4. Combinatorics and Mathematical Induction [4.2 Fundamental Principles of Counting, 4.3 Factorials]

EXERCISE 4.1

1) (i) A person went to a restaurant for dinner. In the menu card, the person saw 10 indian and 7 chinese food items. In how many ways the person can select either an Indian or a chinese food?

Soln: Number of ways of select an Indian or a chinese food $= 10+7 = 17$ ways.

(ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?

Soln: Number of ways of select one car and one train $= 3\times2 = 6$ ways

(iii) How many two digit numbers can be formed using 1,2,3,4,5 without repetition of digits?

Soln:

Tens	unit
5 ways	4 ways

Number of ways of filling two digit numbers $= 5\times4 = 20$ ways

(iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?

Soln:

Person1	Person2	Person3
10 ways	9 ways	8 ways

Number of ways $= 10\times9\times8 = 720$ ways

(v) In how many ways 5 persons can be seated in a row?

Soln:

Person1	Person2	Person3	Person4	Person5
5 ways	4 ways	3 ways	2 ways	1 way

Number of ways $= 5\times4\times3\times2\times1 = 120$ ways

2) (i) A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the pass code?

Soln: Number of digits $= 10$

10 ways	9 ways	8 ways	7 ways	6 ways	5 ways

Maximum number of attempts $= 10\times9\times8\times7\times6\times5 = 151200$ ways

(ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other?

Soln:

→ 4 ways
→ 3 ways
→ 2 ways

Number of ways $= 4\times3\times2 = 24$ ways

3) Four children are running a race
(i) In how many ways can the first two places be filled?
(ii) In how many different ways could they finish the race?

Soln:

(i)

Place1	Place2
4 ways	3 ways

Number of ways $= 4\times3 = 12$ ways

(ii)

Place1	Place2	Place3	Place4
4 ways	3 ways	2 ways	1 way

Number of ways $= 4\times3\times2\times1 = 24$ ways

4) Count the number of three-digit numbers which can be formed from the digits 2,4,6,8 if (i) repetitions of digits is allowed (ii) repetitions of digits is not allowed.

Soln:

(i)

Hundreds	Tens	Unit
4 ways	4 ways	4 ways

Number of ways $= 4\times4\times4 = 64$ ways

(ii)

Hundreds	Tens	Unit
4 ways	3 ways	2 ways

Number of ways $= 4\times3\times2 = 24$ ways

5) How many three-digit numbers are there with 3 in the unit place?
(i) with repetition (ii) without repetition.

Soln:

(i)

Hundreds	Tens	Unit
		3
9 ways	10 ways	1 way

Number of ways $= 9\times10\times1 = 90$ ways

(ii)

Hundreds	Tens	Unit
		3
8 ways	8 ways	1 way

Number of ways $= 8\times8\times1 = 64$ ways

6) How many numbers are there between 100 and 500 with the digits 0,1,2,3,4,5? if (i) repetitions of digits allowed (ii) the repetition of digits is not allowed.

Soln:

(i)

Hundreds	Tens	Unit
4 ways	6 ways	6 ways

100 is one way. This is Except.

Number of ways $= (4\times6\times6)-1 = 144-1 = 143$ ways

(ii)

Hundreds	Tens	Unit
4 ways	5 ways	4 ways

Number of ways $= 4\times5\times4 = 80$ ways

7) How many three-digit odd numbers can be formed by using the digits 0,1,2,3,4,5 if (i) the repetition of digits is not allowed (ii) the repetition of digits is allowed.

Soln:

(i)

Hundreds	Tens	Unit
		1 (or) 3 (or) 5
4 ways	4 ways	3 ways

Number of ways $= 4\times4\times3 = 48$ ways

(ii)

Hundreds	Tens	Unit
		1 (or) 3 (or) 5
5 ways	6 ways	3 ways

Number of ways $= 5\times6\times3 = 90$ ways

8) Count the numbers between 999 and 10000 subject to the condition that there are (i) no restriction (ii) no digit is repeated (iii) atleast one of the digits is repeated.

Soln:

(i)

Thousands	Hundreds	Tens	Unit
9 ways	10 ways	10 ways	10 ways

Number of numbers $= 9\times10\times10\times10$
$= 9000$ ways

(ii)

Thousands	Hundreds	Tens	Unit
9 ways	9 ways	8 ways	7 ways

Number of numbers $= 9\times9\times8\times7$
$= 4536$ ways

(iii) Required number of numbers $= 9000 - 4536 = 4464$ ways

9) How many three-digit numbers, which are divisible by 5, can be formed using the digits 0, 1, 2, 3, 4, 5 if
(i) repetition of digits are not allowed?
(ii) repetition of digits are allowed?

Soln:
(i)

Hundreds	Tens	Unit
		0
5 ways	4 ways	1 way

Number of numbers $= 5\times4\times1 = 20$ ways

Hundreds	Tens	Unit
		5
4 ways	4 ways	1 way

Number of numbers $= 4\times4\times1 = 16$ ways

The number of number divisible by 5 $= 20+16 = 36$ ways

(ii)

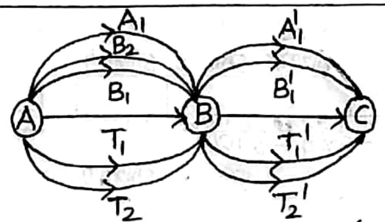
Hundreds	Tens	Unit
		0 (or) 5
5 ways	6 ways	2 ways

Number of numbers $= 5\times6\times2 = 60$ ways

The number of number divisible by $5 = 60$ ways.

10) To travel from a place A to place B, there are two different bus routes B_1, B_2, two different train routes T_1, T_2 and one air route A_1. From place B to place C there is one bus route say B_1', two different train routes say T_1', T_2' and one air route A_1'. Find the number of routes of commuting from place A to place C via place B without using similar mode of transportation.

Soln:



Number of routes $= (2\times3)+(2\times2)+(1\times3)$
$= 6+4+3 = 13$ routes

11) How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?

Soln: Total numbers $= 1000$

divisible by 2

		2 (or) 4 (or) 6 (or) 8 (or) 0
10 ways	10 ways	5 ways

number of ways $= 10\times10\times5 = 500$
Exclude "0" $= 1$ (−)
499
Include "1000" $= 1$ (+)
500
$\therefore n(A) = 500$

divisible by 5

		0 (or) 5
10 ways	10 ways	2 ways

number of ways $= 10\times10\times2 = 200$
Exclude "0" $= 1$ (−)
199
Include "1000" $= 1$ (+)
200
$\therefore n(B) = 200$

divisible by 10

		0
10 ways	10 ways	1 way

number of ways $= 10\times10\times1 = 100$
Exclude "0" $= 1$ (−)
99
Include "1000" $= 1$ (+)
100
$\therefore n(A\cap B) = 100$

$\therefore n(A\cup B) = n(A)+n(B)-n(A\cap B)$
$= 500+200-100 = 600$

number of numbers divisible neither by 2 nor by $5 = 1000-600 = 400$

12) How many strings can be formed using the letters of the word LOTUS if the word (i) either starts with L or ends with S? (ii) neither starts with L nor ends with S?

Soln: Given word "LOTUS"

5 ways	4 ways	3 ways	2 ways	1 way

Total $= 5\times4\times3\times2\times1 = 120$ ways

(i)

L				
1 way	4 ways	3 ways	2 ways	1 way

number of ways $= 1\times4\times3\times2\times1 = 24$ ways
$\therefore n(A) = 24$

				S
1 way	2 ways	3 ways	4 ways	1 way

number of ways $= 1\times2\times3\times4\times1 = 24$ ways
$\therefore n(B) = 24$

L				S
1 way	3 ways	2 ways	1 way	1 way

number of ways $= 1\times3\times2\times1\times1 = 6$ ways
$\therefore n(A\cap B) = 6$

$n(A\cup B) = n(A)+n(B)-n(A\cap B)$
$n(A\cup B) = 24+24-6 = 42$
$\therefore n(A\cup B) = 42$

(ii) neither starts with L nor ends with S $=$ Total $- n(A\cup B) = 120-42 = 78$

13) (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices
(ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes?
(iii) Find the number of ways of distributing 12 distinct prizes to 10 students?

Soln: (i) number of Questions $= 10$
number of choice $= 4$
Total number of ways $= 4\times4\times4\times4\times4\times4$
$= 4^6$

(ii) number of pigeons $= 10$
number of pigeon holes $= 3$
Total number of ways $= 3\times3\times3\times3\times3\times3\times3\times3\times3\times3 = 3^{10}$

(iii) number of prizes $= 12$
number of students $= 10$
Total number of ways $= 10\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10 = 10^{12}$

14) Find the value of (i) $6!$ (ii) $4!+5!$ (iii) $3!-2!$ (iv) $3!\times4!$ (v) $\frac{12!}{9!\times3!}$ (vi) $\frac{(n+3)!}{(n+1)!}$

Soln: (i) $6! = 6\times5\times4\times3\times2\times1 = 720$

(ii) $4!+5! = (4\times3\times2\times1)+(5\times4\times3\times2\times1)$
$= 24+120 = 144$

(iii) $3!-2! = (3\times2\times1)-(2\times1) = 6-2 = 4$

(iv) $3!\times4! = (3\times2\times1)\times(4\times3\times2\times1)$
$= 6\times24 = 144$

(v) $\frac{12!}{9!\times3!} = \frac{\cancel{12}^{2}\times11\times10\times\cancel{9!}}{\cancel{9!}\times\cancel{6}} = 2\times11\times10 = 220$

(vi) $\frac{(n+3)!}{(n+1)!} = \frac{(n+3)(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} = (n+3)(n+2)$

15) Evaluate $\frac{n!}{r!(n-r)!}$ when
(i) $n=6, r=2$ (ii) $n=10, r=3$
(iii) For any n with $r=2$

Soln:

(i) $\frac{n!}{r!(n-r)!} = \frac{6!}{2!(6-2)!} = \frac{\cancel{6}^{3}\times5\times\cancel{4!}}{\cancel{2}\times\cancel{4!}}$

$= 3\times5 = 15$

(ii) $\frac{n!}{r!(n-r)!} = \frac{10!}{3!(10-3)!} = \frac{10\times9\times8\times\cancel{7!}}{6\times\cancel{7!}}$

$= \frac{\cancel{720}\,120}{\cancel{6}} = 120$

(iii) $\frac{n!}{r!(n-r)!} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}}$

$= \frac{n(n-1)}{2}$

1b) Find the value of n if

(i) $(n+1)! = 20(n-1)!$

(ii) $\frac{1}{8!} + \frac{1}{9!} = \frac{n}{10!}$

Soln:

(i) $(n+1)! = 20(n-1)!$

$(n+1)n\cancel{(n-1)!} = 20\cancel{(n-1)!} \Rightarrow (n+1)n = 20$

$n^2+n-20=0 \Rightarrow (n-4)(n+5)=0$

$n-4=0,\ n+5=0$

$n=4,\ n=-5$ is not possible

$\therefore n=4$

(ii) $\frac{1}{8!} + \frac{1}{9!} = \frac{n}{10!}$

$\frac{1}{8!} + \frac{1}{9\times8!} = \frac{n}{10\times9\times8!}$

$\frac{1}{\cancel{8!}}\left[1+\frac{1}{9}\right] = \frac{n}{10\times9\times\cancel{8!}}$

$\frac{10}{9} = \frac{n}{90} \Rightarrow n = \frac{10\times\cancel{90}^{10}}{\cancel{9}} \Rightarrow n=100$

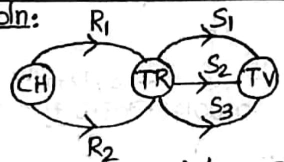
$\therefore n=100$

EX4.1 Suppose one girl or one boy has to be selected for a competition from a class comprising 17 boys and 29 girls. In how many different ways can this selection be made?

Soln: The number of selected ways $=17+29=46$ ways

EX4.2 Consider the 3 cities Chennai, Trichy and Tirunelveli. In order to reach Tirunelveli from Chennai, one has to pass through Trichy. There are 2 roads connecting Chennai with Trichy and there are 3 roads connecting Trichy with Tirunelveli. What are the total number of ways of travelling from Chennai to Tirunelveli?

Soln:



Chennai to Trichy = 2 roads

Trichy to Tirunelveli = 3 roads

$\therefore$ The number of ways travelling from Chennai to Tirunelveli $= 2\times3 = 6$ ways

EX4.3 A school library has 75 books on Mathematics, 35 books on Physics. A student can choose only one book. In how many ways a student can choose a book on Mathematics OR Physics?

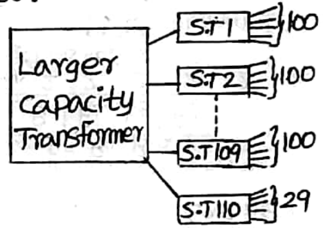
Soln: Choose Maths book = 75 ways

choose Physics book = 35 ways

number of ways a student can choose a book $= 75+35 = 110$ ways

EX4.4 If an electricity consumer has the consumer number say 238:110:29 then describe the linking and count the number of house connections up to 29th consumer connection linked to the larger capacity transformer number 238, subject to the condition that each smaller capacity transformer can have a maximal consumer link of say 100.

Soln:



The total number of consumers linked to the 238th larger capacity transformer $= (109\times100)+29$

$= 10900+29 = 10929$

EX4.5 A person wants to buy a car. There are two brands of car available in the market and each brand has 3 variant models and each model comes in five different colours as in Fig 4.2. In how many ways she can choose a car to buy?

Soln: car chosen = 2 ways

model chosen = 3 ways

colour chosen = 5 ways

The person can buy a car chosen $= 2\times3\times5 = 30$ different ways.

EX4.6 A women wants to select one silk saree and one sungudi saree from a textile shop located at Kancheepuram. In that shop, there are 20 different varieties of silk sarees and 8 different varieties of sungudi sarees. In how many ways she can select her sarees?

Soln:

select a silk sarees = 20 ways

select a sungudi sarees = 8 ways

She can select her sarees $= 20\times8 = 160$ ways.

EX4.7 In a village, out of the total number of people, 80 percentage of the people own coconut groves and 65 percent of the people own paddy fields. What is the minimum percentage of people own both?

Soln: Given: $n(C)=80,\ n(P)=65$ and $n(C\cup P)=100$

$n(C\cup P) = n(C)+n(P)-n(C\cap P)$

$100 = 80+65-n(C\cap P)$

$n(C\cap P) = 145-100 = 45 \Rightarrow n(C\cap P)=45$

$\therefore$ The minimum percentage of the people own both $=45$

EX4.8 (i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD without repetition of the letters.

(ii) How many strings of length 5 can be formed out of the letters of the word PRIME taking all the letters at a time without repetition.

Soln:

(i)

4 ways	3 ways	2 ways	1 way

The required number of strings $= 4\times3\times2\times1 = 24$ ways

(ii)

5 ways	4 ways	3 ways	2 ways	1 way

The required number of strings $= 5\times4\times3\times2\times1 = 120$ ways

Ex 4.9 How many strings of length 6 can be formed using letters of the word FLOWER if (i) either starts with F or ends with R? (ii) neither starts with F nor ends with R?

Soln:

(i)

F					
1 way	5 ways	4 ways	3 ways	2 ways	1 way

number of strings $= 1\times5\times4\times3\times2\times1$

$= 120$ ways

					R
1 way	2 ways	3 ways	4 ways	5 ways	1 way

number of strings $= 1\times2\times3\times4\times5\times1$

$= 120$ ways

F					R
1 way	4 ways	3 ways	2 ways	1 way	1 way

number of strings $= 1\times4\times3\times2\times1\times1$

$= 24$ ways

$\therefore$ The required number of strings

$= 120+120-24 = 216$

(ii) Total number of strings

$= 6\times5\times4\times3\times2\times1 = 720$ ways

neither starts with F nor ends with R $= 720 - 216 = 504$ ways.

Ex 4.10 How many licence plates may be made using either two distinct letters followed by four digits or two digits followed by 4 distinct letters where all digits and letters are distinct?

Soln: case (i) The number of licence plates $= 26\times25\times10\times9\times8\times7 = 32{,}76{,}000$

case (ii) The number of licence plates $= 10\times9\times26\times25\times24\times23 = 3{,}22{,}92{,}000$

Total number of licence plates

$= 32{,}76{,}000 + 3{,}22{,}92{,}000$

$= 3{,}55{,}68{,}000$

Ex 4.11 Count the number of positive integers greater than 7000 and less than 8000 which are divisible by 5, provided that no digits are repeated?

Soln:

7			0 (or) 5
1 way	8 ways	7 ways	2 ways

The required number of numbers

$= 1\times8\times7\times2 = 112$ ways

Ex 4.12 How many 4-digit even numbers can be formed using the digits 0, 1, 2, 3 and 4 if repetition of digits are not permitted?

Soln: case (i)

			0
4 ways	3 ways	2 ways	1 way

number of 4-digit even numbers

$= 4\times3\times2\times1 = 24$ ways

case (ii)

			2 (or) 4
3 ways	3 ways	2 ways	2 ways

number of 4-digit even numbers

$= 3\times3\times2\times2 = 36$ ways

The required number of 4-digit even numbers $= 24+36 = 60$ ways

Ex 4.13 Find the total number of outcomes when 5 coins are tossed once.

Soln:

C_1	C_2	C_3	C_4	C_5
2 ways	2 ways	2 ways	2 ways	2 ways

The total number of outcomes

$= 2\times2\times2\times2\times2 = 2^5 = 32$

Ex 4.14 In how many ways (i) 5 different balls be distributed among 3 boxes? (ii) 3 different balls be distributed among 5 boxes?

Soln:

(i)

Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
3 ways	3 ways	3 ways	3 ways	3 ways

The required number of ways

$= 3\times3\times3\times3\times3 = 3^5 = 243$ ways.

(ii)

Ball 1	Ball 2	Ball 3
5 ways	5 ways	5 ways

The required number of ways

$= 5\times5\times5 = 5^3 = 125$ ways

Ex 4.15 There are 10 bulbs in a room. Each one of them can be operated independently. Find the number of ways in which the room can be illuminated.

Soln: The 10 bulbs are $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}$

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways

number of blubs illuminated $= 2\times2\times2\times2\times2\times2\times2\times2\times2\times2 = 2^{10} = 1024$

number of bulbs not illuminated $= 1$

The total number of ways room can be illuminated $= 1024 - 1 = 1023$

Ex 4.16 Find the value of (i) $5!$ (ii) $6! - 5!$ (iii) $\frac{8!}{5!\,2!}$

Soln:

(i) $5! = 5\times4\times3\times2\times1 = 120$

(ii) $6! - 5! = (6\times5\times4\times3\times2\times1) - (5\times4\times3\times2\times1)$

$= 720 - 120 = 600$

(iii) $\frac{8!}{5!\,2!} = \frac{8\times7\times\cancel{6}^{3}\times\cancel{5!}}{\cancel{5!}\times\cancel{2}} = 8\times7\times3$

$= 168$

Ex 4.17 Simplify $\frac{7!}{2!}$

Soln: $\frac{7!}{2!} = \frac{7\times6\times5\times4\times3\times\cancel{2!}}{\cancel{2!}} = 2520$

Ex 4.18 Evaluate $\frac{n!}{r!(n-r)!}$ when (i) $n=7, r=5$ (ii) $n=50, r=47$ (iii) For any n with $r=3$

Soln: (i) $\frac{n!}{r!(n-r)!} = \frac{7!}{5!(7-5)!} = \frac{7\times6\times\cancel{5!}}{\cancel{5!}\times2!}$

$= \frac{7\times\cancel{6}^{3}}{\cancel{2}} = 7\times3 = 21$

(ii) $\frac{n!}{r!(n-r)!} = \frac{50!}{47!(50-47)!} = \frac{50\times49\times48\times\cancel{47!}}{\cancel{47!}\times3!}$

$= \frac{50\times49\times\cancel{48}^{8}}{\cancel{6}} = 50\times392 = 19600$

(iii) $\frac{n!}{r!(n-r)!} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{6\times\cancel{(n-3)!}}$

$= \frac{n(n-1)(n-2)}{6}$

Ex 4.19 Let N denote the number of days, If the value of $N!$ is equal to the total number of hours in N days then find the value of N?

Soln: Given: $N! = 24N \Rightarrow N! = 4!\,N$

$\cancel{N}(N-1)(N-2)(N-3)(N-4) = 4\times3\times2\times1\times\cancel{N}$

$(N-1)(N-2)(N-3)(N-4) = 4\times3\times2\times1$

Compare, $N-1 = 4 \Rightarrow N = 5$

EX 4.20 If $\frac{6!}{n!}=6$ then find the value of n

Soln: Given: $\frac{6!}{n!}=6 \Rightarrow \frac{\cancel{6}\times 5!}{n!}=\cancel{6}$

$\Rightarrow n! = 5! \Rightarrow n=5$

EX 4.21 If $n!+(n-1)!=30$ then find the value of n

Soln: Given: $n!+(n-1)!=30$

$n(n-1)!+(n-1)!=30$

$(n+1)(n-1)!=5\times 6 \Rightarrow (n+1)(n-1)!=5\times 3!$

Compare, $n+1=5 \Rightarrow n=4$

EX 4.22 What is the unit digit of the sum $2!+3!+4!+\cdots+22!$?

Soln: Given: $2!+3!+4!+\cdots+22!$

$=2+6+24+120+720+\cdots+22!$

$=32+120+720+\cdots+22!$

$\therefore$ The required unit digit is 2

EX 4.23 If $\frac{1}{7!}+\frac{1}{8!}=\frac{A}{9!}$ then find the value of A.

Soln: Given: $\frac{1}{7!}+\frac{1}{8!}=\frac{A}{9!}$

$\frac{1}{7!}+\frac{1}{8\times 7!}=\frac{A}{9\times 8\times 7!}$

$\frac{1}{\cancel{7!}}\left[1+\frac{1}{8}\right]=\frac{A}{9\times 8\times \cancel{7!}} \Rightarrow \frac{9}{\cancel{8}}=\frac{A}{9\times \cancel{8}}$

$\Rightarrow 9=\frac{A}{9} \Rightarrow A=81$

EX 4.24 Prove that $\frac{(2n)!}{n!}=2^n(1\cdot 3\cdot 5\cdots\cdots(2n-1))$

Soln: $\frac{(2n)!}{n!}=\frac{2n(2n-1)(2n-2)\ldots 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{n!}$

$=\frac{(1\cdot 3\cdot 5\ldots(2n-1))(2\cdot 4\cdot 6\cdots(2n-2)2n)}{n!}$

$=\frac{(1\cdot 3\cdot 5\ldots(2n-1))2^n(1\cdot 2\cdot 3\cdots(n-1)n)}{n!}$

$=\frac{(1\cdot 3\cdot 5\cdots(2n-1))2^n \cancel{n!}}{\cancel{n!}}$

$=2^n(1\cdot 3\cdot 5\cdots\cdots(2n-1))$

4.4 Permutations

EXERCISE 4.2

1) If $(n-1)P_3 : nP_4 = 1:10$ find n

Soln: Using $nP_r=\frac{n!}{(n-r)!}$

Given: $(n-1)P_3 : nP_4 = 1:10$

$\frac{(n-1)P_3}{nP_4}=\frac{1}{10} \Rightarrow \frac{\frac{(n-1)!}{\cancel{(n-4)!}}}{\frac{n!}{\cancel{(n-4)!}}}=\frac{1}{10}$

$\frac{(n-1)!}{n!}=\frac{1}{10} \Rightarrow \frac{\cancel{(n-1)!}}{n\cancel{(n-1)!}}=\frac{1}{10} \Rightarrow \frac{1}{n}=\frac{1}{10}$

$\Rightarrow n=10$

$\therefore n=10$

2) If $10P_{r-1}=2\times 6P_r$ find r

Soln: using $nP_r=\frac{n!}{(n-r)!}$

Given: $10P_{r-1}=2\times 6P_r$

$\frac{10!}{(10-(r-1))!}=2\times\frac{6!}{(6-r)!}$

$\frac{10!}{(11-r)!}=2\times\frac{6!}{(6-r)!}$

$\frac{5\,\cancel{10}\times 9\times 8\times 7\times \cancel{6!}}{(11-r)(10-r)(9-r)(8-r)(7-r)\cancel{(6-r)!}}=\cancel{2}\times\frac{\cancel{6!}}{\cancel{(6-r)!}}$

$5\times 3\times 3\times 2\times 4\times 7=(11-r)(10-r)(9-r)(8-r)(7-r)$

$7\times 6\times 5\times 4\times 3=(11-r)(10-r)(9-r)(8-r)(7-r)$

Compare, $7=11-r \Rightarrow r=11-7 \Rightarrow r=4$

$\therefore r=4$

3)(i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?

(ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

Soln:

(i) $8P_3=\frac{8!}{(8-3)!}=\frac{8\times 7\times 6\times \cancel{5!}}{\cancel{5!}}=336$

(ii) $4P_3\times 5P_3\times 6P_3$

$=\frac{4!}{(4-3)!}\times\frac{5!}{(5-3)!}\times\frac{6!}{(6-3)!}=\frac{4!}{1!}\times\frac{5!}{2!}\times\frac{6!}{3!}$

$=\frac{4\times 3\times 2\times 1}{1}\times\frac{5\times 4\times 3\times \cancel{2}\times 1}{\cancel{2}}\times\frac{\cancel{6}\times 5\times 4\times 3\times 2\times 1}{\cancel{6}}$

$=24\times 60\times 120=172800$

4) Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?

Soln: number of letters in the word SIMPLE $=6$

$\therefore 6P_6=6!=6\times 5\times 4\times 3\times 2\times 1=720$

5) A test consists of 10 multiple choice questions. In how many ways can the test be answered if (i) Each question has four choices? (ii) The first four questions have three choices and the remaining have five choices? (iii) Question number n has n+1 choices?

Soln:

(i)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
4 ways	4 ways	4 ways	4 ways	4 ways	4 ways	4 ways	4 ways	4 ways	4 ways

Total number of ways $=4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4=4^{10}$

(ii)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
3 ways	3 ways	3 ways	3 ways	5 ways	5 ways	5 ways	5 ways	5 ways	5 ways

Total number of ways $=3\times 3\times 3\times 3\times 5\times 5\times 5\times 5\times 5\times 5=3^4\times 5^6$

(iii)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
2 ways	3 ways	4 ways	5 ways	6 ways	7 ways	8 ways	9 ways	10 ways	11 ways

Total number of ways $=2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11=11!$

6) A student appears in an objective test which contain 5 multiple choice questions. Each question has four choices out of which one correct answer (i) what is the maximum number of different answers can the students give? (ii) How will the answer change if each question may have more than one correct answers?

Soln:

(i)

Q1	Q2	Q3	Q4	Q5
4 ways	4 ways	4 ways	4 ways	4 ways

The Maximum number of different answers $=4\times 4\times 4\times 4\times 4=4^5$

(ii) number of ways of answering each question $=4C_1+4C_2+4C_3+4C_4$

$=\frac{4}{1}+\frac{\cancel{2}4\times 3}{\cancel{2}\times 1}+\frac{4\times \cancel{3}\times \cancel{2}}{\cancel{3}\times \cancel{2}\times 1}+\frac{\cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}}$

$=4+6+4+1=15$

Q1	Q2	Q3	Q4	Q5
15 ways	15 ways	15 ways	15 ways	15 ways

Total number of ways $=15\times 15\times 15\times 15\times 15=15^5$

7) How many strings can be formed from the letters of the word ARTICLE so that vowels occupy the even places?

Soln: vowels are A, I, E

consonents are R, T, C, L

Total letters: 7

C	V	C	V	C	V	C

3 vowels are occupy 3 even places $= 3P_3 = 3! = 6$ ways

4 consonents are occupy remaining 4 places $= 4P_4 = 4! = 24$ ways

Total number of strings $= 6 \times 24 = 144$ ways

8) 8 women and 6 men are standing in a line (i) How many arrangements are possible if any individual can stand in any position?
(ii) In how many arrangements will all 6 men be standing next to one another?
(iii) In how many arrangements will no two men be standing next to one another?

Soln: Given: women $= 8$ and men $= 6$

Total number of persons $= 8+6 = 14$

(i) The number of arrangements $= 14P_{14} = 14!$

(ii) Consider 6 men as one unit

$\therefore 8+1 = 9!$ ways

But 6 men arranged in $6!$ ways

Total number of arrangements $= 9! \times 6!$

(iii) $- W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 - W_8 -$

Total number of arrangements $= 9P_6 \times 8P_8 = 9P_6 \times 8!$

9) Find the distinct permutations of the letters of the word MISSISSIPPI?

Soln: Given word: MISSISSIPPI

Number of I's $= 4$, Number of s's $= 4$

Number of p's $= 2$, Number of M's $= 1$

Number of letters in the word $= 11$

$\therefore$ The required number of arrangements $= \frac{11!}{4! \times 4! \times 2! \times 1!}$

$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 24 \times 2 \times 1}$

$= 11 \times 10 \times 9 \times 7 \times 5 = 34650$

10) How many ways can the product $a^2 b^3 c^4$ be expressed without exponents?

Soln: Given: $a^2 b^3 c^4 = aabbbcccc$

Number of a's $= 2$, Number of b's $= 3$

Number of c's $= 4$

Total number of letters $= 9$

$\therefore$ The required number of arrangements $= \frac{9!}{2! \times 3! \times 4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2 \times 6 \times 4!}$

$= 9 \times 4 \times 7 \times 5 = 1260$

11) In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together?

Soln:

4 subject books arranged in $4!$ ways

4 mathematics books arranged in $4!$ ways.

3 physics books arranged in $3!$ ways

2 chemistry books arranged in $2!$ ways

1 biology book arranged in $1!$ ways

Total number of ways of arranging the books $= 4! \times 4! \times 3! \times 2! \times 1!$

$= 24 \times 24 \times 6 \times 2 \times 1$

$= 6912$

12) In how many ways can the letters of the word SUCCESS be arranged so that all Ss are together?

Soln: Given word: SUCCESS

Number of S's (treated as 1 unit) $= 1$, Number of C's $= 2$, Number of U's $= 1$, Number of E's $= 1$

$\therefore$ The required number of arrangements $= \frac{5!}{2!} = \frac{\cancel{120}^{60}}{\cancel{2}} = 60$

13) A coin is tossed 8 times (i) How many different sequences of heads and tails are possible?
(ii) How many different sequences containing six heads and two tails are possible?

Soln: Sample space $= \{H, T\}$, $n(S) = 2$

(i)

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways

The number of possible $= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$

(ii) Number of heads $= 6$

Number of tails $= 2$

Total $= 8$

Total number of possible $= \frac{8!}{6! \times 2!}$

$= \frac{\cancel{8}^{4} \times 7 \times \cancel{6!}}{\cancel{6!} \times \cancel{2}}$

$= 4 \times 7$

$= 28$ ways

14) How many strings are there using the letters of the word INTERMEDIATE if (i) The vowels and consonents are alternative (ii) All the vowels are together (iii) Vowels are never together (iv) No two vowels are together

Soln: Given word: INTERMEDIATE

Total number of letters $= 12$

Vowels are A, E, E, E, I, I

Total number of vowels $= 6$

Consonents are N, T, R, M, D, T

Total number of consonents $= 6$

(i)

V	C	V	C	V	C	V	C	V	C	V	C

number of arrangements $= \frac{6!}{3! \times 2!} \times \frac{6!}{2!}$

C	V	C	V	C	V	C	V	C	V	C	V

number of arrangements $= \frac{6!}{2!} \times \frac{6!}{3! \times 2!}$

Total number of arrangements $= \frac{6!}{3! \times 2!} \times \frac{6!}{2!} + \frac{6!}{2!} \times \frac{6!}{3! \times 2!} = 2 \times \frac{6!}{3! \times 2!} \times \frac{6!}{2!}$

$= 2 \times \frac{\cancel{720}^{120}}{\cancel{6 \times 2}} \times \frac{\cancel{720}^{360}}{\cancel{2}} = 120 \times 360 = 43200$

(ii) Take all the vowels as 1 unit

$\therefore$ Vowels + Consonents $= 1+6 = 7$

The required number of arrangements $= \frac{7!}{2!} \times \frac{6!}{3! \times 2!} = \frac{\cancel{5040}^{2520}}{\cancel{2}} \times \frac{\cancel{720}^{60}}{\cancel{6 \times 2}}$

$= 2520 \times 60 = 151200$

(iii) The required number of arrangements $=$ Total arrangements $-$ Vowels are together.

$= \frac{12!}{3! \times 2! \times 2!} - \left(\frac{7!}{2!} \times \frac{6!}{3! \times 2!}\right)$

$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{6 \times 2 \times 2} - \left(\frac{\cancel{5040}^{2520}}{\cancel{2}} \times \frac{\cancel{720}^{60}}{\cancel{6 \times 2}}\right)$

$= 19958400 - 151200 = 19807200$

(iv)

V	N	V	T	V	R	V	M	V	D	V	T	V

The required number of arrangements $= \frac{7P_6}{3! \times 2!} \times \frac{6!}{2!} = \frac{7!}{3! \times 2} \times \frac{\cancel{720}^{360}}{\cancel{2}}$

$= \frac{7\times6\times5\times\cancel{4}^{2}\times\cancel{3!}}{\cancel{3!}\times\cancel{2}}\times360 = 420\times360$

$=151200$

15) Each of the digits 1,1,2,3,3 and 4 is written on a separate card. The six cards are then laid out in a row to form a 6-digit number.
(i) How many distinct 6-digit numbers are there?
(ii) How many of these 6-digit numbers are even?
(iii) How many of these 6-digit numbers are divisible by 4?

Soln: Given digits are 1,1,2,3,3 and 4

Total number of digits = 6

(i) number of distinct 6-digit numbers $= \frac{6!}{2!\times2!} = \frac{\cancel{720}^{180}}{\cancel{2\times2}} = 180$

(ii)

					2(or)4
1way	2ways	3ways	4ways	5ways	2ways

number of 6-digit even numbers $= \frac{2\times5\times4\times3\times2\times1}{2!\times2!} = \frac{\cancel{2}\times5\times4\times3\times\cancel{2}\times1}{\cancel{2\times2}} = 60$

(iii)

				1	2
1way	2ways	3ways	4ways	1way	1way

(or)

				2	4
1way	2ways	3ways	4ways	1way	1way

(or)

				3	2
1way	2ways	3ways	4ways	1way	1way

number of 6-digit numbers divisible by 4 $= \frac{1\times2\times3\times4\times1\times1}{2!} + \frac{1\times2\times3\times4\times1\times1}{2!\times2!} + \frac{1\times2\times3\times4\times1\times1}{2!}$

$= \frac{\cancel{24}^{12}}{\cancel{2}} + \frac{\cancel{24}^{6}}{\cancel{2\times2}} + \frac{\cancel{24}^{12}}{\cancel{2}} = 12+6+12 = 30$

16) If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the rank of the words (i) GARDEN (ii) DANGER

Soln: No Repeated Letters (ie) distinct letters

(i) Given word GARDEN

Alphabetical order ⇒ A D E G N R

G	A	R	D	E	N	
5!	4!	3!	2!	1!	0!	Add GARDEN ⇒ 1 way
3	0	3	0	0	0	

∴ Rank of GARDEN $= (5!\times3)+(3!\times3)+1$

$= (120\times3)+(6\times3)+1 = 360+18+1 = 379$

∴ Rank = 379

(ii) Given word DANGER

Alphabetical order ⇒ A D E G N R

D	A	N	G	E	R	
5!	4!	3!	2!	1!	0!	Add DANGER ⇒ 1 way
1	0	2	1	0	0	

∴ Rank of DANGER $= (5!\times1)+(3!\times2)+(2!\times1)+1$

$= (120\times1)+(6\times2)+(2\times1)+1 = 120+12+2+1 = 135$

∴ Rank = 135

17) Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the 85th string?

Soln: Given word THING

Number of strings $= 5! = 120$

Alphabetical order ⇒ G H I N T

G _ _ _ _ ⇒ 4! = 24
H _ _ _ _ ⇒ 4! = 24
I _ _ _ _ ⇒ 4! = 24
N G _ _ _ ⇒ 3! = 6
N H _ _ _ ⇒ 3! = 6
Also, NIGHT ⇒ 1
85

∴ 85^{th} String is NIGHT

18) If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order. Find the rank of the word FUNNY

Soln: Given word FUNNY. This word no distinct letters (ie) Repeated letters allowed

1	3	2	2	4	
F	U	N	N	Y	Add FUNNY ⇒ 1 way
$\frac{0}{2!}$	$\frac{2}{2!}$	$\frac{0}{2!}$	$\frac{0}{1!}$	$\frac{0}{1!}$	
4!	3!	2!	1!	0!	

∴ Rank of FUNNY $= \left(\frac{2}{2!}\times3!\right)+1$

$= \left(\frac{\cancel{2}}{\cancel{2}}\times6\right)+1 = 6+1 = 7$

∴ Rank = 7

19) Find the sum of all 4-digit numbers that can be formed using digits 1,2,3,4 and 5 repetitions not allowed?

Soln: [Using Thm 4·1]

Put $r=4$, $n=5$

Sum of all r-digit numbers $= (n-1)P_{(r-1)} \times$ Sum of the digits $\times 111\ldots1$ (r times)

$= (5-1)P_{(4-1)}\times(1+2+3+4+5)\times1111$

$= 4P_3\times15\times1111 = (4\times3\times2)\times15\times1111$

$= 24\times15\times1111 = 399960$

20) Find the sum of all 4-digit numbers that can be formed using digits 0,2,5,7,8 without repetition?

Soln: [Using Thm 4·2]

Put $r=4$, $n=5$

Sum of all r-digit numbers $= \{(n-1)P_{(r-1)}\times$ Sum of the digits $\times 111\ldots1$ (r times)$\} - \{(n-2)P_{(r-2)}\times$ Sum of the digits $\times 111\ldots1$ ($(r-1)$ times)$\}$

$= \{(5-1)P_{(4-1)}\times(0+2+5+7+8)\times1111\} - \{(5-2)P_{(4-2)}\times(0+2+5+7+8)\times111\}$

$= \{4P_3\times22\times1111\} - \{3P_2\times22\times111\}$

$= \{(4\times3\times2)\times22\times1111\} - \{(3\times2)\times22\times111\}$

$= \{24\times22\times1111\} - \{6\times22\times111\}$

$= 586608 - 14652 = 571956$

Ex 4·25 Evaluate (i) $4P_4$ (ii) $5P_3$ (iii) $8P_4$ (iv) $6P_5$

Soln: (i) $4P_4 = 4\times3\times2\times1 = 24$

(ii) $5P_3 = 5\times4\times3 = 60$

(iii) $8P_4 = 8\times7\times6\times5 = 1680$

(iv) $6P_5 = 6\times5\times4\times3\times2 = 720$

Ex 4·26 If $(n+2)P_4 = 42\times nP_2$ find n?

Soln: $(n+2)P_4 = 42\times nP_2$

$(n+2)(n+1)\cancel{n(n-1)} = 42\times\cancel{n(n-1)}$

$(n+2)(n+1) = 42$

$n^2+3n+2-42=0$

$n^2+3n-40=0$

$(n-5)(n+8)=0$

$n=5$, $n=-8$ is not possible

∴ $n=5$

EX 4.27 If $^{10}P_r = {}^{7}P_{r+2}$ find r.

Soln: $\frac{10!}{(10-r)!} = \frac{7!}{(5-r)!}$

$$\frac{10\times9\times8\times\cancel{7!}}{(10-r)(9-r)(8-r)(7-r)(6-r)\cancel{(5-r)!}} = \frac{\cancel{7!}}{\cancel{(5-r)!}}$$

$(10-r)(9-r)(8-r)(7-r)(6-r) = 10\times9\times8$

$(10-r)(9-r)(8-r)(7-r)(6-r) = 2\times5\times3\times3\times4\times2$

$(10-r)(9-r)(8-r)(7-r)(6-r) = 6\times5\times4\times3\times2$

Compare, $10-r=6 \Rightarrow r=10-6 \Rightarrow r=4$

$\therefore r=4$

EX 4.28 How many "letter strings" together can be formed with the letters of the word "VOWELS" so that (i) the strings begin with E (ii) the strings begin with E and end with W.

Soln: Given word: VOWELS

Total letters $=6$

(i)

E					
1way	5ways	4ways	3ways	2ways	1way

Number of letter strings $=1\times5\times4\times3\times2\times1=120$

(ii)

E					W
1way	4ways	3ways	2ways	1way	1way

Number of letter strings $=1\times4\times3\times2\times1\times1=24$

EX 4.29 A number of four different digits is formed with the use of the digits 1,2,3,4 and 5 in all possible ways. Find the following
(i) How many such numbers can be formed
(ii) How many of these are even?
(iii) How many of these are exactly divisible by 4?

Soln: (i)

5ways	4ways	3ways	2ways

The required number of numbers $=5\times4\times3\times2=120$

(ii)

			2(or)4
4ways	3ways	2ways	2ways

The required number of numbers $=4\times3\times2\times2=48$

(iii)

		1	2
3ways	2ways	1way	1way

(or)

		2	4
3ways	2ways	1way	1way

(or)

		3	2
3ways	2ways	1way	1way

(or)

		5	2
3ways	2ways	1way	1way

The required number of numbers
$=(3\times2\times1\times1)+(3\times2\times1\times1)+(3\times2\times1\times1)+(3\times2\times1\times1)$
$=6+6+6+6=24$

EX 4.30 How many different strings can be formed together using the letters of the word "EQUATION" so that (i) the vowels always come together? (ii) the vowels never come together?

Soln: Given word: EQUATION

(i) Vowels are A, E, I, O, U
consider 5 vowels as one letter
consonents are Q, T, N
$\therefore$ Letters are arrangement $=4!$
Vowels are arrangement $=5!$
The required number of words $=4!\times5!$
$=24\times120=2880$

(ii) Total letters are arrangement $=8!=8\times7\times6\times5\times4\times3\times2\times1=40320$
The vowels never come together
= Total letters are arrangement − the vowels always come together
$=40320-2880=37440$

EX 4.31 There are 15 candidates for an examination, 7 candidates are appearing for mathematics examination while the remaining 8 are appearing for different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?

Soln: $\_O_1\_O_2\_O_3\_O_4\_O_5\_O_6\_O_7\_O_8\_$

The required number of ways
$=8!\times{}^9P_7=8!\times\frac{9!}{(9-7)!}=\frac{8!\times9!}{2!}$

EX 4.32 In how many ways 5 boys and 4 girls can be seated in a row so that no two girls are together?

Soln: $\_B_1\_B_2\_B_3\_B_4\_B_5\_$

The required number of ways
$=5!\times{}^6P_4=5!\times\frac{6!}{(6-4)!}=5!\times\frac{6!}{2!}=120\times\frac{\cancel{720}^{360}}{\cancel{2}}$
$=120\times360=43200$

EX 4.33 4 boys and 4 girls from a line with the boys and girls alternating. Find the number of ways making this line.

Soln: $B_1\_B_2\_B_3\_B_4\_$
(or) $G_1\_G_2\_G_3\_G_4\_$
Total number of arrangement
$=(4!\times4!)+(4!\times4!)=2\times4!\times4!=2\times24\times24=1152$

EX 4.34 A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members $F, M, S_1, S_2, S_3, D_1, D_2$. How many ways can the family sit in the van if (i) There are no restriction (ii) Either F or M drives the van? (iii) D_1, D_2 sits next to a window and F is driving?

Soln: Total seats $=8$

(i) Total number of ways $={}^7P_1\times{}^7P_6$
$=7\times7\times6\times5\times4\times3\times2=7\times5040=35280$

(ii) Total number of ways $={}^2P_1\times{}^7P_6$
$=2\times7\times6\times5\times4\times3\times2=2\times5040=10080$

(iii) Total number of ways $={}^5P_2\times{}^1P_1\times{}^5P_4$
$=5\times4\times1\times5\times4\times3\times2=20\times120=2400$

EX 4.35 If the letters of the word TABLE are permuted in all possible ways and the words thus formed are arranged in the dictionary order (alphabetical order) find the rank of the words (i) TABLE (ii) BLEAT.

Soln: [distinct letters]

(i) Given word TABLE
Alphabetical order ⇒ A B E L T

T	A	B	L	E
4!	3!	2!	1!	0!
4	0	0	1	0

Add TABLE ⇒ 1 way

Rank of TABLE $=(4!\times4)+(1!\times1)+1$
$=(24\times4)+(1\times1)+1=96+1+1=98$
$\therefore$ Rank $=98$

(ii) Given word BLEAT
Alphabetical order ⇒ A B E L T

$$\begin{array}{ccccc|l} B & L & E & A & T & \\ 4! & 3! & 2! & 1! & 0! & \text{Add BLEAT} \Rightarrow 1 \text{ way} \\ 1 & 2 & 1 & 0 & 0 & \end{array}$$

Rank of BLEAT $= (4!\times1)+(3!\times2)+(2!\times1)+1 = (24\times1)+(6\times2)+(2\times1)+1 = 24+12+2+1 = 39$

$\therefore$ Rank $= 39$

EX 4.36 Find the number of ways of arranging the letters of the word BANANA.

Soln: Given word BANANA

Total letters $= 6$

The number of ways of arrangement $= \frac{6!}{3!\times2!} = \frac{6\times5\times\cancel{4}^{2}\times\cancel{3!}}{\cancel{3!}\times\cancel{2}} = 6\times5\times2 = 60$

EX 4.37 Find the number of ways arranging the letters of the word RAMANUJAN so that the relative positions of vowels and consonents are not changed.

Soln: Given word RAMANUJAN

The vowels are A, A, U, A

The consonents are R, M, N, J, N

The number of ways of arrangement $= \frac{4!}{3!}\times\frac{5!}{2!} = \frac{\cancel{24}^{4}}{\cancel{6}}\times\frac{120}{\cancel{2}} = 2\times120 = 240$

EX 4.38 Three twins pose a photograph standing in a line. How many arrangements are there (i) when there are no restrictions (ii) when each person is standing next to his or her twin?

Soln: (i) The number of arrangements $= 6P_6 = 6! = 6\times5\times4\times3\times2\times1 = 720$ ways

(ii) Each twin is consider as a single unit.

The number of arrangements $= 3!\times2!\times2!\times2! = 6\times2\times2\times2 = 48$ ways

EX 4.39 How many numbers can be formed using the digits 1, 2, 3, 4, 2, 1 such that, even digits occupies even place?

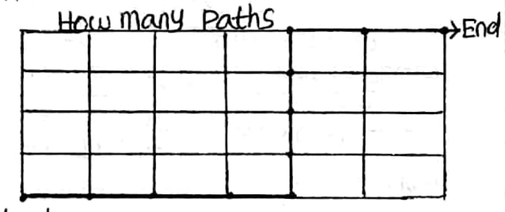
Soln: Given digits are 1, 2, 3, 4, 2, 1

Even digits are 2, 4, 2

odd digits are 1, 3, 1

The required number of numbers $= \frac{3!}{2!}\times\frac{3!}{2!} = \frac{3\cancel{6}}{\cancel{2}}\times\frac{3\cancel{6}}{\cancel{2}} = 3\times3 = 9$

EX 4.40 How many paths are there from start to end on a 6x4 grid as shown in the picture?



Soln: Horizontal length $= 6$ units

Vertical length $= 4$ units

Each path length $= 10$ units

The total number of path $= \frac{10!}{6!\times4!} = \frac{10\times\cancel{9}^{3}\times\cancel{8}\times7\times\cancel{6!}}{\cancel{6!}\times\cancel{24}\cancel{3}} = 10\times3\times7 = 210$

EX 4.41 If the different permutations of all letters of the word BHASKARA are listed as in a dictionary, how many strings are there in this list before the first word starting with B?

Soln: Given word BHASKARA

Starting letter with A $\Rightarrow$ A, A, A, B, H, K, R, S

A	A	A	B	H	K	R	S

1 way, 7! ways

The required number of strings $= \frac{7!}{2!} = \frac{7\times6\times5\times4\times3\times\cancel{2!}}{\cancel{2!}} = 2520$

EX 4.42 If the letters of the word IITJEE are permuted in all possible ways and the strings thus formed are arranged in the lexicographic order. Find the rank of the word IITJEE

Soln: [No distinct letters]

Given word IITJEE

$$\begin{array}{cccccc|l} 2 & 2 & 4 & 3 & 1 & 1 & \\ I & I & T & J & E & E & \text{Add IITJEE} \\ \frac{2}{2!\times2!} & \frac{2}{2!} & \frac{3}{2!} & \frac{2}{2!} & \frac{0}{2!} & \frac{0}{1!} & \Rightarrow 1 \text{ way} \\ 5! & 4! & 3! & 2! & 1! & 0! & \end{array}$$

Rank of IITJEE $= \left(5!\times\frac{2}{2!\times2!}\right)+\left(4!\times\frac{2}{2!}\right)+\left(3!\times\frac{3}{2!}\right)+\left(2!\times\frac{2}{2!}\right)+1$

$= \left(\cancel{120}^{60}\times\frac{\cancel{2}}{\cancel{2\times2}}\right)+\left(24\times\frac{\cancel{2}}{\cancel{2}}\right)+\left(\cancel{6}^{3}\times\frac{3}{\cancel{2}}\right)+\left(2\times\frac{\cancel{2}}{\cancel{2}}\right)+1$

$= 60+24+9+2+1 = 96$

$\therefore$ Rank $= 96$

EX 4.43 Find the sum of all 4-digit numbers that can be formed using the digits 1, 2, 4, 6, 8

Soln: [Using Thm 4.1], Put $r=4$, $n=5$

Sum of all r-digit numbers $= (n-1)P_{(r-1)}\times(\text{Sum of the digits})\times111\ldots1\,(r\text{ times})$

$= (5-1)P_{(4-1)}\times(1+2+4+6+8)\times1111$

$= 4P_3\times21\times1111 = (4\times3\times2)\times21\times1111$

$= 24\times21\times1111 = 559944$

4.5 Combinations

EXERCISE 4.3

1) If $nC_{12} = nC_9$ find $21C_n$

Soln: Using property, $nC_r = nC_{n-r}$ and $nC_n = 1$

Given: $nC_{12} = nC_9 \Rightarrow nC_{12} = nC_{n-9}$

Compare, $n-9 = 12 \Rightarrow n = 21$

$\therefore 21C_n = 21C_{21} = 1$

2) If $15C_{2r-1} = 15C_{2r+4}$ find r.

Soln: Using property, $nC_r = nC_{n-r}$

Given: $15C_{2r-1} = 15C_{2r+4}$

$15C_{2r-1} = 15C_{15-(2r+4)}$

Compare, $2r-1 = 15-(2r+4)$

$2r-1 = 15-2r-4 \Rightarrow 4r = 15+1-4 \Rightarrow \cancel{4}r = \cancel{12}^{3}$

$\therefore r = 3$

3) If $nP_r = 720$ and $nC_r = 120$ find n, r

Soln: Given: $nP_r = 720$ and $nC_r = 120$

$\frac{n!}{(n-r)!} = 720$ —① and $\frac{n!}{(n-r)!\,r!} = 120$ —②

$\frac{①}{②} \Rightarrow \frac{\frac{n!}{\cancel{(n-r)!}}}{\frac{n!}{\cancel{(n-r)!}\,r!}} = \frac{\cancel{720}^{6}}{\cancel{120}} \Rightarrow \cancel{n!}\times\frac{r!}{\cancel{n!}} = 6$

$\Rightarrow r! = 3! \Rightarrow r = 3$

Given: $nP_r = 720 \Rightarrow nP_3 = 720$

$n(n-1)(n-2) = 10\times9\times8$

Compare, $n = 10$

$\therefore n = 10$, $r = 3$

4) Prove that $15C_3+2\times15C_4+15C_5 = 17C_5$

Soln:

Using Property, $nC_r + nC_{r-1} = n+1C_r$

Given: $15C_3 + 2\times 15C_4 + 15C_5$

$= 15C_3 + 15C_4 + 15C_4 + 15C_5$

$= 15C_4 + 15C_3 + 15C_5 + 15C_4$

$= 16C_4 + 16C_5 = 16C_5 + 16C_4 = 17C_5$

5) Prove that $35C_5 + \sum_{r=0}^{4}(39-r)C_4 = 40C_5$

Soln: Using property, $nC_r + nC_{r-1} = n+1C_r$

$35C_5 + \sum_{r=0}^{4}(39-r)C_4 = 35C_5 + 39C_4 + 38C_4 + 37C_4 + 36C_4 + 35C_4$

$= 35C_5 + 35C_4 + 36C_4 + 37C_4 + 38C_4 + 39C_4$

$= 36C_5 + 36C_4 + 37C_4 + 38C_4 + 39C_4$

$= 37C_5 + 37C_4 + 38C_4 + 39C_4$

$= 38C_5 + 38C_4 + 39C_4 = 39C_5 + 39C_4 = 40C_5$

6) If $(n+1)C_8 : (n-3)P_4 = 57:16$ Find the value of n.

Soln: Given: $(n+1)C_8 : (n-3)P_4 = 57:16$

$\frac{(n+1)C_8}{(n-3)P_4} = \frac{57}{16} \Rightarrow (n+1)C_8 = \frac{57}{16}(n-3)P_4$

$\frac{(n+1)!}{\cancel{(n-7)!}\,8!} = \frac{57}{16}\frac{(n-3)!}{\cancel{(n-7)!}}$

$\frac{(n+1)n(n-1)(n-2)\cancel{(n-3)!}}{8!} = \frac{57}{16}\cancel{(n-3)!}$

$(n+1)n(n-1)(n-2) = \frac{57}{16}\times 8!$

$(n+1)n(n-1)(n-2) = \frac{57}{\cancel{16}}\times\cancel{8}\times7\times6\times5\times4\times3\times\cancel{2}\times1$

$(n+1)n(n-1)(n-2) = 19\times(3\times7)\times6\times(5\times4)\times3$

$(n+1)n(n-1)(n-2) = 21\times20\times19\times18$

Compare, $n=20$

7) Prove that $2nC_n = \frac{2^n\times1\times3\times\cdots(2n-1)}{n!}$

Soln: $2nC_n = \frac{2n!}{n!n!}$

$= \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\cdots4\cdot3\cdot2\cdot1}{n!n!}$

$= \frac{(2n(2n-2)(2n-4)\cdots4\cdot2)((2n-1)(2n-3)\cdots3\cdot1)}{n!n!}$

$= \frac{2^n(n(n-1)(n-2)\cdots2\cdot1)(1\cdot3\cdots(2n-1))}{n!n!}$

$= \frac{2^n\cancel{n!}(1\times3\times\cdots(2n-1))}{\cancel{n!}n!}$

$= \frac{2^n\times(1\times3\times\cdots(2n-1))}{n!}$

8) Prove that if $1\le r\le n$ then $n\times(n-1)C_{r-1} = (n-r+1)nC_{r-1}$

Soln: $(n-r+1)nC_{r-1} = (n-r+1)\frac{n!}{(n-r+1)!(r-1)!}$

$= \frac{\cancel{(n-r+1)}n(n-1)!}{\cancel{(n-r+1)}(n-r)!(r-1)!} = n\times(n-1)C_{r-1}$

9)(i) A kabadi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?

Soln:

Number of different teams $= 14C_7$

$= \frac{14!}{(14-7)!7!} = \frac{14!}{7!7!} = \frac{14\times13\times12\times11\times10\times9\times8\times7!}{7\times6\times5\times4\times3\times2\times1\times7!}$

$= 13\times2\times11\times2\times3\times2 = 3432$

(ii) There are 15 persons in a party and if each 2 of them shakes hands with each other. how many handshakes happen in the party?

Soln: Number of handshakes $= 15C_2$

$= \frac{15!}{(15-2)!2!} = \frac{15\times14\times\cancel{13!}}{\cancel{13!}\times2} = 15\times7 = 105$

(iii) How many chords can be drawn through 20 points on a circle.

Soln: Number of chords $= 20C_2$

$= \frac{20!}{(20-2)!2!} = \frac{20\times19\times\cancel{18!}}{\cancel{18!}\times2} = 10\times19 = 190$

(iv) In a parking lot one hundred, one year old cars are parked. out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars can be chosen?

Soln: Number of cars chosen $= 100C_5$

$= \frac{100!}{(100-5)!5!} = \frac{100\times99\times98\times97\times96\times\cancel{95!}}{\cancel{95!}\times120}$

$= 5\times99\times98\times97\times16 = 75287520$

(v) How many ways can a team of 3 boys 2 girls and 1 transgender be selected from 5 boys, 4 girls and 2 transgenders?

Soln: Total number of ways of selection

$= 5C_3\times4C_2\times2C_1 = \frac{5!}{(5-3)!3!}\times\frac{4!}{(4-2)!2!}\times\frac{2!}{(2-1)!1!}$

$= \frac{5\times4\times\cancel{3!}}{2!\times\cancel{3!}}\times\frac{4\times3\times\cancel{2!}}{2!\times\cancel{2!}}\times\frac{2!}{1!1!} = \frac{20}{2}\times\frac{12}{2}\times\frac{2}{1\times1}$

$= 10\times12\times1 = 120$

10) Find the total number of subsets of a set with (i) 4 elements (ii) 5 elements (iii) n elements [Hint: $nC_0+nC_1+nC_2+\cdots+nC_n=2^n$]

Soln: (i) $4C_0+4C_1+4C_2+4C_3+4C_4 = 2^4 = 16$

(ii) $5C_0+5C_1+5C_2+5C_3+5C_4+5C_5 = 2^5 = 32$

(iii) $nC_0+nC_1+nC_2+\cdots+nC_n = 2^n$

11) A trust has 25 members (i) How many ways 3 officers can be selected? (ii) In how many ways can a President, Vice President and a Secretary be selected?

Soln: (i) Number of ways of selection

$= 25C_3$

$= \frac{25!}{(25-3)!3!} = \frac{25\times24\times23\times\cancel{22!}}{\cancel{22!}\times6} = 25\times4\times23$

$= 2300$

(ii) Total number of ways of selection

$= 25C_1\times24C_1\times23C_1 = 25\times24\times23$

$= 25P_3 = 13800$

12) How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?

Soln: Total number of ways of chosen

$= 10C_1\times9C_1\times8C_4 = 10\times9\times\frac{8!}{(8-4)!4!}$

$= 90\times\frac{8\times7\times6\times5\times\cancel{4!}}{\cancel{4!}\times24} = 90\times7\times2\times5$

$= 6300$

13) How many different selections of 5 books can be made from 12 different books if, (i) Two particular books are always selected? (ii) Two particular books are never selected?

Soln:

(i) $(12-2)C_{(5-2)} = 10C_3 = \frac{10!}{(10-3)!3!}$

$= \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}\times6} = 10\times3\times4 = 120$

(ii) $(12-2)C_5 = 10C_5 = \frac{10!}{(10-5)!5!}$

$= \frac{10\times9\times8\times7\times6\times\cancel{5!}}{\cancel{5!}\times120} = 9\times4\times7 = 252$

14) There are 5 teachers and 20 students. out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees

(i) a particular teacher is included (ii) a particular student is excluded?

Soln: Number of ways of committees

$= 5C_2 \times 20C_3 = \frac{5!}{(5-2)!2!} \times \frac{20!}{(20-3)!3!}$

$= \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!} \times \cancel{2}} \times \frac{20 \times 19 \times \cancel{18}^{3} \times \cancel{17!}}{\cancel{17!} \times \cancel{6}} = 5 \times 2 \times 20 \times 19 \times 3$

$= 200 \times 57 = 11400$

(i) Number of ways of committees

$= 4C_1 \times 20C_3 = 4 \times \frac{20!}{(20-3)!3!}$

$= 4 \times \frac{20 \times 19 \times \cancel{18}^{3} \times \cancel{17!}}{\cancel{17!} \times \cancel{6}} = 4 \times 20 \times 19 \times 3 = 4560$

(ii) Number of ways of committees

$= 5C_2 \times 19C_3 = \frac{5!}{(5-2)!2!} \times \frac{19!}{(19-3)!3!}$

$= \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!} \times \cancel{2}} \times \frac{19 \times \cancel{18}^{3} \times 17 \times \cancel{16!}}{\cancel{16!} \times \cancel{6}}$

$= 5 \times 2 \times 19 \times 3 \times 17 = 9690$

15) In an examination a student has to answers 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?

Soln: Number of ways can answer the questions $= (9-2)C_{(5-2)} = 7C_3 = \frac{7!}{(7-3)!3!}$

$= \frac{7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{6}} = 7 \times 5 = 35$

16) Determine the number of 5 card combinations out of a deck of 52 cards, if there is exactly three aces in each combination

Soln:

Ace cards	Remaining cards	Total cards
4	48	52

The required number of ways of selection $= 4C_3 \times 48C_2$

$= \frac{4!}{(4-3)!3!} \times \frac{48!}{(48-2)!2!} = \frac{4 \times \cancel{3!}}{1 \times \cancel{3!}} \times \frac{\cancel{48}^{24} \times 47 \times \cancel{46!}}{\cancel{46!} \times \cancel{2}}$

$= \frac{4}{1} \times 24 \times 47 = 4 \times 1128 = 4512$

17) Find the number of ways of forming a committee of 5 members out of 7 Indians and 5 Americans so that always Indians will be the majority in the committee

Soln:

Indians (7)	5	4	3
Americans (5)	0	1	2

Number of ways of committees

$= 7C_5 \times 5C_0 + 7C_4 \times 5C_1 + 7C_3 \times 5C_2$

$= \frac{7!}{2!5!} \times 1 + \frac{7!}{3!4!} \times 5 + \frac{7!}{4!3!} \times \frac{5!}{3!2!}$

$= \frac{7 \times \cancel{6}^{3} \times \cancel{5!}}{\cancel{2} \times \cancel{5!}} \times 1 + \frac{7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{6} \times \cancel{4!}} \times 5 + \frac{7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{6}} \times \frac{5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!} \times \cancel{2}}$

$= (7 \times 3 \times 1) + (7 \times 5 \times 5) + (7 \times 5 \times 5 \times 2)$

$= 21 + 175 + 350 = 546$

18) A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of (i) exactly 3 women (ii) atleast 3 women (iii) atmost 3 women

Soln: (i) The required number of ways

$= 8C_4 \times 4C_3 = \frac{8!}{4!4!} \times \frac{4!}{1!3!}$

$= \frac{8 \times 7 \times \cancel{6}^{2} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{24}} \times \frac{4 \times \cancel{3!}}{1 \times \cancel{3!}} = 7 \times 2 \times 5 \times 4$

$= 280$

(ii)

Men (8)	4	3
Women (4)	3	4

The required number of ways

$= 8C_4 \times 4C_3 + 8C_3 \times 4C_4$

$= \frac{8!}{4!4!} \times \frac{4!}{1!3!} + \frac{8!}{5!3!} \times 1$

$= \frac{\cancel{8}^{2} \times 7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{24}_{4} \times \cancel{4!}} \times \frac{4 \times \cancel{3!}}{1 \times \cancel{3!}} + \frac{8 \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times \cancel{6}} \times 1$

$= (2 \times 7 \times 5 \times 4) + (8 \times 7 \times 1) = 280 + 56$

$= 336$

(iii)

Men (8)	7	6	5	4
Women (4)	0	1	2	3

The required number of ways

$= 8C_7 \times 4C_0 + 8C_6 \times 4C_1 + 8C_5 \times 4C_2 + 8C_4 \times 4C_3$

$= \frac{8!}{1!7!} \times 1 + \frac{8!}{2!6!} \times 4 + \frac{8!}{3!5!} \times \frac{4!}{2!2!} + \frac{8!}{4!4!} \times \frac{4!}{1!3!}$

$= \frac{8 \times \cancel{7!}}{1 \times \cancel{7!}} \times 1 + \frac{\cancel{8}^{4} \times 7 \times \cancel{6!}}{\cancel{2} \times \cancel{6!}} \times 4 + \frac{8 \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{6} \times \cancel{5!}} \times \frac{\cancel{24}^{6}}{\cancel{2 \times 2}}$

$+ \frac{\cancel{8}^{2} \times 7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{24}_{4} \times \cancel{4!}} \times \frac{4 \times \cancel{3!}}{1 \times \cancel{3!}}$

$= (8 \times 1) + (4 \times 7 \times 4) + (8 \times 7 \times 6) + (2 \times 7 \times 5 \times 4)$

$= 8 + 112 + 336 + 280 = 736$

19) 7 relatives of a man comprises 4 ladies and 3 gentle men, his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can then invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of men's relative and 3 of the wife's relatives?

Soln:

Husband		Wife	
Gentle men (3)	ladies (4)	Gentle men (4)	ladies (3)
3	0	0	3
2	1	1	2
1	2	2	1
0	3	3	0

The required number of ways

$= (3C_3 \times 4C_0 \times 4C_0 \times 3C_3) + (3C_2 \times 4C_1 \times 4C_1 \times 3C_2)$

$+ (3C_1 \times 4C_2 \times 4C_2 \times 3C_1) + (3C_0 \times 4C_3 \times 4C_3 \times 3C_0)$

$= (1 \times 1 \times 1 \times 1) + (3 \times 4 \times 4 \times 3) + (3 \times 6 \times 6 \times 3)$

$+ (1 \times 4 \times 4 \times 1)$

$= 1 + 144 + 324 + 16 = 485$

20) A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if atleast one black ball is to be included in the draw?

Soln:

White ball (2)	Black ball (3)	Red ball (4)
2	1	0
1	1	1
0	1	2
1	2	0
0	2	1
0	3	0

The required number of ways

$= (2C_2 \times 3C_1 \times 4C_0) + (2C_1 \times 3C_1 \times 4C_1)$

$+ (2C_0 \times 3C_1 \times 4C_2) + (2C_1 \times 3C_2 \times 4C_0)$

$+ (2C_0 \times 3C_2 \times 4C_1) + (2C_0 \times 3C_3 \times 4C_0)$

$=(1\times3\times1)+(2\times3\times4)+(1\times3\times6)$
$+(2\times3\times1)+(1\times3\times4)+(1\times1\times1)$
$=3+24+18+6+12+1=64$

21) Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION?

Soln: Given word EXAMINATION
Distinct letters are E, X, M, T, O
Repeated letters are AA, II, NN

(i) letters are distinct

[E, X, M, T, O, A, I, N]

The required number of strings
$=8C_4\times4!=\frac{8!}{4!\cancel{4!}}\times\cancel{4!}=\frac{8\times7\times6\times5\times\cancel{4!}}{\cancel{4!}}$
$=8\times7\times6\times5=1680$

(ii) 2 letters are same and other two letters are same

[AAII (or) AANN (or) IINN]

The required number of strings
$=3C_2\times\frac{4!}{2!2!}=3\times\frac{\cancel{24}^{6}}{\cancel{2\times2}}=3\times6=18$

(iii) 2 letters are same and other two letters are distinct

[AAEX (or) IIMT]

The required number of strings
$=3C_1\times7C_2\times\frac{4!}{2!}=3\times\frac{7!}{5!2!}\times\frac{\cancel{24}^{12}}{\cancel{2}}$
$=3\times\frac{7\times\cancel{6}^{3}\times\cancel{5!}}{\cancel{5!}\times\cancel{2}}\times12=3\times7\times3\times12$
$=756$

Total number of strings
$=1680+18+756=2454$

22) How many triangles can be formed by joining 15 points on the plane in which no line joining any three points.

Soln: The required number of triangles $=15C_3=\frac{15!}{12!3!}=\frac{\cancel{15}^{5}\times\cancel{14}^{7}\times13\times\cancel{12!}}{\cancel{12!}\times\cancel{6}}$
$=5\times7\times13=455$

23) How many triangles can be formed by 15 points in which 7 of them lie on one line and the remaining 8 on another parallel line?

Soln:

line 1 7 points	line 2 8 points
1	2
2	1

The required number of triangles
$=7C_1\times8C_2+7C_2\times8C_1$
$=7\times\frac{8!}{6!2!}+\frac{7!}{5!2!}\times8$
$=7\times\frac{\cancel{8}^{4}\times7\times\cancel{6!}}{\cancel{6!}\times\cancel{2}}+\frac{7\times\cancel{6}^{3}\times\cancel{5!}}{\cancel{5!}\times\cancel{2}}\times8$
$=(7\times4\times7)+(7\times3\times8)$
$=196+168=364$

24) There are 11 points in a plane. No three of these lies in the same straight line except 4 points which are collinear. Find (i) the number of straight lines that can be obtained from the pairs of these points?
(ii) the number of triangles that can be formed for which the points are their vertices?

Soln: (i) The required number of straight lines $=11C_2-4C_2+1$
$=\frac{11!}{9!2!}-\frac{4!}{2!2!}+1=\frac{11\times\cancel{10}^{5}\times\cancel{9!}}{\cancel{9!}\times\cancel{2}}-\frac{\cancel{24}^{6}}{\cancel{2\times2}}+1$
$=55-6+1=50$

(ii) The required number of triangles $=11C_3-4C_3=\frac{11!}{8!3!}-\frac{4!}{1!3!}$
$=\frac{11\times\cancel{10}^{5}\times\cancel{9}^{3}\times\cancel{8!}}{\cancel{8!}\times\cancel{6}}-\frac{4\times\cancel{3!}}{1\times\cancel{3!}}=(11\times5\times3)-4$
$=165-4=161$

25) A polygon has 90 diagonals. Find the number of its sides?

Soln: The number of diagonals of n sides polygon is $\frac{n(n-3)}{2}$

Given: $\frac{n(n-3)}{2}=90\Rightarrow n^2-3n=180$
$n^2-3n-180=0\Rightarrow(n-15)(n+12)=0$
$n-15=0,\ n+12=0$
$n=15,\ n=-12$ is not possible.
$\therefore$ The required number of sides $=15$

Ex 4.44 Evaluate the following
(i) $10C_3$ (ii) $15C_{13}$ (iii) $100C_{99}$ (iv) $50C_{50}$

Soln: (i) $10C_3=\frac{10!}{7!3!}=\frac{\cancel{10}^{5}\times\cancel{9}^{3}\times8\times\cancel{7!}}{\cancel{7!}\times\cancel{6}_{2}}$
$=5\times3\times8=120$

(ii) $15C_{13}=\frac{15!}{2!13!}=\frac{15\times\cancel{14}^{7}\times\cancel{13!}}{\cancel{2}\times\cancel{13!}}=15\times7=105$

(iii) $100C_{99}=\frac{100!}{1!99!}=\frac{100\times\cancel{99!}}{1\times\cancel{99!}}=100$

(iv) $50C_{50}=\frac{\cancel{50!}}{0!\cancel{50!}}=\frac{1}{1}=1$

Ex 4.45 Find the value of $5C_2$ and $7C_3$ using the property 5

Soln: Using property, $nC_r=\frac{n}{r}\times(n-1)C_{r-1}$
(i) $5C_2=\frac{5}{2}\times4C_1=\frac{5}{\cancel{2}}\times\cancel{4}^{2}=5\times2=10$
(ii) $7C_3=\frac{7}{3}\times6C_2=\frac{7}{\cancel{3}}\times\frac{\cancel{6}}{\cancel{2}}\times5C_1=7\times5$
$=35$

Ex 4.46 If $nC_4=495$, what is n?

Soln: Given: $nC_4=495$
$\frac{n!}{(n-4)!4!}=495\Rightarrow\frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}\times4\times3\times2\times1}=495$
$n(n-1)(n-2)(n-3)=495\times4\times3\times2\times1$
$n(n-1)(n-2)(n-3)=5\times3\times3\times11\times4\times3\times2\times1$
$n(n-1)(n-2)(n-3)=12\times11\times10\times9$

Compare, $n=12$
$\therefore n=12$

5	495
3	99
3	33
11	11
	1

Ex 4.47 If $nP_r=11880$ and $nC_r=495$. Find n and r

Soln: Given $nP_r=11880$ and $nC_r=495$
$\frac{n!}{(n-r)!}=11880$ — ① and $\frac{n!}{(n-r)!r!}=495$ — ②
substitute ① in ②
$\frac{11880}{r!}=495\Rightarrow r!=\frac{\cancel{11880}^{24}}{\cancel{495}}\Rightarrow r!=24$
$\Rightarrow r!=4!\Rightarrow r=4$

Given ${}^nC_r = 495 \Rightarrow {}^nC_4 = 495$

$\frac{n!}{(n-4)!4!} = 495 \Rightarrow \frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}4!} = 495$

$n(n-1)(n-2)(n-3) = 495 \times 4!$

$n(n-1)(n-2)(n-3) = 5\times3\times3\times11\times4\times3\times2\times1$

$n(n-1)(n-2)(n-3) = 12\times11\times10\times9$

Compare, $n=12$

$\therefore r=4$ and $n=12$

<u>EX 4.48</u> <u>Prove that</u> $24C_4 + \sum_{r=0}^{4}(28-r)C_3 = 29C_4$

<u>Soln</u>: Using <u>Property</u>, ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$

$24C_4 + \sum_{r=0}^{4}(28-r)C_3 = 24C_4 + 28C_3 + 27C_3 + 26C_3 + 25C_3 + 24C_3$

$= 24C_4 + 24C_3 + 25C_3 + 26C_3 + 27C_3 + 28C_3$

$= 25C_4 + 25C_3 + 26C_3 + 27C_3 + 28C_3$

$= 26C_4 + 26C_3 + 27C_3 + 28C_3$

$= 27C_4 + 27C_3 + 28C_3 = 28C_4 + 28C_3 = 29C_4$

<u>EX 4.49</u> <u>Prove that</u> $10C_2 + 2\times10C_3 + 10C_4 = 12C_4$

<u>Soln</u>: <u>Using Property</u>, ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$

$10C_2 + 2\times10C_3 + 10C_4 = 10C_2 + 10C_3 + 10C_3 + 10C_4$

$= 10C_3 + 10C_2 + 10C_4 + 10C_3 = 11C_3 + 11C_4 = 11C_4 + 11C_3$

$= 12C_4$

<u>EX 4.50</u> If ${}^{(n+2)}C_7 : {}^{(n-1)}P_4 = 13:24$ <u>Find n</u>.

<u>Soln</u>: Given: ${}^{(n+2)}C_7 : {}^{(n-1)}P_4 = 13:24$

$\frac{(n+2)C_7}{(n-1)P_4} = \frac{13}{24} \Rightarrow \frac{\frac{(n+2)!}{\cancel{(n-5)!}7!}}{\frac{(n-1)!}{\cancel{(n-5)!}}} = \frac{13}{24}$

$\frac{(n+2)!}{(n-1)!7!} = \frac{13}{24} \Rightarrow \frac{(n+2)(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} = 7!\cdot\frac{13}{24}$

$(n+2)(n+1)n = 7\times6\times5\times\cancel{4!}\times\frac{13}{\cancel{4!}}$

$(n+2)(n+1)n = 7\times2\times3\times5\times13$

$(n+2)(n+1)n = 14\times15\times13$

$(n+2)(n+1)n = 15\times14\times13$

Compare, $n=13$

$\therefore n=13$

<u>EX 4.51</u> A Salad at a certain restaurant consists of 4 of the following fruits: apple, banana, guava, pomegranate, grapes, papaya and pineapple. <u>Find the total possible number of fruit Salads</u>

<u>Soln</u>: The total possible number of fruit Salads $= 7C_4 = \frac{7!}{(7-4)!4!} = \frac{7\times6\times5\times\cancel{4!}}{3!\cancel{4!}}$

$= \frac{7\times\cancel{6}\times5}{\cancel{6}} = 35$

<u>EX 4.52</u> A mathematics club has 15 members, In that 8 are girls, 6 of the members are to be selected for a competition and half of them should be girls. <u>How many ways of these Selections are possible?</u>

<u>Soln</u>: Total members $= 15$

Girls $= 8$ (−)

Boys $= 7$

Number of ways of selection $= 8C_3 \times 7C_3$

$= \frac{8!}{5!3!} \times \frac{7!}{4!3!} = \frac{8\times7\times\cancel{6}\times\cancel{5!}}{\cancel{5!}\times\cancel{6}} \times \frac{7\times\cancel{6}\times5\times\cancel{4!}}{\cancel{4!}\times\cancel{6}}$

$= 56\times35 = 1960$

<u>EX 4.53</u> In rating 20 brands of cars, a car magazine picks a first, second, third, fourth and fifth best brand and then 7 more as acceptable. <u>In how many ways can it be done?</u>

<u>Soln</u>: The number of ways can it be done $= 20P_5 \times 15C_7$ ways

<u>EX 4.54</u> From a class of 25 students, 10 students are to be chosen for an excursion party. There are 4 students who decide that either all of them will join or none of them will join. <u>In how many ways can the excursion party be chosen?</u>

<u>Soln</u>: The total number of ways

$= (25-4)C_{(10-4)} + (25-4)C_{10}$

$= 21C_6 + 21C_{10} = \frac{21!}{15!\times6!} + \frac{21!}{11!\times10!}$

<u>EX 4.55</u> A box of one dozen apple contains a rotten apple. If we are choosing 3 apples simultaneously <u>in how many ways, one can get only good apples.</u>

<u>Soln</u>: Total Apples $= 12$ (−)

Rotten Apple $= 1$

Remaining Apples $= 11$

The required number of ways one can get only good apples $= 11C_3$

$= \frac{11!}{8!3!} = \frac{11\times\overset{5}{\cancel{10}}\times\overset{3}{\cancel{9}}\times\cancel{8!}}{\cancel{8!}\times\cancel{6}_{2}} = 11\times5\times3 = 165$

<u>EX 4.56</u> An Exam paper contains 8 questions 4 in part A and 4 in part B. Examiners are required to answer 5 questions. <u>In how many ways can this be done if (i) There are no restrictions of choosing a number of questions in either parts</u>

<u>(ii) Atleast two questions from Part A must be answered?</u>

<u>Soln</u>:

(i)

Part A	Part B	Total
4	4	8

The total number of ways $= 8C_5$

$= \frac{8!}{3!5!} = \frac{8\times7\times\cancel{6}\times\cancel{5!}}{\cancel{6}\times\cancel{5!}} = 8\times7 = 56$

(ii)

Part A (4)	2	3	4
Part B (4)	3	2	1

The total number of ways

$= 4C_2\times4C_3 + 4C_3\times4C_2 + 4C_4\times4C_1$

$= (6\times4) + (4\times6) + (1\times4) = 24+24+4$

$= 52$

<u>EX 4.57</u> out of 7 consonents and 4 vowels, <u>how many strings of 3 consonents and 2 vowels can be formed?</u>

<u>Soln</u>: The required number of ways $= 7C_3 \times 4C_2 \times 5! = \frac{7!}{4!3!} \times \frac{4!}{2!2!} \times 5!$

$= \frac{7\times\cancel{6}\times5\times\cancel{4!}}{\cancel{4!}\times\cancel{6}} \times \frac{\cancel{24}6}{\cancel{2\times2}} \times 120 = 35\times6\times120$

$= 25200$

<u>EX 4.58</u> <u>Find the number of strings of 5 letters</u> that can be formed <u>with the letters of the word</u> PROPOSITION.

<u>Soln</u>: Given word PROPOSITION

(O,O,O) ⇒ 1 set of three letters

(PP, II) ⇒ 2 sets of two letters

R, S, T, N ⇒ 4 distinct letters

(i) Letters are distinct

[R,S,T,N,O,P,I]

The required number of strings

$= 7C_5 \times 5! = \frac{7!}{2!5!} \times 5! = \frac{7 \times \cancel{6}^{3} \times \cancel{5!}}{\cancel{2} \times \cancel{5!}} \times 120$

$= 7 \times 3 \times 120 = 2520$

(ii) Letters are 1 set of three letters and 2 sets of two letters

[(O,O,O) and (PP,II)]

The required number of strings

$= 1C_1 \times 2C_1 \times \frac{5!}{3!2!} = 1 \times 2 \times \frac{\cancel{120}^{20}}{\cancel{6} \times \cancel{2}}$

$= 1 \times 20 = 20$

(iii) Letters are 1 set of three letters and 2 distinct letters

[(O,O,O) and R,S,T,N,P,I]

The required number of strings

$= 1C_1 \times 6C_2 \times \frac{5!}{3!} = 1 \times \frac{6!}{4!2!} \times \frac{5!}{3!}$

$= 1 \times \frac{\cancel{6}^{3} \times 5 \times \cancel{4!}}{\cancel{4!} \times \cancel{2}} \times \frac{\cancel{120}^{20}}{\cancel{6}} = 1 \times 3 \times 5 \times 20$

$= 300$

(iv) Letters are 2 sets of two letters and 1 distinct letters

[(PP,II,OO) and R,S,T,N,O]

The required number of Strings

$= 3C_2 \times 5C_1 \times \frac{5!}{2!2!} = 3 \times 5 \times \frac{\cancel{120}^{30}}{\cancel{2} \times \cancel{2}}$

$= 3 \times 5 \times 30 = 450$

(v) Letters are 2 sets of one letter and 3 distinct letters

[(PP,II,OO) and R,S,T,N,I,O (or) R,S,T,N,O,P (or) R,S,T,N,P,I]

The required number of strings

$= 3C_1 \times 6C_3 \times \frac{5!}{2!} = 3 \times \frac{6!}{3!3!} \times \frac{5!}{2!}$

$= 3 \times \frac{\cancel{6} \times 5 \times 4 \times \cancel{3!}}{\cancel{6} \times \cancel{3!}} \times \frac{\cancel{120}^{60}}{\cancel{2}}$

$= 3 \times 5 \times 4 \times 60 = 3600$

The total number of strings

$= 2520 + 20 + 300 + 450 + 3600$

$= 6890$

EX 4.59 If a set of m parallel lines intersect another set of n parallel lines (not parallel to the lines in the first set) then find the number of parallelograms formed in this lattice structure

Soln: The number of parallelograms

$= mC_2 \times nC_2$

EX 4.60 How many diagonals are there in a polygon with n sides?

Soln: Number of diagonals of the polygon = Total number of lines − n sides $= nC_2 - n = \frac{n!}{(n-2)!2!} - n$

$= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!} \times 2} - n = \frac{n(n-1)}{2} - n$

$= \frac{n(n-1)-2n}{2} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$

4.6 Mathematical Induction

EXERCISE 4.4

1) By the principle of mathematical induction, Prove that, for $n \geq 1$,

$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Soln: $P(n) = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$P(1) = (1)^3 = \left(\frac{1(1+1)}{2}\right)^2 \Rightarrow 1 = \left(\frac{\cancel{2}}{\cancel{2}}\right)^2 \Rightarrow 1 = 1$

$\therefore$ P(1) is true.

Assume that P(k) is true.

$P(k) = 1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$ — ①

To prove: P(k+1) is also true.

$P(k+1) = 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$

Consider,

$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$

$= \frac{(k+1)^2}{4}(k^2 + 4(k+1)) = \frac{(k+1)^2}{4}(k^2 + 4k + 4)$

$= \frac{(k+1)^2}{4}(k+2)^2 = \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2$

$\therefore$ P(k+1) is also true

$\therefore$ By mathematical induction, P(n) is true for all $n \in N$

2) By the principle of mathematical induction, Prove that, for $n \geq 1$,

$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Soln: $P(n) = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

$P(1) = (1)^2 = \frac{1(1)(\cancel{3})}{\cancel{3}} \Rightarrow 1 = 1$, $\therefore$ P(1) is true

Assume that P(k) is true

$P(k) = 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

To prove: P(k+1) is also true

$P(k+1) = 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

Consider, $1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2$

$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 = \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$

$= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} = \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$

$= \frac{(2k+1)(2k^2 + 5k + 3)}{3} = \frac{(2k+1)(2k^2 + 2k + 3k + 3)}{3}$

$= \frac{(2k+1)(2k(k+1) + 3(k+1))}{3} = \frac{(2k+1)(k+1)(2k+3)}{3}$

$= \frac{(k+1)(2k+1)(2k+3)}{3}$

$\therefore$ P(k+1) is also true.

$\therefore$ By mathematical induction, P(n) is true for all $n \in N$

3) Prove that the sum of the first n non-zero even numbers is $n^2 + n$

Soln: $P(n) = 2 + 4 + 6 + \dots + 2n = n^2 + n$

$P(1) = 2(1) = (1)^2 + 1 \Rightarrow \mathbf{2 = 2}$, $\therefore$ P(1) is true

Assume that P(k) is true

$P(k) = 2 + 4 + 6 + \dots + 2k = k^2 + k$

To Prove: P(k+1) is also true

$P(k+1) = 2 + 4 + 6 + \dots + 2(k+1) = (k+1)^2 + (k+1)$

$P(k+1) = 2 + 4 + 6 + \dots + 2(k+1) = (k+1)(k+1+1)$

$P(k+1) = 2 + 4 + 6 + \dots + 2(k+1) = (k+1)(k+2)$

Consider,

$2 + 4 + 6 + \dots + 2k + 2(k+1) = k^2 + k + 2(k+1)$

$= k(k+1) + 2(k+1) = (k+1)(k+2)$

$\therefore$ P(k+1) is also true

$\therefore$ By mathematical induction, P(n) is true for all $n \in N$

4) By the principle of mathematical induction, Prove that, for $n \geq 1$,

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$

Soln: $P(n) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$

$P(1) = 1\cdot2 = \frac{1(2)(\cancel{3})}{\cancel{3}} \Rightarrow 2 = 2$

$\therefore P(1)$ is true

Assume that $P(k)$ is true

$P(k) = 1\cdot2 + 2\cdot3 + 3\cdot4 + \dots + k\cdot(k+1) = \frac{k(k+1)(k+2)}{3}$

To Prove: $P(k+1)$ is also true

$P(k+1) = 1\cdot2 + 2\cdot3 + 3\cdot4 + \dots + (k+1)\cdot(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

Consider,

$1\cdot2 + 2\cdot3 + 3\cdot4 + \dots + k\cdot(k+1) + (k+1)\cdot(k+2)$

$= \frac{k(k+1)(k+2)}{3} + (k+1)\cdot(k+2)$

$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$

$\therefore P(k+1)$ is also true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \in N$

5) Using the mathematical induction, Show that for any natural number $n \geq 2$, $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$

Soln:

$P(n) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$

$P(2) = \left(1-\frac{1}{2^2}\right) = \frac{3}{4} \Rightarrow \left(1-\frac{1}{4}\right) = \frac{3}{4} \Rightarrow \frac{3}{4} = \frac{3}{4}$

$\therefore P(2)$ is true.

Assume that $P(k)$ is true.

$P(k) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$

To Prove: $P(k+1)$ is also true.

$P(k+1) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$

Consider,

$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) = \left(\frac{\cancel{k+1}}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}\right)$

$= \left(\frac{1}{2k}\right)\left(\frac{k^2+2k+\cancel{1}-\cancel{1}}{k+1}\right) = \left(\frac{1}{2\cancel{k}}\right)\left(\frac{\cancel{k}(k+2)}{k+1}\right)$

$= \frac{k+2}{2(k+1)} = \frac{k+2}{2k+2}$

$\therefore P(k+1)$ is also true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \geq 2$

6) Using the mathematical induction, Show that for any natural numbers $n \geq 2$, $\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+n} = \frac{n-1}{n+1}$

Soln: $P(n) = \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+n} = \frac{n-1}{n+1}$, $P(2) = \frac{1}{1+2} = \frac{1}{2+1} \Rightarrow \frac{1}{3} = \frac{1}{3}$

$\therefore P(2)$ is true.

Assume that $P(k)$ is true.

$P(k) = \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+k} = \frac{k-1}{k+1}$

To Prove: $P(k+1)$ is also true

$P(k+1) = \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+k+1} = \frac{k}{k+2}$

Consider,

$\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+k} + \frac{1}{1+2+3+\dots+k+1} = \frac{k-1}{k+1} + \frac{1}{\frac{(k+1)(k+2)}{2}}$

$= \frac{k-1}{k+1} + \frac{2}{(k+1)(k+2)} = \frac{(k-1)(k+2)+2}{(k+1)(k+2)}$

$= \frac{k^2+k-\cancel{2}+\cancel{2}}{(k+1)(k+2)} = \frac{k\cancel{(k+1)}}{\cancel{(k+1)}(k+2)} = \frac{k}{k+2}$

$\therefore P(k+1)$ is also true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \geq 2$

7) Using the mathematical induction, Show that for any natural numbers n, $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Soln: $P(n) = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{n\cdot(n+1)\cdot(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

$P(1) = \frac{1}{1\cdot2\cdot3} = \frac{1(\cancel{4})}{\cancel{4}(2)(3)} \Rightarrow \frac{1}{6} = \frac{1}{6}$

$\therefore P(1)$ is true

Assume that $P(k)$ is true.

$P(k) = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{k\cdot(k+1)\cdot(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

To Prove: $P(k+1)$ is also true.

$P(k+1) = \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{(k+1)\cdot(k+2)\cdot(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

Consider,

$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{k\cdot(k+1)\cdot(k+2)} + \frac{1}{(k+1)\cdot(k+2)\cdot(k+3)} = \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)} = \frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$

$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 9 & 4 \\ & 0 & -1 & -5 & -4 \\ \hline -1 & 1 & 5 & 4 & 0 \\ & 0 & -1 & -4 & \\ \hline & 1 & 4 & 0 & \end{array}$$

$= \frac{(k+1)^{\cancel{2}}(k+4)}{4\cancel{(k+1)}(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$\therefore P(k+1)$ is also true

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \in N$

8) Using the mathematical induction, Show that for any natural number n, $\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

Soln:

$P(n) = \frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

$P(1) = \frac{1}{(2)(5)} = \frac{1}{6+4} \Rightarrow \frac{1}{10} = \frac{1}{10}$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k) = \frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$

To Prove: $P(k+1)$ is also true.

$P(k+1) = \frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{(3k+2)(3k+5)} = \frac{k+1}{6k+10}$

Consider,

$\frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$

$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{k(3k+5)+2}{2(3k+2)(3k+5)}$

$= \frac{3k^2+5k+2}{2(3k+2)(3k+5)} = \frac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}$

$= \frac{3k(k+1)+2(k+1)}{2(3k+2)(3k+5)} = \frac{\cancel{(3k+2)}(k+1)}{2\cancel{(3k+2)}(3k+5)}$

$= \frac{k+1}{2(3k+5)} = \frac{k+1}{6k+10}$ $\therefore P(k+1)$ is also true

$\therefore$ By mathematical induction, $P(n)$ is true for all $n \in N$

9) Prove by mathematical induction, Show that $1!+(2\times2!)+(3\times3!)+\dots+(n\times n!)=(n+1)!-1$

Soln: $P(n)=1!+(2\times2!)+(3\times3!)+\dots+(n\times n!)=(n+1)!-1$

$P(1)=(1\times1!)=(1+1)!-1 \Rightarrow 1!=2!-1$

$\Rightarrow 1=2-1 \Rightarrow 1=1$, $\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k)=1!+(2\times2!)+(3\times3!)+\dots+(k\times k!)$
$=(k+1)!-1$

To Prove : $P(k+1)$ is also true.

$P(k+1)=1!+(2\times2!)+(3\times3!)+\dots+((k+1)\times(k+1)!)=(k+2)!-1$

Consider,

$1!+(2\times2!)+(3\times3!)+\dots+(k\times k!)+((k+1)\times(k+1)!)=(k+1)!-1+((k+1)\times(k+1)!)$

$=(k+1)!+((k+1)\times(k+1)!)-1$

$=(k+1)!(1+k+1)-1$

$=(k+1)!(k+2)-1$

$=(k+2)(k+1)!-1=(k+2)!-1$

$\therefore P(k+1)$ is also true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n\in N$.

10) Using the mathematical induction, Show that for any natural number n, $x^{2n}-y^{2n}$ is divisible by $x+y$.

Soln: $P(n)=x^{2n}-y^{2n}$ is divisible by $x+y$. $P(1)=x^2-y^2=(x-y)(x+y)$ is divisible by $x+y$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k)=x^{2k}-y^{2k}$ is divisible by $x+y$

Since $x^{2k}-y^{2k}=m(x+y)$

$x^{2k}=y^{2k}+m(x+y)$ — (1)

To Prove : $P(k+1)$ is also true.

$P(k+1)=x^{2(k+1)}-y^{2(k+1)}$ is divisible by $x+y$.

Consider, $x^{2(k+1)}-y^{2(k+1)}=x^{2k+2}-y^{2k+2}$

$=x^{2k}\cdot x^2-y^{2k}\cdot y^2$

$=[y^{2k}+m(x+y)]\cdot x^2-y^{2k}\cdot y^2$

$=y^{2k}\cdot x^2+m(x+y)\cdot x^2-y^{2k}\cdot y^2$

$=y^{2k}\cdot x^2-y^{2k}\cdot y^2+m(x+y)\cdot x^2$

$=y^{2k}\cdot(x^2-y^2)+m(x+y)\cdot x^2$

$=y^{2k}\cdot(x-y)(x+y)+m(x+y)\cdot x^2$

$=(x+y)(y^{2k}(x-y)+mx^2)$ is divisible by $x+y$.

$\therefore P(k+1)$ is also true.

$\therefore$ By mathematical induction, $P(n)$ is true for all $n\in N$

11) By the Principle of mathematical induction, Prove that, for $n\geq1$, $1^2+2^2+3^2+\dots+n^2>\frac{n^3}{3}$

Soln: $P(n)=1^2+2^2+3^2+\dots+n^2>\frac{n^3}{3}$

$P(1)=(1)^2>\frac{(1)^3}{3}\Rightarrow 1>\frac{1}{3}$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k)=1^2+2^2+3^2+\dots+k^2>\frac{k^3}{3}$

To Prove : $P(k+1)$ is also true.

$P(k+1)=1^2+2^2+3^2+\dots+(k+1)^2>\frac{(k+1)^3}{3}$

consider,

$1^2+2^2+3^2+\dots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

$=\frac{(k+1)(2k^2+7k+6)}{6}=\frac{2k^3+7k^2+6k+2k^2+7k+6}{6}$

$=\frac{2k^3+9k^2+13k+6}{6}=\frac{2(k^3+\frac{9}{2}k^2+\frac{13}{2}k+3)}{6\,3}$

$=\frac{k^3+3k^2+3k+1+\frac{9}{2}k^2+\frac{13}{2}k+3-3k^2-3k-1}{3}$

$=\frac{(k+1)^3+\frac{9}{2}k^2-3k^2+\frac{13}{2}k-3k+3-1}{3}$

$=\frac{(k+1)^3+\frac{3}{2}k^2+\frac{7}{2}k+2}{3}$

$=\frac{(k+1)^3}{3}+\frac{1}{3}(\frac{3}{2}k^2+\frac{7}{2}k+2)>\frac{(k+1)^3}{3}$

$\therefore 1^2+2^2+3^2+\dots+(k+1)^2>\frac{(k+1)^3}{3}$

$\therefore P(k+1)$ is also true

By mathematical induction, $P(n)$ is true for all $n\geq1$

12) Use induction to Prove that n^3-7n+3 is divisible by 3 for all natural number n

Soln: $P(n)=n^3-7n+3$ is divisible by 3

$P(1)=1-7+3=-3$ is divisible by 3

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k)=k^3-7k+3$ is divisible by 3

Since $k^3-7k+3=3m$ — (1)

To prove : $P(k+1)$ is also true

$P(k+1)=(k+1)^3-7(k+1)+3$ is divisible by 3.

Consider, $(k+1)^3-7(k+1)+3=k^3+3k^2+3k+1-7k-7+3$

$=(k^3-7k+3)+(3k^2+3k-6)$

$=3m+3(k^2+k-2)$

$=3(m+(k^2+k-2))$ is divisible by 3

$\therefore P(k+1)$ is also true.

By mathematical induction, $P(n)$ is true for all $n\in N$

13) Use induction to prove that $5^{n+1}+4\times6^n$ when divided by 20 leaves a remainder 9, for all natural number n

Soln: $P(n)=5^{n+1}+4\times6^n$

$P(1)=5^2+4\times6^1=25+24=49=20(2)+9$ when divided by 20 leaves a remainder 9

$\therefore P(1)$ is true

Assume that $P(k)$ is true.

$P(k)=5^{k+1}+4\times6^k$ when divided by 20 leaves a remainder 9.

Since $5^{k+1}+4\times6^k=20m+9$

$5^{k+1}=20m+9-4\times6^k$ — (1)

To Prove: $P(k+1)$ is also true.

$P(k+1)=5^{(k+1)+1}+4\times6^{k+1}$ when divided by 20 leaves a remainder 9

Consider,

$5^{(k+1)+1}+4\times6^{k+1}=5^{k+1}\cdot5^1+4\times6^k\cdot6$

$=(20m+9-4\times6^k)5+24\times6^k$

$=100m+45-20\times6^k+24\times6^k$

$=100m+45+4\times6^k$

$=100m+9+36+4\times6^k$

$=100m+9+4(9+6^k)$

$=100m+9+4(15P)$

$=100m+9+60P$

$=100m+60P+9$

$=20(5m+3P)+9$ when divided by 20 leaves a remainder 9

$\therefore P(k+1)$ is also true

By mathematical induction, $P(n)$ is true for all $n \in N$

14) Use induction to <u>prove that</u> $10^n + 3\times4^{n+2} + 5$ is divisible by 9, for all natural numbers n

<u>Soln</u>: $P(n) = 10^n + 3\times4^{n+2} + 5$ is divisible by 9

$P(1) = 10^1 + 3\times4^3 + 5 = 10 + 3\times64 + 5$

$P(1) = 10 + 192 + 5 = 207$

$P(1) = 207$ is divisible by 9

$\therefore P(1)$ is true

Assume that $P(k)$ is true.

$P(k) = 10^k + 3\times4^{k+2} + 5$ is divisible by 9

since $10^k + 3\times4^{k+2} + 5 = 9m$

$10^k = 9m - 3\times4^{k+2} - 5$ —①

To Prove: $P(k+1)$ is also true.

$P(k+1) = 10^{k+1} + 3\times4^{k+3} + 5$ is divisible by 9. consider, $10^{k+1} + 3\times4^{k+3} + 5$

$= 10^k.10 + 3\times4^k4^3 + 5$

$= (9m - 3\times4^{k+2} - 5)10 + 3\times4^k(64) + 5$

$= (9m - 3\times4^k4^2 - 5)10 + 3\times4^k(64) + 5$

$= 90m - 480\times4^k - 50 + 192\times4^k + 5$

$= 90m - 288\times4^k - 45$

$= 9(10m - 32\times4^k - 5)$ is divisible by 9

$\therefore P(k+1)$ is also true

By mathematical induction, $P(n)$ is true for all $n \in N$

15) <u>Prove that</u> using the mathematical induction $\sin(\alpha) + \sin(\alpha + \frac{\pi}{6}) + \sin(\alpha + \frac{2\pi}{6}) + \dots + \sin(\alpha + \frac{(n-1)\pi}{6}) = \frac{\sin(\alpha + \frac{(n-1)\pi}{12}) \times \sin(\frac{n\pi}{12})}{\sin(\frac{\pi}{12})}$

<u>Soln</u>: $P(n) = \sin(\alpha) + \sin(\alpha + \frac{\pi}{6}) + \sin(\alpha + \frac{2\pi}{6}) + \dots + \sin(\alpha + \frac{(n-1)\pi}{6}) = \frac{\sin(\alpha + \frac{(n-1)\pi}{12}) \times \sin(\frac{n\pi}{12})}{\sin(\frac{\pi}{12})}$

$P(1) = \sin\alpha = \frac{\sin\alpha \times \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})} \Rightarrow \sin\alpha = \sin\alpha$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true

$P(k) = \sin\alpha + \sin(\alpha + \frac{\pi}{6}) + \sin(\alpha + \frac{2\pi}{6}) + \dots + \sin(\alpha + \frac{(k-1)\pi}{6}) = \frac{\sin(\alpha + \frac{(k-1)\pi}{12}) \times \sin(\frac{k\pi}{12})}{\sin(\frac{\pi}{12})}$

To Prove: $P(k+1)$ is also true.

$P(k+1) = \sin\alpha + \sin(\alpha + \frac{\pi}{6}) + \sin(\alpha + \frac{2\pi}{6}) + \dots + \sin(\alpha + \frac{k\pi}{6}) = \frac{\sin(\alpha + \frac{k\pi}{12}) \times \sin(\frac{(k+1)\pi}{12})}{\sin(\frac{\pi}{12})}$

consider,

$\sin\alpha + \sin(\alpha + \frac{\pi}{6}) + \sin(\alpha + \frac{2\pi}{6}) + \dots + \sin(\alpha + \frac{(k-1)\pi}{6}) + \sin(\alpha + \frac{k\pi}{6})$

$= \frac{\sin(\alpha + \frac{(k-1)\pi}{12}) \times \sin(\frac{k\pi}{12})}{\sin(\frac{\pi}{12})} + \sin(\alpha + \frac{k\pi}{6})$

$= \frac{\sin(\alpha + \frac{(k-1)\pi}{12}) \times \sin(\frac{k\pi}{12}) + \sin(\alpha + \frac{k\pi}{6}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})}$

$= \frac{\sin(\alpha + \frac{k\pi}{12} - \frac{\pi}{12}) \sin(\frac{k\pi}{12}) + \sin(\alpha + \frac{2k\pi}{12}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})}$

$= \frac{\frac{1}{2}[\cos(\alpha - \frac{\pi}{12}) - \cos(\alpha + \frac{2k\pi}{12} - \frac{\pi}{12})] + \frac{1}{2}[\cos(\alpha + \frac{2k\pi}{12} - \frac{\pi}{12}) - \cos(\alpha + \frac{2k\pi}{12} + \frac{\pi}{12})]}{\sin(\frac{\pi}{12})}$

$= \frac{\frac{1}{2}[\cos(\alpha - \frac{\pi}{12}) - \cancel{\cos(\alpha + \frac{2k\pi}{12} - \frac{\pi}{12})} + \cancel{\cos(\alpha + \frac{2k\pi}{12} - \frac{\pi}{12})} - \cos(\alpha + \frac{2k\pi}{12} + \frac{\pi}{12})]}{\sin(\frac{\pi}{12})}$

$= \frac{\frac{1}{2}[\cos(\alpha - \frac{\pi}{12}) - \cos(\alpha + \frac{2k\pi}{12} + \frac{\pi}{12})]}{\sin(\frac{\pi}{12})}$

$= \frac{\sin(\alpha + \frac{k\pi}{12}) \sin(\frac{k\pi}{12} + \frac{\pi}{12})}{\sin(\frac{\pi}{12})}$

$= \frac{\sin(\alpha + \frac{k\pi}{12}) \times \sin(\frac{(k+1)\pi}{12})}{\sin(\frac{\pi}{12})}$

$\therefore P(k+1)$ is also true.

By mathematical induction, $P(n)$ is true for all $n \in N$

<u>Ex 4.61</u> By the principle of mathematical induction, <u>prove that</u>, for all integers $n \geq 1$, $1+2+3+\dots+n = \frac{n(n+1)}{2}$

<u>Soln</u>: $P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

$P(1) = 1 = \frac{1(2)}{2} \Rightarrow 1 = 1$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k) = 1+2+3+\dots+k = \frac{k(k+1)}{2}$

To prove: $P(k+1)$ is also true.

$P(k+1) = 1+2+3+\dots+k+1 = \frac{(k+1)(k+2)}{2}$

Consider, $1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$

$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

$\therefore P(k+1)$ is also true.

By mathematical induction, $P(n)$ is true for all $n \in N$

<u>Ex 4.62</u> <u>Prove that</u> the sum of first n positive odd number is n^2

<u>Soln</u>: $P(n) = 1+3+5+\dots+(2n-1) = n^2$

$P(1) = 1 = 1$, $\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k) = 1+3+5+\dots+(2k-1) = k^2$

To Prove: $P(k+1)$ is also true.

$P(k+1) = 1+3+5+\dots+(2k+1) = (k+1)^2$

Consider, $1+3+5+\dots+(2k-1)+(2k+1)$

$= k^2+2k+1 = (k+1)^2$

$\therefore P(k+1)$ is also true.

By mathematical induction, $P(n)$ is true for all $n \in N$

<u>Ex 4.63</u> By the Principle of mathematical induction, <u>Prove that</u>, for all integers $n \geq 1$, $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

<u>Soln</u>: $P(n) = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$P(1) = 1^2 = \frac{1(2)(3)}{6} \Rightarrow 1 = 1$

$\therefore P(1)$ is true

Assume that $P(k)$ is true.

$P(k) = 1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

To prove: $P(k+1)$ is also true.

$P(k+1) = 1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Consider, $1^2+2^2+3^2+\dots+k^2+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)(k(2k+1)+6(k+1))}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k^2+4k+3k+6)}{6}$

$= \frac{(k+1)(2k(k+2)+3(k+2))}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$\therefore P(k+1)$ is also true.

By mathematical induction, $P(n)$ is true for all $n \in N$

<u>Ex 4.64</u> Using the mathematical induction, <u>Show that</u> for any natural number n, $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

<u>Soln</u>: $P(n) = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$P(1) = \frac{1}{1(2)} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$, $\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$P(k) = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

To Prove: $P(k+1)$ is also true.

$P(k+1) = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Consider,

$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$

$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}$

$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}=\frac{k+1}{k+2}$

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n\in N$

EX 4.65 Prove that for any natural number n, a^n-b^n is divisible by a−b, where $a>b$

Soln: $P(n)=a^n-b^n$ is divisible by a−b

$P(1)=a^1-b^1=a-b$ is divisible by a−b

$\therefore$ P(1) is true.

Assume that P(k) is true.

$P(k)=a^k-b^k$ is divisible by a−b.

Since $a^k-b^k=\lambda(a-b)$

$a^k=\lambda(a-b)+b^k$ —(1)

To Prove: P(k+1) is also true.

$P(k+1)=a^{k+1}-b^{k+1}$ is divisible by a−b

Consider,

$a^{k+1}-b^{k+1}=a^k\cdot a-b^k\cdot b$

$=(\lambda(a-b)+b^k)a-b^k\cdot b$

$=\lambda a(a-b)+ab^k-b^k\cdot b$

$=\lambda a(a-b)+b^k(a-b)$

$a^{k+1}-b^{k+1}=(a-b)(\lambda a+b^k)$ is divisible by a−b.

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n\in N$

EX 4.66 Prove that $3^{2n+2}-8n-9$ is divisible by 8 for all $n\geq1$

Soln: $P(n)=3^{2n+2}-8n-9$ is divisible by 8.

$P(1)=3^{2+2}-8-9=3^4-17=81-17=64$ is divisible by 8.

$\therefore$ P(1) is true.

Assume that P(k) is true.

$P(k)=3^{2k+2}-8k-9$ is divisible by 8. Since $3^{2k+2}-8k-9=8\lambda$

$3^{2k+2}=8\lambda+8k+9$ —(1)

To Prove: P(k+1) is also true.

$P(k+1)=3^{2k+2+2}-8(k+1)-9$ is divisible by 8.

Consider, $3^{2k+2+2}-8(k+1)-9$

$=3^{2k+2}3^2-8k-8-9$

$=(8\lambda+8k+9)9-8k-17$

$=72\lambda+72k+81-8k-17$

$=72\lambda+64k+64=8(9\lambda+8k+8)$ is divisible by 8

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n\in N$

EX 4.67 Using the mathematical induction, Show that for any integer $n\geq2$, $3n^2>(n+1)^2$

Soln: $P(n)=3n^2>(n+1)^2$

$P(2)=3(2)^2>(2+1)^2\Rightarrow 12>9$

$\therefore$ P(2) is true.

Assume that P(k) is true.

$P(k)=3k^2>(k+1)^2$

To Prove: P(k+1) is also true.

$P(k+1)=3(k+1)^2>(k+2)^2$

Consider, $3(k+1)^2=3(k^2+2k+1)$

$=3k^2+6k+3>(k+1)^2+6k+3$

$=k^2+2k+1+6k+3$

$=k^2+8k+4$

$=k^2+4k+4+4k$

$=(k+2)^2+4k>(k+2)^2$

$\therefore 3(k+1)^2>(k+2)^2$

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n\geq2$

EX 4.68 Using the mathematical induction, Show that for any integer, $n\geq2$, $3^n>n^2$

Soln: $P(n)=3^n>n^2$

$P(2)=3^2>(2)^2\Rightarrow 9>4$

$\therefore$ P(2) is true.

Assume that P(k) is true.

$P(k)=3^k>k^2$

To prove: P(k+1) is also true.

$P(k+1)=3^{k+1}>(k+1)^2$

Consider, $3^{k+1}=3^k\cdot3>3k^2>(k+1)^2$

[Using $n\geq2$, $3n^2>(n+1)^2$]

$\therefore 3^{k+1}>(k+1)^2$

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n\geq2$

EX 4.69 By the principle of mathematical induction, Prove that, for $n\in N$

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots+\cos(\alpha+(n-1)\beta)=\cos\left(\alpha+\frac{(n-1)\beta}{2}\right)\times\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Soln: $P(n)=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots+\cos(\alpha+(n-1)\beta)$

$=\cos\left(\alpha+\frac{(n-1)\beta}{2}\right)\times\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

$P(1)=\cos\alpha=\cos\alpha\times\cancel{\frac{\sin\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}}\Rightarrow\cos\alpha=\cos\alpha$

$\therefore$ P(1) is true.

Assume that P(k) is true.

$P(k)=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots+\cos(\alpha+(k-1)\beta)=\cos\left(\alpha+\frac{(k-1)\beta}{2}\right)\times\frac{\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

To Prove: P(k+1) is also true.

$P(k+1)=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots+\cos(\alpha+k\beta)=\cos\left(\alpha+\frac{k\beta}{2}\right)\times\frac{\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Consider,

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\dots+\cos(\alpha+(k-1)\beta)+\cos(\alpha+k\beta)$

$=\cos\left(\alpha+\frac{(k-1)\beta}{2}\right)\times\frac{\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}+\cos(\alpha+k\beta)$

$=\frac{\cos\left(\alpha+\frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\cos(\alpha+k\beta)\sin\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

$=\frac{\cos\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)+\cos(\alpha+k\beta)\sin\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

$=\frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\frac{1}{2}\left[\sin\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}+\frac{k\beta}{2}\right)-\sin\left(\alpha+\cancel{\frac{k\beta}{2}}-\frac{\beta}{2}-\cancel{\frac{k\beta}{2}}\right)\right]+\frac{1}{2}\left[\sin\left(\alpha+k\beta+\frac{\beta}{2}\right)-\sin\left(\alpha+k\beta-\frac{\beta}{2}\right)\right]\right]$

$=\frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left[\cancel{\sin\left(\alpha+k\beta-\frac{\beta}{2}\right)}-\sin\left(\alpha-\frac{\beta}{2}\right)+\sin\left(\alpha+k\beta+\frac{\beta}{2}\right)-\cancel{\sin\left(\alpha+k\beta-\frac{\beta}{2}\right)}\right]$

$=\frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left[\sin\left(\alpha+k\beta+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)\right]$

$$= \frac{1}{2\sin\left(\frac{\beta}{2}\right)}\left[2\cos\left(\frac{\alpha+k\beta+\cancel{\frac{\beta}{2}}+\alpha-\cancel{\frac{\beta}{2}}}{2}\right) \times \sin\left(\frac{\cancel{\alpha}+k\beta+\frac{\beta}{2}-\cancel{\alpha}+\frac{\beta}{2}}{2}\right)\right]$$

$$= \frac{1}{\cancel{2}\sin\left(\frac{\beta}{2}\right)}\left[\cancel{2}\cos\left(\frac{\cancel{2}\alpha}{\cancel{2}}+\frac{k\beta}{2}\right)\sin\left(\frac{k\beta+\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{(k+1)\beta}{2}\right)\right]$$

$$= \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n \in N$

EX 4.70 Using the mathematical induction, Show that for any natural number n, with the assumption $i^2=-1$
$(r(\cos\theta+i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

Soln:

$P(n) = (r(\cos\theta+i\sin\theta))^n = r^n(\cos n\theta+i\sin n\theta)$

$P(1) = (r(\cos\theta+i\sin\theta))^1 = r^1(\cos\theta+i\sin\theta)$

$P(1) = r(\cos\theta+i\sin\theta) = r(\cos\theta+i\sin\theta)$

$\therefore$ P(1) is true.

Assume that P(k) is true.

$P(k) = (r(\cos\theta+i\sin\theta))^k = r^k(\cos k\theta+i\sin k\theta)$

To Prove : P(k+1) is also true.

$P(k+1) = (r(\cos\theta+i\sin\theta))^{k+1} = r^{k+1}(\cos(k+1)\theta+i\sin(k+1)\theta)$

Consider, $(r(\cos\theta+i\sin\theta))^{k+1}$

$= (r(\cos\theta+i\sin\theta))^k (r(\cos\theta+i\sin\theta))$

$= r^k(\cos k\theta+i\sin k\theta)(r(\cos\theta+i\sin\theta))$

$= r^{k+1}(\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)$

$= r^{k+1}((\cos k\theta\cos\theta-\sin k\theta\sin\theta) + i(\sin k\theta\cos\theta+\cos k\theta\sin\theta))$

$= r^{k+1}(\cos(k\theta+\theta)+i\sin(k\theta+\theta))$

$= r^{k+1}(\cos(k+1)\theta+i\sin(k+1)\theta)$

$\therefore$ P(k+1) is also true.

By mathematical induction, P(n) is true for all $n \in N$

EXERCISE 4.5

1) The sum of the digits at the 10^{th} place of all numbers formed with the help of 2,4,5,7 taken all at a time is

Soln:

Thousands	Hundreds	Tens	Unit
			2/4/5/7
3 ways	2 ways	1 way	1 way

Number of ways $= 3\times2\times1\times1 = 6$

Sum of the digits at the 10^{th} place $= 6(2+4+5+7) = 6(18) = 108$

Ans: (2) 108

2) In an examination there are three multiple choice questions and each question has 5 choices Number of ways in which a Student can fail to get all answer correct is

Soln:

Q_1	Q_2	Q_3
5 ways	5 ways	5 ways

Number of ways $= 5\times5\times5 = 125$

The required number of ways $= 125-1 = 124$

Ans: (2) 124

3) The number of ways in which the following Prize be given to a class of 30 boys first and second in mathematics first and second in physics, first in chemistry and first in English is

Soln:

M_1	M_2	P_1	P_2	C_1	E_1
30 ways	29 ways	30 ways	29 ways	30 ways	30 ways

Number of ways $= 30\times29\times30\times29\times30\times30$
$= 30^4 \times 29^2$

Ans: (1) $30^4 \times 29^2$

4) The number of 5 digit numbers all digits of which are odd is

Soln: The odd numbers are 1,3,5,7,9

5 ways	5 ways	5 ways	5 ways	5 ways

Number of odd numbers $= 5\times5\times5\times5\times5 = 5^5$. Ans (2) 5^5

5) In 3 fingers, the number of ways four rings can be worn is ___ ways

Soln:

Finger 1	Finger 2	Finger 3
4 ways	4 ways	4 ways

Number of ways $= 4\times4\times4 = 64$

Ans: (4) 64

6) If $^{(n+5)}P_{(n+1)} = \left(\frac{11(n-1)}{2}\right) {}^{(n+3)}P_n$ then the value of n are

Soln:

$^{(n+5)}P_{(n+1)} = \left(\frac{11(n-1)}{2}\right) {}^{(n+3)}P_n$

$\frac{(n+5)!}{4!} = \frac{11n-11}{2} \cdot \frac{(n+3)!}{3!}$

$\frac{(n+5)(n+4)\cancel{(n+3)!}}{2 \cancel{4}\times\cancel{3!}} = \frac{11n-11}{\cancel{2}} \cdot \frac{\cancel{(n+3)!}}{\cancel{3!}}$

$(n+5)(n+4) = 22n-22$

$n^2+9n+20-22n+22 = 0$

$n^2-13n+42=0$

$42 \to -6, -7$

$(n-6)(n-7)=0$

$n-6=0, n-7=0$

$n=6, n=7$

Ans: (2) 6 and 7

7) The Product of r consecutive Positive integer is divisible by

Soln:

Ans: (1) $r!$

8) The number of five digit telephone numbers having atleast one of their digits repeated is

Soln: The digits are 0,1,2,3,4,5,6,7,8,9

Number of numbers atleast one of their digits repeated $= (10\times10\times10\times10\times10) - (10\times9\times8\times7\times6) = 100000 - 30240$
$= 69,760$

Ans: (4) 69,760

9) If $a^2 - {}^aC_2 = a^2 - {}^aC_4$ then the value of 'a' is

Soln: Given: $a^2 - {}^aC_2 = a^2 - {}^aC_4$

$\cancel{a^2} - {}^aC_2 = \cancel{a^2} - {}^aC_{a-4}$ $(\because {}^nC_r = {}^nC_{n-r})$

$2 = a^2-a-4 \Rightarrow a^2-a-6=0$ $\quad -6 \to -3, 2$

$(a-3)(a+2)=0 \Rightarrow a-3=0, a+2=0$

$\Rightarrow \boxed{a=3}, a=-2$

Ans: (2) 3

10) There are 10 Points in a plane and 4 of them are collinear. The number of straight lines joining any two Points is

Soln: Number of straight lines

$= 10C_2 - 4C_2 + 1 = \frac{10!}{8!2!} - \frac{4!}{2!2!} + 1$

$= \frac{5\cancel{10}\times9\times\cancel{8!}}{\cancel{8!}\times\cancel{2}} - \frac{\cancel{24}6}{\cancel{2\times2}} + 1 = (5\times9) - 6 + 1$

$= 45 - 5 = 40$, Ans: (2) 40

11) The number of ways in which a host lady invite 8 People for a Party of 8 out of 12 People of whom two

do not want to attend the party together is

Soln: Number of ways $= 12C_8 - (12-2)C_{(8-2)}$

$= 12C_8 - 10C_6$

Ans: (3) $12C_8 - 10C_6$

12) The number of Parallelogram that can be formed from a set of four Parallel lines intersecting another set of three parallel lines

Soln: Number of Parallelogram $= 4C_2 \times 3C_2$

$= \frac{4!}{2!2!} \times 3 = \frac{6\cancel{24}}{\cancel{2\times2}} \times 3 = 6\times3 = 18$

Ans: (4) 18

13) Everybody in a room Shakeshand with everybody else. The total number of Shakeshands is 66. The number of Persons in the room is

Soln: number of handshakes $= nC_2 = 66$

$\frac{n(n-1)}{2} = 66 \Rightarrow n(n-1) = 132 \Rightarrow n^2 - n - 132 = 0$

$\Rightarrow (n-12)(n+11) = 0 \Rightarrow n-12=0, n+11=0$

$\Rightarrow n = 12, n = -11$

$\therefore n = 12$, Ans: (2) 12

14) number of sides of a Polygon having 44 diagonals is

Soln: Number of diagonals $= nC_2 - n = 44$

$\frac{n(n-1)}{2} - n = 44 \Rightarrow \frac{n(n-1)-2n}{2} = 44$

$\Rightarrow n^2 - n - 2n = 88 \Rightarrow n^2 - 3n - 88 = 0$

$\Rightarrow (n-11)(n+8) = 0 \Rightarrow n-11=0, n+8=0$

$\Rightarrow n = 11, n = -8 \therefore n = 11$, Ans: (3) 11

15) If 10 lines are drawn in a plane such that no two of them are parallel and no three are concurrent then the total number of Points of intersection are

Soln: Number of Points of intersection $= 10C_2$

$= \frac{10!}{8!2!} = \frac{5\cancel{10}\times9\times\cancel{8!}}{\cancel{8!}\times\cancel{2}} = 5\times9 = 45$

Ans: (1) 45

16) In a plane there are 10 Points are there out of which 4 Points are collinear then the number of triangles formed

Soln: Number of triangles $= 10C_3 - 4C_3$

$= \frac{10!}{7!3!} - 4 = \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}\times6} - 4 = \frac{\cancel{720}^{120}}{\cancel{6}} - 4$

$= 120 - 4 = 116$, Ans: (4) 116

17) In $2nC_3 : nC_3 = 11:1$ then n is

Soln: $\frac{2nC_3}{nC_3} = \frac{11}{1} \Rightarrow \frac{\frac{2n!}{(2n-3)!\cancel{3!}}}{\frac{n!}{(n-3)!\cancel{3!}}} = 11$

$\frac{2n!}{(2n-3)!} \times \frac{(n-3)!}{n!} = 11$

$\frac{2\cancel{n}(2n-1)(2n-2)\cancel{(2n-3)!}}{\cancel{(2n-3)!}} \times \frac{\cancel{(n-3)!}}{\cancel{n}(n-1)(n-2)\cancel{(n-3)!}} = 11$

$\frac{2(2n-1)2\cancel{(n-1)}}{\cancel{(n-1)}(n-2)} = 11 \Rightarrow 4(2n-1) = 11(n-2)$

$\Rightarrow 8n - 4 = 11n - 22 \Rightarrow 22 - 4 = 11n - 8n \Rightarrow \cancel{18}^{6} = \cancel{3}n$

$\Rightarrow n = 6$, Ans: (2) 6

18) $(n-1)C_r + (n-1)C_{(r-1)}$ is

Soln: Given: $(n-1)C_r + (n-1)C_{(r-1)}$

$= \frac{(n-1)!}{(n-r-1)!r!} + \frac{(n-1)!}{(n-r)!(r-1)!}$

$= \frac{(n-1)!}{(n-r-1)!r(r-1)!} + \frac{(n-1)!}{(n-r)(n-r-1)!(r-1)!}$

$= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r}\right]$

$= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{n-\cancel{r}+\cancel{r}}{r(n-r)}\right] = \frac{n(n-1)!}{(n-r)(n-r-1)!\,r(r-1)!}$

$= \frac{n!}{(n-r)!r!} = nC_r$, Ans: (3) nC_r

19) The number of ways of choosing 5 cards out of a deck of 52 cards which include atleast one king is

Soln: $52C_5$ include all possibilities 0 king, 1 king, 2 king, 3 king, 4 king

$48C_5$ has no kings

Number of ways $= 52C_5 - 48C_5$

Ans: (4) $52C_5 - 48C_5$

20) The number of rectangles that a chessboard has ——

Soln: Number of rectangles $= 9C_2 \times 9C_2$

$= \frac{9!}{7!2!} \times \frac{9!}{7!2!} = \frac{9\times8\times\cancel{7!}}{\cancel{7!}\times2} \times \frac{9\times8\times\cancel{7!}}{\cancel{7!}\times2}$

$= \frac{\cancel{72}36}{\cancel{2}} \times \frac{\cancel{72}36}{\cancel{2}} = 36\times36 = 1296$

Ans: (3) 1296

21) The number of 10 digit number that can be written by using the digits 2 and 3 is

Soln:

2/3	2/3	2/3	2/3	2/3	2/3	2/3	2/3	2/3	2/3
2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways	2 ways

Number of numbers $= 2\times2\times2\times2\times2\times2\times2\times2\times2\times2 = 2^{10}$. Ans: (2) 2^{10}

22) If P_r Stands for rP_r then the Sum of the series $1 + P_1 + 2P_2 + 3P_3 + \dots + nP_n$ is

Soln: $1 + P_1 + 2P_2 + 3P_3 + \dots + nP_n$ $\boxed{P_r = rP_r \\ P_r = r!}$

$= 1 + (1)(1!) + 2(2!) + 3(3!) + \dots + n(n!)$

$= (n+1)P_{(n+1)} = P_{n+1}$

Ans: (1) P_{n+1}

23) The Product of first n odd natural numbers equals

Soln: using EX 4.24

$\frac{(2n)!}{n!} = 2^n(1\cdot3\cdot5\cdots\cdots(2n-1))$

$\Rightarrow 1\cdot3\cdot5\cdots(2n-1) = \frac{(2n)!}{n!\,2^n} = \frac{1}{2^n}\cdot\frac{(2n)!}{n!}$

$= \left(\frac{1}{2}\right)^n \cdot 2nC_n \times nP_n$

Ans: (2) $\left(\frac{1}{2}\right)^n \cdot 2nC_n \times nP_n$

24) If nC_4, nC_5, nC_6 are in A.P the value of n can be

Soln: $nC_5 - nC_4 = nC_6 - nC_5$

$2nC_5 = nC_4 + nC_6$

$2\cdot\frac{n!}{(n-5)!5!} = \frac{n!}{(n-4)!4!} + \frac{n!}{(n-6)!6!}$

$\frac{2n!}{(n-5)(n-6)!5\times4!} = \frac{n!}{(n-4)(n-5)(n-6)!4!} + \frac{n!}{(n-6)!6\times5\times4!}$

$\frac{\cancel{n!}}{\cancel{(n-6)!4!}}\left[\frac{2}{5(n-5)}\right] = \frac{\cancel{n!}}{\cancel{(n-6)!4!}}\left[\frac{1}{(n-4)(n-5)} + \frac{1}{30}\right]$

$\frac{2}{5(n-5)} - \frac{1}{(n-4)(n-5)} = \frac{1}{30}$

$\frac{2(n-4) - 5}{\cancel{5}(n-4)(n-5)} = \frac{1}{\cancel{30}6}$

$\frac{2n-8-5}{n^2-9n+20} = \frac{1}{6} \Rightarrow \frac{2n-13}{n^2-9n+20} = \frac{1}{6}$

$12n - 78 = n^2 - 9n + 20 \Rightarrow n^2 - 21n + 98 = 0$

$(n-14)(n-7) = 0 \Rightarrow n-14=0, n-7=0$

$n = 14, n = 7 \therefore$ Ans: (1) 14

25) $1+3+5+7+\dots+17$ is equal to

Soln: $1+3+5+7+\dots+(2n-1) = n^2$

$\therefore 2n - 1 = 17 \Rightarrow \cancel{2}n = \cancel{18}9 \Rightarrow n = 9$

$1+3+5+7+\dots+17 = (9)^2 = 81$

Ans: (2) 81

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