Remark. Thomas O'Hare was a collaborator.

Problem 1. Let $\mathcal{L}^1(X,\mu)$ (shortened to $\mathcal{L}^1(X)$) be the space of integrable functions on X w.r.t. μ. Show that $\mathcal{L}^1(X)$ is a vector space and that the integral defines a linear functional on it.

Proof. We note that $\mathcal{L}^1(X) \subset \mathrm{Fun}(X,\mathbb{C})$ and we have that $\mathrm{Fun}(X,\mathbb{C})$ is a vector space, so it suffices to show that $\mathcal{L}^1(X)$ is a vector subspace. To see that it is a vector subspace, we need to show that it's closed under scaling and under addition. Let $f,g \in \mathcal{L}^1(X)$, then we have by the linearity of the integral and the triangle inequality that

$$\int |f+g| \leq \int (|f|+|g|) = \int |f| + \int |g| < \infty,$$

so $f+g \in \mathcal{L}^1(X)$. Next, if $r \in \mathbb{C}$, $f \in \mathcal{L}^1(X)$, we have again by the linearity of the integral that

$$\int |rf| = \int |r||f| = |r| \int |f| < \infty,$$

so $rf \in \mathcal{L}^1(X)$. Hence, this is a vector subspace, and so a vector space. The integral is linear (by last semester/**Chapter 2** material) and so we see that it is a linear functional on $\mathcal{L}^1(X)$. $\square$

Problem 2. If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

Proof. ($\implies$): Assume that $\int f = 0$. Let

$$M_{\geq 1/n} := \left\{ x \ : \ f(x) \geq \frac{1}{n} \right\},$$

$$M := \{x \ : \ f(x) > 0\} = \bigcup_{n\in\mathbb{N}} M_{\geq 1/n}.$$

Notice that

$$0 = \int f \geq \int_{M_{\geq 1/n}} f \geq \int_{M_{\geq 1/n}} \frac{1}{n} = \mu(M_{\geq 1/n})\frac{1}{n},$$

so for every n we have that $\mu(M_{\geq 1/n}) = 0$. Now, notice that

$$\mu(M) = \mu\left(\bigcup_{n\in\mathbb{N}} M_{\geq 1/n} \right) \leq \sum_{n\in\mathbb{N}} \mu(M_{\geq 1/n}) = 0,$$

so that $\mu(M) = 0$. Hence, we have that $f = 0$ a.e.
($\impliedby$) Assume $f = 0$ a.e. Then we have that $E := \{x \ : \ f(x) \neq 0\}$ is such that $\mu(E) = 0$. Notice that we can write

$$\int f = \int_E f + \int_{E^c} f = \int_E f,$$

since $f = 0$ on E^c. Since $f \in L^+$, we can construct a sequence of simple functions φ_n such that $\varphi_n \nearrow f$. Hence, by the monotone convergence theorem, we have

$$\int_E f = \lim_{n\to\infty} \int_E \varphi_n.$$

Each simple function is bounded above by something, say M_n, and so we have that

$$\int_E \varphi_n \leq \int_E M_n = M_n \mu(E) = 0$$

for all n. Thus, we get

$$\int_E f = \lim_{n\to\infty} \int_E \varphi_n = 0,$$

and hence $\int f = 0$. $\square$

Problem 3. Show that equality holds in Hölder iff $\alpha|f|^p = \beta|g|^q$ a.e. for some $\alpha, \beta \in \mathbb{C}$ such that $(\alpha, \beta) \neq (0, 0)$.

Proof. Recall that Hölder's inequality states that if $1 < p < \infty$ and $1/p + 1/q = 1$, f, g measurable functions on X, then

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

In the proof of Hölder's inequality, we used a lemma which states that if $a \ge 0$, $b \ge 0$, $0 < \lambda < 1$, then we have

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda) b.$$

We wish to show the conditions for equality here. In the case that $b = 0$, we see that we have equality iff $a = 0$, so we assume that $b \neq 0$. Dividing both sides by b grants us

$$(a/b)^\lambda \le \lambda(a/b) + (1-\lambda).$$

Let $t = a/b$. Then we may rewrite this as

$$t^\lambda \le \lambda t + (1-\lambda).$$

Subtracting from both sides λt, we get

$$t^\lambda - \lambda t \le 1 - \lambda.$$

We wish to find the value of t when the left hand side is maximized for $0 \le t$. We see that taking the derivative and setting it to zero gives

$$\lambda(t^{\lambda-1} - 1) = 0,$$

and so we have a critical point at $t = 1$. Since $0 < \lambda < 1$, we get that for $t < 1$ this is increasing (the derivative will be positive), for $t > 1$ this will be decreasing (the derivative will be negative), so we have that $t = 1$ is the maximum; i.e. for $t = 1$ we have equality. Notice that $t = 1 \implies a/b = 1 \implies a = b$. So we have equality here iff $a = b$.

Going back to Hölder, we see that equality is clear when $\|f\|_p = 0$, $\|g\|_q = 0$, $\|f\|_p = \infty$, $\|g\|_q = \infty$, so it suffices to assume that this is not the case. Going through the proof of Hölder, we can scale f and g via α and β (non-zero by assumption) respectively so that $f' = \alpha f$, $g' = \beta g$ is such that $\|f'\|_p = \|g'\|_q = 1$ (i.e., we normalize these functions with regards to the p and q norm). Thus, we get Hölder by using the Calculus lemma to get

$$|f'(x)g'(x)| \le p^{-1}|f'(x)|^p + q^{-1}|g'(x)|^q, \tag{1}$$

we integrate both sides to get

$$\|f'g'\|_1 \le p^{-1}\|f'\|_p^p + q^{-1}\|g'\|_q^q = p^{-1} + q^{-1} = 1 = \|f'\|_p \|g'\|_q,$$

and rewriting this we have

$$|\alpha\beta| \|fg\|_1 \le |\alpha\beta| \|f\|_p \|g\|_q.$$

Since α, β non-zero, we can divide this out to get the desired inequality. Notice then that we have equality in Equation (1) a.e. iff we have equality in Hölder, and by the observation earlier we have equality in Equation (1) a.e. iff $|f'|^p = |g'|^q$. Writing out the definitions of f', g', we get that this is true iff $|\alpha|^p|f|^p = |\beta|^q|g|^q$, and so redefining α and β accordingly gives us that we have equality iff $\alpha|f|^p = \beta|g|^q$. $\square$

In the following problems, we use Chebychev's inequality, which we will prove here.

Claim. For f a measurable function on X, $0 < p < \infty$ and $t > 0$, we have

$$\mu(\{x \ : \ |f(x)| \geq t\}) \leq \frac{1}{t^p} \int_{|f| \geq t} |f|^p \leq \frac{1}{t^p} \|f\|_p^p.$$

Proof. Notice that

$$\int_{|f| \geq t} |f|^p \geq \int_{|f| \geq t} t^p = \mu(\{x \ : \ |f(x)| \geq t\}) t^p,$$

so rewriting gives us the desired result. For the final inequality, we simply note that

$$\frac{1}{t^p} \int_{|f| \geq t} |f|^p \leq \frac{1}{t^p} \int |f|^p = \frac{1}{t^p} \|f\|_p^p.$$

□

Problem 4 (Folland 6.3)**.** If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm given by $\|f\| = \|f\|_p + \|f\|_r$, and the inclusion map $L^p \cap L^r \to L^q$ is continuous for $p < q < r$.

Proof. We check that we have closure under finite addition and closure under multiplication by scalars. This gives us that it is a vector subspace of L^p, and so it is a vector space. We first check that it is closed under addition. Since $\| \cdot \|_p, \| \cdot \|_r$ are norms, taking $f, g \in L^p \cap L^r$, we have that

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty,$$

$$\|f + g\|_r \leq \|f\|_r + \|g\|_r < \infty,$$

so $f + g \in L^p \cap L^r$ as desired. For closure under scalars, let $r \in \mathbb{C}$, $f \in L^p \cap L^r$. We have that

$$\|rf\|_p = |r| \|f\|_p < \infty,$$

since $f \in L^p$, and likewise with r, so $rf \in L^p \cap L^r$. Hence, we have that $L^p \cap L^r$ is a vector space.

Next, we wish to show that the function $\| \cdot \|$ defined above is a norm. We need to show that the four axioms are satisfied:

(1) Notice that $\| \cdot \|_p, \| \cdot \|_r \geq 0$, so we have that $\| \cdot \| = \| \cdot \|_p + \| \cdot \|_r \geq 0$.

(2) We need to show that $\|f + g\| \leq \|f\| + \|g\|$. Writing this out, we have

$$\|f + g\| = \|f + g\|_p + \|f + g\|_r,$$

and using the fact that these are norms we get

$$\|f + g\| \leq \|f\|_p + \|g\|_p + \|f\|_r + \|g\|_r.$$

Regrouping gives

$$\|f + g\| \leq (\|f\|_p + \|f\|_r) + (\|g\|_p + \|g\|_r) = \|f\| + \|g\|.$$

(3) For scalars $r \in \mathbb{C}$, we need to show that

$$\|rf\| = |r| \|f\|.$$

Using again that $\| \cdot \|_p, \| \cdot \|_r$ are norms, we get that

$$\|rf\| = \|rf\|_p + \|rf\|_r = |r| \|f\|_p + |r| \|f\|_r = |r| \left(\|f\|_p + \|f\|_r \right) = |r| \|f\|.$$

(4) We need to show that $\|f\| = 0$ if and only if $f = 0$ a.e. Notice that $\|f\| = 0$ implies that $\|f\|_p = 0$, $\|f\|_r = 0$ (since these are non-negative), and so we have that $f = 0$ a.e. since $\| \cdot \|_p$ and $\| \cdot \|_q$ are norms. The other direction is clear from this as well; if $f = 0$æ, then $\|f\|_p = \|f\|_r = 0$, so $\|f\| = \|f\|_p + \|f\|_r = 0$.

Hence, we have that $L^p \cap L^r$ is a vector space, and $\|\cdot\|$ is a norm. To show that this is a Banach space, we need to show that the norm is complete. Let $(f_n) \subset \mathcal{L}^p \cap L^r$ be a Cauchy sequence. That is to say, for all $\epsilon > 0$, there exists an N such that for all $n, m \geq N$, we have

$$\|f_n - f_m\| < \epsilon.$$

Since $\|\cdot\|_p$, $\|\cdot\|_q$ are non-negative, we have that $\|f\| \geq \|f\|_p$, $\|f\| \geq \|f\|_r$. In other words, we have that the sequence $(f_n) \subset L^p \cap L^r$ is Cauchy with respect to both the p and r norm. Since these are Banach spaces (**Theorem 6.6**/Lecture notes), we have that there is a limit with respect to these norms. Let g be such that $f_n \to g$ in L^p, h be such that $f_n \to h$ in L^r. We now use Chebychev's inequality to see that

$$\mu\left(\{|f_n - g| > \epsilon\}\right) \leq \frac{1}{\epsilon^p}\|f_n - g\|_p^p.$$

Since $f_n \to g$ in L^p, we get that $f_n \to g$ in measure by the above. Assuming $r < \infty$, we get that $f_n \to h$ in measure as well. We have that we can construct a subsequence $f_{n_k} \to g$ a.e., and we can then refine this subsequence to get that $f_{n_{k_j}} \to h$ a.e. as well. Hence, $g = h$ a.e. Now assume that $r = \infty$. We get that $f_n \to h$ in L^∞, which means that $\|f_n - h\|_\infty \to 0$. Notice that

$$\|f_n - h\|_\infty = \inf\left\{a \geq 0 \;:\; \mu\left(\{x \;:\; |f_n(x) - h(x)| > a\}\right) = 0\right\} \to 0.$$

Hence, for all $\epsilon > 0$, there exists an N such that for all $n \geq N$, $\|f_n - h\|_\infty < \epsilon$. Notice that this means that $|f_n - h| < \epsilon$ a.e., so we have that

$$\mu(\{x \;:\; |f_n(x) - h(x)| \geq \epsilon\}) = 0.$$

This applies for all $\epsilon > 0$, and so we have that $f_n \to h$ in measure. Hence, we can construct a subsequence which converges a.e. and the same argument before gives us that $g = h$ a.e. In other words, we get that $g \in L^p \cap L^r$. We now wish to show that $f_n \to g$ with regards to the norm; that is,

$$\|f_n - g\| \to 0.$$

But this follows, since

$$\|f_n - g\| = \|f_n - g\|_p + \|f_n - g\|_r \to 0,$$

since $f_n \to g$ in L^p and in L^r. Hence, we get that the norm is complete.

Finally, we need to show that the inclusion map $L^p \cap L^r \to L^q$ is continuous for $p < q < r$. Let $f \in L^p \cap L^r$ be such that $\|f\| = 1$. We then wish to show that $\|f\|_q \leq \|f\| = 1$, which gives us that the inclusion is bounded, and so by **Proposition 5.2** that it is continuous (if $\|f\| = 0$, then $\|f\|_q = 0$, and otherwise we can normalize for the case where $\|f\| \neq 1$ to get the desired result). **Proposition 6.10** (or the following Remark/Claim) gives us that there is a $\lambda \in (0,1)$ with

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}.$$

Again, using the fact that $\|f\|_p \leq \|f\| = 1$, $\|f\|_r \leq \|f\| = 1$, we get that $\|f\|_q \leq 1$, as desired. □

Remark. Since **Proposition 6.10** wasn't proven in class, we show it here. That is, we prove the following claim:

Claim. For $1 \leq p < q < r \leq \infty$, we have

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda},$$

where $\lambda \in (0,1)$ is defined by

$$\lambda := \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}.$$

Proof. For $r = \infty$, we have
$$|f|^q = |f|^p|f|^{q-p} \leq |f|^p\|f\|_\infty^{q-p},$$
and we have
$$\lambda := \frac{q^{-1}}{p^{-1}} = \frac{p}{q}.$$
Integrating gives
$$\|f\|_q^q \leq \|f\|_p^p\|f\|_\infty^{q-p},$$
and taking qth roots gives
$$\|f\|_q \leq \|f\|_p^{p/q}\|f\|_\infty^{(q-p)/q} = \|f\|_p^\lambda\|f\|_\infty^{1-\lambda},$$
as desired. For $r < \infty$, we write
$$\|f\|_q^q = \int |f|^q = \int |f|^{\lambda q}|f|^{(1-\lambda)q}.$$
We can now apply Hölder with conjugate pairs given by $p/\lambda q$, $r/(1-\lambda)q$. This gives us
$$\|f\|_q^q \leq \||f|^{\lambda q}\|_{p/\lambda q} \cdot \||f|^{(1-\lambda)q}\|_{r/(1-\lambda)q}$$
$$= \left(\int |f|^p\right)^{\lambda q/p}\left(\int |f|^r\right)^{(1-\lambda)q/r} = \|f\|_p^{\lambda q}\|f\|_r^{(1-\lambda)q}.$$
Now taking qth roots, we have
$$\|f\|_q \leq \|f\|_p^\lambda\|f\|_r^{(1-\lambda)}$$
as desired.

□

Problem 5 (Folland 6.9)**.** Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure and a subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^P$ for all n, then $\|f_n - f\|_p \to 0$.

Proof. We use Chebychev's inequality here. We have that
$$\mu(\{x \ : \ |f_n(x) - f(x)| > \epsilon\}) \leq \frac{1}{\epsilon^p}\|f_n - f\|_p^p \to 0.$$
Hence, it converges in measure. For the remainder, we use **Theorem 2.30** to deduce there is a subsequence converging to f a.e.

On the other hand, assume that $f_n \to f$ in measure and that $|f_n| \leq g \in L^p$ for all n. Recall from **Folland Exercise 2.34 (a)** that if $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure, then
$$\int f = \lim \int f_n.$$
Notice that $|f_n - f|^p \leq 2^p(|f|^p + |f_n|^p) \leq 2^{p+1}|g|^p$. Notice as well that $f_n \to f$ in measure implies that $|f_n - f|^p \to 0$ in measure as well. Since $g \in L^p$, we have that
$$\left(\int |g|^p\right)^{1/p} < \infty \implies \int |g|^p < \infty \implies |g|^p \in L^1,$$
and moreover $2^{p+1}|g|^p \in L^1$. So $|f_n - f|^p \leq 2^{p+1}|g|^p$, and we apply **Folland Exercise 2.34 (a)** to get
$$0 = \lim \int |f_n - f|^p,$$
and so
$$\lim \left(\int |f_n - f|^p\right)^{1/p} = \lim \|f_n - f\|_p = 0$$

$\square$

Remark. The following is the exercise (and solution) which was used implicitly in the prior problem.

Claim. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure.
(1) Show that if $f_n \geq 0$ everywhere, then $\int f \leq \liminf \int f_n$.
(2) Suppose $|f_n| \leq g \in \mathcal{L}^1$. Prove that $\int f = \lim \int f_n$ and $f_n \to f$ in $\mathcal{L}^1$.

Proof. (1) Fatou's Lemma gives

$$\int \liminf f_n \leq \liminf \int f_n,$$

since $f_n \geq 0$. Thus, we can construct a subsequence $f_{n_j} \to \liminf f_n$, and so we get

$$\int \lim_j f_{n_j} = \int \liminf f_n \leq \liminf \int f_n.$$

Now, since $f_n \to f$ in measure, we have $f_{n_j} \to f$ in measure as well, so we can construct a subsequence $f_{n_{j_k}} \to f$ almost everywhere. Hence, we have

$$\int f = \int \lim_k f_{n_{j_k}} = \int \lim_j f_{n_j} \leq \liminf \int f_n.$$

(2) It suffices to do this for real valued functions, since if $f_n \to f$ in measure, we have

$$|f_n - f| \leq |\mathrm{Re}(f_n) - \mathrm{Re}(f)| + |\mathrm{Im}(f_n) - \mathrm{Im}(f)| \leq 2|f_n - f|,$$

and so $f_n \to f$ in measure if and only if $\mathrm{Re}(f_n) \to \mathrm{Re}(f)$ and $\mathrm{Im}(f_n) \to \mathrm{Im}(f)$ converge in measure, and so we can consider both separately.

If $|f_n| \leq g \in \mathcal{L}^1$, we have $f_n \leq g$ and $-f_n \leq g$, or in other words, $0 \leq g - f_n$ and $0 \leq g + f_n$. Using (1), we get

$$\int g - \int f = \int (g - f) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n,$$

and

$$\int g + \int f = \int (g + f) \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n.$$

Since $g \in \mathcal{L}^1$, we can subtract it from both sides and rearrange terms to get

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n,$$

or that

$$\lim \int f_n = \int f.$$

To see that $f_n \to f$ in $\mathcal{L}^1$, we need to show that $\int |f_n - f| \to 0$. Notice that $f_n \to f$ in measure implies $|f_n - f| \to 0$ in measure as well, and so we can use this and $h = g + |f| \geq |f_n| + |f| \geq |f_n - f|$ to get that, by what we've just shown,

$$\lim \int |f_n - f| = \int 0 = 0.$$

Hence, $f_n \to f$ in $\mathcal{L}^1$.

$\square$

Problem 6 (Folland 6.10)**.** Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

Proof. ($\Longrightarrow$): We use the reverse triangle inequality to get that

$$|\|f_n\|_p - \|f\|_p| \leq \|f - f_n\|_p \to 0,$$

and so $\|f_n\|_p \to \|f\|_p$.
($\Longleftarrow$): We use the inequality (introduced on page 181 and in the lecture notes)

$$|f - f_n|^p \leq 2^p(|f|^p + |f_n|^p).$$

Integrating both sides gives

$$\int |f - f_n|^p \leq 2^p \left(\int |f_n|^p + |f|^p \right).$$

Moving things around, we can rewrite this as

$$0 \leq 2^p \int |f_n|^p + 2^p \int |f|^p - \int |f - f_n|^p.$$

We apply Fatou's Lemma to get

$$\int \liminf_{n\to\infty} 2^p \left(|f_n|^p + |f|^p\right) - |f - f_n|^p = \int 2^{p+1}|f|^p \leq \liminf_{n\to\infty} \int 2^p \left(|f_n|^p + |f|^p\right) - |f - f_n|^p.$$

Distributing and using linearity gives us

$$2^{p+1}\|f\|_p^p \leq 2^p \liminf_{n\to\infty} \|f_n\|_p^p + 2^p\|f\|_p^p - \limsup_{n\to\infty} \int |f - f_n|^p.$$

Since $\|f_n\|_p \to \|f\|_p$, we get as well that $\|f_n\|_p^p \to \|f\|_p^p$, so that

$$2^{p+1}\|f\|_p^p \leq 2^{p+1}\|f\|_p^p - \limsup_{n\to\infty} \int |f - f_n|^p.$$

That is,

$$\limsup_{n\to\infty} \int |f - f_n|^p \leq 0.$$

Since $|f - f_n|^p \geq 0$, this gives us that

$$\lim_{n\to\infty} \int |f - f_n|^p = 0,$$

or in other words,

$$\|f_n - f\|_p^p \to 0 \iff \|f_n - f\|_p \to 0.$$

$\square$

Remark. Thomas O'Hare was a collaborator.

Problem 7. Let $0 < \alpha < 1$ and (X, μ) be a σ-finite measure space. Set

$$L^\alpha(X) := \{u : X \to \mathbb{R} \ : \ u \text{ is measurable and } |u|^\alpha \in L^1(X)\}$$

and

$$[u]_\alpha = \left(\int |u|^\alpha\right)^{1/\alpha}.$$

(a) Show that L^α is a vector space and if $u, v \in L^\alpha(X)$, $u \geq 0$æ and $v \geq 0$æ, then

$$[u+v]_\alpha \geq [u]_\alpha + [v]_\alpha,$$

hence $[\cdot]_\alpha$ is not a norm.

(b) Prove that for all $u, v \in L^\alpha(X)$,

$$[u+v]_\alpha^\alpha \leq [u]_\alpha^\alpha + [v]_\alpha^\alpha.$$

Proof. (a) We first show that L^α is a vector space. Notice that $L^\alpha \subset \text{Fun}(X, \mathbb{R})$, which is a vector space, so it suffices to show that it's closed under scaling and addition. For addition, let $u, v \in L^\alpha$. Then we have $|u|^\alpha, |v|^\alpha \in L^1$, and we notice that

$$|u+v|^\alpha \leq (|u| + |v|)^\alpha.$$

We then claim that, for $a, b \geq 0$, we have

$$(a+b)^\alpha \leq a^\alpha + b^\alpha.$$

If $a = 0$, we are done. Otherwise, we can divide by a^p to get

$$\left(1 + \frac{a}{b}\right)^\alpha \leq 1 + \left(\frac{a}{b}\right)^\alpha.$$

Let $t = a/b$, then we can write this as

$$(1+t)^\alpha \leq 1 + t^\alpha,$$

with $t \geq 0$. If $t = 0$, we have that both sides are equal, and taking the derivative we see that

$$\frac{d}{dt}((1+t)^\alpha - t^\alpha) = \alpha\left((1+t)^{\alpha-1} - t^{\alpha-1}\right),$$

which we see is less than 0 for $t > 0$. In other words, we get that it decreases, so we have the desired inequality. Hence, going back, we have

$$|u+v|^\alpha \leq (|u| + |v|)^\alpha \leq |u|^\alpha + |v|^\alpha,$$

and integrating it gives us that $|u+v|^\alpha \in L^1$; in other words, $u + v \in L^\alpha$. For scalars, we see that for $r \in \mathbb{R}$, $u \in L^\alpha$, we have

$$|ru|^\alpha = |r|^\alpha |u|^\alpha,$$

and integrating gives us that $|ru|^\alpha \in L^1$, as desired. Hence, $ru \in L^\alpha$. So L^α is a vector space.

We now show that this is not a norm.[1] Assume $u \geq 0$æ, $v \geq 0$æ We have then that

$$[u+v]_\alpha = \left(\int |u+v|^\alpha\right)^{1/\alpha}.$$

[1]The following argument was adapted from a related Stackexchange post (I don't think it was proving this inequality necessarily but something similar), but I now can't find a link to said post. If you see the link somewhere, please let me know.

Since both functions are greater than or equal to 0 a.e., we have that this reduces to

$$[u+v]_\alpha = \left(\int (u+v)^\alpha\right)^{1/\alpha}.$$

We first remark on two different cases. If $[u]_\alpha = 0$, this implies that $u = 0$æ, and so we trivially get equality. It is the same with $[v]_\alpha = 0$. Hence, assume both of these are non-zero. Notice that we can write

$$(u+v)^\alpha = \left(t\frac{u}{t} + (1-t)\frac{v}{1-t}\right)^\alpha,$$

where $t \in (0,1)$. Since x^α is a concave function, we get

$$(u+v)^\alpha \geq t\frac{u^\alpha}{t^\alpha} + (1-t)\frac{v^\alpha}{(1-t)^\alpha}.$$

Choose

$$t = \frac{[u]_\alpha}{[v]_\alpha + [u]_\alpha},$$

then

$$1 - t = \frac{[v]_\alpha}{[v]_\alpha + [u]_\alpha}.$$

Hence, after integrating this, we get

$$[u+v]_\alpha^\alpha \geq t([v]_\alpha + [u]_\alpha)^\alpha + (1-t)([v]_\alpha + [u]_\alpha)^\alpha = ([v]_\alpha + [u]_\alpha)^\alpha.$$

Hence, taking αth roots on both sides, we get

$$[u+v]_\alpha \geq [v]_\alpha + [u]_\alpha,$$

contradicting the triangle inequality.

(b) By our observation prior, we have that

$$[u+v]_\alpha^\alpha = \int |u+v|^\alpha \leq \int (|u|+|v|)^\alpha \leq \int (|u|^\alpha + |v|^\alpha) = \int |u|^\alpha + \int |v|^\alpha = [u]_\alpha^\alpha + [v]_\alpha^\alpha.$$

Notice this holds for all $u, v \in L^\alpha$.

□

Problem 8. If $p \neq 2$, the L^p norm does not arise from an inner product on L^p except in trivial cases $\dim(L^p) \leq 1$.

Proof. We proceed as Folland suggests: we wish to show that the parallelogram law does not hold; i.e.,

$$\|f+g\|_p^2 + \|f-g\|_p^2 \neq 2(\|f\|_p^2 + \|g\|_p^2).$$

To do so, we consider some cases. At first, let's consider $1 \leq p < \infty$, $p \neq 2$. If we can find sets $A, B \in \mathcal{M}$ with $A \cap B = \varnothing$, $0 < \mu(A), \mu(B) < \infty$, then we have that by setting $f = \mu(A)^{-1/p}\chi_A$, $g = \mu(B)^{-1/p}\chi_B$,

$$\|f+g\|_p^2 = \left(\int |f+g|^p\right)^{2/p} = 2^{2/p},$$

$$\|f-g\|_p^2 = \left(\int |f-g|^p\right)^{2/p} = 2^{2/p},$$

and so we have

$$\|f+g\|_p^2 + \|f-g\|_p^2 = 2^{2/p+1},$$

while on the other hand, we have

$$\|f\|_p^2 = \left(\int |f|^p\right)^{2/p} = 1,$$

$$\|g\|_p^2 = \left(\int |g|^p\right)^{2/p} = 1,$$

and so

$$2(\|f\|_p^2 + \|g\|_p^2) = 2(1+1) = 4.$$

Hence, we see we have equality iff $p = 2$. For the case $p = \infty$, we simply take $f = \chi_A$, $g = \chi_B$, and we note that

$$\|f+g\|_\infty^2 = 1,$$
$$\|f-g\|_\infty^2 = 1,$$
$$\|f\|_\infty^2 = 1,$$
$$\|g\|_\infty^2 = 1,$$

so we have

$$1 + 1 \neq 2(1+1)$$

as desired.

Now, we must show that we can always find these two disjoint sets. Assume first that we only have the case that, for all $A \in \mathcal{M}$, $\mu(A) = 0$ or $\mu(A) = \infty$. Then we note that the only simple functions in L^p are the trivial ones, and so we have that $\dim(L^p) = 0$. Assume now there exists only one A such that $0 < \mu(A) < \infty$, and for all other $B \in \mathcal{M}$ we have that $\mu(B) = 0$ or ∞. For there to be $f \in L^p$ non-trivial, it must be non-trivial on this A. In other words, we have that $f = a \cdot \chi_A$æ, so that $\dim(L^p) = 1$ in this case. We remark here that if there are two disjoint sets A and B, but either $\mu(A) = 0$ or $\mu(B) = 0$, then we are still in the case of dimension 1; this is because L^p is defined up to almost everywhere equivalence. Finally, we have the case where we do have two disjoint sets A and B such that $0 < \mu(A), \mu(B) < \infty$. As shown above, this gives us that the parallelogram law fails, and so it is not induced by an inner product. □

Problem 9. Show that $L^p(\mathbb{R}^n)$ is separable for $1 \leq p < \infty$. Show that $L^\infty(\mathbb{R}^n)$ is not separable.

Proof. Let D be the collection of simple functions with rational coefficients over intervals with rational endpoints. This is clearly countable, so we check that this is dense. We have that simple functions with compact support are dense, so it suffices to check that this is dense in this space. Let $f = \sum_1^n a_i \chi_{E_i}$ be a function in this space. We wish to show that, for all $\epsilon > 0$, there is a $g \in D$ with

$$\|f - g\|_p < \epsilon.$$

Notice that, for each E_i, we can choose a G_i which is a union of rectangles so that we have

$$\mu(E_i \triangle G_i) < \epsilon'$$

for any $\epsilon' > 0$. Moreover, we can choose rational endpoints sufficiently close to the normal endpoints (call this new set F_i) so that, for each $\epsilon'' > 0$.

$$\mu(G_i \triangle F_i) < \epsilon''.$$

Combining these two facts together, we get that for any $\gamma > 0$, we can choose F_i so that

$$\mu(F_i \triangle E_i) < \gamma.$$

Thus, for each $1 \leq i \leq n$, choose

$$\gamma = \frac{\epsilon^p}{2n(\max\{|a_i|\}_{1 \leq i \leq n})^p}$$

Notice as well for each coefficient a_i, we can choose a coefficient $b_i \in \mathbb{Q}(i)$ such that $|a_i - b_i| < \beta$, for any $\beta > 0$. For each i, then, choose

$$\beta = \frac{\epsilon}{2n(\max\{\mu(F_i)\}_{1\leq i\leq n})^{1/p}}$$

Writing this out then, for $g \in D$, we have

$$\begin{aligned}
\|f-g\|_p &= \left\| \sum_1^n a_i\chi_{E_i} - \sum_1^n b_i\chi_{F_i} \right\|_p \\
&= \left\| \sum_1^n a_i\chi_{E_i} - \sum_1^n a_i\chi_{F_i} + \sum_1^n a_i\chi_{F_i} - \sum_1^n b_i\chi_{F_i} \right\|_p \\
&\leq \left\| \sum_1^n a_i(\chi_{E_i} - \chi_{F_i}) \right\|_p + \left\| \sum_1^n (a_i - b_i)\chi_{F_i} \right\|_p \\
&\leq \sum_1^n |a_i|\|\chi_{E_i} - \chi_{F_i}\|_p + \sum_1^n |a_i - b_i|\|\chi_{F_i}\|_p.
\end{aligned}$$

We see that

$$|a_i|\|\chi_{E_i} - \chi_{F_i}\|_p = |a_i| \left(\int |\chi_{E_i} - \chi_{F_i}|^p \right)^{1/p} = |a_i|\mu(E_i \triangle F_i)^{1/p} < \frac{\epsilon}{2n}$$

for each $1 \leq i \leq n$, and we have

$$|a_i - b_i|\|\chi_{F_i}\|_p < \frac{\epsilon}{2n}$$

for each $1 \leq i \leq n$. Hence, we get that

$$\|f-g\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We can do this for all ϵ, so we get that D is dense in the space of simple functions with compact support, and so D is dense in all of $L^p(\mathbb{R}^n)$.

We now want to see that L^∞ is not separable. Consider the family of functions $\mathcal{F} = \{\chi_{[0,t]^n}\}$. We have

$$\|\chi_{[0,t)^n} - \chi_{[0,s)^n}\|_\infty = 1, \ t \neq s.$$

If there were a countable dense subset, we could take a ball of radius $1/3$ around the points in this set (say D). The union of these balls would then be the whole space, and we have that each of these balls can only contain 1 function from the family, due to the fact that the L^∞ norm is 1. Therefore, we must have that the set is uncountable, a contradiction. $\square$

Problem 10. Suppose $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$æ.

(a) If $1 < p < \infty$, then $f_n \to f$ weakly in L^p.
(b) The result in (a) is false in general for $p = 1$. It is true for $p = \infty$ if μ is σ-finite and weak convergence is replaced by weak* convergence.

Proof. (a) We follow the hint in Folland. That is, we wish to show three facts: Given $g \in L^q$ and $\epsilon > 0$ we want to show

(1) There exists a $\delta > 0$ such that $\int_E |g|^q < \epsilon$ whenever $\mu(E) < \delta$.
(2) There exists a $A \subset X$ such that $\mu(A) < \infty$ and $\int_{X-A} |g|^q < \epsilon$.
(3) For the A in (2), there exists a $B \subset A$ such that $\mu(A - B) < \delta$ and $f_n \to f$ uniformly on B.

We wish to first show that there exists a $\delta > 0$ such that $\int_E |g|^q < \epsilon$ whenever $\mu(E) < \delta$. To do this, we apply absolute continuity (**Corollary 3.6** in Folland). Next, we wish to show that there exists a $A \subset X$ such that $\mu(A) < \infty$ and

$$\int_{X-A} |g|^q < \epsilon.$$

Define $B = \{x \ : \ |g(x)| > 0\}$. We can write $B_n = \{x \ : \ |g(x)| > n^{-1}\}$, and we have that $B_n \nearrow B$. Notice that

$$\lim_{n\to\infty} \int_{X-B_n} |g|^q = \lim_{n\to\infty} \left(\int |g|^q - \int_{B_m} |g|^q \right) = \int |g|^q - \int_B |g|^q = 0.$$

Hence, setting $A = B_n$, we must have that there is a B_n so that

$$\int_{X-A} |g|^q < \epsilon.$$

This is finite, since

$$\infty > \int |g|^q \geq \int_{B_n} |g|^q \geq \int_{B_n} \frac{1}{n^q} = \frac{\mu(B_n)}{n^q},$$

so that

$$\mu(B_n) = \mu(A) \leq n^q \int |g|^q < \infty.$$

Finally, we need to show that there is a $B \subset A$ such that $\mu(A - B) < \delta$ and $f_n \to f$ uniformly on B. We have that $f_n \to f$æ on A, so applying Egoroff we find that there is a measurable subset $C \subset A$ such that $\mu(C) < \epsilon$ and $f_n \to f$ uniformly on $A - C$; denote $A - C$ as B.

We now combine all of the ingredients to show the result. We wish to show that, for all $\epsilon > 0$, we have that there is an N such that for all $n \geq N$,

$$\left| \int f_n g - \int fg \right| < \epsilon.$$

Notice that we can write

$$\left| \int f_n g - \int fg \right| = \left| \int g(f_n - f) \right| \leq \int |g||f_n - f|.$$

Choose the A in (2) such that

$$\int_{X-A} |g|^q < \frac{\epsilon^q}{2 \cdot 6^q M^q}.$$

Choose the B in (3) so that $\mu(A - B) < \delta$, which forces

$$\int_E |g|^q < \frac{\epsilon^q}{2 \cdot 6^q M^q}.$$

We can now consider

$$\int |g||f_n - f| = \int_B |g||f_n - f| + \int_{B^c} |g||f_n - f|.$$

Notice that

$$\int_B |g||f_n - f| \leq \left(\int_B |f_n - f|^p \right)^{1/p} \|g\|_q.$$

Since $f_n \to f$ uniformly on B, we can choose N sufficiently large so that this is as small as we wish. That is, we can choose N so that for all $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon^p}{3\mu(B)\|g\|_q^p}.$$

Notice that this gives us
$$\int_B |g||f_n - f| < \frac{\epsilon}{3}.$$
On the other hand, examining the other integral, we get
$$\int_{B^C} |g||f_n - f| \leq \left(\int_{B^c} |g|^q\right)^{1/q} \|f_n - f\|_p.$$
Since $B \subset A$, we have $X - A \subset X - B$. Furthermore, we can write $X - B = X - A \sqcup A - B$. So we write this as
$$\left(\int_{B^c} |g|^q\right)^{1/q} \|f_n - f\|_p = \left(\int_{X-A} |g|^q + \int_{A-B} |g|^q\right)^{1/q} \|f_n - f\|_p.$$
We notice that
$$\|f_n - f\|_p \leq \|f_n\|_p + \|f\|_p,$$
and since $\sup_n \|f_n\|_p = M < \infty$, we have that (by Fatou)
$$\int |f|^p \leq \liminf_{n\to\infty} \int |f_n|^p \leq M^p \implies \|f\|_p \leq M.$$
So in particular,
$$\begin{aligned}\left(\int_{X-A} |g|^q + \int_{A-B} |g|^q\right)^{1/q} \|f_n - f\|_p &\leq \left(\int_{X-A} |g|^q + \int_{A-B} |g|^q\right)^{1/q} 2M \\ &< \left(\frac{\epsilon^q}{6^q M^q}\right)^{1/q} 2M = \frac{\epsilon}{3}.\end{aligned}$$
Hence, for the choice of N, we have that for all $n \geq N$,
$$\left|\int g(f_n - f)\right| < \frac{2\epsilon}{3} < \epsilon.$$

(b) We wish to find a counterexample for L^1. Consider $f_n = n \cdot \chi_{[0,1/n]}$, $f = 0$ (See **Folland 6.22 (b)**). We have that $f_n \to f$ almost everywhere, and $\sup_n \|f_n\|_1 = 1$. We wish to show now that $f_n \not\to f$ weakly. Consider $g = 1$. Then we have that
$$\int f_n g = \int n\chi_{[0,1/n]} = 1,$$
so that
$$\lim_{n\to\infty} \int f_n g = 1,$$
but
$$\int fg = \int 0 = 0.$$

Now, assume $\sup_n \|f_n\|_\infty = M < \infty$, $f_n \to f$ a.e., μ is σ-finite. Then we wish to show that $f_n \to f$ in the weak* sense. First, we remark that σ-finite gives that $(L^1)^* = L^\infty$. Notice that we have
$$\left|\int f_n g - \int fg\right| \leq \int |g||f_n - f|,$$
where $g \in L^1$. Since $\|f_n\|_\infty \leq M$ for all n, we have that $\mu(\{x \,:\, |f_n(x)| > M\}) = 0$. Consider now $\mu(\{x \,:\, |f(x)| > M\})$. Since $f_n \to f$ almost everywhere, we get that this must also have measure 0 (we could write the inside as a union of the set where $f_n \to f$ and where it doesn't,

and where it does will have measure zero since it holds for all n, and where it doesn't will have measure zero since it converges a.e.), so that $\|f\|_\infty \leq M$. We use this now to note that

$$|g||f_n - f| \leq |g|\|f_n - f\|_\infty \leq 2M|g| \text{æ},$$

and since $g \in L^1$ we have that the dominated convergence theorem gives us that

$$\lim_{n\to\infty} \int |g||f_n - f| = 0.$$

In other words, we have weak* convergence.

□

Problem 11. Complete the proof of **Theorem 6.18** for the cases $p = 1$ and $p = \infty$.

Proof. We first do the case where $p = 1$. We wish to show that $Tf \in L^1(\mu)$, $\|Tf\|_1 \leq C\|f\|_1$. To see this, notice that

$$\|Tf\|_1 = \int |Tf(x)| d\mu(x) = \int \left| \int K(x,y) f(y) d\nu(y) \right| d\mu(x) \leq \int \int |K(x,y)||f(y)| d\nu(y) d\mu(x).$$

Tonelli applies here to give us

$$\int \int |K(x,y)||f(y)| d\nu(y) d\mu(x) = \int \left(\int |K(x,y)| d\mu(x) \right) |f(y)| d\nu(y) \leq \int C|f(y)| d\nu(y)$$
$$= C \int |f(y)| d\nu(y) = C\|f\|_1.$$

Hence, we get that $Tf \in L^1$ and satisfies the desired inequality. Fubini tells us that it converges absolutely for almost every $x \in X$, as desired.

We now do the case where $p = \infty$. We wish to show that $Tf \in L^\infty(\mu)$, $\|Tf\|_\infty \leq C\|f\|_\infty$. However, this follows simply by noting that we have

$$|Tf(x)| = \left| \int K(x,y) f(y) d\nu(y) \right| \leq \int |K(x,y)| \|f\|_\infty d\nu(y) \leq C\|f\|_\infty \text{æ},$$

so we must have that

$$\|Tf\|_\infty \leq C\|f\|_\infty.$$

We have that Tf converges absolutely for almost every $x \in X$, as desired. □

Remark. Thomas O'Hare was a collaborator.

Problem 12 (Folland 6.35)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in (0, \infty)$. Then weak L^p is a quasi-normed vector space with

$$[f]_{p,\infty} := \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p}.$$

Proof. Recall that a norm function is said to be a *quasinorm* if we replace the triangle inequality with

$$\|f + g\| \leq K(\|f\| + \|g\|).$$

To check that weak L^p is a vector space, we just check that it is a subspace of $\mathrm{Fun}(X, \mathbb{C})$. That is, it's closed under scalar multiplication and addition. In doing so, we will also prove two of the required axioms for showing that the norm function given is a quasinorm. We first check closure under scalars. Let $k \in \mathbb{C}$. We have then that

$$[kf]_{p,\infty} = \left(\sup_{\alpha > 0} \alpha^p \lambda_{kf}(\alpha) \right)^{1/p}.$$

Recall that

$$\lambda_{kf}(\alpha) := \mu\left(\{x \ : \ |kf(x)| > \alpha\}\right)$$
$$= \mu\left(\left\{x \ : \ |f(x)| > \frac{\alpha}{|k|}\right\}\right) = \lambda_f\left(\frac{\alpha}{|k|}\right).$$

Hence, rewriting the above, we have

$$[kf]_{p,\infty} = \left(\sup_{\alpha > 0} \alpha^p \lambda_{kf}(\alpha) \right)^{1/p} = \left(\sup_{\alpha > 0} \alpha^p \lambda_f \left(\frac{\alpha}{|k|}\right) \right)^{1/p}.$$

Defining $\beta = \alpha/|k|$, we get that $\alpha = |k|\beta$, so we can rewrite this as

$$[kf]_{p,\infty} = \left(\sup_{|k|\beta > 0} (|k|\beta)^p \lambda_f\left(\beta\right) \right)^{1/p} = |k| \left(\sup_{\beta > 0} \beta^p \lambda_f\left(\beta\right) \right)^{1/p} = |k|[f]_{p,\infty}.$$

So we have closure under scalar multiplication; if f in weak L^p, then kf in weak L^p, since

$$[kf]_{p,\infty} = |k|[f]_{p,\infty} < \infty.$$

Next, we show the triangle inequality. We write out

$$[f + g]_{p,\infty} = \left(\sup_{\alpha > 0} \alpha^p \lambda_{f+g}(\alpha) \right)^{1/p}.$$

Recall that

$$\lambda_{f+g}(\alpha) \leq \lambda_f(\alpha/2) + \lambda_g(\alpha/2)$$

by **Proposition 6.22 (d)** from Folland. So we have that

$$\begin{aligned}
[f+g]_{p,\infty} = \left(\sup_{\alpha>0} \alpha^p \lambda_{f+g}(\alpha)\right)^{1/p} &\leq \left(\sup_{\alpha>0} \alpha^p (\lambda_f(\alpha/2) + \lambda_g(\alpha/2))\right)^{1/p} \\
&\leq \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha/2) + \sup_{\alpha>0} \alpha^p \lambda_g(\alpha/2))\right)^{1/p} \\
&= \left(\sup_{\alpha>0} \alpha^p \lambda_{2f}(\alpha) + \sup_{\alpha>0} \alpha^p \lambda_{2g}(\alpha))\right)^{1/p} \\
&= \left([2f]_{p,\infty}^p + [2g]_{p,\infty}^p\right)^{1/p} = 2([f]_{p,\infty}^p + [g]_{p,\infty}^p)^{1/p},
\end{aligned}$$

using the same tricks as above. Hence, we get that weak L^p is a vector space, since this gives us closure under addition.

Notice that it's clear that this function is positive. If $[f]_{p,\infty} = 0$, we see that for all $\alpha > 0$, $\lambda_f(\alpha) = 0$, which implies that $f = 0$ almost everywhere. Likewise, if $f = 0$ almost everywhere, we get that $[f]_{p,\infty} = 0$. So it suffices to finish showing the quasi-triangle inequality. Notice that we have

$$[f+g]_{p,\infty} \leq 2([f]_{p,\infty}^p + [g]_{p,\infty}^p)^{1/p} \leq 2^{1+1/p}([f]_{p,\infty} + [g]_{p,\infty})$$

via the usual inequality (see page 181 of Folland), and so setting $K = 2^{1+1/p}$, we get the desired result. □

Problem 13 (Folland 6.36)**.** If f in weak L^p and $\mu(\{x : f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if f in weak L^p and L^∞, then $f \in L^q$ for all $q > p$.

Proof. First assume that f in weak L^p and $\lambda_f(0) < \infty$. We wish to show that

$$\int |f|^q d\mu < \infty.$$

We have that f in weak L^p implies that

$$[f]_{p,\infty} = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p} < \infty.$$

We have by **Proposition 6.24** that

$$\int |f|^q d\mu = q \int_0^\infty \alpha^{q-1} \lambda_f(\alpha) d\alpha.$$

Since $p > q$, write

$$\alpha^{q-1} = \alpha^{q-p-1}\alpha^p,$$

so that

$$\int |f|^q d\mu = q \int_0^\infty \alpha^{q-p-1}\alpha^p \lambda_f(\alpha) d\alpha.$$

We would like to use Hölder's inequality and conclude, however there is an issue at 0 if we leave it as is. We then break up the integral to get

$$\int |f|^q d\mu = q \left(\int_0^1 \alpha^{q-1} \lambda_f(\alpha) d\alpha + \int_1^\infty \alpha^{q-p-1}\alpha^p \lambda_f(\alpha) d\alpha\right).$$

For the integral on the right, we use Hölder to get

$$\int_1^\infty \alpha^{q-p-1}\alpha^p \lambda_f(\alpha) d\alpha \leq \|\alpha^p \lambda_f(\alpha)\|_\infty \int_1^\infty \alpha^{q-p-1} d\alpha < \infty.$$

For the integral on the left, we have $\lambda_f(\alpha)$ is bounded on $[0,1]$, so we get that

$$\int_0^1 \alpha^{q-1}\lambda_f(\alpha)d\alpha \le \left(\sup_{\alpha\in[0,1]} \lambda_f(\alpha)\right)\int_0^1 \alpha^{q-1}d\alpha < \infty.$$

Taking qth roots gives us that $f \in L^q$.

Now, assume f in weak L^p and L^∞. Then we have that $\|f\|_\infty = M < \infty$. We wish to show that $f \in L^q$ for all $q > p$. We again use the proposition which gives

$$\int |f|^q d\mu = q\int_0^\infty \alpha^{q-1}\lambda_f(\alpha)d\alpha.$$

Breaking up the integral again, we now break it up at the point M; that is, we have

$$\int |f|^q d\mu = q\left(\int_0^M \alpha^{q-1}\lambda_f(\alpha)d\alpha + \int_M^\infty \alpha^{q-1}\lambda_f(\alpha)d\alpha\right).$$

For $\alpha \ge M$, we have

$$\lambda_f(\alpha) = \mu(\{x \ : \ |f(x)| > \alpha\}) = 0,$$

by definition of essential supremum. So we can rewrite the above integral as

$$\int |f|^q d\mu = q\int_0^M \alpha^{q-1}\lambda_f(\alpha)d\alpha.$$

Again, hit it with Hölder –

$$q\int_0^M \alpha^{q-p-1}\alpha^p\lambda_f(\alpha)d\alpha \le q\|\alpha^p\lambda_f(\alpha)\|_\infty \int_0^M \alpha^{q-p-1}d\alpha < \infty,$$

since f in weak L^p. □

Problem 14 (Folland 6.38)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in (0,\infty)$, $f \ge 0$ a measurable function. Then $f \in L^p$ iff

$$\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty.$$

Proof. Let $F_k = \{x : 2^k \le f(x) < 2^{k+1}\}$. Notice than that we can write $Y = [0,\infty) = \bigsqcup_{k=-\infty}^{\infty} F_k$. So we have that

$$\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha = \sum_{k=-\infty}^{\infty}\int_{F_k} \alpha^{p-1}\lambda_f(\alpha)d\alpha = \sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}} \alpha^{p-1}\lambda_f(\alpha)d\alpha.$$

$(\implies)$: Assume $f \in L^p$. Then we get that

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}} \alpha^{p-1}\lambda_f(\alpha)d\alpha &\ge \sum_{k=-\infty}^{\infty}(2^k)^{p-1}\lambda_f(2^{k+1})\cdot 2^k = \sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^{k+1}) \\ &= 2^{-p}\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\lambda_f(2^{k+1}) = 2^{-p}\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k)\end{aligned}$$

Furthermore, we have

$$p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha = \int f^p d\mu$$

by **Proposition 6.24**, so using our observation above we have that

$$\sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f(2^k) \le \frac{2^p}{p}\int f^p d\mu < \infty.$$

($\Longleftarrow$): Assume that the sum is finite. Then we have that

$$\begin{aligned}\frac{1}{p}\int f^p d\mu = \int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha &\leq \sum_{k=-\infty}^{\infty}(2^{k+1})^{p-1}\lambda_f(2^k)2^k \\ = \sum_{k=-\infty}^{\infty}(2^k)^{p-1}2^{p-1}\lambda_f(2^k)2^k &= 2^{p-1}\sum_{k=-\infty}^{\infty}2^{kp}\lambda_f(2^k) < \infty,\end{aligned}$$

so that we have

$$\int f^p d\mu < \infty \implies f \in L^p.$$

$\square$

Problem 15 (Folland 6.40)**.** If f is a measurable function on X, its decreasing rearrangement is the function $f^* : (0,\infty) \to [0,\infty]$ defined by

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \leq t\}.$$

(a) f^* is nonincreasing. If $f^*(t) < \infty$, then $\lambda_f(f^*(t)) \leq t$ and if $\lambda_f(\alpha) < \infty$ then $f^*(\lambda_f(\alpha)) \leq \alpha$.
(b) $\lambda_f = \lambda_{f^*}$.
(c) If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty}\lambda_f(\alpha) = 0$, and φ is a nonnegative measurable function on $(0,\infty)$, then

$$\int_X \varphi \circ |f| d\mu = \int_0^\infty \varphi \circ f^*(t)dt.$$

In particular, $\|f\|_p = \|f^*\|_p$ for $p \in (0,\infty)$.
(d) For $p \in (0,\infty)$, $[f]_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t)$.
(e) The name rearrangement for f^* comes from the case where f is a nonnegative function on $(0,\infty)$. To see why it is appropriate, pick a step function on $(0,\infty)$ assuming four or five different values and draw the graphs of f and f^*.

Proof. (a) We first wish to show that f^* is nonincreasing; i.e., if $\alpha \geq \beta$, then $f^*(\beta) \leq f^*(\alpha)$. Notice that if $\alpha \geq \beta$, then we have that $\lambda_f(\alpha) \leq \lambda_f(\beta)$, so if t is such that $\lambda_f(\beta) \leq t$, then we have $\lambda_f(\alpha) \leq t$. In other words, by the infimum property, we have $f^*(\alpha) \leq f^*(\beta)$.

Next, assume that $f^*(t) < \infty$. Writing things out, we have

$$\lambda_f(f^*(t)) = \lambda_f(\inf\{\alpha : \lambda_f(\alpha) \leq t\}).$$

Since λ_f is right continuous, letting $M = \inf\{\alpha : \lambda_f(\alpha) \leq t\} = f^*(t)$, we get that for all $\epsilon > 0$,

$$\lambda_f(M + \epsilon) \leq t,$$

and taking $\epsilon \to 0$ gives

$$\lambda_f(M) = \lambda_f(f^*(t)) \leq t,$$

as desired.

Finally, assume that $\lambda_f(\alpha) < \infty$. We have

$$f^*(\lambda_f(\alpha)) = \inf\{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha)\}.$$

We have λ_f is nonincreasing as well, so this implies that $\lambda_f(\beta) \leq \lambda_f(\alpha)$ for all $\beta \geq \alpha$. Thus, the infimum property dictates that

$$f^*(\lambda_f(\alpha)) = \inf\{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha)\} \leq \alpha.$$

(b) We have

$$\lambda_{f^*}(\alpha) = \mu(\{x : f^*(x) > \alpha\}).$$

Notice that we have $f^*(x) > \alpha$ implies $\inf\{t : \lambda_f(t) \leq x\} > \alpha$, and using the fact that λ_f is non-decreasing this tells us that $\lambda_f(\alpha) > x$. Likewise, we have that $\lambda_f(\alpha) > x$ implies $f^*(x) > \alpha$. So using this, we get

$$\{x : f^*(x) > \alpha)\} = \{x : \lambda_f(\alpha) > x\} = (0, \lambda_f(\alpha)),$$

and hence

$$\lambda_{f^*}(\alpha) = \mu((0, \lambda_f(\alpha)) = \lambda_f(\alpha).$$

This holds for all α, so we win.

(c) Notice that the assumption that $\lim_{\alpha\to 0} \lambda_f(\alpha) = 0$ implies $f^*(t) < \infty$ for all $t > 0$. From **Proposition 6.23** and (b), we get

$$\int_X \varphi \circ |f| d\mu = -\int_0^\infty \varphi(\alpha) d\lambda_f(\alpha) = -\int_0^\infty \varphi(\alpha) d\lambda_{f^*}(\alpha) = \int_0^\infty \varphi \circ f^*(t) dt.$$

Thus, taking $\varphi(\alpha) = \alpha^p$, we have

$$\|f\|_p^p = \int |f|^p d\mu = \int_0^\infty (f^*(t))^p dt = \|f^*\|_p^p,$$

and taking pth roots we win.

Remark. The following argument comes from Grafakos' "Classical Fourier Analysis."

(d) We first write out the definition:

$$[f]_{p,\infty} = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p} = \sup_{\alpha>0} \alpha \left(\lambda_f(\alpha)\right)^{1/p}.$$

We first wish to establish that

$$\sup_{t>0} t^{1/p} f^*(t) \leq [f]_{p,\infty}.$$

If $f^*(t) = 0$, we clearly have

$$t^{1/p} f^*(t) = 0 \leq [f]_{p,\infty}.$$

Now, assume that $f^*(t) > 0$, pick $\epsilon > 0$. By definition, this implies that

$$f^*(t) - \epsilon < f^*(t),$$

and as long as $\epsilon < f^*(t)$ we have that so we have that

$$\lambda_f(f^*(t) - \epsilon) > t.$$

Taking everything to the $1/p$ power, we get

$$\lambda_f(f^*(t) - \epsilon)^{1/p} > t^{1/p}.$$

Hence, we have

$$(f^*(t) - \epsilon)\lambda_f(f^*(t) - \epsilon)^{1/p} > t^{1/p}(f^*(t) - \epsilon)$$

Letting $\alpha = f^*(t) - \epsilon$, this gives

$$\alpha\lambda_f(\alpha)^{1/p} \leq \sup_{\alpha>0} \alpha\lambda_f(\alpha)^{1/p} = [f]_{p,\infty}.$$

Since it holds for all such ϵ, we get that

$$t^{1/p} f^*(t) \leq [f]_{p,\infty}.$$

Since it holds for arbitrary $t > 0$ by the cases we considered, we have that

$$\sup_{t>0} t^{1/p} f^*(t) \leq [f]_{p,\infty}.$$

Now, we want to establish the other direction; that is,

$$\sup_{t>0} t^{1/p} f^*(t) \geq [f]_{p,\infty} = \sup_{\alpha>0} \alpha \lambda_f(\alpha)^{1/p}.$$

In the case $\lambda_f(\alpha) = 0$, we clearly have the correct inequality by the same as the above argument, so assume $\lambda_f(\alpha) > 0$. Pick $0 < \epsilon < \lambda_f(\alpha)$, then we have that

$$f^*(\lambda_f(\alpha) - \epsilon) = \inf\{\beta : \lambda_f(\beta) \leq \lambda_f(\alpha) - \epsilon\}.$$

Notice that

$$f^*(\lambda_f(\alpha) - \epsilon) > \alpha;$$

this follows from the fact that if, it were less than or equal to α, we would have that α is in the set, which is a contradiction. Hence, we have

$$f^*(\lambda_f(\alpha) - \epsilon) > \alpha.$$

We use this to then get

$$\alpha \lambda_f(\alpha)^{1/p} < f^*(\lambda_f(\alpha) - \epsilon) \lambda_f(\alpha)^{1/p}.$$

So in general, we have that

$$\alpha(\lambda_f(\alpha) - \epsilon)^{1/p} < f^*(\lambda_f(\alpha) - \epsilon)(\lambda_f(\alpha) - \epsilon)^{1/p},$$

and letting $t = \lambda_f(\alpha) - \epsilon$ we have

$$f^*(t) t^{1/p} \leq \sup_{t>0} f^*(t) t^{1/p}.$$

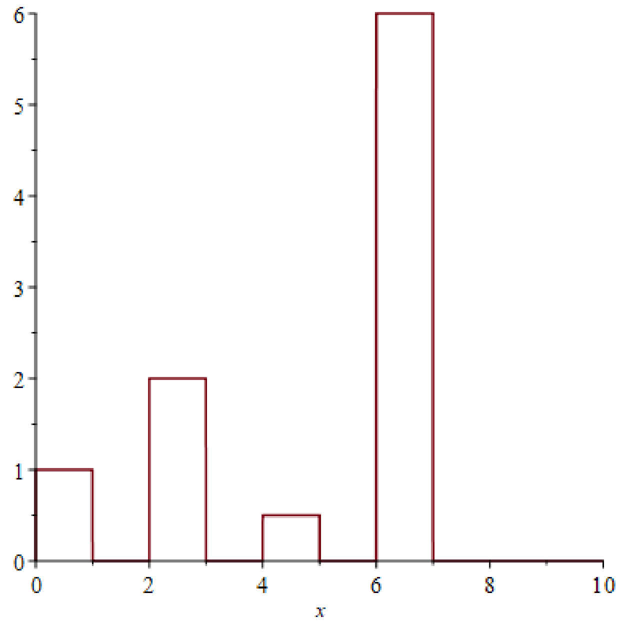
This holds for all $\epsilon > 0$, and so we have that

$$\sup_{t>0} f^*(t) t^{1/p} \geq [f]_{p,\infty}.$$

(e) Consider

$$f(x) = \chi_{(0,1]}(x) + 2\chi_{(2,3]}(x) + 0.5\chi_{(4,5]}(x) + 6\chi_{(6,7]}(x).$$

We have the following graph:



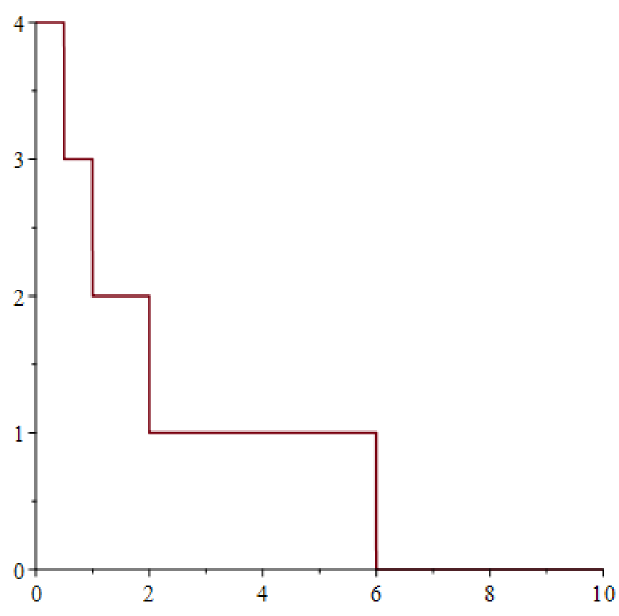
We now wish to calculate the distribution function, $\lambda_f(a)$. Recall that

$$\lambda_f(a) = \mu(\{x : f(x) > a\}),$$

where here μ is Lebesgue measure. We get

$$\lambda_f(a) = \begin{cases} 4 \text{ if } 0 \le a < 0.5, \\ 3 \text{ if } 0.5 \le a < 1 \\ 2 \text{ if } 1 \le a < 2 \\ 1 \text{ if } 2 \le a < 6 \\ 0 \text{ otherwise.} \end{cases}$$

Plotting this, we have



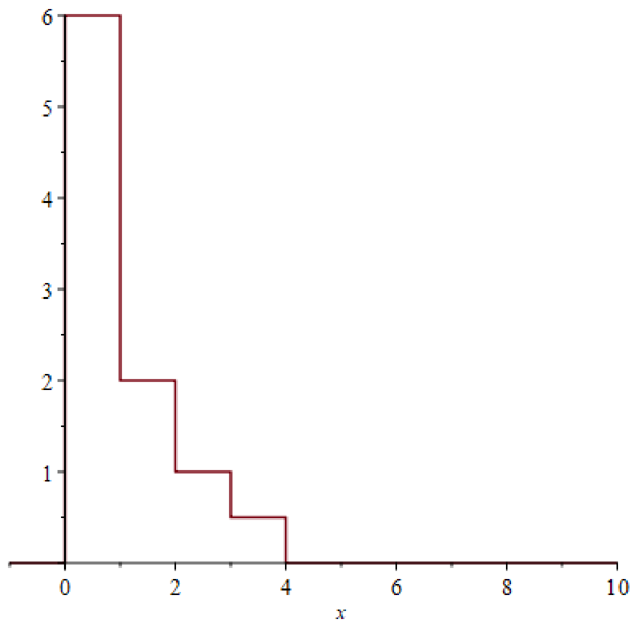
We now wish to calculate

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \le t\}.$$

So calculating things, we have

$$f^*(t) = \begin{cases} 6 \text{ for } 0 \le t < 1 \\ 2 \text{ for } 1 \le t < 2 \\ 1 \text{ for } 2 \le t < 3 \\ 0.5 \text{ for } 3 \le t < 4 \\ 0 \text{ otherwise.} \end{cases}$$

Plotting this, we have



So we see that this function rearranges the values to a decreasing order, hence the name. The Maple code for the plots is given below:

```
f := x-> arrow; piecewise(0 < x and x <= 1,
    1, 2 <= x and x < 3,
    2, 4 <= x and x < 5, .5, 6 <= x and x < 7, 6);
g := x-> piecewise(0 <= x and x < .5,
    4, .5 <= x and x < 1, 3,
    1 <= x and x < 2, 2, 2 <= x and x < 6, 1, 0);
h := x->piecewise(0 <= x and x < 1,
    6, 1 <= x and x < 2, 2,
    2 <= x and x < 3, 1, 3 <= x and x < 4, .5, 0);
plot(f(x),x=0..10);
plot(g(x),x=0..10);
plot(h(x),x=0..10);
```

$\square$

Problem 16. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in (0,\infty)$ and $q \in (0,\infty]$. The *Lorentz space $L^{p,q}(X)$* is defined (identifying functions that are a.e. equal) as the space of all measurable functions f such that

$$[f]_{p,q} = \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty \text{ for } q \in (0, \infty),$$

and

$$[f]_{p,\infty} = \sup \left\{ t^{1/p} f^*(t) : t > 0 \right\} < \infty \text{ for } q = \infty.$$

(a) $L^{p,q}(X)$ is a quasi-normed vector space.
(b) $L^{p,p} = L^p$ and $L^{p,\infty}$ is weak L^p.

Proof. (a) We proceed first for the case $p \in (0, \infty)$, $q \in (0, \infty)$. We do the (now standard) trick of viewing this as a subspace of $\mathrm{Fun}(X, \mathbb{C})$ and showing that it is a subspace. Let $r \in \mathbb{C} - \{0\}$. Then we have

$$[rf]_{p,q} = \left(\int_0^\infty \left(t^{1/p} (rf)^*(t) \right)^q \frac{dt}{t} \right)^{1/q}.$$

Notice that

$$(rf)^*(t) = \inf\{\alpha : \lambda_{rf}(\alpha) \leq t\} = \inf\{\alpha : \lambda_f(\alpha/|r|) \leq t\},$$

and so letting $\beta = \alpha/|r|$, $\alpha = |r|\beta$, we get that

$$(rf)^*(t) = \inf\{|r|\beta \;:\; \lambda_f(\beta) \leq t\} = |r| \inf\{\beta \;:\; \lambda_f(\beta) \leq t\} = |r| f^*(t),$$

so that we have

$$[rf]_{p,q} = \left(\int_0^\infty \left(t^{1/p} |r| f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} = |r| [f]_{p,q}.$$

So it is indeed closed under scalar multiplication for non-zero values. For $r = 0$, notice that

$$(rf)^*(t) = 0,$$

so that

$$[rf]_{p,q} = 0.$$

Hence, it holds for all scalars. Now, we need to show that the triangle inequality is satisfied up to a constant. Notice that we have

$$[f + g]_{p,q} = \left(\int_0^\infty \left(t^{1/p} (f + g)^*(t) \right)^q \frac{dt}{t} \right)^{1/q}.$$

We see that

$$(f + g)^*(t) \leq f^*(t/2) + g^*(t/2).$$

This follows by letting $\alpha = f^*(t/2)$, $\beta = g^*(t/2)$, assuming these values are finite (since otherwise this is trivial), and noting that we have

$$\lambda_{f+g}(\alpha + \beta) \leq \lambda_f(\alpha) + \lambda_g(\beta) = \lambda_f(f^*(t/2)) + \lambda_g(g^*(t/2)) \leq t/2 + t/2 = t,$$

so that

$$(f + g)^*(t) = (f + g)^*(t/2 + t/2) \leq (f + g)^*(\lambda_{f+g}(\alpha + \beta)) \leq \alpha + \beta = f^*(t/2) + g^*(t/2),$$

here abusing property (a) from the last problem. Thus, we have that

$$[f + g]_{p,q} \leq \left(\int_0^\infty \left(t^{1/p} (f^*(t/2) + g^*(t/2)) \right)^q \frac{dt}{t} \right)^{1/q}.$$

Letting $s = t/2$, we have that $2s = t$, $ds = dt/2$, and so

$$\left(\int_0^\infty \left(t^{1/p} (f^*(t/2) + g^*(t/2)) \right)^q \frac{dt}{t} \right)^{1/q} = \left(\int_0^\infty \left((2s)^{1/p} (f^*(s) + g^*(s)) \right)^q \frac{ds}{s} \right)^{1/q}.$$

Now, use the fact that f^*, g^* maps to $[0,\infty)$ (noting the case that f^* or g^* hits ∞ gives us trivially the desired result), we have

$$\left(\int_0^\infty \left((2s)^{1/p}(f^*(s)+g^*(s))\right)^q \frac{ds}{s}\right)^{1/q} \leq \left(\int_0^\infty (2s)^{q/p}2^q(f^*(s)^q+g^*(s)^q)\frac{ds}{s}\right)^{1/q}$$
$$= 2^{1/p+1}([f]_{p,q}^q+[g]_{p,q}^q)^{1/q} \leq 2^{1/p+1/q+1}([f]_{p,q}+[g]_{p,q}).$$

Now, to get that this is a quasinorm, we just need to show that $[f]_{p,q}=0$ implies $f=0$ a.e. (the other properties of a norm follow from the prior arguments). If $[f]_{p,q}=0$, then we get that $f^*(t)=0$ a.e., and this in turn implies that

$$\|f^*\|_p = \|f\|_p = 0,$$

so that $f=0$ a.e. Hence, we get that it's a quasinorm.

Now, let $p\in(0,\infty)$, $q=\infty$. We wish to show that it's a vector space (and in turn, show that this is a norm). Let $r\in\mathbb{C}$, we have that

$$[rf]_{p,\infty} = \sup\left\{t^{1/p}|r|f^*(t): t>0\right\} = |r|\sup\left\{t^{1/p}f^*(t): t>0\right\} = |r|[f]_{p,\infty},$$

so it's closed under scalars. Notice as well we have

$$[f+g]_{p,\infty} = \sup\left\{t^{1/p}(f+g)^*(t): t>0\right\} \leq \sup\left\{t^{1/p}(f^*(t/2)+g^*(t/2)): t>0\right\},$$

and letting $s=t/2$ gives

$$[f+g]_{p,\infty} \leq \sup\left\{(2s)^{1/p}(f^*(s)+g^*(s)): s>0\right\} = 2^{1/p}([f]_{p,\infty}+[g]_{p,\infty}).$$

So it is closed under addition. Furthermore, noticing that

$$[f]_{p,\infty}=0 \implies f^*(t)=0 \implies f=0æ,$$

we have that $[\cdot]_{p,\infty}$ is a quasinorm. That is, we have that $L^{p,q}(X)$ is a quasi-normed vector space for $p\in(0,\infty)$, $q\in(0,\infty]$.

(b) We now wish to show that $L^{p,p}=L^p$, $p\in(0,\infty)$. It suffices to show that $[f]_{p,p}=\|f\|_p$. Notice that

$$[f]_{p,p} = \left(\int_0^\infty f^*(t)pdt\right)^{1/p},$$

and, using property (c) along with the same sort of argument as in **Proposition 6.24**, we have

$$[f]_{p,p} = \left(\int_0^\infty f^*(t)pdt\right)^{1/p} = \left(\int |f|^p\right)^{1/p} = \|f\|_p.$$

Thus, the spaces are equal, since $f\in L^{p,p} \iff [f]_{p,p}<\infty \iff \|f\|_p<\infty \iff f\in L^p$. We now want to show that $L^{p,\infty}$ is weak L^p. To differentiate the notation, let

$$\|f\|_{p,\infty} := \sup_{\alpha>0}\alpha\lambda_f(\alpha)^{1/p}$$

be for weak L^p,

$$[f]_{p,\infty} := \sup_{t>0} t^{1/p}f^*(t)$$

be for the $L^{p,\infty}$ space. We wish to show again that these are equal. This simply follows from part (d) from the last problem. So we get that these spaces are the same, since $f\in$ weak $L^p \iff \|f\|_{p,\infty}<\infty \iff [f]_{p,\infty}<\infty \iff f\in L^{p,\infty}$.

□

Remark. Thomas O'Hare was a collaborator.

The goal of the first four problems is to outline the proof of **Theorem 1** below (the original statement of this theorem is Marcinkiewicz's interpolation theorem).

Theorem. Suppose that (X, μ) and (Y, ν) are measure spaces, with

$$1 \leq p_0 \leq q_0 \leq \infty,\ 1 \leq p_1 \leq q_1 \leq \infty,\ q_0 \neq q_1,\ \text{and}$$

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \text{ and } \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) and (p_1, q_1), then T is strong type (p, q).

We first prove the theorem in the case that $p_0 = p_1$.

Claim (Folland 6.42)**.** The Marcinkiewicz theorem holds in the case $p = p_0 = p_1$.

Remark. The proof comes out of discussions with Thomas and the REU paper "Interpolation Theorems and Applications," by Calista Bernard. The proof in the paper is similar, but doesn't quite finish the idea.

Proof. As remarked by Folland, we show that

$$\lambda_{Tf}(\alpha) \leq \left(\frac{C_j \|f\|_p}{\alpha} \right)^{q_j}.$$

Notice that we have

$$\alpha \lambda_{Tf}(\alpha)^{1/q_j} \leq [Tf]_{q_j} = \sup_{\alpha > 0} \alpha \lambda_{Tf}(\alpha)^{1/q_j} \leq C_j \|f\|_p$$

for all α, so rearranging variables gives us the desired result.

First assume that $q_0 < q_1 < \infty$ (the argument also applies for $q_1 < q_0 < \infty$). Notice that we have (by **Proposition 6.24**)

$$\begin{aligned}
\|Tf\|_q^q &= q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha \\
&= q \int_0^{\|f\|_p} \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha + q \int_{\|f\|_p}^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha \\
&\leq q \int_0^{\|f\|_p} \alpha^{q-1} \left(\frac{C_0 \|f\|_p}{\alpha} \right)^{q_0} d\alpha + q \int_{\|f\|_p}^\infty \alpha^{q-1} \left(\frac{C_1 \|f\|_p}{\alpha} \right)^{q_1} d\alpha \\
&= q C_0^{q_0} \|f\|_p^{q_0} \int_0^{\|f\|_p} \alpha^{q-q_0-1} d\alpha + q C_1^{q_1} \|f\|_p^{q_1} \int_{\|f\|_p}^\infty \alpha^{q-q_1-1} d\alpha \\
&= q C_0^{q_0} \|f\|_p^{q_0} \left(\frac{\alpha^{q-q_0}}{q-q_0} \bigg|_{\alpha=0}^{\|f\|_p} \right) + q C_1^{q_1} \|f\|_p^{q_1} \left(\frac{\alpha^{q-q_1}}{q-q_1} \bigg|_{\alpha=\|f\|_p}^{\infty} \right).
\end{aligned}$$

Here, we note that $q - q_0 > 0$, $q - q_1 < 0$, so that the evaluation of these integrals makes sense. Hence, we have

$$\begin{aligned} qC_0^{q_0}\|f\|_p^{q_0}\left(\frac{\alpha^{q-q_0}}{q-q_0}\bigg|_{\alpha=0}^{\|f\|_p}\right) &+ qC_1^{q_1}\|f\|_p^{q_1}\left(\frac{\alpha^{q-q_1}}{q-q_1}\bigg|_{\alpha=\|f\|_p}^{\infty}\right) \\ &= qC_0^{q_0}\|f\|_p^{q_0}\frac{\|f\|_p^{q-q_0}}{q-q_0} + qC_1^{q_1}\|f\|_p^{q_1}\frac{\|f\|_p^{q-q_1}}{q_1-q} \\ &= \|f\|_p^q q\left(\frac{C_0^{q_0}}{q-q_0} + \frac{C_1^{q_1}}{q_1-q}\right). \end{aligned}$$

Letting B_p be this constant, we have the desired result.

Now, we examine the case where $q_1 = \infty$, $q_0 < \infty$ (the case $q_0 = \infty$, $q_1 < \infty$ is the same). Notice now that we have

$$[Tf]_\infty = \|Tf\|_\infty \le C_1\|f\|_p,$$
$$\lambda_{Tf}(\alpha) \le \left(\frac{C_0\|f\|_p}{\alpha}\right)^{q_0}.$$

Thus, we see that

$$\begin{aligned} \|Tf\|_q^q &= q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \\ &= q\int_0^{\|Tf\|_\infty} \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha + q\int_{\|Tf\|_\infty}^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \\ &= q\int_0^{\|Tf\|_\infty} \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha, \end{aligned}$$

since $\lambda_{Tf}(\alpha) = \mu(\{x : |Tf(x)| > \alpha\}) = 0$ for $\alpha \ge \|Tf\|_\infty$. Continuing on, we have

$$\begin{aligned} q\int_0^{\|Tf\|_\infty} \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha &\le q\int_0^{\|Tf\|_\infty} \alpha^{q-1}\left(\frac{C_0\|f\|_p}{\alpha}\right)^{q_0} d\alpha \\ &= qC_0^{q_0}\int_0^{\|Tf\|_\infty} \alpha^{q-q_0-1}d\alpha = qC_0^{q_0}\left(\frac{\alpha^{q-q_0}}{q-q_0}\bigg|_{\alpha=0}^{\|Tf\|_\infty}\right) \\ &= qC_0^{q_0}\frac{\|Tf\|_\infty^{q-q_0}}{q-q_0} \le qC_0^{q_0}\frac{(C_1\|f\|_p)^{q-q_0}}{q-q_0} \\ &= \frac{qC_0^{q_0}C_1^{q-q_0}}{q-q_0}\|f\|_p^{q-q_0}. \end{aligned}$$

Taking qth roots gives

$$\|Tf\|_q \le \left(\frac{qC_0^{q_0}C_1^{q-q_0}}{q-q_0}\right)^{1/q}\|f\|_p^{(q-q_0)/q}.$$

Now, we take

$$\sup\{\|Tf\|_q : \|f\|_p = 1\} \le \left(\frac{qC_0^{q_0}C_1^{q-q_0}}{q-q_0}\right)^{1/q} = B_p.$$

If $\|f\|_p = 0$, we have trivially that $\|Tf\|_q \le B_p\|f\|_p$. Assume $\|f\|_p \neq 0$, then we can take $\widehat{f} = f/\|f\|_p$. Then $\|\widehat{f}\|_p = 1$, so

$$\|T\widehat{f}\|_q = \frac{\|Tf\|_q}{\|f\|_p} \le B_p \implies \|Tf\|_q \le B_p\|f\|_p,$$

as desired. ∎

Remark. With this, we are now able to assume that $p_0 \neq p_1$ throughout.

Problem 17. Let $A > 0$ and $f \in L^p$. Define

$$h_A = f\chi_{E^c(A)} + A \cdot \operatorname{sgn}(f)\chi_{E(A)},$$
$$g_A = f - h_A.$$

For $p_0 < p_1$, show that

$$\int |g_A|^{p_0} d\mu \leq p_0 \int_A^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta$$

and

$$\int |h_A|^{p_1} d\mu = p_1 \int_0^A \beta^{p_1-1}\lambda_f(\beta)d\beta.$$

Proof. Recall we have the following:

Claim. If g_A, h_A are defined as above, then we have for $E(A) = \{x : |f(x)| > A\}$,

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A),$$
$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) \text{ if } \alpha < A, \\ 0 \text{ if } \alpha \geq A. \end{cases}$$

Proof. Recall that

$$\lambda_{g_A}(\alpha) = \mu(\{x : |g_A(x)| > \alpha\}).$$

We check this statement then on sets. If $x \in E(A)$, then we have that $h_A = A \cdot \operatorname{sgn}(f)$, $|f(x)| > A$, and $f(x) - h_A(x) = f(x) - A \cdot \operatorname{sgn}(f)$, so that $|g_A(x)| = f(x) - A$ if $f(x) > A$ or $|g_A(x)| = f(x) + A$ if $f(x) < -A$. Hence, we can write this as

$$\begin{aligned} \{x : |g_A(x)| > \alpha\} &= \{x : f(x) > A, f(x) - A > \alpha\} \sqcup \{x : f(x) < -A, f(x) + A < \alpha\} \\ &= \{x : f(x) > A, f(x) > \alpha + A\} \sqcup \{x : f(x) < -A, f(x) < \alpha - A\} \\ &= \{x : |f(x)| > A + \alpha\}, \end{aligned}$$

so that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A).$$

For the other hand, notice that

$$\lambda_{h_A}(\alpha) = \mu(\{x : |f(x)| \leq A, f(x) > \alpha\}) + \mu(\{x : |f(x)| > A, A > \alpha\}).$$

For $\alpha < A$, we have that this is equal to

$$\lambda_{h_A}(\alpha) = \mu(\{x : a < |f(x)| \leq A\}) + \mu(\{x : |f(x)| > A\}) = \mu(\{x : a < |f(x)|\}) = \lambda_f(\alpha).$$

For $\alpha \geq A$, we have that

$$\lambda_{h_A}(\alpha) = \mu(\{x : |f(x)| \leq A, f(x) > \alpha\}) + \mu(\{x : |f(x)| > A, A > \alpha\}) = 0 + 0 = 0.$$

∎

Now, we wish to calculate

$$\int |g_A|^{p_0} d\mu.$$

We can use **Proposition 6.24** to write this as

$$\int |g_A|^{p_0} d\mu = p_0 \int_0^\infty \alpha^{p_0-1}\lambda_{g_A}(\alpha)d\alpha.$$

We now use the claim to get that

$$\int |g_A|^{p_0} d\mu = p_0 \int_0^\infty \alpha^{p_0-1} \lambda_f(\alpha + A) d\alpha.$$

Let $u = \alpha + A$, $u - A = \alpha$, $du = d\alpha$, we have that this translates to

$$\int |g_A|^{p_0} d\mu = p_0 \int_A^\infty (u-A)^{p_0-1} \lambda_f(u) du \leq p_0 \int_A^\infty u^{p_0-1} \lambda_f(u) du.$$

Changing variables gives the desired result.

Now, on the other hand, we have (by **Proposition 6.24**)

$$\int |h_A|^{p_1} d\mu = p_1 \int_0^\infty \alpha^{p_1-1} \lambda_{h_A}(\alpha) d\alpha.$$

Using the claim again, we get

$$\int |h_A|^{p_1} d\mu = p_1 \int_0^A \alpha^{p_1-1} \lambda_f(\alpha) d\alpha,$$

as desired. □

Problem 18. Let

$$\sigma = \frac{p_0(q_0 - q)}{q_0(p_0 - p)},$$

and χ_0, χ_1 characteristic functions of $\{(\alpha, \beta) : \beta > \alpha^\sigma\}$ and $\{(\alpha, \beta) : \beta < \alpha^\sigma\}$ respectively. Show that

$$\|Tf\|_q^q \leq \sum_{j=0}^{1} C_j \left[\int_0^\infty \left[\int_0^\infty \varphi_j(\alpha, \beta)^{q_j/p_j} d\alpha \right]^{p_j/q_j} d\beta \right]^{q_j/p_j}, \tag{2}$$

where

$$\varphi_j(\alpha, \beta) = \chi_j(\alpha, \beta) \alpha^{(q - q_j - 1)p_j/q_j} \beta^{p_j - 1} \lambda_f(\beta).$$

Note that this reduces the problem of estimating $\|Tf\|_q$ to estimating the expression on the right hand side of (1) for φ_j.

Proof. We have

$$\|Tf\|_q^q = \int |Tf|^q d\mu.$$

We use **Proposition 6.24** again to get

$$\int |Tf|^q d\mu = q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha.$$

Now, we remark that by construction we have $g_A = f - h_A$ so that $g_A + h_A = f$. Hence, we get

$$|T(f)| \leq |T(h_A)| + |T(g_A)|$$

by sublinearity. Using **Proposition 6.22(d)**, **Proposition 6.22(b)**, and this fact, we have

$$\lambda_{Tf}(\alpha) \leq \lambda_{Th_A}(\alpha/2) + \lambda_{Tg_A}(\alpha/2).$$

Now we use this to write

$$\int |Tf|^q d\mu \leq q \int_0^\infty \alpha^{q-1} \left(\lambda_{Th_A}(\alpha/2) + \lambda_{Tg_A}(\alpha/2) \right) d\alpha.$$

Let $\beta = \alpha/2$, then $2\beta = \alpha$, $2d\beta = d\alpha$, so we have

$$\|Tf\|_q^q \leq q \int_0^\infty (2\beta)^{q-1} (\lambda_{Th_A}(\beta) + \lambda_{Tg_A}(\beta)) 2d\beta = 2^q q \int_0^\infty \beta^{q-1} (\lambda_{Th_A}(\beta) + \lambda_{Tg_A}(\beta)) d\beta.$$

Now, from the assumptions in the theorem, we have that

$$[Tg_A]_{q_0} = \sup_{\beta>0} \beta\lambda_{Tg_A}(\beta)^{1/q_0} \leq C_0\|g_A\|_{p_0},$$

$$[Th_A]_{q_1} = \sup_{\beta>0} \beta\lambda_{Th_A}(\beta)^{1/q_1} \leq C_1\|h_A\|_{p_1},$$

so taking everything to the appropriate power, we get

$$\sup_{\beta>0} \lambda_{Tg_A}(\beta) \leq \sup_{\beta>0} \left(\frac{C_0\|g_A\|_{p_0}}{\beta}\right)^{q_0},$$

$$\sup_{\beta>0} \lambda_{Th_A}(\beta) \leq \sup_{\beta>0} \left(\frac{C_1\|h_A\|_{p_1}}{\beta}\right)^{q_1}.$$

Hence, we have that

$$\|Tf\|_q^q \leq 2^q q \int_0^\infty \beta^{q-1} \left[\left(\frac{C_0\|g_A\|_{p_0}}{\beta}\right)^{q_0} + \left(\frac{C_1\|h_A\|_{p_1}}{\beta}\right)^{q_1}\right] d\beta.$$

Since everything so far is true independent of $\alpha > 0$ and $A > 0$, it is fine to take A to depend on α. Choosing $A = \alpha^\sigma$, where σ defined above, we have that this in conjunction with **Problem 1** gives

$$\begin{aligned}
\|Tf\|_q^q &\leq 2^q q \int_0^\infty \beta^{q-1} \left[\left(\frac{C_0\|g_A\|_{p_0}}{\beta}\right)^{q_0} + \left(\frac{C_1\|h_A\|_{p_1}}{\beta}\right)^{q_1}\right] d\beta \\
&= 2^q q C_0^{q_0} \int_0^\infty \beta^{q-q_0-1}\|g_A\|_{p_0}^{q_0} d\beta + 2^q q C_1^{q_1} \int_0^\infty \beta^{q-q_1-1}\|h_A\|_{p_1}^{q_1} d\beta \\
&= 2^q q C_0^{q_0} \int_0^\infty \beta^{q-q_0-1} \left(\int |g_A|^{p_0}\right)^{q_0/p_0} d\beta + 2^q q C_1^{q_1} \int_0^\infty \beta^{q-q_1-1} \left(\int |h_A|^{p_1}\right)^{q_1/p_1} d\beta \\
&\leq 2^q q C_0^{q_0} \int_0^\infty \beta^{q-q_0-1} \left(p_0 \int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha) d\alpha\right)^{q_0/p_0} d\beta + 2^q q C_1^{q_1} \int_0^\infty \beta^{q-q_1-1} \left(p_1 \int_0^A \alpha^{p_1-1}\lambda_f(\alpha) d\alpha\right)^{q_1/p_1} d\beta \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \beta^{q-q_0-1} \left(\int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha) d\alpha\right)^{q_0/p_0} d\beta \\
&\quad + 2^q q C_1^{q_1} p_1^{q_1/p_1} \int_0^\infty \beta^{q-q_1-1} \left(\int_0^A \alpha^{p_1-1}\lambda_f(\alpha) d\alpha\right)^{q_1/p_1} d\beta \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \left(\int_0^\infty \varphi_0(\beta,\alpha) d\alpha\right)^{q_0/p_0} d\beta + 2^q q C_1^{q_1} p_1^{q_1/p_1} \int_0^\infty \left(\int_0^\infty \varphi_1(\beta,\alpha) d\alpha\right)^{q_1/p_1} d\beta \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \left(\left[\int_0^\infty \left(\int_0^\infty \varphi_0(\beta,\alpha) d\alpha\right)^{q_0/p_0} d\beta\right]^{p_0/q_0}\right)^{q_0/p_0} \\
&\quad + 2^q q C_1^{q_1} p_1^{q_1/p_1} \left(\left[\int_0^\infty \left(\int_0^\infty \varphi_1(\beta,\alpha) d\alpha\right)^{q_1/p_1} d\beta\right]^{p_1/q_1}\right)^{q_1/p_1}
\end{aligned}$$

Now hit it with Minkowski's inequality for integrals (and rearrange the variables so that they're in the right order) to get

$$\|Tf\|_q^q \leq 2^q q C_0^{q_0} p_0^{q_0/p_0} \left[\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0} d\alpha\right)^{p_0/q_0} d\beta\right]^{q_0/p_0}$$
$$+2^q q C_1^{q_1} p_1^{q_1/p_1} \left[\int_0^\infty \left(\int_0^\infty \varphi_1(\alpha,\beta)^{q_1/p_1} d\alpha\right)^{p_1/q_1} d\beta\right]^{q_1/p_1}.$$

In other words, we have

$$\|Tf\|_q^q \leq \sum_{j=0}^{1} C_j \left[\int_0^\infty \left[\int_0^\infty \varphi_j(\alpha,\beta)^{q_j/p_j} d\alpha\right]^{p_j/q_j} d\beta\right]^{q_j/p_j}$$

as desired. □

Problem 19. Show that

$$\int_0^\infty \left[\int_0^\infty \varphi_j(\alpha,\beta)^{q_j/p_j} d\alpha\right]^{p_j/q_j} d\beta = k_j \|f\|_p^p,$$

where

$$k_j = \frac{1}{p|q-q_j|^{p_j/q_j}},$$

and prove **Theorem** for $q_0, q_1 < \infty$.

Proof. We first show it for $j = 0$. Assuming $q_1 > q_0$, $\sigma > 0$, we have that $\beta > \alpha^\sigma$ is equivalent to $\beta^{1/\sigma} > \alpha$, and so we can leverage this to get

$$\int_0^\infty \left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta$$
$$= \int_0^\infty \left[\int_0^\infty \left(\chi_0(\alpha,\beta)\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta)\right)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta$$
$$= \int_0^\infty \left[\int_0^{\beta^{1/\sigma}} \left(\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta)\right)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta$$
$$= \int_0^\infty \beta^{p_0-1}\lambda_f(\beta) \left[\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1} d\alpha\right]^{p_0/q_0} d\beta.$$

Integrating the inside, we get

$$\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1} d\alpha = \frac{1}{q-q_0}\alpha^{q-q_0}\Big|_{\alpha=0}^{\beta^{1/\sigma}} = \frac{\beta^{(q-q_0)/\sigma}}{q-q_0}.$$

Hence, we have the above is equal to

$$(q-q_0)^{-p_0/q_0} \int_0^\infty \beta^{p_0-1+((q-q_0)p_0)/(\sigma q_0)}\lambda_f(\beta) d\beta.$$

Recall that

$$\sigma = \frac{p_0(q_0-q)}{q_0(p_0-p)},$$

so

$$\frac{p_0(q-q_0)}{\sigma q_0} = \frac{p_0(q-q_0)q_0(p_0-p)}{p_0(q_0-q)q_0} = p - p_0,$$

and hence the above is now equal to

$$|q-q_0|^{-p_0/q_0}\int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta.$$

We have **Proposition 6.24** gives us

$$\int |f|^p = p\int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta,$$

so that

$$\frac{\|f\|_p^p}{p} = \int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta.$$

Substituting this in then gives

$$\int_0^\infty \left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0}d\alpha\right]^{p_0/q_0} d\beta = \frac{\|f\|_p^p}{p|q-q_0|^{p_0/q_0}} = k_0\|f\|_p^p.$$

Next, assuming $q_0 > q_1$, $\sigma < 0$, we have that the inequality goes the other direction, i.e., $\beta > \alpha^\sigma$ is equivalent to $\beta^{1/\sigma} < \alpha$. Using this, we then get

$$\begin{aligned}
&\int_0^\infty \left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0}d\alpha\right]^{p_0/q_0} d\beta \\
&= \int_0^\infty \left[\int_0^\infty \left(\chi_0(\alpha,\beta)\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta)\right)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta \\
&= \int_0^\infty \left[\int_{\beta^{1/\sigma}}^\infty \left(\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta)\right)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta \\
&= \int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\left[\int_{\beta^{1/\sigma}}^\infty \alpha^{q-q_0-1}d\alpha\right]^{p_0/q_0} d\beta.
\end{aligned}$$

Integrating the inside, we get

$$\int_{\beta^{1/\sigma}}^\infty \alpha^{q-q_0-1}d\alpha = \frac{1}{q-q_0}\alpha^{q-q_0}\Big|_{\alpha=\beta^{1/\sigma}}^\infty = \frac{\beta^{(q-q_0)/\sigma}}{q_0-q}.$$

Hence, we have the above is equal to

$$|q_0-q|^{-p_0/q_0}\int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta.$$

From the reductions before, we have that

$$\int_0^\infty \left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0}d\alpha\right]^{p_0/q_0} d\beta = \frac{\|f\|_p^p}{p|q-q_0|^{p_0/q_0}} = k_0\|f\|_p^p$$

again.

Following the same path with the other integral, we have

$$\int_0^\infty \left[\int_0^\infty \varphi_1(\alpha,\beta)^{q_0/p_0}d\alpha\right]^{p_0/q_0} d\beta = |q-q_1|^{-p_1/q_1}p^{-1}\|f\|_p^p = k_1\|f\|_p^p.$$

Thus, combining everything, we have

$$\sup\{\|Tf\|_q : \|f\|_p = 1\} \le B_p = 2q^{1/q}\left[\sum_{j=0}^1 C_j^{q_j}(p_j/p)^{q_j/p_j}|q-q_j|^{-1}\right]^{1/q}.$$

Hence, if $\|f\|_p = 0$, we get the desired inequality, and in the case where $\|f\|_p \neq 0$ we can normalize it to via

$$\widehat{f} = \frac{f}{\|f\|_p},$$

so that $\|\widehat{f}\|_p = 1$ and

$$\|T\widehat{f}\|_q = \frac{\|Tf\|_q}{\|f\|_p} \leq B_p \implies \|Tf\|_q \leq B_p \|f\|_p.$$

□

Problem 20. Prove **Theorem 1** in the three exceptional cases:

(i) $p_1 = q_1 = \infty$,
(ii) $p_1 < \infty$, $q_1 = \infty$,
(iii) $p_1 < \infty$, $q_0 = \infty$.

Proof. (i) Assume $p_1 = q_1 = \infty$. Instead of taking $A = \alpha^\sigma$, take $A = \alpha/C_1$. Then we have (by the assumptions of the theorem) that

$$\|Th_A\|_\infty \leq C_1 \|h_A\|_\infty \leq \alpha,$$

so that

$$\lambda_{Th_A}(\alpha) = \mu(\{x : |Th_A(x)| > \alpha\}) = 0.$$

We then apply the same argument as in **Problem 2** to get

$$\int |Tf|^q d\mu \leq q \int_0^\infty \alpha^{q-1} \left(\lambda_{Th_A}(\alpha/2) + \lambda_{Tg_A}(\alpha/2)\right) d\alpha = q \int_0^\infty \alpha^{q-1} \lambda_{Tg_A}(\alpha/2) d\alpha.$$

Again, letting $\beta = \alpha/2$, we have $2\beta = \alpha$, $2d\beta = d\alpha$, so

$$\int |Tf|^q d\mu \leq 2^q q \int_0^\infty \beta^{q-1} \lambda_{Tg_A}(\beta) d\beta.$$

Using the assumptions in the theorem, we get

$$\begin{aligned}
\|Tf\|_q^q &\leq 2^q q \int_0^\infty \beta^{q-1} \left(\frac{C_0 \|g_A\|_{p_1}}{\beta}\right)^{q_1} d\beta \\
&= 2^q q C_0^{q_1} \int_0^\infty \beta^{q-q_0-1} \left(\int |g_A|^{p_0}\right)^{q_0/p_0} d\beta \\
&\leq 2^q q C_0^q \int_0^\infty \beta^{q-q_0-1} \left(p_0 \int_A^\infty \alpha^{p_0-1} \lambda_f(\alpha) d\alpha\right)^{q_0/p_0} d\beta \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \beta^{q-q_0-q} \left(\int_A^\infty \alpha^{p_0-1} \lambda_f(\alpha) d\alpha\right)^{q_0/p_0} d\beta, \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \left(\int_0^\infty \varphi_0(\beta, \alpha) d\alpha\right)^{q_0/p_0} d\beta \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \left(\left[\int_0^\infty \left(\int_0^\infty \varphi_0(\beta, \alpha) d\alpha\right)^{q_0/p_0} d\beta\right]^{p_0/q_0}\right)^{q_0/p_0}.
\end{aligned}$$

where we now redefine

$$\varphi_0(\alpha, \beta) = \chi_0(\alpha, \beta) \alpha^{(q-q_0-1)p_0/q_0} \beta^{p_0-1} \lambda_f(\beta),$$

where $\chi_0(\alpha, \beta)$ is the characteristic function of $\{(\alpha, \beta) : \beta > \alpha/C_1\}$. Moving things around (including changing the order of α and β for notational simplicity), we then have

$$\|Tf\|_q^q \leq 2^q q C_0^{p_0} p_0^{q_0/p_0} \left(\left[\int_0^\infty \left(\int_0^\infty \varphi_0(\beta, \alpha) d\alpha \right)^{q_0/p_0} d\beta \right]^{p_0/q_0} \right)^{q_0/p_0}.$$

Going through the motions, we get again from Minkowski that

$$\|Tf\|_q^q \leq 2^q q C_1^{q_0} p_0^{q_0/p_0} \left[\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha, \beta)^{q_0/p_0} d\alpha \right)^{p_0/q_0} d\beta \right]^{q_0/p_0}.$$

We now examine the inner integral, noticing that we have

$$\begin{aligned} &\int_0^\infty \left(\int_0^\infty \varphi_0(\alpha, \beta)^{q_0/p_0} d\alpha \right)^{p_0/q_0} d\beta \\ &= \int_0^\infty \left(\int_0^\infty \left(\chi_0(\alpha, \beta) \alpha^{(q-q_0-1)p_0/q_0} \beta^{p_0-1} \lambda_f(\beta) \right)^{q_0/p_0} d\alpha \right)^{p_0/q_0} d\beta \\ &= \int_0^\infty \beta^{p_0-1} \lambda_f(\beta) \left(\int_0^{C_1\beta} \alpha^{(q-q_0-1)} d\alpha \right)^{p_0/q_0} d\beta. \end{aligned}$$

Evaluating the inner integral, we get

$$\int_0^{C_1\beta} \alpha^{q-q_0-1} d\alpha = \frac{1}{q-q_0} \alpha^{q-q_0} \Big|_0^{C_1\beta} = \frac{(C_1\beta)^{q-q_0}}{|q-q_0|},$$

so going back we have

$$\begin{aligned} &\int_0^\infty \beta^{p_0-1} \lambda_f(\beta) \left(\int_0^{C_1\beta} \alpha^{(q-q_0-1)} d\alpha \right)^{p_0/q_0} d\beta \\ &= \int_0^\infty \beta^{p_0-1} \lambda_f(\beta) \left(\frac{(C_1\beta)^{q-q_0}}{|q-q_0|} \right)^{p_0/q_0} d\beta \\ &= |q-q_0|^{-p_0/q_0} C_1^{(q-q_0)p_0/q_0} \int_0^\infty \beta^{p_0-1+(q-q_0)p_0/q_0} \lambda_f(\beta) d\beta \\ &= |q-q_0|^{-p_0/q_0} C_1^{p_0 q/q_0 - p_0} \int_0^\infty \beta^{p_0 q/q_0 - 1} \lambda_f(\beta) d\beta. \end{aligned}$$

Now, since we have that (for $0 < t < 1$),

$$\frac{1}{q} = \frac{t}{q_0},$$

we get that

$$\frac{q_0}{q} = t.$$

Furthermore, we have that

$$\frac{1}{p} = \frac{t}{p_0}$$

so that

$$p = \frac{p_0}{t} = \frac{p_0 q}{q_0}.$$

Substituting this in, then, gives

$$\begin{aligned}|q-q_0|^{-p_0/q_0}C_1^{p_0q/q_0-p_0}\int_0^\infty \beta^{p_0q/q_0-1}\lambda_f(\beta)d\beta\\ = |q-q_0|^{-p_0/q_0}C_1^{p-p_0}\int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta\\ = |q-q_0|^{-p_0/q_0}C_1^{p-p_0}p^{-1}\|f\|_p^p.\end{aligned}$$

Going back, we have

$$\begin{aligned}\|Tf\|_q^q &\le 2^q qC_0^{q_0}p_0^{q_0/p_0}\left(|q-q_0|^{-p_0/q_0}C_1^{p-p_0}p^{-1}\|f\|_p^p\right)^{q_0/p_0}\\ &= 2^q qC_0^{q_0}(p_0/p)^{q_0/p_0}C_1^{(p-p_0)q_0/p_0}|q-q_0|^{-1}\|f\|_p^{pq_0/p_0}\\ &= 2^q qC_0^{q_0}(p_0/p)^{q_0/p_0}C_1^{q-q_0}|q-q_0|^{-1}\|f\|_p^q.\end{aligned}$$

Taking qth roots gives then

$$\|Tf\|_q \le 2\left[C_0^{q_0}(p_0/p)^{q_0/p_0}C_1^{q-q_0}|q-q_0|^{-1}\right]^{1/q}\|f\|_p,$$

as desired. Using the same argument as the last problem, we then have

$$\sup\{\|Tf\|_q : \|f\|_p = 1\} \le 2\left[C_0^{q_0}(p_0/p)^{q_0/p_0}C_1^{q-q_0}|q-q_0|^{-1}\right]^{1/q} = B_p,$$

and so if $\|f\|_p \neq 0$ we normalize to get $\widehat{f} = f/\|f\|_p$, and we have

$$\|T\widehat{f}\|_q = \frac{\|Tf\|_q}{\|f\|_p} \le B_p \implies \|Tf\|_q \le B_p\|f\|_p.$$

Hence, it holds for all f.

(ii) Now choose $A = (\alpha/d)^\sigma$, where

$$d = C_1[p_1\|f\|_p^p/p]^{1/p_1},$$

and

$$\sigma = \frac{p_1}{p_1-p}.$$

Since $p_1 > p$ by assumption, we have

$$\|Th_A\|_\infty^{p_1} \le C_1^{p_1}\|h_A\|_{p_1}^{p_1} = C_1^{p_1}p_1\int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha \le C_1^{p_1}p_1A^{p_1-p}\int_0^A \alpha^{p-1}\lambda_f(\alpha)d\alpha = \alpha^{p_1}.$$

We get again that

$$\lambda_{Th_A}(\alpha) = 0,$$

so we have that $\varphi_1 = 0$, and hence going through the the exact same argument as before, we get

$$\|Tf\|_q^q \le 2^q qC_0^{p_0}p_0^{q_0/p_0}\left(\int_0^\infty\left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0}d\alpha\right]^{p_0/q_0}d\beta\right)^{q_0/p_0},$$

noting again that

$$\varphi_0(\alpha,\beta) = \chi_0(\alpha,\beta)\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta),$$

where now the characteristic function is over the set $\{(\alpha,\beta) \ : \ \beta > \alpha^\sigma\}$. Examining the inner integral, we have that

$$\int_0^\infty \left[\int_0^\infty \varphi_0(\alpha,\beta)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta$$

$$= \int_0^\infty \left[\int_0^\infty \left(\chi_0(\alpha,\beta)\alpha^{(q-q_0-1)p_0/q_0}\beta^{p_0-1}\lambda_f(\beta)\right)^{q_0/p_0} d\alpha\right]^{p_0/q_0} d\beta$$

$$= \int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\left[\int_0^{\beta^{1/\sigma}} \alpha^{(q-q_0-1)} d\alpha\right]^{p_0/q_0} d\beta.$$

Solving, we get

$$\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1} d\alpha = \frac{1}{q-q_0}\alpha^{q-q_0}\Big|_{\alpha=0}^{\beta^{1/\sigma}} = \frac{\beta^{(q-q_0)/\sigma}}{|q-q_0|},$$

so substituting this in we have

$$\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\left[\int_0^{\beta^{1/\sigma}} \alpha^{(q-q_0-1)} d\alpha\right]^{p_0/q_0} d\beta$$

$$= \int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\left[\frac{\beta^{(q-q_0)/\sigma}}{|q-q_0|}\right]^{p_0/q_0} d\beta$$

$$= |q-q_0|^{-p_0/q_0}\int_0^\infty \beta^{p_0-1+(q-q_0)p_0/(\sigma q_0)}\lambda_f(\beta) d\beta.$$

Now, we have that

$$\frac{1}{q} = \frac{1-t}{q_0} \implies \frac{q_0}{q} = 1-t,$$

$$p = \frac{p_0p_1}{p_0t - p_1t + p_1},$$

$$p - p_0 = \frac{p_0t(p_1-p_0)}{p_0t - p_1t + p_1}$$

so that

$$\frac{(q-q_0)p_0}{\sigma q_0} = \frac{(q-q_0)p_0(p_1-p)}{p_1q_0} = \frac{tp_0(p-p_1)}{(t-1)p_1}$$

$$= \frac{tp_0(p_1-p_0)}{p_0t-p_1t+p_1} = p - p_0.$$

Substituting this in, we get

$$|q-q_0|^{-p_0/q_0}\int_0^\infty \beta^{p_0-1+(q-q_0)p_0/(\sigma q_0)}\lambda_f(\beta) d\beta$$

$$= |q-q_0|^{-p_0/q_0}\int_0^\infty \beta^{p-1}\lambda_f(\beta) d\beta = |q-q_0|^{-p_0/q_0}p^{-1}\|f\|_p^p.$$

Now bounding the original equation, we have

$$\|Tf\|_q^q \le 2^q q C_0^{p_0} p_0^{q_0/p_0}\left(|q-q_0|^{-p_0/q_0}p^{-1}\|f\|_p^p\right)^{q_0/p_0}$$

$$= 2^q q C_0^{p_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\|f\|_p^{pq_0/p_0}.$$

Taking qth roots gives

$$\|Tf\|_q \leq 2\left[C_0^{p_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\right]^{1/q}\|f\|_p^{pq_0/(qp_0)},$$

and hence

$$\sup\{\|Tf\|_q : \|f\|_p = 1\} \leq 2\left[C_0^{p_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\right]^{1/q} = B_p.$$

Normalizing f when $\|f\|_p \neq 0$, we set $\widehat{f} = f/\|f\|_p$ and get

$$\|T\widehat{f}\|_q = \frac{\|Tf\|_q}{\|f\|_p} \leq B_p \implies \|Tf\|_q \leq B_p\|f\|_p,$$

as desired.

(iii) We now have $q_1 < q_0 = \infty$, $p_0 < p_1 < \infty$. We want to do the same thing we did in part (ii), except we now want to choose our d so that $\lambda_{Tg_A}(\alpha) = 0$. Setting

$$d = C_0[p_0\|f\|_p^p/p]^{1/p_0},$$

$$\sigma = \frac{p_0}{p_0 - p}$$

we have

$$\begin{aligned}
\|Tg_A\|_\infty^{p_0} \leq C_0^{p_0}\|g_A\|_{p_0}^{p_0} &\leq C_0^{p_0}p_0\int_A^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta \\
&= C_0^{p_0}p_0\int_A^\infty \beta^{p_0-p}\beta^{p-1}\lambda_f(\beta)d\beta \\
&\leq C_0^{p_0}p_0A^{p_0-p}\int_A^\infty \beta^{p-1}\lambda_f(\beta)d\beta \\
&= C_0^{p_0}\frac{p_0}{p}\left(\frac{\alpha}{d}\right)^{\sigma(p_0-p)}\|f\|_p^p \\
&= C_0^{p_0}\frac{p_0}{p}\left(\frac{\alpha}{d}\right)^{p_0}\|f\|_p^p = \alpha^{p_0}.
\end{aligned}$$

Hence, we get that $\lambda_{Tg_A}(\alpha) = 0$. This time, we get that $\varphi_0 = 0$, and so going back through the same argument we have that

$$\|Tf\|_q^q \leq 2^q q C_1^{p_1}p_1^{q_1/p_1}\left(\int_0^\infty\left[\int_0^\infty \varphi_1(\alpha,\beta)^{q_1/p_1}d\alpha\right]^{p_1/q_1}d\beta\right)^{q_1/p_1},$$

here φ_1 is defined to be

$$\varphi_1(\alpha,\beta) = \chi_1(\alpha,\beta)\alpha^{(q-q_1-1)p_1/q_1}\beta^{p_1-1}\lambda_f(\beta),$$

with $\chi_1(\alpha,\beta)$ the characteristic function of the set $\{(\alpha,\beta) : \beta < \alpha^\sigma\}$. Examining the integral on the right, we have

$$\begin{aligned}
&\int_0^\infty\left[\int_0^\infty \varphi_1(\alpha,\beta)^{q_1/p_1}d\alpha\right]^{p_1/q_1}d\beta \\
&= \int_0^\infty\left[\int_0^\infty \left(\chi_1(\alpha,\beta)\alpha^{(q-q_1-1)p_1/q_1}\beta^{p_1-1}\lambda_f(\beta)\right)^{q_1/p_1}d\alpha\right]^{p_1/q_1}d\beta \\
&= \int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\left[\int_{\beta^{1/\sigma}}^\infty \alpha^{q-q_1-1}d\alpha\right]^{p_1/q_1}d\beta.
\end{aligned}$$

Evaluating the integral on the inside, we have

$$\int_{\beta^{1/\sigma}}^{\infty} \alpha^{q-q_1-1} d\alpha = -\frac{1}{q-q_1}\alpha^{q-q_1}\Big|_{\alpha=\beta^{1/\sigma}}^{\infty} = \frac{\beta^{(q-q_1)/\sigma}}{|q-q_1|}.$$

Substituting this in, we have

$$\begin{aligned} &\int_0^{\infty} \beta^{p_1-1}\lambda_f(\beta)\left[\int_{\beta^{1/\sigma}}^{\infty} \alpha^{q-q_1-1}d\alpha\right]^{p_1/q_1} d\beta \\ &= \int_0^{\infty} \beta^{p_1-1}\lambda_f(\beta)\left[\frac{\beta^{(q-q_1)/\sigma}}{|q-q_1|}\right]^{p_1/q_1} d\beta \\ &= |q-q_1|^{-p_1/q_1}\int_0^{\infty} \beta^{p_1-1+p_1(q-q_1)/(\sigma q_1)}\lambda_f(\beta)d\beta. \end{aligned}$$

Again, we note that

$$\frac{1}{q} = \frac{t}{q_1} \implies \frac{q_1}{q} = t,$$

$$p - p_1 = \frac{(t-1)p_1(p_1-p_0)}{p_0 t - p_1 t + p_1},$$

so that

$$\begin{aligned} \frac{p_1(q-q_1)}{q_1\sigma} = \frac{(q_1-q)p_1(p-p_0)}{p_0 q_1} &= \frac{(t-1)p_1(p-p_0)}{tp_0} \\ &= \frac{(t-1)p_1(p_1-p_0)}{p_0 t - p_1 t + p_1} = p - p_1. \end{aligned}$$

Substituting this in, we have

$$\begin{aligned} &|q-q_1|^{-p_1/q_1}\int_0^{\infty} \beta^{p_1-1+p_1(q-q_1)/(\sigma q_1)}\lambda_f(\beta)d\beta \\ &= |q-q_1|^{-p_1/q_1}\int_0^{\infty} \beta^{p-1}\lambda_f(\beta)d\beta = |q-q_1|^{-p_1/q_1}p^{-1}\|f\|_p^p. \end{aligned}$$

Hence, going back, we have

$$\begin{aligned} \|Tf\|_q^q &\le 2^q q C_1^{p_1} p_1^{q_1/p_1}\left(|q-q_1|^{-p_1/q_1}p^{-1}\|f\|_p^p\right)^{q_1/p_1} \\ &= 2^q q C_1^{p_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\|f\|_p^{pq_1/p_1}, \end{aligned}$$

so taking qth roots we get

$$\|Tf\|_q \le 2\left[qC_1^{p_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\right]^{1/q}\|f\|_p^{pq_1/(p_1 q)}.$$

Again, we have

$$\sup\{\|Tf\|_q : \|f\|_p = 1\} \le 2\left[qC_1^{p_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\right]^{1/q} = B_p,$$

so again, as long as $\|f\|_p \neq 0$, we can normalize to get $\widehat{f} = f/\|f\|_p$, and we have

$$\|T\widehat{f}\|_q = \frac{\|Tf\|_q}{\|f\|_p} \le B_p \implies \|Tf\|_q \le \|f\|_p B_p.$$

Thus, we have the desired result.

□

Problem 21. If $f \in \mathcal{S}$ (the Schwartz space), then $\partial^\alpha f \in L^p$ for all α multi-index and all $p \in [1,\infty]$.

Proof. Let $p \in [1, \infty]$. We wish to show that

$$\|\partial^\alpha f\|_p^p = \int |\partial^\alpha f|^p < \infty$$

for all α. Recall that $f \in \mathcal{S}$ implies that $\|f\|_{(N,\alpha)} < \infty$ for all N, α; that is,

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)| < \infty.$$

Let $C_N = \|f\|_{(N,\alpha)}$. We have then that

$$|\partial^\alpha f(x)| \le C_N (1 + |x|)^{-N}$$

for all x, using the fact that the supremum of the product is the product of the supremum and

$$(1 + |x|)^N \le \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \text{ for all } x \in \mathbb{R}^n,$$

so

$$1 \le \frac{\sup_{x \in \mathbb{R}^n} (1 + |x|)^N}{(1 + |x|)^N},$$

and hence

$$|\partial^\alpha f(x)| \le \frac{\sup_{x \in \mathbb{R}^n} (1 + |x|)^N}{(1 + |x|)^N} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| = C_N (1 + |x|)^{-N}.$$

Hence, we have

$$\|\partial^\alpha f\|_p^p = \int |\partial^\alpha f(x)|^p dx \le C_N^p \int (1 + |x|)^{-pN} dx.$$

By **Corollary 2.52 (b)** (i.e. polar coordinates), we have that the integral on the right is in L^1 if $pN > n$; in other words, if $N > n/p$ (n here the dimension of $\mathbb{R}^n$). Since the above holds true for all $N \ge 0$ and α multi-indices, we can choose N sufficiently large so that this holds. ☐

Problem 22. If $1 \le p < \infty$, then translation is continuous in the L^p norm, i.e., if $f \in L^p$ and $z \in \mathbb{R}^n$, then

$$\lim_{y \to 0} \|\tau_{y+z} f - \tau_z f\|_p = 0.$$

Proof. We use **Proposition 7.9**, which says that continuous functions with compact support are dense in L^p. First, let's show the result for continuous functions with compact support. We have that

$$\|\tau_{y+z} f - \tau_z f\|_p^p = \int |\tau_{y+z} f(x) - \tau_z f(x)|^p dx = \int |f(x + y + z) - f(x + z)|^p dx.$$

Let $u = x + z$, then $du = dx$, and so we have that the transformation gives

$$\|\tau_{y+z} f - \tau_z f\|_p^p = \int |f(u + y) - f(u)|^p du.$$

Since we're assuming that f is a continuous function with compact support, we have that there is a compact set K which contains the support of $f(u + y)$ for all $|y| \le 1$. We want to let $y \to 0$, so it suffices to consider such y. Thus, we can write

$$\int |f(u+y) - f(u)|^p du = \int_K |f(u+y) - f(u)|^p du \le \int_K \|f(u+y) - f(u)\|_u^p du = \mu(K) \|f(u+y) - f(u)\|_u^p.$$

By uniform continuity of continuous functions with compact support (see **Lemma 8.4**), we get that this goes to 0 as $y \to 0$. That is, we have that

$$\|\tau_{y+z} f - \tau_z f\|_p^p \to 0 \text{ as } y \to 0,$$

so

$$\|\tau_{y+z} f - \tau_z f\|_p \to 0 \text{ as } y \to 0.$$

Now the result holds for continuous functions with compact support. We use the density in L^p to get the result for general functions. Let $f \in L^p$ and fix $\epsilon > 0$. There is a g which is continuous and has compact support so that $\|g - f\|_p < \epsilon/3$ by density, so we have

$$\|\tau_{y+z}f - \tau_z f\|_p \leq \|\tau_{y+z}(f-g)\|_p + \|\tau_{y+z}g - \tau_z g\|_p + \|\tau_z(g-f)\|_p < \frac{2}{3}\epsilon + \|\tau_{y+z}g - \tau_z g\|_p.$$

Using the result prior, we get that we can take y sufficiently small so that

$$\|\tau_{y+z}g - \tau_z g\|_p < \epsilon/3,$$

and we get that

$$\|\tau_{y+z}f - \tau_z f\|_p < \epsilon,$$

as desired. □

Problem 23 (Folland 8.4)**.** If $f \in L^\infty$ and

$$\|\tau_y f - f\|_\infty \to 0$$

as $y \to 0$, then f agree a.e. with a uniformly continuous function.

Proof. Since $f \in L^\infty$, we have $f \in L^1_{\text{loc}}$, since for any K bounded we have that

$$\int_K f(x) \leq \|f\|_\infty \int_K = \mu(K)\|f\|_\infty < \infty.$$

We then follow the hint. Let

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy.$$

We wish to first establish that $A_r f$ is uniformly continuous for $r > 0$. Notice that we have

$$\begin{aligned}
|A_r f(x) - A_r f(y)| &= \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z)dz - \frac{1}{m(B(r,y))} \int_{B(r,y)} f(z)dz \right| \\
&= \left| \frac{1}{m(B(r,y))} \int_{B(r,y)} f(z+x-y)dz - \frac{1}{m(B(r,y))} \int_{B(r,y)} f(z)dz \right|,
\end{aligned}$$

where we shift things around so both are at y. Now, we have

$$\begin{aligned}
&\left| \frac{1}{m(B(r,y))} \int_{B(r,y)} f(z+x-y)dz - \frac{1}{m(B(r,y))} \int_{B(r,y)} f(z)dz \right| \\
&= \left| \frac{1}{m(B(r,y))} \int_{B(r,y)} [f(z+x-y) - f(z)]dz \right| \\
&\leq \frac{1}{m(B(r,y))} \int_{B(r,y)} |f(z+x-y) - f(z)|dz \\
&= \frac{1}{m(B(r,y))} \int_{B(r,y)} |\tau_{x-y}f(z) - f(z)|dz \\
&\leq \frac{1}{m(B(r,y))} \int_{B(r,y)} \|\tau_{x-y}f - f\|_\infty dz \\
&= \frac{\|\tau_{x-y}f - f\|_\infty}{m(B(r,y))} \int_{B(r,y)} dz = \|\tau_{x-y}f - f\|_\infty.
\end{aligned}$$

So since $|A_r f(x) - A_r f(y)| \leq \|\tau_{x-y}f - f\|_\infty$, we can find δ sufficiently small so that for $|x-y| < \delta$, $|A_r f(x) - A_r f(y)| \leq \|\tau_{x-y}f - f\|_\infty < \epsilon$. Hence, it's uniformly continuous.

Next, we need to show that $A_r f$ is uniformly Cauchy as $r \to 0$. Let $r = 1/n$, then we wish to show that as $n \to \infty$, $A_{1/n} f$ is uniformly Cauchy. That is, for every $\epsilon > 0$, there exists an $N > 0$ so that for all x, $n, m \geq N$, we have that

$$|A_{1/n} f(x) - A_{1/m}(x)| < \epsilon.$$

Plugging in things directly, letting $r = 1/n$ and $s = 1/m$, we have

$$|A_r f(x) - A_s f(x)| = |A_r f(x) - f(x) + f(x) - A_s f(x)| \leq |A_r f(x) - f(x)| + |f(x) - A_s f(x)|.$$

We now examine the first inequality; we have

$$\begin{aligned} |A_r f(x) - f(x)| &= \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z) dz - f(x) \right| \\ &= \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z) dz - \frac{f(x)}{m(B(r,x))} \int_{B(r,x)} dz \right| \\ &\leq \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(z) - f(x)| dz. \end{aligned}$$

Now we notice that

$$\tau_{x-z} f(z) = f(z + x - z) = f(x),$$

so that we have

$$\frac{1}{m(B(r,x))} \int_{B(r,x)} |f(z) - f(x)| dz = \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(z) - \tau_{x-z} f(z)| dz.$$

Notice here that $z \in B(r,x) \cup B(s,x)$. Choosing r and s sufficiently small (i.e., n, m sufficiently large), we have from above that $\|f - \tau_{x-z} f\|_\infty < \epsilon/2$ for all such z. Hence, we get that we can bound the above by $\epsilon/2$. The second inequality is analogous, so we have that

$$|A_r f(x) - A_s f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall x \in \mathbb{R}^n.$$

Hence, we have that it's uniformly Cauchy.

Since $(A_{1/n} f)$ is uniformly Cauchy, it converges uniformly to some function g, and this function is uniformly continuous since the $A_{1/n} f$ are all uniformly continuous. Hence, applying **Theorem 3.18**, we get that $g = f$ almost everywhere, with g uniformly continuous. $\square$

Remark. Thomas O'Hare was a collaborator.

Problem 24. If $K \subset \mathbb{R}^n$ is compact and U is an open set containing K, there exists $f \in C_c^\infty$ such that $0 \leq f \leq 1$, $f = 1$ on K, $\text{supp}(f) \subset U$.

Proof. We first must establish the existence of smooth bump functions (**Equation 8.1** in Folland and the remarks before (pg. 236), **Exercise 8.3** in Folland). Let $\eta(t) := e^{-1/t}\chi_{(0,\infty)}(t)$. We wish to show that this is a smooth function. First, we claim that for $k \in \mathbb{N}$, we have that $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$ when $t > 0$, where P_k is a polynomial of degree $2k$. To see this, we simply induct. The case $k = 0$ is clear, and so assume it holds for $k-1$. Then we have

$$\eta^{(k-1)}(t) = P_{k-1}(1/t)e^{-1/t},$$

and so taking derivatives on each side we have

$$\begin{aligned}\eta^{(k)}(t) &= P_{k-1}'(1/t)(-1/t^2)e^{-1/t} + P_{k-1}(1/t)e^{-1/t}(1/t)^2 \\ &= e^{-1/t}(1/t^2)\left[P_{k-1}(1/t) - P_{k-1}'(1/t)\right].\end{aligned}$$

Letting $x = 1/t$, we can rewrite this as

$$e^{-x}x^2\left[P_{k-1}(x) - P_{k-1}'(x)\right].$$

Now, notice that the degree of the polynomial $P_{k-1}(x) - P_{k-1}'(x)$ is going to be $2k-2$, since derivatives drop a degree, and hence multiplying it by x^2 we have that the polynomial has degree $2k$. In other words, after rewriting terms, we get that

$$\eta^{(k)}(t) = e^{-1/t}P_k(1/t).$$

Notice that, for all $k \in \mathbb{N}$, we have that

$$\lim_{n\to\infty} n^k e^{-n} = 0;$$

to see this, we rewrite this as

$$\lim_{n\to\infty} \frac{n^k}{e^n}$$

and use L'Hospitals. We proceed by induction on k. For $k = 0$, we get this clearly is 0, so assume it holds for $k-1$. Applying L'Hospital here, we have

$$\lim_{n\to\infty} \frac{n^k}{e^n} = \lim_{n\to\infty} \frac{kn^{k-1}}{e^n} = 0$$

by the induction hypothesis. Thus, we have the desired result.

Going back, we now wish to show that $\eta^{(k)}(0) = 0$ for all k. Again, going by induction on k, we have that the case $k = 0$ is by definition, and so we assume it holds for $k-1$. For the derivative on the left, it's clear that we will have 0 (since this is the constant 0 function), so it suffices to consider the derivative on the right; i.e., consider the limit as $h \to 0^+$. Thus, we have

$$\eta^{(k)}(0) = \lim_{h\to 0^+} \frac{\eta^{(k-1)}(h) - \eta^{(k-1)}(0)}{h} = \lim_{h\to 0^+} \frac{\eta^{(k-1)}(h)}{h}.$$

Using what we derived prior, we have that writing $h = 1/n$ and taking $n \to \infty$, this can be written as

$$\eta^{(k)}(0) = \lim_{n\to\infty} n\eta^{(k-1)}(1/n) = \lim_{n\to\infty} nP_{k-1}(n)e^{-n} = 0,$$

using the fact that $n^k e^{-n} \to 0$ as $n \to \infty$. Hence, the derivative on the right is also 0, and so we have the desired result; that is, $\eta(t) \in C^\infty$.

From here on, we follow Folland. To make a smooth bump function, we define

$$\psi(x) = \eta(1 - |x|^2) = e^{1/(|x|^2-1)}\chi_{(0,1)}(|x|).$$

We have then that $0 \le \psi(x) \le 1$, and furthermore $\psi \in C_c^\infty(\mathbb{R}^n)$, since the composition of smooth functions is smooth. We let $\delta = d(K, U^c) > 0$ (note this is positive since K compact). Let $V = \{x : d(x,K) < \delta/3\}$. We then wish to construct a nonnegative $\varphi \in C_c^\infty$ with $\int \varphi = 1$ and $\varphi(x) = 0$ for $|x| \ge \delta/3$. Noting that $\psi \in L^1$ (clearly, since it has compact support and is continuous) we have that a good candidate is

$$\varphi(x) = \frac{3^n\psi(3x/\delta)}{\delta^n\left(\int\psi\right)}.$$

Notice that

$$\int \varphi = \int \frac{3^n\psi(3x/\delta)}{\delta^n\left(\int\psi\right)} = \frac{\int\psi}{\int\psi} = 1$$

after a change of variables ($u = 3x/\delta$, $du = (3/\delta)^n dx)$), and notice that, since ψ is non-zero for $|x| < 1$, we have φ is non-zero for $|x| < \delta/3$. That is, φ is 0 for $|x| \ge \delta/3$. This then fits the criteria of what we want.

Now, we can set $f = \chi_V * \varphi$. Notice that $f \in C^\infty$, since $\varphi \in C_c^\infty$ and $\chi_V \in L^1$, so we can apply **Proposition 8.10** to get the desired result. Notice as well that f has compact support by **Proposition 8.6 (d)**. Hence, $f \in C_c^\infty$. We then check that all the properties for Urysohn are satisfied:

(1) We see that $0 \le f \le 1$, since (letting V'_x be V shifted by x) we have

$$f(x) = \int \chi_V(x-y)\varphi(y)dy = \int_{V'_x}\varphi(y)dy,$$

and by what we've done earlier we have that

$$0 \le \int_{V'_x}\varphi(y)dy \le \int\varphi(y)dy = 1,$$

so that

$$0 \le f(x) \le 1$$

for all x.

(2) Taking $z \in K$ now, we have

$$f(z) = \int_V \varphi(z-y)dy = \int_V \frac{3^n\psi(3(z-y)/\delta)}{\delta^n\left(\int\psi\right)}dy.$$

Let $u = 3(z-y)/\delta$, then $du = (-3/\delta)^n dy$, so we have

$$f(z) = -\int_V \frac{\psi(u)}{\int\psi}du = -\int_{\mathbb{R}^n}\frac{\psi(u)}{\int\psi}du = \int_{\mathbb{R}^n}\frac{\psi(u)}{\int\psi}du = 1.$$

So $f(z) = 1$ for all $z \in K$.

(3) Finally, we note that **Proposition 8.6 (d)** gives

$$\operatorname{supp}(f) \subset \overline{\{x + y : x \in \operatorname{supp}(\chi_V), y \in \operatorname{supp}(\varphi)\}} \subset U.$$

To see this, simply note that

$$\overline{\operatorname{supp}(\chi_V) + \operatorname{supp}(\varphi)} = \overline{V + \{x : |x| \le \delta/3\}} \subset \{x : d(x,K) \le 2\delta/3\} \subset U,$$

since $V = K + \{x : |x| < \delta/3\}$, so $V + \{x : |x| \le \delta/3\} \subset K + \{x : |x| \le 2\delta/3\} = \{x : d(x,K) \le 2\delta/3\}$, and taking the closure of everything preserves containment.

Hence, we have that f satisfies all of the criteria. $\square$

Problem 25. Suppose that $p \in (1, \infty)$ and $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y \to 0} \left\| \frac{\tau_{-y} f - f}{y} - h \right\|_p = 0,$$

we call h the *strong* L^p *derivative of* f. Suppose that p and q are conjugate exponents, $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, and the L^p derivative Df exists. Then $D(f * g)$ exists (in the ordinary sense) and is equal to $(Df) * g$.

Proof. We wish to show that $D(f * g) = (Df) * g$. To see this, we pick a point x and examine

$$|D(f * g)(x) - (Df) * g(x)|.$$

We write out the function to get

$$\begin{aligned}
&\lim_{y \to 0} \left| \frac{\tau_{-y}(f * g)(x) - (f * g)(x)}{y} - ((Df) * g)(x) \right| \\
&= \lim_{y \to 0} \left| \frac{((\tau_{-y} f) * g)(x) - (f * g)(x)}{y} - ((Df) * g)(x) \right| \\
&= \lim_{y \to 0} \left| \frac{(((\tau_{-y} f) - f) * g)(x)}{y} - ((Df) * g)(x) \right| \\
&= \lim_{y \to 0} \left| \left(\frac{((\tau_{-y} f) - f)}{y} * g \right)(x) - ((Df) * g)(x) \right| \\
&= \lim_{y \to 0} \left| \left[\left(\frac{((\tau_{-y} f) - f)}{y} - Df \right) * g \right](x) \right| \\
&= \lim_{y \to 0} \left| \int g(x - z) \left(\frac{((\tau_{-y} f) - f)}{y} - Df \right)(z) dz \right| \\
&\leq \lim_{y \to 0} \int |g(x - z)| \left| \left(\frac{((\tau_{-y} f) - f)}{y} - Df \right)(z) \right| dz \\
&\leq \lim_{y \to 0} \|g\|_q \left\| \frac{((\tau_{-y} f) - f)}{y} - Df \right\|_p = 0.
\end{aligned}$$

Hence, we have $D(f * g)(x) = (Df) * g(x)$ for all x, and so we have the desired result. $\square$

Problem 26. For $p \in (1, \infty)$ if $f \in L^p(\mathbb{R})$ then the L^p derivative h of f exists iff f is absolutely continuous on every bounded interval up to a modification on a null set and its pointwise derivative f' is in L^p, in which case $h = f'$ a.e.

Proof. ($\Longrightarrow$): Assume f has L^p derivative h. We follow the hint in Folland. Choose $g \in C_c(\mathbb{R})$ with $\int g = 1$. Define $g_t(x)$ in the usual way. By the prior problem, we have $D(f * g_t) = D(f) * g_t = h * g_t$, so this is differentiable with derivative $h * g_t$. Notice now that for every pair $x \leq y \in \mathbb{R}$, we get that

$$(f * g_t)(y) - (f * g_t)(x) = \int_x^y (h * g_t)(z) dz.$$

Now, by **Theorem 8.14 (a)**, we note that $h * g_t \to h$ as $t \to 0$ in the L^p norm, and $f * g_t \to f$ in the L^p norm as $t \to 0$. We note that we also have

$$\lim_{t \to 0} \int_x^y h * g_t(z) = \int_x^y h(z) dz.$$

To see this, use the L^p convergence and Hölder to get that

$$\|(h * g_t)\chi_{[x,y]} - h\chi_{[x,y]}\| = \|[(h * g_t) - h]\chi_{[x,y]}\| \leq \|h * g_t - h\|_p \|\chi_{[x,y]}\|_q \to 0.$$

Writing things out, we have

$$\left|\int_x^y [(h*g_t)(z) - h(z)]dz\right| \to 0,$$

giving us the desired result.

From prior homework (**Homework 1, Problem 5** or **Folland 6.9**), we have that $f*g_t \to f$ in the L^p norm implies there is a subsequence (t_j) so that $f*g_{t_j} \to f$ almost everywhere. Fix some x so that $f*g_j(x) \to f(x)$. For all y, then, we have

$$(f*g_{t_j})(y) = (f*g_{t_j})(x) + \int_x^y (h*g_{t_j})(z)dz.$$

Define a function

$$p(y) = \lim_{j\to\infty}\left[f*g_{t_j}(x) + \int_x^y (h*g_{t_j})(z)dz\right] = f(x) + \int_x^y h(z)dz.$$

Noting that $h \in L^1([a,b])$ for any bounded interval $[a,b]$ (since things are finite), we get that p is absolutely continuous on any bounded interval by the **Fundamental Theorem of Lebesgue Integrals** (**Theorem 3.35**). Furthermore, $p = f$ almost everywhere, and so $p' = h = f'$ pointwise almost everywhere (**Corollary 3.31**). So redefining f to be p on some null set, we get that f is absolutely continuous, pointwise differentiable, and the derivative will be h.
($\Longleftarrow$): We again follow Follands hint. Assume that f is absolutely continuous on every bounded interval and its pointwise derivative is f', which is in L^p. We wish to show that the L^p derivative of f exists. Note that

$$\frac{f(x+y)-f(y)}{y} - f'(x) = \frac{1}{y}\int_0^y f'(x+t)dt - f'(x)$$
$$= \frac{1}{y}\int_0^y f'(x+t)dt - \frac{1}{y}\int_0^y f'(x)dt = \frac{1}{y}\int_0^y [f'(x+t)-f'(x)]dt.$$

Taking the p norms of both sides gives

$$\begin{aligned}\left\|\frac{f(x+y)-f(x)}{y} - f'(x)\right\|_p &= \left(\int\left|\frac{1}{y}\int_0^y [f'(x+t)-f'(x)]dt\right|^p dx\right)^{1/p}\\ &\leq \left(\left(\frac{1}{y}\int_0^y |f'(x+t)-f'(x)|dt\right)^p dx\right)^{1/p}\\ &\leq \frac{1}{y}\int_0^y\left(\int |f'(x+t)-f'(x)dx\right)^{1/p} dt\\ &= \frac{1}{y}\int_0^y \|\tau_{-t}(f')-f'\|_p dt,\end{aligned}$$

where the first inequality follows from the triangle inequality and the second from Minkowski's inequality for integrals. Fix $\epsilon > 0$. By **Proposition 8.5**, since $f \in L^p$, we can find $\delta > 0$ so that for $|t| < \delta$, we have $\|\tau_{-t}(f') - f'\|_p < \epsilon$. Choosing $|y| < \delta$, we have

$$\left\|\frac{f(x+y)-f(x)}{y} - f'(x)\right\|_p \leq \frac{1}{y}\int_0^y \epsilon dt = \epsilon.$$

Letting $\epsilon \to 0$ gives us the desired result. $\square$

Problem 27. Show that $\{e^{2\pi i\kappa\cdot x}\}_{\kappa\in\mathbb{Z}^n}$ is an orthonormal basis in $L^2(\mathbb{T}^n)$.

Proof. We first show that this is an orthonormal set. Let $\kappa, \gamma \in \mathbb{Z}^n$. We have that

$$\langle e^{2\pi i\kappa\cdot x}, e^{2\pi i\gamma\cdot x}\rangle = \int_{\mathbb{T}^n} e^{2\pi i\kappa\cdot x}e^{-2\pi i\gamma\cdot x}dx = \int_{\mathbb{T}^n} e^{2\pi i(\kappa-\gamma)\cdot x}dx.$$

If $\kappa = \gamma$, then $\kappa - \gamma = 0$, and so we are left with

$$\int_{\mathbb{T}^n} dx = 1.$$

That is, $\|e^{2\pi i\kappa\cdot x}\| = 1$. Now, if $\kappa \neq \gamma$, we have that $\kappa - \gamma = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}^n$, and the non-equality forces $\alpha_i \neq 0$ for some i. Since $|e^{2\pi i(\kappa-\gamma)\cdot x}| \leq 1$, we have that the function is in L^1, and so Fubini applies to give us

$$\int_{\mathbb{T}^n} e^{2\pi i(\kappa-\gamma)\cdot x}dx = \int_0^1 e^{2\pi i\alpha_1 x_1}dx_1 \cdots \int_0^1 e^{2\pi i\alpha_n x_n}dx_n.$$

Assume without loss of generality that $\alpha_1 \neq 0$. Letting $u = 2\pi i\alpha_1 x_1$, $du = 2\pi i\alpha_1 dx_1$ gives us

$$\int_0^1 e^{2\pi i\alpha_1 x_1}dx_1 = \frac{1}{2\pi i\alpha_1}\int_0^{2\pi i\alpha_1} e^u du = \frac{e^{2\pi i\alpha_1} - 1}{2\pi i\alpha_1},$$

and we note that for any integer $k \in \mathbb{Z}$, we have

$$e^{2\pi ik} = 1,$$

which we can deduce easily from DeMoivre:

$$e^{2\pi ik} = \cos(2\pi ik) + i\sin(2\pi ik) = 1.$$

Hence, we have that

$$\int_0^1 e^{2\pi i\alpha_1 x_1}dx_1 = 0,$$

and so our entire integral is 0. That is, we have that this is indeed an orthonormal set.

Next, we wish to apply Stone-Weierstrass to get that this is a dense set in $L^2(\mathbb{T}^n)$. To do so, we need to verify that the span of these forms an algebra. First, notice that we have it's a vector subspace; multiplying still gives us a linear combination, and adding linear combinations still gives us finite linear combinations. Next, we need to show that the product of finite linear combinations is still in the space. Notice that for $\kappa, \gamma \in \mathbb{Z}^n$ we have

$$e^{2\pi i\kappa\cdot x} \cdot e^{2\pi i\gamma\cdot x} = e^{2\pi i(\kappa+\gamma)\cdot x},$$

and $\kappa + \gamma \in \mathbb{Z}^n$ still. Expanding over products of linear combinations, then, still gives us a linear combination. So it's an algebra.

We then need that it separates points. Let $x \neq y \in \mathbb{T}^n$. Notice that this implies that there is a dimension where these two points disagree; i.e., this reduces down to just considering $x \neq y$ on S^1. Using DeMoivre's again, we see this boils down to using the fact that sine and cosine are projections onto the x and y axis (viewing S^1 in $\mathbb{R}^2$), and so if $x \neq y$ on the circle, we have that at least one of $\sin(x) \neq \sin(y)$ or $\cos(x) \neq \cos(y)$. Hence, there is a trigonometric polynomial f where $f(x) \neq f(y)$.

Now, $e^{2\pi i0\cdot x} = 1$ is in in this algebra, so all constant functions are in the algebra, and we have that the complex conjugation of $\overline{e^{2\pi i\kappa\cdot x}} = e^{-2\pi i\kappa\cdot x}$ which is still in this algebra, since $-\kappa$ is an integer still. Hence, it's closed under complex conjugation, and so if we have that the algbera is $\mathcal{A}$, we have that $\overline{\mathcal{A}}^{\|\cdot\|_u} = C(\mathbb{T}^n)$; i.e. they are dense in the space of continuous functions in the uniform norm. Using **Proposition 7.9**, we have $C_c(\mathbb{T}^n) \subset C(\mathbb{T}^n)$ is dense in $L^2(\mathbb{T}^n)$. By last semesters notes (https://people.math.osu.edu/penneys.2/6211/FunctionalAnalysis.pdf, pg. 20, last theorem) we have the being dense is equivalent to being an orthonormal basis, and so this set is indeed an orthonormal basis. $\square$

Problem 28. The Fourier transform maps the Schwarts class $\mathcal{S}$ continuously into itself.

Remark. I tried following Folland but have only a slight idea on how he got what he did. The first proof is based on https://math.stackexchange.com/questions/78441/fourier-transform-of-schwartz-space?noredirect=1&lq=1. The second proof is based on a long discussion with Thomas, and hopefully mimics whatever Folland was trying to say (the solution on Stackexchange that follows this is mine).

Proof. We first wish to show that $\widehat{\cdot}: \mathcal{S} \to \mathcal{S}$ is linear. It's clear that it is finitely additive, since

$$\widehat{f+g}(m) = \int_{\mathbb{R}^n} (f+g)(x)e^{-2\pi i m\cdot x}dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i m\cdot x}dx + \int_{\mathbb{R}^n} g(x)e^{-2\pi i m\cdot x}dx = \widehat{f}(m) + \widehat{g}(m),$$

and so we just need to check that if $r \in \mathbb{C}$ is a scalar, then $\widehat{rf} = r\widehat{f}$. Notice that

$$\widehat{rf}(m) = \int_{\mathbb{R}^n} (rf)(x)e^{-2\pi i m\cdot x}dx = r\int_{\mathbb{R}^n} f(x)e^{-2\pi i m\cdot x}dx = r\widehat{f}(m).$$

So the operator is linear.

Now, we have a family of (semi)norms on $\mathcal{S}$ given by $\|\cdot\|_{(N,\alpha)}$. We wish to show that for each (N,α), there is some constant C so that

$$\|\widehat{f}\|_{(N,\alpha)} \leq C\sum_{1}^{k} \|f\|_{(N_k,\alpha_k)}.$$

This will give us continuity by **Proposition 5.15**. Writing out thedefinition, we have

$$\|\widehat{f}\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n}(1+|x|)^N|\partial^\alpha \widehat{f}|.$$

We now follow the proof of **Proposition 8.3** to get a bound. Notice that $\sum_1^n |x_j|^N$ is strictly positive on $|x|=1$, so admits a positive minimum $\delta > 0$ (since this is compact and the function continuous), and so since $|x/|x||=1$, we get

$$\sum_1^n \left|\frac{x_j}{|x|}\right|^N > \delta \implies \sum_1^n \frac{|x_j|^N}{|x|^N} > \delta \implies \sum_1^n |x_j|^N > \delta|x|^N.$$

Hence, we have

$$(1+|x|)^N \leq 2^N(1+|x|^N) \leq 2^N\left[1+\delta^{-1}\sum_1^N |x_j|^N\right] \leq 2^N\delta^{-1}\sum_{|\beta|\leq N}|x^\beta|.$$

Substituting this in, we get

$$\|\widehat{f}\|_{(N,\alpha)} \leq \sup_{x\in\mathbb{R}^n}\left[2^N\delta^{-1}\sum_{|\beta|\leq N}|x^\beta\partial^\alpha\widehat{f}|\right] \leq 2^N\delta^{-1}\sum_{|\beta|\leq N}\|x^\beta\partial^\alpha\widehat{f}\|_u.$$

We now turn to inspecting $x^\beta\partial^\alpha\widehat{f}$. Fixing x and using **Proposition 8.22 (d)** we have

$$\begin{aligned}
|x^\beta\partial^\alpha\widehat{f}(x)| &= |x^\beta[\widehat{(-2\pi i t)^\alpha f(x)}]| \\
&= \left|x^\beta\int_{\mathbb{R}^n}(-2\pi i t)^\alpha f(t)e^{-2\pi i x\cdot t}dt\right| \\
&= \left|\int_{\mathbb{R}^n}x^\beta(-2\pi i t)^\alpha f(t)e^{-2\pi i x\cdot t}dt\right| \\
&= (2\pi)^\alpha\left|\int_{\mathbb{R}^n}x^\beta t^\alpha f(t)e^{-2\pi i x\cdot t}dt\right|.
\end{aligned}$$

Now examine the inner integral. We have

$$\int_{\mathbb{R}^n} x^\beta t^\alpha f(t) e^{-2\pi i x\cdot t} dt.$$

Let's first normalize; letting $u = -2\pi x$, we may rewrite the integral as

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\frac{u}{-2\pi}\right)^\beta t^\alpha f(t) e^{u\cdot t} dt \\ &= \frac{1}{(-2\pi)^{|\beta|}} \int_{\mathbb{R}^n} u^\beta t^\alpha f(t) e^{iu\cdot t} dt.\end{aligned}$$

Now notice that

$$\partial^\beta e^{iu\cdot t} = i^{|\beta|} u^\beta e^{iu\cdot t};$$

to prove this, we go by induction. For one variable, we have

$$\begin{aligned}&\frac{d}{dt_1} e^{iu\cdot t} = \frac{d}{dt_1} e^{i(u_1 t_1 + \cdots + u_n t_n)} \\ &= \frac{d}{dt_1} e^{iu_1 t_1} \cdots e^{iu_n t_n} = iu_1 e^{iu\cdot t}.\end{aligned}$$

Inducting for n is clear based off of this. We have

$$\frac{d^n}{dt_1^n} e^{iu\cdot t} = \frac{d}{dt}\frac{d^{n-1}}{dt_1^{n-1}} e^{iu\cdot t} = \frac{d}{dt} i^{n-1} u_1^{n-1} e^{iu\cdot t} = i^n u_1^n e^{iu\cdot t}.$$

Inducting on the length of β follows as well. Assuming we have it for length $n-1$, write $\beta = (\beta_1, \ldots, \beta_n) = (\beta_1, \ldots, \beta_{n-1}, 0) + (0, \ldots, \beta_n)$, letting $\gamma = (\beta_1, \ldots, \beta_{n-1}, 0)$. Applying the induction hypothesis then gives

$$\partial^\beta e^{iu\cdot t} = \frac{d^{\beta_n}}{dt_n^{\beta_n}} \partial^\gamma e^{iu\cdot t} = \frac{d^{\beta_n}}{dt_n^{\beta_n}} i^{|\gamma|} u^\gamma e^{iu\cdot t} = i^{|\gamma|+\beta_n} u^{\gamma+(0,\ldots,\beta_n)} e^{iu\cdot t} = i^{|\beta|} u^\beta e^{iu\cdot t}.$$

So we have

$$\frac{\partial^\beta e^{iu\cdot t}}{i^{|\beta|}} = u^\beta e^{iu\cdot t},$$

and so substituting this in gives us

$$\frac{1}{(-2\pi i)^{|\beta|}} \int_{\mathbb{R}^n} (\partial^\beta e^{iu\cdot t}) f(t) t^\alpha dt.$$

Examine this in just one variable. We have then

$$\int_{\mathbb{R}} \left(\frac{d^n}{dt^n} e^{iut}\right) f(t) t^\alpha dt.$$

We then do integration by parts here, letting $dv = (d^n/dt^n)e^{iut}dt$, $v = (d^{n-1}/dt^{n-1})e^{iut}$, $u = f(t)t^\alpha$, $du = ((d/dt)f(t)t^\alpha + \alpha t^{\alpha-1} f(t))dt$. We can then rewrite the above as

$$\begin{aligned}&\frac{d^{n-1}}{dt^{n-1}} e^{iut} f(t) t^\alpha \bigg|_{t=-\infty}^{\infty} - \int \left[\frac{d^{n-1}}{dt^{n-1}} e^{iut}\right] ((d/dt)f(t)t^\alpha + \alpha t^{\alpha-1} f(t))dt \\ &= -\int \left[\frac{d^{n-1}}{dt^{n-1}} e^{iut}\right] ((d/dt)f(t)t^\alpha + \alpha t^{\alpha-1} f(t))dt,\end{aligned}$$

since f is a Schwarz function and $|(d^{n-1}/dt^{n-1})e^{iut}| \leq C < \infty$. Iterating and using the product rule then gives us

$$(-1)^n \int e^{iut} \left[\sum_{k=0}^n \frac{n!}{k!(n-k)!} f^{(k)} \left(\frac{d^{(n-k)}}{dt^{(n-k)}} t^\alpha\right).\right]$$

Now, since $|\partial^\alpha e^{iu\cdot t}| \leq C < \infty$, $f(t)t^\alpha \in L^1(\mathbb{R}^n)$ since f is Schwarz (use the fact that $|f(t)| \leq C_N(1+|t|)^{-N}$ for all N and pick N sufficiently large), and $\partial^\alpha f \in \mathcal{S}$ for all α, we get that Tonelli tells us Fubini applies, and so we can use this to iterate the integral above. Moreover, we get that we can iterate integration by parts. Using what we proved for one variable above, then, we get that the integral is equal to

$$\frac{1}{(2\pi i)^{|\beta|}}\int_{\mathbb{R}^n} e^{iu\cdot t}\left[\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}(\partial^\eta f)(\partial^\gamma t^\alpha)dt\right]$$
$$= \frac{1}{(2\pi i)^{|\beta|}}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} e^{iu\cdot t}(\partial^\eta f)t^{\alpha-\gamma}dt\right].$$

Substituting back in $u = -2\pi x$, multiplying in the $(2\pi)^{|\alpha|}$, and taking the absolute value, we get that this is

$$\left|(2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} e^{-2\pi i x\cdot t}(\partial^\eta f)t^{\alpha-\gamma}dt\right]\right|$$
$$\leq (2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} |\partial^\eta f||t|^{\alpha-\gamma}dt\right].$$

Divide and multiply by $(1+|t|)^{n+1}$ on the inside of the integral. This give

$$(2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} \frac{(1+|t|)^{n+1}}{(1+|t|)^{n+1}}|\partial^\eta f||t|^{\alpha-\gamma}dt\right]$$
$$\leq (2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} \frac{1}{(1+|t|)^{n+1}}dt\right]\||\partial^\eta f(t)|(1+|t|)^{n+1}|t|^{\alpha-\gamma}\|_u$$
$$\leq (2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\sum_{\eta+\gamma=\beta}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} \frac{1}{(1+|t|)^{n+1}}dt\right]\||\partial^\eta f(t)|(1+|t|)^{n+1+|\alpha-\gamma|}\|_u$$
$$= \sum_{\eta+\gamma=\beta}(2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} \frac{1}{(1+|t|)^{n+1}}dt\right]\|f\|_{n+1+|\alpha-\gamma|,\eta}.$$

Letting

$$C_{\eta,\gamma} = (2\pi)^{|\alpha|-|\beta|}\frac{\alpha!}{(\alpha-\beta)!}\frac{\beta!}{\eta!\gamma!}\left[\int_{\mathbb{R}^n} \frac{1}{(1+|t|)^{n+1}}dt\right],$$

we can more concisely write this as

$$\sum_{\eta+\gamma=\beta} C_{\eta,\gamma}\|f\|_{n+1+|\alpha-\gamma|,\eta}.$$

Notice as well this does not depend on x, so we get that

$$\|x^\beta\partial^\alpha \widehat{f}\|_u \leq \sum_{\eta+\gamma=\beta} C_{\eta,\gamma}\|f\|_{n+1+|\alpha-\gamma|,\eta},$$

and so substituting this in to the above, we have

$$\|\widehat{f}\|_{(N,\alpha)} \leq 2^N\delta^{-1}\sum_{|\beta|\leq N}\sum_{\eta+\gamma=\beta} C_{\eta,\gamma}\|f\|_{n+1+|\alpha-\gamma|,\eta}.$$

These are all finite sums, so choosing C' to be the maximum of $C_{\eta,\gamma}$ over all γ, η, β and letting $C = 2^N \delta^{-1} C'$, we have

$$\|\widehat{f}\|_{(N,\alpha)} \leq C \sum_{\substack{|\beta| \leq N \\ \eta+\gamma=\beta}} \|f\|_{n+1+|\alpha-\gamma|,\eta},$$

which satisfies the criteria for **Proposition 5.15**. Thus, the Fourier transform is continuous on $\mathcal{S}$. □

Proof following Folland. Recall from **Folland Exercise 8.1** that we have

$$\partial^\alpha (x^\beta f) = x^\beta \partial^\alpha f + \sum c_{\gamma\delta} x^\delta \partial^\gamma f,$$

where the $c_{\gamma\delta}$ vanish for $|\gamma| \geq |\alpha|$, $|\delta| \geq |\beta|$. To prove this, we use the product rule (which we discussed in recitation). We induct on the length of β. First, assume that β has length 1. Using general Leibniz rule (without loss of generality assuming that β is non-zero in the first coordinate), we have

$$\partial^\alpha (x_1^k f) = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} (\partial^\gamma f)(\partial^\delta x_1^k).$$

This will evaluate to 0 for all δ which are non-zero outside of the first coordinate, and so we get that this amounts to

$$\begin{aligned}\partial^\alpha (x^k f) &= x^k \partial^\alpha f + \sum_{\substack{\gamma+(j,0,\ldots,0)=\alpha \\ j\neq 0}} \frac{\alpha!}{\gamma! j!} (\partial^\gamma f) \left(\frac{d^j}{dx_1^j} x^k \right) \\ &= x^k \partial^\alpha f + \sum_{\substack{\gamma+(j,0,\ldots,0)=\alpha \\ j\neq 0}} \frac{\alpha!}{\gamma! j!} (\partial^\gamma f) \left(\frac{k!}{(k-j)!} x^{k-j} \right) \\ &= x^k \partial^\alpha f + \sum c_{\gamma\delta} (\partial^\gamma f) x^\delta.\end{aligned}$$

Assuming it holds for length of β being $n-1$, we have that getting it for n is a matter of writing $\beta = (\beta_1, \ldots, \beta_{n-1}, 0) + (0, \ldots, 0, \beta_n)$ and letting $\gamma = (\beta_1, \ldots, \beta_{n-1}, 0)$. This then gives us

$$\partial^\alpha (x^\beta f) = \partial^\alpha (x_n^{\beta_n} x^\gamma f) = x_n^{\beta_n} \partial^\alpha (x^\gamma f) + \sum c_{\delta\eta} (\partial^\delta f) x^\eta.$$

Using the induction hypothesis now, we get

$$\partial^\alpha (x^\beta f) = x_n^{\beta_n} x^\gamma (\partial^\alpha f) + \sum c_{\delta\eta} (\partial^\delta f) x^\eta = x^\beta (\partial^\alpha f) + \sum c_{\delta'\eta'} (\partial^{\delta'} f) x^{\eta'},$$

as desired. Note that the properties on the constants vanishing are clear by the derivation.

By definition, we have that

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f|,$$

and using what we've derived earlier we have

$$\|f\|_{(N,\alpha)} \leq \sup_{x\in\mathbb{R}^n} \left[2^N \eta^{-1} \sum_{|\beta|\leq N} |x^\beta \partial^\alpha f| \right],$$

where we now write η instead of δ for the minimum for notational convenience. Notice now that

$$
\begin{aligned}
\|\widehat{f}\|_{(N,\alpha)} &\leq \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}|x^\beta\partial^\alpha\widehat{f}|\right] \\
(\text{Use Folland 8.22 (d) on } \partial^\alpha\widehat{f}) &= \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}|x^\beta\widehat{(-2\pi ix)^\alpha f}|\right] \\
(\text{Use linearity and the absolute value to pull out constants}) &= \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|}|x^\beta\widehat{x^\alpha f}|\right] \\
(\text{Use Folland 8.22 (e) on } x^\beta\widehat{x^\alpha f}\text{, pull constants out with linearity}) &= \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}|\widehat{\partial^\beta x^\alpha f}|\right] \\
(\text{Use the product rule to rewrite } \partial^\beta x^\alpha f) &= \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\left|\left[x^\alpha\partial^\beta f+\sum c_{\gamma\delta}x^\delta\partial^\gamma f\right]^\wedge\right|\right] \\
(\text{Use linearity of Fourier transform}) &= \sup_{x\in\mathbb{R}^n}\left[2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\left|\widehat{x^\alpha\partial^\beta f}+\sum c_{\gamma\delta}\widehat{x^\delta\partial^\gamma f}\right|\right] \\
(\text{T.I.}) &\leq 2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\left[\|\widehat{x^\alpha\partial^\beta f}\|_u+\sum c_{\gamma\delta}\|\widehat{x^\delta\partial^\gamma f}\|_u\right] \\
(\text{Inequality from Folland}) &\leq 2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\left[C\|(1+|x|)^{n+1}x^\alpha\partial^\beta f\|_u+\sum c_{\gamma\delta}C\|(1+|x|)^{n+1}x^\delta\partial^\gamma f\|_u\right] \\
(\text{Use the fact that } |x|^\delta\leq(1+|x|)^{|\alpha|}) &\leq 2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\Big[C\|(1+|x|)^{n+|\alpha|+1}\partial^\beta f\|_u+ \\
&\qquad \sum c_{\gamma\delta}C\|(1+|x|)^{n+1+|\alpha|}\partial^\gamma f\|_u\Big] \\
(\text{By definition}) &= 2^N\eta^{-1}\sum_{|\beta|\leq N}(2\pi)^{|\alpha|-|\beta|}\left[C\|f\|_{(n+|\alpha|+1,\beta)}+\sum c_{\gamma\delta}C\|f\|_{(n+|\alpha|+1,\gamma)}\right],
\end{aligned}
$$

where the constants in the last sum are such that $c_{\gamma\delta}=0$ unless $|\gamma|<|\beta|$ and $|\delta|<|\alpha|$. Now, absorbing constants and maybe letting some constants be zero, we can write this as

$$\|\widehat{f}\|_{(N,\alpha)}\leq\sum_{\substack{|\gamma|\leq|\beta|\leq N\\ |\delta|\leq|\alpha|}}C_{\beta,\delta,\gamma}\|f\|_{(|\alpha|+n+1,\gamma)}.$$

This is a finite sum, so we can find some constant denoted by $C_{N,\alpha}$ which bounds above all constants and gets rid of multiplicity, and this gives us

$$\|\widehat{f}\|_{(N,\alpha)}\leq C_{N,\alpha}\sum_{|\gamma|\leq N}\|f\|_{|\alpha|+n+1,\gamma},$$

which is now what Folland has. To see where the inequality comes from, notice that we have

$$\|\widehat{x^\alpha\partial^\beta f}\|_u=\|\widehat{x^\alpha\partial^\beta f}\|_\infty\leq\|x^\alpha\partial^\beta f\|_1,$$

and furthermore calculating the last part, we have

$$\int_{\mathbb{R}^n} |x^\alpha \partial^\beta f| = \int_{\mathbb{R}^n} \frac{(1+|x|)^{n+1}}{(1+|x|)^{n+1}} |x^\alpha \partial^\beta f| dx \leq \left(\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{n+1}} \right) \|(1+|x|)^{n+1} x^\alpha \partial^\beta f\|_u.$$

Setting

$$C = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{n+1}} < \infty,$$

we have the desired inequality. □

Problem 29. Let $f, g \in L^1(\mathbb{T}^n)$, $m \in \mathbb{Z}^n$, and $y \in \mathbb{T}^n$. Then we have

(1) $\widehat{(f+g)}(m) = \widehat{f}(m) + \widehat{g}(m)$,
(2) $\widehat{\tau_y f}(m) = \widehat{f}(m) e^{-2\pi i m \cdot y}$,
(3) $\widehat{f * g}(m) = \widehat{f}(m)\widehat{g}(m)$.

Proof. We first check that if $f, g \in L^1(\mathbb{T}^n)$, then $f * g$ is periodic with period 1. Writing things out, we have

$$(f * g)(x) = \int_{\mathbb{T}^n} f(x-y) g(y) dy.$$

Now, we check that this has period 1. We see that

$$(f * g)(x+1) = \int_{\mathbb{T}^n} f(x+1-y) g(y) dy = \int_{\mathbb{T}^n} f(x-y) g(y) dy = (f * g)(x),$$

since f is a function with period 1. So the convolution is still in the space of functions on $\mathbb{T}^1$.

(1) We have

$$\begin{aligned} \widehat{(f+g)}(m) &= \int_{\mathbb{T}^n} (f+g)(x) e^{-2\pi i m \cdot x} dx \\ &= \int_{\mathbb{T}^n} \left[f(x) e^{-2\pi i m \cdot x} + g(x) e^{-2\pi i m \cdot x} \right] dx \\ &= \int_{\mathbb{T}^n} \left[f(x) e^{-2\pi i m \cdot x} dx + \int_{\mathbb{T}^n} g(x) e^{-2\pi i m \cdot x} \right] dx \\ &= \widehat{f}(m) + \widehat{g}(m). \end{aligned}$$

(2) Notice that

$$\widehat{\tau_y f}(m) = \int_{\mathbb{T}^n} (\tau_y f)(x) e^{-2\pi i m \cdot x} dx = \int_{\mathbb{T}^n} f(x-y) e^{-2\pi i m \cdot x} dx,$$

where here we use the alternate definition of translation given in the chapter so that the equation makes sense (i.e. we now have $\tau_y f(x) = f(x-y)$ rather than $\tau_y f(x) = f(x+y)$). Let $u = x - y$, $du = dx$, then we can rewrite the above as

$$\int_{\mathbb{T}^n} f(u) e^{-2\pi i m \cdot (u+y)} du = e^{-2\pi i m \cdot y} \widehat{f}(m).$$

(3) We have

$$\begin{aligned} \widehat{f * g}(m) = \int_{\mathbb{T}^n} (f * g)(x) e^{-2\pi i m \cdot x} dx &= \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} f(x-y) g(y) dy \right) e^{-2\pi i m \cdot x} dx \\ &= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} f(x-y) g(y) e^{-2\pi i m \cdot x} dy dx. \end{aligned}$$

Taking the absolute value, we see that by Tonelli we have

$$\iint |f(x-y)||g(y)|d(y \times x) = \int \left(\int |f(x-y)|dx \right) |g(y)|dy = \|f\|_1 \|g\|_1 < \infty,$$

so the functions are in $L^1(y \times x)$. Hence we can use Fubini to change the order of integration. Thus, we have

$$\begin{aligned} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} f(x-y)g(y)e^{-2\pi i m\cdot(x-y)}e^{-2\pi i m\cdot y}dxdy &= \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} f(x-y)e^{-2\pi i m\cdot(x-y)}dx \right) e^{-2\pi i m\cdot y}g(y)dy \\ &= \int_{\mathbb{T}^n} \widehat{f}(m)e^{-2\pi i m\cdot y}g(y)dy = \widehat{f}(m)\widehat{g}(m). \end{aligned}$$

□

Remark. Thomas O'Hare was a collaborator.

Remark. I guess there is a discrepancy between Grafakos' second and third edition, I am using the third edition (which can be found easily through a google search).

Problem 30 (Folland 9.16)**.** Let

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

(a) If $a > 0$, then $(\chi_{[-a,a]})^\wedge(x) = (\chi_{[-a,a]})^\vee(x) = 2a\operatorname{sinc}(2ax)$.

(b) Let

$$H_a = \{f \in L^2 : \widehat{f}(\zeta) = 0 \text{ a.e. for } |\zeta| > a\}.$$

Then H_a is a Hilbert space and

$$\{\sqrt{2a}\operatorname{sinc}(2ax - k) : k \in \mathbb{Z}\}$$

is an orthonormal basis for H_a.

(c) If $f \in H_a$, then $f \in C_0$ after a modification on a null set, and

$$f(x) = \sum_{-\infty}^{\infty} f(k/2a)\operatorname{sinc}(2ax - k),$$

where the series converge both uniformly and in L^2.

Proof. (a) Assume $x \neq 0$. We have

$$\begin{aligned}(\chi_{[-a,a]})^\wedge(x) &= \int_{\mathbb{R}} \chi_{[-a,a]}(z)e^{-2\pi i x \cdot z}dz \\ &= \int_{-a}^{a} e^{-2\pi i x \cdot z}dz.\end{aligned}$$

Letting $u = -2\pi i x z$, $du = -2\pi i x dz$, we have that the above integral is equal to

$$\left.\frac{e^{-2\pi i x z}}{-2\pi i x}\right|_{z=-a}^{a} = \frac{-e^{-2\pi i x a} + e^{2\pi i x a}}{2\pi i x}.$$

By DeMoivre,

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x), \\ e^{-ix} &= \cos(x) - i\sin(x), \end{aligned}$$

and subtracting the second from the first and dividing by $2i$ gives

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin(x).$$

Using this, we can rewrite the above as

$$\frac{-e^{-2\pi i x a} + e^{2\pi i x a}}{2\pi i x} = \frac{2\sin(2\pi x a)}{2\pi x} = \frac{2a\sin(2\pi x a)}{2\pi x a} = \frac{2a\sin(\pi[2ax])}{\pi[2ax]} = 2a\operatorname{sinc}(2ax),$$

as desired. Similarly,

$$\begin{aligned}(\chi_{[-a,a]})^\vee(x) = (\chi_{[-a,a]})^\wedge(-x) &= \int_{\mathbb{R}} \chi_{[-a,a]}(z)e^{2\pi i x z}dz \\ &= \int_{-a}^{a} e^{2\pi i x z}dz.\end{aligned}$$

Letting $u = 2\pi ixz$, $du = 2\pi ixdz$, we have that the integral evaluates to

$$\left.\frac{e^{2\pi ixz}}{2\pi ix}\right|_{z=-a}^{a} = \frac{e^{2\pi ixa} - e^{-2\pi ixa}}{2\pi ix},$$

and the same argument as above gives that this comes out to $2a\text{sinc}(2ax)$. For $x = 0$, we have that

$$\left(\chi_{[-a,a]}\right)^{\wedge}(0) = \int_{-a}^{a} dx = 2a = 2a\text{sinc}(0) = \left(\chi_{[-a,a]}\right)^{\vee}(0).$$

Hence, we have $(\chi_{[-a,a]})^{\wedge}(x) = (\chi_{[-a,a]})^{\vee}(x) = 2a\text{sinc}(2ax)$ for all x.

(b) We recall that L^2 is a Hilbert space from prior homework, and so it suffices to show that this is a closed subspace. First, if $f, g \in H_a$, then we have that

$$\widehat{f+g}(\zeta) = \widehat{f}(\zeta) + \widehat{g}(\zeta) = 0 \text{ a.e. for } |\zeta| > a,$$

so $f + g \in H_a$. If r is a scalar, we see that

$$\widehat{rf}(\zeta) = r\widehat{f}(\zeta) = 0 \text{ a.e. for } |\zeta| > a,$$

so it is a subspace.

Next, we wish to show that it is closed. Let $(f_n) \subset H_a$ be a sequence such that $f_n \to f$ in L^2. The goal is to show that $f \in H_a$; that is, $\widehat{f}(\zeta) = 0$ a.e. for all $|\zeta| > a$. Notice that **Plancherel's theorem** says that $\widehat{\cdot}$ extends uniquely to a unitary isomorphism on L^2, so we have that $\widehat{f_n} \to \widehat{f}$ in L^2; furthermore, we can extract some subsequence $\widehat{f_{n_j}} \to \widehat{f}$ almost everywhere. Hence, we have $\widehat{f}(\zeta) = 0$ a.e. for $|\zeta| > a$.

Now, we need to show that

$$\{\sqrt{2a}\text{sinc}(2ax - k) : k \in \mathbb{Z}\}$$

is an orthonormal basis for H_a. First, we show that the set is actually orthonormal. Letting $E_k(x) = \sqrt{2a}\text{sinc}(2ax - k)$, we note that this is the same as showing that

$$\langle E_k(x), E_{k'}(x)\rangle = \begin{cases} 0 \text{ if } k \neq k' \\ 1 \text{ if } k = k'. \end{cases}$$

Since the Fourier transform is a unitary isomorphism, we get

$$\langle E_k, E_{k'}\rangle = \langle \widehat{E_k}, \widehat{E_{k'}}\rangle.$$

Recall that part (a) tells us that

$$(\chi_{[-a,a]})^{\vee}(x) = 2a\text{sinc}(2ax),$$

where here we interpret the Fourier inverse in terms of the L^2 Fourier transform as well as in terms of the L^1 Fourier transform (so that we may ignore issues of the transform being L^1 for Fourier inversion). We note such an interpretation is valid by Plancherel.

Taking the Fourier transform (to help with notation, we will denote the Fourier transform with $\mathcal{F}$ instead of $\widehat{\cdot}$ and the inverse Fourier transform with $\mathcal{G}$ instead of $\cdot^{\vee}$) we have that

$$\mathcal{F}(2a\text{sinc}(2ax - k)) = 2a\mathcal{F}(\text{sinc}(2a(x - k/2a))).$$

Writing this in terms of the translation function gives

$$\left(\tau_{k/2a}\mathcal{F}(\mathcal{G}(\chi_{[-a,a]}))\right) = 2a\mathcal{F}(\tau_{k/2a}\text{sinc}(2ax)) = 2a\mathcal{F}(\text{sinc}(2ax - k)).$$

We must be careful here. We'd like to just use **Theorem 8.22 (a)** and conclude the desired result, however this only works for L^1 functions and, as we've noted in recitation, sinc is **not** an L^1 function. We can salvage this by extending the result to a.e. equivalence (or L^2 equivalence) in the following way.

Claim. If $f \in L^2(\mathbb{R}^n)$, we have that

$$\mathcal{F}(\tau_y f)(m) = e^{-2\pi i m\cdot y}\mathcal{F}(f)(m) \text{ as functions in } L^2.$$

Proof. Let $f \in L^2$ and take $(f_n) \subset L^1 \cap L^2$ such that $f_n \to f$ in L^2. Let y be some point in $\mathbb{R}^n$. Note that

$$\mathcal{F}(\tau_y f_n)(m) = e^{-2\pi i m\cdot y}\mathcal{F}(f_n)(m)$$

by **Theorem 8.22 (a)**. From this, we conclude that

$$\mathcal{F}(\tau_y f_n) \to e^{-2\pi i m\cdot y}\mathcal{F}(f)$$

in L^2. Furthermore, we note that

$$\|\tau_y(f_n) - \tau_y(f)\|_2 = \|\tau_y(f_n - f)\|_2 = \|f_n - f\|_2 \to 0.$$

So Plancherel tells us that

$$\mathcal{F}(\tau_y(f_n)) \to \mathcal{F}(\tau_y(f))$$

in L^2 as well. Hence, *as functions in* L^2, we have

$$\mathcal{F}(\tau_y(f)) = e^{-2\pi i m\cdot y}\mathcal{F}(f).$$

□

Using this claim, we get that

$$\mathcal{F}(\tau_{k/2a}\mathcal{G}(\chi_{[-a,a]}))(x) = e^{-2\pi i m\cdot y}\chi_{[-a,a]}(x) \text{ as functions in } L^2.$$

So, we have that

$$\mathcal{F}(2a\mathrm{sinc}(2ax-k)) = e^{-2\pi i m\cdot y}\chi_{[-a,a]}(x) \text{ as functions in } L^2.$$

Now, using that the Fourier transform is unitary, we have

$$\langle E_k, E_{k'}\rangle = \langle \widehat{E_k}, \widehat{E_{k'}}\rangle.$$

Writing out the right hand side and using the result from the claim (which is valid, since we only need up to L^2 equivalence here), we have

$$\frac{1}{2a}\int_{\mathbb{R}} \chi_{[-a,a]}(x)e^{-2\pi i x(k-k')/2a}dx = \frac{1}{2a}\int_{-a}^{a} e^{-2\pi i x(k-k')/2a}dx.$$

If $k = k'$, we get 1 since

$$\frac{1}{2a}\int_{-a}^{a} dx = \frac{2a}{2a} = 1.$$

Assume then that $k \neq k'$. Letting $u = -2\pi i x(k-k')/2a$, we get $du = -dx2\pi i(k-k')/2a$, so we have

$$\frac{1}{2\pi i(k'-k)}\int_{\pi i(k-k')}^{-\pi i(k-k')} e^u du = \frac{e^{-\pi i(k-k')} - e^{\pi i(k-k')}}{2\pi i(k-k')} = -\sin(\pi(k-k')) = 0.$$

So the set is orthogonal, as desired.

We now need to show completeness of the span of this set. That is, we wish to show that if $\langle g, E_k\rangle = 0$ for all k, then $g = 0$ almost everywhere. Notice that

$$\begin{aligned} 0 = \langle g, E_k\rangle = \langle \widehat{g}, \widehat{E_k}\rangle &= \int \widehat{g}(x)\overline{\widehat{E_k(x)}}dx \\ = \int \widehat{g}(x)\sqrt{2a}\cdot\overline{\mathcal{F}(\mathrm{sinc}(2ax-k))}dx = \frac{1}{\sqrt{2a}}\int &\widehat{g}(x)2a\overline{\mathcal{F}(\mathrm{sinc}(2ax-k))}dx \\ &= \frac{1}{\sqrt{2a}}\langle \widehat{g}, e^{2\pi i xk/2a}\chi_{[-a,a]}\rangle. \end{aligned}$$

Since this holds for all $k \in \mathbb{Z}$, we claim that this shows that $\widehat{g} = 0$ on $[-a, a]$. To show this, we follow the guideline outlined in recitation. On $L^2([-1/2, 1/2]) = L^2(\mathbb{T}^1)$, **Theorem 8.20** tells us that $\{e^{2\pi ikx}\}$ is an orthonormal basis. To adjust this for $L^2([-a, a]) = \{f \in L^2 : f = f\chi_{[-a,a]}æ\} = H_a' = \mathcal{F}(H_a)$, we simply scale. Consider the map $T : L^2([-a, a]) \to L^2(\mathbb{T}^1)$ via $Tf = f(x/2a)$. This is bijective, with inverse $T^{-1} : L^2(\mathbb{T}^1) \to L^2([-a, a])$ given by $T^{-1}f \mapsto f(2ax)$, and furthermore we see that

$$\langle Tf, Tg\rangle = \int_{-1/2}^{1/2} f(x/2a)\overline{g}(x/2a)dx = \int_{-a}^{a} f(x)\overline{g}(x)dx = \langle f, g\rangle$$

by a simple change of variables. So it is a unitary map. Now, if g is such that $\langle g, E_k\rangle = 0$ for all E_k, we get that it is such that $\langle g, E_k\rangle = \langle \widehat{g}, \widehat{E_k}\rangle = \langle T\widehat{g}, T\widehat{E_k}\rangle = 0$ for all $T\widehat{E_k}$. Since this is an orthonormal basis of $L^2(\mathbb{T}^1)$, we get that this forces $T\widehat{g} = 0$. Using that T and $\widehat{\cdot}$ are unitary, this then implies that $g = 0$ on $L^2([-a, a])$, or $g = 0$ almost everywhere on $[-a, a]$, and by virtue of g being in H_a, we get that $g = 0$ almost everywhere as desired. So it is indeed an orthonormal basis.

(c) Let $f \in H_a$. We wish to show that $f \in C_0$ (after modifying on a null set), so it suffices to show there is a $h \in C_0$ such that $f = h$ almost everywhere. Since $f \in H_a$, we get that $f^\wedge$ is such that it is supported in $[-a, a]$ and is in L^2 using Plancherel. Now, since it is supported in $[-a, a]$, we get that $f^\wedge \in L^1 \cap L^2$ (**Proposition 6.12**). Taking the inverse Fourier transform and invoking Plancherel (that is, interpreting this in terms of the L^2 Fourier transform rather than using Fourier inversion), we get that $(f^\wedge)^\vee = f$ in L^2 (i.e. almost everywhere), and Riemann-Lebesgue says that $(f^\wedge)^\vee = h \in C_0$ (by Plancherel, the L^2 Fourier transform agrees with the usual Fourier transform on $L^1 \cap L^2$). To see this more explicitly, notice that $(f^\wedge)^\vee(x) = (f^\wedge)^\wedge(-x)$, since $f^\wedge \in L^1 \cap L^2$. Now, $C_0 = C_0(\mathbb{R})$ is the space of functions where $f(x) \to 0$ as $|x| \to 0$. Riemann-Lebesgue says that $(f^\wedge)^\wedge \in C_0(\mathbb{R})$, so using this we have that $(f^\wedge)^\wedge(-x) \to 0$ as $|x| \to \infty$ as well; in other words, $(f^\wedge)^\vee \in C_0(\mathbb{R})$. So $f = h = (f^\wedge)^\vee$ almost everywhere, where $h \in C_0$.

Now, we get from the orthonormal basis that

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, E_k\rangle E_k(x).$$

Let $c_k = \langle f, E_k\rangle$. We then get that

$$f(x) = \sum_{k\in\mathbb{Z}} c_k E_k(x).$$

We wish to determine what the value for c_k is. Plugging in $r/2a$, we have

$$\begin{aligned} f(r/2a) &= \sum_{k\in\mathbb{Z}} c_k E_k(r/2a) \\ &= \sum_{k\in\mathbb{Z}} c_k \sqrt{2a}\operatorname{sinc}(r-k). \end{aligned}$$

Recall that for $r \neq k$, $\operatorname{sinc}(r-k) = 0$, since for $\kappa \in \mathbb{Z} - \{0\}$ we have $\operatorname{sinc}(\kappa) = \sin(\pi\kappa)/(\pi\kappa) = 0$, while for $r - k = 0$ we have $\operatorname{sinc}(r - k) = 1$. So we see that

$$f(r/2a) = \sqrt{2a}c_r,$$

so that

$$c_r = \frac{f(r/2a)}{\sqrt{2a}}.$$

Thus, we have

$$f(x) = \sum_{k \in \mathbb{Z}} f(k/2a)\text{sinc}(2ax - k)$$

where the sum converges in L^2.

We wish to finally show that the series converges uniformly as well. Letting

$$g_N = \sum_{k=-N}^{N} f(k/2a)\text{sinc}(2ax - k),$$

we wish to show that

$$\|f - g_N\|_u \to 0.$$

Since we have an isometry, we get that

$$\|f - g_N\|_2 \to 0 \implies \|\widehat{f} - \widehat{g_N}\|_2 \to 0.$$

Furthermore, since the Fourier transform of f is in

$$\mathcal{F}(H_a) = H_a' = \{f \in L^2 : f = f\chi_{[-a,a]} \text{ a.e.}\} = L^2([-a,a]),$$

we get that this is converging in L^2 on a space of finite measure $[-a,a]$, and so using **Proposition 6.12** we get that

$$\frac{1}{\sqrt{2a}}\|\widehat{f} - \widehat{g_N}\|_1 \leq \|\widehat{f} - \widehat{g_N}\|_2 \to 0,$$

so that

$$\|\widehat{f} - \widehat{g_N}\|_1 \to 0$$

as well. That is, these converge in the L^1 norm as well. Now, using that $\|\widehat{f}\|_u \leq \|f\|_1$, we take the inverse Fourier transform[2] to get that

$$\|(f^\wedge)^\vee - (g_N{}^\wedge)^\vee\|_u \leq \|\widehat{f} - \widehat{g_N}\|_1 \to 0,$$

and so Plancherel gives us that

$$\|f - g_N\|_u \leq \|\widehat{f} - \widehat{g_N}\|_1 \to 0.$$

Hence, this sum converges uniformly as well.

$\square$

Problem 31 (Folland Lemma 8.34)**.** If $f, g \in L^2(\mathbb{R}^n)$, then $(\widehat{f}\widehat{g})^\vee = f * g$.

Proof. We first must show that $\widehat{f}\widehat{g} \in L^1$. First, by Plancherel we see that $f, g \in L^2$ implies that $\widehat{f}, \widehat{g} \in L^2$. Now, Hölder tells us that

$$\|\widehat{f}\widehat{g}\|_1 \leq \|\widehat{f}\|_2\|\widehat{g}\|_2 < \infty,$$

so we see that $\widehat{f}\widehat{g} \in L^1$. Thus, we have that $(\widehat{f}\widehat{g})^\vee$ makes sense in terms of using the formula.

Now, fix $x \in \mathbb{R}^n$ and let $h(y) = \overline{g(x-y)}$. Consider $g_n \to g$ in $L^1 \cap L^2$. Defining correspondingly $h_n = \overline{g_n(x-y)}$, and note that $h_n \to h$ in L^2. We see that for all n, we have

$$\begin{aligned}\widehat{h_n}(m) = \int h_n(y)e^{-2\pi i m\cdot y}dy &= \int \overline{g_n(x-y)}e^{-2\pi i m\cdot y}dy \\ &= e^{-2\pi i m\cdot x}\int \overline{g_n(x-y)e^{-2\pi m\cdot(x-y)}}dy \\ &= e^{-2\pi i m\cdot x}\overline{\widehat{g_n}(m)}.\end{aligned}$$

[2]Again, in terms of L^2/Plancherel so that we do not need to worry about conditions for L^1 Fourier inversion.

So using Plancherel, we see that $\widehat{h_n} \to \widehat{h}$ in L^2, $e^{-2\pi im\cdot x}\overline{\widehat{g_n}} \to e^{2\pi im\cdot x}\overline{\widehat{g}}$ in L^2, and thus we have $\widehat{h} = \overline{\widehat{g}}e^{-2\pi im\cdot x}$ almost everywhere.[3]

Putting things together, we see that for this fixed x, we have

$$\begin{aligned} f * g(x) = \int f(y)g(x-y)dy &= \int f(y)\overline{h(y)} \\ &= \langle f, h\rangle, \end{aligned}$$

so since the Fourier transform is unitary we get

$$\begin{aligned} f * g(x) &= \langle f, h\rangle = \langle \widehat{f}, \widehat{h}\rangle \\ = \int \widehat{f}(y)\overline{\widehat{h}(y)}dy &= \int \widehat{f}(y)\overline{e^{-2\pi iy\cdot x}\overline{\widehat{g}(y)}}dy \\ = \int \widehat{f}(y)\widehat{g}(y)e^{2\pi iy\cdot x}dy &= \int \widehat{f}(y)\widehat{g}(y)e^{-2\pi y\cdot(-x)}dy \\ &= (\widehat{f}\widehat{g})^\wedge(-x) = (\widehat{f}\widehat{g})^\vee(x). \end{aligned}$$

Thus we have equality. $\square$

Problem 32 (Grafakos Example 3.1.5)**.** Let

$$P(x) = \sum_{m\in\mathbb{Z}^n} a_m e^{2\pi im\cdot x}$$

be a trigonometric polynomial on $\mathbb{T}^n$, where $(a_m)_{m\in\mathbb{Z}^n}$ is a finitely supported sequence in $\mathbb{Z}^n$. Then:

(a) We have

$$P(x) = \sum \widehat{P}(m)e^{2\pi im\cdot x}.$$

(b) If $f \in L^1(\mathbb{T}^n)$, then

$$(f * P)(x) = \sum \widehat{P}(m)\widehat{f}(m)e^{2\pi im\cdot x}.$$

Proof. (a) Since the sequence (a_m) is finitely supported, we note that the series is going to be a finite linear combination of $L^1 \cap L^2$ functions, and so it will be $L^1 \cap L^2$ as well. Hence, the Fourier transform of the polynomial will be

$$\begin{aligned} \widehat{P}(k) &= \int_{\mathbb{T}^n} P(x)e^{-2\pi ik\cdot x}dx \\ &= \int_{\mathbb{T}^n} \left[\sum_{m\in\mathbb{Z}^n} a_m e^{2\pi im\cdot x}\right] e^{-2\pi ik\cdot x}dx \\ &= \sum_{m\in\mathbb{Z}^n} a_m \int_{\mathbb{T}^n} e^{2\pi i(m-k)\cdot x}, \end{aligned}$$

where the last equality follows since the sum is finite. Recall that $(e^{\pi i\kappa\cdot x})_{\kappa\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, so for $k \neq m$ we have that the integral is zero, and otherwise the integral will be 1. This then gives us

$$\widehat{P}(k) = a_k.$$

By direct substitution, then, we see that

$$P(x) = \sum_{m\in\mathbb{Z}^n} \widehat{P}(m)e^{2\pi im\cdot x}.$$

[3]What Folland has is technically incorrect; we cannot get equality without the functions being in L^1, since we don't have a formula which applies for all of L^2.

(b) If $f \in L^1(\mathbb{T}^n)$, we see that

$$
\begin{aligned}
(f * P)(x) = \int_{\mathbb{T}^n} f(x-y)P(y)dy = \int_{\mathbb{T}^n} f(x-y)\left[\sum_{m\in\mathbb{Z}^n} \widehat{P}(m)e^{2\pi i m\cdot y}\right]dy \\
= \sum_{m\in\mathbb{Z}^n} \widehat{P}(m)\int_{\mathbb{T}^n} f(x-y)e^{2\pi i m\cdot y}dy \\
= \sum_{m\in\mathbb{Z}^n} \widehat{P}(m)\int_{\mathbb{T}^n} f(x-y)e^{-2\pi i m\cdot(x-y)}e^{2\pi i m\cdot x}dy \\
= \widehat{P}(m)\widehat{f}(m)e^{2\pi i m\cdot x}.
\end{aligned}
$$

□

Problem 33 (Grafakos Exercise 3.1.4)**.** On $\mathbb{T}$ define the *de la Vallée Poussin kernel*:

$$V_N(x) = 2F_{2N+1}(x) - F_N(x)$$

where F_N denotes the Fejér kernel.

(a) Show that the sequence (V_N) is an approximate identity.

(b) Prove that $\widehat{V}_N(m) = 1$ when $|m| \le N+1$, and $\widehat{V}_N(m) = 0$ when $|m| \ge 2N+2$.

Proof. (a) Recall that a sequence is an approximate identity if it satisfies three properties:

(1) We first want to show that $\sup_N \|V_N\|_1 < \infty$. We see that, for all N, we have

$$\|V_N\|_1 = \|2F_{2N+1} - F_N\|_1 \le 2\|F_{2N+1}\|_1 + \|F_N\|_1,$$

and since the Fejér kernel is an approximate identity (by the lecture notes or **Proposition 3.1.10** in Grafakos), we see that

$$\sup_N \|V_N\|_1 \le 2\sup_N \|F_{2N+1}\|_1 + \sup_N \|F_N\|_1 < \infty.$$

(2) We now want to show that

$$\int V_N(x)dx = 1$$

for all N. Fixing an N and using that the Fejér kernel is an approximate identity, we see that

$$\int V_N(x)dx = \int (2F_{2N+1}(x) - F_N(x))dx = 2\int F_{2N+1}(x)dx - \int F_N(x)dx = 2 - 1 = 1.$$

Hence, we have the desired result.

(3) Finally, we wish to show that for any neighborhood V^c of 0, we have that

$$\int_V |V_N|dx \to 0.$$

Since we're on the torus, it suffices to show that for all $\delta > 0$,

$$\int_{\delta\le|x|\le 1/2} |V_N|dx \to 0.$$

Again, we use that $|V_N(x)| \le 2|F_{2N+1}(x)| + |F_N(x)|$, so that if $V = \{x : \delta \le |x| \le 1/2\}$,

$$0 \le \int_V |V_N|dx \le 2\int_V |F_{2N+1}(x)|dx + \int_V |F_N(x)|dx.$$

Taking the limit as $N \to \infty$ of both sides gives us

$$0 \le \lim_{N\to\infty}\int_V |V_N|dx \le 2\left[\lim_{N\to\infty}\int_V |F_{2N+1}(x)|dx\right] + \lim_{N\to\infty}\int_V |F_N(x)|dx,$$

and since (F_N) is an approximate identity, we have that

$$\lim_{N\to\infty}\int_V |V_N|dx = 0.$$

So this holds for any neighborhood of 0.

Hence, (V_N) is an approximate identity.

(b) We have that

$$\widehat{V_N}(m) = 2\widehat{F_{2N+1}}(m) - \widehat{F_N}(m)$$

by linearity of the Fourier transform. Utilizing **Proposition 3.1.7** in Grafakos, we see that

$$\widehat{F_{2N+1}(m)} = 1 - \frac{|m|}{2N+2}$$

if $|m| \le 2N+1$ and 0 otherwise, and likewise

$$\widehat{F_N}(m) = 1 - \frac{|m|}{N+1}$$

if $|m| \le N$ and 0 otherwise. Clearly, then, for $|m| \ge 2N+2$, we get that $\widehat{V_N}(m) = 0$, since both components will be 0 in this range. Now, for $|m| \le N$, we see that

$$\begin{aligned}\widehat{V_N}(m) &= 2\left(1 - \frac{|m|}{2N+2}\right) - \left(1 - \frac{|m|}{N+1}\right)\\ &= \left(\frac{4N+4-2|m|}{2N+2}\right) - \left(\frac{N+1-|m|}{N+1}\right)\\ &= \frac{4N+4-2|m|-2N-2+2|m|}{2N+2}\\ &= \frac{2N+2}{2N+2} = 1.\end{aligned}$$

For $|m| = N+1$, we see that we have

$$\widehat{V_N}(m) = 2\left(1 - \frac{|m|}{2N+2}\right) = 2\left(\frac{2N+2-|m|}{2N+2}\right)$$
$$= \frac{4N+4-2|m|}{2N+2} = \frac{4N+4-2N+2}{2N+2} = \frac{2N+2}{2N+2} = 1.$$

So if $|m| \le N+1$, we have $\widehat{V_N}(m) = 1$, as desired.

□

Remark. **Proposition 3.1.7** of Grafakos claims that

$$F_N(x) = \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)e^{2\pi i j x},$$

which we derived in our class notes. Furthermore, he uses this to note that $\widehat{F_N}(m) = 1 - \frac{|m|}{N+1}$ if $|m| \le N$ and zero otherwise, which is the property used in this problem. To see this, we use linearity of the Fourier transform to note that

$$\begin{aligned}\widehat{F_N}(m) &= \sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)\widehat{e^{2\pi ijx}}(m)\\ &= \sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)\int_{\mathbb{T}} e^{2\pi ijx}e^{-2\pi ixm}dx\\ &= \sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)\operatorname{sinc}(j-m)\\ &= \begin{cases}1-\frac{|m|}{N+1} \text{ if } |m|\le N\\ 0 \text{ otherwise.}\end{cases}\end{aligned}$$

Remark. Thomas O'Hare was a collaborator.

Remark. I guess there is a discrepancy between Grafakos' second and third edition, I am using the third edition (which can be found easily through a google search).

Problem 34 (Grafakos Theorem 3.3.4)**.** Let $(d_m)_{m\in\mathbb{Z}^n}$ be a sequence of positive real numbers with $d_m \to 0$ as $|m| \to \infty$. Then there exists $g \in L^1(\mathbb{T}^n)$ such that $|\widehat{g}(m)| \geq d_m$ for all $m \in \mathbb{Z}^n$. In other words, given any rate of decay there exists an integrable function on the torus whose Fourier coefficients have slower rate of decay.

Proof. To build this, we will use **Lemma 3.3.2** and **Lemma 3.3.3**. Recall these lemmas.

Lemma (Grafakos Lemma 3.3.2)**.** Given a sequence of positive real numbers $(a_m)_{m=0}^\infty$ that tends to zero as $m \to \infty$, there exists a sequence $(c_m)_{m=0}^\infty$ that satisfies the following three conditions:
(1) $a_m \leq c_m$,
(2) $c_m \searrow 0$,
(3) $c_{m+2} + c_m \geq 2c_{m+1}$
for all $m \in \mathbb{Z}_{\geq 0}$. A sequence (c_m) satisfiying these conditions is called convex.

Lemma (Grafakos Lemma 3.3.3)**.** Given a convex decreasing sequence $(c_m)_{m=0}^\infty$ of positive real numbers satisfying $\lim_{m\to\infty} c_m = 0$ and a fixed integer $s \geq 0$, we have that

$$\sum_{r=0}^\infty (r+1)(c_{r+s} + c_{r+s+2} - 2c_{r+s+1}) = c_s.$$

We omit the proofs of these, since they are in Grafakos. Consider first the case $n = 1$. We have a sequence of positive numbers $(d_m)_{m\in\mathbb{Z}}$ such that $d_m \to 0$ as $|m| \to \infty$. We can consider the sequence $(d_m+d_{-m})_{m=0}^\infty$, which is still a sequence of positive real numbers such that $d_m+d_{-m} \to 0$ as $m \to \infty$. Thus, we can apply **Lemma 3.3.2** to extract a convex sequence (c_m) so that $c_m \geq d_m + d_{-m}$, $c_m \searrow 0$, and $c_{m+2} + c_m \geq 2c_{m+1}$. We extend this to all integers by setting $c_{-m} := c_m$ for $m > 0$.

Our goal, then, is to create a function $f \in L^1(\mathbb{T}^1)$ with this sequence $(c_m)_{m\in\mathbb{Z}}$ so that the $|f(m)| \geq d_m$ for all m. **Lemma 3.3.3** suggest that we choose

$$f(x) = \sum_{r=0}^\infty (r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x),$$

where $F_r(x)$ is the one-dimensional Fejér kernel – that is,

$$F_r(x) = \sum_{j=-r}^{r} \left(1 - \frac{|j|}{r+1}\right) e^{2\pi ijx}.$$

Since F_r is periodic, we get that f is periodic. We then want to see whether it has finite L^1 norm on the torus. Checking this, we get

$$\begin{aligned}\|f\|_1 &= \int \left|\sum_{r=0}^\infty (r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x)\right| \\ &\leq \int \sum_{r=0}^\infty (r+1)(c_r + c_{r+2} - 2c_{r+1})|F_r(x)| \\ &= \int \sum_{r=0}^\infty (r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x),\end{aligned}$$

since $F_r(x)$ is positive. To see this, we have the equivalent definition of the Fejer kernel given by

$$F_r(x) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2.$$

Now we use Tonelli, noting that everything is positive and interpreting the sum as an integral with respect to counting measure, in order to switch the integral and sum (alternatively, one could invoke **Theorem 2.25** using the following facts). This gives us

$$\|f\|_1 \leq \sum_{r=0}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1})\|F_r\|_1 = c_0 < \infty$$

by **Lemma 3.3.3**, noting that $\|F_r\|_1 = 1$ by the proof of **Proposition 3.1.10**. Thus, $f \in L^1(\mathbb{T}^1)$.

Now, we wish to show that $\widehat{f}(m) \geq d_m$ for all $m \in \mathbb{Z}$. Note that the series converges to f in L^1; using techniques above, we have

$$\begin{aligned}\left\|f(x) - \sum_{r=0}^{N}(r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x)\right\|_1 &= \left\|\sum_{r=N+1}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x)\right\|_1 \\ &\leq \sum_{r=N+1}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1}) \to 0 \text{ as } N \to \infty,\end{aligned}$$

since convergence of the series implies the tail goes to 0. Denote the partial sums as

$$f_N(x) = \sum_{r=0}^{N}(r+1)(c_r + c_{r+2} - 2c_{r+1})F_r(x).$$

Since these functions are in L^1, we have that the Fourier transforms will be

$$\widehat{f}(m) = \int_{\mathbb{T}} f(x)e^{-2\pi imx}dx,$$

$$\widehat{f_N}(m) = \int_{\mathbb{T}} f_N(x)e^{-2\pi imx}dx.$$

Since the series converges in L^1, we see that

$$\begin{aligned}\lim_{N\to\infty}|\widehat{f}(m) - \widehat{f_N}(m)| &= \lim_{N\to\infty}\left|\int_{\mathbb{T}}(f(x) - f_N(x))e^{-2\pi imx}dx\right| \\ &\leq \lim_{N\to\infty}\int_{\mathbb{T}}|f(x) - f_N(x)|dx = \lim_{N\to\infty}\|f - f_N\|_1 = 0.\end{aligned}$$

So

$$\widehat{f}(m) = \lim_{N\to\infty}\widehat{f_N}(m) = \sum_{r=0}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1})\widehat{F_r}(m),$$

using the linearity of the Fourier transform on finite sums. Now, recall that

$$\widehat{F_r}(m) = \begin{cases}1 - \frac{|m|}{r+1} \text{ if } |m| \leq r \\ 0 \text{ otherwise,}\end{cases}$$

either from the last homework or from **Proposition 3.1.7**. Hence, we get that

$$\begin{aligned}
\widehat{f}(m) &= \sum_{r=0}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1})\widehat{F_r}(m) \\
&= \sum_{r=|m|}^{\infty}(r+1)(c_r + c_{r+2} - 2c_{r+1})\left(1 - \frac{|m|}{r+1}\right) \\
&= \sum_{r=0}^{\infty}(r+|m|+1)(c_{r+|m|} + c_{r+|m|+2} - 2c_{r+|m|+1})\left(1 - \frac{|m|}{r+|m|+1}\right) \\
&= \sum_{r=0}^{\infty}(r+|m|+1)(c_{r+|m|} + c_{r+|m|+2} - 2c_{r+|m|+1})\left(\frac{r+1}{r+|m|+1}\right) \\
&= \sum_{r=0}^{\infty}(c_{r+|m|} + c_{r+|m|+2} - 2c_{r+|m|+1})(r+1).
\end{aligned}$$

Now, we can hit it with **Lemma 3.3.3** with $s = |m|$ to get

$$\widehat{f}(m) = c_{|m|} = c_m.$$

Since the coefficients are positive, we see that

$$\widehat{f}(m) = c_m \geq d_m,$$

as desired.

Now, we wish to show this for general $n > 1$. We first need the following claim.

Claim (Grafakos Exercise 3.3.2)**.** Given a positive sequence $(d_m)_{m\in\mathbb{Z}^n}$ with $d_m \to 0$ as $|m| \to \infty$, there exists a positive sequence $(a_j)_{j\in\mathbb{Z}}$ with $a_{m_1}\cdots a_{m_n} \geq d_{(m_1,\ldots,m_n)}$ and $a_j \to 0$ as $|j| \to \infty$.

Remark. Note that in Grafakos, for $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we have $|m| = \sqrt{m_1^2 + \cdots + m_n^2}$.

Proof. The case $n = 1$ is clear; just take $d_m = a_m$ for all m.

Now consider the general case of $n > 1$. Let

$$A_r^{(1)} = \max_{m\in\mathbb{Z}^{n-1}} \sqrt[n]{d_{(r,m)}},$$

$$A_r^{(2)} = \max_{k\in\mathbb{Z},m\in Z^{n-2}} \sqrt[n]{d_{(k,r,m)}},$$

$$\vdots$$

$$A_r^{(n)} = \max_{k\in\mathbb{Z}^{n-1}} \sqrt[n]{d_{(k,r)}}.$$

Note that these maximums are finite by the decay of the coefficients. Let $a_r = \max_{i=1,\ldots,n}\{A_r^{(i)}\}$. Then we see that

$$a_{m_1}\cdots a_{m_n} \geq A_{m_1}^{(1)}\cdots A_{m_n}^{(n)} \geq \sqrt[n]{d_{(m_1,\ldots,m_n)}}\cdots\sqrt[n]{d_{(m_1,\ldots,m_n)}} = d_{(m_1,\ldots,m_n)}.$$

Fixing $\epsilon > 0$, we choose R sufficiently large so that for $|(m_1, \ldots, m_n)| \geq R$, we have that $d_{(m_1,\ldots,m_n)} < \epsilon^n$. Hence, we have that for $|n| \geq R$, $|m| \geq R$ for $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, $r = m_j$ for some j, $m_k \in \mathbb{Z}$ for all $k \neq j$. Notice that for all $i = 1, \ldots, n$, we have

$$A_r^{(i)} = \max_{k\in\mathbb{Z}^{i-1},m\in Z^{n-i}} \sqrt[n]{d_{(k,r,m)}} < \sqrt[n]{\epsilon^n} = \epsilon,$$

so we get that

$$a_r = \max_{i=1,\ldots,n}\{A_r^{(i)}\} < \sqrt[n]{\epsilon^n} = \epsilon.$$

Thus, $a_r \to 0$ as $|r| \to \infty$, as desired.

□

Using this claim, we can construct a desired sequence $(a_m)_{m\in\mathbb{Z}}$ for the $(d_m)_{m\in\mathbb{Z}^n}$. Let

$$g(x_1,\ldots,x_n) = f(x_1)\cdots f(x_n),$$

where f is the function previously constructed when $n = 1$ so that $\widehat{f}(m) \geq a_m$. Then we see that

$$\begin{aligned}\widehat{g}(m_1,\ldots,m_n) &= \int_{\mathbb{T}^n} g(x)e^{-2\pi i m\cdot x}dx \\ &= \int_{\mathbb{T}^1}\cdots\int_{\mathbb{T}^1} f(x_1)\cdots f(x_n)e^{-2\pi i m_1 x_1}\cdots e^{-2\pi i m_n x_n}dx_1\cdots dx_n \\ &= \prod_{j=1}^n \int_{\mathbb{T}^1} f(x_j)e^{-2\pi i m_j x_j}dx_j \\ &= \widehat{f}(m_1)\cdots\widehat{f}(m_n) \geq a_{m_1}\cdots a_{m_n} \geq d_{(m_1,\ldots,m_n)},\end{aligned}$$

as desired. Note that the justification of iterating the integral comes from the fact that the inside of the integral is L^1, since

$$\int_{\mathbb{T}^n}|g(x)||e^{-2\pi i m\cdot x}|dx = \int_{\mathbb{T}^n}|g(x)|dx = \int_{\mathbb{T}^1}\cdots\int_{\mathbb{T}^1} f(x_1)\cdots f(x_n)dx_1\cdots dx_n = \prod_{j=1}^n\int_{\mathbb{T}^1} f(x_j)dx_j < \infty,$$

where we note that the f are all positive and we use Tonelli to iterate this integral. So Fubini applies, and we can iterate the above integral to get the desired result. □

Recall that we say $F \in \mathrm{BV}$ (F is a function of bounded variation) if $T_F(\infty) = \lim_{x\to\infty} T_F(x)$ is finite, where

$$T_F(x) = \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$

Problem 35 (Folland Lemma 3.26)**.** If $F \in \mathrm{BV}$ real valued, then $T_F + F$ and $T_F - F$ are increasing.

Proof. Fix $\epsilon > 0$, $x < y$. Choose $x_0 < \cdots < x_n = x$ so that

$$\sum_1^n |F(x_j) - F(x_{j-1})| \geq T_F(x) - \epsilon.$$

Notice that we can do this since T_F is defined via a supremum; that is, we have

$$T_F(x) = \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}.$$

Now, we have that

$$x_0 < \cdots < x_n = x < y,$$

so we get that

$$\sum_1^n |F(x_j) + F(x_{j-1})| + |F(y) - F(x)|$$

is in the collection

$$T_F(y) = \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = y\right\}.$$

Notice as well that we can write

$$F(y) = [F(y) - F(x)] + F(x).$$

Hence, we have that

$$T_F(y) \pm F(y) \geq \sum_1^n |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \pm [F(y) - F(x)] \pm F(x).$$

If $x \in \mathbb{R}$, we have that

$$|x| + x = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases},$$

so that $|x| + x \geq 0$. Hence, we have that

$$T_F(y) \pm F(y) \geq \sum_1^n |F(x_j) - F(x_{j-1})| \pm F(x) \geq T_F(x) - \epsilon \pm F(x).$$

This holds for all $\epsilon > 0$, so letting it go to 0 gives

$$T_F(y) \pm F(y) \geq T_F(x) \pm F(x).$$

The choice of $x < y$ was arbitrary, so we get that $T_F \pm F$ is an increasing function. □

Problem 36 (Folland Theorem 3.27)**.** (a) $F \in BV$ iff $\mathrm{Re}(F)$ and $\mathrm{Im}(F) \in \mathrm{BV}$.
(b) If $F : \mathbb{R} \to \mathbb{R}$, then $F \in \mathrm{BV}$ iff F is the difference of two bounded increasing functions.
(c) If $F \in \mathrm{BV}$, the set of points at which F is discontinuous is countable.
(d) If $F \in BV$ and $G(x) = F(x+)$, then F' and G' exist and are equal a.e.

Proof. (a) ($\implies$): Assume $F \in \mathrm{BV}$. Then we have

$$T_F(\infty) = \lim_{x\to\infty} \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\} < \infty.$$

Notice we may write $F = \mathrm{Re}(F) + i\mathrm{Im}(F)$. Substituting this in, then, we have

(3)
$$\lim_{x\to\infty} \sup\left\{\sum_1^n |[\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})] + i[\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})]| : -\infty < x_0 < \cdots < x_n = x\right\}.$$

Notice that for all x, this bounds above

$$\sup\left\{\sum_1^n |\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$

and

$$\sup\left\{\sum_1^n |\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\},$$

since we have

$$\sum_1^N |[\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})] + i[\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})]|$$
$$= \sum_1^N \sqrt{[\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})]^2 + [\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})]^2}$$
$$\geq \max\left\{\sum_1^N |\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})|, \sum_1^N |\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})|\right\}$$

for any partition. Taking the limit, then, we get that each of these are finite, and so $\mathrm{Re}(F)$ and $\mathrm{Im}(F) \in \mathrm{BV}$.

$(\Longleftarrow)$: Assume $\mathrm{Re}(F)$ and $\mathrm{Im}(F) \in \mathrm{BV}$. Applying a triangle inequality in (1), we get

$$\sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$\leq \sup\left\{\sum_1^n |\mathrm{Re}(F)(x_j) - \mathrm{Re}(F)(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$+ \sup\left\{\sum_1^n |\mathrm{Im}(F)(x_j) - \mathrm{Im}(F)(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\},$$

and taking the limit as $x \to \infty$ implies $T_F(\infty)$ is finite.

(b) For this, we need a few facts.

(1) Notice that if $F : \mathbb{R} \to \mathbb{R}$ is bounded and increasing, then $F \in \mathrm{BV}$. This is because, for all x, we have

$$T_F(x) = \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$= \sup\left\{\sum_1^n F(x_j) - F(x_{j-1}) : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$= \sup\{F(x) - F(x_0) : n \in \mathbb{N}, -\infty < x_0 < x\} \leq 2M,$$

where here we used the fact that F was increasing, so $|F(x_j) - F(x_{j-1})| = F(x_j) - F(x_{j-1})$ for $x_{j-1} < x_j$, and we used that M bounded F. So $T_F(\infty) \leq 2M < \infty$.

(2) Notice that if $F, G \in \mathrm{BV}$ and $a, b \in \mathbb{C}$, then $aF + bG \in \mathrm{BV}$. This follows by the triangle inequality, since

$$T_{aF+bG}(x)$$
$$= \sup\left\{\sum_1^n |aF(x_j) + bG(x_j) - aF(x_{j-1}) - bG(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$\leq |a| \sup\left\{\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$+ |b| \sup\left\{\sum_1^n |G(x_j) - G(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}$$
$$= |a| T_F(x) + |b| T_G(x).$$

Thus, taking the limit as $x \to \infty$, we have

$$T_{aF+bG}(\infty) \leq |a|T_F(\infty) + |b|T_G(\infty) < \infty.$$

($\Longrightarrow$): Assume that $F \in \text{BV}$. The prior problem tells us that $T_F + F$ and $T_F - F$ are increasing functions. Furthermore, we have that they are bounded; we see that (for $y > x$)

$$T_F(y) \pm F(y) \geq T_F(x) \pm F(x)$$

implies that

$$|F(y) - F(x)| \leq T_F(y) - T_F(x) \leq T_F(\infty) - T_F(-\infty) < \infty,$$

since

$$\begin{aligned} T_F(y) - T_F(x) &\geq \pm F(x) \mp F(y) \\ \Longrightarrow T_F(y) - T_F(x) &\geq \max\{F(y) - F(x), F(x) - F(y)\} \\ \Longrightarrow T_F(y) - T_F(x) &\geq |F(y) - F(x)|, \end{aligned}$$

and we use the fact that T_F is an increasing function (see **Equation (3.24)** and the remark after). This implies that F is bounded, and so furthermore we have $T_F \pm F$ is bounded. Thus, we have that

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$

is a difference of bounded increasing functions.

($\Longleftarrow$): Assume that F is the difference of two bounded increasing functions. Since they are bounded and increasing, we have that (1) tells us that they are in BV, and (2) tells us that linear combinations of BV functions are in BV. So F is in BV.

(c) Let $F \in \text{BV}$. By **Theorem 3.23**, the set of discontinuities of an increasing function is countable. Write $F = \text{Re}(F) + i\text{Im}(F)$. Since $\text{Re}(F), \text{Im}(F) : \mathbb{R} \to \mathbb{R}$, we can use (b) to express them as the difference of two increasing functions, that is, we have that $\text{Re}(F) = G_1 - G_2$, $\text{Im}(F) = H_1 - H_2$. So $F = (G_1 - G_2) + i(H_1 - H_2)$. Since these have a countable number of discontinuities, this implies that F has a countable number of discontinuities (the set of discontinuities for F is the union of the sets of discontinuities for each function, and a union of a finite number of countable sets is countable), as desired.

(d) Assume F is real valued. By (b), we have that $F \in \text{BV}$ implies that F is the difference of two increasing functions, say H and K for notational simplicity. So $F(x) = H(x) - K(x)$. Let $S(x) = H(x+)$, and $T(x) = K(x+)$. By **Theorem 3.23**, we have that H and K are differentiable almost everywhere (and hence, F is differentiable almost everywhere), and $S' = H'$, $T' = K'$ almost everywhere. Furthermore, if $F(x) = H(x) - K(x)$, we have $G(x) = F(x+) = H(x+) - K(x+) = S(x) - T(x)$ is such that $G' = S' - T'$, and so $G' = H' - K' = F'$ almost everywhere.

We have it holds for real valued functions, so consider now $F \in \text{BV}$ such that it is complex valued. Write $F = \text{Re}(F) + i\text{Im}(F)$. Let $G_1(x) = \text{Re}(F)(x+)$, $G_2(x) = \text{Im}(F)(x+)$. Then $G(x) = F(x+) = G_1(x) + iG_2(x)$, so $G_1 = \text{Re}(G)$, $G_2 = \text{Im}(G)$. By prior, we have $\text{Re}(F)' = G_1'$ and $\text{Im}(F)' = G_2'$ exist and are equal a.e., so we get that $F' = \text{Re}(F)' + i\text{Im}(F)' = G_1' + iG_2'$ exists and they are equal almost everywhere.

$\square$

Remark. Thomas O'Hare was a collaborator.

Problem 37. Suppose that $f_1, f_2, \dots,$ and f are in $L^1_{\text{loc}}(U)$. The condition in (a) and (b) below imply that $f_n \to f$ in $\mathcal{D}'(U)$, but the condition in (c) does not.
(a) $f_n \in L^p(U)$ $(1 \le p \le \infty)$ and $f_n \to f$ in the L^p norm or weakly in L^p.
(b) For all n, $|f_n| \le g$ for some $g \in L^1_{\text{loc}}(U)$, and $f_n \to f$ a.e.
(c) $f_n \to f$ pointwise.

Proof. Recall throughout that a sequence $(F_n) \subset \mathcal{D}'(U)$ converges to F in $\mathcal{D}'(U)$ if, for all $\varphi \in C_c^\infty(U)$, we have $\langle F_n, \varphi\rangle \to \langle F, \varphi\rangle$ (i.e., pointwise convergence).
(a) First, assume that $f_n \to f$ in the L^p norm. Let $\varphi \in C_c^\infty(U)$ arbitrary. Then we have that

$$\langle f_n, \varphi\rangle = \int f_n\varphi.$$

So

$$|\langle f_n, \varphi\rangle - \langle f, \varphi\rangle| = \left|\int (f_n - f)\varphi\right| \le \int |f_n - f||\varphi|.$$

Since the φ are functions with compact support which are bounded, we see that they are in L^q for q such that $(p, q) = 1$ with $1 \le p \le \infty$ (this follows from **Proposition 6.12**). We can apply Hölder to get that

$$\int |f_n - f||\varphi| \le \|f_n - f\|_p \|\varphi\|_q$$

with $\|\varphi\|_q < \infty$. Since $f_n \to f$ in L^p, this implies that this goes to 0. Thus, we have

$$\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$$

for all φ, so we have that they converge in $\mathcal{D}'(U)$. If we assume that they converge weakly, then this implies that

$$\lim_{n\to\infty} \int f_n\varphi = \int f\varphi$$

for all $\varphi \in L^q(U)$, with $(p, q) = 1$. Notice that $C_c^\infty(U) \subset L^q(U)$, so we get that it holds for all $\varphi \in C_c^\infty(U)$, and thus we have convergence in $\mathcal{D}'(U)$, as desired.
(b) Choose $\varphi \in \mathcal{D}(U)$ arbitrarily. Since $g \in L^1_{\text{loc}}(U)$, we get that for all $\varphi \in \mathcal{D}(U)$, $g\varphi \in L^1$ (utilizing the fact here that φ has compact support). Notice that

$$|f_n\varphi| \le g|\varphi| \in L^1(U),$$

so we see that DCT applies here. In other words, we have that

$$\lim_{n\to\infty} \langle f_n, \varphi\rangle = \lim_{n\to\infty} \int f_n\varphi = \int \lim_{n\to\infty} f_n\varphi = \int f\varphi = \langle f, \varphi\rangle.$$

The choice of φ was arbitrary, so we get that it converges weakly – i.e., as distributions, $f_n \to f$.
(c) We see that $f_t(x) = t^{-1}\chi_{(0,t)}(x) \in L^1_{\text{loc}}(U)$, $f_t \to 0$ pointwise as $t \to 0$. Notice that we can write $f_t(x) = t^{-1}\chi_{(0,1)}(x/t)$, and so this is an approximate identity for $f(x) = \chi_{(0,1)}(x)$. We have $\int f = 1$, and applying **Proposition 9.1**, we see that $f_t \to \delta$ in $\mathcal{D}'$ as $t \to 0$. Notice that $\delta \ne 0$ as distributions (take any function which is non-zero at the origin), and so we have the desired result.

□

Problem 38. The product rule for derivatives is valid for products of smooth functions and distributions.

Proof. The goal is to first show that

$$\langle (\psi F)', \varphi \rangle = \langle \psi F', \varphi \rangle + \langle \psi' F, \varphi \rangle,$$

for all $\varphi \in \mathcal{D}(U)$, F a distribution, ψ a smooth function. We will then use induction to show that it holds for general products.

Fixing a φ, we see that the usual product rule gives us

$$\begin{aligned} \langle \psi F', \varphi \rangle &= \langle F', \varphi\psi \rangle = -\langle F, (\varphi\psi)' \rangle \\ &= -\langle F, \varphi'\psi + \varphi\psi' \rangle = -\langle F, \varphi'\psi \rangle - \langle F, \varphi\psi' \rangle \\ &= -\langle \psi F, \varphi' \rangle - \langle \psi' F, \varphi \rangle. \end{aligned}$$

We have as well that

$$\langle (\psi F)', \varphi \rangle = -\langle \psi F, \varphi' \rangle,$$

so substituting this in, we get

$$\langle \psi F', \varphi \rangle = \langle (\psi F)', \varphi \rangle - \langle \psi' F, \varphi \rangle.$$

Rearranging, we have

$$\langle (\psi F)', \varphi \rangle = \langle \psi F', \varphi \rangle + \langle \psi' F, \varphi \rangle,$$

as desired. The choices of $F \in \mathcal{D}'(U)$, $\psi \in C^\infty(U)$, and $\varphi \in \mathcal{D}(U)$ were all arbitrary, and so we have the product rule.

For the generalized product rule, we induct the usual product rule. Assume it holds for multi-indices of magnitude up to $k-1$. Let α be a multi-index such that $|\alpha| = k$ and write $\alpha = (\alpha_1, \ldots, \alpha_n)$. Without loss of generality, assume $\alpha_1 \geq 1$ (Pigeonhole principle says that there must be one coordinate greater than or equal to 1; if α_1 is not, then apply the following argument to an index that is). Let $\beta = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$, then $|\beta| = k-1$, and furthermore we have

$$\partial^\alpha(\psi F) = \frac{d}{dx_1}\partial^\beta(\psi F).$$

By the induction hypothesis, we get

$$\frac{d}{dx_1}\left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\delta!\gamma!}(\partial^\gamma \psi)(\partial^\delta F)\right).$$

Notice that the derivative is linear with respect to distributions, since for f, g distributions, φ a test function, we have

$$\begin{aligned} \langle (f+g)', \varphi \rangle = -\langle f+g, \varphi' \rangle = -\langle f, \varphi' \rangle - \langle g, \varphi' \rangle = \langle f', \varphi \rangle + \langle g', \varphi \rangle \\ = \langle f' + g', \varphi \rangle, \end{aligned}$$

and for a a constant, f a distribution, φ a test function, we have

$$\langle (af)', \varphi \rangle = -\langle af, \varphi' \rangle = -a\langle f, \varphi' \rangle = a\langle f', \varphi \rangle.$$

Thus, using the linearity, we get

$$\frac{d}{dx_1}\left(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\delta!\gamma!}(\partial^\gamma \psi)(\partial^\delta F)\right) = \sum_{\gamma+\delta=\beta} \frac{\beta!}{\delta!\gamma!}\frac{d}{dx_1}\left[(\partial^\gamma \psi)(\partial^\delta F)\right].$$

Using the base case, this gives

$$\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!}\left[\left(\frac{d}{dx_1}\partial^\gamma \psi\right)(\partial^\delta F) + (\partial^\gamma \psi)\left(\frac{d}{dx_1}\partial^\delta F\right)\right].$$

Simplifying this (akin to the usual product rule), we get

$$\sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!}(\partial^\gamma \psi)(\partial^\delta F).$$

So we see that the generalized product rule follows for products of distributions and smooth functions. □

Problem 39. A distribution F on $\mathbb{R}^n$ is called *homogeneous* of degree λ if $F \circ S_r = r^\lambda F$ for all $r > 0$, where $S_r(x) = rx$.

(a) Show that δ is homogeneous of degree $-n$.
(b) If F is homogeneous of degree λ, then $\partial^\alpha F$ is homogeneous of degree $\lambda - |\alpha|$.
(c) The distribution $(d/dx)[\chi_{(0,\infty)}(x)\log(x)]$ discussed in recitation is not homogeneous, although it agrees on $\mathbb{R} \setminus \{0\}$ with a "function" that is homogeneous of degree -1.

Proof. (a) Fixing $\varphi \in \mathcal{D}(U)$ arbitrary, we have that

$$\langle \delta \circ S_r, \varphi\rangle = r^{-n}\langle \delta, \varphi \circ S_r^{-1}\rangle = r^{-n}\varphi(0/r) = r^{-n}\varphi(0) = \langle r^{-n}\delta, \varphi\rangle.$$

Hence, as distributions, $\delta \circ S_r = r^{-n}\delta$, and so δ is homogeneous of degree $-n$.

(b) Since F is homogenous of degree λ, we see that

$$\langle F \circ S_r, \varphi\rangle = \langle r^\lambda F, \varphi\rangle$$

for all test functions φ. Applying things directly, we see that we have

$$\begin{aligned}\langle \partial^\alpha F \circ S_r, \varphi\rangle &= r^{-n}\langle \partial^\alpha F, \varphi \circ S_r^{-1}\rangle \\ &= (-1)^{|\alpha|} r^{-n}\langle F, \partial^\alpha(\varphi \circ S_r^{-1})\rangle.\end{aligned}$$

Now, we have that

$$\partial^\alpha \varphi(x/r) = r^{-|\alpha|}\varphi'(x/r),$$

where $\varphi' = \partial^\alpha \varphi$. To see this, we proceed by induction. For single derivatives, this is simply the chain rule; we have

$$\frac{d}{dx_i}\varphi(x/r) = r^{-1}\frac{d\varphi}{dx_i}(x/r).$$

Assuming it holds up for indices of magnitude up to $n-1$. Let α be a multi-index such that $|\alpha| = n$. We have then that

$$\partial^\alpha \varphi(x/r) = \frac{d^{\alpha_1}}{dx_1^{\alpha_1}} \cdots \frac{d^{\alpha_n}}{dx_n^{\alpha_n}}\varphi(x/r).$$

If $\alpha_n = n$, we rewrite this as

$$\frac{d^{\alpha_n - 1}}{dx_n^{\alpha_n - 1}}\frac{d}{dx_n}\varphi(x/r),$$

and we use the fact that it works for $n = 1$ to get that this is equal to

$$\frac{d^{\alpha_n - 1}}{dx_n^{\alpha_n - 1}} r^{-1}\frac{d\varphi_1}{dx_n}(x/r).$$

We then use the induction hypothesis along with linearity of the derivative to then get that this is equal to

$$r^{-\alpha_n}\frac{d^{\alpha_n}\varphi}{dx_n^{\alpha_n}}(x/r).$$

If $\alpha_n < n$, we can simply invoke the induction hypothesis to get the same result (that is, let $\beta = (0, \dots, 0, \alpha_n)$, then since $\alpha_n < n$ we have $|\beta| < n$, so we can apply the induction hypothesis to $\partial^\beta \varphi(x/r)$ and we get the result). We proceed in the same fashion for the other $\alpha_1, \dots, \alpha_{n-1}$ indices (if they are equal to n, do the argument above, otherwise invoke induction hypothesis). This leaves us with

$$r^{-\alpha_1 - \alpha_2 - \cdots - \alpha_n} \varphi'(x/r) = r^{-|\alpha|} \varphi'(x/r)$$

using the fact that

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Thus, we have

$$\begin{aligned}
\langle \partial^\alpha F \circ S_r, \varphi \rangle &= -r^{-n-|\alpha|} \langle F, \varphi' \circ S_r^{-1} \rangle \\
\text{(Flipping the } S_r \text{ back)} &= (-1)^{|\alpha|} r^{-|\alpha|} \langle F \circ S_r, \varphi' \rangle \\
\text{(Using the fact that F is homogeneous)} &= (-1)^{|\alpha|} r^{-|\alpha|} \langle r^\lambda F, \varphi' \rangle \\
\text{(Moving the derivative back over, using linearity)} &= \langle r^{\lambda - |\alpha|} \partial^\alpha F, \varphi \rangle.
\end{aligned}$$

Hence, since the choice of test function was arbitrary, we have that $\partial^\alpha F \circ S_r = r^{\lambda - |\alpha|} \partial^\alpha F$, so $\partial^\alpha F$ is homogeneous of degree $\lambda - |\alpha|$.

(c) We follow an argument similar to the book. Consider $F(x) = \chi_{(0,\infty)}(x) \log(x)$, $F_\epsilon(x) = \chi_{(\epsilon,\infty)}(x) \log(x)$, where $\epsilon > 0$. We remark that $F(x) \in L^1_{\text{loc}}(\mathbb{R})$. The only non-trivial part for this is examining the integral over intervals $[a, b] \subset \mathbb{R}$ which contain the interval; if we show that it's integrable on these, we get that it is integrable on all compact sets $K \subset \mathbb{R}$. Notice that if $[a, b]$ contains the interval, then $a < 0$, $b > 0$, so we have

$$\int_{[a,b]} F(x) dx = \int_0^b \log(x) dx = \lim_{\epsilon \to 0} \left(x \log(x) - x \Big|_{x=\epsilon}^{b} \right).$$

Notice that

$$\lim_{\epsilon \to 0} \epsilon \log(\epsilon) = \lim_{\epsilon \to 0} \frac{\log(\epsilon)}{\frac{1}{\epsilon}},$$

and applying L'Hospital we have this is equal to

$$\lim_{\epsilon \to 0} \frac{\frac{1}{\epsilon}}{-\frac{1}{\epsilon^2}} = \lim_{\epsilon \to 0} -\epsilon = 0.$$

Hence, the above integral is

$$\int_{[a,b]} F(x) dx = b \log(b) - b < \infty.$$

We deduce then that $F(x) \in L^1_{\text{loc}}(\mathbb{R})$. A similar argument shows that $F_\epsilon \in L^1_{\text{loc}}(\mathbb{R})$.

The goal is to use DCT to deduce that $F_\epsilon \to F$ in $\mathcal{D}'(\mathbb{R})$. Fixing $\varphi \in \mathcal{D}(\mathbb{R})$ a test function, we have that

$$\lim_{\epsilon \to 0} \langle F_\epsilon, \varphi \rangle = \lim_{\epsilon \to 0} \int F_\epsilon(x) \varphi(x) dx = \lim_{\epsilon \to 0} \int \chi_{(\epsilon,\infty)}(x) \log(x) \varphi(x) dx$$

Notice that

$$|\chi_{(\epsilon,\infty)}(x) \log(x) \varphi(x)| \leq |\log(x)| \chi_{K \cap (0,\infty)}(x) M,$$

where $|\varphi(x)| \leq M < \infty$ (we know this exists since φ has compact support and is smooth). If we show this is in L^1, we can apply DCT to move the limit inside. Thus, we wish to show that

$$M \int_{(0,\infty) \cap K} |\log(x)| dx < \infty.$$

Since K compact, it is closed and bounded; that is, there exist R sufficiently large so that $K \subset [-R, R]$. We then can bound this integral by examining

$$M \int_{(0,\infty)\cap K} |\log(x)|dx \leq M \int_0^R |\log(x)|dx = -M \int_0^1 \log(x)dx + M \int_1^R \log(x)dx$$
$$= R\log(R) + M(2 - R) < \infty.$$

Thus, this is in $L^1(\mathbb{R})$, so we can apply DCT to get

$$\lim_{\epsilon \to 0} \langle F_\epsilon, \varphi \rangle = \lim_{\epsilon \to 0} \int \chi_{(\epsilon,\infty)}(x) \log(x)\varphi(x)dx = \int \chi_{(0,\infty)}(x) \log(x)\varphi(x)dx = \langle F, \varphi \rangle.$$

Notice the choice of test function was arbitrary, so $F_\epsilon \to F$ in $\mathcal{D}'(\mathbb{R})$. Consequently, we have that $F'_\epsilon \to F'$ in $\mathcal{D}'(\mathbb{R})$. Fixing $\varphi \in \mathcal{D}(\mathbb{R})$, we have that

$$\lim_{\epsilon \to 0} \langle F'_\epsilon, \varphi \rangle = -\lim_{\epsilon \to 0} \langle F_\epsilon, \varphi' \rangle = -\langle F, \varphi' \rangle = \langle F', \varphi \rangle$$

since $F_\epsilon \to F$. The choice of test function was arbitrary, and so we have the desired convergence.

Now, fixing $\epsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle F'_\epsilon \circ S_r, \varphi \rangle = -r^{-1} \langle F_\epsilon, (\varphi \circ S_r^{-1})' \rangle = -r^{-1} \int_\epsilon^\infty \log(x)[\varphi(x/r)]'dx,$$

using the fact that F_ϵ is locally integrable. Expanding the derivative on the inside using the chain rule, we get that this is equal to

$$-r^{-2} \int_\epsilon^\infty \log(x)\varphi'(x/r)dx.$$

Now we integrate by parts. Let $dv = \varphi'(x/r)dx, v = r\varphi(x/r)$, $u = \log(x)$, $du = dx/x$. Then we have that the above is equal to

$$-r^{-2} \left[r\log(x)\varphi(x/r) \Big|_{x=\epsilon}^\infty - \int_\epsilon^\infty \frac{r\varphi(x/r)}{x} dx \right].$$

Since φ has compact support, we can evaluate this; we get that this is equal to

$$r^{-2} \left[r\log(\epsilon)\varphi(\epsilon/r) + \int_\epsilon^\infty \frac{r\varphi(x/r)}{x} dx \right].$$

Thus, pulling out a constant r, we have that this is equal to

$$r^{-1} \left[\log(\epsilon)\varphi(\epsilon/r) + \int_\epsilon^\infty \frac{\varphi(x/r)}{x} dx \right].$$

Letting $\epsilon' = \epsilon/r$, $u = x/r$, $du = dx/r$, we can rewrite this as

$$r^{-1} \left[\log(r\epsilon')\varphi(\epsilon') + \int_{\epsilon'}^\infty \frac{\varphi(u)}{u} du \right] = r^{-1} \left[\log(r)\varphi(\epsilon') + \log(\epsilon')\varphi(\epsilon') + \int_{\epsilon'}^\infty \frac{\varphi(u)}{u} du \right]$$
$$= r^{-1} \left[\log(r)\varphi(\epsilon') + \langle F'_{\epsilon'}, \varphi \rangle \right].$$

Taking $\epsilon \to 0$, we get $\epsilon' \to 0$, and by what we've noted earlier we have this converges to

$$r^{-1}\langle F', \varphi \rangle + \frac{\log(r)}{r}\varphi(0) = r^{-1}\langle F', \varphi \rangle + \frac{\log(r)}{r}\langle \delta, \varphi \rangle = \left\langle r^{-1}F' + \frac{\log(r)}{r}\delta, \varphi \right\rangle.$$

Thus, since the choice of test function was arbitrary, we have that as distributions

$$F' \circ S_r = r^{-1}F' + \frac{\log(r)}{r}\delta.$$

So F' is not homogeneous.

Now, for $\varphi \in \mathcal{D}(\mathbb{R} - \{0\})$, we note that $\langle \varphi, \delta \rangle = 0$. So, for all $r > 0$, $\varphi \in \mathcal{D}(\mathbb{R} - \{0\})$, we see that

$$F' \circ S_r = r^{-1} F'.$$

Hence, away from the origin, we have that F is homogeneous of degree -1. The function with which it agrees with is the one from recitation/page 288; that is, it agrees with $f(x) = x^{-1}\chi_{(0,\infty)}(x)$.

□

Problem 40. Define G on $\mathbb{R}^n \times \mathbb{R}$ by $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \chi_{(0,\infty)}(t)$.
(a) $(\partial_t - \Delta)G = \delta$, where Δ is the Laplacian on $\mathbb{R}^n$.
(b) If $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$, the function $f = G * \varphi$ satisfies $(\partial_t - \Delta)f = \varphi$.

Proof. (a) We follow Folland's hint. Fix $\epsilon > 0$. Let $G^\epsilon(x,t) = G(x,t)\chi_{(\epsilon,\infty)}(t)$. Notice that $G^\epsilon \to G$ in $\mathcal{D}'$; we show this via an application of the DCT. Fix a compact $K \subset \mathbb{R}^n \times \mathbb{R}$. Since $G(x,t)$ is always positive, we can iterate the integral by Tonelli. Let K' denote the projection of K onto the t-coordinate (the last/time coordinate). We integrate over K to get an upper bound of

$$\int_K G(x,t) \leq \int_{K'} \int_{\mathbb{R}^n} G(x,t) dx dt.$$

Writing things, we have that the right hand side is equal to

$$\int_{K' \cap (0,\infty)} (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x|^2/4t} dx dt.$$

Using **Proposition 2.53**, since t here is positive, we get that

$$\int_{\mathbb{R}^n} e^{-|x|^2/4t} dx = (4\pi t)^{n/2},$$

so we have that the integral above evaluates to

$$\int_{K' \cap (0,\infty)} 1 dt.$$

This is finite since K' is bounded (if it was not, we would have that K is unbounded in the last coordinate, contradicting the fact that K is compact). Hence, $G(x,t)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$. Furthermore, the same argument gives that $G^\epsilon(x,t)$ is locally integrable for all $\epsilon > 0$. Now, fixing $\varphi \in \mathcal{D}$, we have that

$$\langle G^\epsilon, \varphi \rangle = \int G^\epsilon(x,t)\varphi(x,t).$$

Since $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$, it has compact support and is smooth. Thus, it is bounded. Furthermore, letting $K = \mathrm{supp}(\varphi)$ and letting M be such that $|\varphi(x,t)| \leq M$ for all x, t, we have that $|G^\epsilon(x,t)\varphi(x,t)| \leq MG(x,t)\chi_K$, and since $MG(x,t)\chi_K \in L^1$, we can invoke DCT to bring the limit as $\epsilon \to 0$ on the inside of the integral; in other words,

$$\lim_{\epsilon \to 0} \int G^\epsilon(x,t)\varphi(x,t) = \int G(x,t)\varphi(x,t).$$

This holds for all $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$, so we get that $G^\epsilon \to G$ as distributions as $\epsilon \to 0$.

Recall from the lecture notes that we have

$$\partial_t G = \Delta G,$$

where

$$\Delta = \sum_1^n \frac{d^2}{dx_j^2}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ (that is, Green's function satisfies the heat equation). We will deduce a weaker result of a.e. convergence by using the Fourier transform (which is easier than actually calculating it, and we only need it almost everywhere). Transforming with respect to the space variable, we see that we have

$$[\partial_t G]^\wedge(m,t) = \partial_t \widehat{G}(m,t).$$

This is a consequence of **Theorem 2.27 (b)**. Notice that, by **Proposition 8.24**, we get that the Fourier transform of

$$G(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

maps to

$$\widehat{G}(m,t) = e^{-4\pi^2|m|^2 t}$$

after letting $a = 1/(4\pi t)$. Hence,

$$\partial_t \widehat{G}(m,t) = -4\pi^2|m|^2 e^{-4\pi^2|m|^2 t} = -4\pi^2|m|^2 \widehat{G}(m,t).$$

Now, we note that

$$\begin{aligned}
\widehat{\Delta G}(m,t) &= \left[\sum_1^n \frac{d^2}{dx_j^2} G\right]^\wedge (m,t) \\
&= \sum_1^n \left[\frac{d^2}{dx_j^2} G\right]^\wedge (m,t) \\
&= \sum_1^n (-4\pi^2) m_j^2 \widehat{G}(m,t) \\
&= -4\pi^2 |m|^2 \widehat{G}(m,t).
\end{aligned}$$

Thus, see that

$$[\partial_t G]^\wedge(m,t) = \widehat{\Delta G}(m,t).$$

We have then that

$$[\partial_t G - \Delta G]^\wedge = 0,$$

so they are equal almost everywhere. Furthermore, Green's function is continuous on $\mathbb{R}^n \times (0,\infty)$, so we have that they are honestly equal, although almost everywhere equality is sufficient for what we're doing.

Now, notice that

$$\langle (\partial_t - \Delta) G^\epsilon, \varphi \rangle = \langle \partial_t G^\epsilon, \varphi \rangle - \langle \Delta G^\epsilon, \varphi \rangle.$$

Notice as well that

$$\begin{aligned}
\langle \Delta G^\epsilon, \varphi \rangle &= \left\langle \sum_1^n \frac{d^2}{dx_j^2} G^\epsilon, \varphi \right\rangle \\
&= \sum_1^n \left\langle \frac{d^2}{dx_j^2} G^\epsilon, \varphi \right\rangle, \\
&= \sum_1^n \left\langle G^\epsilon, \frac{d^2}{dx_j^2} \varphi \right\rangle \\
&= \langle G^\epsilon, \Delta \varphi \rangle,
\end{aligned}$$

$$\langle \partial_t G^\epsilon, \varphi \rangle = -\langle G^\epsilon, \partial_t \varphi \rangle,$$

so we get

$$\langle(\partial_t - \Delta)G^\epsilon, \varphi\rangle = -\langle G^\epsilon, (\partial_t + \Delta)\varphi\rangle.$$[4]

Using the definition of G^ϵ and the fact that this is locally integrable, this gives

$$\langle(\partial_t - \Delta)G^\epsilon, \varphi\rangle = -\langle G^\epsilon, (\partial_t + \Delta)\varphi\rangle = -\int_{\mathbb{R}^n\times(\epsilon,\infty)} G(x,t)(\partial_t + \Delta)\varphi(x,t).$$

Since this is integrable (using the fact that G is locally integrable and φ is bounded, smooth, and has compact support, so all of its derivatives are also smooth, bounded, and have compact support), we can change this to an iterated integral using Fubini; thus, we have

$$-\int_{\mathbb{R}^n}\int_\epsilon^\infty G(x,t)(\partial_t + \Delta)\varphi(x,t)dtdx.$$

Now, we expand the integral with linearity to get

$$-\int_{\mathbb{R}^n}\int_\epsilon^\infty G(x,t)\partial_t\varphi(x,t)dtdx - \int_{\mathbb{R}^n}\int_\epsilon^\infty G(x,t)\Delta\varphi(x,t)dtdx.$$

Notice that the integral on the left can be integrated with respect to t via integration by parts. Letting $u = G(x,t)$, $du = \partial_t G(x,t)dt$, $dv = \partial_t\varphi(x,t)dt$, $v = \varphi(x,t)$, we get

$$\begin{aligned}\int_\epsilon^\infty G(x,t)\partial_t\varphi(x,t)dt &= G(x,t)\varphi(x,t)\Big|_{t=\epsilon}^\infty - \int_\epsilon^\infty \partial_t G(x,t)\varphi(x,t)dt\\ &= -G(x,\epsilon)\varphi(x,\epsilon) - \int_\epsilon^\infty \partial_t G(x,t)\varphi(x,t)dt.\end{aligned}$$

Substituting this back in, we have

$$\int_{\mathbb{R}^n} G(x,\epsilon)\varphi(x,\epsilon)dx + \int_{\mathbb{R}^n}\int_\epsilon^\infty \partial_t G(x,t)\varphi(x,t)dtdx - \int_{\mathbb{R}^n}\int_\epsilon^\infty G(x,t)\Delta\varphi(x,t)dtdx.$$

Recalling that $\partial_t G = \Delta G$, we get

$$\int_{\mathbb{R}^n}\int_\epsilon^\infty \partial_t G(x,t)\varphi(x,t)dtdx = \int_{\mathbb{R}^n}\int_\epsilon^\infty \Delta G(x,t)\varphi(x,t)dtdx.$$

Recall as well

$$\Delta G(x,t) = \sum_1^n \frac{d}{dx_j^2}G(x,t).$$

Hence, we can rewrite this as

$$\sum_1^n \int_{\mathbb{R}^n}\int_\epsilon^\infty \frac{d^2}{dx_j^2}G(x,t)\varphi(x,t)dtdx.$$

Applying Fubini, this is the same as

$$\sum_1^n \int_\epsilon^\infty\int_{\mathbb{R}^n} \frac{d^2}{dx_j^2}G(x,t)\varphi(x,t)dxdt.$$

Without loss of generality, we examine $j = 1$; the argument will be the same for all other j. Here, we iterate the integral again;

$$\int_\epsilon^\infty\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \frac{d^2}{dx_1^2}G(x_1,y,t)\varphi(x_1,y,t)dx_1dydt.$$

[4]As an aside for my future notes, the reason this doesn't end up being the 0 distribution is because of the issue at the origin for Green's function. Like Folland remarks, issues with continuity leads to δ functions.

Let $dv = \frac{d^2}{dx_1^2}G(x_1, y, t)dx_1$, $v = \frac{d}{dx_1}G(x_1, y, t)$, $u = \varphi(x_1, y, t)$, $du = \frac{d}{dx_1}\varphi(x_1, y, t)dx_1$. Then integration by parts tells us that the integral is the same as

$$\begin{aligned}\int_{\epsilon}^{\infty}\int_{\mathbb{R}^{n-1}}\left[\frac{d}{dx_1}G(x_1,y,t)\varphi(x_1,y,t)\Big|_{x_1=-\infty}^{\infty} - \int_{\mathbb{R}}\frac{d}{dx_1}G(x_1,y,t)\frac{d}{dx_1}\varphi(x_1,y,t)dx_1\right]dydt\\ = -\int_{\epsilon}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\frac{d}{dx_1}G(x_1,y,t)\frac{d}{dx_1}\varphi(x_1,y,t)dx_1dydt,\end{aligned}$$

where we use the fact that φ has compact support. Integrating by parts again, letting $dv = \frac{d}{dx_1}G(x_1, y, t)dx_1$, $v = G(x_1, y, t)$, $u = \frac{d}{dx_1}\varphi(x_1, y, t)$, $du = \frac{d^2}{dx_1^2}\varphi(x_1, y, t)dx_1$, we get (after simplifying like above)

$$\int_{\epsilon}^{\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} G(x_1,y,t)\frac{d^2}{dx_1^2}\varphi(x_1,y,t)dx_1dydt.$$

Hence, we have that it comes out to

$$\int_{\epsilon}^{\infty}\int_{\mathbb{R}^n} G(x,t)\frac{d^2}{dx_1^2}\varphi(x,t)dxdt.$$

Repeating this same argument for all of the other variables, we get that this comes out to

$$\sum_{1}^{n}\int_{\epsilon}^{\infty}\int_{\mathbb{R}^n} G(x,t)\frac{d^2}{dx_j^2}\varphi(x,t)dxdt,$$

which is the same as

$$\int_{\epsilon}^{\infty}\int_{\mathbb{R}^n} G(x,t)\Delta\varphi(x,t)dxdt = \int_{\mathbb{R}^n}\int_{\epsilon}^{\infty} G(x,t)\Delta\varphi(x,t)dtdx.$$ [5]

Substituting this into our original integral, we are left with

$$\langle(\partial_t - \Delta)G^\epsilon, \varphi\rangle = \int_{\mathbb{R}^n} G(x,\epsilon)\varphi(x,\epsilon)dx.$$

That is, we have it is equal to

$$(4\pi\epsilon)^{-n/2}\int_{\mathbb{R}^n} e^{-|x|^2/4\epsilon}\varphi(x,\epsilon)dx.$$

Letting $x = 2t\sqrt{\epsilon}$, we have

$$\pi^{-n/2}\int_{\mathbb{R}^n} e^{-|t|^2}\varphi(2t\sqrt{\epsilon},\epsilon)dt.$$

Now, we can apply dominated convergence theorem here, since $\varphi(2t\sqrt{\epsilon}, \epsilon)$ has compact support K and upper bound M, so

$$|e^{-|t|^2}\varphi(2t\sqrt{\epsilon},\epsilon)| \le Me^{-|t|^2}\chi_K(t) \in L^1(\mathbb{R}^n).$$

Bringing the limit inside, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\langle(\partial_t - \Delta)G^\epsilon, \varphi\rangle = \langle(\partial_t - \Delta)G, \varphi\rangle = \lim_{\epsilon\to 0}\pi^{-n/2}\int_{\mathbb{R}^n} e^{-|t|^2}\varphi(2t\sqrt{\epsilon},\epsilon)dt\\ = \pi^{-n/2}\varphi(0,0)\int_{\mathbb{R}^n} e^{-|t|^2}dt\\ = \varphi(0,0) = \langle\delta,\varphi\rangle,\end{aligned}$$

[5] This is a long detailed argument to conclude that $\langle\Delta G^\epsilon, \varphi\rangle = \langle G^\epsilon, \Delta\varphi\rangle$, which we could've simplified greatly by just noticing the above.

where here we again use **Proposition 2.53**. Since this applies for all $\varphi \in \mathcal{D}$, we get that as distributions $(\partial_t - \Delta)G = \delta$.

(b) Here, we have

$$\begin{aligned}(\partial_t - \Delta)(G * \varphi)(x) &= ((\partial_t - \Delta)G) * \varphi(x) = \langle (\partial_t - \Delta)G, \tau_x \widetilde{\varphi} \rangle \\ &= \langle \delta, \tau_x \widetilde{\varphi} \rangle = \tau_x \widetilde{\varphi}(0) = \widetilde{\varphi}(-x) = \varphi(x),\end{aligned}$$

where the first equality comes from **Proposition 9.3**. Hence, we have that as functions,

$$(\partial_t - \Delta)(G * \varphi) = \varphi.$$

□

Remark. Thomas O'Hare was a collaborator.

Problem 41. Suppose that $F \in \mathcal{S}'$. Then

(a) $(\tau_y F)^\wedge = e^{-2\pi i m\cdot y}\widehat{F}$.
 $\tau_y \widehat{F} = (e^{2\pi i m\cdot y}F)^\wedge$.

(b) $\partial^\alpha \widehat{F} = [(-2\pi i x)^\alpha F]^\wedge$
 $\widehat{\partial^\alpha F} = (2\pi i m)^\alpha \widehat{F}$.

(c) For $T \in \mathrm{GL}_n(\mathbb{R})$, $(F\circ T)^\wedge = |\det T|^{-1}\widehat{F}\circ (T^*)^{-1}$.

(d) $(F * \psi)^\wedge = \widehat{\psi}\widehat{F}$ for $\psi \in \mathcal{S}$.

Recall the following first.

Proposition (Properties of Fourier Transform, **Proposition 8.22**). For $f \in \mathcal{S}$, we have the following:

(a) $(\tau_y f)^\wedge(m) = e^{-2\pi i m\cdot y}\widehat{f}(m)$.

(b) $\tau_y(\widehat{f}) = [hf]^\wedge$, where $h(x) = e^{2\pi i m\cdot x}$.

(c) If $T \in \mathrm{GL}_n(\mathbb{R})$, then $(f\circ T)^\wedge = |\det T|^{-1}\widehat{f}\circ (T^*)^{-1}$.

(d) $(f * g)^\wedge = \widehat{f}\widehat{g}$.

(e) $\partial^\alpha \widehat{f} = [(-2\pi i x)^\alpha f]^\wedge$.

(f) $(\partial^\alpha f)^\wedge = (2\pi i m)^\alpha \widehat{f}(m)$.

Let $F \in \mathcal{S}'$, $\varphi \in \mathcal{S}$, then we have the following properties (as a modification of these properties for distributions, see **Folland page 284-285**):

(i) $\langle \partial^\alpha F, \varphi\rangle = (-1)^{|\alpha|}\langle F, \partial^\alpha \varphi\rangle$.

(ii) For $\psi \in \mathcal{S}$, we have $\langle \psi F, \varphi\rangle = \langle F, \psi\varphi\rangle$.

(iii) $\langle \tau_y F, \varphi\rangle = \langle F, \tau_{-y}\varphi\rangle$.

(iv) For $T \in \mathrm{GL}_n(\mathbb{R})$ we have $\langle F\circ T, \varphi\rangle = |\det T|^{-1}\langle F, \varphi\circ T^{-1}\rangle$.

(v) If $\psi \in \mathcal{S}$, then $F * \psi(x) = \langle F, \tau_x \widetilde{\psi}\rangle$, where $\widetilde{\psi}(x) = \psi(-x)$.

Recall as well we define the Fourier transform on distributions $F \in \mathcal{S}'$ via

$$\langle \widehat{F}, \varphi\rangle = \langle F, \widehat{\varphi}\rangle,$$

where $\varphi \in \mathcal{S}$.

Proof. (a) Let $\varphi \in \mathcal{S}$. Letting $h(y) = e^{-2\pi i m\cdot y}$, we have

$$\langle (\tau_y F)^\wedge, \varphi\rangle = \langle \tau_y F, \widehat{\varphi}\rangle = \langle F, \tau_{-y}\widehat{\varphi}\rangle = \langle F, [h\varphi]^\wedge\rangle = \langle \widehat{F}, h\varphi\rangle = \langle h\widehat{F}, \varphi\rangle.$$

The first equality holds by definition of the Fourier transform on distributions. The second equality follows from how translations act on distributions [property (iii) above]. The third follows from properties of the Fourier transform on $f \in \mathcal{S}$ [property (b) above]. The fourth follows from the definition of the Fourier transform on distributions. The final equality follows from how Schwarz functions multiply with distributions [property (ii) above]. Since this holds for all $\varphi \in \mathcal{S}$, we have that

$$(\tau_y F)^\wedge = h\widehat{F} = e^{-2\pi i m\cdot y}\widehat{F}$$

as distributions.

We now prove the second equality. Let $\varphi \in \mathcal{S}$. Letting $h(y) = e^{2\pi i m\cdot y}$, we have

$$\langle \tau_y \widehat{F}, \varphi\rangle = \langle \widehat{F}, \tau_{-y}\varphi\rangle = \langle F, [\tau_{-y}\varphi]^\wedge\rangle = \langle F, h\widehat{\varphi}\rangle = \langle hF, \widehat{\varphi}\rangle = \langle \widehat{hF}, \varphi\rangle.$$

The first equality follows from how translations act on distributions [property (iii) above]. The second follows from the definition of the Fourier transform on distributions. The third follows from how translations relate with the Fourier transform [property (a) above]. The fourth follows from how Schwarz functions multiply with distributions [property (ii) above]. The last follows from the definition of Fourier transform on distributions. Thus, as distributions,

$$\tau_y \widehat{F} = (e^{2\pi i m \cdot y} F)^\wedge.$$

(b) Let $\varphi \in \mathcal{S}$. Then we see that

$$\begin{aligned}\langle \partial^\alpha \widehat{F}, \varphi\rangle &= (-1)^{|\alpha|}\langle \widehat{F}, \partial^\alpha \varphi\rangle = (-1)^{|\alpha|}\langle F, \widehat{\partial^\alpha \varphi}\rangle = (-1)^{|\alpha|}\langle F, (2\pi i m)^\alpha \widehat{\varphi}\rangle \\ &= (-1)^{|\alpha|}\langle (2\pi i m)^\alpha F, \widehat{\varphi}\rangle = \langle (-2\pi i m)^\alpha F, \widehat{\varphi}\rangle = \langle [(-2\pi i m)^\alpha F]^\wedge, \varphi\rangle.\end{aligned}$$

The first equality follows from how derivatives act on distributions [property (i) above]. The second follows from the definition of the Fourier transform on distributions. The third follows from how derivatives interact with the Fourier transform [property (f) above]. The fourth follows from how Schwarz functions multiply with distributions [property (ii) above]. The fifth follows from using linearity in the first coordinate. The final follows from the definition of Fourier transform on distributions. Thus, as distributions, we have

$$\partial^\alpha \widehat{F} = [(-2\pi i x)^\alpha F]^\wedge.$$

We now prove the second equality. Let $\varphi \in \mathcal{S}$. Then we have

$$\begin{aligned}\langle \widehat{\partial^\alpha F}, \varphi\rangle = \langle \partial^\alpha F, \widehat{\varphi}\rangle &= (-1)^{|\alpha|}\langle F, \partial^\alpha \widehat{\varphi}\rangle = (-1)^{|\alpha|}\langle F, [(-2\pi i x)^\alpha \varphi]^\wedge\rangle \\ &= (-1)^{|\alpha|}\langle \widehat{F}, (-2\pi i x)^\alpha \varphi\rangle = \langle (2\pi i x)^\alpha \widehat{F}, \varphi\rangle\end{aligned}.$$

The first equality follows from the definition of Fourier transform on distributions. The second follows from how the derivative interacts with distributions [property (i) above]. The third follows from how derivatives interact with the Fourier transform [property (e) above]. The fourth follows from the definition of the Fourier transform on distributions. The final follows from using linearity and how Schwarz functions multiply with distributions [property (ii) above]. Thus, as distributions, we have

$$\widehat{\partial^\alpha F} = (2\pi i x)^\alpha \widehat{F}.$$

(c) Let $T \in \mathrm{GL}_n(\mathbb{R})$. Then we have

$$\begin{aligned}\langle (F\circ T)^\wedge, \varphi\rangle = \langle F \circ T, \widehat{\varphi}\rangle &= |\det T|^{-1}\langle F, \widehat{\varphi}\circ T^{-1}\rangle = |\det T|^{-1}\langle F, |\det T|(\varphi \circ T^*)^\wedge\rangle \\ &= \langle F, (\varphi\circ T^*)^\wedge\rangle = \langle \widehat{F}, \varphi \circ T^*\rangle = |\det T|^{-1}\langle \widehat{F}\circ (T^*)^{-1}, \varphi\rangle.\end{aligned}$$

The first equality follows from the definition of the Fourier transform on distributions. The second follows from how invertible linear functions interact with distributions [property (iv) above]. The third follows from how the Fourier transform interacts with composition of invertible linear functions [property (c) above]. The fourth follows from linearity and how Schwarz functions multiply with distributions (in order to move $|\det T|$ to the other side) [property (ii) above]. The fifth follows from the definition of the Fourier transform on distributions. The sixth follows from how distributions interact with invertible linear functions [property (iv) above, with slight modification]. Thus, as distributions, we have

$$(F\circ T)^\wedge = |\det T|^1 \widehat{F}\circ (T^*)^{-1}.$$

(d) This is the more interesting property. Recall that for $\psi \in \mathcal{S}$, we have

$$\langle F * \psi, \varphi\rangle = \int (F*\psi)\varphi = \langle F, \varphi * \widetilde{\psi}\rangle$$

by **Proposition 9.10**. Taking the Fourier transform, we have

$$\langle \widehat{F * \psi}, \varphi \rangle = \langle F * \psi, \widehat{\varphi} \rangle.$$

Next, we wish to show the following identity:

$$[\widehat{\psi}]^{\wedge} = \widetilde{\psi}.$$

To see the identity, recall that for Schwarz functions we have

$$\psi^{\vee}(x) = \widehat{\psi}(-x) = \widetilde{\widehat{\psi}}(x).$$

We remark that

$$\widehat{\widetilde{\psi}}(x) = \widetilde{\widehat{\psi}}(x).$$

This follows by the fact that $\mathcal{F}$ is an isomorphism on $\mathcal{S}$ (**Corollary 8.28**), since if $\mathcal{G}$ denotes the inverse Fourier transform, $\mathcal{F}$ the usual Fourier transform, P the reflection function (i.e. $P(\psi)(x) = \psi(-x)$), we have

$$\mathcal{G} \circ \mathcal{F}(\psi) = \mathcal{F} \circ \mathcal{G}(\psi) = \psi,$$
$$\mathcal{G}(\psi) = P \circ \mathcal{F}(\psi),$$

so

$$P \circ \mathcal{F} \circ \mathcal{F}(\psi) = \mathcal{F} \circ P \circ \mathcal{F}(\psi).$$

Since $\mathcal{F}$ is an isomorphism on $\mathcal{S}$, we have that for all $\psi \in \mathcal{S}$ there exists a φ so that $\mathcal{G}(\varphi) = \psi$. The above holds for all ψ, so in particular we have

$$P \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{G}(\varphi) = P \circ \mathcal{F}(\varphi) = \mathcal{F} \circ P \circ \mathcal{F} \circ \mathcal{G}(\varphi) = \mathcal{F} \circ P(\varphi).$$

Hence, for all $\psi \in \mathcal{S}$, we have

$$\mathcal{F} \circ P(\psi) = P \circ \mathcal{F}(\psi).$$

That is, we have that reflection commutes with the Fourier transform.

Now, notice that taking the Fourier transform of both sides of

$$\mathcal{G}(\psi) = P \circ \mathcal{F}(\psi),$$

gives us

$$\psi = \mathcal{F} \circ P \circ \mathcal{F}(\psi),$$

so that, using the commutativity of P and $\mathcal{F}$, we have

$$\psi = P \circ \mathcal{F}^{(2)}(\psi).$$

Note that $P \circ P = \mathrm{Id}$, so taking P of both sides, we have

$$P(\psi) = \mathcal{F}^{(2)}(\psi);$$

that is, reverting to old notation, we have

$$[\widehat{\psi}]^{\wedge} = \widetilde{\psi}.$$

This coupled with **Proposition 9.10** gives us that

$$\langle F * \psi, \widehat{\varphi} \rangle = \langle F, \widehat{\varphi} * [\widehat{\psi}]^{\wedge} \rangle$$

Next, we note that

$$\widehat{\varphi} * [\widehat{\psi}]^{\wedge} = [\varphi \widehat{\psi}]^{\wedge}.$$

To see this, note that property (d) above gives

$$\mathcal{F}^{(2)}(\varphi) \mathcal{F}^{(3)}(\psi) = \mathcal{F}(\mathcal{F}(\varphi) * \mathcal{F}^{(2)}(\psi)).$$

Hence, we have

$$\mathcal{G}(\mathcal{F}^{(2)}(\varphi) \mathcal{F}^{(3)}(\psi)) = \mathcal{F}(\varphi) * \mathcal{F}^{(2)}(\psi)$$

Note that

$$P(\psi\varphi) = P(\psi)P(\varphi);$$

this is due to the fact that

$$P(\psi\varphi)(x) = (\psi\varphi)(-x) = \psi(-x)\varphi(-x) = (P(\psi)P(\varphi))(x).$$

Hence, we have

$$\begin{aligned}\mathcal{G}(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) = (P \circ \mathcal{F})(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) = (\mathcal{F} \circ P)(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi))\\ = (\mathcal{F} \circ P)(P(\varphi)(P \circ \mathcal{F})(\psi)) = \mathcal{F}(P^{(2)}(\varphi)(P^{(2)} \circ \mathcal{F})(\psi))\\ = \mathcal{F}(\varphi\mathcal{F}(\psi)).\end{aligned}$$

The first equality here follows by expanding out the definition of $\mathcal{G}$, the second follows from using the fact that P and F commute, the third follows from using the identity $\mathcal{F}^{(2)} = P$ and writing $\mathcal{F}^{(3)} = \mathcal{F}^{(2)} \circ \mathcal{F}$, the fourth follows from the fact that P distributes over multiplication, and the last follows from the fact that $P^{(2)} = \mathrm{Id}$. In other words, reverting to old notation, we have that

$$\widehat{\varphi} * [\widehat{\psi}]^\wedge = [\varphi\widehat{\psi}]^\wedge.$$

This gives us

$$\langle F, \widehat{\varphi} * [\widehat{\psi}]^\wedge\rangle = \langle F, [\varphi\widehat{\psi}]^\wedge\rangle.$$

Using the definition of the Fourier transform for tempered distributions as well as how Schwarz functions multiply with distributions [property (ii) above], we get

$$\langle F, [\varphi\widehat{\psi}]^\wedge\rangle = \langle \widehat{F}, \varphi\widehat{\psi}\rangle = \langle \widehat{\psi}\widehat{F}, \varphi\rangle$$

Thus, we have

$$\langle \widehat{F * \psi}, \varphi\rangle = \langle \widehat{F}\widehat{\psi}, \varphi\rangle.$$

Since the choice of φ was arbitrary, we have that as distributions,

$$\widehat{F * \psi} = \widehat{F}\widehat{\psi}.$$

□

Problem 42 (Folland Proposition 9.7)**.** Let $U \subset \mathbb{R}^n$ be open. Let $\{V_n\}$ be an increasing family of precompact open sets whose union is U. We have that $C_c^\infty(U)$ is dense in $C^\infty(U)$ with respect to the topology generated by the family of seminorms $\|\cdot\|_{[m,\alpha]}$, where for $m \in \mathbb{N}$, α a multi-index, we have

$$\|f\|_{[m,\alpha]} = \sup_{x \in \overline{V_m}} |\partial^\alpha f(x)|.$$

Proof. Recall **Proposition 4.39**, which says that if X is a σ-compact LCH space, then there is an increasing sequence of precompact open sets such that $\overline{V}_n \subset V_{n+1}$ for all n and $X = \bigcup_1^\infty V_n$. Let $\{V_n\}$ be such a family for $\mathbb{R}^n$ which generates the family of seminorms. Notice that any other choice for such a family will generate the same topology, since we have that $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on compact sets for all α if and only if $\|f_j - f\|_{[m,\alpha]} \to 0$ for all m, α (see the remark before **Proposition 9.7** in **Folland page 291**). The goal, then, is to show that for all $\varphi \in C^\infty(U)$, there exists a $(\kappa_n) \subset C_c^\infty(U)$ so that $\|\kappa_n - \varphi\|_{[m,\alpha]} \to 0$ for all m, α. Using C^∞ Urysohn's lemma (**Proposition 8.18**) we can find ψ_n so that $0 \le \psi_n \le 1$, $\psi_n = 1$ on $\overline{V_n}$, and $\psi_n = 0$ outside of V_{n+1}. Let $\kappa_n = \varphi\psi_n$. Then this is smooth, since a product of smooth functions is smooth, and it has compact support due to ψ_n, so $\kappa_n \in C_c^\infty(U)$. Furthermore, we have that

$$\|\kappa_n - \varphi\|_{[m,\alpha]} = \sup_{x \in \overline{V_m}} |\partial^\alpha(\kappa_n - \varphi)(x)| = \sup_{x \in \overline{V_m}} |\partial^\alpha(\psi_n\varphi)(x) - \partial^\alpha\varphi(x)|.$$

For $m \leq n$, we see that $\kappa_n = \varphi$ on $\overline{V_m}$, since $x \in \overline{V_m} \subset \overline{V_n}$ implies that $\kappa_n(x) = \psi_n(x)\varphi(x) = \varphi(x)$. Thus, for $m \leq n$, we have that $\|\kappa_n - \varphi\|_{[m,\alpha]} = 0$. So, for all m, α fixed, we can choose n sufficiently large so that $\kappa_n = \varphi$ on $\overline{V_m}$, and hence $\|\kappa_n - \varphi\|_{[m,\alpha]} = 0$. Thus, we get that $\kappa_n \to \varphi$ in $C^\infty(U)$. The choice of φ was arbitrary, so we get that $C_c^\infty(U)$ is dense in $C^\infty(U)$. $\square$

Problem 43 (Folland Theorem 9.8)**.** Let $U \subset \mathbb{R}^n$ be open. We have that $\mathcal{E}'(U)$ is the dual space of $C^\infty(U)$.

Proof. We prove what Folland states afterwards; that is, we prove

(1) If $F \in \mathcal{E}'(U)$, then F extends uniquely to a continuous linear functional on $C^\infty(U)$.
(2) If G is a continuous linear functional on $C^\infty(U)$, then $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$.

For (1), let $F \in \mathcal{E}'(U)$. We need to show that we can extend this, and this extension is unique. Recall that the support of F is defined to be the complement of the maximal open set where F vanishes, and we have that F vanishes on an open set U if $\langle F, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(U)$. Recall as well that $\mathcal{E}'(U)$ is the space of distributions with compact support.

Since the support is compact, we can invoke C^∞ Urysohn to find ψ so that $\psi = 1$ on $\mathrm{supp}(F)$. Define a linear functional G on $C^\infty(U)$ where

$$\langle G, \varphi\rangle = \langle F, \varphi\psi\rangle.$$

First, note that G is a linear functional, since

$$|\langle G, \varphi\rangle| \leq C' \sum_{|\alpha|\leq N} \|\partial^\gamma \varphi\|_{u(\mathrm{supp}(\psi))} \leq C' \sum_{|\alpha|\leq N} \|\varphi\|_{[m,\alpha]}.$$

where φ, κ are test functions, r some scalar constant, and we use the fact that F is a linear functional. Next, notice that if $\varphi \in C_c^\infty(U)$, then we have

$$\langle G, \varphi\rangle = \langle F, \varphi\psi\rangle = \langle F, \varphi\rangle,$$

so G indeed extends F. Next, we need to show that the extension is continuous. Since it is defined on a dense subset (by the prior problem), the fact that it is continuous implies that it is a unique extension of F, and so we'll have our result.

To see continuity, we invoke **Proposition 5.15**. Note that F is continuous on $C_c^\infty(\mathrm{supp}(\psi))$. The topology on $C_c^\infty(\mathrm{supp}(\psi))$ is defined by $\|\partial^\alpha \varphi\|_u$, so applying the proposition tells us that we have

$$|\langle G, \varphi\rangle| = |\langle F, \psi\varphi\rangle| \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha(\psi\varphi)\|_u$$

for some constant C, using the fact that F is continuous. Notice that the product rule tells us that

$$\partial^\alpha(\psi\varphi) = \sum_{\beta+\gamma=\alpha} (\partial^\beta \psi)(\partial^\gamma \varphi).$$

Hence, we have

$$|\langle G, \varphi\rangle| \leq C \sum_{|\alpha|\leq N} \sum_{\beta+\gamma=\alpha} \|\partial^\beta \psi\|_u \|\partial^\gamma \varphi\|_u,$$

where the uniform norm is taken over the set $\mathrm{supp}(\psi)$. Choose m large enough so that $\mathrm{supp}(\psi) \subset V_m$; we know such an m exists, since $\bigcup V_k = \mathbb{R}^n$ and the set is increasing. Doing so and letting C' absorb constants, we have that

$$|\langle G, \varphi\rangle| \leq C' \sum_{|\alpha|\leq N} \|\partial^\alpha \varphi\|_u \leq C' \sum_{|\alpha|\leq N} \|\varphi\|_{[m,\alpha]},$$

where again the uniform norm in the second inequality is over $\operatorname{supp}(\psi)$. Hence, **Proposition 5.15** tells us that this is continuous, and we see that from the remark earlier we get that this is the unique extension of F.

Now we show (2). Let G be a continuous linear functional on $C^\infty(U)$. Again, using **Proposition 5.15**, we get

$$|\langle G, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|\varphi\|_{[m,\alpha]}$$

for all $\varphi \in C^\infty(U)$. Notice that we have $\|\varphi\|_{[m,\alpha]} \leq \|\partial^\alpha \varphi\|_u$, so restricting our view to $C_c^\infty(K)$ for arbitrary compact $K \subset U$, we get that G is continuous on $C_c^\infty(K)$. Thus, $G|_{C_c^\infty(U)} \in \mathcal{D}'(U)$ by **Folland page 282 (ii)**. The goal from here is to show that G has compact support. Notice that if $\operatorname{supp}(\varphi) \cap \overline{V_m} = \varnothing$, then

$$|\langle G, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|\varphi\|_{[m,\alpha]} = 0,$$

so $\langle G, \varphi \rangle = 0$. Thus, following the definition of support, we have that $\operatorname{supp}(G) \subset \overline{V_m}$; that is, $\operatorname{supp}(G)$ is compact (its a closed set contained in a compact set). So $G|_{C_c^\infty(U)} \subset \mathcal{E}'(U)$.

We now note that the above proves that $\mathcal{E}'(U)$ is the dual space of $C^\infty(U)$. Let G be in its dual space. Then we see that using (2) and restricting it to the space of test functions it agrees with some $F \in \mathcal{E}'(U)$. Thus, for all $\varphi \in C_c^\infty(U)$, we have

$$\langle F, \varphi \rangle = \langle G, \varphi \rangle.$$

Since F extends uniquely to some continuous linear functional on $C^\infty(U)$, we must have that this linear functional is G by the density argument prior. Thus, we can define F on all $\varphi \in C^\infty(U)$ by

$$\langle F, \varphi \rangle = \langle G, \varphi \rangle.$$

Since this extension is unique, we can without loss of generality label G as F. Going the other direction, for $F \in \mathcal{E}'(U)$, we can identify it uniquely with a continuous linear functional on $C^\infty(U)$ by (1). Hence, on all $C^\infty(U)$, we have that F is a continuous linear functional, so we see that the dual space can be identified as $\mathcal{E}'(U)$. $\square$

Remark. Thomas O'Hare was a collaborator.

We note the following lemma (left as an exercise in the class notes, also remarked by Folland on **Folland page 301**).

Lemma. Let $k \in \mathbb{N}$. The following are equivalent:
(a) $f \in H^k(\mathbb{R}^n)$,
(b) $|m^\alpha|\widehat{f}(m) \in L^2(\mathbb{R}^n)$ for $|\alpha| \le k$,
(c) $(1+|m|^2)^{k/2}\widehat{f}(m) \in L^2(\mathbb{R}^n)$.

Proof. We first show $(a) \iff (b)$. By Plancherel, we have that

$$\sum_{|\alpha|\le k} \|\partial^\alpha f\|_2^2 = \sum_{|\alpha|\le k} \|(\partial^\alpha f)^\wedge\|_2^2 = \sum_{|\alpha|\le k} \|(2\pi i m)^\alpha \widehat{f}\|_2^2,$$

where we use the Fourier transform on distributions properties (see **Homework 9, Exercise 1**), as well as the fact that the Fourier transform of a distribution defined by a function agrees with the Fourier transform of the function (see **Folland page 295**). Thus, we see that

$$\|f\|_{(k)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha f\|_2^2\right)^{1/2} = \left(\sum_{|\alpha|\le k} \|(2\pi i m)^\alpha \widehat{f}\|_2^2\right)^{1/2}.$$

For $(b) \iff (c)$, we need to first find constants C_1 and C_2 greater than 0 so that

$$C_1(1+|m|^2)^{k/2} \le \sum_{|\alpha|\le k} |m^\alpha| \le C_2(1+|m|^2)^{k/2}.$$

Recall from **Folland Proposition 8.3** that we have

$$|m^\alpha| \le (1+|m|)^k$$

for $|\alpha| \le k$, since

$$|m^\alpha| = |m_1|^{\alpha_1}\cdots|m_n|^{\alpha_n} \le (1+|m|)^{\alpha_1}\cdots(1+|m|)^{\alpha_n} = (1+|m|)^{\sum\alpha_i} = (1+|m|)^{|\alpha|} \le (1+|m|)^k.$$

Now, notice that

$$|m^\alpha|^2 = |m_1|^{2\alpha_1}\cdots|m_n|^{2\alpha_n} \le (1+|m|^2)^{|\alpha|} \le (1+|m|^2)^k.$$

Taking square roots of both sides leaves us with

$$|m^\alpha| \le (1+|m|^2)^{k/2}.$$

Hence, we can choose C_2 to be $\sum_{|\alpha|\le k} 1$ and we get the desired upper bound.

For the lower bound, recall that we have $\sum_1^n |m_i|^k$ is strictly positive on the unit sphere $|m| = 1$. Since this is compact and the function continuous, it admits a minimum $\delta > 0$. Thus, for all m, we have

$$\sum_1^n \left|\frac{m_i}{|m|}\right|^k \ge \delta \implies \sum_1^n |m_i|^k \ge |m|^k\delta.$$

Now, we have

$$(1+|m|^2)^{k/2} \le 2^{k/2}\left(1+|m|^k\right) \le 2^{k/2}\left[1+\delta^{-1}\sum_1^n |m_i^k|\right] \le 2^{k/2}\delta^{-1}\sum_{|\alpha|\le k} |m^\alpha|.$$

Choosing $C_1 = 2^{-k/2}\delta$, we get the desired lower bound. Since these are constants, $|m^\alpha|\widehat{f}(m) \in L^2(\mathbb{R}^n)$ if and only if $(1+|m|^2)^{k/2}\widehat{f}(m) \in L^2(\mathbb{R}^n)$. $\square$

Problem 44. For $s \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$, let

$$\omega_s(\zeta) = (1+|\zeta|^2)^{s/2}.$$

Show that $L^2(\mathbb{R}^n, \omega_s)$ equipped with the inner product

$$(f,g) = (f,g)_{L^2(\mathbb{R}^n,\omega_s)} = \langle \omega_s f, \omega_s g\rangle_{L^2(\mathbb{R}^n)} = \int (\omega_s(x)f(x))(\overline{\omega_s(x)g(x)})dx$$

is a Hilbert space. Moreover, show that

$$\mathcal{S} \subset L^2(\mathbb{R}^n, \omega_s) \subset \mathcal{S}',$$

and show that $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}$ are dense in $L^2(\mathbb{R}^n, \omega_s)$.

Proof. We remark that

$$L^2(\mathbb{R}^n, \omega_s) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \omega_s f \in L^2(\mathbb{R}^n) \right\}.$$

First, we wish to show that the prescribed inner product is indeed an inner product. There are three properties we must show.

(1) First, we see that it is linear in the first component. Let $a, b \in \mathbb{C}$, $f, g, h \in L^2(\mathbb{R}^n, \omega_s)$. Then we need to show that

$$(af+bg, h) = a(f,h) + b(g,h).$$

Notice that

$$\begin{aligned}(af+bg,h) &= \int \left(\omega_s(x)\left(af+bg\right)(x)\right)(\overline{\omega_s(x)h(x)})dx \\ &= \int \left[\omega_s(x)(af(x)) + \omega_s(x)(bg(x))\right](\overline{\omega_s(x)h(x)})dx \\ &= a\int (\omega_s(x)f(x))(\overline{\omega_s(x)h(x)})dx + b\int (\omega_s(x)g(x))(\overline{\omega_s(x)h(x)})dx \\ &= a(f,h) + b(g,h).\end{aligned}$$

(2) Let $f, g \in L^2(\mathbb{R}^n, \omega_s)$. We wish to show that

$$(g,f) = \overline{(f,g)}.$$

This follows from noting that

$$\overline{(f,g)} = \overline{\int (\omega_s(x)f(x))(\overline{\omega_s(x)g(x)})dx} = \int (\overline{\omega_s(x)f(x)})(\omega_s(x)g(x))dx = (g,f).$$

(3) Finally, let $f \in L^2(\mathbb{R}^n, \omega_s)$. We wish to show that $(f,f) \geq 0$, with equality if and only if $f = 0$ almost everywhere. Notice that we have

$$(f,f) = \int (\omega_s(x)f(x))(\overline{\omega_s(x)f(x)})dx = \int |\omega_s(x)f(x)|^2dx = \int (1+|x|^2)^s|f(x)|^2dx \geq 0.$$

If the integral is equal to 0, we have that $(1+|x|^2)^s|f(x)|^2 = 0$ almost everywhere, which implies that $f(x) = 0$ almost everywhere. If $f = 0$ almost everywhere, we see clearly that the integral evaluates to 0. Hence, we have the desired result.

We then get an associated norm

$$\|f\|^2 = (f, f) = \int (1 + |x|^2)^s |f(x)|^2 dx.$$

We wish to show that $L^2(\mathbb{R}^n, \omega_s)$ is complete with respect to this norm. For $f \in L^2(\mathbb{R}^n, \omega_s)$, we have that $\|f\| < \infty$ so that $\omega_s f \in L^2(\mathbb{R}^n)$. Now, notice that $(f_n) \subset L^2(\mathbb{R}^n, \omega_s)$ Cauchy implies $(\omega_s f_n) \subset L^2(\mathbb{R}^n)$ is Cauchy, since

$$\|f_n - f_m\|^2 = (f_n - f_m, f_n - f_m) = \langle \omega_s(f_n - f_m), \omega_s(f_n - f_m)\rangle = \|\omega_s(f_n - f_m)\|_2^2.$$

Since $L^2(\mathbb{R}^n)$ is complete, we get that $\omega_s f_n \to g$ in $L^2(\mathbb{R})$. Defining $f = \omega_s^{-1} g$, we see that $\omega_s f_n \to \omega_s f$ in $L^2(\mathbb{R})$, but this then means that

$$\|f_n - f\|^2 = \int \omega_s^2(x) |f_n(x) - f(x)|^2 dx = \int |\omega_s(x) f_n(x) - g(x)|^2 dx \to 0.$$

Hence, $f_n \to f$ in $L^2(\mathbb{R}^n, \omega_s)$, so the space is complete. Thus, $L^2(\mathbb{R}^n, \omega_s)$ is a Hilbert space.

We need to show that if $\varphi \in \mathcal{S}$, then $\varphi \in L^2(\mathbb{R}^n, \omega_s)$. Notice that $\varphi \in \mathcal{S} \subset L^1(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ (**Folland Proposition 8.17**). We then need to show that $\omega_s \varphi \in L^2(\mathbb{R}^n)$; by the remarks earlier, this implies that $\varphi \in L^2(\mathbb{R}^n, \omega_s)$. That is, the goal is to show that

$$\|\varphi\| = \int (1 + |x|^2)^s |\varphi(x)|^2 dx < \infty.$$

Note that for $s \geq 0$, we have

$$(1 + |x|^2)^s \leq (1 + |x|)^{2s},$$

since

$$(1 + |x|)^{2s} = [(1 + |x|)^2]^s = (1 + 2|x| + |x|^2)^s \geq (1 + |x|^2)^s.$$

For $s < 0$, let $k = -s$ so that $k > 0$. Then we have

$$(1 + |x|^2)^{-k} \leq C(1 + |x|)^{-2k} \leftrightarrow (1 + |x|)^{2k} \leq C(1 + |x|^2)^k,$$

and this follows from **Folland page 181**, where we see that we can set $C = 4^k$. Combining these facts, we have

$$\int (1 + |x|^2)^s |\varphi(x)|^2 dx \leq C \int (1 + |x|)^{2s} |\varphi(x)|^2 dx$$

for some constant C which depends on s. Since $\varphi \in \mathcal{S}$, we have that $\|\varphi\|_{(N,\alpha)} < \infty$ for all (N, α). Thus, there's some constant C_N so that

$$|\varphi(x)| \leq C_N (1 + |x|)^{-N}$$

for all N natural numbers. Squaring, we have

$$|\varphi(x)|^2 \leq C_N^2 (1 + |x|)^{-2N}.$$

Thus, for some constant C' which depends on N, we have

$$\int (1 + |x|^2)^s |\varphi(x)|^2 dx \leq C' \int (1 + |x|)^{2(s-N)} dx$$

for all N. Choose N so that

$$N > n/2 + s,$$

then we get that this integral is finite (**Folland Corollary 2.52**), so that $\omega_s \varphi \in L^2$. Hence, $\varphi \in L^2(\mathbb{R}^n, \omega_s)$, and since the choice of $\varphi \in \mathcal{S}$ was arbitrary we have that $\mathcal{S} \subset L^2(\mathbb{R}^n, \omega_s)$.

We now wish to show that $f \in L^2(\mathbb{R}^n, \omega_s) \subset \mathcal{S}'$. In other words, the goal is to show that if $f \in L^2(\mathbb{R}^n, \omega_s)$, then it defines a tempered distribution. Letting $f \in L^2(\mathbb{R}^n, \omega_s)$, we wish to show that for all $\varphi \in \mathcal{S}$, we have

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx < \infty.$$

Notice that we have

$$\int f(x)\varphi(x)dx \leq \left| \int f(x)\varphi(x)dx \right| \leq \int |f(x)||\varphi(x)|dx$$

using the triangle inequality. Next, since φ is Schwarz, we get that

$$(1+|x|)^N |\varphi(x)| \leq C_N$$

for some constant C_N and all natural numbers N. Thus,

$$|\varphi(x)| \leq C_N (1+|x|)^{-N},$$

so for all natural numbers N, we have

$$\int |f(x)||\varphi(x)|dx \leq C_N \int |f(x)|(1+|x|)^{-N}dx = C_N \int |f(x)|(1+|x|)^{-N/2}(1+|x|)^{-N/2}dx.$$

Using Hölder's inequality, we have

$$\int |f(x)||\varphi(x)|dx \leq C_N \left(\int |f(x)|^2 (1+|x|)^{-N}dx \right)^{1/2} \left(\int (1+|x|)^{-N}dx \right)^{1/2}.$$

Now, we can choose $N = \max\{R, 2|s|\}$ so that $-N \leq 2s$ and $N \geq R$, where R is chosen sufficiently large so that $N > n$; that is, R is chosen so that **Folland Corollary 2.52** applies, and hence the integral on the right is finite. Since $-N \leq 2s$, we get that $(1+|x|)^{-N} \leq (1+|x|)^{2s}$. If $s \geq 0$, then note that

$$(1+|x|)^{2s} \leq 2^{2s}(1+|x|^2)^s,$$

since

$$(1+|x|)^2 \leq 2^2(1+|x|^2).$$

If $s < 0$, then we note that

$$(1+|x|)^{2s} \leq (1+|x|^2)^s,$$

since if $s = -k$ for some $k > 0$, we have

$$(1+|x|)^{-2k} \leq (1+|x|^2)^{-k} \leftrightarrow (1+|x|^2)^k \leq (1+|x|)^{2k},$$

and the inequality on the right follows from our observation above; that is, it follows from

$$(1+|x|)^{2k} = [(1+|x|)^2]^k = (1+2|x|+|x|^2)^k \geq (1+|x|^2)^k.$$

The integral on the left is now bounded above by

$$C \left(\int |f(x)|^2 (1+|x|^2)^s dx \right)^{1/2} < \infty$$

for some constant C depending on s. Since in either case it's finite (since $f \in L^2(\mathbb{R}^n, \omega_s)$), we have that

$$\int f(x)\varphi(x)dx < \infty,$$

so that f defines a tempered distribution. Thus, $L^2(\mathbb{R}^n, \omega_s) \subset \mathcal{S}'$.

Finally, we need to show that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \omega_s)$ (the fact that $\mathcal{S}$ will be dense follows from this). This follows from the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (**Folland Proposition 8.17**). For $\omega_s f \in L^2$ we can find $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that

$$\|\omega_s f - \varphi\|_2^2 < \epsilon.$$

Letting $\psi = \varphi\omega_s^{-1}$, we get that

$$\|\omega_s f - \omega_s\psi\|_2^2 < \epsilon,$$

and we still have that $\psi \in C_c^\infty(\mathbb{R}^n)$ (the product of smooth functions is smooth, and φ has compact support, so the product will be a smooth function with compact support). By what we've observed earlier, we see that

$$\|f - \psi\|^2 = \|\omega_s f - \omega_s\psi\|_2^2 < \epsilon.$$

Hence, for all $\epsilon > 0$, we can find $\psi \in C_c^\infty(\mathbb{R}^n)$ so that

$$\|f - \psi\| < \epsilon,$$

and thus $C_c^\infty(\mathbb{R}^n)$, and so $\mathcal{S}$, is dense in $L^2(\mathbb{R}^n, \omega_s)$. $\square$

Problem 45. Show that the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ generate unitary maps of $H^k(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n, \omega_k)$:

$$\mathcal{F}H^k(\mathbb{R}^n) = \mathcal{F}^{-1}H^k(\mathbb{R}^n) = L^2(\mathbb{R}^n, \omega_k).$$

Proof. Recall that

$$H^k = \{f \in \mathcal{S}' : \Lambda_k f \in L^2\},$$

and

$$\Lambda_k f = [(1+|m|^2)^{k/2}\widehat{f}]^\vee.$$

Recall that a map is unitary if it is surjective and preserves the inner product. Note that the inner product on H^k is defined by

$$(f,g)_{(k)} = \int (\Lambda^k f)(\overline{\Lambda^k g})dx = \int (1+|m|^2)^k \mathcal{F}(f)(m)\overline{\mathcal{F}(g)(m)}dm.$$

We'll first show that $H^k \subset \mathcal{F}^{-1}(L^2(\mathbb{R}^n, \omega_s))$, and then show that $L^2(\mathbb{R}^n, \omega_s) \subset \mathcal{F}(H^k)$. Doing so, we get that $L^2(\mathbb{R}^n, \omega_s) = \mathcal{F}(H^k)$ (i.e. the Fourier transform is surjective onto $L^2(\mathbb{R}^n, \omega_s)$). Let $g \in H^k$, then g is a tempered distribution such that

$$\Lambda_k g = \left[(1+|m|^2)^{k/2}\widehat{g}\right]^\vee \in L^2.$$

Using Plancherel and applying the Fourier transform, we have that this implies that

$$(1+|m|^2)^{k/2}\widehat{g} = \omega_k\widehat{g} \in L^2.$$

We have then that there is some $h \in L^2$ so that

$$\omega_k\widehat{g} = h \text{ almost everywhere.}$$

Thus, we have that, almost everywhere, $\widehat{g} = \omega_k^{-1}h$. Note that $\widehat{g}$ is both a tempered distribution *and* a function in L^2 (i.e. an equivalence class of a function in L^2), and as a function in L^2 it agrees with $\omega_k^{-1}h$.[6] So we can define a distribution $f = \widehat{g}$ such that it is equal to $\omega_k^{-1}h$ as a function (in L^2). Furthermore, we have that $f^\vee = g$ as distributions (since the Fourier transform is an isomorphism on tempered distributions), and we have that $\omega_k f \in L^2$.

If we have that f is locally integrable, then we win, since $f \in L^2(\mathbb{R}^n, \omega_k)$ such that $f^\vee = g$, and since the choice of g was arbitrary we have $H^k \subset \mathcal{F}^{-1}(L^2(\mathbb{R}^n, \omega_k))$. To prove that it's locally

[6] The remark here is just to be extra careful about our interpretation of what's going on, since we're juggling distributions and functions.

integrable, it suffices to prove that it's integrable on sets of the form $E_R := \{x : |x| < R\}$. Thus, we wish to show that

$$\int_{E_R} |f(x)| dx < \infty.$$

Using the fact that as functions we have that $f = \omega_k^{-1} h$, we can write this as

$$\int_{E_R} (1+|x|^2)^{-k/2} |h(x)| dx.$$

If $k \geq 0$, we have $(1+|x|^2)^{-k/2} \leq 1$, so we can bound this above by

$$\int_{E_R} |h(x)| dx < \infty,$$

since $h \in L^2$ implies that $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ by **Folland Proposition 6.12**. If $k < 0$, we have that over E_R, $(1+|x|^2)^{-k/2} \leq (1+R^2)^{-k/2} = C$ a constant. Thus, we see that we can bound the integral above by

$$C \int_{E_R} |h| < \infty.$$

Thus, for all $k \in \mathbb{R}$, we get that as a function f is locally integrable, so $f \in L^2(\mathbb{R}^n, \omega_k)$ and it is such that $f^\vee = g$. Hence, we have $H^k \subset \mathcal{F}^{-1}(L^2(\mathbb{R}^n, \omega_k))$, or $\mathcal{F}(H^k) \subset L^2(\mathbb{R}^n, \omega_k)$.

We now wish to show surjectivity. Let $g \in L^2(\mathbb{R}^n, \omega_k)$. We wish to find $f \in H^k(\mathbb{R}^n)$ so that $\mathcal{F}(f) = g$. Notice that $g \in L^2(\mathbb{R}^n, \omega_k)$ implies that

$$\omega_k g = (1+|x|^2)^{k/2} g \in L^2.$$

By Plancherel, we see that

$$[(1+|x|^2)^{k/2} g]^\vee \in L^2.$$

By the first problem, we know that $g \in L^2(\mathbb{R}^n, \omega_k)$ defines a tempered distribution, and hence we can use the Fourier transform of distributions to define $f = g^\vee$. We see then that

$$\Lambda_k f = [(1+|x|^2)^{k/2} \widehat{f}]^\vee = [(1+|x|^2)^{k/2} g]^\vee \in L^2,$$

and f is a tempered distribution, so $f \in H^k$. Furthermore, $\mathcal{F}(f) = g$, so we get that the Fourier transform is surjective and $\mathcal{F}(H^k) = L^2(\mathbb{R}^n, \omega_k)$.

We now need to show that it's unitary; in other words, we wish to show that

$$(\mathcal{F}(f), \mathcal{F}(g)) = (f, g)_{(k)}.$$

Note that

$$(f,g)_{(k)} = \int (1+|m|^2)^k \mathcal{F}(f)(m) \overline{\mathcal{F}(g)(m)} dm$$
$$= \int (\omega_k(x) \mathcal{F}(f)(x)) (\overline{\omega_k(x) \mathcal{F}(g)(x)}) dx = (\mathcal{F}(f), \mathcal{F}(g)).$$

Thus, the Fourier transform is unitary. So $\mathcal{F}$ is a unitary isomorphism from H^k to $L^2(\mathbb{R}^n, \omega_k)$.

Notice as well that the Fourier inverse is a unitary isomorphism. We have that

$$(f,g)_{(k)} = \int (1+|m|^2)^k \mathcal{F}(f)(m) \overline{\mathcal{F}(g)(m)} dm$$
$$= \int (1+|m|^2)^k \mathcal{F}^{(2)}(f)(m) \overline{\mathcal{F}^{(2)}(g)(m)} dm = \int (1+|m|^2)^k \mathcal{F}^{(3)}(f)(m) \overline{\mathcal{F}^{(3)}(g)(m)} dm$$
$$= \int (1+|m|^2)^k \mathcal{F}^{-1}(f)(m) \overline{\mathcal{F}^{-1}(g)(m)} dm = (\mathcal{F}^{-1}(f), \mathcal{F}^{-1}(g))$$

using Plancherel and the properties of the Fourier transform from the last homework. Hence, it still preserves the inner product. The argument that $\mathcal{F}^{-1}$ is surjective follows from the same kind of argument outlined in the last paragraph. Alternatively, we could use the fact that $\mathcal{F}H^k = L^2(\mathbb{R}^n, \omega_k)$ and use the periodicity of the Fourier transform (which follows from the periodicity of the Fourier transform for functions, proven on the last homework) to get $\mathcal{F}^{(3)}H^k = \mathcal{F}^{-1}H^k = \mathcal{F}^{(2)}L^2(\mathbb{R}^n, \omega_k)$. Even though this is the Fourier transform as a distribution, we have that $L^2(\mathbb{R}^n, \omega_k)$ consists of locally integrable functions, so distributions defined by functions. Using the fact that the Fourier transform as a distribution agrees with the Fourier transform as a function, we can interpret $\mathcal{F}^{(2)}$ to be reflection, which is a bijection of $L^2(\mathbb{R}^n, \omega_k)$. Thus, $\mathcal{F}^{-1}H^k = L^2(\mathbb{R}^n, \omega_k)$, and the Fourier inverse is surjective. Thus, we also have that the inverse Fourier transform is a unitary isomorphism. □

Remark. Note that this problem gives us that $\mathcal{S} \subset H^s$ is dense in the topology of H^s.

Problem 46. For $t < s$, show that H^s is a dense subspace of H^t (in the topology of H^t).

Proof. We first must show that $H^s \subset H^t$. Letting $f \in H^s$, we have that as a function it is such that $\Lambda_s f \in L^2$. Taking the Fourier transform, we have that

$$\mathcal{F}(\Lambda_s f) = (1 + |m|^2)^{s/2}\widehat{f} \in L^2,$$

so

$$\int (1 + |m|^2)^s |\widehat{f}(m)|^2 dm < \infty.$$

Since $t < s$, we see that $(1 + |m|^2)^t < (1 + |m|^2)^s$, so

$$\int (1 + |m|^2)^t |\widehat{f}(m)|^2 dm < \int (1 + |m|^2)^s |\widehat{f}(m)|^2 dm < \infty.$$

Thus, we have $\mathcal{F}(\Lambda_t f) \in L^2$, and by Plancherel this implies that $\Lambda_t f \in L^2$. So $f \in H^t$. Thus, $H^s \subset H^t$.

For density, we remark that $\mathcal{F}(H^s) = L^2(\mathbb{R}^n, \omega_s) \subset \mathcal{F}(H^t) = L^2(\mathbb{R}^n, \omega_t)$ by the prior problem. We showed that $\mathcal{S} \subset L^2(\mathbb{R}^n, \omega_s)$ in the first problem, so we have $\mathcal{S} \subset L^2(\mathbb{R}^n, \omega_s) \subset L^2(\mathbb{R}^n, \omega_t)$. Now, the Fourier transform maps Schwarz functions to themselves, and so since the Fourier transform is a unitary isomorphism we have that $\mathcal{S} \subset H^s \subset H^t$, and we have that $\mathcal{S}$ is dense in H^t in the topology of H^t. Thus, we get that H^s is dense in H^t in the topology of H^t, as desired. □

Problem 47. Prove the following:

(a) ∂^α is a bounded linear map from H^s to $H^{s-|\alpha|}$ for $s \in \mathbb{R}$, α a multi-index.
(b) Λ_t is a unitary isomorphism from H^s to H^{s-t} for $s, t \in \mathbb{R}$.

Proof. (a) Note that we're interpreting this as the *distributional derivative* as opposed to an honest derivative. We have that ∂^α is a linear map defined on $\mathcal{S}'$, so we restrict it to H^s to get a linear map. Furthermore, we see that the image will be

$$\partial^\alpha(H^s) = \{\partial^\alpha f : f \in \mathcal{S}', \Lambda_s f \in L^2\}.$$

We wish to show that this is $H^{s-|\alpha|}$. The derivative of a tempered distribution is still a tempered distribution, so we just need to show that

$$\Lambda_{s-|\alpha|}\partial^\alpha f \in L^2.$$

Note that, expanding this out and taking the Fourier transform, we have that this is equivalent to showing that

$$(1 + |m|^2)^{(s-|\alpha|)/2}(\partial^\alpha f)^\wedge \in L^2.$$

By the prior homework, we see that as distributions we have

$$(\partial^\alpha f)^\wedge = (2\pi i m)^\alpha \widehat{f}(m).$$

Hence, this reduces to showing that

$$(1+|m|^2)^{(s-|\alpha|)/2}(2\pi i)^{|\alpha|}m^\alpha \widehat{f} \in L^2.$$

Taking this integral, we have

$$\int (1+|m|^2)^{s-|\alpha|}(2\pi i)^{2|\alpha|}|m^\alpha|^2|\widehat{f}(m)|^2 dm.$$

In the inequality in the above Lemma, we have that there is some constant C so that

$$|m^\alpha| \leq C(1+|m|^2)^{|\alpha|/2}.$$

Substituting this in, we have that the above integral is bounded above by

$$C' \int (1+|m|^2)^{s-|\alpha|}(1+|m|^2)^{|\alpha|}|\widehat{f}(m)|^2 dm = C' \int (1+|m|^2)^{s}|\widehat{f}(m)|^2 dm < \infty$$

for some constant C', since $\Lambda_s f \in L^2$. By Plancherel, then, we see that this forces

$$\Lambda_{s-|\alpha|}\partial^\alpha f \in L^2,$$

so the image of ∂^α is $H^{s-|\alpha|}$. Finally, we wish to show that this map is bounded; in other words, for all f, we have that

$$\|\partial^\alpha f\|_{(s-|\alpha|)} \leq C\|f\|_{(s)}.$$

Notice that

$$\begin{aligned}
\|\partial^\alpha f\|_{(s-|\alpha|)} &= \left[\int |\widehat{\partial^\alpha f}(m)|^2(1+|m|^2)^{s-|\alpha|}dm\right]^{1/2} \\
&\leq (2\pi)^{|\alpha|}\left[\int |\widehat{f}(m)|^2(1+|m|^2)^{s}dm\right]^{1/2} \\
&= (2\pi)^{|\alpha|}\|f\|_{(s)}
\end{aligned}$$

by the arguments above (and implicitly using the properties of the Fourier transform on distributions, and then using that the Fourier transform of a distribution and a function which defines the distribution agree). Hence, the mapping is bounded.

(b) We now claim that Λ_t is a unitary isomorphism from H^s to H^{s-t}. First, we show that $\Lambda_t(H^s) \subset H^{s-t}$. Let $f \in H^s$, then the goal is to show that $\Lambda_t(f)$ is a tempered distribution such that $\Lambda_{s-t}\Lambda_t f \in L^2$. To see that $\Lambda_t f$ is a tempered distribution, we need to show that for all $\varphi \in \mathcal{S}$,

$$\int (\Lambda_t f)\varphi dm < \infty.$$

We have that for some $\psi, \kappa \in \mathcal{S}$ with $\overline{\kappa} = \varphi$, $\overline{\widehat{\overline{\kappa}}} = \psi$, this integral is equivalent to

$$\int [(1+|m|^2)^{t/2}\widehat{f}(m)]^\vee \varphi dm = \int [(1+|m|^2)^{t/2}\widehat{f}(m)]^\vee \overline{\kappa} dm = \int (1+|m|^2)^{t/2}\widehat{f}(m)\psi(m)dm$$

by Plancherel. Next, since we have $(1+|m|^2) \geq 1$, we get that $(1+|m|^2)^{t/2} < (1+|m|^2)^{s/2}$, since $t < s$. So we can bound this above by

$$\int (1+|m|^2)^{s/2}\widehat{f}(m)\psi(m)dm.$$

Finally, we remark that $\Lambda_s f \in L^2$, so Plancherel says $\mathcal{F}(\Lambda_s f) \in L^2$, and

$$\mathcal{F}(\Lambda_s f) = (1+|m|^2)^{s/2}\widehat{f} \in L^2.$$

Using Hölder and the triangle inequality, we have that

$$\int (1+|m|^2)^{s/2}\widehat{f}(m)\psi(m)dm \leq \int (1+|m|^2)^{s/2}|\widehat{f}(m)||\psi(m)|dm \leq \|\Lambda_s f\|_2\|\psi\|_2 < \infty.$$

Thus, $\Lambda_t f$ defines a tempered distribution.

Next, the goal is to show that $\Lambda_{s-t}\Lambda_t f \in L^2$. We remark that $\Lambda_s\Lambda_t = \Lambda_{s+t}$; to see this, notice that for any f, we have

$$\begin{aligned}\Lambda_s(\Lambda_t(f)) &= \Lambda_s\left(\left[(1+|m|^2)^{t/2}\widehat{f}\right]^{\vee}\right)\\ &= \left[(1+|m|^2)^{s/2}\left[\left((1+|m|^2)^{t/2}\widehat{f}\right)^{\vee}\right]^{\wedge}\right]^{\vee}\\ &= \left[(1+|m|^2)^{(s+t)/2}\widehat{f}\right]^{\vee} = \Lambda_{s+t}(f).\end{aligned}$$

Thus, we see that $\Lambda_{s-t}\Lambda_t f = \Lambda_s f \in L^2$ by assumption. Consequently, $\Lambda_t^{-1} = \Lambda_{-t}$, since $\Lambda_t\Lambda_{-t} = \Lambda_0$, and $\Lambda_0 f = [\widehat{f}]^{\vee} = f$ almost everywhere, so they are equal as distributions, and so Λ_0 is the identity. Likewise, $\Lambda_{-t}\Lambda_t = \Lambda_0$ which is the identity, so Λ_t is invertible with inverse Λ_{-t}. Thus, for $f \in H^s$, we have $\Lambda_{s-t}\Lambda_t f = \Lambda_s f \in L^2$, as desired. So $\Lambda_t(H^s) \subset H^{s-t}$.

To see that Λ_t is surjective, then, we simply note that for $g \in H^{s-t}$, we have that $\Lambda_{-t}g \in H^s$, since $\Lambda_s\Lambda_{-t}g = \Lambda_{s-t}g \in L^2$. Thus, $\Lambda_t(H^s) = H^{s-t}$.

Finally, we need to show that it preserves the inner product; that is,

$$(\Lambda_t f, \Lambda_t g)_{(s-t)} = (f,g)_{(s)}.$$

Notice that

$$(\Lambda_t f, \Lambda_t g)_{(s-t)} = \int (\Lambda_{s-t}\Lambda_t f)(\overline{\Lambda_{s-t}\Lambda_t g}) = \int (\Lambda_s f)(\overline{\Lambda_s g}) = (f,g)_{(s)}.$$

Thus, it is a unitary isomorphism.

□

Problem 48. If $k \in \mathbb{N}$, then H^k is the space of all $f \in L^2$ that posses strong L^2 derivatives $\partial^\alpha f$, as defined in **Folland Exercise 8.8** for $|\alpha| \leq k$ and these strong derivatives coincide with distribution derivatives.

Proof. Recall that in **Folland Exercise 8.8**, we defined the strong partial derivative of $f \in L^p$ as a function $h \in L^p$ such that

$$\lim_{y\to 0} \|y^{-1}(\tau_{-ye_j}f - f) - h\|_p = 0.$$

We generally denote h as $\partial_j f$ when it is unambiguous. For multi-indices, we have that $\partial^\alpha f$ as a strong L^p derivative corresponds to the iterative definition; i.e., if $\alpha = \beta + e_j$, then $\partial^\alpha f = \partial_j\partial^\beta f = \partial^\beta\partial_j f$ as strong L^p derivatives. Since it's defined iteratively, the approach is going to be induction.

The goal is to show $H^k = \widehat{H^k}$, where we define

$$\widehat{H^k} := \{f \in L^2 : f \text{ posses strong } L^2 \text{ derivatives for } |\alpha| \leq k\}.$$

The first step is to show that $H^k \subset \widehat{H^k}$, and that, when defined, the distributional derivative coincides with the strong L^p derivative. The strategy is to do something like **Folland Exercise 8.18 (a)** (which we did in recitation).

We note that $f \in H^k$, $k \geq 0$ implies that $f \in L^2$ as a function by the Lemma above (by the equivalence, we see that $|m^\alpha|\widehat{f}(m) \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$, which implies that $\widehat{f}(m) \in L^2(\mathbb{R}^n)$, and so Plancherel tells us that $f(m) \in L^2(\mathbb{R}^n)$; see also **Folland page 302**, the remark right after property (vi)). We now take the distributional derivative, labeling it as $\partial^\alpha f$. Taking the Fourier transform of this, we have that the last homework gives us

$$\widehat{\partial^\alpha f} = (2\pi i m)^\alpha \widehat{f}.$$

The Fourier transform of f as a function and as a distribution agree, so it suffices to now view f as a function. Notice that the Lemma above tells us that the quantity above is in L^2 for $|\alpha| \leq k$. We will first show that

$$\lim_{y\to 0} \|y^{-1}(\tau_{-ye_j} f - f)^\wedge - (2\pi i m_j)\widehat{f}\|_2 = 0,$$

where we view $f \in H^k$ as a function (instead of as the distribution it defines). We now recall that

$$(\tau_{-ye_j} f)^\wedge = e^{2\pi i m\cdot(ye_j)}\widehat{f}(m).$$

Thus, squaring this, we're examining

$$\lim_{y\to 0} \int \left(\left| \frac{e^{2\pi i m\cdot(e_j y)} - 1}{y} - 2\pi i m_j \right| |\widehat{f}(m)| \right)^2 dm.$$

Notice that

$$\left| \frac{e^{2\pi i m\cdot(e_j y)} - 1}{y} - 2\pi i m_j \right| \leq \left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} \right| + 2\pi |m_j|.$$

Notice as well (from the recitation earlier or from a MVT application, see **Rudin's Principles of Mathematical Analysis, Theorem 5.19** with a slight modification) we have

$$\left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} \right| \leq 2\pi |m_j|.$$

Note that we have $|m^\alpha|\widehat{f} \in L^2$ for all $|\alpha| \leq k$. Thus, for all $|\alpha| \leq k$, we see that we can apply the DCT to bring the limit inside, giving us that this goes to 0 as $y \to 0$. Hence, we have

$$\lim_{y\to 0} \|\mathcal{F}(y^{-1}(\tau_{-ye_j} f - f) - \partial_j f)\|_2^2 = \lim_{y\to 0} \|y^{-1}(\tau_{-ye_j} f - f) - \partial_j f\|_2^2 = 0,$$

so

$$\lim_{y\to 0} \|y^{-1}(\tau_{-ye_j} f - f) - \partial_j f\|_2 = 0.$$

Thus, the distributional derivative $\partial_j f$ is the strong L^p derivative.

For general multi-indices, we induct on $|\alpha|$. Thus, we assume that we can show it for $|\alpha| < k$, and we wish to show it for $|\alpha| + 1 \leq k$. Letting $\beta = \alpha + e_j$, $1 \leq j \leq n$, the goal is to show it holds for β assuming that it holds for α. We have the same set up as above then, with the minor change that we replace f with $\partial^\alpha f$; thus, we have

$$\lim_{y\to 0} \int \left(\left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} - (2\pi i)m_j \right| |\widehat{\partial^\alpha f}(m)| \right)^2 dm.$$

Again, we see that the inside will be bounded by

$$\left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} - 2\pi i m_j \right| \leq \left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} \right| + 2\pi |m_j|,$$

and we now have the bound given by

$$\left| \frac{e^{2\pi i m\cdot(ye_)j)} - 1}{y} \right| \leq 2\pi |m_j|,$$

so the same result applies as above to give us that the limit is 0. In other words, this establishes that

$$\lim_{y\to 0}\|y^{-1}(\tau_{-ye_j}\partial^\alpha f-\partial^\alpha f)-\partial^\beta f\|_2=0,$$

where $\beta=\alpha+e_j$, so that the distributional derivative $\partial^\beta f$ is the same as the strong L^2 derivative $\partial^\beta f$.

The goal now is to show that $\widehat{H^k}\subset H^k$. Assume $f\in\widehat{H^k}$. Again, we try using the strategy from **Folland Exercise 8.18 (a)**. Assume that the partial L^2 derivative of f, denoted $\partial_j f$, exists. Then the goal is to show that $m_j\widehat{f}\in L^2$ for $1\le j\le n$. This will give us that $f\in H^1$ by the Lemma above. Proceeding iteratively, we get that ∂^α implies that $|m^\alpha|\widehat{f}(m)\in L^2(\mathbb{R}^n)$ for $|\alpha|\le k$, and so we apply the Lemma above to get that $f\in H^k(\mathbb{R}^n)$.

Denoting the partial L^2 derivative as $\partial_j f$, we have

$$\lim_{y\to 0}\left\|y^{-1}(\tau_{-ye_j}f-f)-\partial_j f\right\|_2=0.$$

Applying Plancherel and taking the Fourier transform, we have that

$$\lim_{y\to 0}\|y^{-1}(\mathcal{F}(\tau_{-ye_j}f)-\mathcal{F}(f))-\mathcal{F}(\partial_j f)\|_2=0.$$

By the results above, we see that this implies that

$$\lim_{y\to 0}\left\|\mathcal{F}(f)\left(\frac{e^{2\pi i m\cdot(ye_j)}-1}{y}\right)-\mathcal{F}(\partial_j f)\right\|_2=0.$$

Using the reverse triangle inequality, we get that

$$\lim_{y\to 0}\left\|\mathcal{F}(f)\left(\frac{e^{2\pi i m\cdot(ye_j)}-1}{y}\right)\right\|_2=\|\mathcal{F}(\partial_j f)\|_2=\|\partial_j f\|_2<\infty.$$

Squaring everything, we see that this is the same as

$$\lim_{y\to 0}\int|\widehat{f}(m)|^2\left|\frac{e^{2\pi i m\cdot(ye_j)}-1}{y}\right|^2dm=\int|\partial_j f(x)|^2dx<\infty.$$

The goal now is to find a bound based on this.

Let $y=1/n$,

$$f_n(m)=\left|\widehat{f}(m)\left(n(e^{2\pi i m_j/n}-1)\right)\right|^2,$$

we note that

$$\liminf_{n\to\infty}f_n=\lim_{n\to\infty}f_n=4\pi^2|\widehat{f}(m)m_j|^2,$$

since this is a derivative. Recalling Fatou's Lemma, we see that

$$\int|\widehat{f}(m)|^2 4\pi^2|m_j|^2dm\le\liminf_{n\to\infty}\int f_n(m)dm=\lim_{n\to\infty}\int f_n(m)dm<\infty,$$

so we have that $\widehat{f}(m)m_j\in L^2$ for $1\le j\le n$. Thus, $f\in H^1$, and from prior we have that the distributional derivative $\partial_j f$ coincides with the strong L^2 derivative $\partial_j f$.

Now, assume that we have shown that it holds for multi-indices $|\alpha|=r<k$, we wish to show it for multi-indices of the form $|\beta|=r+1\le k$; that is, multi-indices of the form $\beta=\alpha+e_j$, where $1\le j\le n$. In other words, we have that $\widehat{f}(m)|m^\alpha|\in L^2$, and the goal is to show that $\widehat{f}(m)|m^\beta|\in L^2$. Notice that $\partial^\beta f$ as a strong L^2 derivative exists, and since it is defined iteratively we have that it is defined so that

$$\lim_{y\to 0}\|y^{-1}(\tau_{-ye_j}\partial^\alpha f-\partial^\alpha f)-\partial^\beta f\|_2=0.$$

We remark here that we have that it holds for multi-indices $|\alpha| \le r$, so $\widehat{f}(m)|m^\alpha| \in L^2$ for $|\alpha| \le r$, so $f \in H^r$. By the prior result, we have then that $\partial^\alpha f$ as a distributional derivative agrees with the strong L^2 derivative $\partial^\alpha f$, so we can unambiguously refer to it as either. This is important, since when we take the Fourier transform of $\partial^\alpha f$, it wouldn't make sense as an L^2 derivative (in the sense that we wish to use Fourier properties to rewrite it), but as a distributional derivative we can apply the problem from the last homework to get our desired result.

Applying Plancherel, we see that we have

$$\lim_{y\to 0} \|y^{-1}(\mathcal{F}(\tau_{-ye_j}\partial^\alpha f) - \mathcal{F}(\partial^\alpha f)) - \mathcal{F}(\partial^\beta f)\|_2,$$

and applying the reverse triangle inequality we have that

$$\lim_{y\to 0} \left\| \mathcal{F}(\partial^\alpha f) \left(\frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} \right) \right\|_2 = \left\| \mathcal{F}(\partial^\beta f) \right\|_2 = \|\partial^\beta f\|_2 < \infty.$$

Now, by the remark earlier, we can interpret $\partial^\alpha f$ as the distributional derivative, so we are able to rewrite it as

$$\mathcal{F}(\partial^\alpha f) = (2\pi i m)^\alpha \mathcal{F}(f),$$

and using the fact that the distributional Fourier transform agrees with the functional Fourier transform, we have that

$$\mathcal{F}(\partial^\alpha f) = (2\pi i m)^\alpha \widehat{f}(m).$$

Substituting this in, we get

$$(4\pi^2)^{|\alpha|} \lim_{y\to 0} \int |m^\alpha|^2 |\widehat{f}(m)|^2 \left| \frac{e^{2\pi i m\cdot(ye_j)} - 1}{y} \right|^2 dm = \|\partial^\beta f\|_2 < \infty.$$

Again, a Fatou argument applies, so letting $y = 1/n$ and

$$f_n(m) = \left| m^\alpha \widehat{f}(m) \left(n(e^{2\pi i m_j/n} - 1) \right) \right|^2,$$

we have

$$\liminf_{n\to\infty} f_n(m) = \lim_{n\to\infty} f_n(m) = 4\pi^2 |\widehat{f}(m) m^\alpha m_j|^2,$$

and so

$$4\pi^2 \int |m^\alpha|^2 |\widehat{f}(m)|^2 |m_j|^2 dm \le \liminf_{n\to\infty} \int f_n(m) dm = \lim_{n\to\infty} \int f_n(m) dm < \infty.$$

Thus, we have $|m^\alpha||m_j|\widehat{f}(m) \in L^2$, or in other words $|m^\beta|\widehat{f}(m) \in L^2$. Thus, the inductive argument applies, giving us that $|m^\alpha|\widehat{f}(m) \in L^2$ for $|\alpha| \le k$, so $f \in H^k$. By prior, we see that the distributional derivative coincides with the strong L^2 derivative. Thus, we have that $\widehat{H^k} = H^k$ and the derivatives coincide, as desired. $\square$

Remark. Thomas O'Hare was a collaborator.

Remark. A family member is going through health issues due to the pandemic, so I will not be super focused on the next few homework assignments. I'm sorry for the quality.

Problem 49. Prove that if $H^s \subset C_0^k$, then $s > k + n/2$. (Note this is a converse to the Sobolev embedding theorem.)

Proof. We follow Folland's hint. Assume that $H^s \subset C_0^k$. First, remark that s is such that $s \geq 0$, since for $s < 0$ we have that elements in H^s may not be functions, so this inclusion doesn't make sense (by the remark in **Folland page 302**). Note that the identity function is a linear map, clearly, and we have that H^s and C_0^k are Banach spaces with respect to their norms. Thus, we are in an appropriate setting to apply the Closed Graph theorem (**Folland Theorem 5.12**). The first step, then, is to show that the identity map is closed. Let $(f_n) \subset H^s$ be such that $f_n \to f$ in H^s, and let g be such that $\mathrm{Id}(f_n) = f_n \to g$ in C_0^k. The goal is to show that $\mathrm{Id}(f) = f = g$ in C_0^k. Notice $f_n \to f$ in H^s if

$$\|f_n - f\|_{(s)}^2 = \int (1 + |m|^2)^s |\widehat{f_n}(m) - \widehat{f}(m)|^2 dm \to 0.$$

fSince $s \geq 0$ by assumption, we get that

$$\|f_n - f\|_2^2 = \|\widehat{f_n} - \widehat{f}\|_2^2 = \int |\widehat{f_n}(m) - \widehat{f}(m)|^2 dm \leq \int (1 + |m|^2)^s |\widehat{f_n}(m) - \widehat{f}(m)|^2 dm,$$

so $f_n \to f$ in the L^2 norm, and hence almost everywhere. Since these are in C_0^k, we have that $f_n \to f$ pointwise. Notice that if $f_n \to g$ in C_0^k, then we have that for all α such that $|\alpha| \leq k$,

$$\|\partial^\alpha f_n - \partial^\alpha g\|_u \to 0.$$

In particular, we get that $f_n \to g$ pointwise, so we have that $f = g$ pointwise. Since $f = g$ as functions, we have that $\mathrm{Id}(f) = f = g$ in C_0^k. So the identity mapping is closed, hence continuous.

The goal now is to show that $\partial^\alpha \delta \in (H^s)^*$ for $|\alpha| \leq k$. From what we've shown above, this equivalently follows from showing that $\partial^\alpha \delta \in (C_0^k)^*$ for $|\alpha| \leq k$, and this follows since this is just taking the derivative and evaluating at 0. Since $H^s \subset C_0^k$ and it continuously embeds, we can define a functional by $f \mapsto \partial^\alpha f(0)$ for $|\alpha| \leq k$; this is a continuous linear functional on H^s, since we map f into its continuous function analogue using the identity function, and then we use the fact that $\partial^\alpha \delta$ is a linear functional on C_0^k for $|\alpha| \leq k$. By **Folland Proposition 9.16**, we have that $(H^s)^* \cong H^{-s}$. So unambiguously referring to this new linear functional by the same name, we have $\partial^\alpha \delta \in H^{-s}$ for $|\alpha| \leq k$. Note that this means that

$$\int (1 + |m|^2)^{-s} |\widehat{\partial^\alpha \delta}(m)|^2 dm < \infty.$$

Using properties of the Fourier transform, we recall that $\widehat{\partial^\alpha \delta} = (2\pi i m)^\alpha \widehat{\delta}$ and $\widehat{\delta} = 1$. So we have that

$$4\pi^2 \int (1 + |m|^2)^{-s} |m^\alpha| dm < \infty.$$

Notice that we can use the lemma from the last homework to bound below by

$$C \int (1 + |m|^2)^{-s+k} dm \leq 4\pi^2 \int (1 + |m|^2)^{-s} |m^\alpha| dm < \infty$$

for some constant C. Recall from **Folland Corollary 2.52** that the integral on the left is finite iff $-s+k<-n/2$, or in other words, $s>k+n/2$. Thus, if $H^s\subset C_0^k$, then we must have $s>k+n/2$. ☐

Problem 50. Suppose $s_0\le s_1$, $t_0\le t_1$, and $0\le\lambda\le 1$. Let

$$s_\lambda=(1-\lambda)s_0+\lambda s_1,\quad t_\lambda=(1-\lambda)t_0+\lambda t_1.$$

Show that if T is a bounded linear map from H^{s_0} to H^{t_0} whose restriction to H^{s_1} is bounded from H^{s_1} to H^{t_1}, then the restriction of T to H^{s_λ} is bounded from H^{s_λ} to H^{t_λ} for $0\le\lambda\le 1$.

Remark. I've found that this is a proposition in one of Folland's other books (**Introduction to Partial Differential Equations, Second Edition**), and that the hint follows the proof of the theorem pretty closely modulo actually showing that $F(z)$ is holomorphic (what he says seems kinda bogus and I don't see how you can fix it). If you ignore wanting to establish an actual upper bound based on λ, you can show that it's bounded without invoking the Three-Lines Lemma, which means avoiding the issue of holomorphicity and dealing with complex numbers. I already did a lot of the work to apply the Three-Lines lemma, so there's some unnecessary things here, but the end result should still be correct.

Proof. We first claim that T is bounded from H^s to H^t iff $\Lambda_t T\Lambda_{-s}$ is bounded on L^2. This follows from the fact that Λ_t, Λ_{-s} are unitary isomorphisms (see the last homework), and $\Lambda_{-s}:H^0\to H^s$, $\Lambda_t:H^t\to H^0$. Note that we can identify L^2 with H^0. To see this, by definition we have that

$$H^0=\{f\in\mathcal{S}':\Lambda_0 f\in L^2\},$$

and so remarking that

$$\Lambda_0 f=[(1+|m|^2)^0\widehat{f}]^\vee=f,$$

we see

$$H^0=\{f\in\mathcal{S}':f\in L^2\}.$$

We note that $f\in L^2$ defines a tempered distribution by Hölder's inequality, since for $\varphi\in\mathcal{S}$ we have

$$|\langle f,\varphi\rangle|\le\int|f||\varphi|\le\|f\|_2\|\varphi\|_2<\infty,$$

so in fact we have $L^2=H^0$. Hence, we really have $\Lambda_{-s}:L^2\to H^s$ and $\Lambda_t:H^t\to L^2$, so $T:H^s\to H^t$ is bounded iff $\Lambda_t T\Lambda_{-s}:L^2\to L^2$ is bounded.

Next, we observe that Λ_z is well-defined for all $z\in\mathbb{C}$, where we note that we define it for complex numbers in the analogous way;

$$\Lambda_z(f)=\left[(1+|m|^2)^{z/2}\widehat{f}\right]^\vee.$$

If $f=g$, then

$$\mathcal{F}(\Lambda_z(f))=(1+|m|^2)^{z/2}\widehat{f}=(1+|m|^2)^{z/2}\widehat{g}=\mathcal{F}(\Lambda_z(g)),$$

and so since the Fourier transform is an isometric isomorphism we have $\Lambda_z(f)=\Lambda_z(g)$. Furthermore, assume that $z=bi\in\mathbb{C}$, where $b\in\mathbb{R}$ (that is $\Re(z)=0$). We wish to show that $\Lambda_z:H^s\to H^s$ is unitary for all $s\in\mathbb{R}$. Note that

$$\Lambda_z(H^s)=\left\{\Lambda_z f:f\in\mathcal{S}',\Lambda_s f\in L^2\right\}.$$

We first show that $\Lambda_s(\Lambda_z f)\in L^2$, establishing that $\Lambda_z(H^s)\subset H^s$. Note that $(1+|m|^2)^{bi/2}$ is slowly increasing. We prove this by induction on $|\alpha|$. For the case $|\alpha|=0$, we have

$$|(1+|m|^2)^{bi/2}|=1\le(1+|n|)^0$$

by the above result. Now, assume that it is slowly increasing for $|\alpha| = k-1$. To deduce the result for $|\alpha| = k$, we let $\beta = \alpha + e_j$, $1 \le j \le n$, and we note that

$$\begin{aligned}
|\partial^\beta (1+|m|^2)^{bi/2}| &= |\partial^\alpha \partial_j (1+|m|^2)^{bi/2}| \\
&= \left| \left(\frac{bi}{2}\right) \partial^a (1+|m|^2)^{bi/2} \left(\frac{2m_j}{1+|m|^2}\right) \right| \\
&= \frac{b}{2} \left| \partial^a (1+|m|^2)^{bi/2} \left(\frac{2m_j}{1+|m|^2}\right) \right|.
\end{aligned}$$

We now invoke the general product rule as well as the induction hypothesis (since $|\delta| \le |\alpha| = k-1$) to get that this is equal to

$$\begin{aligned}
&\frac{b}{2} \left| \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!} \partial^\delta \left((1+|m|^2)^{bi/2}\right) \partial^\gamma \left(\frac{2m_j}{1+|m|^2}\right) \right| \\
\le{}& \frac{b}{2} \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!} \left| \partial^\delta \left((1+|m|^2)^{bi/2}\right) \right| \left| \partial^\gamma \left(\frac{2m_j}{1+|m|^2}\right) \right| \\
\le{}& \frac{b}{2} \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!} C_\delta (1+|m|)^{N(\delta)} \left| \partial^\gamma \left(\frac{2m_j}{1+|m|^2}\right) \right|.
\end{aligned}$$

Invoking product rule again for the derivative on the right, we have

$$2\left|\partial^\gamma (m_j)(1+|m|^2)^{-1}\right| \le 2 \sum_{\eta+\zeta=\gamma} \frac{\gamma!}{\eta!\zeta!} |\partial^\eta m_j| |\partial^\zeta (1+|m|^2)^{-1}|.$$

Notice that $\partial^\gamma m_j = 1$ if $\gamma = e_j$, m_j if $\gamma = 0$, and 0 otherwise. So there is a constant C' so that we can bound this above by

$$C' \left[|m_j| |\partial^\gamma (1+|m|^2)^{-1}| + |\partial^{\gamma - e_j} (1+|m|^2)^{-1}| \right].$$

We now claim that $(1+|m|^2)^{-1}$ is slowly increasing. If this is true, then noting that $|m_j| \le (1+|m_j|) \le (1+|m|)$, we can find constants C_γ and $N(\gamma)$ so that this is bounded above by $C_\gamma (1+|m|)^{N(\gamma)}$. Substituting this into the original sum, we have an upper bound of

$$|\partial^\beta (1+|m|^2)^{bi/2}| \le \frac{b}{2} \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\gamma!\delta!} (C_\delta (1+|m|)^{N(\delta)})(C_\gamma (1+|m|)^{N(\gamma)}).$$

Taking maximums and absorbing constants, we see that there are constants C_β and $N(\beta)$ so that

$$|\partial^\beta (1+|m|^2)^{bi/2}| \le C_\beta (1+|m|)^{N(\beta)}.$$

So, by induction, we get that this is slowly increasing.

In this, we assumed that $(1+|m|^2)^{-1}$ is slowly increasing. We now prove this, using induction again. Notice that

$$|(1+|m|^2)^{-1}| \le 1 = (1+|m|)^0,$$

so we have it holds for the base case $|\alpha| = 0$. Assume it holds for $|\alpha| = k-1$, then we wish to show it holds for $|\beta| = k$, where $\beta = \alpha + e_j$, $1 \le j \le n$. Again, we have

$$|\partial^\beta (1+|m|^2)^{-1}| = |\partial^\alpha \partial_j (1+|m|^2)^{-1}| = 2|\partial^\alpha (1+|m|^2)^{-2} m_j| = 2|\partial^\alpha (1+|m|^2)^{-1} [(1+|m|^2)^{-1} m_j].$$

We now invoke product rule on this to get that it is equal to

$$\begin{aligned}
&2\left|\sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}\left[\partial^{\gamma}(1+|m|^2)^{-1}\right]\left[\partial^{\delta}(1+|m|^2)^{-1}m_j\right]\right| \\
&\leq 2\sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}\left|\partial^{\gamma}(1+|m|^2)^{-1}\right|\left|\partial^{\delta}(1+|m|^2)^{-1}m_j\right|.
\end{aligned}$$

Note that $|\gamma| \leq |\alpha| = k-1$ and $|\delta| \leq |a| = k-1$ by assumption. Thus, we can use the induction hypothesis to find constants $C_\gamma, N(\gamma)$ so that

$$\left|\partial^{\gamma}(1+|m|^2)^{-1}\right| \leq C_\gamma(1+|m|)^{N(\gamma)}.$$

Next, we expand $\partial^{\delta}(1+|m|^2)^{-1}m_j$ with the product rule to get

$$\left|\partial^{\delta}(1+|m|^2)^{-1}m_j\right| = \left|\sum_{\eta+\zeta=\delta}\frac{\delta!}{\eta!\zeta!}(\partial^{\eta}(1+|m|^2)^{-1})(\partial^{\zeta}m_j)\right|.$$

Recall that $\partial^{\zeta}m_j = 1$ if $\zeta = e_j$, m_j if $\zeta = 0$, and 0 otherwise, so there is a constant C' so that this is bounded above by

$$C'\left[|\partial^{\delta-e_j}(1+|m|^2)^{-1}||m_j| + |\partial^{\delta}(1+|m|^2)^{-1}|\right].$$

Since $|\delta| \leq |\alpha| = k-1$, and we have the observation that $|m_j| \leq (1+|m_j|) \leq (1+|m|)$, we can use the induction hypothesis plus the observation to get that there are constants C_δ, $N(\delta)$ so that this is bounded above by $C_\delta(1+|m|)^{N(\delta)}$. Hence, absorbing constants and maximizing again, we get that there are constants C_β, $N(\beta)$ so that

$$|\partial^{\beta}(1+|m|^2)^{-1}| \leq C_\beta(1+|m|)^{N(\beta)}.$$

Thus, by induction we have $(1+|m|^2)^{-1}$ is slowly increasing. Consequently, we have $(1+|m|^2)^{bi/2}$ is slowly increasing as well.

Since this is slowly increasing, we note that Λ_z applied to a distribution is still a tempered distribution, and so applying Λ_s to this makes sense. Now, notice that

$$\Lambda_s(\Lambda_z f) = \Lambda_s\left[(1+|m|^2)^{bi/2}\widehat{f}\right]^{\vee} = \left[(1+|m|^2)^{(s+bi)/2}\widehat{f}\right]^{\vee}.$$

We remark here that this sort of trick also establishes that

$$\Lambda_s\Lambda_z = \Lambda_{s+z}.$$

By Plancherel, it suffices to show that

$$\int |(1+|m|^2)^{(s+bi)/2}|^2|\widehat{f}(m)|^2 dm < \infty.$$

Notice that this is equal to

$$\int (1+|m|^2)^{s}|\widehat{f}(m)|^2 dm < \infty$$

since the modulus of a positive real number to a complex power is the positive real number to the power of the real part. We know that the latter integral *is* finite, since $f \in H^s$, so we have that $\Lambda_z(H^s) \subset H^s$.

Next, for surjectivity, let $g \in H^s$. We wish to find $f \in H^s$ such that $\Lambda_z f = g$. Utilizing the fact that the Fourier transform is an isomorphism on tempered distributions, we get that there is a distribution f so that $\widehat{f} = (1+|m|^2)^{-bi/2}\widehat{g}$. Thus, we have

$$\Lambda_z f = \left[(1+|m|^2)^{bi/2}\widehat{f}\right]^{\vee} = \left[(1+|m|^2)^{bi/2}\left[(1+|m|^2)^{-bi/2}\widehat{g}\right]\right]^{\vee} = g.$$

Since the choice of g was arbitrary, we have that Λ_z is surjective.

Next, we wish to show that Λ_z preserves the inner product on H^s. To see this, note that by definition and the observation we made earlier on complex powers and the modulus, we have

$$\begin{aligned}
(\Lambda_z f, \Lambda_z g)_{(s)} &= \int (\Lambda_s \Lambda_z f)\overline{(\Lambda_s \Lambda_z g)} dm \\
&= \int \left|(1+|m|^2)^{(s+bi)/2}\right|^2 \widehat{f}(m)\overline{\widehat{g}}(m) dm \\
&= \int (1+|m|^2)^s \widehat{f}(m)\overline{\widehat{g}}(m) dm = (f,g)_{(s)}.
\end{aligned}$$

Thus, if $\Re(z) = 0$, then Λ_z is a unitary map on H^s, as desired.

Define $s(z) = (1-z)s_0 + zs_1$, $t(z) = (1-z)t_0 + zt_1$. For $z \in \mathbb{C}$ with $0 \le \Re(z) \le 1$ and $\varphi, \psi \in \mathcal{S}$, we define

$$F(z) := \int \left[\Lambda_{t(z)} T \Lambda_{-s(z)} \varphi\right] \overline{\psi}.$$

We now diverge from the hint, with the goal only being to show that the function is bounded. Note apriori we have

$$\|Tf\|_{(t_0)} \le C_0 \|f\|_{(s_0)}, \quad \|Tf\|_{(t_1)} \le C_1 \|f\|_{(s_1)}.$$

Notice that for $\Re(z) = 0$, we get (using Plancherel, Hölder, and definitions/observations)

$$\begin{aligned}
|F(z)| \le \int |\Lambda_{-t(z)} T \Lambda_{-s(z)} \varphi||\psi| &\le \left(\int |\Lambda_{-t(z)} T \Lambda_{-s(z)} \varphi|^2\right)^{1/2} \left(\int |\psi|^2\right)^{1/2} \\
&= \left(\int \left|\left[(1+|m|^2)^{t(z)/2} \mathcal{F}(T\Lambda_{-s(z)}\varphi)\right]^{\vee}\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int \left|(1+|m|^2)^{t(z)/2} \mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int (1+|m|^2)^{t_0} \left|\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \|T\Lambda_{-s(z)}\varphi\|_{(t_0)} \|\psi\|_2 \le C_0 \|\Lambda_{-s(z)}\varphi\|_{(s_0)} \|\psi\|_2 \\
&= C_0 \left(\int (1+|m|^2)^{s_0} |\mathcal{F}(\Lambda_{-s(z)}\varphi)|^2\right)^{1/2} \|\psi\|_2 \\
&= C_0 \left(\int (1+|m|^2)^{s_0} \left|(1+|m|^2)^{-s(z)/2} \widehat{\varphi}\right|^2\right)^{1/2} \|\psi\|_2 \\
&= C_0 \left(\int |\widehat{\varphi}|^2\right)^{1/2} \|\psi\|_2 = C_0 \|\varphi\|_2 \|\psi\|_2.
\end{aligned}$$

Similarly, for $\Re(z) = 1$, we get that (using Plancherel, Hölder, and definitions/observations)

$$\begin{aligned}
|F(z)| \le \int |\Lambda_{-t(z)} T \Lambda_{-s(z)}\varphi||\psi| &\le \left(\int |\Lambda_{-t(z)} T \Lambda_{-s(z)}\varphi|^2\right)^{1/2} \left(\int |\psi|^2\right)^{1/2} \\
&= \left(\int \left|\left[(1+|m|^2)^{t(z)/2}\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right]^{\vee}\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int \left|(1+|m|^2)^{t(z)/2}\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int (1+|m|^2)^{t_1}\left|\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \|T\Lambda_{-s(z)}\varphi\|_{(t_1)}\|\psi\|_2 \le C_1\|\Lambda_{-s(z)}\varphi\|_{(s_1)}\|\psi\|_2 \\
&= C_1\left(\int (1+|m|^2)^{s_1}|\mathcal{F}(\Lambda_{-s(z)}\varphi)|^2\right)^{1/2} \|\psi\|_2 \\
&= C_1\left(\int (1+|m|^2)^{s_1}\left|(1+|m|^2)^{-s(z)/2}\widehat{\varphi}\right|^2\right)^{1/2} \|\psi\|_2 \\
&= C_1\left(\int |\widehat{\varphi}|^2\right)^{1/2} \|\psi\|_2 = C_1\|\varphi\|_2\|\psi\|_2.
\end{aligned}$$

For $0 < \Re(z) < 1$, we get that

$$\begin{aligned}
|F(z)| \le \int |\Lambda_{-t(z)} T \Lambda_{-s(z)}\varphi||\psi| &\le \left(\int |\Lambda_{-t(z)} T \Lambda_{-s(z)}\varphi|^2\right)^{1/2} \left(\int |\psi|^2\right)^{1/2} \\
&= \left(\int \left|\left[(1+|m|^2)^{t(z)/2}\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right]^{\vee}\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int \left|(1+|m|^2)^{t(z)/2}\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&= \left(\int (1+|m|^2)^{\Re(t(z))}\left|\mathcal{F}(T\Lambda_{-s(z)}\varphi)\right|^2\right)^{1/2} \|\psi\|_2 \\
&\le \|T\Lambda_{-s(z)}\varphi\|_{(t_1)}\|\psi\|_2 \le C_1\|\Lambda_{-s(z)}\varphi\|_{(s_1)}\|\psi\|_2 \\
&= C_1\left(\int (1+|m|^2)^{s_1}|\mathcal{F}(\Lambda_{-s(z)}\varphi)|^2\right)^{1/2} \|\psi\|_2 \\
&= C_1\left(\int (1+|m|^2)^{s_1}\left|(1+|m|^2)^{-s(z)/2}\widehat{\varphi}\right|^2\right)^{1/2} \|\psi\|_2 \\
&\le C_1\left(\int (1+|m|^2)^{s_1-s_0}\,|\widehat{\varphi}|^2\right)^{1/2} \|\psi\|_2 = C_1\|\varphi\|_{(s_1-s_0)}\|\psi\|_2 < \infty.
\end{aligned}$$

Remark. If you can show that $F(z)$ is holomorphic, we have at this point the Three-Lines lemma would apply and you could actually get an upper bound based on λ for $0 \le \lambda \le 1$. Thomas suggested something like Morea's theorem but I don't see how to apply that here.

Thus, the function is bounded for $0 \le \Re(z) \le 1$ by taking the maximum between these three constants. Using duality (**Folland Theorem 6.14**), $\Lambda_{t(z)}T\Lambda_{-s(z)}\varphi$ is a bounded map on L^2. By the remark earlier, we have that T is a bounded map from H^{s_λ} to H^{t_λ} for $0 \le \lambda \le 1$. □

For $s \in \mathbb{R}$, the periodic Sobolev spaces are defined as

$$H^s(\mathbb{T}^n) := \left\{ f \in \mathcal{D}'(\mathbb{T}^n) : \sum (1+|m|^2)^s |\widehat{f}(m)|^2 < \infty \right\}.$$

We define the periodic Sobolev norm as

$$\|f\|_{H^t(\mathbb{T}^n)} := \left(\sum (1+|m|^2)^s |\widehat{f}(m)|^2 \right)^{1/2}.$$

Problem 51. For $s, t \in \mathbb{R}$ and $s \geq t$, show that the space $H^s(\mathbb{T}^n)$ is continuously and densely embedded in $H^t(\mathbb{T}^n)$, and

$$\|f\|_{H^t(\mathbb{T}^n)} \leq \|f\|_{H^s(\mathbb{T}^n)} \text{ for all } f \in H^s(\mathbb{T}^n).$$

Proof. Notice that

$$\|f\|^2_{H^t(\mathbb{T}^n)} = \sum (1+|m|^2)^t |\widehat{f}(m)|^2,$$

and since $t \leq s$, we get that $(1+|m|^2)^t \leq (1+|m|^2)^s$, hence

$$\|f\|^2_{H^t(\mathbb{T}^n)} = \sum (1+|m|^2)^t |\widehat{f}(m)|^2 \leq \sum (1+|m|^2)^s |\widehat{f}(m)|^2 = \|f\|^2_{H^s(\mathbb{T}^n)}.$$

In other words,

$$\|f\|_{H^t(\mathbb{T}^n)} \leq \|f\|_{H^s(\mathbb{T}^n)}.$$

Next, we wish to show that the embedding $\mathrm{Id} : H^s(\mathbb{T}^n) \hookrightarrow H^t(\mathbb{T}^n)$ has dense image. Let $f \in H^t(\mathbb{T}^n)$ be fixed. Define f_M a distribution so that

$$\widehat{f_M}(m) = \begin{cases} \widehat{f}(m) \text{ if } |m| < |M| \\ 0 \text{ if } |m| \geq M. \end{cases}$$

We first remark that this indeed defines a distribution. Let

$$F(x) := \sum_{m \in \mathbb{Z}^n} \widehat{f_M}(m) E_m(x),$$

where $E_m(x) = e^{2\pi i m \cdot x}$. Since $f \in H^t(\mathbb{T}^n)$, we note that

$$\sum (1+|m|^2)^t \widehat{f}(m) < \infty,$$

which forces $\widehat{f}(m) < \infty$ for all $m \in \mathbb{Z}^n$. Since F is a finite sum of things, we have that it defines a function in $L^2(\mathbb{T}^n)$, so a distribution. Furthermore, this distribution is such that $\widehat{F}(m) = \widehat{f_M}(m)$. We label the distribution f_M to be the $L^2(\mathbb{T}^n)$ function F, so without ambiguity we can just refer to it as f_M. Notice that $f_M \in H^s(\mathbb{T}^n)$, since

$$\|f_M\|_{H^s(\mathbb{T}^n)} = \sum_{m \in \mathbb{Z}^n} (1+|m|^2)^s |\widehat{f}(m)|^2 = \sum_{\substack{m \in \mathbb{Z}^n \\ |m| < M}} (1+|m|^2)^s |\widehat{f}(m)|^2 < \infty,$$

which we note is finite. Notice that $f_M \to f$ in $H^t(\mathbb{T}^n)$, since

$$\|f_M - f\|_{H^t(\mathbb{T}^n)} = \sum_{m \in \mathbb{Z}^n} (1+|m|^2)^t |\widehat{f_M}(m) - \widehat{f}(m)|^2 = \sum_{\substack{m \in \mathbb{Z}^n \\ |m| \geq M}} (1+|m|^2)^t |\widehat{f}(m)|^2 \to 0,$$

since the series is convergent. Thus, $H^s(\mathbb{T}^n) \subset H^t(\mathbb{T}^n)$ is dense. □

Problem 52. Show that the dual $(H^s(\mathbb{T}^n))^*$ of $H^s(\mathbb{T}^n)$ is isometrically isomorphic to $H^{-s}(\mathbb{T}^n)$.

Proof. We proceed in two steps. First, let $f \in H^{-s}(\mathbb{T}^n)$. We wish to show that $\varphi \mapsto \langle f, \varphi \rangle$ is a continuous linear functional on $H^s(\mathbb{T}^n)$, where $\varphi \in H^s(\mathbb{T}^n)$. Notice that we have

$$\langle f, \varphi \rangle = \langle f^\vee, \widehat{\varphi} \rangle = \sum_{m \in \mathbb{Z}^n} f^\vee(m) \widehat{\varphi}(m).$$

Hence,

$$|\langle f, \varphi \rangle| \leq \sum_{m \in \mathbb{Z}^n} (1 + |m|^2)^{-s/2} |f^\vee(m)| (1 + |m|^2)^{s/2} |\widehat{\varphi}(m)|,$$

and applying the Schwarz inequality we have

$$|\langle f, \varphi \rangle| \leq \|f\|_{H^{-s}(\mathbb{T}^n)} \|\varphi\|_{H^s(\mathbb{T}^n)}.$$

So the linear functional is bounded, with norm at most $\|f\|_{(-s)} < \infty$. We note that the linear functional has norm equal to $\|f\|_{(-s)}$, since choosing g to be a distribution so that $\widehat{g}(m) = \|f\|_{(-s)}^{-1} (1 + |m|^2)^{-s} \overline{\widehat{f}(m)}$ (the case where $\|f\|_{(-s)} = 0$ follow trivially since the norm of the linear map needs to be greater than or equal to 0, so we can assume $\|f\|_{(-s)} \neq 0$), we have

$$\langle f, g \rangle = \frac{1}{\|f\|_{(-s)}} \sum_{m \in \mathbb{Z}^n} |\widehat{f}(m)|^2 (1 + |m|^2)^s = \|f\|_{(-s)}.$$

Hence, it is continuous, and so it is in the dual of $H^s(\mathbb{T}^n)$, with norm equal to its H^{-s} norm.

Next, let $G \in (H^s(\mathbb{T}^n))^*$. The goal is to show that it agrees with some $f \in H^{-s}$. Notice that, via Fourier series, we can identify $H^s(\mathbb{T}^n)$ with

$$l^2_s = \{(a_m)_{m \in \mathbb{Z}^n} : a_m \in \mathbb{C} \text{ for all } m \in \mathbb{Z}^n, \sum (1 + |m|^2)^s |a_m|^2 < \infty\},$$

where $s \in \mathbb{R}$. In other words, l^2_s is the equivalent of $L^2(\mathbb{R}^n, \omega_s)$ for little l^2. As a result, we equip it with the inner product

$$((a_m), (b_m))_{(2,s)} = \sum (1 + |m|^2)^s a_m \overline{b_m},$$

and hence it has associated norm

$$\|(a_m)\|_{(2,s)} = \|a_m\|_{(2,s)} = \left(\sum (1 + |m|^2)^s |a_m|^2 \right)^{1/2}.$$

We note that l^2_s is a Banach space (and hence a Hilbert space) with respect to this norm. To see that it is a vector space, notice that $(a_i), (b_i) \in l^2_s$ tells us that

$$\begin{aligned}
\|(a_m + b_m)\|^2_{(2,s)} = \sum (1 + |m|^2)^s |a_m + b_m|^2 &\leq \sum (1 + |m|^2)^s (|a_m| + |b_m|)^2 \\
&= \sum (1 + |m|^2)^s (|a_m|^2 + 2|a_m||b_m| + |b_m|^2) \\
= \|a_m\|^2_{(2,s)} + 2 \sum [(1 + |m|^2)^{s/2} |a_m|] &[(1 + |m|^2)^{s/2} |b_m|] + \|b_m\|^2_{(2,s)}.
\end{aligned}$$

We now recall Cauchy's inequality for general l^2, giving us that this is bounded above by

$$\begin{aligned}
\|a_m\|^2_{(2,s)} + 2 \left(\sum (1 + |m|^2)^s |a_m|^2 \right)^{1/2} &\left(\sum (1 + |m|^2)^s |b_m|^2 \right)^{1/2}] + \|b_m\|^2_{(2,s)} \\
&= \|a_m\|^2_{(2,s)} + 2 \|a_m\|_{(2,s)} \|b_m\|_{(2,s)} + \|b_m\|^2_{(2,s)} \\
&= (\|a_m\|_{(2,s)} + \|b_m\|_{(2,s)})^2,
\end{aligned}$$

so

$$\|a_m + b_m\|_{(2,s)} \leq \|a_m\|_{(2,s)} + \|b_m\|_{(2,s)} < \infty,$$

hence $(a_m + b_m) \subset l^2_s$. For scalars c, we have

$$\|ca_m\|_{(2,s)} = \left(\sum (1+|m|^2)^s |ca_m|^2\right)^{1/2} = |c|\|a_m\|_{(2,s)} < \infty.$$

Thus, $(ca_m) \subset l^2_s$ as well. Therefore l^2_s a vector space.

We've shown the triangle inequality and the scalar property for norms, so all that remains is $\|a_m\|_{(2,s)} = 0$ iff $a_m = 0$ for all $m \in \mathbb{Z}^n$. The converse is clear, so we show the implication. If $\|a_m\|_{(2,s)} = 0$, then

$$0 \leq (1+|m|^2)^s |a_m|^2 \leq \sum (1+|m|^2)^s |a_m|^2 = 0,$$

and hence, we have $(1+|m|^2)^s|a_m|^2 = 0$ for all $m \in \mathbb{Z}^n$, which is only possible if $a_m = 0$ for all $m \in \mathbb{Z}^n$.

The last thing to check is that it is complete with respect to its norm. Let $(a^n_m) \subset l^2_s$ be a Cauchy sequence; in other words, for all $\epsilon > 0$, there exists an N so that for $n, r \geq N$, we have

$$\|a^n_m - a^r_m\|_{(2,s)} < \epsilon.$$

Fixing m, we have that a^n_m is therefore a Cauchy sequence in the underlying field (say $\mathbb{C}$), since

$$(1+|m|^2)^s |a^n_m - a^r_m|^2 \leq \|a^n_m - a^r_m\|_{(2,s)}.$$

Thus, $a^n_m \to a_m$ for fixed m. The goal is to show that $a^n_m \to a_m$ in the l^2_s norm. Notice first that $(a_m) \subset l^2_s$, since

$$\sup_{n\in\mathbb{N}} \|a^n_m\|_{(2,s)} \leq C,$$

and hence for all n we have

$$\sum_{|m|\leq M} (1+|m|^2)^s |a^n_m|^2 \leq C.$$

Let $M, n \to \infty$ to get

$$\|a_m\|_{(2,s)} \leq C.$$

Now, we show that $\|a^n_m - a_m\|_{(2,s)} \to 0$. We do the same trick, namely we have that for all $\epsilon > 0$, we can find N so that for $n, r \geq N$,

$$\sum_{|m|\leq M} (1+|m|^2)^s |a^n_m - a^r_m|^2 \leq \|a^n_m - a^r_m\|_{(2,s)} < \epsilon.$$

Letting $M, r \to \infty$, this tells us that for all $\epsilon > 0$, we can find an N so that for $n \geq N$,

$$\|a^n_m - a_m\|_{(2,s)} < \epsilon.$$

Thus, $a^n_m \to a_m$ in l^2_s, and so l^2_s is a Banach space with respect to the norm.

We now show that it is a Hilbert space with respect to the prescribed inner product. Notice that

$$(ca_m + kb_m, d_m)_{(2,s)} = \sum (1+|m|^2)^s (ca_m + kb_m)\overline{d_m} = c(a_m, d_m)_{(2,s)} + k(b_m, d_m)_{(2,s)},$$

$$\overline{(a_m, b_m)_{(2,s)}} = \overline{\sum (1+|m|^2)^s a_m \overline{b_m}} = (b_m, a_m)_{(2,s)},$$

$$(a_m, a_m) = \sum (1+|m|^2)^s |a_m|^2 \geq 0.$$

Hence, we have that it is an inner product, so l^2_s is a Hilbert space. Since it is a Hilbert space, we have that it is self dual (**Folland Theorem 5.25**), so $(l^2_s)^* = l^2_s$.

Consider $\mathcal{G} : H^s(\mathbb{T}^n) \to l^2_s$ defined by $\mathcal{G}(f) = (\widehat{f}(m))_{m\in\mathbb{Z}^n}$; in other words, the Fourier transform as distributions. We've already seen that the Fourier transform is defined on $\mathcal{D}'(\mathbb{T}^n)$, since this is the space of distributions with compact support. It suffices to show that this is a unitary isomorphism of spaces, then. Notice that, by construction, $\mathcal{G}(H^s(\mathbb{T}^n)) \subset l^2_s$. We then need to prove surjectivity. By the discussion on **Folland page 297-298** (for details, check Folland's **Fourier Analysis and**

its Applications, specifically **Theorem 9.6** and the discussion before), we have that the Fourier transform on periodic distributions is an isomorphism. Take a sequence $(a_m)_{m\in\mathbb{Z}^n} \subset l^2_s$. Define a distribution $F \in \mathcal{D}'(\mathbb{T}^n)$ to be such that $\widehat{F}(m) = a_m$; by the discussion in Folland, we know such a distribution exists. We then check that this distribution is in $H^s(\mathbb{T}^n)$. If it is, we have that $\mathcal{G}(F) = (a_m)_{m\in\mathbb{Z}^n}$, proving surjectivity. The fact that it is in $H^s(\mathbb{T}^n)$ follows by construction, though, since

$$\sum (1+|m|^2)^s |\widehat{F}(m)|^2 = \sum (1+|m|^2)^s |a_m|^2 < \infty$$

since $(a_m) \subset l^2_s$. Thus, the mapping is surjective.

To see that it preserves the inner product, notice that

$$(\mathcal{G}(F), \mathcal{G}(G))_{(2,s)} = \sum (1+|m|^2)^s \widehat{F}(m)\overline{\widehat{G}(m)} = (F,G)_{(s)}.$$

Thus, $\mathcal{G}$ is a unitary isomorphism between Hilbert spaces.

Since $G \circ \mathcal{G}^{-1} : l^2_s \to \mathbb{C}$ is a bounded operator on l^2_s, using the fact that G is bounded and $\mathcal{G}$ is a unitary isomorphism, we get that duality tells us there is a sequence $(b_m)_{m\in\mathbb{Z}^n} \in l^2_s$ so that

$$G \circ \mathcal{G}^{-1}(a_m) = \sum_{m\in\mathbb{Z}^n} (1+|m|^2)^s a_m b_m.$$

Define g to be the distribution in $\mathcal{D}'(\mathbb{T}^n)$ so that $g^\vee(m) = (1+|m|^2)^s b_m$. In other words, g is the distribution defined by

$$\langle g, \varphi\rangle = \langle g^\vee, \widehat{\varphi}\rangle = \sum (1+|m|^2)^s \widehat{\varphi}(m) b_m.$$

Notice that

$$\begin{aligned} G(\varphi) = G \circ \mathcal{G}^{-1} \circ \mathcal{G}(\varphi) = G \circ \mathcal{G}^{-1}((\widehat{\varphi}(m))) &= \sum (1+|m|^2)^s \widehat{\varphi}(m) b_m \\ &= \sum_{m\in\mathbb{Z}^n} g^\vee(m)\widehat{\varphi}(m) = \langle g^\vee, \varphi^\wedge\rangle = \langle g, \varphi\rangle. \end{aligned}$$

Thus, these define the same distribution. Furthermore, $g \in H^{-s}(\mathbb{T}^n)$, since

$$\|g\|^2_{(-s)} = \sum_{m\in\mathbb{Z}^n} |\widehat{g}(m)|^2 (1+|m|^2)^{-s} = \sum_{m\in\mathbb{Z}^n} |(1+|m|^2)^s b_m|^2 (1+|m|^2)^{-s} = \sum_{m\in\mathbb{Z}^n} (1+|m|^2)^s |b_m|^2 < \infty.$$

So we have that every element in $(H^{(s)})^*$ can be identified (uniquely) with a distribution in $H^{(-s)}$. So the map from $H^{(-s)}$ to $(H^{(s)})^*$ given by $f \mapsto \langle f, \cdot\rangle$ is bijective, isometric, and linear (since the inverse Fourier transform is linear and integration is linear), so its an isometric isomorphism. $\square$

Problem 53. Suppose $s > k + n/2$. Show that $\mathrm{Id} : H^s(\mathbb{T}^n) \hookrightarrow C^k(\mathbb{T}^n)$.

Proof. Let $f \in H^s$. Then we have that

$$\sum_{m\in\mathbb{Z}^n} (1+|m|^2)^s |\widehat{f}(m)|^2 < \infty.$$

From prior discussions, we have that

$$F = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m) e^{2\pi i m\cdot x}$$

defines a distribution. We note that, as distributions, $f = F$. This follows, since

$$\langle F, \varphi\rangle = \langle \widehat{F}, \varphi^\vee\rangle = \int \widehat{F}(m)\varphi^\vee(m)dm = \int \widehat{f}(m)\varphi^\vee(m)dm = \langle \widehat{f}, \varphi^\vee\rangle = \langle f, \varphi\rangle.$$

We see that

$$|\partial^\alpha f(x)| = (2\pi)^{|\alpha|}\left|m^\alpha \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x}\right| \leq (2\pi)^{|\alpha|}|m^\alpha| \sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)|$$
$$\leq C \sum_{m\in\mathbb{Z}^n} (1+|m|^2)^{k/2}|\widehat{f}(m)|,$$

where C is some constant (which we found from the Lemma from the prior homework). Using the Schwarz inequality, we have that this is bounded above by

$$C\left(\sum_{m\in\mathbb{Z}^n}(1+|m|^2)^s|\widehat{f}(m)|^2\right)^{1/2}\left(\sum_{m\in\mathbb{Z}^n}(1+|m|^2)^{k-s}\right)^{1/2} = C\|f\|_{(s)}\left(\sum_{m\in\mathbb{Z}^n}(1+|m|^2)^{k-s}\right)^{1/2}.$$

Now, since $s > k + n/2$, we have that $2(k-s) < -n$; in other words, we have that the series on the right converges (by a generalized integral test, see this for an example). Thus, the Sobolev embedding theorem holds. $\square$

Problem 54. Let $1 \leq p < q < \infty$ and (X, μ) be a finite measure space.

(a) What is the domain of the identity map from $L^p(\mu)$ to $L^q(\mu)$?
(b) Is the identity map from part (a) continuous on its domain? Prove or disprove.

Remark. I'm going off of the definition that the domain of a map is the space of elements where the function can be uniquely defined. That is, if $T : X \to Y$ is a map, we define the domain of T to be the maximal subset $D \subset X$ so that, for all $x \in D$, $T(x) \in Y$ is defined uniquely. I tried looking in Folland for a definition of domain but the one there doesn't line up with what the problem is asking for (Grafakos seemed to implicitly be using this one, but never formally stated what he meant by domain).

Proof. (a) Let $T : L^p(\mu) \to L^q(\mu)$ be the map such that $T(f) = f$ (i.e. the identity map). The domain, then, is the collection of maps $f \in L^p(\mu)$ such that $T(f) = f \in L^q(\mu)$; that is, the collection of maps which are in $L^p(\mu)$ and which are in $L^q(\mu)$. Thus, the domain of the map is $L^p(\mu) \cap L^q(\mu)$. Recall from **Folland Proposition 6.12** that, on finite measure spaces, we have that $L^q(\mu) \subset L^p(\mu)$; hence, the domain will be $L^p(\mu) \cap L^q(\mu) = L^q(\mu)$, equipped with the p norm. To prevent confusion, let's denote the domain with

$$\mathcal{D} = \{f \in L^p(\mu) : f \in L^q(\mu)\};$$

then $\mathcal{D} = L^q(\mu)$ as sets and we have that $\mathcal{D}$ is equipped with the p norm.

(b) The question is whether $T : \mathcal{D} \to L^q(\mu)$ is continuous on its domain $\mathcal{D}$. Inspired by **Homework 1, Problem 4**, we first show that T is a closed map. Assume that $T(f_n) = f_n \to g$ in $L^q(\mu)$, $f_n \to f$ in $\mathcal{D}$. The goal, then, is to show that $f = g$ as functions in $L^q(\mu)$. Since $f_n \to f$ in $\mathcal{D}$, which is equipped with the p norm, we have that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \frac{1}{\epsilon^p} \|f_n - f\|_p^q \to 0$$

by Chebychev's inequality (**Folland Theorem 6.17**); thus, $f_n \to f$ in measure. Likewise, $T(f_n) = f_n \to g$ in $L^q(\mu)$ equipped with the q norm, so we have that another application of Chebychev gives us

$$\mu(\{x : |f_n(x) - g(x)| > \epsilon\}) \leq \frac{1}{\epsilon^q} \|f_n - g\|_q^q \to 0.$$

Hence $f_n \to g$ in measure. Invoking **Folland Theorem 2.30**, we see that $f = g$ almost everywhere, and since we are viewing these are functions in $L^q(\mu)$, we have that $T(f) = f = g$ as functions in $L^q(\mu)$. Thus, $T(f_n) \to T(f)$ in $L^q(\mu)$, so that T is a closed map.

Next, we note that the domain $\mathcal{D}$ is a dense subset. Using **Folland Proposition 6.7**, we have that the simple functions are in $\mathcal{D}$ (since they are in $L^q(\mu)$), the simple functions are in $L^p(\mu)$, and they are dense in $L^p(\mu)$. Thus, letting Σ denote the set of simple functions, we have $\Sigma \subset \mathcal{D} \subset L^p(\mu)$ so that $\mathcal{D}$ is a dense subset of L^p.

If T were continuous on its domain $\mathcal{D}$, then we claim that its domain would be all of $L^p(\mu)$. Since T is a closed map and $\mathcal{D}$ is dense, taking arbitrary $f \in L^p(\mu)$, we can find a sequence $(f_n) \subset L^p(\mu)$ such that $f_n \to f$, so

$$T(f) = f = \lim_{n \to \infty} f_n = \lim_{n \to \infty} T(f_n)$$

is uniquely defined, so $f \in \mathcal{D}$. If the domain is equal to $L^p(\mu)$, we have that $L^p(\mu) = L^q(\mu)$. This, however, is not always true; following **Folland page 185** and taking, for example, $p = 1$

and $q = 2$, we have $((0,1), \lambda)$ (λ the Lebesgue measure) is a finite measure space. Notice that $f(x) = x^{-1/2}$ is in L^1 since

$$\|f\|_1 = \int_0^1 x^{-1/2} d\lambda(x) = 2x^{1/2}\Big|_{x=0}^{1} = 2,$$

but

$$\|f\|_2^2 = \int_0^1 x^{-1} d\lambda(x) = \ln(x)\Big|_{x=0}^{1} = \infty.$$

So $f \notin \mathcal{D}$, since $T(f) = f \notin L^2(\lambda)$, and hence the containment $L^2(\lambda) \subsetneq L^1(\lambda)$ is strict. Thus, the identity function $T : L^p(\mu) \to L^q(\mu)$ need not be continuous.

$\square$

Problem 55. For $0 < p < 1$, let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and f a measurable function on the product space $X \otimes Y$. Then

$$\int_X \left[\int_Y |f(x,y)|^p d\nu(y)\right]^{1/p} d\mu(x) \leq \left[\int_Y \left(\int_X |f(x,y)| d\mu(x)\right)^p d\nu(y)\right]^{1/p}.$$

Proof. We remark that this is the "flipped" version of Minkowski's inequality for integrals. We somewhat expect this to hold, since in general these kinds inequalities flip for $0 < p < 1$ (see **Homework 2 Problem 1 (a)**, for example). To prove this, we will follow Grafakos' hint for proving the usual Minkowski's integral inequality (**Grafakos Exercise 1.1.6 (a)**, the same hint is also in **Measure and Integral: An Introduction to Real Analysis** by **Wheeden and Zygmund**, **Exercise 8.8**) and use a sort of "flipped" Hölder in place for Hölder (**Grafakos Exercise 1.1.2 (c)**). We first prove the desired lemma.

Lemma ("Flipped" Hölder, **Grafakos Exercise 1.1.2 (c)**)**.** For $r < 0$ and $g > 0$ almost everywhere, define

$$\|g\|_r = \|g^{-1}\|_{|r|}^{-1}.$$

In other words, we define it as we have for the positive numbers:

$$\|g\|_r = \left[\left(\int (g^{-1})^{|r|}\right)^{1/|r|}\right]^{-1} = \left(\int g^{-|r|}\right)^{-1/|r|} = \left(\int g^r\right)^{1/r}.$$

We define L^r for $r < 0$ in the usual way, which is that L^r is the space of functions where the $\|\cdot\|_r$ norm is finite.

Let $0 < p < 1$, $q = p/(p-1)$. If g is strictly positive almost everywhere and lies in L^q and f is measurable such that fg is in L^1, then

$$\|fg\|_1 \geq \|f\|_p \|g\|_q.$$

The goal is to prove this by applying the usual Hölder (**Grafakos Exercise 1.1.2 (a)** with $k = 2$, **Folland Theorem 6.2**). That is, the goal is to use the following.

Theorem (Hölder's Inequality)**.** Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. If f and g are measurable functions on X, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof of Lemma. Let $r = 1/p$ so that $1 < r < \infty$. We examine

$$\|f\|_p^p = \int |f|^p = \int |f|^{1/r}.$$

Multiplying and dividing by $|g|^{1/r}$ (and thus using the fact that g is positive), we get

$$\int |fg|^{1/r} |g|^{-1/r}.$$

We now can apply Hölder using r and q', where q' is chosen so that $1/r + 1/q' = 1$. This gives us

$$\int |fg|^{1/r}|g|^{-1/r} \leq \left(\int |fg|\right)^{1/r} \left(\int |g|^{-q'/r}\right)^{1/q'}.$$

Notice that $q' = r/(r-1)$, and substituting in $1/p = r$, we have

$$q' = \frac{1/p}{(1/p - 1)} = \frac{1}{p((1-p)/p)} = \frac{1}{1-p}.$$

Hence, we have

$$\frac{-q'}{r} = \frac{1/(p-1)}{1/p} = \frac{p}{p-1} = q,$$

and so

$$\|f\|_p^p \le \left(\int |fg|\right)^p \left(\int |g|^q\right)^{-(p-1)}.$$

Multiplying both sides by $\left(\int |g|^q\right)^{(p-1)}$, we have

$$\left(\int |g|^q\right)^{p-1} \|f\|_p^p \le \left(\int |fg|\right)^p = \|fg\|_1^p,$$

where we note it's valid to do since $g \in L^q$. Taking pth roots of both sides, we get

$$\left(\int |g|^q\right)^{(p-1)/p} \|f\|_p = \left(\int |g|^q\right)^{1/q} \|f\|_p = \|g\|_q \|f\|_p \le \|fg\|_1,$$

as desired. □

Assume without loss of generality that $f(x,y) \ge 0$ (to simplify notation). We rule out some extraneous cases first in the hopes of getting the appropriate assumptions for the above lemma. First, if there is some positive measurable set $E \subset Y$ so that for all $y \in E$, we have

$$\int_X f(x,y)d\mu(x) = \infty,$$

then we see that the inequality clearly holds; taking the pth power of the right hand side, we have

$$\infty = \int_E \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) \le \int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y).$$

So we assume that for almost every $y \in Y$,

$$\int_X f(x,y)d\mu(x) < \infty.$$

Similarly, if we have

$$\left[\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right]^{1/p} = \infty,$$

then the inequality trivially holds again, so assume that

$$\left[\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right]^{1/p} < \infty.$$

Using the fact that $f(x,y) \ge 0$, we have that for almost every $y \in Y$,

$$0 \le \int_X f(x,y)d\mu(x) < \infty.$$

The goal now is to show that we can reduce to the case where for almost every $y \in Y$ we have

$$0 < \int_X f(x,y)d\mu(x) < \infty.$$

Let $E \subset Y$ be the collection

$$\begin{aligned} E &:= \left\{y \in Y : \int_X f(x,y)d\mu(x) = 0\right\} \\ &= \{y \in Y : f(x,y) = 0 \text{ for almost every x}\}, \end{aligned}$$

where the second equality comes from **Folland Proposition 2.16**; that is, for fixed $y \in Y$,

$$\int_X f(x,y)d\mu(x) = 0 \iff f(x,y) = 0 \text{ for almost every x}$$

Observe that we can write the left hand side as

$$\int_X \left[\int_Y f(x,y)^p d\nu(y)\right]^{1/p} d\mu(x) = \int_X \left[\int_E f(x,y)^p d\nu(y) + \int_{E^c} f(x,y)^p d\nu(y)\right]^{1/p} d\mu(x).$$

Since for $y \in E$, $f(x,y) = 0$ for almost every $x \in X$, we have that

$$\int_X \left[\int_Y f(x,y)^p d\nu(y)\right]^{1/p} d\mu(x) = \int_X \left[\int_{E^c} f(x,y)^p d\nu(y)\right]^{1/p} d\mu(x).$$

Finally, observe that the right hand side of the inequality is equal to

$$\begin{aligned}\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) &= \int_E \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) + \int_{E^c} \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\\ &= \int_{E^c} \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y).\end{aligned}$$

So proving the inequality reduces to proving it in the case where

$$\int_X f(x,y)d\mu(x) > 0$$

for all $y \in Y$ (for the case where this does not hold for all of Y, this holds on E^c, and by what we've shown above showing it for E^c is sufficient). Thus, we may assume that for almost every $y \in Y$, we have

$$0 < \int_X f(x,y)d\mu(x) < \infty,$$

with

$$\left[\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right]^{1/p} < \infty.$$

For notational simplicity, let

$$h(y) := \int_X f(x,y)d\mu(x) < \infty.$$

Translating, the assumptions then state that $0 < h(y)$ for all $y \in Y$, $h(y) < \infty$ for almost every y, and $\|h\|_p < \infty$.

Going back to the inequality, notice that the pth power of the right hand side can be written as

$$\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y).$$

By the assumptions we made earlier (that is, $0 < h(y) < \infty$), we see that we can write this as

$$\begin{aligned}\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) &= \int_Y \left(\int_X f(z,y)d\mu(z)\right)^{p-1} \left(\int_X f(x,y)d\mu(x)\right) d\nu(y)\\ = \int_Y \int_X \left(\int_X f(z,y)d\mu(z)\right)^{p-1} f(x,y)d\mu(x)d\nu(y) &= \int_Y \int_X h(y)^{p-1} f(x,y)d\mu(x)d\nu(y).\end{aligned}$$

We see that Tonelli applies, since $h(y)^{p-1}f(x,y) \geq 0$, hence we can rewrite this as

$$\int_Y \int_X h(y)^{p-1} f(x,y)d\mu(x)d\nu(y) = \int_X \int_Y h(y)^{p-1} f(x,y)d\nu(y)d\mu(x).$$

Now, the goal is to use the flipped Hölder lemma above. By our notes earlier, we note that $g(y) := h(y)^{p-1}$ is such that $g > 0$ almost everywhere (the notation is changed in order match the lemma). Notice as well that we assumed that $\|h\|_p < \infty$ which tells us that $g(y)f(x,y) \in L^1(\nu(y))$ for almost every x; to see this, fix $x \in X$ and note that we have

$$\int_Y g(y)f(x,y)d\nu(y) = \int_Y \left(\int_X f(z,y)d\mu(z)\right)^{p-1} f(x,y)d\nu(y).$$

Integrating this with respect to x, we have

$$\begin{aligned}
&\int_X \left[\int_Y \left(\int_X f(z,y)d\mu(z)\right)^{p-1} f(x,y)d\nu(y)\right] d\mu(x) \\
&= \int_X \int_Y \left(\int_X f(z,y)d\mu(z)\right)^{p-1} f(x,y)d\nu(y)d\mu(x) \\
&= \int_Y \int_X \left(\int_X f(z,y)d\mu(z)\right)^{p-1} f(x,y)d\mu(x)d\nu(y) \\
&= \int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) = \|h\|_p^p < \infty,
\end{aligned}$$

so **Folland Proposition 2.20** tells us that, for almost every x, we must have

$$\int_Y g(y)f(x,y)d\nu(y) < \infty.$$

Since we will be integrating this with respect to X after applying the inequality, we remark that $gf \in L^1(\nu)$ for almost every x is sufficient.

Finally, we need to check that $g \in L^q(\nu)$, recalling $q = p/(p-1)$. Notice that

$$\|g\|_q^q = \int_Y g(y)^q d\nu(y) = \int \left(h(y)^{p-1}\right)^q d\nu(y) = \int h(y)^p d\nu(y) < \infty.$$

Hence, $\|g\|_q < \infty$, so $g \in L^q(\nu)$. The conditions are then met to apply the lemma. Fixing $x \in X$ where $gf \in L^1(\nu)$, we see that

$$\int_Y g(y)f(x,y)d\nu(y) \geq \left(\int_Y f(x,y)^p d\nu(y)\right)^{1/p} \left(\int_Y g(y)^q d\nu(y)\right)^{1/q}$$

for almost every x. Using the fact that $q = p/(p-1)$, we see that the right most value will be

$$\begin{aligned}
\left(\int_Y g(y)^q d\nu(y)\right)^{1/q} &= \left(\int_Y g(y)^{p/(p-1)} d\nu(y)\right)^{(p-1)/p} \\
&= \left(\int_Y \left(\int_X f(z,y)d\mu(z)\right)^p d\nu(y)\right)^{(p-1)/p},
\end{aligned}$$

so after integrating both sides with respect to X, we have

$$\begin{aligned}
&\int_X \int_Y g(y)f(x,y)d\nu(y)d\mu(x) = \int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y) \\
&\geq \int_X \left[\left(\int_Y \left(\int_X f(z,y)d\mu(z)\right)^p d\nu(y)\right)^{(p-1)/p} \left(\int_Y f(x,y)^p d\nu(y)\right)^{1/p}\right] d\mu(x) \\
&= \left(\int_Y \left(\int_X f(z,y)d\mu(z)\right)^p d\nu(y)\right)^{(p-1)/p} \int_X \left(\int_Y f(x,y)^p d\nu(y)\right)^{1/p} d\mu(x).
\end{aligned}$$

Dividing both sides by
$$\left(\int_Y \left(\int_X f(z,y)d\mu(z)\right)^p d\nu(y)\right)^{(p-1)/p},$$
which we remark is valid to do by our assumptions prior, we get
$$\left(\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{1/p} \geq \int_X \left(\int_Y f(x,y)^p d\nu(y)\right)^{1/p} d\mu(x),$$
as desired. $\square$

Problem 56. For $1 < p_0 < p < p_1 < \infty$, let $L^{p,\infty}$ denote weak L^p. Then

$$L^{p,\infty} \subset L^{p_0} + L^{p_1}.$$

Proof. Recall that

$$L^{p,\infty} = \{f \in \mathrm{Fun}(X, \mathbb{C}) : [f]_p < \infty\},$$

where

$$[f]_p = \sup_{\alpha>0} \alpha\lambda_f(\alpha)^{1/p}.$$

The goal is to write $f = g + h$, where $g \in L^{p_0}$ and $h \in L^{p_1}$. Fix some constant $A > 0$ (say $A = 1$ if you like). Let $E = \{x : |f(x)| > A\}$. Recall the functions

$$h_A = f\chi_{E^c} + A(\mathrm{sgn} f)\chi_E, \quad g_A = f - h_A.$$

Notice these functions are such that $f = g_A + h_A$. If we show that $g_A \in L^{p_0}$ and $h_A \in L^{p_1}$, then we win.

The idea comes from the proof of the Marcinkiewicz Interpolation theorem (**Folland Theorem 6.28**). Recall **Folland Proposition 6.24**, which says that if $0 < p < \infty$, then

$$\int |f|^p d\mu = p \int_0^\infty \alpha^{p-1}\lambda_f(\alpha) d\alpha.$$

Recall as well **Folland Proposition 6.25**, in which we have that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \quad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) \text{ if } \alpha < A, \\ 0 \text{ if } \alpha \geq A. \end{cases}$$

The proof of the prior proposition was **Quiz 2**. Combining these two results, we have that

$$\int |g_A|^{p_0} d\mu = p_0 \int_0^\infty \alpha^{p_0-1}\lambda_{g_A}(\alpha) d\alpha$$
$$= p_0 \int_0^\infty \alpha^{p_0-1}\lambda_f(\alpha + A) d\alpha \leq p_0 \int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha), d\alpha$$

where to get the inequality, we use a change of variables $\beta = \alpha + A$ and note that $(\beta - A)^{p_0-1} \leq \beta^{p_0-1}$, and then relabel the βs as αs. We also have that

$$\int |h_A|^{p_1} d\mu = p_1 \int_0^\infty \alpha^{p_1-1}\lambda_{h_A}(\alpha) d\alpha = p_1 \int_0^A \alpha^{p_1-1}\lambda_f(\alpha) d\alpha.$$

These match the results on **Folland page 204**.

Let k be such that

$$[f]_p = \sup_{\alpha>0} \alpha\lambda_f(\alpha)^{1/p} = k < \infty.$$

Then for any α, we have

$$\alpha\lambda_f(\alpha)^{1/p} \leq \sup_{\alpha>0} \alpha\lambda_f(\alpha)^{1/p} = k \implies \lambda_f(\alpha) \leq \frac{k^p}{\alpha^p}.$$

Thus, substituting this into the first equation, we have

$$\begin{aligned}\int |g_A|^{p_0} d\mu &\leq p_0 \int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha \\ &\leq k^p p_0 \int_A^\infty \alpha^{p_0-p-1}d\alpha \\ &= k^p p_0 \int_A^\infty \alpha^{p_0-p-1}d\alpha \\ &= k^p p_0 \left[\frac{\alpha^{p_0-p}}{p_0-p}\bigg|_{\alpha=A}^\infty \right] \\ &= k^p p_0 \frac{A^{p_0-p}}{p-p_0} < \infty,\end{aligned}$$

where here we use the fact that $p_0 < p$, so $\lim_{\alpha\to\infty} \alpha^{p_0-p} = 0$. Thus, taking p_0th roots, we have that $\|g_A\|_{p_0} < \infty$, giving us that $g_A \in L^{p_0}$.

Similarly, substituting the above in the second equation, we see that this gives

$$\begin{aligned}\int |h_A|^{p_1} d\mu &= p_1 \int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha \\ &\leq k^p p_1 \int_0^A \alpha^{p_1-p-1}d\alpha \\ &= k^p p_1 \left[\frac{\alpha^{p_1-p}}{p_1-p}\bigg|_{\alpha=0}^A \right] \\ &= k^p p_1 \frac{A^{p_1-p}}{p_1-p} < \infty,\end{aligned}$$

where here we use the fact that $p < p_1$, so there are no issues for α^{p_1-p} at 0. Thus, taking p_1th roots, we have that $\|h_A\|_{p_1} < \infty$, giving us that $h_A \in L^{p_1}$. Hence, $f \in L^{p_0} + L^{p_1}$. The choice of $f \in L^{p,\infty}$ was arbitrary, so we get that $L^{p,\infty} \subset L^{p_0} + L^{p_1}$. $\square$

Problem 57. For $x \in \mathbb{T}$, let

$$P_N(x) = 2F_{2N+1}(x) - F_N(x),$$

where F_N denotes the Féjer kernel.[7]

(a) Prove that the sequence P_N is an approximate identity.

(b) Prove that $\widehat{P_N}(m) = 1$ when $|m| \leq N+1$ and $\widehat{P_N}(m) = 0$ when $|m| \geq 2N+2$.

Proof. (a) Recall that a sequence is an approximate identity if it satisfies three properties:

(1) We first want to show that $\sup_N \|P_N\|_1 < \infty$. We see that, for all N, we have

$$\|P_N\|_1 = \|2F_{2N+1} - F_N\|_1 \leq 2\|F_{2N+1}\|_1 + \|F_N\|_1$$

using Minkowski's inequality (**Folland Proposition 6.5**) and the linearity of the integral (to pull out the 2). Since the Fejér kernel is an approximate identity (by the lecture notes from 2/10 or **Grafakos Proposition 3.1.10**), we see that

$$\sup_N \|P_N\|_1 \leq 2 \sup_N \|P_{2N+1}\|_1 + \sup_N \|P_N\|_1 < \infty.$$

(2) We now want to show that

$$\int P_N(x) dx = 1$$

for all N. Fixing an N and using that the Fejér kernel is an approximate identity, we see that

$$\int P_N(x)dx = \int (2F_{2N+1}(x) - F_N(x))dx = 2\int F_{2N+1}(x)dx - \int F_N(x)dx = 2 - 1 = 1.$$

Hence, we have the desired result.

(3) Finally, we wish to show that for any neighborhood V^c of 0, we have that

$$\int_V |P_N| dx \to 0.$$

Since we're on the torus, it suffices to show that for all $\delta > 0$,

$$\int_{\delta \leq |x| \leq 1/2} |P_N| dx \to 0.$$

Again, we use that $|P_N(x)| \leq 2|F_{2N+1}(x)| + |F_N(x)|$, so that if $V = \{x : \delta \leq |x| \leq 1/2\}$,

$$0 \leq \int_V |P_N| dx \leq 2\int_V |F_{2N+1}(x)|dx + \int_V |F_N(x)|dx.$$

Taking the limit as $N \to \infty$ of both sides gives us

$$0 \leq \lim_{N\to\infty} \int_V |P_N| dx \leq 2\left[\lim_{N\to\infty} \int_V |F_{2N+1}(x)|dx\right] + \lim_{N\to\infty} \int_V |F_N(x)|dx = 0$$

since (F_N) is an approximate identity. Thus, we have that

$$\lim_{N\to\infty} \int_V |P_N| dx = 0,$$

as desired.

Hence, (P_N) is an approximate identity.

[7]This is called the *de la Vallée Poussin Kernel* – see **Grafakos Exercise 3.1.4**.

(b) We have that

$$\widehat{P_N}(m) = 2\widehat{F_{2N+1}}(m) - \widehat{F_N}(m)$$

by linearity of the Fourier transform. Utilizing **Grafakos Proposition 3.1.7** (the details from which will be shown after the problem), we see that

$$\widehat{F_{2N+1}(m)} = \begin{cases} 1 - \frac{|m|}{2N+2} \text{ if } |m| \leq 2N+1 \\ 0 \text{ otherwise.} \end{cases}$$

Likewise,

$$\widehat{F_N}(m) = \begin{cases} 1 - \frac{|m|}{N+1} \text{ if } |m| \leq N \\ 0 \text{ otherwise.} \end{cases}$$

Thus, for $|m| \geq 2N+2$, we get that $\widehat{P_N}(m) = 0$, since both components will be 0 in this range. Now, for $|m| \leq N$, we see that

$$\begin{aligned}\widehat{P_N}(m) &= 2\left(1 - \frac{|m|}{2N+2}\right) - \left(1 - \frac{|m|}{N+1}\right) \\ &= \left(\frac{4N+4-2|m|}{2N+2}\right) - \left(\frac{N+1-|m|}{N+1}\right) \\ &= \frac{4N+4-2|m|-2N-2+2|m|}{2N+2} \\ &= \frac{2N+2}{2N+2} = 1.\end{aligned}$$

For $|m| = N+1$, we see that we have

$$\widehat{P_N}(m) = 2\left(1 - \frac{|m|}{2N+2}\right) = 2\left(\frac{2N+2-|m|}{2N+2}\right)$$
$$= \frac{4N+4-2|m|}{2N+2} = \frac{4N+4-2N+2}{2N+2} = \frac{2N+2}{2N+2} = 1.$$

So if $|m| \leq N+1$, we have $\widehat{P_N}(m) = 1$, as desired.

$\square$

Remark. Grafakos Proposition 3.1.7 claims that

$$F_N(x) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j x}.$$

To see this, recall that we define the Féjer kernel by

$$F_N(x) = \frac{1}{N+1}[D_0(x) + \cdots + D_N(x)],$$

where

$$D_j(x) = \sum_{|m| \leq j} e^{2\pi i m \cdot x}$$

denotes the Dirichlet kernel. Notice that we have

$$F_N(x) = \frac{1}{N+1}\sum_{j=0}^{N} D_j(x) = \frac{1}{N+1}\sum_{k=0}^{N}\sum_{|j|\leq k} e^{2\pi i j x} = \frac{1}{N+1}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} e^{2\pi i j x}\chi_E(k,j),$$

where

$$E = \{(k,j) \in \mathbb{Z}^2 : |j| \leq k \leq N\}.$$

We apply Tonelli (with respect to counting measures) to get

$$\frac{1}{N+1}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} e^{2\pi ijx}\chi_E(k,j) = \frac{1}{N+1}\sum_{j=-\infty}^{\infty} e^{2\pi ijx}\sum_{k=-\infty}^{\infty}\chi_E(k,j)$$
$$= \frac{1}{N+1}\sum_{j=-\infty}^{\infty} e^{2\pi ijx}|E_k(j)|,$$

where $E_k(j) = \{k \in \mathbb{Z} : |j| \leq k \leq N\}$. Notice that for fixed j we have

$$|E_k(j)| = \begin{cases} N+1-|j| \text{ for } |j| \leq N \\ 0 \text{ otherwise,} \end{cases}$$

since this is just counting the number of integers between $|j|$ and N. Substituting this in, we have

$$F_N(x) = \sum_{|j|\leq N} \frac{N+1-|j|}{N+1} e^{2\pi ijx} = \sum_{|j|\leq N}\left(1 - \frac{|j|}{N+1}\right) e^{2\pi ijx},$$

as desired.

Grafakos then uses this to note that

$$\widehat{F_N}(m) = \begin{cases} 1 - \frac{|m|}{N+1} \text{ if } |m| \leq N \\ 0 \text{ otherwise.} \end{cases}$$

To see this, we use linearity of the Fourier transform to note that

$$\widehat{F_N}(m) = \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)\widehat{e^{2\pi ijx}}(m)$$
$$= \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)\int_{\mathbb{T}} e^{2\pi ijx}e^{-2\pi ixm}dx$$
$$= \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)\int_{\mathbb{T}} e^{2\pi ix(j-m)}dx = \begin{cases} 1 - \frac{|m|}{N+1} \text{ if } |m| \leq N \\ 0 \text{ otherwise,} \end{cases}$$

since

$$\int_{\mathbb{T}} e^{2\pi ix(j-m)}dx = \begin{cases} 1 \text{ if } j = m \\ 0 \text{ otherwise.} \end{cases}$$

To see the last identity, note that if $j \neq m$,

$$\int_{\mathbb{T}} e^{2\pi ix(j-m)}dx = \frac{1}{2\pi ix(j-m)}e^{2\pi ix(j-m)}\Big|_{x=-1/2}^{1/2}$$
$$= \frac{1}{\pi(j-m)}\frac{e^{\pi i(j-m)} - e^{-\pi i(j-m)}}{2i}$$
$$= \frac{\sin(\pi(j-m))}{\pi(j-m)}$$

using DeMoivre. Since $j - m \in \mathbb{Z} - \{0\}$, we have that this evaluates to 0, as desired. Note that if $j = m$, we get that the integral is

$$\int_{\mathbb{T}} dx = 1,$$

as desired.

Problem 58. Prove the Fourier inversion theorem: If $f, \widehat{f} \in L^1$, then f agrees almost everywhere with a continuous function f_0, and $(\widehat{f})^\vee = (f^\vee)^\wedge = f_0$.

Proof. We follow Folland's proof (see **Folland Theorem 8.26**). Fix $t > 0$ and $x \in \mathbb{R}^n$. Set

$$\varphi(m) = e^{2\pi i m\cdot x - \pi t^2 |m|^2}.$$

Using **Folland Theorem 8.22 (a)** and **Folland Proposition 8.24**, we see that

$$\begin{aligned}\widehat{\varphi}(z) &= \mathcal{F}\left(e^{2\pi i m\cdot x} e^{-\pi t^2|m|^2}\right)(z) = \tau_x \mathcal{F}(e^{-\pi t^2|m|^2}))(z) \\ &= \tau_x\left(\left(t^2\right)^{-n/2} e^{-\pi|z|^2/t^2}\right) = t^{-n} e^{-\pi|x-z|^2/t^2}.\end{aligned}$$

Let $g(x) = e^{-\pi|x|^2}$. Then recall that for $t > 0$, we defined the approximate identity $g_t(x) = t^{-n} g(x/t) = t^{-n} e^{-\pi|x|^2/t^2}$. Hence, we see that

$$\widehat{\varphi}(z) = g_t(x - z).$$

By **Folland Lemma 8.25**, we see that

$$\int \varphi(m)\widehat{f}(m)dm = \int \widehat{\varphi}(z) f(z)dz = \int g_t(x-z)f(z)dz = g_t * f(x) = f * g_t(x).$$

Note that

$$\int g(x)dx = \int e^{-\pi|x|^2}dx = \left(\frac{\pi}{\pi}\right)^{n/2} = 1$$

by **Folland Proposition 2.53**. Hence, invoking **Folland Theorem 8.14 (a)/Quiz 4**, we see that $f * g_t \to f$ in the L^1 norm as t tends to 0. We now utilize the fact that $\widehat{f} \in L^1$ to apply the dominated convergence theorem (**Folland Theorem 2.24**, abbreviated as DCT). That is, examining the integral

$$\lim_{t\to 0} \int e^{-\pi t^2|m|^2} e^{2\pi m\cdot x}\widehat{f}(m)dm,$$

we see that the absolute value of the inside of the integral comes out to $|e^{-\pi t^2|m|^2}\widehat{f}(m)| \le |\widehat{f}(m)| \in L^1$, so we can apply DCT to move the limit inside. Thus, we have

$$\begin{aligned}\lim_{t\to 0}\int e^{-\pi t^2|m|^2} e^{2\pi m\cdot x}\widehat{f}(m)dm = \int \lim_{t\to 0} e^{-\pi t^2|m|^2} e^{2\pi m\cdot x}\widehat{f}(m)dm &= \int e^{2\pi m\cdot x}\widehat{f}(m)dm \\ &= [\widehat{f}]^\vee(x).\end{aligned}$$

Since these converge to the same thing, we see that $[\widehat{f}]^\vee = f$ almost everywhere. Using Riemann-Lebesgue lemma (**Folland Theorem 8.22 (f)**), we recall that the Fourier transform will be a continuous function which vanishes at zero, and so there is some (unique) $f_0 \in C_0(\mathbb{R}^n)$ so that $[\widehat{f}]^\vee = f_0$. Thus, $f = f_0$ almost everywhere (which also gives uniqueness; if g_0 were another function, we would have $f_0 = g_0$ almost everywhere, but since these are continuous this implies $g_0 = f_0)$[8].

Finally, we need to show that $[f^\vee]^\wedge = [f^\wedge]^\vee$. The idea is to replace f in the equation with $\widetilde{f}$ (as mentioned in **Grafakos Theorem 2.2.14**). Doing so grants us $[(\widetilde{f})^\wedge]^\vee(x) = \widetilde{f}_0(x) = f_0(-x)$,

[8]Briefly, two continuous functions which are equal almost everywhere are equal in fact everywhere by a contradiction argument; if they weren't equal everywhere, there would be some point at which they differ, and we can find a very small open ball around that point which would have positive measure in Lebesgue measure, and so we have a contradiction to the fact that they are equal almost everywhere. Grafakos uses this fact a few times without proof, and I don't see a proof in Folland anywhere.

so applying a change of variables we see that $[(\widetilde{f})^{\wedge}]^{\vee}(-x) = f_0(x)$. Notice that $[(\widetilde{f})^{\wedge}]^{\vee}(-x) = [(\widetilde{f})^{\wedge}]^{\wedge}(x)$ by definition. Finally, we claim that $\widetilde{\widehat{f}} = \widehat{\widetilde{f}} = f^{\vee}$. To see this, notice that

$$\widehat{\widetilde{f}}(m) = \int \widetilde{f}(x) e^{-2\pi i m \cdot x} dx = \int f(-x) e^{-2\pi i m \cdot x} dx.$$

We now preform a change of variables using **Folland Theorem 2.44**. Let $T(x) = -x$ (this is in $\mathrm{GL}_n(\mathbb{R})$ since $\det(T) = (-1)^n \neq 0$) and let $q(x) = f(x)e^{2\pi i m \cdot x}$ (this is integrable, since $\int |f(x)e^{2\pi i m \cdot x}| dx \leq \int |f(x)| dx < \infty$). The theorem gives us

$$\begin{aligned}\int f(-x) e^{-2\pi i m \cdot x} dx = \int q(-x) dx = \int q \circ T(x) dx = |\det(T)|^{-1} \int q(x) dx \\ = |(-1)^n|^{-1} \int f(x) e^{2\pi i m \cdot x} dx = \int f(x) e^{2\pi i m \cdot x} dx.\end{aligned}$$

Hence, we have

$$f^{\vee}(m) = \widetilde{\widehat{f}}(m) = \widehat{f}(-m) = \int f(x) e^{2\pi i m \cdot x} dx = \int f(-x) e^{-2\pi i m \cdot x} dx = \widehat{\widetilde{f}}(m).$$

Using this claim, we see that

$$[(\widetilde{f})^{\wedge}]^{\wedge}(x) = [\widetilde{(f^{\wedge})}]^{\wedge}(x) = [f^{\vee}]^{\wedge}(x).$$

Thus,

$$[f^{\vee}]^{\wedge} = f_0$$

and we deduce that

$$[f^{\vee}]^{\wedge} = [f^{\wedge}]^{\vee} = f_0.$$

$\square$

Problem 59. Suppose that $F \in \mathcal{S}'$, $G \in \mathcal{E}'$. Prove the following.
(a) $\widehat{F}\widehat{G}$ is in $\mathcal{S}'$.
(b) If $\psi \in \mathcal{S}$, then $G * \psi \in \mathcal{S}$.

Proof. Recall the notation: $F \in \mathcal{S}'$ means F is a tempered distribution (that is, a continuous linear functional on $\mathcal{S}$) and $G \in \mathcal{E}'$ is a distribution with compact support.

(a) By **Folland Proposition 9.11**, we see that $\widehat{G}$ is a slowly increasing C^∞ function (defined by $\widehat{G}(m) = \langle G, e^{2\pi i m \cdot x}\rangle$). Based on how we defined the Fourier transform of tempered distributions, we have that $\widehat{F}$ is still a tempered distribution defined by

$$\langle \widehat{F}, \varphi \rangle = \langle F, \widehat{\varphi} \rangle$$

for all $\varphi \in \mathcal{S}$ (by the discussion on **Folland page 295** and the lecture notes from 3/2). It suffices, then, to show that if F is a tempered distribution, ψ is a slowly increasing function, then $F\psi$ is a well-defined element of $\mathcal{S}'$, where we will define $F\psi$ via

$$\langle F\psi, \varphi \rangle = \langle F, \psi\varphi \rangle;$$

in other words, where we define it the usual way (see the discussion on **Folland page 294**). The result is then proven by showing that the product of a Schwarz function and a slowly increasing function is a Schwarz function (matching the discussion in **Grafakos Definition 2.3.15**). To see that $\psi\varphi$ is a Schwarz function, we merely need to check that for all N, α,

$$\|\psi\varphi\|_{(N,\alpha)} < \infty.$$

Fixing N and α, we have

$$\begin{aligned} \|\psi\varphi\|_{(N,\alpha)} &= \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha (\psi\varphi)(x)| \\ &= \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \left| \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta \psi(x))(\partial^\gamma \varphi(x)) \right| \\ &\le \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} |\partial^\beta \psi(x)||\partial^\gamma \varphi(x)|. \end{aligned}$$

Since ψ is slowly increasing, we have that

$$|\partial^\beta \psi(x)| \le C_\beta (1 + |x|)^{N(\beta)},$$

where $N(\beta)$ is some positive integer depending on β, C_β is some constant. Substituting this into the above, we have

$$\begin{aligned} \|\psi\varphi\|_{(N,\alpha)} &\le \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left(C_\beta (1 + |x|)^{N(\beta)} \right) |\partial^\gamma \varphi(x)| \\ &\le \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} C_\beta \left[\sup_{x \in \mathbb{R}^n} (1 + |x|)^{N(\beta)} |\partial^\gamma \varphi(x)| \right] \\ &= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} C_\beta \|\varphi\|_{(N(\beta),\gamma)} < \infty \end{aligned}$$

since $\varphi \in \mathcal{S}$. The choice of N and α was arbitrary, so we get that $\psi\varphi$ is a Schwarz function. Hence, $\langle \psi F, \varphi \rangle = \langle F, \varphi\psi \rangle$ is well-defined for all φ, so ψF is a tempered distribution. Translating this back, we have that $\widehat{F}\widehat{G}$ is a tempered distribution, defined by $\langle \widehat{F}\widehat{G}, \varphi \rangle = \langle \widehat{F}, \widehat{G}\varphi \rangle$ for all $\varphi \in \mathcal{S}$.

(b) The goal now is to show that $G * \psi$ is a Schwarz function, where $\psi \in \mathcal{S}$. Note apriori we know that $G*\psi$ is a slowly increasing function by **Folland Proposition 9.10**, using the fact $\mathcal{E}' \subset \mathcal{S}'$ (lecture notes 3/2, **Grafakos page 120**). Notice that we have a Fourier transform defined on $\mathcal{E}'$, simply by defining it to be the usual Fourier transform on $\mathcal{S}'$. Using the properties of the Fourier transform on $\mathcal{S}'$ and $\mathcal{S}$, the goal is to show that $\mathcal{F}(G * \psi) \in \mathcal{S}$, and hence use **Folland Corollary 8.28** to get that $G * \psi \in \mathcal{S}$.

Using properties of the Fourier transform on tempered distributions (which is **Homework 9, Problem 1**; see the remark afterwards for a proof), we recall that for $\psi \in \mathcal{S}$, we have as distributions,

$$\mathcal{F}(G * \psi) = \mathcal{F}(G)\mathcal{F}(\psi).$$

By **Folland Proposition 9.11**, $\mathcal{F}(G)$ is a slowly increasing function, and we note that $\mathcal{F}(\psi)$ is a Schwartz function (using **Folland Corollary 8.28**). In (a), we showed that the product of a Schwarz function and a slowly increasing function is a Schwarz function, so we see that $\mathcal{F}(G)\mathcal{F}(\psi)$ as a function is a Schwarz function. Applying the inverse Fourier transform (denoted by $\mathcal{G}$) to both sides, we have that

$$G * \psi = \mathcal{G}(\mathcal{F}(G)\mathcal{F}(\psi))$$

as distributions in $\mathcal{S}'$. Since the Fourier transform on $\mathcal{S}$ is an isomorphism (**Folland Corollary 8.28**), we have that as a function, the right hand side is a Schwarz function (see the third claim following this proof). Note that two functions define the same distribution iff they are equal almost everywhere (see the second claim following this proof). Thus, $G * \psi = \mathcal{G}(\mathcal{F}(G)\mathcal{F}(\psi))$ almost everywhere (with respect to the Lebesgue measure). But as noted earlier, these are both continuous, so in fact they are equal everywhere. Thus, $G * \psi$ is a Schwarz function.

Alternatively, we can prove it directly in the following way (**Grafakos Theorem 2.3.20**). Notice that $G * \psi(x) = \langle G, \tau_x \widetilde{\psi} \rangle$. Since G is a continuous linear functional, we have that

$$|\langle G, \tau_x \widetilde{\psi} \rangle| \leq C \sum_{|\alpha| \leq k} \sup_{y \in \overline{V_N}} |\partial^\alpha \varphi(x - y)|,$$

by the proof of **Folland Theorem 9.8**, where we have that the (V_n) are an increasing sequence of precompact open subset of $\mathbb{R}^n$ whose union is $\mathbb{R}^n$. Choosing an M large enough, we have $\overline{V_N} \subset \{y \in \mathbb{R}^n : |y| \leq M\}$. Thus, we can rewrite this as

$$|\langle G, \tau_x \widetilde{\psi} \rangle| \leq C \sum_{|\alpha| \leq k} \sup_{|y| \leq M} |\partial^\alpha \varphi(x - y)|.$$

Since $\varphi \in \mathcal{S}$, we have that for any integer J,

$$\sup_{(x-y) \in \mathbb{R}^n} (1 + |x - y|)^J |\partial^\alpha \varphi(x - y)| = C_{\alpha,J} < \infty.$$

For $|x| \geq 2M$, we observe that

$$\sup_{|y| \leq M} |\partial^\alpha \varphi(x - y)| \leq \sup_{|y| \leq M} C_{\alpha,J}(1 + |x - y|)^{-J} \leq C'(1 + |x|)^{-J}$$

for some constant C'. Substituting this in, we see that

$$|G * \psi(x)| = |\langle G, \tau_x \widetilde{\psi} \rangle| \leq C'' \sum_{|\alpha| \leq k} (1 + |x|)^{-J} = C'''(1 + |x|)^{-J}$$

for constants C'', C'''. Notice this holds for all J, and so we see that $\|G * \psi(x)\|_{(N,0)} < \infty$ for all N. To get it for all multi-indices α, simply notice that $\partial^\alpha(G * \varphi) = G * (\partial^\alpha \varphi)$, and so applying the same argument to this gives us that $\|G * \psi(x)\|_{(N,\alpha)} < \infty$ for all (N, α). Hence, it is Schwarz.

$\square$

Remark. In the last problem, we used the properties of the Fourier transform on tempered distributions (**Homework 9, Problem 1**, see also **Folland page 295**, **Folland Exercise 9.17**). Since we technically haven't proven this yet, we submit a proof here.

Claim. For $F \in \mathcal{S}'$, $\psi \in \mathcal{S}$, we have

$$\widehat{F * \psi} = \widehat{\psi}\widehat{F}.$$

Proof. Recall that for $\psi \in \mathcal{S}$, we have

$$\langle F * \psi, \varphi \rangle = \int (F * \psi)\varphi = \langle F, \varphi * \widetilde{\psi} \rangle$$

by **Folland Proposition 9.10**. Taking the Fourier transform of $F * \psi$ and taking arbitrary $\varphi \in \mathcal{S}$, we have

$$\langle \widehat{F * \psi}, \varphi \rangle = \langle F * \psi, \widehat{\varphi} \rangle.$$

Now, we want to use the following identity:

$$[\widehat{\psi}]^\wedge = \widetilde{\psi},$$

To see this identity, recall that for Schwarz functions we have

$$\psi^\vee(x) = \widehat{\psi}(-x) = \widetilde{\widehat{\psi}}(x).$$

We recall that in the proof of **Problem 5** we showed that

$$\widetilde{\widehat{\psi}}(x) = \widehat{\widetilde{\psi}}(x).$$

For notational simplicity, let the reflection function be denoted by $P(f) = \widetilde{f}$, let the Fourier transform be denoted by $\mathcal{F}(f) = \widehat{f}$, and let its inverse be denoted by $\mathcal{G}(f) = f^\vee$. Notice that taking the Fourier transform of both sides of

$$\mathcal{G}(\psi) = P \circ \mathcal{F}(\psi)$$

gives us

$$\psi = \mathcal{F} \circ P \circ \mathcal{F}(\psi),$$

so that, using the commutativity of P and $\mathcal{F}$, we have

$$\psi = P \circ \mathcal{F}^{(2)}(\psi).$$

Note that $P \circ P = \mathrm{Id}$, so taking P of both sides, we have

$$P(\psi) = \mathcal{F}^{(2)}(\psi);$$

that is, reverting to old notation, we have

$$[\widehat{\psi}]^\wedge = \widetilde{\psi}.$$

Using this and **Folland Proposition 9.10**, we get that

$$\langle F * \psi, \widehat{\varphi} \rangle = \langle F, \widehat{\varphi} * \widetilde{\psi} \rangle = \langle F, \widehat{\varphi} * [\widehat{\psi}]^\wedge \rangle$$

Next, we'd like to use the identity

$$\widehat{\varphi} * [\widehat{\psi}]^\wedge = [\varphi\widehat{\psi}]^\wedge.$$

To see this, note that **Folland 8.22 (d)** gives us

$$\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi) = \mathcal{F}(\mathcal{F}(\varphi) * \mathcal{F}^{(2)}(\psi)).$$

Hence, we have

$$\mathcal{G}(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) = \mathcal{F}(\varphi) * \mathcal{F}^{(2)}(\psi)$$

Note that
$$P(\psi\varphi) = P(\psi)P(\varphi);$$
this is due to the fact that
$$P(\psi\varphi)(x) = (\psi\varphi)(-x) = \psi(-x)\varphi(-x) = (P(\psi)P(\varphi))(x).$$
Hence, we have
$$\begin{aligned}\mathcal{G}(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) = (P \circ \mathcal{F})(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) &= (\mathcal{F} \circ P)(\mathcal{F}^{(2)}(\varphi)\mathcal{F}^{(3)}(\psi)) \\ = (\mathcal{F} \circ P)(P(\varphi)(P \circ \mathcal{F})(\psi)) &= \mathcal{F}(P^{(2)}(\varphi)(P^{(2)} \circ \mathcal{F})(\psi)) \\ &= \mathcal{F}(\varphi\mathcal{F}(\psi)).\end{aligned}$$
The first equality here follows by expanding out the definition of $\mathcal{G}$, the second follows from using the fact that P and F commute, the third follows from using the identity $\mathcal{F}^{(2)} = P$ and writing $\mathcal{F}^{(3)} = \mathcal{F}^{(2)} \circ \mathcal{F}$, the fourth follows from the fact that P distributes over multiplication, and the last follows from the fact that $P^{(2)} = \mathrm{Id}$. In other words, reverting to old notation, we have that
$$\widehat{\varphi} * [\widehat{\psi}]^\wedge = [\varphi\widehat{\psi}]^\wedge.$$
Thus, we have that
$$\langle F, \widehat{\varphi} * [\widehat{\psi}]^\wedge \rangle = \langle F, [\varphi\widehat{\psi}]^\wedge \rangle$$
Now, by how Fourier transforms work for tempered distributions, we have
$$\langle F, [\varphi\widehat{\psi}]^\wedge \rangle = \langle \widehat{F}, \varphi\widehat{\psi} \rangle.$$
Finally, by how Schwarz functions multiply with tempered distributions, we have that
$$\langle \widehat{F}, \varphi\widehat{\psi} \rangle = \langle \widehat{\psi}\widehat{F}, \varphi \rangle.$$
Putting this all together, we get that
$$\langle \widehat{F * \psi}, \varphi \rangle = \langle \widehat{F}\widehat{\psi}, \varphi \rangle.$$
Since the choice of $\varphi \in \mathcal{S}$ was arbitrary, we have that as tempered distributions,
$$\widehat{F * \psi} = \widehat{F}\widehat{\psi}.$$
$\square$

Remark. Recall that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ defines a distribution, via
$$\langle f, \varphi \rangle = \int f\varphi.$$
In this case, we say that the function f defines the distribution. In the prior problem, we used the following claim.

Claim. Two functions define the same distribution if and only if they are equal almost everywhere (note that we are working over $\mathbb{R}^n$ with the Lebesgue measure).

Proof. ($\Longleftarrow$): This direction is clear; if $f = g$ almost everywhere, then for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have
$$\int f\varphi = \int g\varphi,$$
so they define the same distribution.
($\Longrightarrow$): This is the less trivial direction, which we took advantage of in the problem. Suppose that f and g define the same distribution, so that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have
$$\langle f, \varphi \rangle = \int f\varphi = \int g\varphi = \langle g, \varphi \rangle.$$

In other words, we have that

$$\int (f-g)\varphi = 0$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, so it suffices to show that if h is such that it defines the 0 distribution, then h is 0 almost everywhere. We first want to show that, for all rectangles $E = [a_1, b_1] \times \cdots \times [a_n, b_n] \in \Pi$,

$$\int_E h = 0.$$

Using the C^∞ Urysohn (**Folland Proposition 8.18**), we can construct a sequence $(\varphi_n) \subset C_c^\infty(\mathbb{R}^n)$ so that $\varphi_n \to \chi_E$ pointwise, $0 \le \varphi_n \le 1$ for all n (take a sequence of open intervals U_n which decrease to E, have φ_n be 1 on E and 0 outside of U_n). Notice that we have

$$|h\varphi_n| \le |h\chi_U| \le |h\chi_{\overline{U}}|,$$

and since $h \in L^1_{\text{loc}}(\mathbb{R}^n)$, we have that this is in $L^1(\mathbb{R}^n)$. Thus, we can use DCT to get

$$0 = \lim_{n\to\infty} \int h\varphi_n = \int h\chi_E = \int_E h.$$

Thus, $\int_E h = 0$ on rectangles. We can then take a family of cubes (rectangles where the side lengths are the same) such that $x \in Q_r$ for all r, $l(Q_r) \to 0$ (l here is the length function, which returns the length of one of the sides), and we see that

$$\lim_{r\to 0} \frac{1}{\lambda(Q_r)} \int_{Q_r} f(y)dy = f(x) = 0$$

for almost every $x \in \mathbb{R}^n$ by **Folland Theorem 3.21** (Lebesgue Differentiation Theorem), so $h = 0$ almost everywhere, as desired.[9] $\square$

Note as well that we implicitly used the fact that the Fourier transform of a distribution defined by a function agrees with the Fourier transform of the function (so that the Fourier transform is what you want it to be). We prove this as follows.

Claim. Let F be a distribution defined by $f \in \mathcal{S}$, then $\widehat{F}$ agrees with the distribution defined by $\widehat{f} \in \mathcal{S}$. In other words, we have that $\widehat{F}$ is defined by $\widehat{f} \in \mathcal{S}$.

Proof. Let $\varphi \in \mathcal{S}$, then we wish to show that for all such φ,

$$\langle \widehat{F}, \varphi \rangle = \langle \widehat{f}, \varphi \rangle.$$

Note that

$$\langle \widehat{F}, \varphi \rangle = \langle F, \widehat{\varphi} \rangle = \int f\widehat{\varphi} = \int \widehat{f}\varphi = \langle \widehat{f}, \varphi \rangle$$

using **Folland Lemma 8.25**. Thus, as distributions they agree, so $\widehat{F}$ is defined by $\widehat{f} \in \mathcal{S}$. $\square$

So taking the Fourier transform of a distribution defined as a function, we can take the Fourier transform as a function or as a distribution and get the same result. Since the Fourier transform is an isomorphism on both $\mathcal{S}$ and $\mathcal{S}'$, taking the inverse Fourier transform yields the same result.

[9] This was how we learned Lebesgue differentiation in 6211 last semester; the claim in Folland seems to differ slightly in the sense that it's any sequence of sets which are shrinking "nicely" to a point x.

References

[1] Gerald Folland. *Real Analysis: Modern Techniques and Their Applications*. Second Edition. OSU Catalog link.

[2] Loukas Grafakos. *Classical Fourier Analysis*. Third Edition. OSU Catalog link.[10]

[3] Richard Wheeden, Antoni Zygmund. *Measure and Integral: An Introduction to Real Analysis*. Second Edition. OSU Catalog link.

[10] I was only able to find the second edition in the library, however all of my notes have used the third edition. The second edition essentially seems to have all of the same information; the differences (that I've noticed) are that the hint for the usual Minkowski is not there, **Proposition 3.1.10** is instead **Remark 3.1.9**, and **Proposition 3.1.7** is a remark on **page 167**. A link to the third edition can be found by googling "Grafakos Classical Fourier Analysis" (it is the first link, it sadly changes so I can't link it to you here).

Problem 60 (Quiz 1)**.** Suppose that $p, q \in (0, \infty)$ and $1/p + 1/q = 1$. Prove that if $f_n \to f$ in $L^p(\mathbb{R})$ and $g_n \to g$ in $L^q(\mathbb{R})$, then $f_n g_n \to fg$ in $L^1(\mathbb{R})$.

Proof. First, we remark that since $1/p + 1/q = 1$, we cannot have $0 < p, q < 1$. Assume that $0 < p < 1$. Then this implies that $1/p > 1$, and so there is no $0 < q < \infty$ so that $1/p + 1/q = 1$. An analogous argument applies for $0 < q < 1$.

The goal, then, is to show

$$\|f_n g_n - fg\|_1 \to 0.$$

Notice that we can add and subtract by fg_n to get

$$\|f_n g_n - fg\|_1 = \|f_n g_n - fg_n + fg_n - fg\|_1 \leq \|(f_n - f)g_n\|_1 + \|f(g_n - g)\|_1.$$

Apply Hölders to this to get

$$\|(f_n - f)g_n\|_1 \leq \|f_n - f\|_p \|g_n\|_q,$$
$$\|f(g_n - g)\|_1 \leq \|f\|_p \|g_n - g\|_q.$$

Now, if $g_n \to g$ in L^q, we have

$$\|g_n - g\|_q \to 0.$$

We can use the reverse triangle inequality here to get

$$|\|g_n\|_q - \|g\|_q| \leq \|g_n - g\|_q \to 0,$$

so we have that $\|g_n\|_q \to \|g\|_q < \infty$. Hence, taking the limit, we have

$$\|f_n - f\|_p \|g_n\|_q \to 0,$$
$$\|f\|_p \|g_n - g\|_q \to 0,$$

so we get that

$$\|f_n g_n - fg\|_1 \to 0,$$

as desired. $\square$

Problem 61 (Quiz 2)**.** Suppose that f a measurable function and $A > 0$. Let

$$E(A) := \{x : |f(x)| > A\}$$

and let

$$h_A := f\chi_{E^c(A)} + A\text{sgn}(f)\chi_{E(A)},$$
$$g_A := f - h_A.$$

Show that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A),$$

and

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) \text{ if } \alpha < A \\ 0 \text{ if } \alpha \geq A. \end{cases}$$

Proof. Recall that

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}) = \mu(E(\alpha)).$$

Hence,

$$\lambda_{g_A}(\alpha) = \mu(\{x : |f(x) - h_A(x)| > \alpha\}).$$

Examining the set, notice that we can write

$$\begin{aligned} \{x : |f(x) - h_A(x)| > \alpha\} &= \{x : f(x) - h_A(x) > \alpha\} \sqcup \{x : f(x) - h_A(x) < -\alpha\} \\ &= \{x : f(x) - h_A(x) > \alpha\} \sqcup \{x : f(x) < -\alpha + h_A(x)\}. \end{aligned}$$

Since $E(A), E^c(A)$ are disjoint, we can write this as

$$\begin{aligned}\{x : |f(x) - h_A(x)| > \alpha\} &= \{x : x \in E(A), f(x) > \alpha + h_A(x)\} \sqcup \{x : x \in E^c(A), f(x) > \alpha + h_A(x)\} \\ &\sqcup \{x : x \in E(A), f(x) < -\alpha + h_A(x)\} \sqcup \{x : x \in E^c(A), f(x) < -\alpha + h_A(x)\}\end{aligned}.$$

If $x \in E(A)$, we get that $h_A(x) = A\text{sgn}(f)$, and if $x \in E^c(A)$, we get that $h_A(x) = f$. Hence, we can rewrite the above as

$$\begin{aligned}\{x : |f(x) - h_A(x)| > \alpha\} &= \{x : f(x) > A + \alpha\} \sqcup \{x : f(x) < -A - \alpha\} \\ &= \{x : |f(x)| > A + \alpha\}\end{aligned}.$$

Thus, we get that

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A).$$

Similarly, examine

$$\begin{aligned}\{x : |h_A(x)| > \alpha\} &= \{x : h_A(x) > \alpha\} \sqcup \{x : h_A(x) < -\alpha\} \\ &= \{x : x \in E(A), h_A(x) > \alpha\} \sqcup \{x : x \in E^c(A), h_A(x) > \alpha\} \\ &\sqcup \{x : x \in E(A), h_A(x) < -\alpha\} \sqcup \{x : x \in E^c(A), h_A(x) < -\alpha\}.\end{aligned}$$

If $x \in E(A)$, we see that $h_A(x) = A\text{sgn}(f)$. If $x \in E^c(A)$, we get that $h_A(x) = f(x)$. Thus, we rewrite the above as

$$\{x : |h_A(x)| > \alpha\} = \{x : A > f(x) > \alpha, f(x) \geq A\} \sqcup \{x : -A < f(x) < -\alpha, f(x) \leq -A\}$$

If $\alpha \geq A$, we see that this will be 0. If $\alpha < A$, we see that this will be the same as $\{x : |f(x)| > \alpha\}$. Hence, we have

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) \text{ if } \alpha < A \\ 0 \text{ if } \alpha \geq A. \end{cases}$$

□

Problem 62 (Quiz 3)**.** Let $f, g \in L^1(\mathbb{R})$. Prove the following:

(a) $f * (g * h)(x) = (f * g) * h(x)$ almost everywhere.

(b) $f * (g + h)(x) = f * g(x) + f * h(x)$.

Proof. (a) This one is trickier than it seems. Recall that $f * g(x) = \int f(x - y)g(y)dy$. Using this, we write out

$$\begin{aligned}f * (g * h)(x) &= \int f(x - y)(g * h)(y)dy \\ &= \int f(x - y)\left[\int g(y - z)h(z)dz\right] dy \\ &= \iint f(x - y)g(y - z)h(z)dzdy.\end{aligned}$$

We now wish to iterate the integral. To do so, we need to check that Fubini applies. Write

$$k(x) = \iint f(x - y)g(y - z)h(z)dzdy.$$

Then we wish to check that $k(x) \in L^1(\mathbb{R})$. We have

$$\begin{aligned}\int |k(x)|dx = \int \left|\int \int f(x - y)g(y - z)h(z)dzdy\right| dx &\leq \iiint |f(x - y)||g(y - z)||h(z)|dzdydx \\ &\leq \|f\|_1\|g\|_1\|h\|_1 < \infty\end{aligned}$$

iterating Young's inequality. Hence, $k \in L^1(\mathbb{R})$, so for almost every x, we have that

$$\iint f(x-y)g(y-z)h(z)dzdy < \infty;$$

Hence, taking absolute values, we get for almost every x,

$$\iint |f(x-y)||g(y-z)||h(z)|dzdy \le \|f\|_1\|g\|_1\|h\|_1,$$

so Fubini applies. Thus, switching the order, we get

$$\iint f(x-y)g(y-z)h(z)dydz = \int \left(\int f(x-y)g(y-z)dy\right) h(z)dz.$$

Doing a change of variables ($u = y - z$, $y = u + z$, $du = dy$), we get

$$\int \left(\int f(x-u-z)g(u)du\right) h(z)dz = \int (f * g)(x-z)h(z)dz = (f * g) * h(x).$$

(b) This is just an application of linearity of the integral.

□

Problem 63 (Quiz 4)**.** Suppose $\varphi \in L^1(\mathbb{R})$ is such that $\int \varphi(x)dx = a$, and define $\varphi_t(x) = t^{-1}\varphi(x/t)$. If $f \in L^p(\mathbb{R})$, $p \in [1, \infty)$, then $f * \varphi_t \to af$ in the L^p norm as $t \to 0$.

Proof. Note that

$$\int \varphi_t(x)dx = \int t^{-1}\varphi(x/t)dx = \int \varphi(u)du = a \text{ for all } t > 0,$$

preforming the change of variable $u = x/t$, $du = dx/t$. The same change of variables gives us $\|\varphi_t\|_1 < \infty$. Thus, we see that

$$f * \varphi_t(x) - af(x) = \int f(x-y)\varphi_t(y)dy - \int f(x)\varphi_t(y)dy = \int [\tau_y f(x) - f(x)]\varphi_t(y)dy.$$

We take the p norm to get

$$\begin{aligned}
\|f * \varphi_t(x) - af(x)\|_p &= \left(\int \left|\int [\tau_y f(x) - f(x)]\varphi_t(y)dy\right|^p dx\right)^{1/p} \\
&\le \left(\int \left(\int |\tau_y f(x) - f(x)||\varphi_t(y)|dy\right)^p dx\right)^{1/p} \\
&\le \int \left(\int |\tau_y f(x) - f(x)|^p |\varphi_t(y)|^p dx\right)^{1/p} dy \\
&= \int |\varphi_t(y)|\|\tau_y f - f\|_p dy,
\end{aligned}$$

where the first inequality comes from the triangle inequality and the second from Minkowski for integrals. Notice that the inside is bounded by $|\varphi_t(y)|2\|f\|_p \in L^1(\mathbb{R})$, so DCT applies to bring the limit inside. In other words, we have

$$\limsup_{t\to 0} \|f * \varphi_t(x) - af(x)\|_p \le \lim_{t\to 0} \int |\varphi_t(y)|\|\tau_y f - f\|_p dy = \int \lim_{t\to 0} |\varphi_t(y)|\|\tau_y f - f\|_p dy.$$

Now, writing out the inside, we have

$$\int \lim_{t\to 0} |\varphi_t(y)|\|\tau_y f - f\|_p dy = \int \lim_{t\to 0} t^{-1}|\varphi(y/t)|\|\tau_y f - f\|_p dy.$$

Let $z = y/t$, $dz = dy/t$, so we get that this is equal to

$$\int \lim_{t\to 0} |\varphi(z)| \|\tau_{zt} f - f\|_p dz.$$

Note the L^p norm is continuous with respect to translation, so we get that the inside will be 0. In other words, we have that the integral comes out to zero, so

$$\liminf_{t\to 0} \|\varphi_t * f - af\|_p = \limsup_{t\to 0} \|\varphi_t * f - af\|_p = 0.$$

Thus, we have that it converse in the p norm. □

Problem 64 (Quiz 5)**.** Suppose $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are complex valued functions and $\widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then

(a) $\int \widehat{f} g = \int f \widehat{g}$,
(b) $\langle f, g\rangle_2 = \langle \widehat{f}, \widehat{g}\rangle_2$.

Proof. Note that Plancherel says the Fourier transform on L^2 agrees with the usual Fourier transform on L^1 on the set $L^1 \cap L^2$.

(a) Using the above remark, we can use the usual Fourier transform definition for L^1. Thus, we have

$$\begin{aligned}\int \widehat{f}(y) g(y) dy &= \int \left(\int f(z) e^{-2\pi i y\cdot z} dz \right) g(y) dy \\ &= \iint f(z) g(y) e^{-2\pi i y \cdot z} dz dy.\end{aligned}$$

Using Tonelli and taking absolute values, we note this is integrable, so we see that we can use Fubini to iterate the integral. We have that the above is equal to

$$\iint f(z) g(y) e^{-2\pi i y\cdot z} dy dz = \int f(z) \widehat{g}(z) dz.$$

(b) One could just say this follows by Plancherel. If you want to see the calculation, we set $h = \overline{\widehat{g}}$, and we note that

$$\begin{aligned}\widehat{h}(y) = \int h(z) e^{-2\pi i z\cdot y} dz &= \int \overline{\widehat{g}}(z) e^{-2\pi i z \cdot y} dz \\ &= \overline{\int \widehat{g}(z) e^{2\pi i z\cdot y} dz} \\ &= \overline{\widehat{g}^{\vee}}(y) = \overline{g}(y)\end{aligned}$$

where the last equality is interpreted as almost everywhere equivalence, and the second to last equality is a result of Fourier inversion. So using (a) and this, we see that

$$\langle f, g\rangle_2 = \int f\overline{g} = \int f \widehat{h} = \int \widehat{f} h = \int \widehat{f}\overline{\widehat{g}} = \langle \widehat{f}, \widehat{g}\rangle_2.$$

□

Problem 65 (Quiz 6)**.** Suppose $f \in L^1(\mathbb{T})$. Then $|\widehat{f}(m)| \to 0$ as $|m| \to \infty$.

Proof. By a consequence of Fejérs theorem, we have that trigonometric polynomials are dense in $L^1(\mathbb{T})$ (alternatively, invoke Stone-Weierstrass). Fix $\epsilon > 0$. By the prior remark, there exists a trigonometric polynomial P such that $\|f - P\|_1 < \epsilon$. Let M denote the degree of P. By hypothesis,

$M < \infty$. Notice now that for $|m| > M$, we have that $\widehat{P}(m) = 0$ (this is a consequence of problems from prior homeworks). Hence, we see that

$$|\widehat{f}(m)| = |\widehat{f}(m) - \widehat{P}(m)| \leq \|f - P\|_1 < \epsilon,$$

using the fact that $\|\widehat{f}\|_u < \|f\|_1$. We can do this for all $\epsilon > 0$, so we see that $|\widehat{f}(m)| \to 0$ as $|m| \to \infty$. $\square$

Remark. Note that this just says it goes to 0, not how fast it tends to 0. We can make this go to 0 arbitrarily slow by future homeworks.