

# ISLR | Chapter 7 Exercises

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## Conceptual

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- **A.** The cubic piecewise polynomial:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - \xi)_+^3 \quad \text{where} \quad (x - \xi)_+^3 = \begin{cases} 0, & x \leq \xi \\ (x - \xi)^3, & \text{otherwise} \end{cases}$$

...can be broken up and rewritten to be:

$$f(x) = \begin{cases} f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, & x \leq \xi \\ f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3, & \text{otherwise} \end{cases}$$

In  $f_1(x)$ , since  $(x - \xi)_+^3 = 0$  (because  $x \leq \xi$ ), the fifth term (of  $f(x)$ ) zeroes out and the coefficients can be expressed as  $a_1 = \beta_0$ ,  $b_1 = \beta_1$ ,  $c_1 = \beta_2$  and  $d_1 = \beta_3$ .

- **B.** Expanding the fifth term in  $f(x)$  allows for the various powers of  $x$  to be grouped together and then recondensed.  $a_2$ ,  $b_2$ ,  $c_2$  and  $d_2$  are expressed in terms of the coefficients below.

$$f_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - \xi)^3 \quad (1)$$

$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - \xi)(x - \xi)(x - \xi) \quad (2)$$

$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^2 - 2x\xi + \xi^2)(x - \xi) \quad (3)$$

$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^3 - x^2\xi - 2x^2\xi + 2x\xi^2 + \xi^2 x - \xi^3) \quad (4)$$

$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^3 - 3x^2\xi + 3x\xi^2 - \xi^3) \quad (5)$$

$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^3 - \beta_4 3x^2\xi + \beta_4 3x\xi^2 - \beta_4 \xi^3 \quad (6)$$

$$= (\beta_0 - \beta_4 \xi^3) + (\beta_1 x + \beta_4 3x\xi^2) + (\beta_2 x^2 - \beta_4 3x^2\xi) + (\beta_3 x^3 + \beta_4 x^3) \quad (7)$$

$$= (\beta_0 - \beta_4 \xi^3) + (\beta_1 + 3\beta_4 \xi^2)x + (\beta_2 - 3\beta_4 \xi)x^2 + (\beta_3 + \beta_4)x^3 \quad (8)$$

$$f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3 \quad \text{where} \quad \begin{cases} a_2 = \beta_0 - \beta_4 \xi^3 \\ b_2 = \beta_1 + 3\beta_4 \xi^2 \\ c_2 = \beta_2 - 3\beta_4 \xi \\ d_2 = \beta_3 + \beta_4 \end{cases} \quad (9)$$

- **C.** Showing that  $f(x)$  is continuous at  $\xi$  is illustrated by showing that  $f(\xi)_1 = f(\xi)_2$ .

$$f_1(\xi) = a_1 + b_1(\xi) + c_1(\xi)^2 + d_1(\xi)^3 \quad (10)$$

$$= \beta_0 + \beta_1(\xi) + \beta_2(\xi)^2 + \beta_3(\xi)^3 \quad (11)$$

$$(12)$$

$$f_2(\xi) = a_2 + b_2(\xi) + c_2(\xi)^2 + d_2(\xi)^3 \quad (13)$$

$$= (\beta_0 - \beta_4\xi^3) + (\beta_1 + 3\beta_4\xi^2)(\xi) + (\beta_2 - 3\beta_4\xi)(\xi)^2 + (\beta_3 + \beta_4)(\xi)^3 \quad (14)$$

$$= (\beta_0 - \beta_4\xi^3) + (\beta_1\xi + 3\beta_4\xi^3) + (\beta_2\xi^2 - 3\beta_4\xi^3) + (\beta_3\xi^3 + \beta_4\xi^3) \quad (15)$$

$$= \beta_0 - \beta_4\xi^3 + \beta_1\xi + 3\beta_4\xi^3 + \beta_2\xi^2 - 3\beta_4\xi^3 + \beta_3\xi^3 + \beta_4\xi^3 \quad (16)$$

$$= \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 + 3\beta_4\xi^3 - 3\beta_4\xi^3 + \beta_4\xi^3 - \beta_4\xi^3 \quad (17)$$

$$= \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 + (3\beta_4\xi^3 - 3\beta_4\xi^3) + (\beta_4\xi^3 - \beta_4\xi^3) \quad (18)$$

$$f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 \quad (19)$$

$$f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 = f_1(\xi)$$

- **D.** In order to show that  $f'_1(\xi) = f'_2(\xi)$ , we must first find  $f'(x)$  with respect to  $x$  and then simplify both  $f'_1(\xi)$  and  $f'_2(\xi)$ .

$$f(x) = a_1 + b_1x + c_1x^2 + d_1x^3 \quad (20)$$

$$f'(x) = b_1 + 2c_1x + 3d_1x^2 \quad (21)$$

Therefore, substituting the necessary coefficients in for  $b_1$ ,  $c_1$  and  $d_1$  in both  $f'_1(\xi)$  and  $f'_2(\xi)$ , we get:

$$f'(x) = b_1 + 2c_1x + 3d_1x^2 \quad \text{then} \quad \begin{cases} f'_1(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 \\ f'_2(\xi) = (\beta_1 + 3\beta_4\xi^2) + 2(\beta_2 - 3\beta_4\xi)\xi + 3(\beta_3 + \beta_4)\xi^2 \end{cases} \quad (22)$$

$$f'_2(\xi) = (\beta_1 + 3\beta_4\xi^2) + 2(\beta_2 - 3\beta_4\xi)\xi + 3(\beta_3 + \beta_4)\xi^2 \quad (23)$$

$$= \beta_1 + 3\beta_4\xi^2 + 2\beta_2\xi - 6\beta_4\xi^2 + 3\beta_3\xi^2 + 3\beta_4\xi^2 \quad (24)$$

$$= \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 + (3\beta_4\xi^2 + 3\beta_4\xi^2 - 6\beta_4\xi^2) \quad (25)$$

$$= \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 + (6\beta_4\xi^2 - 6\beta_4\xi^2) \quad (26)$$

$$f'_2(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 \quad (27)$$

We now see that the derivative  $f'(x)$  is continuous at knot  $\xi$ , which is to say  $f'_1(\xi) = f'_2(\xi)$ :

$$f'_2(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 = f'_1(\xi)$$

- **E.** In order to show that  $f_1''(\xi) = f_2''(\xi)$ , we must first find  $f''(x)$  with respect to  $x$  and then simplify both  $f_1''(\xi)$  and  $f_2''(\xi)$ .

$$f(x) = a_1 + b_1x + c_1x^2 + d_1x^3 \quad (28)$$

$$f'(x) = b_1 + 2c_1x + 3d_1x^2 \quad (29)$$

$$f''(x) = 2c_1 + 6d_1x \quad (30)$$

Therefore, substituting the necessary coefficients in for  $c_1$  and  $d_1$  in both  $f_1''(\xi)$  and  $f_2''(\xi)$ , we come to:

$$f''(x) = 2c_1 + 6d_1x \quad \text{then} \quad \begin{cases} f_1''(\xi) = 2\beta_2 + 6\beta_3\xi \\ f_2''(\xi) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)\xi \end{cases} \quad (31)$$

$$f_2''(\xi) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)\xi \quad (32)$$

$$= 2\beta_2 - 6\beta_4\xi + 6\beta_3\xi + 6\beta_4\xi \quad (33)$$

$$= 2\beta_2 + 6\beta_3\xi + (6\beta_4\xi - 6\beta_4\xi) \quad (34)$$

$$f_2''(\xi) = 2\beta_2 + 6\beta_3\xi \quad (35)$$

We now see that the second derivative  $f''(x)$  is continuous at knot  $\xi$ , which is to say  $f_1''(\xi) = f_2''(\xi)$ :

$$f_2''(\xi) = 2\beta_2 + 6\beta_3\xi = f_1''(\xi)$$