ACI650 - Modelos y Simulación

Generating Continuous Random Variables

Mario González
Facultad de Ingeniería y Ciencias Ambientales
Centro de Investigación, Estudios y Desarrollo de Ingeniería (CIEDI)

UNIVERSIDAD DE LAS AMÉRICAS
Laureate International Universities®

April 25, 2016

Learning Objectives

- Generate continuous probability distributions.
- Hands-on workshop.
- Simulation examples.

The Inverse Probability Transform

- **Universality of the Uniform**. Let F be a CDF which is a continuous function and strictly increasing on the support of the distribution. This ensures that the inverse function F^{-1} exists, as a function from (0, 1) to $\mathbb{R}$. We then have the following results.
  1. Let $U \sim Unif(0,1)$ and $X = F^{-1}(U)$. Then X is an r.v. with CDF F.
  2. Let X be an r.v. with CDF F. Then $F(X) \sim Unif(0,1)$.
- Prove numerically the Universality of the Uniform:
  - Let $f(x) = \frac{x^3}{4}$ for $0 < x < 2$.
  - Thus $F(x) = \frac{x^4}{16}$ and $F^{-1}(y) = 2 \times y^{1/4}$.

Generating Continuous Random Variables

- To generate continuous random variables all we need is to compute the inverse c.d.f. (F^{-1}), this is not always an easy task.
- Example: Simulating an exponential r.v. ($X \sim Expo(\lambda)$)
- Remember that a continuous exponential r.v. X has a p.d.f.:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

- And c.d.f.:

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

- To apply the inverse transform method we need to invert $F(x)$, that is, we need to solve for x the equation

$$u = F(x).$$

Exponential Random Variables I

- Let $F(x) = 1 - e^{-\lambda x}, \quad x > 0$:
- Solving $u = F(x)$ for x, we have:

$$u = 1 - e^{-\lambda x}$$

$$1 - u = e^{-\lambda x}$$

$$ln(1 - u) = -\lambda x$$

$$x = \frac{-1}{\lambda} ln(1 - u)$$

Exponential Random Variables II

- Then
$$X = \frac{-1}{\lambda} ln(1-U)$$
is a r.v. with exponential distribution with parameter λ.
- **Note that** $1-U$ is also a uniform random variable over $[0,1]$, thus:
$$X = \frac{-1}{\lambda} ln(U)$$
is also an exponential r.v. with parameter λ.
- Pseudo-code:

```
function Expo(lambda)
    return -1 / lambda ln(rand())
```

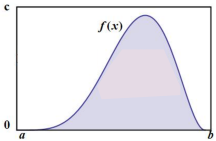
Exponential Random Variables III

- The Exponential distribution has a very special property called the memoryless property, which says that even if you've waited for hours or days without success, the success isn't any more likely to arrive soon. In fact, you might as well have just started waiting 10 seconds ago.
- Applications:
  1. The time until a radioactive particle decays, or the time between clicks of a geiger counter.
  2. The time it takes before your next telephone call.
  3. In queuing theory, the service times of agents in a system (e.g. how long it takes for a bank teller etc. to serve a customer) are often modeled as exponentially distributed variables.
  4. In hydrology, the exponential distribution is used to analyze extreme values of such variables as monthly and annual maximum values of daily rainfall and river discharge volumes

The Rejection Method I

- Sometimes it is not possible to compute easily the c.d.f., and therefore computing its inverse is likely to be even harder.
- However, we often have access to the density $f(x)$ of the random variable. In this case we can use what is known as the rejection method.
- Let's make the assumption that the density $f(x)$ is bounded and had finite support (meaning the set where the density if different than zero is bounded), that is:

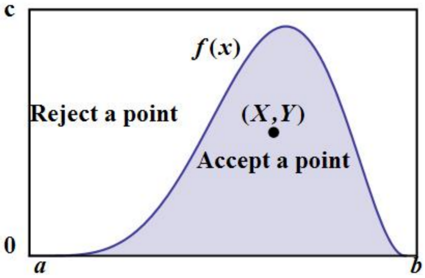
- Assume we have numbers a, b and c such that
  - I $0 \leq f(x) \leq c$ for all $x \in \mathbb{R}$
  - II $f(x) = 0$ if $x < a$ or $x > b$



All we have to do now is to generate points uniformly at random in the shadowed area.

The Rejection Method II

- The Rejection Method:
  i Find a, b, c such that for $a \leq x \leq b$ we have $0 \leq f(x) \leq c$ and for all other values of x we have $f(x) = 0$.
  ii Generate two standard uniform random variables U and V.
  iii Define $X = a + (b - a)U$ and $Y = cV$. Note that (X, Y) are uniform over the rectangle in the figure below.
  iv If $Y > f(X)$ reject the sample and return to step (ii). Otherwise return the sample X.



The Rejection Method III

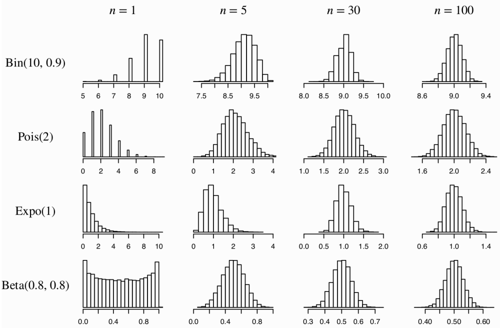
- Example: Draw a sample of size n from a r.v. X with p.d.f. $f(x) = \frac{X^3}{4}$ using the rejection method.

Exercises I

- In practice we will use predefined functions for r.v. generation with any given p.d.f.
- In python we will use scipy.stats, which implements the common probability distributions.
- Build the Standard Normal Distribution $N(0,1)$ Table using scipy.stats.
- Prove (numerically) that $V = Z_1^2 + \cdots + Z_n^2$ where $Z_1, Z_2, ..., Z_n$ are i.i.d. $N(0,1)$ has the Chi-Square distribution with n degree of freedom. We write this as $V \sim \chi_n^2$

Assignment (10 %)

- Demonstrate the Central limit theorem by simulation.
  - Assume we have i.i.d. $X_1, X_2, X_3, \ldots$ with mean μ variance σ^2. The law of large numbers says that as $n \to \infty$, $\bar{X}_n$ converges to the constant μ.
  - But what is its distribution along the way to becoming a constant?
  - Central limit theorem: For large n, the distribution of $\bar{X}_n$ is approximately $N(\mu, \sigma^2/n)$



Sources and resources

- Introduction to the Central Limit Theorem.
- Sampling distribution.
- Chi-squared distribution.
- Statistical functions (scipy.stats)