$r=m=n$ $\quad R=I$ $\quad$ 1 solution to $Ax=b$

$r=n<m$ $\quad R=\begin{bmatrix} I \\ 0 \end{bmatrix}$ $\quad$ 0 or 1 solution

$r=m<n$ $\quad R=[I \; F]$ $\quad$ 1 or ∞ solution

$r<m, r<n$ $\quad R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $\quad$ 0 or ∞ solution

§9 线性相关性.基.维数

Independence: Vectors $x_1, x_2 \cdots x_n$ are independent if no combination gives zero vector (except the zero comb, all $c_i=0$)

$c_1x_1+c_2x_2+\cdots+c_nx_n \neq 0$

When $V_1, V_2 \cdots V_n$ are columns of A

They are independent if null space of A is {zero vector}

They are dependent if $Ac=0$ for some non zero c

Vectors: $V_1, V_2 \cdots V_l$ span a space

means: The space consists of all combs of those vectors

Basis for a space is a sequence of vectors $V_1, V_2 \cdots V_d$ with 2 properties

1) They are independent

2) They span the space

Examples space is R^3 one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$

R^n n vectors give basis if the $n\times n$ matrix with those cols is invertible 可逆

Every basis for the space has the same number of vectors

Dimension of the space

$\text{rank}(A) = \#$ pivot cols $=$ dimension of $C(A)$

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§10 四个基本子空间

4 subspaces: column space $C(A)$ R^m nullspace $N(A)$ R^n

row space = all combs of rows = all combs of columns of $A^T = C(A^T)$ R^n

nullspace of $A^T = N(A^T)$ = left null space of A R^m

R^n $\begin{cases} \text{row space} & \dim C(A^T) = r \\ \text{null space} & \dim N(A) = n-r \end{cases}$ R^m $\begin{cases} \text{column space} & \dim C(A) = r \text{ (rank)} \\ \text{null space of } A^T & \dim N(A^T) = m-r \end{cases}$

	$C(A)$	$N(A)$
basis	pivot cols	special solution
dimension	r	n-r

$$\underbrace{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}}_{A} \longleftrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

§11 矩阵空间、秩1矩阵和小世界图

$S \cap U$ = symmetric & upper triangular = diagonal 3×3

$\dim(S\cap U) = 3$ $\dim(S+U) = 9$

sum $S+U$ = any element of S + any element of U = all 3×3's

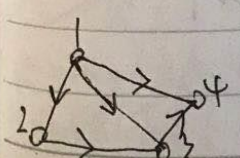
$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$ rank 1 matrix $A = uv^T$

S = all v in R^4 with $v_1+v_2+v_3+v_4 = 0$ rank = 1

S = null space of $A = [1\ 1\ 1\ 1]$ $\dim N(A) = n-r = 4-1 = 3$

Graph = {nodes, edges}

§12 图和网络



$n=4$ nodes

$m=5$ edges

$$A = \begin{array}{c} \text{node } 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$Ax=0$

$$Ax=\begin{bmatrix}x_2-x_1\\x_3-x_2\\x_3-x_1\\x_4-x_1\\x_4-x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$x = x_1, x_2, x_3, x_4$ potentials at nodes

$A\downarrow$

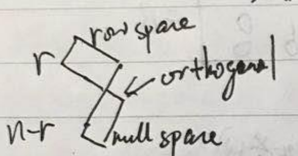
x_2-x_1, etc. potential differences

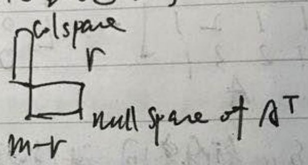
rank = 3 $\qquad A^Ty=0$

$$\begin{bmatrix}-1&0&-1&-1&0\\1&-1&0&0&0\\0&1&1&0&-1\\0&0&0&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

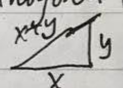
$\dim N(A^T)=m-r=5-3=2$

§1.4 正交向量与子空间





orthogonal vectors



$x^Ty=0 \qquad \|x\|^2+\|y\|^2=\|x+y\|^2$

$x=\begin{bmatrix}1\\2\\3\end{bmatrix} \quad y=\begin{bmatrix}2\\-1\\0\end{bmatrix} \quad x+y=\begin{bmatrix}3\\1\\3\end{bmatrix}$

$\|x\|^2=14 \quad \|y\|^2=5 \quad \|x+y\|^2=19$

证明: $\cancel{x^Tx+y^Ty}=(x+y)^T(x+y)=\cancel{x^Tx+y^Ty}+2x^Ty+y^Tx$

$\Rightarrow x^Ty=0$

~~Step~~ subspace S is orthogonal to subspace T

means: every vector in S is orthogonal to every vector in T

nullspace & row space are orthogonal complements in R^n

$\Rightarrow$ Null space contains all vectors $\perp$ row space

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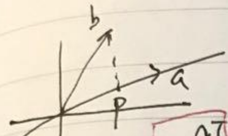
$N(A^TA)=N(A)$ rank of A^TA = rank of A

A^TA is invertible exactly if A has independent columns.

§15 子空间投影

p - projection of b

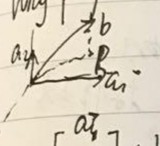


$a^T(b-xa)=0$

$x a^Ta = a^Tb$ $x=\frac{a^Tb}{a^Ta}$ $p=ax$

$p=a\boxed{\frac{a^Tb}{a^Ta}}$ ★ $P^T=P$ $P^2=P$ 投影公式

Why project? $Ax=b$ may have no solution



$e=b-p$ $p=A\hat{x}$ find $\hat{x}$ key: $b-A\hat{x}$ is perp. to plane

$a_1^T(b-A\hat{x})=0$ $a_2^T(b-A\hat{x})=0$

$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b-A\hat{x})=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $A^T\underbrace{(b-A\hat{x})}_{e}=0$ $e\perp C(A)$

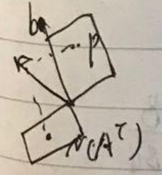
$A^TA\hat{x}=A^Tb$ $\hat{x}=(A^TA)^{-1}A^Tb$ $p=A\hat{x}=A(A^TA)^{-1}A^Tb$

Least square fitting by a line

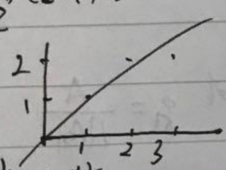
§16 投影矩阵和最小二乘

Projection matrix: $P=A(A^TA)^{-1}A^T$

If b in column space $Pb=b$ If $b\perp$ column space $Pb=0$



$p+e$ $e=(I-P)b$



$y=C+DX$ 方程无解 但可有最优解

$\begin{cases} C+D=1 \\ C+2D=2 \\ C+3D=2 \end{cases}$ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

minimize $\|Ax-b\|^2=\|e\|^2$

find $\hat{x}=\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P $\boxed{A^TA\hat{x}=A^Tb}$

$(C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2$

$\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\left[\begin{array}{cc|c}1&1&1\\1&2&2\\1&3&2\end{array}\right]=\left[\begin{array}{cc|c}3&6&5\\6&14&11\end{array}\right]$ $\begin{cases}3C+6D=5\\6C+14D=11\end{cases}\Rightarrow\begin{cases}C=\frac{2}{3}\\D=\frac{1}{2}\end{cases}$

最优线: $\frac{2}{3}+\frac{1}{2}t=y$ $e=[-\frac{1}{6}\ \frac{2}{6}\ -\frac{1}{6}]$

$b=p+e$ $\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}7/6\\10/6\\13/6\end{bmatrix}+\begin{bmatrix}-\frac{1}{6}\\\frac{2}{6}\\-\frac{1}{6}\end{bmatrix}$

If A has independant columns, then A^TA is invertible.

suppose $A^TAX=0$ $X^TA^TAX=(AX)^TAX\Rightarrow AX=0\Rightarrow X=0$

Columns definitely independant if they are perp. unit vectors

orthonormal vectors 标准正交

§17 正交矩阵和Gram-Schmidt正交化

orthonormal vectors: $q_i^Tq_j=\begin{cases}0 & \text{if } i\neq j\\1 & \text{if } i=j\end{cases}$

$Q=[q_1\ q_2\cdots q_n]$ $Q^TQ=\begin{bmatrix}q_1^T\\\vdots\\q_n^T\end{bmatrix}[q_1 q_2\cdots q_n]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

If Q is square then $Q^TQ=I$ $Q^T=Q^{-1}$

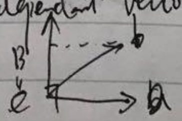
Example: $Q=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$ $Q=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

Q has orthonormal columns Project onto its column space

$P=Q(Q^TQ)^{-1}Q^T=QQ^T$

Gram-Schmidt

independant vectors $a, b\to$ orthonormal $q_1=\frac{A}{\|A\|}$ $q_2=\frac{B}{\|B\|}$



$B=b-\frac{A^Tb}{A^TA}A$

$A^TB=A^T(b-\frac{A^Tb}{A^TA}A)=0$

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$a=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $b=\begin{bmatrix}1\\0\\2\end{bmatrix}$ $B=\begin{bmatrix}1\\0\\2\end{bmatrix}-\frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$A=\begin{bmatrix}1&1\\1&0\\1&2\end{bmatrix}$ $Q=[q_1\ q_2]=\begin{bmatrix}1/\sqrt{3}&0\\1/\sqrt{3}&-\frac{1}{\sqrt{2}}\\1/\sqrt{3}&1/\sqrt{2}\end{bmatrix}$ $A\perp B.$

§18 行列式及其性质

Determinants $\det A=|A|$

① $\det I=1$ ② Exchange rows: reverse sign of det

③ $\begin{vmatrix}ta&tb\\c&d\end{vmatrix}=t\begin{vmatrix}a&b\\c&d\end{vmatrix}$ $\begin{vmatrix}a+a'&b+b'\\c&d\end{vmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}+\begin{vmatrix}a'&b'\\c&d\end{vmatrix}$

④ 2 equal rows $\rightarrow \det=0$

⑤ substract l row from row k. DET doesn't change

$\begin{vmatrix}a&b\\c-la&d-lb\end{vmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}+\begin{vmatrix}a&b\\-la&-lb\end{vmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}-l\begin{vmatrix}a&b\\a&b\end{vmatrix}=0$

⑥ Row of zeros $\rightarrow \det A=0$

⑦ $\det U=\begin{vmatrix}d_1&*&*&*\\0&d_2&*&*\\0&0&d_3&*\\0&0&0&d_4\end{vmatrix}=d_1d_2\cdots d_n$. product of pivots

⑧ $\det A=0$ when A is singular

$\det A\neq 0$ when A is invertible

⑨ $\det AB=(\det A)(\det B)$ $\det 2A=2^n\det A$

⑩ $\det A^T=\det A$ $|A^T|=|A|$ $|U^TL^T|=|LU|$ $|U^T||L^T|=|L||U|$

§19 行列式公式和代数余子式

$\begin{vmatrix}a&b\\c&d\end{vmatrix}=\begin{vmatrix}a&0\\c&d\end{vmatrix}+\begin{vmatrix}0&b\\c&d\end{vmatrix}=\begin{vmatrix}a&0\\c&0\end{vmatrix}+\begin{vmatrix}a&0\\0&d\end{vmatrix}+\begin{vmatrix}0&b\\c&0\end{vmatrix}+\begin{vmatrix}0&b\\0&d\end{vmatrix}=ad-\cancel{bd}bc$

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Date big formula

$\det A = \sum_{n!\text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$

$(\alpha, \beta, \gamma \cdots \omega) = $ perm of $(1, 2, \cdots n)$

Cofactors 代数余子式

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Cofactor of $a_{ij} = \pm \det\begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{ erased} \\ \text{col } j \end{pmatrix} = C_{ij}$

$i+j$ even $(+)$ $i+j$ odd $(-)$

Cofactor formula (along row 1)

$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$

§20 Cramer's 法则 逆矩阵 体积 $|\det A| = $ Volume of box

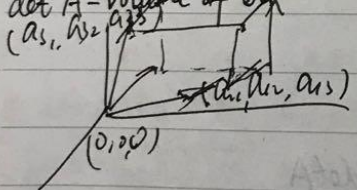
$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{1}{ad-bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$ $A^{-1} = \frac{1}{\det A} C^T$

$Ax = b$ $x = A^{-1} b = \frac{1}{\det A} C^T b$

Cramer's rule: $x_1 = \frac{\det B_1}{\det A}$ $x_j = \frac{\det B_j}{\det A}$

A with col 1 replaced by $b = B_1 = [b \;\; n-1 \text{ columns of } A]$

$\det A = $ ~~volume of box~~



area $= \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ for paral...

$\frac{1}{2}(ad-bc)$ for △

§21 特征向量/值

Ax parallel to x $Ax = \lambda x$

If A is singular, $\lambda = 0$ is eigen value

Any x in plane, $Px = x$ $\lambda = 1$

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Any $x \perp$ plane: $Px = 0x$ $\lambda = 0$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 1$ $Ax = x$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda = -1$ $Ax = -x$

How to solve $Ax = \lambda x$? $\underset{\text{singular}}{(A - \lambda I)} x = 0$

$\det(A - \lambda I) = 0$

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

特征向量 $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Example $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\det(Q - \lambda I) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$ $\left\}\begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}\right.$

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = (3 - \lambda)^2$ $\lambda_1 = 3$ $\lambda_2 = 3$ 特征向量 $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

退化矩阵 degenerate matrix

§22 对角化和A的幂

suppose n indep. eigenvectors of A. Put them in columns of S

$AS = A[x_1 x_2 \cdots x_n] = [\lambda_1 x_1 \cdots \lambda_n x_n] = [x_1 x_2 \cdots x_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \lambda_2 \cdots & 0 \\ 0 & \cdots & \lambda_n \end{bmatrix} = S\Lambda$ diagnal matrix

eigen value

$AS = S\Lambda$ $S^{-1}AS = \Lambda$ $A = S\Lambda S^{-1}$

If $Ax = \lambda x$ $A^2x = \lambda Ax = \lambda^2 x$

$\to A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$ $A^k = S\Lambda^k S^{-1}$

Theorem $A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$

A is sure to have n indep evectors (and can be diagonalizable)

if all the λ's are different (no repeated λ's)

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Repeated eigenvalues // may or may not have n indep eVectors

$A=\begin{bmatrix}2&1\\0&2\end{bmatrix}$ $\det(A)=\begin{vmatrix}2-\lambda&1\\0&2-\lambda\end{vmatrix}=(2-\lambda)^2$ $A-2I=\begin{bmatrix}0&1\\0&0\end{bmatrix}$

$\lambda=2$ $x_1=\begin{bmatrix}1\\0\end{bmatrix}$

Equation $u_{k+1}=Au_k$ start with given vector u_0

$u_1=Au_0$, $u_2=A^2u_0$ $u_k=A^ku_0$

To really solve: write $u_0=c_1x_1+c_2x_2+\dots+c_nx_n$

$Au_0=c_1\lambda_1x_1+c_2\lambda_2x_2+\dots+c_n\lambda_nx_n=\Lambda^{10}Sc=u_n$

Fibonacci: 0, 1, 1, 2, 3, 5, 8, … $F_{100}=?$

$F_{k+2}=F_{k+1}+F_k$ $u_k=\begin{bmatrix}F_{k+1}\\F_k\end{bmatrix}$ $u_{k+1}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\underbrace{\begin{bmatrix}F_{k+1}\\F_k\end{bmatrix}}_{u_k}$

$F_{k+1}=F_{k+1}$

§2.3 微分方程和exp(At)

$\frac{du}{dt}=Au$ $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ $\frac{du_1}{dt}=-u_1+2u_2$ $\frac{du_2}{dt}=u_1-2u_2$

$\to A=\begin{bmatrix}-1&2\\1&-2\end{bmatrix}$ $|A-\lambda I|=\begin{vmatrix}-1-\lambda&2\\1&-2-\lambda\end{vmatrix}=\lambda^2+3\lambda=0$

$\lambda=0,-3$ $\lambda_1=0$ $x_1=\begin{bmatrix}2\\1\end{bmatrix}$ $\lambda_2=-3$ $x_2=\begin{bmatrix}1\\-1\end{bmatrix}$

Solution: $u(t)=c_1e^{\lambda_1 t}x_1+c_2e^{\lambda_2 t}x_2$

At $t=0$ $c_1\begin{bmatrix}2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ $c_1=c_2=\frac{1}{3}$

Steady state $=\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}$

① Stability $u(t)\to 0$ need $e^{\lambda t}\to 0$, $\lambda<0$

② steady ~~status~~ state $\lambda_1=0$ and other Real part $\lambda<0$

③ Blow up if any Re $\lambda>0$

$\frac{du}{dt}=Au$ $\frac{dv}{dt}=S^{-1}ASv=\Lambda v$ $v(t)=e^{\Lambda t}v(0)$

$u(t)=Se^{\Lambda t}S^{-1}u(0)$