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On a Conjecture by Gale about One-Sided Matching Problems

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This paper proves the following result on one-sided matching problems: when there are n objects to be assigned to n agents, for $n \geqslant 3$, there exists no mechanism that satisfies symmetry, Pareto optimality, and strategy-proofness. Examples of mechanisms are presented to show the independence of the conditions, which also illustrate the well-known tradeoff between equity and efficiency in the framework of matching problems. Finally, some extensions of the result to more general matching problems are considered. *Journal of Economic Literature* Classification Numbers: 022, 025.

1. INTRODUCTION

Many decision problems involve the assignment of a set of objects with finite capacities to some agents. I will refer to them as one-sided matching problems, or simply matching problems. Examples of such problems include the assignment of office spaces to faculty members, the matching of first-year undergraduate students to different micro-economics classes, etc. In practice, these problems are usually dealt with by means of different ad hoc rules. Are there better or more systematic ways of dealing with them? Hylland and Zeckhauser [7] investigated a matching problem in detail and proposed a procedure for solving it. However, like Gale and Shapley's [3] work on a similar problem (the marriage problem), Hylland and Zeckhauser's work does not explicitly consider the incentive problems associated with the revelation of agents' private information.[1] Recently,

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[1] In fact, Hylland and Zeckhauser were aware of the problem of eliciting true preferences of the agents. However, they did not address it directly. Instead they proposed a procedure

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Gale [2] raised the issue of whether a "nice" solution can be found for matching problems when the agents' strategic behavior is taken into consideration.

Informally, a mechanism is a rule for matching objects to agents, and matchings are permitted to be random in the sense that a matching may be a probability distribution over pure matchings. We are concerned with the existence of mechanisms that lead to satisfactory matchings. A standard requirement for a mechanism to be satisfactory is Pareto optimality. Obviously, Pareto optimality is not difficult to achieve if one knows the preferences of all agents over objects. However, in most cases true preferences are known to agents themselves only; therefore, a mechanism has to work with the announced, rather than the true, preferences. When agents realize that the preferences they announce will be used to determine the outcome, they will in general misrepresent their true preferences in order to obtain outcomes that are more favorable to them. This phenomenon, and the possibility of designing mechanisms in its presence, have been extensively investigated by many researchers during the last 20 years (e.g., Gibbard [4, 5], Groves and Ledyard [6]). The challenge is to design mechanisms that lead to Pareto optimal matching and yet do not require an agent to report his true preferences when it is not in his interest to do so. Mechanisms which induce truthful reporting of preferences as optimal actions are called strategy-proof. For matching problems at hand there are mechanisms that are both Pareto optimal and strategy-proof (see Example 4 in the next section). Nevertheless, as observed by Gale, they fail to treat agents with identical preferences equally. Under these mechanisms, although two agents have exactly the same preferences, they do not necessarily have equal chances of getting various objects. In other words, they fail to satisfy the property of anonymity, a condition that is considered to be an important characteristic of fairness. Gale further conjectured that for matching problems in which there are equal numbers of agents and objects, and this number is greater than two, there exists no mechanism that satisfies anonymity, Pareto optimality, and strategy-proofness.

The purpose of this paper is to discuss issues related to Gale's conjecture. Section 2 is devoted to formal analysis, in which I prove a result that is even stronger than Gale's original conjecture. Section 3 deals with some extensions of the analysis in Section 2 to more general matching problems.

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which, under the assumption that each individual truthfully reveals his preferences, leads to a "pseudomarket" allocation. Then they asserted that this further justifies their adopted assumption, at least when there are many agents. However, in our view, this is hardly a valid argument. On the other hand, the strategic aspect of Gale and Shapley's marriage problem is discussed in Roth [10].

2. Gale's Conjecture

We start with the simplest matching problem. There are n agents and n objects. The set of agents is $\mathbf{A} = \{A_1, A_2, ..., A_n\}$, and the set of objects $\mathbf{O} = \{O_1, O_2, ..., O_n\}$. A *pure matching* p is a bijection from $\mathbf{O}$ to $\mathbf{A}$. It can be represented by a permutation matrix $P = (p_{ij})$, in which $p_{ij} = 1$ if p assigns the ith object to agent j, and $p_{ij} = 0$ otherwise. There are $n!$ different pure matchings. A *mixed matching* m is a lottery over pure matchings, $m = (m_1, m_2, ..., m_{n!})$, with $\sum_k m_k = 1$ and $m_k \geqslant 0$ for all k.

Each agent is assumed to be selfish; that is, he only cares about which object he receives. Each agent thus has a *von Neumann–Morgenstern utility vector* $\mathbf{u} = (u_1, u_2, ..., u_n)$ over $\mathbf{O}$. These vectors are normalized so that $\max u_i = 1$, and $\min u_i = 0$. A utility matrix $U = (u_{ij})$ is an n square whose ith row $\mathbf{u}_i$ is the ith agent's normalized von Neumann–Morgenstern utility vector. The set of all such U's is denoted by $\mathbf{U}$. For any U in $\mathbf{U}$ and any $\mathbf{u}_i'$ a utility vector for agent i, $U \mid \mathbf{u}_i'$ represents the matrix obtained from U by replacing $\mathbf{u}_i$ with $\mathbf{u}_i'$.

When a mixed matching m is used, each agent i evaluates his expected utility by the following formula:

$$E_i = \sum_k m_k \left(\sum_j p_{ji}^k u_{ij} \right) = \sum_j \left(\sum_k m_k p_{ji}^k \right) u_{ij}.$$

Hence two mixed matchings m and m' are payoff equivalent if and only if the following two matrices are the same:

$$\sum_k m_k P^k = \sum_k m_k' P^k.$$

Since it is assumed that the agents only care about their own expected payoffs, two mixed matchings are regarded the same if they are payoff equivalent. Therefore, we identify each mixed matching m with the corresponding matrix $L = \sum_k m_k P^k$. It is easy to verify that such a matrix must be a *bi-stochastic matrix*, in which all entries are nonnegative, and all row sums and columns sums are equal to 1. On the other hand, by a well-known theorem of Birkhoff [1], any bi-stochastic matrix can be decomposed as a nonnegative weighted sum of some permutation matrices. Hence we will think of $\mathbf{L}$, the set of all bi-stochastic matrices, as the set of all mixed matchings. Given any $L = (l_{ij})$ in $\mathbf{L}$, l_{ij} is the probability that the ith object goes to the jth agent, hence the jth column $\mathbf{l}^j$ is the jth agent's probability distribution over $\mathbf{O}$.

Given a utility matrix U and a mixed matching L, the ith agent's expected utility under L is $u_i(L) = \sum_j u_{ij} l_{ji} = \mathbf{u}_i \mathbf{l}^i$. L is Pareto optimal for U

if there is no other L' in $\mathbf{L}$ such that $u_i(L') \geqslant u_i(L)$ for all agent i, with at least one strict inequality for some agent.[2]

An *allocation mechanism*, or simply a *mechanism*, is a mapping σ from $\mathbf{U}$ to $\mathbf{L}$, which specifies a mixed matching $L = \sigma(U)$ for each utility matrix U.

We consider some basic properties of a mechanism σ. The first concerns its efficiency.

DEFINITION 1. A mechanism σ is *Pareto optimal* if, for any U in $\mathbf{U}$, $\sigma(U)$ is Pareto optimal.

We are also interested in a mechanism's equity performance. Conventionally, we say that a mechanism σ is *anonymous* if for any U and any U' obtained from U by a permutation of row vectors, σ maps U' to L', which is obtained from $L = \sigma(U)$ by applying the same permutation to column vectors. Gale [2] used only part of the above condition; he called a mechanism σ anonymous if for any U in $\mathbf{U}$, any pair of agents i and j, $\mathbf{u}_i = \mathbf{u}_j$ implies $\mathbf{l}^i = \mathbf{l}^j$, where $L = \sigma(U)$. However, from a utilitarian point of view, an even weaker condition is more appropriate to capture the idea of fairness.

DEFINITION 2. A mechanism σ is *symmetric* if for any U in $\mathbf{U}$, any pair of agents i and j, $\mathbf{u}_i = \mathbf{u}_j$ implies $u_i(L) = u_j(L)$, where $L = \sigma(U)$.

The last condition deals with the agents' strategic behavior. In general, individuals are not expected to reveal true preferences unless a mechanism provides them with the right incentive. Serious problems arise when agents start to misrepresent their preferences. For example, a mechanism which is both efficient and equitable when applied to announced preferences is no longer so with respect to true preferences (see Muller and Satterthwaite [9] for a more detailed discussion). To avoid such problems, we impose the following requirement on a mechanism:

DEFINITION 3. A mechanism σ is *strategy-proof* if, for any U in $\mathbf{U}$, any agent i, and any utility vector $\mathbf{u}_i'$,

$$u_i(\sigma(U)) \geqslant u_i(\sigma(U \mid \mathbf{u}_i')).$$

A mechanism would be ideal if it could possess all three properties. However, our main theorm shows that such a mechanism does not exist:

[2] More precisely, what we have defined is the concept of ex ante Pareto optimality. Also related is a weaker concept—ex post Pareto optimality: a mixed matching is ex post Pareto optimal if each pure matching in its support is Pareto optimal. Since mixed matchings are considered in the paper, it is natural that ex ante Pareto optimality is chosen as a welfare criterion.

THEOREM 1. *When $n \geqslant 3$, there is no mechanism that satisfies symmetry, Pareto optimality, and strategy-proofness.*[3]

Proof. We start with the case of $n=3$. The proof proceeds as follows. Assuming that there exists such a mechanism σ, we will be able to determine the values of σ for some utility matrices by using the restrictions imposed on σ by the three properties. However, the resulting mechanism σ violates strategy-proofness.

Let a, b, c be such that $0<a<b<c<\frac{1}{6}$. We look at several utility matrices and the mixed matchings assigned to them by σ. First we consider

$$U_1=\begin{pmatrix} 1 & b & 0 \\ 1 & b & 0 \\ 1 & b & 0 \end{pmatrix} \rightarrow L_1=\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \tag{1}$$

Although we cannot determine the exact values of all entries of L_1, we can establish certain relationships among them. Since $p_{11}+p_{12}+p_{13}+b(p_{21}+p_{22}+p_{23})=1+b$, symmetry implies

$$p_{11}+bp_{21}=p_{12}+bp_{22}=p_{13}+bp_{23}=(1+b)/3, \tag{2}$$

and thus

$$p_{11}>\tfrac{1}{6}, \qquad p_{12}>\tfrac{1}{6}, \qquad \text{and} \qquad p_{13}>\tfrac{1}{6}. \tag{3}$$

Second we consider

$$U_2=\begin{pmatrix} 1 & a & 0 \\ 1 & b & 0 \\ 1 & b & 0 \end{pmatrix} \rightarrow L_2=\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}.$$

We claim $q_{12} \neq 0$ and $q_{13} \neq 0$. (If $q_{12}=0$, then, by symmetry, $bq_{22}=q_{13}+bq_{23}$, which further implies $q_{13} \leqslant b<\frac{1}{6}$ and $q_{11}>\frac{5}{6}>(1+b)/3$. Thus agent 1 in situation U_1 would find it profitable to misrepresent himself as an agent of type a—that is, an agent with a utility representation $(1, a, 0)$. This contradicts that σ is strategy-proof. Assuming $q_{12}=0$ leads to the same contradiction.) On the other hand, either $q_{32} \neq 0$ or $q_{33} \neq 0$. (Otherwise, $q_{31}=1$ and agent 1 in situation U_2 would lie about his type.) Without loss of generality we assume $q_{32} \neq 0$. Now we claim $q_{21}=0$.

[3] In Gale's original conjecture, the condition of anonymity instead of symmetry is imposed. Since the latter is weaker than the former, as a negative result, Theorem 1 is stronger than Gale's conjecture.

Suppose $q_{21} \neq 0$; we define M by

$$M = \begin{pmatrix} b & -b & 0 \\ -1 & 1 & 0 \\ 1-b & -1+b & 0 \end{pmatrix}.$$

It is easy to see that for sufficiently small $s > 0$, $L_s = L_2 + sM$ is a matching that Pareto dominates L_1. This is a contradiction. Therefore, we have

$$L_1 = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ 0 & q_{22} & q_{23} \\ 1-q_{11} & q_{32} & q_{33} \end{pmatrix}.$$

Now let us fix q_{11}. We compare situations U_1 and U_2. That σ is strategy-proof implies that in both situations agent 1 does not get higher payoffs by misrepresenting his own type. This leads to the following two inequalities:

$$p_{11} + bp_{21} \geqslant q_{11}, \qquad \text{and} \qquad q_{11} \geqslant p_{11} + ap_{21}.$$

Note that q_{11}, in general, is a function of both a and b, but here it does not depend on a. This is because L_1's are of the same form for all $a < b$. Strategy-proofness thus requires that q_{11} be independent of a. (Otherwise, agent 1 of type a with $a < b$ would find it profitable to claim to be of type a_0, where $a_0 = \operatorname{argmax} q_{11}$). Since a can be arbitrarily close to b, these two inequalities imply that $q_{11} = p_{11} + bp_{21} = (1+b)/3$. We now can write

$$U_2 = \begin{pmatrix} 1 & a & 0 \\ 1 & b & 0 \\ 1 & b & 0 \end{pmatrix} \to L_2 = \begin{pmatrix} (1+b)/3 & q_{12} & q_{13} \\ 0 & q_{22} & q_{23} \\ (2-b)/3 & q_{32} & q_{33} \end{pmatrix}. \tag{4}$$

Furthermore, since $q_{12} + q_{13} + b(q_{22} + q_{23}) = (2-b)/3 + b = 2(1+b)/3$, symmetry implies

$$q_{12} + bq_{22} = q_{13} + bq_{23} = (1+b)/3, \tag{5}$$

and thus

$$q_{12} > \tfrac{1}{6}, \qquad \text{and} \qquad q_{13} > \tfrac{1}{6}. \tag{6}$$

Applying a similar argument, we can get

$$U_3 = \begin{pmatrix} 1 & b & 0 \\ 1 & b & 0 \\ 1 & c & 0 \end{pmatrix} \to L_3 = \begin{pmatrix} r_{11} & r_{12} & (1-2b)/3(1-b) \\ r_{21} & r_{22} & (2-b)/6(1-b) \\ r_{31} & r_{32} & 0 \end{pmatrix}, \tag{7}$$

in which

$$r_{11}+br_{21}=r_{12}+br_{22}=(1+b)/3, \tag{8}$$

and thus

$$r_{11}>\tfrac{1}{6}, \qquad \text{and} \qquad r_{12}>\tfrac{1}{6}. \tag{9}$$

Finally we consider the situation

$$U_4=\begin{pmatrix}1 & a & 0\\ 1 & b & 0\\ 1 & c & 0\end{pmatrix}\to L_4=\begin{pmatrix}s_{11} & s_{12} & s_{13}\\ s_{21} & s_{22} & s_{23}\\ s_{31} & s_{32} & s_{33}\end{pmatrix}.$$

As before, either $s_{13}=0$ or $s_{33}=0$ by Pareto optimality. We claim $s_{33}=0$. (Suppose $s_{13}=0$. The utility level agent 3 achieves by telling the truth is $c(1-s_{33})$. But if he pretends to be of type b, according to (4) and (6), the utility level he can achieve is $q_{13}+cq_{23}$, which is greater than $q_{13}>\frac{1}{6}>c(1-s_{33})$. This contradicts that σ is strategy-proof.) Pareto optimality also requires that either $s_{12}=0$, or $s_{21}=0$, or $s_{32}=0$. (Otherwise, for sufficiently small $s>0$, L_4+sM—see above for the definition of M—is a matching that Pareto dominates L_4.) But both $s_{12}\neq 0$ and $s_{32}\neq 0$. (If $s_{32}=0$, then $s_{31}=1$ and agent 1 certainly would lie in this case. And if $s_{12}=0$, then the utility payoff for agent 2 is bs_{22}, which is less than $\frac{1}{6}<q_{12}+bq_{22}$, the utility level he can achieve by misrepresenting himself as of type c according to (4) and (6). Both contradict the assumption that σ is strategy-proof.) Therefore $s_{21}=0$. Now L_4 can be written as

$$L_4=\begin{pmatrix}s_{11} & 1-s_{11}-s_{13} & s_{13}\\ 0 & s_{13} & 1-s_{13}\\ 1-s_{11} & s_{11} & 0\end{pmatrix}.$$

In order to fix s_{11}, we apply strategy-proofness to agent 1 in U_3 and U_4 to get two inequalities. Once again using the fact that s_{11} is independent of a, we get $s_{11}=(1+b)/3$. And s_{13} can be fixed the same way: $s_{13}=(1-2b)/3(1-b)$. Thus we have

$$U_4=\begin{pmatrix}1 & a & 0\\ 1 & b & 0\\ 1 & c & 0\end{pmatrix}\to$$

$$L_4=\begin{pmatrix}(1+b)/3 & (b^2-b+1)/3(1-b) & (1-2b)/3(1-b)\\ 0 & (1-2b)/3(1-b) & (2-b)/3(1-b)\\ (2-b)/3 & (1+b)/3 & 0\end{pmatrix}. \tag{10}$$

But (10) implies that σ is manipulable by agent 2. To see this, we consider two numerical examples:

$$U_5=\begin{pmatrix}1 & 4/30 & 0\\ 1 & 2/30 & 0\\ 1 & 1/30 & 0\end{pmatrix}\rightarrow L_5=\begin{pmatrix}* & 211/630 & *\\ * & 195/630 & *\\ * & 224/630 & *\end{pmatrix},$$

and

$$U_6=\begin{pmatrix}1 & 4/30 & 0\\ 1 & 3/30 & 0\\ 1 & 1/30 & 0\end{pmatrix}\rightarrow L_6=\begin{pmatrix}* & 91/270 & *\\ * & 80/270 & *\\ * & 99/270 & *\end{pmatrix}.$$

Agent 2 in U_5 gets $211/630+(2/30)(195/630)=224/630$. But if he claims to be of type 3/30, he can get $91/270+(2/30)(80/270)=289/810$, which is larger than 224/630. Thus σ is manipulable. This completes our proof for the case $n=3$.

Now let us consider the general case in which $n>3$. Suppose that there exists an allocation mechanism φ that satisfies the three conditions of the theorem. We consider utility matrices of the form

$$U=\begin{pmatrix}u_{11} & u_{12} & u_{13} & 0 & \cdots & 0\\ u_{21} & u_{22} & u_{23} & 0 & \cdots & 0\\ u_{31} & u_{32} & u_{33} & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

Pareto optimality requires that for $i>3$, agent i should get the ith object with probability one. Thus φ will induce an allocation mechanism σ for the three-agent-three-object case. Furthermore, σ inherits from φ the properties of symmetry, Pareto optimality, and strategy-proofness. However, we have already shown that such σ does not exist. This is a contradiction. Thus we complete the proof of the theorem. Q.E.D.

The conditions required in Theorem 1 are tight. If one of them is relaxed, then there exists a mechanism that satisfies the others. We first look at the case $n=2$, which is not covered by Theorem 1. Contrary to our general negative result, there does exist a unique mechanism satisfying all the conditions in Theorem 1.

EXAMPLE 1. Suppose that objects O_1 and O_2 are to be assigned to agents A_1 and A_2. The mechanism works as follows: if A_1 prefers O_1 to O_2

(or O_2 to O_1), and A_2 prefers O_2 to O_1 (or O_1 to O_2), then each gets what he prefers. If both prefer the same object, then assignment will be determined by tossing a fair coin.

In reality we usually have to deal with more than two agents and two objects, in which case Theorem 1 is applicable. The following examples are mechanisms for which one of the three conditions of Theorem 1 is dropped.

EXAMPLE 2 (Pareto optimality and symmetry). For any $U \in \mathbf{U}$, we first find all those pure matchings which maximize, among all mixed matchings, the sum of the expected utilities of all agents. This is the so-called assignment problem in operational research. Then we define $\sigma(U)$ as the uniformly weighted average of all such pure matchings. It is clear that σ is Pareto optimal. σ is also symmetric. This can be derived from the following observation. When $\mathbf{u}_i = \mathbf{u}_j$, if a pure matching p maximizes the sum of the expected utilities of all agents, so does the pure matching q that is obtained from p by switching the objects assigned to agents i and j.

EXAMPLE 3 (symmetry and strategy-proofness). The simplest mechanism in this case is the mechanism which always assigns the uniform lottery to any utility matrix U,

$$\eta(U) = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}.$$

There may exist more complicated mechanisms as well. For instance, if we restrict our attention to the subset $\mathbf{U}'$ of utility matrices, in wich all preferences have the same order over all objects, then the following mechanism is both symmetric and strategy-proof (for simplicity we let $n = 3$):[4]

$$\begin{pmatrix} 1 & a & 0 \\ 1 & b & 0 \\ 1 & c & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1/3 - (2a^2 - b^2 - c^2)/12 & 1/3 - (2b^2 - a^2 - c^2)/12 & 1/3 - (2c^2 - a^2 - b^2)/12 \\ 1/3 + (2a - b - c)/6 & 1/3 + (2b - c - a)/6 & 1/3 + (2c - a - b)/6 \\ * & * & * \end{pmatrix}$$

EXAMPLE 4 (Pareto optimality and strategy-proofness). We rank all agents according to some pre-determined order, say alphabetically by their names. Following this order, we assign to each agent his most preferred

[4] This mechanism is also a negative answer for another conjecture in Gale [2].

object among those which have not yet been assigned. If at some stage the agent who is to be assigned an object has several most preferred objects, we will let him wait and assign objects to agents succeeding him until there is only one of his most preferred objects left and then assign it to him. Under this mechanism, no agent has an incentive to misrepresent his preferences since by telling the truth he always gets his most preferred object among those available. Hence the mechanism is strategy-proof. The tie-breaker rule guarantees that the mechanism is also Pareto optimal.

A more general form of such mechanism can be described as follows. One particular agent is always assigned his most preferred object according to his announced preference relation. His announced preference relation also determines who is the second agent to be considered. That agent is assigned his most preferred object among those not assigned to the first agent. The announced preferences of both agents then will determine the next agent to consider, and so on. The tie-breaker rule is the same as above. The reason why it is strategy-proof and Pareto optimal is also the same.

The economic implications of Theorem 1 and the above examples are important. Theorem 1 reveals an essential difficulty in designing satisfactory mechanisms for even the simplest one-sided matching problem. Example 2 demonstrates that this difficulty arises mainly from incentive problems, since it shows that the tradeoff between equity and efficiency would not be an issue if incentive problems were absent. However, as long as agents do have private information and they do act in their own interests, Theorem 1 asserts that symmetry and Pareto optimality cannot be achieved simultaneously. Hence there exists a tradeoff between equity and efficiency when we consider mechanisms that are incentive-compatible (i.e., strategy-proof). What can we say about this tradeoff? Example 3 and 4 provide two extreme cases. For mechanisms in Example 3, especially the uniform lottery, while equity (i.e. symmetry) is achieved, efficiency (i.e. Pareto optimality) is almost totally lost.[5] On the other hand, for mechanisms in Example 4, although efficiency is always achieved, the power structure of the agents (the probability distributions of agents' getting their most preferred objects) is obviously lopsided. We can construct various weighted averages of both types of mechanisms to get intermediate mechanisms, which represent compromises between the consideration of equity and that of efficiency.[6]

Finally, we discuss a positive result which arises when the condition of

[5] Gale proves that if we restrict the agents' preferences in U', then uniform lottery is never Pareto optimal unless all agents' preferences are identical.

[6] Since the expectation operator is linear, a weighted average of several strategy-proof mechanisms is still strategy-proof.

ex ante Pareto optimality in Theorem 1 is replaced by the condition of ex post Pareto optimality. We take those mechanisms described in the first half of Example 4 with all possible orders of agents and then construct the uniform lottery over these mechanisms. The resulting mechanism δ satisfies symmetry, ex post Pareto optimality, and strategy-proofness. It can be easily implemented in many ways such as the popular first-come-first-served rule (if one believes that the order of agents' arrivals is purely random). Let Σ be the set of all mechanisms satisfying symmetry, ex post Pareto optimality, and strategy-proofness. Given our general negative result, mechanisms in Σ certainly have a special appeal to some people. Then how to choose among Σ presents another intriguing problem. In our view, one should have at least a partial answer to the following question before a satisfactory choice is made: what mechanisms are ex ante Pareto optimal among Σ? However, we do not know any answer yet. In fact, whether δ is ex ante Pareto optimal among Σ still remains an open question.

3. Some Extensions

In the matching problems we have considered so far there are three basic assumptions: first, every agent can receive at most one object; second, each object can be assigned to at most one agent; and third, there are equal numbers of agents and objects. In general matching problems, none of these assumptions necessarily holds. All micro-economics classes have to admit more than, say five, students; the number of the belongings of a suddenly deceased millionaire can easily exceed the number of his heirs; and so on. Thus it is important to know whether our previous analysis extends to some more general matching problems than that in Section 2.

Extension 1. The first and the second assumptions are still maintained, while the third dropped. There are n agents and m objects, where n can be different from m. The set of agents is $\mathbf{A} = \{A_1, A_2, ..., A_n\}$, and that of objects $\mathbf{O} = \{O_1, O_2, ..., O_m\}$. When $n < m$, Theorem 1 is still true. To see this, one can apply the same tactics used in the proof of Theorem 1 when we proceed from the 3×3 case to general $n \times n$ case. When $n > m \geqslant 2$, we add $n - m$ artificial objects $O_{m+1}, ..., O_n$ so that an agent receives nothing if and only if he receives one of these objects. We also have to specify the utility levels of the agents when they receive nothing. If it is allowed that an agent may prefer receiving nothing over receiving any real object, then Theorem 1 can be proved in the same manner. To sum up, Theorem 1 is still valid whenever each agent can receive at most one object and each object can be assigned to at most one agent.

EXTENSION 2. The first assumption only is maintained. Again there are n agents and m objects, and there are also m positive numbers $n_1, n_2, ..., n_m$, with each n_i representing the capacity of the ith object. Since we can always introduce an artificial object with arbitrarily large capacity O_{m+1} so that an agent gets O_{m+1} when he actually gets nothing, we only need to consider cases in which $\sum n_i \geqslant n$.

Each agent here is assumed to be not only selfish but also insensitive, i.e., he is indifferent to the alternative sets of agents with whom he might share an object. Hence each agent has a von Neumann–Morgenstern utility function over **O**. A utility matrix is defined the same way as in Section 2. Other terms can also be defined similarly.

When $n_i + n_j \geqslant n$ for all i and j $(1 \leqslant i, j \leqslant m)$, the situation is not much different from the 2×2 case in Section 2. We consider the following mechanism. Each object is assigned to agents who put it at the top of their preference relations unless its capacity falls short. If the capacity of one particular object is less than the number of agents who want it (since $n_i + n_j \geqslant n$ there is at most one such object), then a fair lottery is drawn among those agents. Winners are given this object while losers get their second choice. It is not difficult to verify that this is the unique mechanism that satisfies symmetry, Pareto optimality, and strategy-proofness.

A negative result arises when there exist some i and j $(1 \leqslant i, j \leqslant m)$ such that $n_i + n_j < n$. It is, however, not as strong as Theorem 1. We say that a mechanism is *k-coalitional-strategy-proof* if in any situaion and for any coalition C of agents of size up to k, it is impossible for the agents in C to coordinate their announced preferences to achieve a Pareto improvement over the allocation that will be adopted if they all tell the truth. Theorem 1 is generalized in the following manner. Let $n^* = \min\{\min n_i, 2\}$, then we have

THEOREM 2. *Assume $\sum n_i \geqslant n$. If there exists at least one pair i and j, $1 \leqslant i, j \leqslant m$, such that $n_i + n_j < n$, then there is no mechanism that satisfies symmetry, Pareto optimality, and n^*-coalitional-strategy-proofness.*

The proof of Theorem 2 is similar to that of Theorem 1, which we choose not to repeat here. Although Theorem 2 does generalize Theorem 1, its use of the condition of coalitional-strategy-proofness weakens the result substantially. Our conjecture is that Theorem 2 still holds even if the condition of simple strategy-proofness only is required.

It is more difficult to extend our analysis to matching problems in which all three assumptions are dropped. An important example of such problems is the allocation of several indivisible private goods among some individuals. It is clear that more work needs to be done on this subject.

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