Homework 2

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1 Part 1

1.1 a

The new objective function, where α_i is a lagrangian multiplier for each constraint, will be

$$min\ L = \frac{1}{2}w^2 - \sum_{i=1}^{n} \alpha_i(y_i(w^T\phi(x_i)) - 1)$$

1.2 b

Taking the derivative of L with respect to w we get

$$\frac{\partial}{\partial w}L = w - \sum_{i=1}^{n} \alpha_i y_i \phi(x_i)$$

1.3 c

Setting the derivative to zero and solving for w gives us

$$\frac{\partial}{\partial w}L = w - \sum_{i=1}^{n} \alpha_i y_i \phi(x_i) = 0$$

$$w = \sum_{i=1}^{n} \alpha_i y_i \phi(x_i)$$

This allows us to replace w in our objective function as a linear combination of the data points in feature space, $\phi(x_i)$, and the signed Lagrange multipliers, $\alpha_i y_i$ as coefficients.

1.4 d

Substituting w into our objective function will give us

$$L = \frac{1}{2}(\sum_{i=1}^{n} \alpha_i y_i \phi(x_i))^2 - \sum_{i=1}^{n} \alpha_i (y_i (\sum_{j=1}^{n} \alpha_j y_j \phi(x_j))\phi(x_i)) - 1)$$
$$= \frac{1}{2}\sum_{i=1}^{n} \alpha_i y_i \phi(x_i) \sum_{j=1}^{n} \alpha_j y_j \phi(x_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \phi(x_i)^T \phi(x_j) - \alpha_i$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \phi(x_i)^T \phi(x_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \phi(x_j)^T \phi(x_i) + \sum_{i=1}^{n} \alpha_i$$
$$= \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \phi(x_i)^T \phi(x_j)$$

So the new objective function will be

$$max_\alpha \ L_{dual} = \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j K(x_i, x_j)$$

subject to $\alpha_i \geq 0 \ \forall i \in D$

1.5 e

let $J(\alpha) = L_{dual}$ then taking the partial derivative with repect to one of the Lagrange multipliers we get

$$\frac{\partial J(\alpha_k)}{\partial \alpha_k} = 1 - y_k \sum_{i=1}^{n} \alpha_i y_i K(x_i, x_k)$$

1.6 f

After calculating all of the Lagrange multipliers we can compute w using the formula we derived from part (c)

$$w = \sum_{i=1}^{n} \alpha_i y_i \phi(x_i)$$

We can then calculate the class of a new point z as

$$y = sign(w^T \phi(z)) = sign(\sum_{\alpha_i \geq 0} \alpha_i y_i K(x_i, z))$$

1.7 g

Picture (a) can not represent hard-margin kernel SVM since the dataset is non-seperable. I.e. there is a red data point inside the circle with the blue data points.
Picture (b) can represent hard-margin kernel SVM since the dataset is seperable and the kernel will just be a linear kernel.
Picture (c) can represent hard-margin kernel SVM since the dataset is seperable and the kernel will just be a quadratic kernel.
Picture (d) can not represent hard-margin kernel SVM since the dataset is non-seperable. I.e. There are 2 data points which lie on the seperating hyperplane so they cannot be classified.

2 Part 2

2.1 a

(1)

$$W(S_1, T_1) = 4$$
$$W(S_2, T_2) = 2 + 3 = 5$$
$$W(S_3, T_3) = 10 + 10 = 20$$

(2)

$$\frac{W(S_1, T_1)}{|S_1|} + \frac{W(S_1, T_1)}{|T_1|} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \approx 2.6667$$
$$\frac{W(S_2, T_2)}{|S_2|} + \frac{W(S_2, T_2)}{|T_2|} = \frac{5}{1} + \frac{5}{5} = 5 + 1 = 6$$
$$\frac{W(S_3, T_3)}{|S_3|} + \frac{W(S_3, T_3)}{|T_3|} = \frac{20}{4} + \frac{20}{2} = 5 + 10 = 15$$

(3)

$$\frac{W(S_1, T_1)}{vol(S_1)} + \frac{W(S_1, T_1)}{vol(T_1)} = \frac{4}{3+2+3+4} + \frac{4}{10+10+100+4} = \frac{4}{12} + \frac{4}{124} = \frac{34}{93} \approx 0.3656$$
$$\frac{W(S_2, T_2)}{vol(S_2)} + \frac{W(S_2, T_2)}{vol(T_2)} = \frac{5}{3+2} + \frac{5}{3+2+3+10+10+100+4} = \frac{5}{5} + \frac{5}{136} = \frac{141}{136} \approx 1.0368$$
$$\frac{W(S_3, T_3)}{vol(S_3)} + \frac{W(S_3, T_3)}{vol(T_3)} = \frac{20}{3+2+3+10+10+4} + \frac{20}{10+10+100} = \frac{20}{32} + \frac{20}{120} = \frac{19}{24} \approx 0.7917$$

From smallest to largest for cut weight: S_1, S_2, S_3
From smallest to largest for ratio cut: S_1, S_2, S_3
From smallest to largest for normalized cut: S_1, S_3, S_2
In general, the rank from smallest to largest is: S_1, S_2, S_3
The only difference was for the normalized cut since the second cut had only a single point and so $\frac{W(S_2,T_2)}{vol(S_2)}$ was equal to 1.

2.2 b

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

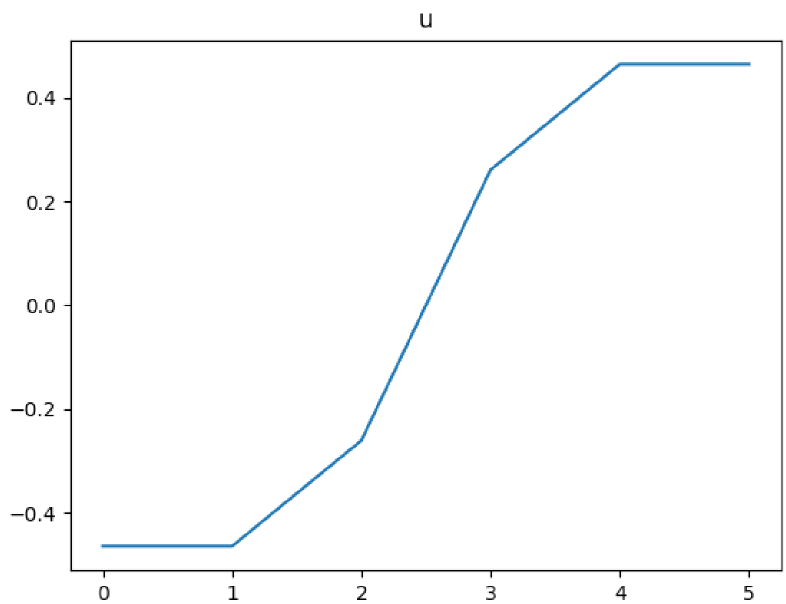
$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

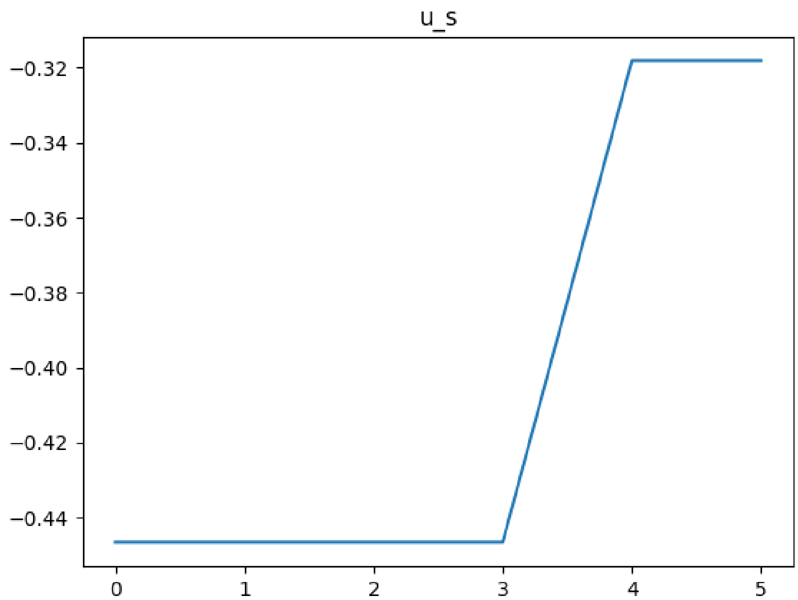
$$L_s = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \odot \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \odot \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix} \odot \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{-1}{2} & \frac{-1}{3} & 0 & 0 & 0 \\ \frac{-1}{2} & 2 & \frac{-1}{3} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 3 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & \frac{-1}{3} & 3 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{-1}{3} & 2 & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{-1}{3} & \frac{-1}{2} & 2 \end{bmatrix}$$

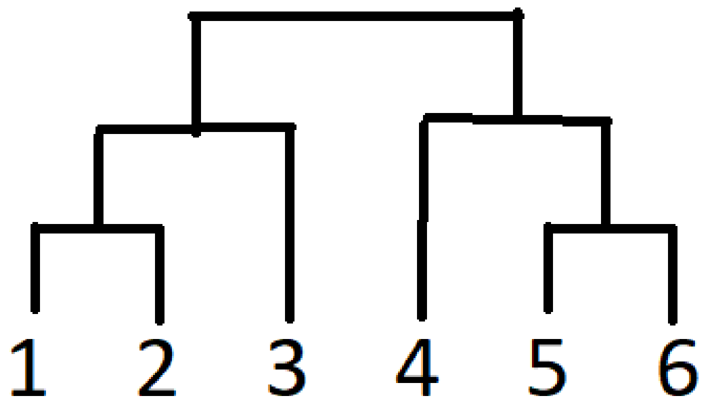
2.3 c





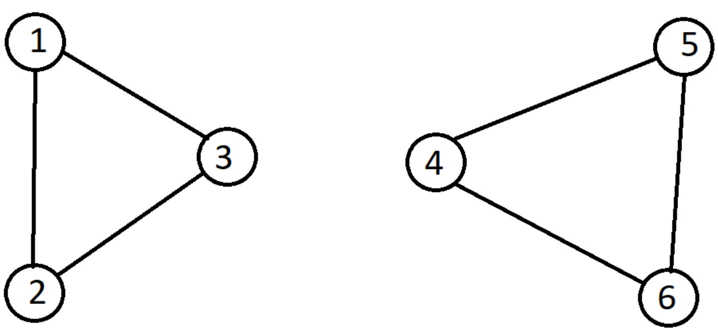
2.4 d

Dendogram for u is on the next page. The dendogram for u_s doesn't seem to work since the furthest away points are connected first. However, if these points are not allowed to connect then the dendogram would be similar to the one for u.



2.5 e

Partition for u The cut weights, normalized cuts, and ratio cuts for u are the



exact same as the partion S_1 in part (a). We obtain this partition because it is equivalent to minimizing the ratio cut.

3 Part 3

For the set, $X = \{0, 1, 2, 2, 10\}$, the mean is

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{5}(0+1+2+2+10) = \frac{15}{5} = 3$$

And the median is the value m where $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$
Which in this case will just be 2 since it is the middle-most value.
so $m = 2$

3.1 a

(1) The sum of squared distances for the mean is

$$\mu = \sum_{i=1}^{n}(x_i - \mu)^2 = [(0-3)^2+(1-3)^2+(2-3)^2+(2-3)^2+(10-3)^2] = (9+4+1+1+49) = 64$$

And the sum of squared distances for the median is

$$\mu = \sum_{i=1}^{n}(x_i - m)^2 = [(0-2)^2+(1-2)^2+(2-2)^2+(2-2)^2+(10-2)^2] = (4+1+0+0+64) = 69$$

Therefore, $\sum_{i=1}^{n}(x_i-\mu)^2 \le \sum_{i=1}^{n}(x_i-m)^2$ Since $64 \le 69$

(2) The sum of absolute distances for the mean is

$$\mu = \sum_{i=1}^{n}|x_i - \mu| = [|0-3|+|1-3|+|2-3|+|2-3|+|10-3|] = (3+2+1+1+7) = 14$$

And the sum of absolute distances for the median is

$$\mu = \sum_{i=1}^{n}|x_i - m| = [|0-2|+|1-2|+|2-2|+|2-2|+|10-2|] = (2+1+0+0+8) = 11$$

Therefore, $\sum_{i=1}^{n}(x_i-m)^2 \le \sum_{i=1}^{n}(x_i-\mu)^2$ Since $11 \le 14$

3.2 b

Let

$$f = argmin_a \sum_{i=1}(x_i - a)^2$$

Also,

$$\sum_{i=1}(x_i - a)^2 = \sum_{i=1} x_i^2 - 2a\sum_{i=1} x_i + na^2$$

f reaches a minimum value when $\frac{\partial f}{\partial a} = 0$.

$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}[\sum_{i=1} x_i^2 - 2a \sum_{i=1} x_i + na^2]$$

$$= -2 \sum_{i=1} x_i + 2na$$

Set to zero.

$$-2 \sum_{i=1} x_i + 2na = 0$$

$$a = \frac{1}{n} \sum_{i=1} x_i$$

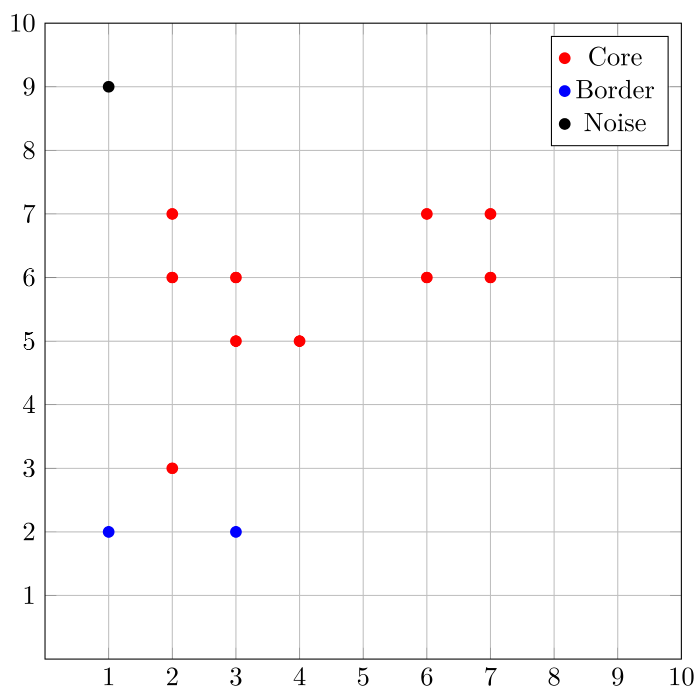
$$= \mu$$

3.3 c

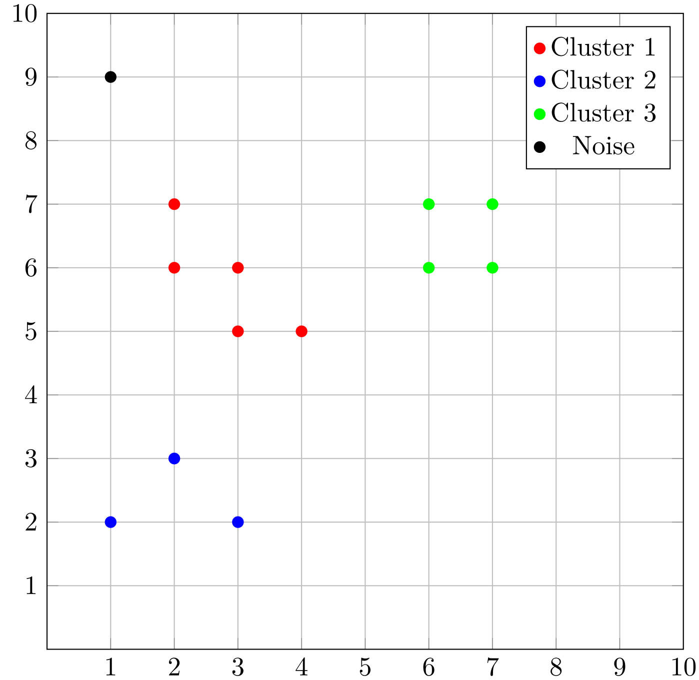
todo

4 Part 4

4.1 a

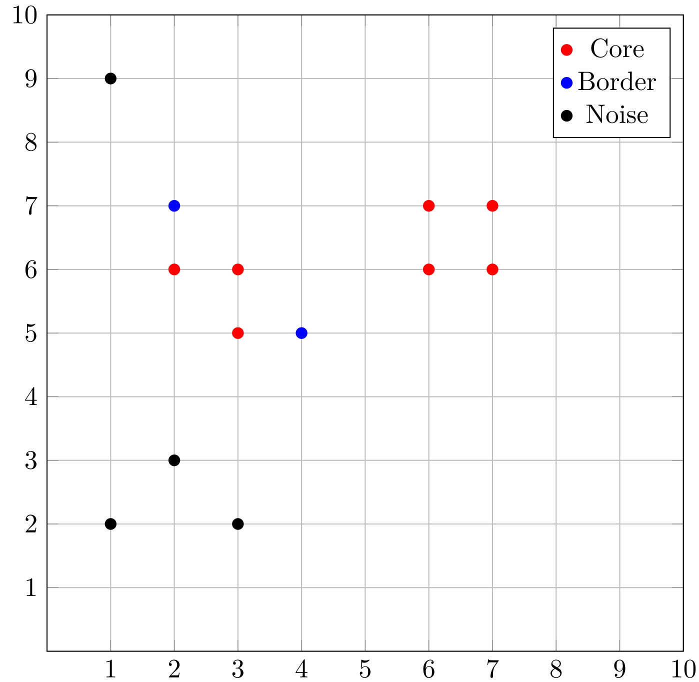


4.2 b



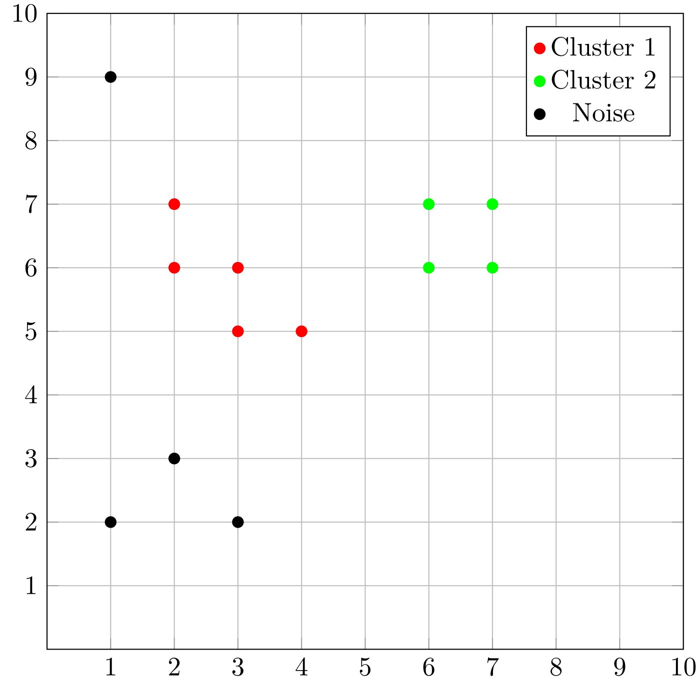
Cluster 1
Cluster 2
Cluster 3
Noise
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10

4.3 c



Compared to part (a), the 3 points in the bottom left have all become outliers since the point (2,3) is no longer within distance of the other 2 points (Since the distance between diagonal points is $\sqrt{2}$ away from each other and that is greater than $\epsilon = 1$). Also, the points (2,7) and (4,5) are now border points because of the same issue with diagonal points. For all of these points, the number of points within distance $\epsilon = 1$ has changed and is now less than $minpts = 3$, causing them to become an outlier if they are within distance of a core point and they become a border point otherwise.

4.4 d



Compared to part (b), the bottom left points are no longer a single cluster since these points became outliers and are not connected.