The Equilibria of the Rank-Sum Mechanism

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Abstract

The University of Waterloo uses a version of a priority matching mechanism, known as the rank-sum mechanism, to match thousands of students to co-op employers every term. This mechanism was proposed as a method to achieve Waterloo's policy goals of efficiency (matching students and employers who want each other the most), strategy-proofness, and balance between student and employer interests. We show that the rank-sum mechanism is neither efficient nor strategy-proof, and that in equilibrium, any matching between students and employers may be an outcome of such a mechanism. This suggests that the rank-sum mechanism may as well be a totally random matching mechanism. Furthermore, we conjecture that Waterloo's policy goals are simultaneously unattainable—an extension of the result of Zhou (1990) for one-sided matching markets—and that the concept of a balanced mechanism necessarily leads to mechanisms which are "meaningless" i.e. have any possible matching as a potential equilibrium outcome.

1 Introduction

1.1 The Rank-Sum Mechanism, and Terminology

The University of Waterloo matches students to employers for co-op terms using a slightly modified version of what we refer to as the rank-sum mechanism. A stylized version of this mechanism proceeds as follows, for N students and employers. Each student assigns each employer a unique rank in $1, \ldots, N$; similarly, each employer assigns each student a unique rank in $1, \ldots, N$. For each student-employer pair, the mutual ranks are added to form the rank-sum of the pair. Then, in increasing order of rank-sum, matches are made.

In other words, first, any $1-1$ mutually ranked pair is matched. These matched students and employers are removed from the system, and any remaining $1-2$ and $2-1$ mutually ranked pair is matched, and so on, until all students and employers are matched. Ties in rank-sum are broken randomly.

We say a matching mechanism is **efficient** if the outcomes of the mechanism are such that no one can be re-assigned at a personal gain without making someone else worse off. We say a matching mechanism is **strategy-proof** if it is always in the best interest of every agent to report their true preferences.

1.2 Preliminaries

Consider a one-to-one matching market of students S and employers E, where students and employers have known, strict, and complete ordinal preferences P and Q respectively over each other, and $|S| = |E| = N$. Under the rank-sum mechanism, an outcome of this matching market is known as a random matching. Formally, a random matching is a random variable $\tilde{\mu} : \Omega \to M$ which realizes a value in the space of matchings M, where a matching $\mu \in M$ is a bijective function $\mu : S \to E$ from the set of students to the set of employers. We will refer to a random matching as a deterministic matching if the random matching realizes a particular matching with probability 1.

Since outcomes of the rank-sum mechanism are random matchings, we must have a way for agents to compare one random matching to another.

Definition 1.1 (Reasonable stochastic P_s-ordering)**.** A reasonable stochastic P_s-ordering $\rhd_{P_s}$ is an ordering over random matches which has its ordering over deterministic matches induced by P_s, and satisfies $\tilde{\mu}_i \rhd_{P_s} \bar{\tilde{\mu}}_{j\leq i}$ where $\tilde{\mu}_i$ deterministically matches s to her ith most favored choice under P_s and $\bar{\tilde{\mu}}_{j\leq i}$ matches s to her jth most favored match under P_s, for $j \leq i$, with equal probability.

We say a random matching $\tilde{\mu}$ stochastically P_s-dominates a random matching $\tilde{\nu}$ in a reasonable way if $\tilde{\mu} \rhd_{P_s} \tilde{\nu}$ for some reasonable stochastic P_s-ordering $\rhd_{P_s}$.

We say a strategy $\mathbb{P}_s$ stochastically P_s-dominates another strategy $\mathbb{P}'_s$ if, for any true preferences (P, Q) and any strategy profile $(\mathbb{P}_{-s}, \mathbb{Q})$, the resulting random matching of $(\mathbb{P}_{-s}, \mathbb{P}_s, \mathbb{Q})$ stochastically P_s-dominates $(\mathbb{P}_{-s}, \mathbb{P}'_s, \mathbb{Q})$ in a reasonable way.

Note that the ordering induced by first-order stochastic P_s-dominance is a reasonable stochastic P_s-ordering.

1.3 Summary of Claims

1. **The rank-sum mechanism is niether efficient nor strategy-proof.** We demonstrate in simulation that there are outcomes of the rank-sum mechanism in which the number of apparent $1-1$ matches is far greater than the number of true $1-1$ matches. Furthermore, we demonstrate that in equilibrium, it is almost never in an agent's interest to tell the truth.

2. **No dominant strategy for any agent.** The first result below says that there is no dominant strategy for any agent in the rank-sum game. The intuition for this result comes from the fact that the mechanism is balanced in some sense—even if I say a particular employer is my favorite, other students and employers can rank each other in a way that completely removes information about my statement of favoritism from the game. So, what I play depends heavily on what everyone else in the game plays.

3. **All deterministic matchings are possible in Nash equilibrium.** The second result below says that for a fixed true preference profile, all deterministic matchings are a weak subset of the set of Nash equilibrium outcomes. The intuition here is that since no agent has a dominant strategy, we have a ton of degrees of freedom to manipulate their reports, and since rank-sum guarantees $1-1$ matches, we can create situations where any particular agent literally cannot change their outcome in equilibrium.

2 Proofs of Claims

Claim 2.1. There is no stochastically dominant strategy for any agent in the preference revelation game induced by the rank-sum mechanism.

Proof. Consider a student s. We show that for any strategy $\mathbb{P}_s$ and any true preferences (P, Q), there exists a strategy profile $(\mathbb{P}_{-s}, \mathbb{Q})$ such that $\mathbb{P}_s$ does not stochastically P_s-dominate all other strategies $\mathbb{P}'_s$.

Fix (P, Q) as true student and employer preferences. We will demonstrate that there is no strategy $\mathbb{P}_s$ that, for any submitted preference profile $(\mathbb{P}_{-s}, \mathbb{Q})$, stochastically P_s-dominates all other strategies $\mathbb{P}'_s \neq \mathbb{P}_s$ in a reasonable way. Consider the strategy profile $\left(\mathbb{P}'_{-s}, \mathbb{Q}'\right)$ such that all employers rank s as their first choice. Then, the strategies in which s ranks her most favored employer as her first choice are the only strategies that stochastically P_s-dominate all other strategies in a reasonable way. Call the set of these strategies ρ. We proceed to construct a strategy profile $\left(\mathbb{P}'_{-s}, \mathbb{Q}'\right)$ such that there exists a strategy $\mathbb{P}'_s$ that $\mathbb{P}_s \in \rho$ does not stochastically P_s-dominate $\mathbb{P}'_s$ in a reasonable way.

To this end, we fix $\mathbb{P}_s \in \rho$. First, let $r\left(e; \mathbb{P}_s\right)$ be the rank that student s assigns to employer e under the strategy $\mathbb{P}_s$, and accordingly $r^{-1}\left(n; \mathbb{P}_s\right)$ be the employer to which s assigns the rank of n under $\mathbb{P}_s$. Then, let every $s' \neq s$ rank $r^{-1}\left(1; P_s\right)$ as her first choice. Finally, each s' must rank the remaning $N-1$ employers with a unique "shift" of $[2, \ldots, N]$ e.g. $[N, 2, \ldots, N-1]$, $[N-1, N, 2, \ldots, N-2]$, $[N-j, \ldots N, 2, \ldots, N-j-1]$. No two students can have the same "shift".

Now, pick the student ς for whom $r\left(e_i; \mathbb{P}'_\varsigma\right) = r\left(e_i; \mathbb{P}_s\right)$ for $2 \leq i \leq N$, and let $r^{-1}\left(1; P_s\right)$ rank ς as her first choice, ranking the other students in an arbitrary order. Let every other employer $e \neq r^{-1}\left(1; P_s\right)$ rank ς as their Nth choice, and otherwise assign ranks so that the rank-sums are equal to $N+1$. It is easily verified that the resulting preference profile $\mathbb{Q}'$ is a valid one.

Under $\left(\mathbb{P}'_{-s}, \mathbb{P}_s, \mathbb{Q}\right)$, s has an equal chance of being matched with employers $e_2, \ldots, e_N$. However, note that if s submits a report $\mathbb{P}'_s$ such that $r^{-1}\left(2; P_s\right)$ is ranked first, and $r^{-1}\left(1; P_s\right)$ is ranked second, s can guarantee a match with e_2. Therefore, $\mathbb{P}_s$ does not stochastically P_s-dominate $\mathbb{P}'_s$ in a reasonable way.

Since we chose $\mathbb{P}_s \in \rho$ arbitrarily, we conclude that there is no single strategy $\mathbb{P}_s$ that stochastically P_s-dominates all other strategies for all strategy profiles $(\mathbb{P}_{-s}, \mathbb{Q})$ in a reasonable way. s was also chosen arbitrarily, so no student has a stochastically dominant strategy. Exactly the same argument follows for a given employer e, and so we conclude that no agent has a stochastically dominant strategy. $\square$

The fact that no agent has a stochastically dominant strategy allows us to show a result that demonstrates the issues associated with the rank-sum mechanism (we assume agents make decisions about the strategies they play based on a reasonable stochastic ordering).

Claim 2.2. Fix any true student and employer preference profiles, and consider the preference revelation game induced by the rank-sum mechanism. Then, the set of possible deterministic matchings is a weak subset of the set of outcomes induced by the Nash equilibrium strategies in this game.

Proof. We first show that any possible deterministic match is also a match that can be induced by a Nash equilibrium profile of the preference revelation game, with respect to fixed true preferences (P, Q). Note that we can consider any valid preference profiles since there exist no stochastically dominant strategies for any agent. Consider any deterministic matching $\tilde{\mu}$ such that $\tilde{\mu}(s_i) = e_j$ with probability 1. Consider any preference profile $(\mathbb{P}, \mathbb{Q})$ in which s_i and e_j rank each other as their first choices. This preference profile guarantees s_i is matched with e_j under the rank-sum mechanism as desired.

Furthermore, this preference profile is a Nash equilibrium with respect to the true preferences P and Q, since for any alternative strategy profile $\mathbb{P}'_s$ resulting in outcome $\tilde{\nu}$, $\tilde{\nu}(s') = \tilde{\mu}(s')$ with probability 1 for all $s' \in S_{-s}$ since $(\mathbb{P}_{-s}, \mathbb{Q})$ is fixed. Then, every student but s is matched with her original match, which forces s to be matched with her original match, and s cannot achieve any other outcome, let alone a better outcome, by deviating.

This demonstrates that the set of possible deterministic matchings is a weak subset of the Nash equilibrium outcomes of the game induced by the rank-sum mechanism for any (P, Q).

Now, we show that there exists at least one non-deterministic random matching that can be induced by a Nash equillibrium strategy for some true student and employer preferences. Consider any strategy profile $(\mathbb{P}, \mathbb{Q})$ that induces a random matching in which every student and every employer is matched to each other with equal probability. It is easy to verify such a profile exists. Then, the true preferences $(P, Q) = (\mathbb{P}, \mathbb{Q})$ allow such a strategy profile to be a Nash equilibrium, and we see that there is some non-deterministic random matching which is also a Nash equilibrium outcome with respect to some set of true preferences. Since any possible deterministic match is also a Nash equilibrium outcome with respect to these preferences, we see that the set of possible deterministic matches is a strict subset of the Nash equilibrium outcomes for at least one true preference profile. $\square$

3 Some Thoughts

1. **Balance and the priority matching mechanisms.** The proofs above seem to be easily generalizable to a broader class of priority matching

mechanisms. For example, in the proof of the first claim, we really need only that the priority matching mechanism has a rank-pair that guarantees the match is made (in this case, $(1, 1)$ does, and more generally, for any monotonic matching mechanism, $(1, 1)$ does as well) and that the students and employers can create a bunch of equal-priority matches (in this case, they ranked each other such that the rank-sums were $N + 1$). The second proof only requires that the priority matching mechanism has a rank-pair that guarantees the match is made. We think it is encouraging that monotonic matching mechanisms satisfy this property, and we think something like, or even broader than, a commutative operation on student and employer rank-sums will guarantee the second property.

2. **Balance and the two-sided Boston mechanism.** It seems that the difference between the rank-sum mechanism, in which all deterministic matchings are Nash equilibrium outcomes, and something like the two-sided Boston mechanism, in which all stable matchings are Nash equilibrium outcomes in undominated strategies as per Ergin and Sonmez (2006), is the fact that rank-sum looks the same from students' and employers' points of view. The two-sided Boston mechanism has truthtelling as a dominant strategy for schools, which prevents huge number of degrees of freedom that are necessary to achieve any possible deterministic match in equilibrium. It seems like this points towards some notion of "balance", and how "balance" might have some unintended consequences by giving in some sense, too much power, to both sides.

3. **Balance and the theory of one-sided matching mechanisms.** Zhou (1990) shows that there is no symmetric, strategy-proof, and Pareto efficient one-sided matching mechanism, a result tantalizingly close to what we would want to show about Waterloo's demands for a balanced, strategy-proof, efficient two-sided matching mechanism. Zhou's definition of symmetry/anonymity seems like a reasonable projection onto the one-sided matching mechanism space of a more general notion of balance or symmetry (i.e. since all agents "have the same say" in detemining matches, if two agents have the same preferences in a one-sided market, they ought to be assigned the same random matching), but we haven't been able to really find anything more, and haven't really been able to think about how to generalize the notion of symmetry/anonymity to the two-sided space, although it seems like this notion must hold within-sides in the two-sided case.