Thesis Proposal.

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Introduction

A natural question that arises in a diverse set of disciplines is how, given an initial distribution $f_{t_0}(x)$, the probability density $p_t(x)$ of a given diffusion process X_t evolves for $t > t_0$. For example, the price of a stock is often modeled as a geometric Brownian motion, and financial market participants may be interested in the distribution of the stock price at some point in the future, given its price today.

The Fokker-Planck partial differential equation associated with a given diffusion process is a key result in answering this question because its solution is the transition density of the diffusion. In particular, for a diffusion process represented by the stochastic differential equation $dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t$, where W_t is a Wiener process, the transition density $p = p(x, t \mid x', t_0)$ for $t > t_0$ of the process X_t is the solution to

$$\frac{\partial}{\partial t} p = -\frac{\partial}{\partial x}\left[\mu(x,t)\,p\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(X_t, t)\,p\right]$$

with initial condition $p(x, t_0 \mid x', t_0) = \delta(x - x')$ [4]. With the transition density p, we can recover the probability density $p_t(x)$ of X_t by computing

$$p_t(x) = \int p(x, t \mid x', t_0)\, f_{t_0}(x')\, dx'$$

Note that for initial distribution $f_{t_0}(x) = \delta(x - x_0)$, we trivially have that the density $p_t(x)$ is equal to the transition density $p(x, t \mid x_0, t_0)$.

Motivation

An extension to the motivating question of the Fokker-Planck equation above is to ask how, given an initial distribution $f_{t_0}(x)$ and final distribution $g_T(x)$, the probability density $p_t(x)$ of a given diffusion process X_t evolves for $t_0 < t < T$. Answering this question is of significant interest in a variety of fields.

For example, in macroeconomics, the European Central Bank releases a discretized distribution of survey responses from professional forecasters for the values of various macroeconomic variables such as inflation and unemployment, at the end of the current year t, and the end of years $t+1, t+2$, and $t+4$ [1]. Given a diffusion model for inflation, a distribution of interest may be the implied inflation distribution at the end of year $t+3$.

Lines of Research

Deriving Densities for Diffusion Bridges The line of research proposed can be seen as a generalization of the study of the Brownian bridge. Recall a Brownian bridge B_t is defined as $B_t := W_t \mid W_1 = 0, t \in [0,1]$ for a standard Wiener process W_t; in other words, a Brownian bridge is simply a Wiener process conditioned on starting and ending values [2]. The motivating question above can be seen as asking how we can study the densities of general diffusion processes, conditioned on starting and ending distributions - in intuitive terms, we are interested in "density bridges".

The simplest case begins with studying the density of a Brownian bridge. It turns out that a Brownian bridge can be represented as an unconditional SDE [2], allowing its transition density to be analytically determined through traditional Fokker-Planck solution methods. While this transition density is already known, my derivation using the Fokker-Planck equation seems to be an unorthodox way of arriving at its form [3, 5].

I hope in my thesis to explore the possible following directions (not necessarily listed in order, and not necessarily with the intention of completing every single item):

1. Find a class of initial and final distributions for which an analytical solution to the density $p_t(x)$ of a Brownian bridge B_t exists.
2. Find the transition density $p(x, t \mid x', t_0)$ of an Ornstein-Uhlenbeck bridge X_t.
   (a) Then, find a class of initial and final distributions for which an analytical solution to the density $p_t(x)$ of an Ornstein-Uhlenbeck bridge X_t exists.
3. Find a class of diffusion bridges X_t which admit an analytical solution for their transition densities $p(x, t \mid x', t_0)$.
   (a) Then, find a class of initial and final distributions for which an analytical solution to the density $p_t(x)$ of the diffusion bridge X_t exists.

The Ornstein-Uhlenbeck process is of particular interest, since its bridge, like the Brownian bridge, admits an unconditional diffusion representation [2] (and therefore whose density can be approached using the Fokker-Planck equation), and the Ornstein-Uhlenbeck process is commonly used to model inflation, unemployment, interest rates, and other variables of interest. The third direction is uniquely challenging, because without an unconditional diffusion representation, I will need to develop technology that gives something like a "forward and backward transition" or "bridge" density $p(x, t \mid x_{t_0}, t_0, x_T, T)$ where $t_0 < t < T$. I have not completed enough research to assess whether this direction is feasible, but I think the first and second points will be fruitful.

Developing a Sampler A somewhat more computational direction I could take this research is towards devising a scheme to sample from $p_t(x)$ for some diffusion X_t, given initial distribution $f_{t_0}(x)$, final distribution $g_T(x)$, and $t < t_0 < T$. I propose a simulation scheme in Algorithm 1, and hope to prove methodologically that it correctly samples from the marginal distribution of interest $p_t(x)$.

Algorithm 1 Sampler for $p_t(x)$ for a diffusion X_t, conditional on initial and final distributions f_{t_0} and g_T

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1. For $n = 1, \ldots, N$:
   (a) Sample (x, y) such that $x \sim f_{t_0}$ and $y \sim g_T$.
   (b) Simulate a bridge $X_t^{(n)}$ from x to y, following the diffusion dynamics specified, for $t_0 < t < T$.
   (c) Store $X_{t_0}^{(n)}$.
2. Return $X_{t_0}^{(1)}, \ldots, X_{t_0}^{(N)}$.

Timeline

As outlined in the Statistics thesis handbook, I plan to have a literature review completed by November 10. I hope to have outlined solutions to points 1. and 2. by the new year, and have an understanding whether goal 3. is feasible. If it is, I will spend the new year developing technology to achieve goal 3. Otherwise, I will attempt to develop a sampler of $p_t(x)$ as described above and prove its correctness.

References

[1] European Central Bank. Ecb survey of professional forecasters. `www.ecb.europa.eu/stats/spf/`.

[2] Yongxin Chen and Tryphon Georgiou. Stochastic bridges of linear systems. *Automatic Control*, PP, 2015.

[3] Crispin Gardiner. *Stochastic Methods: A Handbook for the Natural and Social Sciences*. Springer Berlin Heidelberg, 2009.

[4] Hannes Risken. *The Fokker-Planck Equation: Methods of Solutions and Applications*. Springer, 1996.

[5] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Science & Business Media, 2004.