Vertical Frontal Circulations for Variations in Tropopause Height

Outline

- The Sawyer-Eliassen Equation
- Vertical Flow for a Sloping Tropopause
- Vertical Flow with Tropopause Curvature
- Frontal Interaction with Upper-Level Jet Streak
- Concluding Remarks

The Saywer-Eliassen Equation

- We wish to determine the structure of cross-front vertical flow
- Assume flow is either QG or SG and that flow is symmetric along-front (e.g. in y-direction) - reduces to a 2D problem in x-z plane.
- Streamfunction ψ for nongeostrophic flow (u_{ag}, w) satisfies the Sawyer-Eliassen (SE) equation(s)
- The analysis here closely follows Hakim and Keyser (2001)

- QG: $\boxed{N_0^2\psi_{xx} + f^2\psi_{zz} = -2Q}$
- SG: $\boxed{N^2\psi_{xx} - 2S^2\psi_{xz} + F^2\psi_{zz} = -2Q}$

  $N^2 = N_0^2 + b_z,\ S^2 = fv_{gz} = b_x,\ F^2 = f(f + v_{gx})$

  (b is the buoyancy)
- $Q = -f\frac{\partial(u_g,v_g)}{\partial(x,z)}$ + diabatic and cross-plane (along-front) momentum sources
- $Q > 0$ represents a frontogenetic source, and the SE equation determines the necessary flow to maintain thermal wind balance
- We assume that these sources are highly localized, as if point sources (i.e. δ-functions)

- For QG, scale z by some height H and x by Rossby radius $L_{QG} = N_0 H/f$; Q and ψ are rescaled to be $O(1)$
- SE equation becomes $\psi_{xx} + \psi_{zz} = -2Q$
- For SG, change coordinates to $(X, Z) = (x + v_g/f, z)$, and scale in a similar manner
- SE equation becomes $\psi_{XX} + \psi_{ZZ} = -2Q$
- Both cases reduce to Poisson's equation, which can be solved in a number of ways

- For a localized source Q at $\mathbf{r_o}$ we must solve
  $\psi_{xx} + \psi_{yy} = -2Q_0\delta(\mathbf{r} - \mathbf{r_o})$
- This is a Green function problem, whose solution in 2D is
  $\psi = -\frac{Q_0}{\pi} \log |\mathbf{r} - \mathbf{r_o}|$
- The flow due to a collection of sources can be determined by adding together the contributions from each source

Vertical Flow for a Sloping Tropopause

- We model the effects of sloping by considering a wedge-shaped region
- We assume $\psi = 0$ at the sides so that $\psi = 0$ everywhere if no sources are present
- To determine ψ due to a point source, we use the method of images

The Method of Images

- Method relies on uniqueness: Any solution that satisfies the boundary conditions is the only possible solution
- For a flat plane with a source Q above, it is balanced by an imaginary source $-Q$ directly below, thereby setting $\psi = 0$ along the boundary; ψ must be equivalent to this configuration

- We can extend this to a wedge, if edge angle is $\theta = \frac{2\pi}{n}$ and n is an even integer (In figure, $n = 8$)
- Images can't be used for an arbitrary angle, but an analytic solution exists
- For simplicity, we restrict ourselves to cases supporting images

- Using this method, the solution for ψ due to a single source under the QG approximation in nondimensional form is

$$\psi = -\frac{Q}{\pi}\sum_{k=0}^{\frac{n}{2}-1}\log\left[\frac{(x-a\cos(\frac{4\pi k}{n}+\gamma))^2+(y-a\sin(\frac{4\pi k}{n}+\gamma))^2}{(x-a\cos(\frac{4\pi k}{n}-\gamma))^2+(y-a\sin(\frac{4\pi k}{n}-\gamma))^2}\right]$$

- For SG, approximate v_g by a Taylor expansion about source location (x_i, z_i) and use with $F^2 = f(f+v_{gx})$ and $S^2 = fv_{gz}$, obtaining

$$(X - X_i) = \frac{F^2}{f^2}(x - x_i) + \frac{S^2}{f^2}(z - z_i)$$

- This equation, along with $Z - Z_i = z - z_i$ are used on the dimensional form of ψ to determine SG streamfunction

QG Streamfunctions

SG Streamfunctions

Vertical Flow with Tropopause Curvature

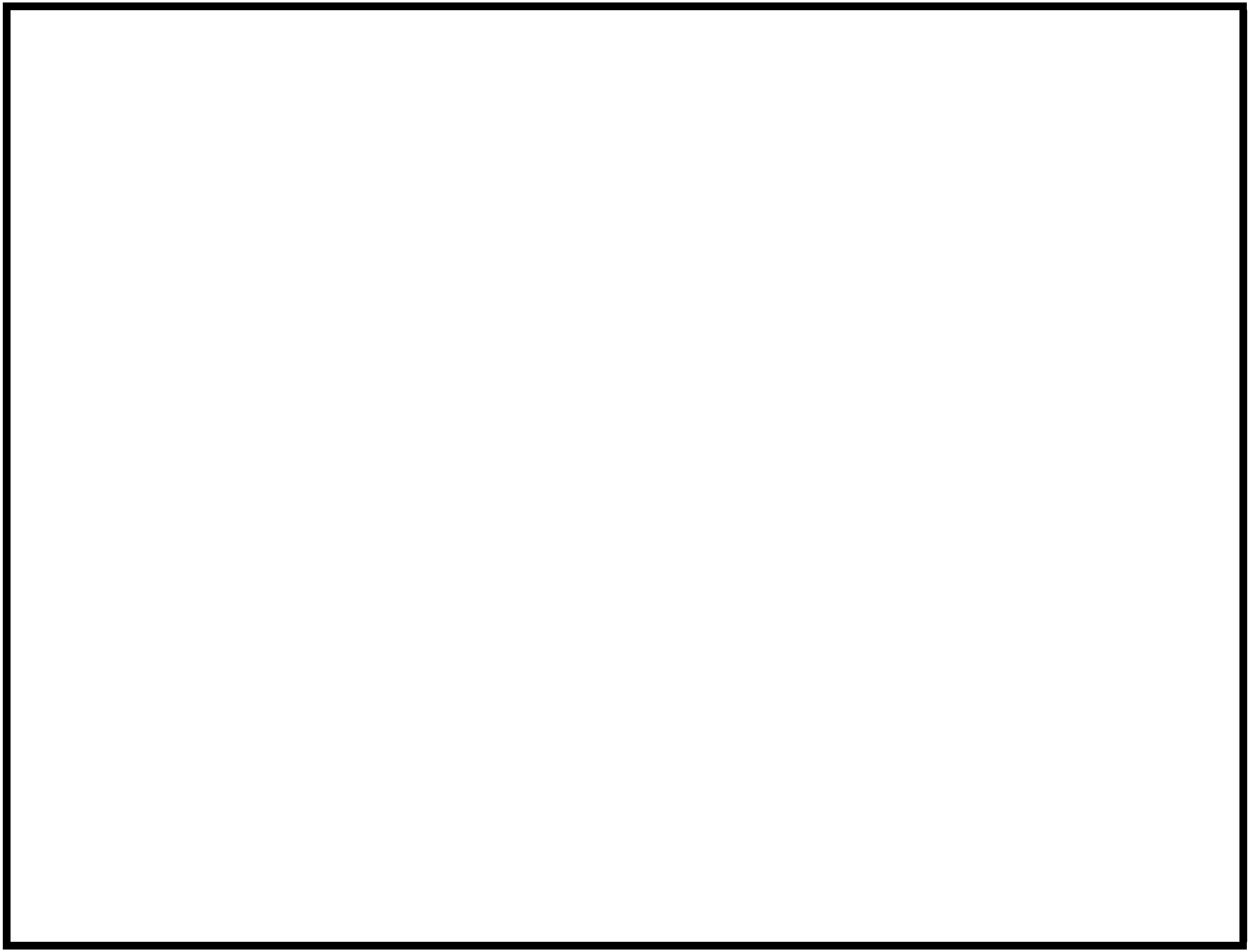
- Curvature effects are modeled by considering a circular cross-section of radius a that sweeps an angle of $2\theta_0$

- This problem is greatly simplified by changing coordinates; we reflect points across a circle of radius $l = 2a \sin \theta_0$, the length of the bottom boundary, by the formula $R = l^2/r$
- Then $(x', z') = l^2 \left(\frac{x}{x^2+z^2}, \frac{z}{x^2+z^2} \right)$
- This is related to a *conformal mapping*, so Poisson's equation holds in both coordinate systems
- The circular arc is transformed into a wedge, which we already have a solution for
- The arc problem has a complete analytic solution!

QG Streamfunctions

SG Streamfunctions

Frontal Interaction with Upper-Level Jet Streak



Concluding Remarks

- Analytic structure of flow can be determined for simple variations in tropopause height
- Slope (first order) and curvature (second order) effects were successfully solved; higher order curvature effects may have analytic solutions as well
- These methods can also apply to variations in topography
- There is potential to produce improved accuracy in vertical frontal flows without appealing to numerical PDE solutions
- Topographic or tropopause variations may generate Rossby waves - such an effect could lead to energy dissipation and frontolytic effects; this would require an unsteady flow analysis