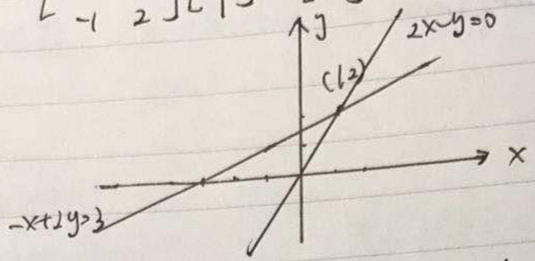
Linear Algebra

§1 方程组的几何解释

$\left\{\begin{array}{l} 2x-y=0 \\ -x+2y=3 \end{array}\right.$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ ~~Ax+b=Y~~ $Ax=b$

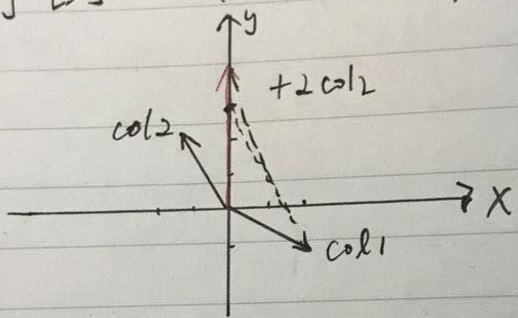
row picture



$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ linear combination of columns

1 2

(1,2)



$\left\{\begin{array}{l} 2x-y=0 \\ -x+2y-z=-1 \\ -3y+4z=4 \end{array}\right.$ $A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$ $b=\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$ (0, 0, 1)

§2 消元法

elimination

first pivot 主元

$\left\{\begin{array}{l} x+2y+z=2 \\ 3x+8y+z=12 \\ 4y+z=2 \end{array}\right.$ $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{(2,1)} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$

$Ax=b$ A b

back - substitution

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$$\begin{matrix} \boxed{1} & 2 & 1 & | & 2 \\ 0 & \boxed{2} & -2 & | & 6 \\ 0 & 0 & \boxed{5} & | & -10 \end{matrix}$$

(3,2) →

u c

$x+2y+z=2$

$2y-2z=6$

$5z=-10$

Matrices. substract 3 + row 1 from row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

E_{21}

Step 2: substract $2 \times$ row 2 from row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

E_{32}

$E_{32}(E_{21}A)=u$ $(E_{32}E_{21})A=u$

Permutation - exchange row 1 & row 2 置换矩阵

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Inverses 逆矩阵

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

§3 乘法和逆矩阵

Standard rule: $C_{34} = (\text{row } 3 \text{ of } A)(\text{col } 4 \text{ of } B) = \sum_{k=1}^{n} a_{3k} b_{k4}$ ①

$$\underset{A_{m\times n}}{[\ \ ]}\ \underset{B_{n\times p}}{[\ \ ]} = \underset{C_{m\times p}}{[\ \ ]}$$

combinations of cols of A — cols of C ②

______ rows of B — rows of C ③

④ AB = sum of (cols of A) × (rows of B)

⑤ block $\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} A_1B_1+A_2B_3 & A_1B_2+A_2B_4 \\ \hline A_3B_1+A_4B_3 & A_3B_2+A_4B_4 \end{array}\right]$

Inverses (square matrices)

$A^{-1}A = I = AA^{-1}$ ~~non sig~~ singular

$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $AX=0$ Singular

Gauss-Jordan (solve 2 equations at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]_{A \quad I} \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]_{I \quad A^{-1}}$$

§4 A的LU分解

$(A^{-1})^T A^T = I = AA^{-1}$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$$

E_{32} E_{21}

$EA=U$

$A=LU$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}=E\ (\text{left of } A)$$

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}=L\ (\text{left of } U)$$

§5 转置—置换—向量空间R

Permutations $A=LU$ $PA=LU$

P – identity matrix with reordered rows

$P^{-1}=P^T$ $P^TP=I$

R^TR is always symmetric.

$$\begin{bmatrix}1&3\\2&3\\4&1\end{bmatrix}\begin{bmatrix}1&2&4\\3&3&1\end{bmatrix}=\begin{bmatrix}10&11&7\\11&17&11\\7&11&17\end{bmatrix}$$

$(R^TR)^T=R^TR$

Vector spaces Examples R^2 = all 2-dim real vectors = x-y plane

R^n = all column vectors with n real components

Subspaces of R^2: ① all of R^2 ② any line through $\begin{bmatrix}0\\0\end{bmatrix}$

③ zero vector only

§6 列空间和零空间

2 subspaces, P & L

$P\cup L$ = all vectors in P or L This is NOT a subspace or both

$P\cap L$ = all vectors in both P & L

Subspaces S & T: $S\cap T$ is a subspace

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Column space of A is subspace of R^4 = all linear combs. of cols.

$$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$ col1 + col2 = col3. col3 线性相关.

pivot cols

Null Space $N(A)$ contains $\begin{bmatrix}0\\0\\0\end{bmatrix}$, $c\begin{bmatrix}1\\1\\-1\end{bmatrix}$

§7 Ax=0 主变量 特解

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\rightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}\rightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

rank of A = # of pivots (2)

$$x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

R = reduced row echelon form

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&2&0&-2\\0&0&\boxed{1}&2\\0&0&0&0\end{bmatrix}\rightarrow R=\mathrm{rref}(A)$$

$$A=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3\\0&2&2\\0&0&0\\0&0&0\end{bmatrix}$$ free

$x_1+2x_2+3x_3=0$

$2x_2+2x_3=0$

$$x=c\begin{bmatrix}-1\\-1\\1\end{bmatrix}=c\begin{bmatrix}-F\\I\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}_{I}&\begin{bmatrix}1\\1\end{bmatrix}\\0\quad 0&0\\0\quad 0&0\end{bmatrix}$$

§8 Ax=b 可解性和解的结构

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$$\begin{cases} x_1+2x_2+2x_3+2x_4=b_1 \\ 2x_1+4x_2+6x_3+8x_4=b_2 \\ 3x_1+6x_2+8x_3+10x_4=b_3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 & 1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 & 3 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 & 0 \end{bmatrix} \quad b=\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

Solvability condition on b

$Ax=b$ solvable when b is in $C(A)$. If a comb. of rows of A gives zero row, then same comb. of entries of b must give 0.

To find complete solutions ~~of~~ to $Ax=b$

① $X_{particular}$: set all free variables to zero. Solve $Ax=b$ for pivot variables

$$\begin{cases} x_1+2x_3=1 \\ 2x_3=3 \end{cases} \quad x_p=\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} \qquad \begin{aligned} Ax_p&=b \\ \underline{Ax_n}&\underline{=0} \\ A(x_p+x_n)&=b \end{aligned}$$

$$X_{complete}=\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}+c_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+c_2\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

m by n matrix A of rank r ($r\le m$, $r\le n$)

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

Full column rank means $r=n$ $\quad N(A)=\{\text{zero vector}\}$

Solution, $Ax=b$: $x=x_p$ unique solutions if it exists (0 or 1 solution)

Full row rank means $r=m$ can solve $Ax=b$ for every b

left with $n-r$ free variables

$$A=\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \qquad R=\begin{bmatrix} 1 & 0 & \overbrace{\begin{matrix} - & - \\ - & - \end{matrix}}^{\text{free}} \end{bmatrix} \text{一般简形}$$

Full rank $m=n=r$

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad R=I \qquad \text{b可以很多值}$$