# ACI650 - Modelos y Simulación

## Monte Carlo Integration

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# Learning Objectives

- How Monte Carlo integration works
- How to do MC integration using the
  - Sampling method
  - Hit and miss method
- Examples of Monte Carlo applications

# Random processes and Monte Carlo simulation

- In a random process it is not possible to predict from the observation of one event, how the next will come out
- However we can predict the expected value.
- Examples:
- Coin: the only prediction about outcome, 50 % the coin will land on its tail
- Dice: In large number of throws, probability 1/6

# Applications for Monte Carlo simulation

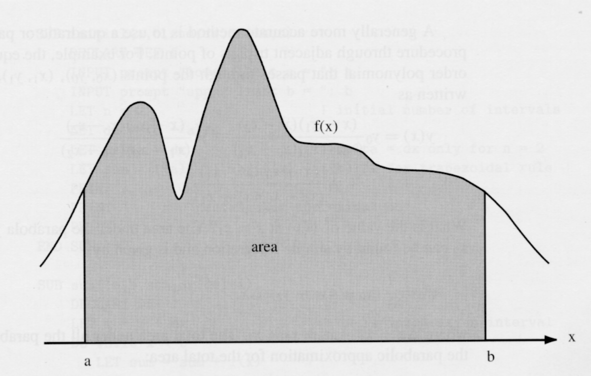
- Stochastic processes
- Complex systems (science)
- **Numerical integration**
- Risk management
- Financial planning

# Monte Carlo integration I

- Many statistical techniques are included in the category of "Monte Carlo methods", however
- Monte Carlo methods were first developed as a method for estimating integrals that could not be evaluated analytically.
- Most of the same principles of Monte Carlo integration hold regardless of whether we are integrating an analytical function or a simulation.

# Monte Carlo integration II

- Let's remember how ordinary numerical integration works
- The problem can be stated in the form that we want to find the area $A$ below an arbitrary curve in some interval $[a, b]$.



# Monte Carlo integration III

- We have that $A$ is a summation over $N$ points at regular interval $\Delta x$ in $x$:

$$A = \sum_{i=1}^{N} f(x_i)\Delta x,$$

- with

$$x_i = a + (i - 0{,}5)\Delta x, \quad \Delta x = \frac{b-a}{N}.$$

- i.e.

$$A = \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)$$

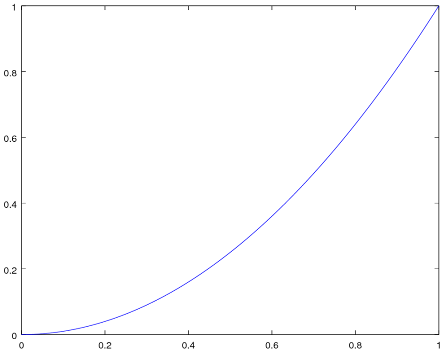
- This takes the value of f from the midpoint of each interval.

# Monte Carlo integration IV

- **Sampling method:**
  - The sampling method for MC integration is very similar to the simple summing rule given above
  - Instead of sampling at regular intervals $\Delta x$, we now sample at random points, and then take the average over these.
  - Say we pick $N$ points $x_i$ in the interval $[a, b]$ in 1D. The integral then becomes
$$A = \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$
  - More generally, in $M$ dimensions we have to pick vectors $x_i = (x_1, x_2, ..., x_M)$.
  - At random in the interval $([a_1, b_1], [a_2, b_2], ..., [a_M, b_M])$ which can be done very easily using uniform random numbers for each dimension at a time.

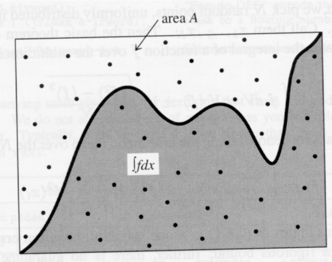
# Monte Carlo integration V

- **Example:** Calculate the area below $f(x) = x^2$ in the interval $[0, 1]$



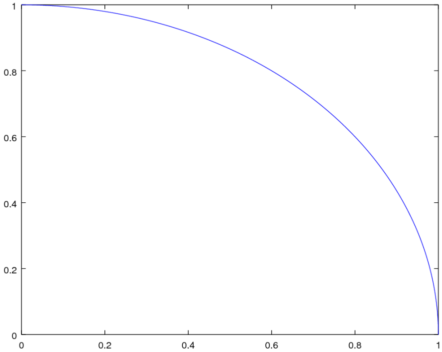
# Monte Carlo integration VI

- **Hit and miss method:**
  - There is another approach to MC integration, which is even simpler than the sampling approach: the hit-and-miss method.
  - The idea is to find some region in space of known area, which encloses the area we want to integrate, then generate random points everywhere in this region, and count the points which actually do hit the area we want to handle:
  - $Area_S = Area_E \times f_h$, where the shadowed area $A_S$ is equal to the enclosing area $Area_E$ times the fraction of hits $f_h = \frac{N_h}{N}$.



# Monte Carlo integration VII

- **Example:** Calculate the area of a circle of $radius = 1$ using the MC hit and miss method, for simplicity use $c = (0, 0)$.

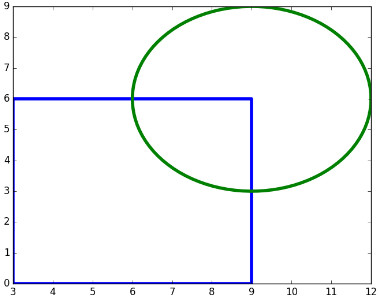


# Monte Carlo Integration Error

- The error in MC Integration is of $O(\frac{1}{\sqrt{N}})$ as $N \to \infty$.
- Monte Carlo computation is poorly suited for problems that must be answered with high precision.
- The error in MC does not depends on the dimension of the problem, only on the number of executions $N$.
- From this MC integration is particularly useful in multidimensions.

# Assignment #2 (7.5 %)

- Calculate the operlapping area between a circle and a rectangle using the MC hit and miss method.
  1. Given the two shapes, may overlap or not.
  2. Build the square, that inscribes the two shapes.
  3. Calculate the overlapping area using the hit and miss MC interpolation.



# Sources and Resources

- Course materials by Kai Nordlund. Monte Carlo simulations.
- Monte Carlo Integration.
- Wolfram. Circle-Circle Intersection.
- Random Walk, Wikipedia.