorithm of your choice. As discussed in class, the idea is to numerically integrate these starting at $\hat{r} = 0 + \epsilon$, (where ϵ is a small number) with the initial conditions $\hat{M} = 0$ and $\hat{\rho} = \hat{\rho}_c$. The radius R of the white dwarf is the value of r at which $\rho = 0$, and $M(R)$ is the total mass at this point.

Once you have the code working, calculate the total masses and radii of white dwarfs with $\hat{\rho}_c$ values ranging from about 0.1 to 10^6. This gives a family of equilibrium configurations. By changing the stepsize h and (maybe) the algorithm used, verify that your solutions are accurate.

Hint: A useful way to display these results is as a plot of radius versus mass. Can you identify the point at which the star can no longer be supported by degenerate electrons? This limiting mass for white dwarfs (with $\alpha = 0.5$, 1 electron per 2 nucleons (1 proton + 1 neutron)) is known as the Chandrasekhar mass. Calculate its value in units of the mass of the sun.