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$e^{At} = S e^{\Lambda t} S^{-1}$ $\qquad e^{At} = I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^n}{n!} + \cdots$

$e^x = \sum_0^\infty \frac{x^n}{n!}$

§24 Markov 矩阵 Fourier 级数

Markov matrices. $A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$

① All entries $\geq 0$ ② All columns add to 1

1. $\lambda = 1$ is an eigenvalue $\qquad$ 2. All other $|\lambda_i| < 1$

$A - 1I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$ $\qquad$ All columns add to zero of $A-I$ → $A-I$ is singular

$\begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix} \begin{bmatrix} 0.6 \\ 33.00 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

projections with orthonormal basis

$V = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$ $\qquad Qx = V$

$[q_1 \; q_2 \cdots q_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = V$ $\qquad x = Q^{-1}V = Q^T V$

Fourier Series $\qquad f(x) = f(x + 2\pi)$ periodic

$f(x) = a_0 1 + a_1 \cos x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

$V^T W = V_1 W_2 + \cdots + V_n W_n$ $\qquad f^T g = \int_0^{2\pi} f(x) g(x)\,dx$

$\int_0^{2\pi} \sin x \cos x\,dx = \frac{1}{2}(\sin x)^2 \Big|_0 = 0$

$a_1 \int_0^{2\pi} (\cos x)^2 dx = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos x\,dx$ $\qquad x = \pi$

§25 对称矩阵及正定性

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$A=A^T$ ① The eigenvalues are real ② Eigenvectors are perpendicular can be chosen

usual case: $A=S\Lambda S^{-1}$ symmetric $A=Q\Lambda Q^{-1}=Q\Lambda Q^T$

对称矩阵可分解成 正交矩阵 × 对角矩阵 × 正交矩阵$^T$

Why real eigenvalues: $Ax=\lambda x \xrightarrow{\text{always}} A\bar{x}=\bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A=\bar{x}^T\bar{\lambda}$

$\bar{x}^T A x=\bar{x}^T\bar{\lambda}x$ $\bar{x}^T A x=\lambda\bar{x}^T x$

$\Rightarrow \lambda\bar{x}^T x=\bar{\lambda}\bar{x}^T x \Rightarrow \lambda=\bar{\lambda}$ $\lambda$ is real

$A=A^T$ $A=Q\Lambda Q^T=\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{bmatrix}\begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$

$=\lambda_1 q_1 q_1^T+\lambda_2 q_2 q_2^T+\cdots$

Every sym matrix is a comb of perp. projection matrix

Signs of pivots same as signs of $\lambda$'s

# pivots = # positive $\lambda$'s

Positive definite symmetric matrix 正定矩阵

All eigenvalues are positive; all pivots are positive

$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ pivots: $5, \frac{11}{5}$ $\lambda=4\pm\sqrt{5}$

all subdeterminants are poltive

复数矩阵和快速傅里叶变换 FFT

$\begin{bmatrix} \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \cdots & \bar{z}_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}=|z_1\ z_2 \cdots|^2$ $\bar{z}_1 z_1=|z_1|^2$

$\bar{z}^T z$ $z^H z=|z_1|^2+|z_2|^2+\cdots+|z_n|^2$

(Hermitian)

symmetric $A^T = A$ real matrice

$\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$ $\bar{A}^T = A$ complex ~ $A^H = A$

perpendicular $q_1, q_2 \dots q_n$

$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $Q^TQ = I = Q^HQ$ 酉矩阵 unitary

$$F_n = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n-1)^2} \end{bmatrix}$$

$F_{ij} = w^{ij}$

$w^n = 1$

$w = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$n=4$ $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$ $F_4^H F = I$

FFT 先需 $n^2$ 次乘法 只需 $\frac{1}{2} n \log_2 n$

## §28 正定矩阵和最小值

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ① $\lambda_1 > 0$. $\lambda_2 > 0$ ② $a > 0$. $ac - b^2 > 0$

③ pivots $a > 0$ $\frac{ac-b^2}{a} > 0$ ④ $x^TAx > 0$

Example: $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ positive semi definite 半正定 $\lambda = 0, 20$

pivots = 2. $\underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{x^T} \underbrace{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = 2x_1^2 + 12x_1x_2 + 18x_2^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ $f(x,y) = 2x^2 + 12xy + 20y^2 = \underline{2}(x + \underline{3}y)^2 + 2y^2$

消元 $\underbrace{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}}_{U}$ $\underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L}$

Example 3×3

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$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ dets 2, 3, 4 pivots 2, $\frac{3}{2}$, $\frac{4}{3}$

eigenvalues: $2-\sqrt{2}$, 2, $2+\sqrt{2}$

$f = x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$

## §28 相似矩阵和若尔当型

A & B are similar $n \times n$ matrices

for some M. $B = M^{-1}AM$

Example: A is similar to $\Lambda$ $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$ $\lambda = 3, 1$

Similar matrices have same $\lambda$'s!

$Ax = \lambda x$ $(M^{-1}AM)M^{-1}x = \lambda M^{-1}x$ $B = M^{-1}AM$

$BM^{-1}x = \lambda M^{-1}x$ Eigenvector of B is $M^{-1}$

$\lambda_1 = \lambda_2 = 4$ one family $M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

big family $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ⇐ Jordan form

Jordan block $J_i = \begin{bmatrix} \lambda_i & 1 & & \\ \vdots & \lambda_i & 1 & \\ 0 & & \ddots & \lambda_i \end{bmatrix}$

Every square A is similar to a Jordan matrix J

$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \ddots \; \boxed{J_d} \end{bmatrix}$ #blocks = # of eigen~~values~~ vectors

## §30 SVD

symmetric positive def $A = Q\Lambda Q^{-1}$

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scaling factor

$A[v_1 v_2 \cdots v_n] = [u_1 u_2 \cdots u_n]\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}$

$AV = U\Sigma \qquad A = U\Sigma V^{-1} = U\Sigma V^T$

$A^TA = V\Sigma^T U^T U \Sigma V = V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \ \sigma_n^2 \end{bmatrix} V^T$

$32\begin{bmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{bmatrix} \quad 18\begin{bmatrix} 1/\sqrt2 \\ -1/\sqrt2 \end{bmatrix}$

exp. $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \qquad ev = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underset{U}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1/\sqrt2 & 1/\sqrt2 \\ 1/\sqrt2 & -1/\sqrt2 \end{bmatrix}}$

$AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

$AA^T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad AA^T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Exp 2 $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \underset{U}{\begin{bmatrix} 1/\sqrt5 & 2/\sqrt5 \\ 2/\sqrt5 & -1/\sqrt5 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}} = \frac{1}{\sqrt5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} \qquad 0, 125$

$v_1 \ldots v_r$ Orthonormal basis for row space $\qquad Av_i = \sigma_i u_i$

$u_1 \ldots u_r$ — — — col space

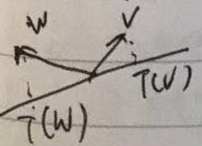
$v_{r+1} \ldots v_n$ — — — null space

$u_{r+1} \ldots u_m$ — — — $N(A^T)$

§7.1 线性变换及对应之矩阵

Rules: $T(v+w) = T(v) + T(w) \qquad T(cv) = cT(v)$

$T(cv+dw) = cT(v) + dT(w)$




Exp 1. Projection. $T: R^2 \to R^2$ 映射

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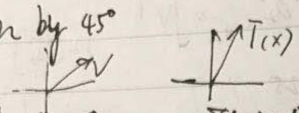
Exp 2: shift whole plane

$T(v) = \|v\|$  $T: R^2 \to R^1$

Exp: Rotation by 45°

$T: R^2 \to R^2$




Exp 3: Matrix $A$  $T(v) = Av$

Info needed to know $T(v)$ for all inputs

$T(v_1), T(v_2) \cdots T(v_n)$ for any input basis $v_1, v_2 \cdots v_n$

Every $v = c_1 v_1 + \cdots + c_n v_n$  know $T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$

Coordinates come from a basis

$v \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\text{input coords}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{output coords}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$  eigenvector basis leads to diagonal matrix $\Lambda$

Rule to find $A$  Given bases $v_1 - v_n$, $w_1 - w_m$

1st col of $A$  Apply $T(v_1) = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

2nd col of $A$  $T(v_2) = a_{12} w_1 + \cdots + a_{m2} w_m$

$T = \frac{d}{dx}$  Input: $c_1 + c_2 x + c_3 x^2$  basis: $1, x, x^2$

Output: $c_2 + 2c_3 x$  $\cdots : 1, x$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$  ~~A = [~~

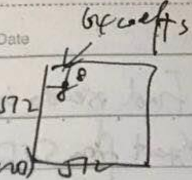
## §31 基变换和图像压缩

standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

Fourier basis 8x8

$[\ \vdots\ ]\ \begin{bmatrix} w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$

signal → coeffs c → compression → $\hat{c}$ (many zeros)




wavelets 小波

$p = c_1 w_1 + \dots + c_8 w_8$

$p = Wc$ $\quad c = W^{-1}p$ $\quad$ Good basis ① fast FFT ② few is enough

change of basis

columns of W = new basis vectors

$[x]$ old basis → $[c]$ new basis $\quad x = Wc$

## §33 左右逆和伪逆

2-sided inverse $\quad AA^{-1} = I = A^{-1}A \quad r = m = n$ full rank

left inverse $\quad$ full col rank $r = n \quad$ null space $= \{0\}$

indep. cols / 0 or 1 solution to $Ax = b \quad r = n - m$

$\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}}A = I \qquad A^{-1}_{left\ n\times m}A_{m\times n} = I$

right inverse. $\quad$ full row rank $\quad r = m < n$

$N(A^T) = \{0\}$ indep rows $\quad \infty$ solutions to $Ax = b$.

$n-m$ free variables $\quad A\underbrace{A^T(AA^T)^{-1}} = I$

$AA^{-1}_{right} = I$

Pseudo inverse.

Rule: If $x \neq y$ both in row space then col space $Ax \neq Ay$

Proof: Suppose $Ax = Ay$ $\quad$ in null space $A(x-y) = 0$ (in row)

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Find pseudo inverse $A^+$

① Start from SVD: $A = U \Sigma V^T$ $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & \cdots & \\ 0 & & \ddots & \\ \vdots & \cdots & & \sigma_r \end{bmatrix}_{m \times n}$

$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 & 0 \\ 0 & 1/\sigma_2 & & \\ 0 & \cdots & & 1/\sigma_n \end{bmatrix}_{n \times m}$ $\Sigma\Sigma^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & \cdots & \\ 0 & 0 & 1 & 0 \end{bmatrix}_{n \times n}$

$A^+ = V \Sigma^+ U^T$

§34 Review

| | | $m$ | $n$ | $r$ |
|---|---|---|---|---|
| 1. Given $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ | no soln | 3 | | $r < m$ |
| $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ | has 1 soln | 3 | | $r = n < 3$ |

$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. TF
1) $\det AA^T = \det A^TA$ ? F
2) $A^TA$ is invertible T if $r = n$
3) $AA^T$ is pos definite F