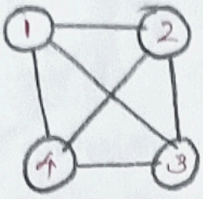
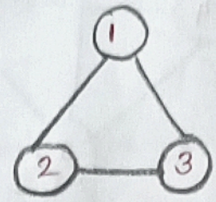
# Clique Decision Problem. (CDP)

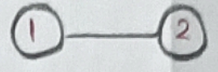
→ A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.



Complete graph of 4 vertices



Complete graph of 3 vertices



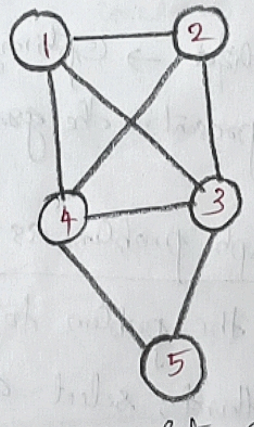
Complete graph of 2 vertices.

Property of Complete graphs

No: of Vertices $= |V| = n$.

Then No: of edges $|E| = \frac{n(n-1)}{2}$

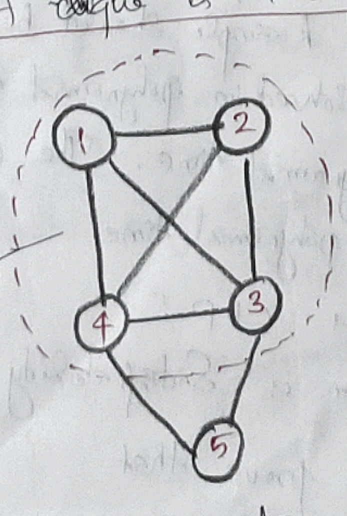
→ If in the above graph if we add one more vertex, let it be 5 and connect few edges,



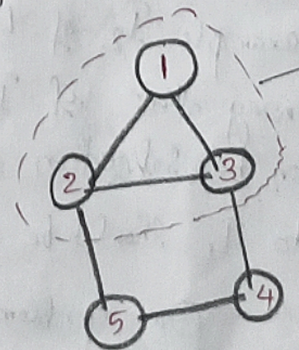
→ This is not a complete graph, but if we consider a subgraph 1-2-3-4, that subgraph will be a complete graph. Such a subgraph is called a clique.

→ ie, A clique is a subgraph of a graph which is complete.

eg:-

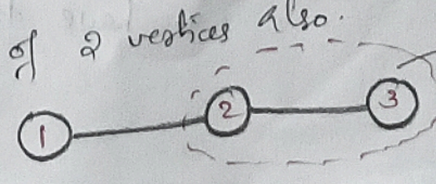


clique.



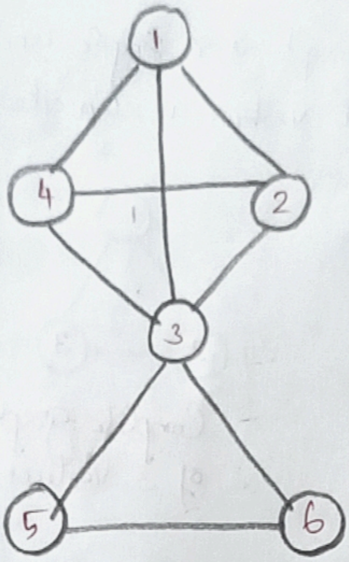
clique.

→ There can be clique of 2 vertices also.



clique.

Let's consider another example.




In this graph, there are cliques with ~~g~~ size $k=4$, $k=3$ and $k=2$. The $max^m$ clique size in this graph is 4 and minimum clique size is 2.

Finding the $max^m$ or $min^m$ clique → Optimization problem.

Finding a clique of size k present in the graph → Decision problem.

Procedure for proving a graph problem as ~~NP~~ NP-Hard

→ ~~Suppose~~ Let $L_2$ be the problem to be proved as NP-Hard

→ To prove it as NP-Hard, select another problem $L_1$ which is already proved as NP-Hard.

→ Then show that $L_1 \propto L_2$

→ For that, take an example $I_1$ of $L_1$, and prepare an example $I_2$ of $L_2$. Example should be taken in such a way that if $I_1$ solved in polynomial time, then $I_2$ also can be solved in polynomial time. The conversion from $I_1$ to $I_2$ should be in polynomial time.

Our problem is CDP.

Known problem is Satisfiability

∴ We have to prove that

Satisfiability $\propto$ CDP.

Proof that boolean satisfiability problem reduces to the clique decision problem.

Let the boolean expression be,

$$F = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge (x_1 \vee x_3) \text{ where}$$

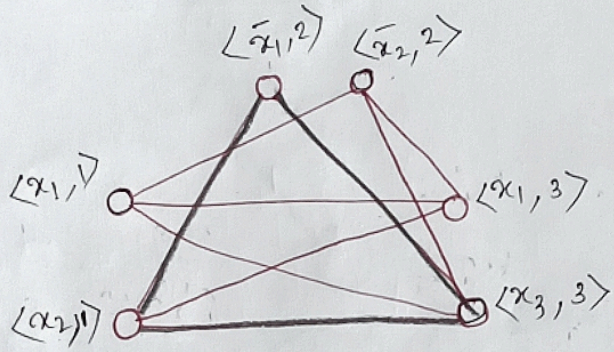
$x_1, x_2, x_3$ are the variables.

Let the expression within each parenthesis be a clause.
Hence we have three clauses — $C_1$, $C_2$ and $C_3$. Consider the vertices $\langle x_1, 1\rangle$ ; $\langle x_2, 1\rangle$ ; $\langle \bar{x}_1, 2\rangle$, $\langle \bar{x}_2, 2\rangle$ ; $\langle x_1, 3\rangle$ ; $\langle x_3, 3\rangle$ where the second term in each vertex denotes the clause number they belong to. We connect these vertices such that

1) No two vertices belonging to the same clause are connected

2) No variable is connected to its complement.

$$F = \bigwedge_{i=1}^{k} C_i$$




Thus, the graph $G(V, E)$ is constructed such that

$$V = \{ \langle a, i\rangle \mid a \in C_i \} \text{ and}$$

$$E = \{ (\langle a, i\rangle, \langle b, j\rangle) \mid i \neq j ; \; b \neq \bar{a} \}.$$

Consider the subgraph of G with the vertices $\langle x_2, 1\rangle$ ; $\langle \bar{x}_1, 2\rangle$ ; $\langle x_3, 3\rangle$. It forms a clique of size 3. Corresponding to this, for the assignment ~~$\langle x_1, x_2, x_3\rangle$ ...~~

$$\langle \bar{x}_1, x_2, x_3\rangle = \langle 0, 1, 1\rangle,$$

F evaluates to true.

Therefore if we have k clauses in our satisfiability ~~express~~ expression, we get a $max^m$ clique of size k and for the corresponding assignment of values, the satisfiability expression evaluates to true. Hence, for a particular instance, the satisfiability pblm is reduced to clique decision pblm.

∴ Clique decision problem is NP-Hard.

<u>Conclusion.</u>

The Clique decision problem is NP and NP-Hard. Therefore, CDP is NP-Complete.