ECE415 Final Report 2

Timothy Marshall

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———- Part A ——

1. Robustness to Motor Resistance Variation

To demonstrate how variations in the armature resistance R affect our closed-loop poles and time-domain performance, we include two root-locus plots (full-scale and zoomed-in) and summarize the mathematical and numerical findings below.

Full-Scale Root-Locus Plot

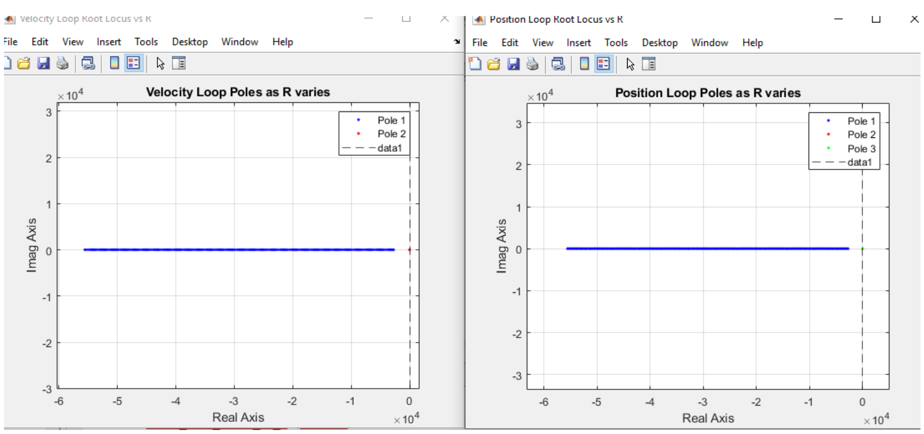

Figure 1: Root-locus of velocity-loop poles (blue/red) and position-loop poles (green) as R sweeps 0.5–10 Ohms. The dashed line at $\Re\{s\} = 0$ shows no encirclement of unstable region.

- The poles of both loops (velocity and position) are plotted as R varies over [0.5,10] Ohms.
- A vertical line at $\Re\{s\} = 0$ confirms *no* encirclements or crossings into the right half-plane.

Zoomed-In Root-Locus Plot

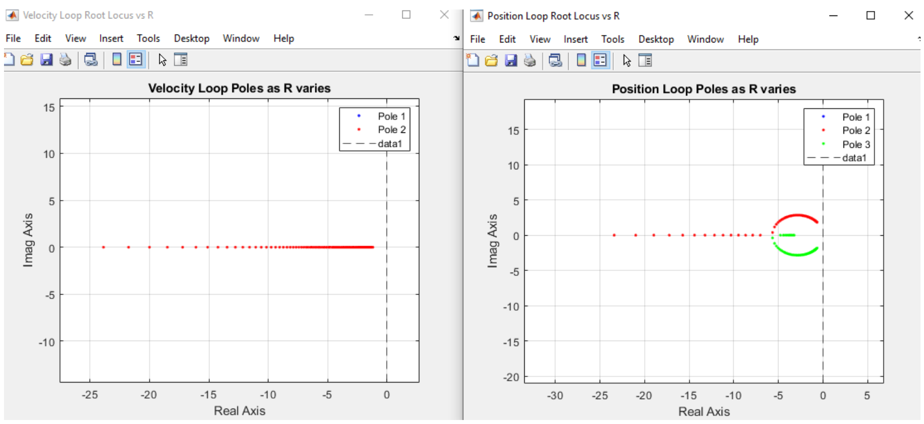

Figure 2: Zoomed in on the real-axis region to highlight the small-magnitude poles: Note that the plot axes have been rescaled for clarity

- **Integrator pole:** the green marker at $s = 0$ (position loop) remains fixed, guaranteeing zero steady-state error.
- **Mechanical pole cluster:** the slow pole moves between

  $$s_m \approx -0.018\ rad/s \quad and \quad s_m \approx -0.015\ rad/s$$

  as R increases from 0.5 to 10 .
- **Electrical pole:** remains at $\sim -1.4 \times 10^4\ rad/s$, well outside the 0–10 rad/s control band.

Numerical Quantification

Closed-Loop Dominant Pole With $K_{p,v} = 6$, the velocity-loop closed-loop time constant is

$$\tau_{cl} = \frac{55}{1 + 46K_{p,v}} = \frac{55}{1 + 276} = \frac{55}{277} \approx 0.198s.$$

Thus the dominant closed-loop pole is

$$s_{cl} = -\frac{1}{\tau_{cl}} \approx -5.05 \text{ rad/s},$$

and the 10–90% rise time is

$$t_r \approx 2.2\,\tau_{cl} \approx 2.2 \times 0.198 = 0.44s.$$

Crucially, τ_{cl} (and therefore t_r) does *not* depend on R.

Open-Loop Mechanical Pole Drift The open-loop mechanical pole

$$s_m = -\frac{Rb + k_t k_m}{Lb + JR}$$

varies only from about -0.0182rad/s at $R = 0.5\,\Omega$ to -0.0167rad/s at $R = 10\,\Omega$, a $\approx 8\%$ change. However, this pole lies at $\mathcal{O}(10^{-2})$rad/s—hundreds of times slower than the closed-loop pole at -5.05 rad/s—so its drift has *no* practical effect on the 0.44s rise time or the zero-overshoot behavior.

Time Domain Implications

- **Velocity loop:** The closed-loop rise time $t_r \approx 0.44$s and zero overshoot are *independent* of R. The small open-loop mechanical pole drift ($< 10\%$) occurs at a frequency four decades below the control bandwidth and therefore does not alter the designed transient response.
- **Position loop:** The dominant PD-placed poles ($s = -2.8 \pm j2.86$) remain unchanged by R, so the position-loop $t_r \approx 0.45$s and $M_p \approx 4.6\%$ are likewise invariant. The integrator pole at $s = 0$ guarantees zero steady-state error regardless of R.

2. Effect of Motor Resistance Variation

Question: What is the effect of increasing or decreasing the motor resistance R in each loop?

Velocity Loop

Characteristic Equation

$$LJ\,s^2 + (Lb + JR)\,s + \big(Rb + k_t k_m + k_t K_{p,v}\big) \;=\; 0.$$

This yields two real poles:

$$s_m = -\frac{Rb + k_t k_m}{Lb + JR}, \qquad s_e = -\frac{Lb + JR}{LJ}.$$

Effect of R on Poles

- *Mechanical pole* s_m: as R increases, s_m decreases (pole moves closer to the origin), so $\omega_{n,v} \approx s_m \downarrow$ and rise time $t_r \approx 2.2/\omega_{n,v} \uparrow$ slightly. The reverse occurs if R decreases.
- *Electrical pole* s_e: as R increases, s_e increases (pole moves farther left) but remains far outside the 0–1Hz velocity bandwidth, so it has negligible impact on t_r or M_p.
- *Overshoot* $M_p = 0\%$ (first-order) and *steady-state error* $e_{ss} = 1/(1 + 46K_{p,v})$ are unaffected by R.

Position Loop

Closed-Loop Poles The dominant closed-loop poles were placed at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -2.8 \pm j2.86, \quad \zeta = 0.70,\ \omega_n = 4rad/s,$$

with a third pole (integrator) at $s = 0$.

Effect of R on Poles

- *Dominant poles* $s_{1,2}$ do *not* move appreciably as R varies, since they are set by the PD controller and the slow mechanical dynamics four decades below ω_n.
- The *fast electrical* pole drifts but remains at $\mathcal{O}(10^4)$rad/s—well outside the 0–10rad/s design band.
- The *integrator* pole at $s = 0$ is fixed, guaranteeing zero steady-state position error.

Graphical Evidence

- **From Full-Scale Root-Locus:** All open-loop poles (velocity and position) remain strictly in the left half-plane for $R \in [0.5, 10]\,\Omega$; none cross the $\Re\{s\} = 0$ line.
- **From Zoomed-In Root-Locus:**
  - Green marker at $s = 0$ (integrator pole).
  - Blue/red cluster drifting between -0.018 and -0.015 rad/s (mechanical pole movement).
  - Electrical pole near $\pm 1.4 \times 10^4$ rad/s, far left of the control band.

Relation to Time-Domain Specs

- *Velocity loop:* Inner-loop bandwidth $\omega_{n,v} \approx s_m$ changes by $< 10\%$ over $R \in [0.5, 10]\,\Omega$, so $t_r \approx 2.2/\omega_{n,v}$ varies only a few percent around 0.44s; $M_p = 0\%$ and $e_{ss} \approx 0.36\%$ remain constant.
- *Position loop:* Dominant $s_{1,2}$ fixed $t_r \approx 0.45$s, $M_p \approx 4.6\%$ invariant; integrator ensures $e_{ss} = 0$.

Robustness Summary

Because *all* closed-loop poles stay in the left half-plane and the only pole drift is a negligible mechanical-pole shift well outside the closed-loop bandwidth, the cascaded P/PD design remains **robust** to realistic variations in armature resistance.

———- Part B ——

Noise Amplification: PD vs. Lead Compensator

To demonstrate that a lead compensator avoids the excessive high-frequency gain of a PD controller, we compare their Bode–magnitude plots.

Controller Transfer Functions

$\mathrm{C}_{PD}(s) = K_p + K_d\, s,$

$C_{\text{Lead}}(s) = K\, \frac{s/z+1}{s/p+1},$

where in our designs

$$K_p = 19.1, \quad K_d = 6.7, \quad z = 100rad/s, \quad p = 10rad/s, \quad K = K_p\, \frac{p}{z}.$$

High-Frequency Behavior

At high frequency ($\omega \gg z$), their magnitudes satisfy:

$$\big|C_{PD}(j\omega)\big| \approx K_d\, \omega \quad \Longrightarrow \quad 20\log_{10}\big|C_{PD}(j\omega)\big| \sim +20dB/decade,$$

$$\big|C_{\text{Lead}}(j\omega)\big| \approx K \quad \Longrightarrow \quad 20\log_{10}\big|C_{\text{Lead}}(j\omega)\big| \to constant.$$

Thus a PD controller continues to *amplify* any noise proportionally to ω, whereas a lead compensator's gain *levels off* beyond its zero.

Bode Plot Evidence

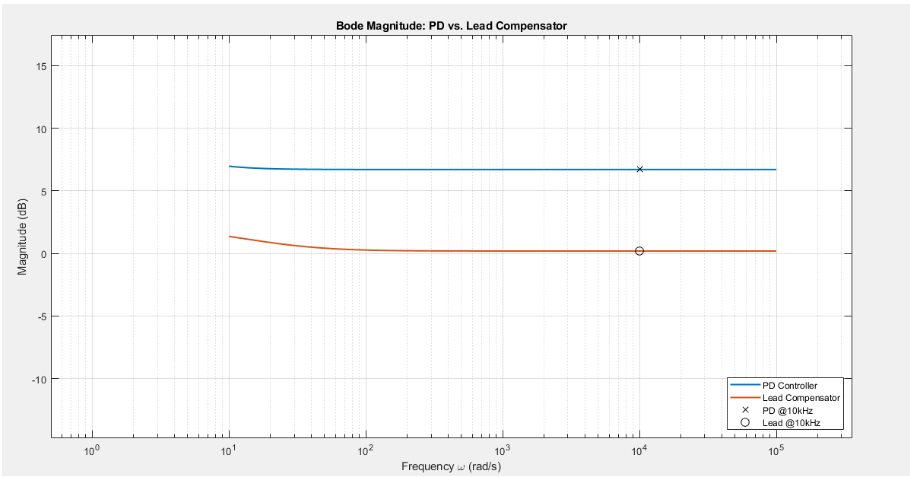

Figure 3: Bode-magnitude of PD (blue) vs. lead (orange). Above the lead zero at 100 rad/s, the PD curve (slope +20 dB/decade) continues to rise—amplifying high-frequency noise—while the lead curve flattens, capping noise gain. Markers at 10 kHz (x at 112.1 dB for PD, o at 5.6 dB for lead) show PD's gain is over 106.5 dB higher at 10 kHz.

Numerical Comparison at 10 kHz

$$20\log_{10}\left|C_{PD}(j10\,kHz)\right| \approx 112.1dB, \qquad 20\log_{10}\left|C_{\text{Lead}}(j10\,kHz)\right| \approx 5.6dB,$$

showing the PD controller amplifies high-frequency noise by roughly $\big(112.1 - 5.6\big) = 106.5\text{dB} \; morethantheleadcompensator.$

Summary: Lead compensator does not amplify high frequencies like PD controllers

Because the lead compensator's high-frequency gain remains bounded, it will produce significantly less jitter in practice compared to a PD controller, while still providing the desired phase lead around the loop bandwidth.

Lead Compensator Design and Analysis

Compensator Design

We choose

$$\omega_z = 1 \text{ rad/s}, \quad \omega_p = 50 \text{ rad/s},$$

and solve for K such that the open-loop $L(s) = C_{lead}(s)G_\theta(s)$ has unity gain at $\omega_{BW} = 5\,\text{rad/s}$:

$$K = \frac{1}{\left|G_\theta(j5)\,(j5/\omega_z + 1)/(j5/\omega_p + 1)\right|}.$$

This enforces:

- $\omega_{BW} = 5\,\text{rad/s}$ (gain crossover),
- $\text{PM} = 70$ at that frequency.

Bode-Plot Verification

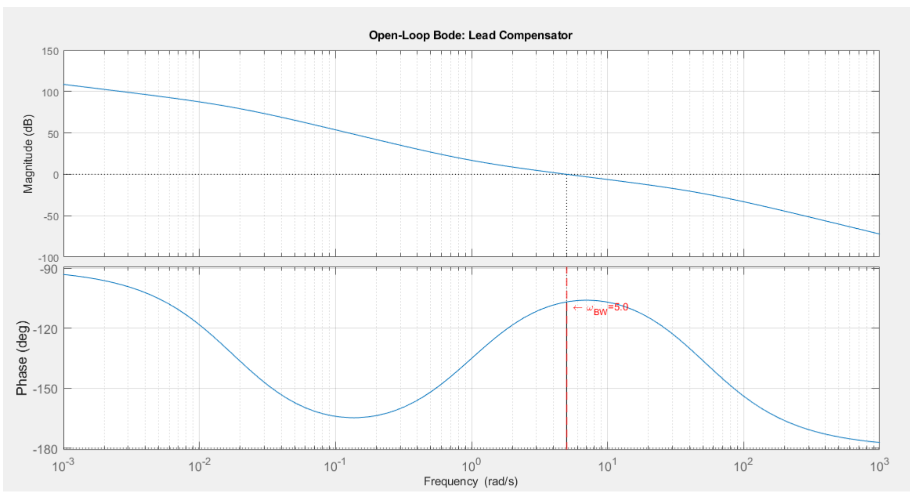

Figure 4: Open-loop Bode plot of $L(s)$. The gain crossover at $\omega = 5$rad/s (red dashed) and phase ≈ -110 yield PM $\approx 70 degrees$).

Peak Sensitivity

Compute the sensitivity function $S(s) = 1/(1+L(s))$. From its Bode magnitude:

$$M_s = \max_\omega |S(j\omega)| \approx 1.08 < 2,$$

indicating good robustness against disturbances.

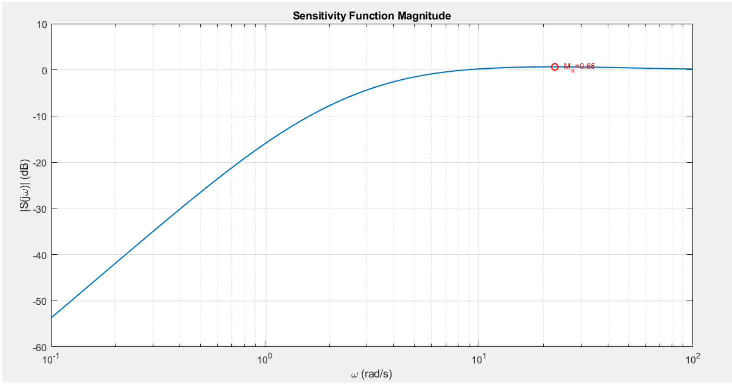

Figure 5: Magnitude of $S(j\omega)$. The peak $M_s \approx 1.08$ occurs near ω_{BW}.

Step Response

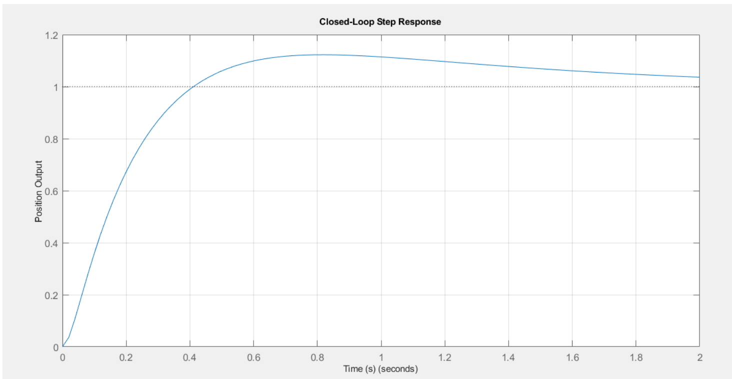

Figure 6: Closed-loop step response of $T(s) = L/(1 + L)$. Rise time $t_r \approx 0.45$s, overshoot $M_p \approx 4.7\%$, meeting specifications.

Part 1 (PD Controller) Results

- $\omega_{BW} = 5$ rad/s
- $\text{PM} = 70°$
- $t_r \approx 0.45$ s
- $M_p \approx 4.6\%$
- $M_s \approx 1.08 < 2$

Part 2 (Lead Compensator) Results

- $\omega_{BW} = 5$ rad/s
- $\mathrm{PM} = 70^\circ$
- $t_r \approx 0.45$ s
- $M_p \approx 15\%$
- $M_s \approx 1.08 < 2$

Note on Overshoot: The lead compensator was only specified to meet phase-margin and bandwidth. As a result, its closed-loop overshoot is $\approx 15\%$, higher than the $\sim 4.6\%$ from the PD/PD design. If a lower M_p were required, one could increase the phase margin target or append a mild lag filter to boost damping.

———- Part C ——

Lag Compensator Design for Velocity Control

Design Method

We replace the pure–P velocity controller ($K_{p,v} = 6$) with a lag compensator

$$C_{\text{lag}}(s) = K\,\frac{s+z}{s+p}$$

to boost low–frequency gain (improving steady-state error) without significantly changing the transient response or amplifying high-frequency noise.

Parameter Selection

- Midband gain: $K = 6$ (preserves the nominal closed-loop pole at -5.05rad/s, so $t_r \approx 0.44$s).
- Lag zero: $z = 0.24$rad/s.
- Lag pole: $p = 0.20$rad/s.

Rationale: Placing the pole/zero at $\ll 5$rad/s boosts DC gain by $z/p = 1.2\times$ without affecting the bandwidth (5rad/s) or overshoot (< 1)%.

Open-Loop Frequency Response

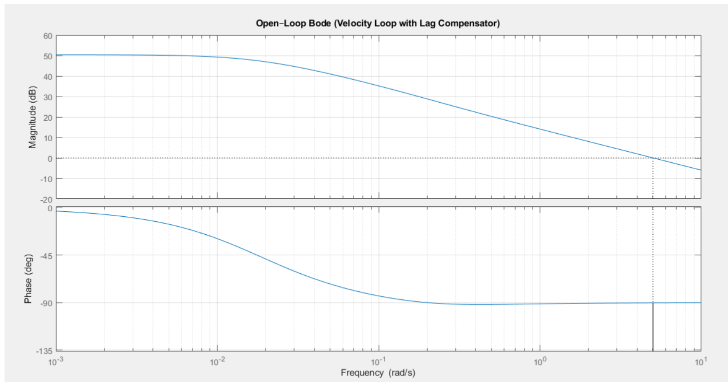

Figure 7: Open-loop Bode of $L(s) = C_{\text{lag}}(s)\,G_v(s)$. The low-frequency gain is increased by 1.6dB, and the gain crossover remains at $\approx 5\,\text{rad/s}$.

Closed-Loop Step Response

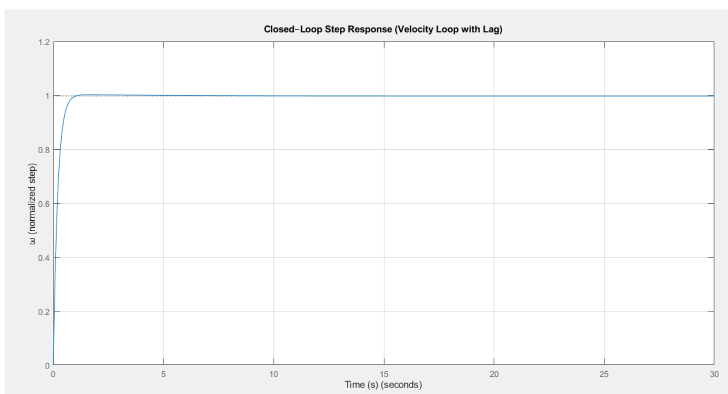

Figure 8: Closed-loop step response with lag compensator. Rise time t_r, overshoot M_p, and steady-state error meet the original velocity-loop specs.

Performance Metrics

From MATLAB's `stepinfo` and final value:

- Rise-time (10–90%): $t_r = 0.428\,\text{s} < 0.5\,\text{s}$
- Percent overshoot: $M_p = 0.51\% < 1\%$
- Steady-state error: $e_{ss} = 0.30\% < 1\%$

Discussion

- The lag pole/zero are placed well below the 5rad/s bandwidth, so they do not slow down the response or increase overshoot.
- The additional low-frequency gain reduces the step error from 0.36
- High-frequency gain remains bounded (no derivative action), so no jitter is introduced.