CHAPTER 1 - SETS, RELATIONS AND FUNCTIONS

LONG ANSWERS: (5 MARKS)

Eg 1.13 In the set Z of integers define mRn if $m-n$ is a multiple of 12. Prove that R is an equivalence relation

(i) $mRm \Rightarrow m-m=0$ is a multiple of 12

$\therefore$ R is reflexive

(ii) $mRn \Rightarrow m-n=12k$

$nRm \Rightarrow n-m=-(m-n)=-12k$ multiple of 12

$\therefore$ R is symmetric.

iii) Let mRn and nRp

$\Rightarrow m-n=12k$ — ① $\quad n-p=12l.$ — ②

① + ② $\Rightarrow m-p=12(k+l)$

$\Rightarrow mRp$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation

Ex 1.2 (9)

In the set Z of integers, define mRn if $m-n$ is divisible by 7. Prove that R is an equivalence relation.

(i) $m-m=0$ which is divisible by 7 $\therefore mRm$

R is reflexive.

(ii) $mRn \Rightarrow m-n$ is divisible by 7

$\Rightarrow m-n=7a$

$\Rightarrow n-m=-7a$ which is divisible by 7

$\therefore nRm$

R is symmetric

(iii) Let mRn and nRp

$\Rightarrow m-n=7a$ — ① and $n-p=7b$ — ②

① + ② $\Rightarrow m-p=7(a+b)$

$\Rightarrow mRp$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation

Eg 1.30: If $f: R \to R$ is defined by $f(x)=2x-3$ prove that f is a bijection and find its inverse.

Let $f(x)=f(y) \Rightarrow 2x-3=2y-3$

$\Rightarrow 2x=2y$

$\Rightarrow x=y$

$\therefore$ f is one-to-one

Let $y \in R$. Let $y=2x-3$

$\Rightarrow x=\frac{y+3}{2}$

$f(x)=2\left(\frac{y+3}{2}\right)-3$

$=y+3-3$

$=y$

$\therefore$ f is onto

$\therefore$ f is a bijection

Let $y=2x-3$

$y+3=2x \Rightarrow x=\frac{y+3}{2}$

$f^{-1}(y)=\frac{y+3}{2}$ by replacing y by x $\quad f^{-1}(x)=\frac{x+3}{2}$

Ex 1.3. (12)

If $f: R \to R$ is defined by $f(x)=3x-5$, Prove that f is a bijection and find its inverse

If $f(x)=f(y)$

$\Rightarrow 3x-5=3y-5$

$\Rightarrow 3x=3y$

$\Rightarrow x=y$

$\therefore f$ is one to one.

Let $y \in R$ such that

$x=\frac{y+5}{3} \Rightarrow f(x)=f\left(\frac{y+5}{3}\right)$

$=3\left(\frac{y+5}{3}\right)-5$

$=y+5-5$

$=y$

$\therefore f$ is onto

$\Rightarrow f$ is bijective.

Let $y=3x-5 \Rightarrow y+5=3x$

$\Rightarrow x=\frac{y+5}{3}$

$f^{-1}(y)=\frac{y+5}{3}$

Replace y by x.

$f^{-1}(x)=\frac{x+5}{3}$

2. Write the values of f at $-4, 1, -2, 7, 0$ if

$$f(x)=\begin{cases} -x+4 & \text{if } -\infty < x \le -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2-x & \text{if } -2 \le x < 1 \\ x-x^2 & \text{if } 1 \le x < 7 \\ 0 & \text{otherwise} \end{cases}$$

$f(-4)=-(-4)+4=4+4=8$

$f(1)=1-(1)^2=1-1=0$

$f(-2)=(-2)^2-(-2)=4+2=6$

$f(7)=0$

$f(0)=(0)^2-0=0$

3. Write the values of f at $-3, 5, 2, -1, 0$ if

$$f(x)=\begin{cases} x^2+x-5 & \text{if } x \in (-\infty, 0) \\ x^2+3x-2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2-3 & \text{otherwise} \end{cases}$$

$f(-3)=(-3)^2+(-3)-5=9-8=1$

$f(5)=(5)^2+3(5)-2=25+15-2=38$

$f(2)=(2^2)-3=4-3=1$

$f(-1)=(-1)^2-1-5=1-6=-5$

$f(0)=(0)^2-3=-3$

8). Find the range of the function $\frac{1}{2\cos x-1}$

$-1 \le \cos x \le 1$

$\times$ by 2

$-2 \le 2\cos x \le 2$

$-2-1 \le 2\cos x-1 \le 2-1$

$-3 \le 2\cos x-1 \le 1$

$-\frac{1}{3} \ge \frac{1}{2\cos x-1} \ge 1$

$\therefore$ The range of f is

$\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

Eg 1.23 Find the range of the function $f(x)=\frac{1}{1-3\cos x}$.

$-1 \le \cos x \le 1$

$-3 \le 3\cos x \le 3$

$3 \ge -3\cos x \ge -3$

$4 \ge 1-3\cos x \ge -2$

$\frac{1}{4} \le \frac{1}{1-3\cos x} \le -\frac{1}{2}$

Range of $f(x)$ is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{4}, \infty\right)$

Eg 1.24 Find the largest possible domain for real valued function $f(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$

Consider $\sqrt{9-x^2}$

If $x>3$, $9-x^2$ is $-ve$

If $x<3$, $9-x^2$ is $-ve$

$\therefore \sqrt{9-x^2}$ is not in R.

$\Rightarrow x \in [-3,3]$ —— ①

Consider $\sqrt{x^2-1}$ between -1 and 1 x^2-1 is $-ve$ at -1 and it is zero both are not possible.

$\therefore x \in (-\infty,-1) \cup (1,\infty)$ —— ②

Combining ① and ②

Domain is $[-3,-1) \cup (1,3]$

CHAPTER 2 : BASIC ALGEBRA

LONG ANSWERS : 5 MARKS :

Ex 2.3 (6)

A manufacturer has 600l of 12% solution of acid. How many litres of 30% acid solution must be added to it so that the acid content in the resulting mixture will be more than 15% but less than 18%. ?

Let x litres of 30% acid be added to 600l of 12% acid solution.

Given

$$(x+600)\frac{15}{100} < \frac{30x}{100} + \frac{600\times 12}{100} < (x+600)\frac{18}{100}$$

$$(x+600)15 < 30x+7200 < (x+600)18$$

$$15x+9000 < 30x+7200 < 18x+10800$$

$$15x+1800 < 30x < 18x+3600$$

$15x+1800<30x$	$30x<18x+3600$
$1800<30x-15x$	$12x<3600$
$1800<15x$	$x<\frac{3600}{12}$
$\frac{1800}{15}<x$	$x<300$
$120<x$ —①	—②

From ① & ②

$$120<x<300$$

Ex 2.3 (8) A model rocket is launched from the ground. The height h reached by the rocket after t seconds from lift off is given by $h(t)=-5t^2+100t$, $0\le t\le 20$. At what time is the rocket 495 feet above the ground

$h(t)=-5t^2+100t$

when $h=495$ $t=?$

$$495=-5t^2+100t$$

$$5t^2-100t+495=0$$

$$t^2-20t+99=0$$

$$(t-9)(t-11)=0 \Rightarrow t=9,\ t=11$$

The rocket is 495 ft above the ground at $t=9$, $t=11$

Ex 2.4 (4) If one root of $k(x-1)^2=5x-7$ is double the other root, show that $k=2$ or -25

$$k(x-1)^2=5x-7$$

$$k(x^2-2x+1)-5x+7=0$$

$$kx^2-2kx+k-5x+7$$

$Kx^2 - (2K+5)x + K + 7 = 0$

$a = K \quad b = -(2K+5) \quad c = K+7$

$\because$ equal roots $\Delta = 0$

$b^2 - 4ac = 0$

$(2K+5)^2 - 4K(K+7) = 0$

$4K^2 + 20K + 25 - 4K^2 - 28K$

Given one root is double the other

$\therefore$ Let $\alpha, 2\alpha$ be the roots

$$\alpha + 2\alpha = -\left(\frac{-(2K+5)}{K}\right)$$

$$3\alpha = \frac{2K+5}{K} \Rightarrow \alpha = \frac{2K+5}{3K} \quad —(1)$$

$$\alpha \cdot 2\alpha = \frac{K+7}{K}$$

$$2\alpha^2 = \frac{K+7}{K} \quad —(2)$$

Subst (1) in (2)

$$2\left(\frac{2K+5}{3K}\right)^2 = \frac{K+7}{K}$$

$$\frac{2(4K^2+20K+25)}{9K^2} = \frac{K+7}{K}$$

$$8K^2 + 40K + 50 = 9K^2 + 63K$$

$$9K^2 - 8K^2 + 63K - 40K - 50 = 0$$

$$K^2 + 23K - 50 = 0$$

$$(K+25)(K-2) = 0$$

$$\Rightarrow K = -25 \quad K = 2$$

Ex 2.4 (7) If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots P.T $ae = 2(b+f)$

$x^2 - ax + b = 0 \; —(1)$; $x^2 - ex + f = 0 \; —(2)$

Let α, β be roots of (1)

Then α, α roots of (2)

(2) has equal roots

$$\therefore e^2 - 4f = 0 \Rightarrow e^2 = 4f \quad —(3)$$

α is common root.

$$\alpha^2 - a\alpha + b = 0$$

$$\alpha^2 - e\alpha + f = 0$$

$$\alpha(e-a) = f - b \quad —(4)$$

In eqn (2)

$$\alpha + \alpha = e \Rightarrow 2\alpha = e.$$

$$\alpha = \frac{e}{2}$$

Subst in (4)

$$\frac{e}{2}(e-a) = f - b$$

$$e^2 - ae = 2f - 2b$$

Subst $e^2 = 4f$ (3)

$$4f - ae = 2f - 2b$$

$$ae = 4f - 2f + 2b$$

$$= 2f + 2b$$

$$ae = 2(f+b)$$

Ex 2.8 (1) find all values of x for which $\frac{x^3(x-1)}{(x-2)} > 0$

CRITICAL NUMBERS $x = 0, 1, 2$.

at 2 $f(x)$ does not exist

INT	x^3	$(x-1)$	$(x-2)$	$f(x)$
$(-\infty, 0)$	$-$	$-$	$-$	$-$ / $+$ ✓
$(0,1)$	$+$	$-$	$-$	$-$ / $+$
$(1,2)$	$+$	$+$	$-$	$-$
$(2,\infty)$	$+$	$+$	$+$	$+$ ✓

$$x \in (0,1) \cup (2, \infty)$$

Ex 2.8 (2) Find all values of x that satisfies the inequality

$$\frac{2x-3}{(x-2)(x-4)} < 0$$

CRITICAL NUMBERS: $x = \frac{3}{2}, 2, 4$

at $2, 4$ $f(x)$ does not exist

$-\infty \quad \frac{3}{2} \quad 2 \quad 4$

INT	$2x-3$	$x-2$	$x-4$	$f(x)$
$(-\infty, \frac{3}{2})$	$-$	$-$	$-$	$-$ ✓
$(\frac{3}{2}, 2)$	$+$	$-$	$-$	$+$
$(2, 4)$	$+$	$+$	$-$	$-$ ✓
$(4, \infty)$	$+$	$+$	$+$	$+$

$x \in (-\infty, \frac{3}{2}) \cup (2, 4)$

Ex 2.8 (3)

Solve $\dfrac{x^2-4}{x^2-2x-15} \leq 0$

$$\frac{(x+2)(x-2)}{(x-5)(x+3)} \leq 0$$

CRITICAL NUMBERS $x = -2, 2, 5, -3$

at $5, -3$ $f(x)$ does not exist

$-\infty \quad -3 \quad -2 \quad 2 \quad 5 \quad \infty$

INT	$(x+2)$	$(x-2)$	$(x-5)$	$(x+3)$	$f(x)$
$(-\infty, -3)$	$-$	$-$	$-$	$-$	$+$
$(-3, -2)$	$-$	$-$	$-$	$+$	$-$ ✓
$(-2, 2)$	$+$	$-$	$-$	$+$	$+$
$(2, 5)$	$+$	$+$	$-$	$+$	$-$ ✓
$(5, \infty)$	$+$	$+$	$+$	$+$	$+$

$\therefore x \in (-3, -2] \cup [2, 5)$

Eg 2.24 Solve $\dfrac{x+1}{x+3} < 3$

$$\frac{x+1}{x+3} < 3 \Rightarrow \frac{x+1}{x+3} - 3 < 0$$

$$\Rightarrow \frac{x+1-3(x+3)}{x+3} < 0$$

$$\Rightarrow \frac{x+1-3x-9}{x+3} < 0$$

$$\Rightarrow \frac{-2x-8}{x+3} < 0$$

$$\div -2 \quad \Rightarrow \frac{x+4}{x+3} > 0$$

CRITICAL NUMBERS $x = -4, -3$

at $x = -3$ $f(x)$ does not exist

$-\infty \quad -4 \quad -3 \quad \infty$

INT	$x+4$	$x+3$	$f(x)$
$(-\infty, -4)$	$-$	$-$	$+$ ✓
$(-4, -3)$	$+$	$-$	$-$
$(-3, \infty)$	$+$	$+$	$+$ ✓

$x \in (-\infty, -4) \cup (-3, \infty)$

Ex 2.9 (3) Resolve into partial fractions $\dfrac{x}{(x^2+1)(x-1)(x+2)}$

$$\frac{x}{(x^2+1)(x-1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+2}$$

$$x = (Ax+B)(x-1)(x+2) + C(x^2+1)(x+2) + D(x^2+1)(x-1)$$

Put $x = 1$

$1 = C(2)(3) \Rightarrow C = \frac{1}{6}$

put $x=-2$

$-2 = D(5)(-3) \Rightarrow D = \frac{2}{15}$

put $x=0$

$0 = B(-1)(2) + C(1)(2) + D(1)(-1)$

$-2B + 2\left(\frac{1}{6}\right) - \frac{2}{15} = 0$

$2B = \frac{2}{6} - \frac{2}{15} = \frac{10-4}{30} = \frac{6}{30}$

$2B = \frac{1}{5} \Rightarrow B = \frac{1}{10}$

put $x=2$.

$2 = (2A+B)(1)(4) + C(5)(4) + D(5)(1)$

$8A + 4B + 20C + 5D = 2$

$8A + \frac{4}{10} + \frac{20}{6} + \frac{10}{15} = 2$

$8A + \frac{12+100+20}{30} = 2$

$8A + \frac{132}{30} = 2$

$4A + \frac{66}{30} = 1$

$4A + \frac{22}{10} = 1$

$4A = 1 - \frac{22}{10} = \frac{-12}{10}$

$A = -\frac{3}{10}$

$\frac{x}{(x^2+1)(x-1)(x+2)} = \frac{-3x+1}{10(x^2+1)} + \frac{1}{6(x-1)} + \frac{2}{15(x+2)}$

Ex 2.9

7) $\frac{x^2+x+1}{x^2-5x+6} = \frac{x^2+x+1}{(x-3)(x-2)}$

$= A + \frac{B}{x-3} + \frac{C}{x-2}$

$x^2+x+1 = A(x-2)(x-3) + B(x-2) + C(x-3)$

Put $x=3$

$9+3+1 = B(3-2) \Rightarrow B = 13$

Put $x=2$

$4+2+1 = C(2-3) \Rightarrow -C = 7$

$\Rightarrow C = -7$

Put $x=0$

$1 = A(-2)(-3) + B(-2) + C(-3)$

$6A - 2B - 3C = 1$

$6A - 26 + 21 = 1$

$6A - 5 = 1$

$6A = 6 \Rightarrow A = 1$

$\frac{x^2+x+1}{x^2-5x+6} = 1 + \frac{13}{x-3} - \frac{7}{x-2}$

Ex2.9

8) $\frac{x^3+2x+1}{x^2+5x+6} = \frac{x^3+2x+1}{(x+3)(x+2)}$

$= (Ax+B) + \frac{C}{x+3} + \frac{D}{x+2}$

$x^3+2x+1 = (Ax+B)(x+3)(x+2) + C(x+2) + D(x+3)$

Put $x=-3$

$-27-6+1 = C(-3+2)$

$-C = -32 \Rightarrow C = 32$

Put $x=-2$

$-8-4+1 = D(-2+3)$

$D = -11$

Put $x=0$

$1 = B(3)(2) + C(2) + D(3)$

$6B + 64 - 33 = 1$

$6B + 31 = 1 \Rightarrow 6B = -30$

$\Rightarrow B = -5$

Put $x=1$

$1+2+1 = (A+B)(4)(3) + C(3) + D(4)$

$12A + 12B + 3C + 4D = 4$

$12A - 60 + 96 - 44 = 4$

$12A - 8 = 4 \Rightarrow 12A = 12 \Rightarrow A = 1$

$\frac{x^3+2x+1}{x^2+5x+6} = x - 5 + \frac{32}{x+3} - \frac{11}{x+2}$

9) $\frac{x+12}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

$x+12 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$

Put $x=-1$

$-1+12 = B(-1-2)$

$-3B = 11 \Rightarrow B = -\frac{11}{3}$

Put $x=2$

$2+12 = A(3)(0) + 9C \Rightarrow 9C = 14$

$\Rightarrow C = \frac{14}{9}$

Put $x=0$

$12 = A(1)(-2) + B(-2) + C$

$-2A + \frac{22}{3} + \frac{14}{9} = 12$

$-A + \frac{11}{3} + \frac{7}{9} = 6$

$-A + \frac{40}{9} = 6$

$A = \frac{40}{9} - 6 = \frac{40-54}{9} = -\frac{14}{9}.$

$\frac{x+12}{(x+1)^2(x-2)} = -\frac{14}{9(x+1)} - \frac{11}{3(x+1)^2} + \frac{14}{9(x-2)}$

Ex 2.9 (2)

~~x+12~~ $\frac{3x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$3x+1 = A(x+1) + B(x-2)$

put $x=2$

$6+1 = A(3) \Rightarrow A = \frac{7}{3}$

Put $x=-1$

$-3+1 = B(-3) \Rightarrow B = \frac{2}{3}$

$\frac{3x+1}{(x-2)(x+1)} = \frac{7}{3(x-2)} + \frac{2}{3(x+1)}$

Ex 2.11

3) If $(x^{1/2} + x^{-1/2})^2 = \frac{9}{2}$, then find the value of $(x^{1/2} - x^{-1/2})$ for $x > 1$

$(x^{1/2} + x^{-1/2})^2 = \frac{9}{2}$

$\left(x^{1/2} + \frac{1}{x^{1/2}}\right)^2 = \frac{9}{2}$

$x + \frac{1}{x} + 2 \cdot x^{1/2} \cdot \frac{1}{x^{1/2}} = \frac{9}{2}$

$x + \frac{1}{x} = \frac{9}{2} - 2 = \frac{5}{2}$ — (1)

$(x^{1/2} - x^{-1/2})^2 = \left(x^{1/2} - \frac{1}{x^{1/2}}\right)^2$

$= x + \frac{1}{x} - 2$

$= \frac{5}{2} - 2$

$= \frac{1}{2}$

$x^{1/2} - x^{-1/2} = \pm \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$ since $x > 1$

Ex 2.11

7) Simplify

$\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$

$\frac{1}{3-\sqrt{8}} \times \frac{3+\sqrt{8}}{3+\sqrt{8}} = \frac{3+\sqrt{8}}{9-8} = 3+\sqrt{8}$

$\frac{1}{\sqrt{8}-\sqrt{7}} \times \frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}-\sqrt{7}} = \frac{\sqrt{8}+\sqrt{7}}{8-7} = \sqrt{8}+\sqrt{7}$

$\frac{1}{\sqrt{7}-\sqrt{6}} \times \frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}} = \frac{\sqrt{7}+\sqrt{6}}{7-6} = \sqrt{7}+\sqrt{6}$

$\frac{1}{\sqrt{6}-\sqrt{5}} \times \frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}} = \frac{\sqrt{6}+\sqrt{5}}{6-5} = \sqrt{6}+\sqrt{5}$

$\frac{1}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{\sqrt{5}+2}{5-4} = \sqrt{5}+2$

$\therefore$

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$

$$=(3+\sqrt{8})-(\sqrt{8}+\sqrt{7})+(\sqrt{7}+\sqrt{6})-(\sqrt{6}+\sqrt{5})+(\sqrt{5}+2)$$

$$=3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2$$

$$=3+2=5$$

Eg 2.37 If $\log_2 x+\log_4 x+\log_{16} x=\frac{7}{2}$ find value of x

$$\log_2 x+\log_4 x+\log_{16} x=\frac{7}{2}$$

$$\log_2 x+\log_{2^2} x+\log_{2^4} x=\frac{7}{2}$$

$$\log_2 x+\frac{1}{2}\log_2 x+\frac{1}{4}\log_2 x=\frac{7}{2}$$

$$\log_2 x\left[1+\frac{1}{2}+\frac{1}{4}\right]=\frac{7}{2}$$

$$\log_2 x\left[\frac{4+2+1}{4}\right]=\frac{7}{2}$$

$$\log_2 x=\frac{7}{2}\cdot\frac{4}{7}=2$$

$$x=2^2=4.$$

Ex 2.12

3) Solve $\log_8 x+\log_4 x+\log_2 x=11$

$$\log_8 x+\log_4 x+\log_2 x=11$$

$$\log_{2^3} x+\log_{2^2} x+\log_2 x=11$$

$$\frac{1}{3}\log_2 x+\frac{1}{2}\log_2 x+\log_2 x=11$$

$$\log_2 x\left[\frac{1}{3}+\frac{1}{2}+1\right]=11$$

$$\log_2 x\left[\frac{2+3+6}{6}\right]=11$$

$$\log_2 x=11\times\frac{6}{11}=6$$

$$\therefore x=2^6=64.$$

Ex 2.12

5. If $a^2+b^2=7ab$, S.T $\log\left(\frac{a+b}{3}\right)=\frac{1}{2}(\log a+\log b)$

Given $a^2+b^2=7ab$

$$a^2+b^2+2ab=7ab+2ab$$

$$(a+b)^2=9ab$$

$$\frac{(a+b)^2}{9}=ab$$

$$\left(\frac{a+b}{3}\right)^2=ab$$

take log on both sides

$$\log\left(\frac{a+b}{3}\right)^2=\log ab$$

$$2\log\left(\frac{a+b}{3}\right)=\log a+\log b$$

$$\log\left(\frac{a+b}{3}\right)=\frac{1}{2}\left[\log a+\log b\right]$$

Ex 2.12

7. P.T $\log 2+16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}=1$

LHS

$$\log 2+16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}$$

$$=\log\left[2\times\left(\frac{16}{15}\right)^{16}*\left(\frac{25}{24}\right)^{12}\times\left(\frac{81}{80}\right)^{7}\right]$$

$$=\log\left[\frac{2\times16^{16}\times25^{12}\times81^{7}}{15^{16}\times24^{12}\times80^{7}}\right]$$

$$=\log\left[\frac{2\times2^{64}\times5^{24}\times3^{28}}{5^{16}\times3^{16}\times2^{36}\times3^{12}\times2^{28}\times5^{7}}\right]$$

$$=\log\left[\frac{2^{65}\times3^{28}\times5^{24}}{2^{64}\times3^{28}\times5^{23}}\right]$$

$$=\log[2\times5]=\log 10=1.\ \text{RHS}$$

Ex 2.12

10. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, then Prove that $xyz = 1$

Let $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y} = k, k \in R$

$\log x = ky - kz$

$\log y = kz - kx$

$\log z = kx - ky$

$\log x + \log y + \log z = 0$

$\log xyz = \log 1$

$\therefore xyz = 1$

CHAPTER 3 : TRIGONOMETRY

Ex 3.4

25. If $\theta+\phi=\alpha$ and $\tan\theta = k\tan\phi$ then Prove that $\sin(\theta-\phi) = \dfrac{k-1}{k+1}\sin\alpha$.

Given $\tan\theta = k\tan\phi$

$K = \dfrac{\tan\theta}{\tan\phi} = \dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi}$

$\dfrac{K-1}{K+1} = \dfrac{\dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi} - 1}{\dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi} + 1}$

$= \dfrac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi}$

$= \dfrac{\sin(\theta-\phi)}{\sin(\theta+\phi)} = \dfrac{\sin(\theta-\phi)}{\sin\alpha}$

$\Rightarrow \dfrac{K-1}{K+1}\sin\alpha = \sin(\theta-\phi)$

Hence proved.

Ex 3.5

6. If $A+B = 45^\circ$, show that $(1+\tan A)(1+\tan B) = 2$

Given $A+B = 45^\circ$

$\tan(A+B) = \tan 45^\circ$

$\dfrac{\tan A + \tan B}{1-\tan A\tan B} = 1$

$\tan A + \tan B = 1 - \tan A\tan B$

$\tan A + \tan B + \tan A\tan B = 1$

Add 1 on both sides

$1 + \tan A + \tan B + \tan A\tan B = 2$

$(1+\tan A) + \tan B(1+\tan A) = 2$

$(1+\tan A)(1+\tan B) = 2$

Ex 3.5

9. show that $\cot 7\tfrac{1}{2}^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

$\cot 7\tfrac{1}{2}^\circ = \cot 15/2^\circ$

$= \dfrac{\cos 15/2^\circ}{\sin 15/2^\circ}$

$= \dfrac{2\cos 15/2 \cdot \cos 15/2}{2\cos 15/2 \cdot \sin 15/2}$

$= \dfrac{2\cos^2 15/2}{\sin 15^\circ}$ [$\sin 2A = 2\sin A\cos A$]

$= \dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

$= \dfrac{1+\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \dfrac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$

$= \dfrac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}$

$= \dfrac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2}$

$= \sqrt{6}+\sqrt{2}+\sqrt{3}+2$

$= \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Ex 3.4

10. P.T $(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta) = \tan 2^n\theta \cot\theta$.

$$1+\sec 2\theta = 1+\frac{1}{\cos 2\theta}$$
$$= 1+\frac{1+\tan^2\theta}{1-\tan^2\theta}$$
$$= \frac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta}$$
$$= \frac{2}{1-\tan^2\theta} = \frac{2\tan\theta\cot\theta}{1-\tan^2\theta}$$
$$= \tan 2\theta \cdot \cot\theta$$
$$1+\sec 4\theta = \tan 4\theta \cot 2\theta$$
$$\vdots \qquad \vdots$$
$$1+\sec 2^n\theta = \tan 2^n\theta \cot 2^{n-1}\theta.$$

$\therefore$

$$(1+\sec 2\theta)(1+\sec 4\theta)\ldots\ldots(1+\sec 2^n\theta)$$
$$= \tan 2\theta\cot\theta \tan 4\theta\cot 2\theta \ldots\ldots \tan 2^n\theta \cot 2^{n-1}\theta$$
$$= \tan 2^n\theta \cot\theta$$

$[\because \tan 2\theta \cot 2\theta = 1 \ldots\ldots \tan 2^{n-1}\theta \cot 2^{n-1}\theta = 1$

Ex 3.5 Prove that

11. $32\sqrt{3}\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6} = 3$

LHS

$$32\sqrt{3}\sin\frac{\pi}{48}\cdot\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$
$$= 16\sqrt{3}\left(2\sin\frac{\pi}{48}\cos\frac{\pi}{48}\right)\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$
$$= 16\sqrt{3}\sin\frac{\pi}{24}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$
$$= 8\sqrt{3}\left(2\sin\frac{\pi}{24}\cos\frac{\pi}{24}\right)\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$
$$= 8\sqrt{3}\sin\frac{\pi}{12}\cos\frac{\pi}{12}\cos\frac{\pi}{6}$$
$$= 4\sqrt{3}\left(2\sin\frac{\pi}{12}\cos\frac{\pi}{12}\right)\cos\frac{\pi}{6}$$
$$= 4\sqrt{3}\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$
$$= 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 3$$

Ex 3.6

4) S.T $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

$$\frac{\pi}{15} = 12^\circ$$

LHS

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$
$$= \cos 12^\circ\cos 24^\circ\cos 36^\circ\cos 48^\circ\cos 60^\circ\cos 72^\circ\cos 84^\circ$$
$$= \frac{1}{2}\cos 48^\circ\cos 12^\circ\cos 72^\circ\cos 36^\circ\cos 24^\circ\cos 84^\circ$$
$$= \frac{1}{2}\cos(60^\circ-12^\circ)\cos 12^\circ\cos(60^\circ+12^\circ)(\cos(60^\circ-24^\circ)\cos 24^\circ(\cos(60^\circ+24^\circ))$$
$$= \frac{1}{2}\cdot\frac{1}{4}\cos 3(12^\circ)\cdot\frac{1}{4}\cos 3\cdot(24^\circ)$$
$$= \frac{1}{32}\cos 36^\circ\cos 72^\circ$$
$$= \frac{1}{32}\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)$$
$$= \frac{1}{128}\cdot\frac{(5-1)}{4} = \frac{1}{128}$$

Ex 3.6

5. S.T $\frac{\sin 8x\cos x-\sin 6x\cos 3x}{\cos 2x\cos x-\sin 3x\sin 4x} = \tan 2x$

LHS

$$\frac{\sin 8x\cos x-\sin 6x\cos 3x}{\cos 2x\cos x-\sin 3x\sin 4x}$$
$$= \frac{\frac{1}{2}[\sin 9x+\sin 7x]-\frac{1}{2}[\sin 9x+\sin 3x]}{\frac{1}{2}[\cos 3x+\cos x]-\frac{1}{2}[\cos x-\cos 7x]}$$
$$= \frac{\sin 9x+\sin 7x-\sin 9x-\sin 3x}{\cos 3x+\cos x-\cos x+\cos 7x}$$
$$= \frac{\sin 7x-\sin 3x}{\cos 3x+\cos 7x}$$
$$= \frac{2\cos\frac{7x+3x}{2}\cdot\sin\frac{7x-3x}{2}}{2\cos\frac{3x+7x}{2}\cos\frac{7x-3x}{2}}$$
$$= \frac{\sin 2x}{\cos 2x} = \tan 2x$$

Ex 3.6

12. P.T $\dfrac{\sin x+\sin 3x+\sin 5x+\sin 7x}{\cos x+\cos 3x+\cos 5x+\cos 7x}=\tan 4x$

LHS

$\dfrac{\sin x+\sin 3x+\sin 5x+\sin 7x}{\cos x+\cos 3x+\cos 5x+\cos 7x}$

$=\dfrac{\sin 7x+\sin x+\sin 5x+\sin 3x}{\cos 7x+\cos x+\cos 5x+\cos 3x}$

$=\dfrac{2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right)+2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right)+2\cos\frac{5x+3x}{2}\cos\frac{5x-3x}{2}}$

$=\dfrac{2\sin 4x\cos 3x+2\sin 4x\cos x}{2\cos 4x\cos 3x+2\cos 4x\cos x}$

$=\dfrac{2\sin 4x(\cos 3x+\cos x)}{2\cos 4x(\cos 3x+\cos x)}$

$=\tan 4x$.

CHAPTER 4: COMBINATORICS AND MATHEMATICAL INDUCTION.

Ex 4.2

14. How many strings are there using the letters of the word INTERMEDIATE, if

(i) The vowels and consonants are alternative

(ii) All vowels are together.

(iii) Vowels are never together

(iv) No two vowels are together.

Vowels = 6 Consonants = 6

I E E I A E NTRMDT

(i) Case(i) starting with vowels

no. of strings $=\dfrac{6!}{2!3!}\cdot\dfrac{6!}{2!}$

Case(ii) starting with consonants

no. of strings $=\dfrac{6!}{2!}\cdot\dfrac{6!}{3!2!}$

Total no. of strings $=2\times\dfrac{6!}{2!3!}\cdot\dfrac{6!}{2!}$

$=43200$

(ii) All vowels are together.

Consider vowels as one unit

∴ No. of objects $=7$ (vowel + 6 consonants)

∴ No. of strings $=\dfrac{7!}{2!}\cdot\dfrac{6!}{3!2!}$

$=151200$

iii) Vowels are never together

No. of strings = Total no. of strings − Vowels are together

Total no. of strings $=\dfrac{12!}{3!.2!2!}$

$=19958400$

Req no. of strings

$=19958400-151200$

$=19807200$

(iv) No vowels are together

Case(i) Starting with vowel

no. of strings $=\dfrac{6!}{2!3!}\cdot\dfrac{6!}{2!}$

Case(ii) starting with consonants

no. of strings $=\dfrac{6!}{2!}\cdot\dfrac{6!}{2!3!}$

Case (iii) $V_1\,CC\,V_2\,C\,V_3\,C\,V_4\,C\,V_5\,C\,V_6$

2 ~~vowe~~ consonants can be together in 5 places

∴ No of strings $=5\cdot\dfrac{6!}{3!2!}\cdot\dfrac{6!}{2!}$

Total $=7\times\dfrac{6!}{3!2!}\cdot\dfrac{6!}{2!}=151200$

Ex 4.3

14. There are 5 teachers and 20 students. Out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees (i) a particular teacher is included? (ii) a particular student is excluded?

No. of committees $= 5C_2 \times 20C_3$

$= \frac{5.\cancel{4}^{2}}{\cancel{2}.1} \times \frac{20.19.\cancel{18}^{3}}{\cancel{3.2}.1}$

$= 11400$

(i) A particular teacher is included

No. of ways $= 4C_1 \times 20C_3$

$= 4 \times \frac{20.19.18}{3.2.1}$

$= 4560$

(ii) A particular student is excluded

No. of ways $= 5C_2 \times \cancel{2}19C_3$

$= \frac{5.\cancel{4}^{2}}{\cancel{2}.1} \times \frac{19.\cancel{18}^{3}.17}{\cancel{3.2}.1}$

$= 9690$

18. A committee of 7 people has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of (i) exactly 3 women? (ii) at least 3 women? (iii) at most 3 women?

(i) MEN – 8 WOMEN – 4

exactly 3 women

No. of ways $= 4C_3 \times 8C_4$

$= \cancel{4} \times \frac{8.7.\cancel{6}.5}{\cancel{4.3.2}.1}$

$= 280$

ii) at least 3 women

W	M	WAYS
3	4	$4C_3 \times 8C_4 = 4 \times \frac{8.7.6.5}{4.3.2} = 280$
4	3	$4C_4 \times 8C_3 = 1 \times \frac{8.7.6}{3.2.1} = 56$

Total $= 280 + 56 = 336$

(iii) at most 3 women

W	M	WAYS
3	4	$4C_3 \times 8C_4 = 280$
2	5	$4C_2 \times 8C_5 = 336$
1	6	$4C_1 \times 8C_6 = 112$
0	7	$4C_0 \times 8C_7 = 8$

Total no. of ways $= 736$

19. 7 relatives of a man comprises 4 ladies and 3 gentlemen his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

MAN		WIFE		No. of ways
M(3)	L(4)	M(4)	L(3)	
0	3	3	0	$3C_0 \times 4C_3 \times 4C_3 \times 3C_0 =$
1	2	2	1	$3C_1 \times 4C_2 \times 4C_2 \times 3C_1$
2	1	1	2	$3C_2 \times 4C_1 \times 4C_1 \times 3C_2$
3	0	0	3	$3C_3 \times 4C_0 \times 4C_0 \times 3C_3$

Total no. of ways

$= 1\times4\times4\times1 + 3\times6\times6\times3 + 3\times4\times4\times3 + 1\times1\times1\times1$

$= 16 + 324 + 144 + 1 = 485$

Ex 4.3

21. Find the number of strings of 4 letters that can be formed with the letters of the word. EXAMINATION ?

E X M T O AA NN II

Case i $\boxed{\checkmark|O|\triangle|\times}$ All distinct

No. of distinct letters = 8

No of ways $= 8C_4 \times 4!$

$= \frac{8.7.6.5}{4!} . 4!$

$= 56\times30 = 1680$

Case ii $\boxed{\checkmark|\checkmark|\times|\times}$ 2 sets of similar letters

No. of ways $= 3C_2 \times \frac{4!}{2!2!}$

$= \frac{3.\overset{2}{\cancel{4}}.3.\cancel{2!}}{\cancel{2}.\cancel{2!}} = 18$

Case iii $\boxed{\checkmark|\checkmark|O|\times}$ 2 similar 2 distinct

No. of ways $= 3C_1 \times 7C_2 \cdot \frac{4!}{2!}$

$= 3 \cdot \frac{7.\cancel{6}^3}{\cancel{2.1}} . \frac{4.3.\cancel{2!}}{\cancel{2!}}$

$= 756$

Total no. of ways $= 1680 + 18 + 756$
$= 2454$

MATHEMATICAL INDUCTION.

Ex 4.4

1. By principal of mathematical induction, prove that, for $n \geq 1$
$1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Step I : Let P(n) be the statement "$1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$"

Step 2 : Let $n=1$

$P(1)$: $1^3 = \left[\frac{1(1+1)}{2}\right]^3$

$1 = 1$

$\therefore P(1)$ is true.

Step 3: Assume P(k) is true.

$1^3+2^3+\cdots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$

Step 4: Consider P(k+1)

$1^3+2^3+3^3+\cdots+k^3+(k+1)^3$

$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$

$= (k+1)^2\left[\left(\frac{k}{2}\right)^2 + k+1\right]$

$= \frac{(k+1)^2(k^2+4k+4)}{4}$

$= \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2$

P(k+1) is true.

$\therefore$ By Mathematical induction P(n) is true for $n \geq 1$

7. Using the Mathematical induction, show that for any natural number n

$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Step 1 : Let P(n) be the statement "$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2}$"

Step 2: Let $n=1$

$P(1): \frac{1}{1(2)(3)} = \frac{1(4)}{4(2)(3)}$

$\frac{1}{6} = \frac{1}{6}$

$\therefore P(1)$ is true.

Step 3: Assume $P(k)$ is true

$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

Step 4: Consider $P(k+1)$

$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4} + \frac{1}{k+3}\right]$

$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)^2 + 4}{4(k+3)}\right]$

$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k^2+6k+9)+4}{4(k+3)}\right]$

$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$\therefore P(k+1)$ is true.

$\Rightarrow$ By mathematical Induction $P(n)$ is true for $n \in N$

Step 1: Let $P(n)$ be the statement "$x^{2n} - y^{2n}$ is divisible by $x+y$, $n \in N$"

Step 2: Let $n=1$

$P(1) = x^2 - y^2 = (x+y)(x-y)$

divisible by $x+y$.

$\therefore P(1)$ is true

Step 3: Assume $P(k)$ is true

$x^{2k} - y^{2k}$ is divisible by $x+y$

(ie) $x^{2k} - y^{2k} = m(x+y)$

$x^{2k} = y^{2k} + m(x+y) \quad \text{—— (1)}$

Step 4: Consider

$P(k+1) = x^{2(k+1)} - y^{2(k+1)}$

$= x^{2k}.x^2 - y^{2k}.y^2$

$= [y^{2k} + m(x+y)]x^2 - y^{2k}.y^2$ (from (1))

$= x^2y^{2k} + mx^2(x+y) - y^{2k}.y^2$

$= y^{2k}(x^2-y^2) + mx^2(x+y)$

$= y^{2k}(x+y)(x-y) + mx^2(x+y)$

$= (x+y)\left[y^{2k}(x-y) + mx^2\right]$

which is divisible by $x+y$

$\therefore P(k+1)$ is true

$\Rightarrow$ By mathematical induction $P(n)$ is true for all $n \in N$

10. Using Mathematical Induction, show that for any natural number n, $x^{2n} - y^{2n}$ is divisible by $x+y$

12. Use induction to prove that $n^3 - 7n + 3$ is divisible by 3 for all natural numbers n

Step 1: Let $P(n)$ be the statement "$n^3 - 7n + 3$ is divisible by 3, $n \in N$"

Step 2: Let $n = 1$

$P(1) = 1 - 7 + 3 = -3$ divisible by 3

$\therefore P(1)$ is true

Step 3: Let $P(k)$ be true

$k^3 - 7k + 3 = 3m$

$k^3 - 7k = 3m - 3 = 3(m-1) = 3\lambda$ —— (1)

Step 4: Consider $P(k+1)$

$P(k+1) = (k+1)^3 - 7(k+1) + 3$

$= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$

$= k^3 - 7k + 3k^2 + 3k - 3$

$= 3\lambda + 3k^2 + 3k - 3$

$= 3(\lambda + k^2 + k - 1)$ which is divisible by 3

$\therefore P(k+1)$ is true

Step 5: By Mathematical Induction $P(n)$ is true for $n \in N$.

13. Use induction to prove that $5^{n+1} + 4 \times 6^n$ when divided by 20 leaves a remainder 9

Step 1: Let $P(n)$ be the statement "$5^{n+1} + 4 \times 6^n$ when divided by 20 leaves a remainder 9"

Step 2: Let $n = 1$

$P(1) = 5^{1+1} + 4 \times 6^1$

$= 25 + 24 = 49$ when divided by 20 leaves a remainder 9

Step 3: Assume $P(k)$ is true

$5^{k+1} + 4 \times 6^k$ when divided by 20 leaves a remainder 9

(ie) $5^{k+1} + 4 \times 6^k = 20\lambda + 9$

Step 4: Consider $P(k+1)$

$P(k+1) = 5^{(k+1)+1} + 4 \times 6^{k+1}$

$= 5^{k+1} \cdot 5 + 4 \times 6^k \cdot 6$

$= (20\lambda + 9 - 4 \times 6^k)5 + 24(6^k)$

$= 100\lambda + 45 - 20 \times 6^k + 24 \times 6^k$

$= 100\lambda + 4 \times 6^k + 45$

$= 100\lambda + 4 \times 6^k + 36 + 9$

$= 100\lambda + 4(6^k + 9) + 9$

$= 100\lambda + 4 \times 15\mu + 9$

$= 20(5\lambda + 3\mu) + 9$ ($6^k + 9$ is a multiple of 15)

which is true

$\therefore$ By Mathematical induction $P(n)$ is true.

CHAPTER 5: BINOMIAL THEOREM SEQUENCES AND SERIES

EX 5.2

10. a, b, c are respectively p^{th}, q^{th} and r^{th} terms of a GP show that
$(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$

$t_p = a = AR^{p-1}$, $t_q = b = AR^{q-1}$

$t_r = c = AR^{r-1}$

$\log a = \log A + (p-1)\log R$

$\log b = \log A + (q-1)\log R$

$\log c = \log A + (r-1)\log R$

LHS

$(q-r)\log a + (r-p)\log b + (p-q)\log c$

$= (q-r)\log A + (q-r)(p-1)\log R +$
$(r-p)\log A + (r-p)(q-1)\log R +$
$(p-q)\log A + (p-q)(r-1)\log R$

$= \log A[q-r+r-p+p-q] +$
$\log R[pq - q - rp + r + qr - r - pq + p + pr - p - qr + q]$

$= (0)\log A + (0)\log R = 0$

EX 5.4

3. Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.

x is large $\Rightarrow \frac{1}{x}$ is small

$\therefore (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$ can be applied for $\frac{1}{x}$

LHS

$\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3} = (x^3+6)^{1/3} - (x^3+3)^{1/3}$

$= x\left(1+\frac{6}{x^3}\right)^{1/3} - x\left(1+\frac{3}{x^3}\right)^{1/3}$

$= x\left[1+\frac{1}{3}\frac{6}{x^3}+\dots\right] - x\left[1+\frac{1}{3}\cdot\frac{3}{x^3}+\dots\right]$

$= \left[x+\frac{2}{x^2}+\dots\right] - \left[x+\frac{1}{x^2}+\dots\right]$

$\simeq x + \frac{2}{x^2} - x - \frac{1}{x^2}$ dropping hr. powers

$= \frac{1}{x^2}$ (approximately)

4. Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1 - x + \frac{x^2}{2}$ when x is small

$\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-x}{1+x}\cdot\frac{1-x}{1-x}}$

$= \sqrt{\frac{(1-x)^2}{1-x^2}}$

$= \frac{1-x}{(1-x^2)^{1/2}}$

$= (1-x)(1-x^2)^{-1/2}$ $\quad (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2}x^2 + \dots$

$= (1-x)\left(1+\frac{1}{2}x^2+\dots\right)$

$= \left(1+\frac{x^2}{2} - x - \frac{x^3}{2} + \dots\right)$

$\simeq \left(1 - x + \frac{x^2}{2}\right)$ (app)

Eg 5.25 Prove that $\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

x is large $\Rightarrow \frac{1}{x}$ is small

$\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4} = (x^3+7)^{1/3} - (x^3+4)^{1/3}$

$= x\left(1+\frac{7}{x^3}\right)^{1/3} - x\left(1+\frac{4}{x^3}\right)^{1/3}$

$= x\left[1+\frac{1}{3}\cdot\frac{7}{x^3}+\dots\right] - x\left[1+\frac{1}{3}\cdot\frac{4}{x^3}+\dots\right]$

$= \left[x+\frac{7}{3x^2}+\dots\right] - \left[x+\frac{4}{3x^2}+\dots\right]$

$\simeq x + \frac{7}{3x^2} - x - \frac{4}{3x^2}$

$= \frac{1}{x^2}$ (approximately)

CHAPTER 6: TWO DIMENSIONAL ANALYTICAL GEOMETRY

Eg 6.19. Express the equation $\sqrt{3}x - y + 4 = 0$ in the following equivalent forms.
(i) slope and intercept form
ii) Intercept form (iii) Normal form

(i) $\sqrt{3}x - y + 4 = 0$ $\boxed{y = mx + b}$

$y = \sqrt{3}x + 4$ $m = \sqrt{3}$ $b = 4$

(ii) $\sqrt{3}x - y = -4$ $\frac{x}{a} + \frac{y}{b} = 1$

$\div -4$

$\frac{\sqrt{3}x}{-4} - \frac{y}{-4} = 1$

$\frac{x}{-4/\sqrt{3}} + \frac{y}{4} = 1$ $a = -4/\sqrt{3}$ $b = 4$

iii) $(-\sqrt{3})x + y = 4$

$A = -\sqrt{3}$ $B = 1$ $\sqrt{A^2+B^2} = \sqrt{3+1} = 2$

$\div$ by 2

$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 2$

Compare with $x\cos\alpha + y\sin\alpha = p$

$\cos\alpha = -\frac{\sqrt{3}}{2}$ $\sin\alpha = \frac{1}{2}$ $p = 2$

clearly α is in second quadrant

$\alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore$ the normal form is

$x\cos\frac{5\pi}{6} + y\sin\frac{5\pi}{6} = 2$

EX 6.2

5. The normal boiling point of water is $100^\circ C$ or $212^\circ F$ and the freezing point of water is $0^\circ C$ or $32^\circ F$ (i) Find the linear relationship between C and F. Find (ii) the value of C for $98.6^\circ F$ and (iii) the value of F for $38^\circ C$.

(i)
Let $(C_1, F_1) = (100^\circ, 212^\circ)$
$(C_2, F_2) = (0^\circ, 32^\circ)$

$\frac{C - C_1}{C_2 - C_1} = \frac{F - F_1}{F_2 - F_1}$

$\frac{C - 100^\circ}{0 - 100^\circ} = \frac{F - 212^\circ}{32^\circ - 212^\circ}$

$\frac{C - 100^\circ}{\cancel{+100}\ 5} = \frac{F - 212^\circ}{\cancel{+180}\ 9}$

$C - 100^\circ = \frac{5F}{9} - \frac{5 \times 212}{9}$

$C = \frac{5}{9}F - \frac{5}{9}(212) + 100$

$C = \frac{5}{9}(F - 32)$

(OR) $F = \frac{9}{5}C + 32$

(ii) If $F = 98.6$

$C = \frac{5}{9}(98.6 - 32^\circ)$

$= \frac{5}{9}(66.6^\circ) = 37^\circ$

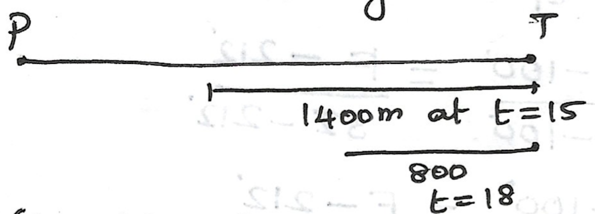
(iii) If $C = 38^\circ$ $F = ?$

$F = \frac{9}{8} \times 38 + 32$

$= 100.4^\circ$

Ex 6.2

6. An object was launched from a place P in constant speed to hit a target. At the 15^{th} second it was 1400m away from the target and at the 18^{th} second 800m away. Find (i) the distance between place and target (ii) the distance covered by it in 15 seconds (iii) time taken to hit the target



$(t_1, d_1) = (15, 1400)$

$(t_2, d_2) = (18, 800)$

Let t be time d be distance

$$\frac{d-d_1}{d_2-d_1} = \frac{t-t_1}{t_2-t_1}$$

$$\frac{d-1400}{800-1400} = \frac{t-15}{18-15}$$

$$\frac{d-1400}{-\cancel{600}\ 200} = \frac{t-15}{\cancel{3}}$$

$$d-1400 = -200t + 3000$$

$$200t + d = 4400 \quad \text{——} \ (1)$$

(i) When $t = 0$ in (1)

$d = 4400m$

This is the distance between the place and target

(ii) when $t = 15$ the distance of the target from object is 1400

$\therefore$ Distance covered by object in 15 seconds

$= 4400 - 1400 = 3000m$
$= 3000$ m

iii) Time taken to hit the target

when object reaches target

$d = 0$ subst in (1)

$200t = 4400$

$t = 22$ sec

Ex 6.3

17) Find the image of the point $(-2,3)$ about the line $x+2y-9=0$

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{-2(-2+6-9)}{1^2+2^2}$$

$$x+2 = \frac{y-3}{2} = \frac{10}{5} = 2$$

$x+2 = 2 \Rightarrow x = 0$

$\frac{y-3}{2} = 2 \Rightarrow y-3 = 4 \Rightarrow y = 7$

Image is $(0,7)$

11. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines

$x \sec\theta + y \operatorname{cosec}\theta = 2a$

$x \cos\theta - y \sin\theta = a\cos 2\theta$

prove that $p_1^2 + p_2^2 = a^2$

$$p_1 = \left| \frac{-2a}{\sqrt{\sec^2\theta + \csc^2\theta}} \right|$$

$$\therefore p_1^2 = \frac{4a^2}{\sec^2\theta + \csc^2\theta}$$

$$= \frac{4a^2}{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}$$

$$= \frac{4a^2\cos^2\theta\sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= 4a^2\cos^2\theta\sin^2\theta$$

$$p_2 = \left| \frac{-a\cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} \right|$$

$$p_2^2 = a^2\cos^2 2\theta.$$

$$= a^2(\cos^2\theta - \sin^2\theta)^2$$

$$p_1^2 + p_2^2 = 4a^2\cos^2\theta\sin^2\theta + a^2(\cos^2\theta - \sin^2\theta)^2$$

$$= a^2\left[4\cos^2\theta\sin^2\theta + (\cos^2\theta - \sin^2\theta)^2\right]$$

$$= a^2(\cos^2\theta + \sin^2\theta)^2$$

$$= a^2$$

hence proved.

Eg 6.38. If the equation $\lambda x^2 - 10xy + 5x - 16y - 3 = 0$ represents a pair of straight lines find (i) the value of λ and the seperate equations of the lines (ii) point of intersection of the lines (iii) angle between the lines.

(i) G.E $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Given $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Comparing co-efficients

$a = \lambda$ $b = 12$ $c = -3$

$h = -5$ $g = \frac{5}{2}$ $f = -8$

Condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(-8)^2 - 12\left(\frac{5}{2}\right)^2 + 3(-5)^2 = 0$$

$$-36\lambda + 200 - 64\lambda - 75 + 75 = 0$$

$$-100\lambda = -200$$

$$\Rightarrow \lambda = 2$$

$\therefore$ The equation is

$$2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$$

(ii) consider

$$2x^2 - 10xy + 12y^2 = (x - 2y)(2x - 6y)$$

The seperate equations are

$$x - 2y + l = 0$$
$$2x - 6y + m = 0$$

$(x - 2y + l)(2x - 6y + m) = 0$ is equivalent to the given equations

Comparing the coeff of x we get

$$2l + m = 5 \quad \text{--- (1)}$$

$$-6l - 2m = -16$$

$$3l + m = 8 \quad \text{--- (2)}$$

(1) $-$ (2)

$$-l = -3 \Rightarrow l = 3$$

substitute $l = 3$ in (1)

$$6 + m = 5 \Rightarrow m = -1$$

$\therefore$ The seperate equations are

$$x - 2y + 3 = 0 \quad \text{--- (3)}$$

$$2x - 6y - 1 = 0 \quad \text{--- (4)}$$

(ii) Pt of $\times$n $\quad x - 2y = -3 \quad \times 2$

$\quad 2x - 6y = 1 \quad \times 1$

$$\begin{array}{r} 2x - 4y = -6 \\ -\,2x \mp 6y = \underline{-}\,1 \\ \hline 2y = -7 \end{array}$$

$$y = -\frac{7}{2}$$

Subst $y=-\frac{7}{2}$ in ①

$$x-\cancel{2}\left(-\frac{7}{\cancel{2}}\right)+3=0$$
$$x+7+3=0$$
$$x=-10$$

$(-10,-7/2)$ is pt of xn.

iii) Angle between the lines

$$\theta=\tan^{-1}\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$
$$=\tan^{-1}\left|\frac{2\sqrt{25-24}}{2+12}\right|=\tan^{-1}\left|\frac{2}{14}\right|$$
$$\theta=\tan^{-1}\left(\frac{1}{7}\right)$$

Ex 6.4.

11. Find p and q, if the following equations represents a pair of perpendicular lines

$$6x^2+5xy-py^2+7x+qy-5=0$$

Comparing with general equation

$a=6$ $b=-p$ $c=-5$

$f=\frac{q}{2}$ $g=\frac{7}{2}$ $h=\frac{5}{2}$

perpendicular condition

$$a+b=0$$
$$6-p=0$$
$$\boxed{p=6}$$

Pair of st. lines condition

$$abc+2fgh-af^2-bg^2-ch^2=0$$
$$30p+\cancel{2}\cdot\frac{q}{\cancel{2}}\cdot\frac{7}{2}\cdot\frac{5}{2}-6\frac{q^2}{4}+6\times\frac{49}{4}+5\cdot\frac{25}{4}=0$$
$$180+\frac{35}{4}q-\frac{6q^2}{4}+\frac{294}{4}+\frac{125}{4}=0$$
$$720+35q-6q^2+294+125=0$$
$$-6q^2+35q+1139=0$$
$$6q^2-35q-1139=0$$
$$(6q+67)(q-17)=0$$
$$\Rightarrow q=-\frac{67}{6} \text{ or } 17$$

Ex 6.4

12. Find the value of k, if the following equation represents a pair of straight lines. Further, find whether these lines are parallel or intersecting

$$12x^2+7xy-12y^2-x+7y+k=0$$

Comparing with general equation

$a=12$ $b=-12$ $c=k$

$f=\frac{7}{2}$ $g=-\frac{1}{2}$ $h=\frac{7}{2}$

$$abc+2fgh-af^2-bg^2-ch^2=0$$
$$-144k+\cancel{2}\left(\frac{7}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{7}{2}\right)-12\left(\frac{49}{4}\right)+12\left(\frac{1}{4}\right)-k\left(\frac{49}{4}\right)=0$$
$$-k\left(144+\frac{49}{4}\right)-\frac{49}{4}-147+3=0$$
$$-k\left(144+\frac{49}{4}\right)=\left(\frac{49}{4}+144\right)$$
$$-k=1$$
$$\boxed{k=-1}$$

$$a+b=12-12=0$$

∴ the lines are perpendicular

∴ they are intersecting

13. For what value of k does the equation $12x^2+2kxy+2y^2+11x-5y+2=0$ represent two straight lines

Comparing with general equation

$a=12$ $b=2$ $c=2$ $f=-\frac{5}{2}$ $g=\frac{11}{2}$

$h=k$

$$abc+2fgh-af^2-bg^2-ch^2=0$$
$$48+\cancel{2}\left(-\frac{5}{2}\right)\left(\frac{11}{2}\right)k-\overset{3}{\cancel{12}}\left(\frac{25}{\cancel{4}}\right)-2\left(\frac{121}{4}\right)-2(k^2)=0$$
$$-2k^2-\frac{55k}{2}-75-\frac{121}{2}+96=0$$
$$-4k^2-55k-195=0$$
$$4k^2+55k+195=0$$
$$(k+5)(4k+35)=0$$

$\therefore k = -5$ or $k = -\frac{35}{4}$.

14. S.T the equation

$9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$

represents a pair of parallel lines. Find the distance between them

Comparing with general equation

$a = 9 \quad b = 16 \quad c = -12$

$h = -12 \quad g = -6 \quad f = 8$

If lines are $\parallel$le $bg^2 = af^2$

LHS $bg^2 = 16(36) = 576$

RHS $af^2 = 9(64) = 576$

$\therefore bg^2 = af^2$

The lines are parallel.

distance between them

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b}}$$

$$= 2\sqrt{\frac{36+108}{9(9+16)}}$$

$$= 2\sqrt{\frac{144}{9 \times 25}}$$

$$= \frac{2 \times \cancel{12}^{4}}{\cancel{3} \times 5} = \frac{8}{5}$$

15. S.T $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines. Find the distance between them

Comparing with the general equation

$a = 4 \quad b = 1 \quad c = -4$

$h = 2 \quad g = -3 \quad f = -\frac{3}{2}$

$bg^2 = (1)(-3)^2 = 9$

$af^2 = (4)(\frac{9}{4}) = 9$

$bg^2 = af^2 \therefore$ the lines are parallel

Distance between them

$$D = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

$$= 2\sqrt{\frac{9+16}{4(5)}}$$

$$= 2\sqrt{\frac{25}{4 \times 5}}$$

$$= 2\frac{\sqrt{5}}{2} = \sqrt{5}$$