$L^2 \times L^2 \times L^2 \to L^{2/3}$ boundedness for trilinear multiplier operator and higher multilinearity

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Abstract

This paper discusses the boundedness of the trilinear multiplier operator, as well as that of the general multilinear multiplier operator, when the multiplier satisfies a certain degree of smoothness but with no decaying condition and is L^q-integrable, with an admissible range of q.

Contents

1 Introduction

Let $d \geqslant 1$. Let $m(\xi, \eta, \delta)$ be a function on $\mathbb{R}^{3d}$. Define an operator T_m as follows.

$$T_m(f, g, h)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(\xi, \eta, \delta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\delta) e^{2\pi i x \cdot (\xi + \eta + \delta)} \, d\xi d\eta d\delta.$$

This paper aims to give an explicit result on the $L^2 \times L^2 \times L^2 \to L^{2/3}$ boundedness of T_m given (restricted) smoothness and integrability of m. The author anticipates the use of such result in an upcoming project. At the time of starting this paper, the author was not aware of any such explicit result on trilinear multiplier operator on the market. This paper also discusses similar boundedness of the l-multilinear multiplier operator T_m when m is a function on $\mathbb{R}^{d \otimes l}, l \geqslant 3, d \geqslant 1$, as well as the optimal range of integrability of m on $\mathbb{R}^{d \otimes l}$. Namely, let $m(\xi_1, \cdots, \xi_l)$ be a function on $\mathbb{R}^{d \otimes l}$. Define an operator T_m as follows.

$$T_m(f_1, \cdots, f_l)(x) = \int_{\mathbb{R}^{d \otimes l}} m(\xi_1, \cdots, \xi_l) \hat{f}_1(\xi_1) \cdots \hat{f}_l(\xi_l) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_l)} \, d\xi_1 \cdots d\xi_l.$$

The crucial point here, that will be needed for later use, is the dependence of either operator norm $\|T_m\|$ on $\|m\|_{L^q}$ (see 3.5, 3.6). Unfortunately, due to the lack of duality theory for $L^s, 0 < s < 1$, one can't extend easily this result to other exponents in the Banach range, $L^{p_1} \times \cdots \times L^{p_l} \to L^r$, with $1/p_1 + \cdots + 1/p_l = 1/r$. Such obstacle is not met in the case of bilinear setting; see [7].

There is a body of literature regarding the boundedness of multiplier operator in the linear and bilinear settings, with various conditions on m, ranging from decay to smoothness to integrability. In the bilinear setting, one classical condition on m to guarantee the boundedness of T_m on the Banach range is the Coifman-Meyer condition [2]:

$$|\partial^\alpha m(\xi, \eta)| \leqslant C_\alpha |(\xi, \eta)|^{-|\alpha|}$$

for sufficiently many α. Moreover, in the bilinear setting, if one is to merely impose uniform bounds on derivatives of m: $\|\partial^\alpha m\|_{L^\infty} \leqslant C_0 < \infty$, then one needs to make other compromises. It was shown in [1] that such uniform boundedness on m alone is not sufficient to make T_m a bounded operator on $L^2 \times L^2 \to L^1$. It was shown in [6] that if one further imposes L^2-integrability of m, one can get back a bounded operator. Moreover, the same authors in [6] showed that there is a short range of integrability that one can impose in order to secure boundedness of T_m. This paper follows the ideas in [6]. That means, the multipliers considered here only have uniform derivative bounds plus some integrability.

There are other venues in the multilinear settings, where positive boundednesss results have been established when the multiplier smoothness is much compromised. See [8] for the case of multipliers from Sobolev spaces, and [5] for the case of L^r-based Sobolev spaces.

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2 Notation explanation

$\mathbb{N}_0$: the set of nonnegative integers

$\ominus$: orthogonal complement

$\{F, M\}^{d*}$: d-tuples whose elements are either F or M and at least one of those must be M

$\mathcal{C}_c^J$: the space of all continuous functions whose derivatives up to, and including, order Jth are continuous and which have compact support

κQ: a cube (interval) that has the same center as the cube (interval) Q and κ times the side-length of Q

$|\cdot|$: either means an absolute value or the Euclidean norm of a vector or the cardinality of a discrete set

3 Some multiresolution analysis background

The approach followed in this discussion requires an understanding of multiresolution analysis and wavelets. In this section, necessary background is introduced. The following facts can be found in [10].

First, one starts with an orthonormal basis of $L^2(\mathbb{R})$.

Definition 1. An (inhomogeneous) multiresolution analysis is a sequence $\{V_j : j \in \mathbb{N}_0\}$ of subspaces of $L^2(\mathbb{R})$ such that
a) $V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots$ spans $\bigcup_{j \geqslant 0} V_j = L^2(\mathbb{R})$.
b) $f \in V_0$ iff $f(x-n) \in V_0$ for any $n \in \mathbb{Z}$.
c) $f \in V_j$ iff $f(2^{-j}\cdot) \in V_0$ for $j \in \mathbb{N}$.
d) There exists $\phi_F \in V_0$ such that $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis in V_0.

Because of the properties $(c), (d)$, ϕ_F is called the *scaling function* or the *father wavelet* of such system.

For $j \in \mathbb{N}_0$, let $W_j = V_{j+1} \ominus V_j$. Let $\phi_M \in W_0$ be such that $\{\phi_M(\cdot - n)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in W_0. Such existence is guaranteed by wavelet theory [10]. This function ϕ_M is called the *mother wavelet* associated with ϕ_F.

One now needs an orthonormal system of $L^2(\mathbb{R}^d)$. Such system can be generated from one of one dimension.

For $n = (n_r)_{1\leqslant r\leqslant d} \in \mathbb{Z}^d$, let

$$\Phi_n(x) = \prod_{r=1}^{d} \phi_F(x_r - n_r),$$

with $x \in \mathbb{R}^d$.

Denote $G = (G_1, \cdots, G_d) \in \{F, M\}^d$ - in other words, a d-tuple of types, F or M (father wavelet or mother wavelet, respectively). Let

$$\Phi_n^G(x) = \prod_{r=1}^{d} \phi_{G_r}(x_r - n_r).$$

Finally, let $G^0 = \{(F, \cdots, F)\}$ and $G^j = \{F, M\}^{d*}$ for all $j \in \mathbb{N}$. Assume that $\|\phi_F\|_{L^2} = \|\phi_M\|_{L^2} = 1$, then one has the following promise.

Proposition 2. The system below forms an orthonormal basis in $L^2(\mathbb{R}^d)$,

$$\Phi_n^{j,G}(x) = \begin{cases} \Phi_n(x) & \text{for } j = 0, G \in G^0, n \in \mathbb{Z}^d \\ 2^{\frac{jd}{2}}\Phi_n^G(2^j x) & \text{for } j \in \mathbb{N}, G \in G^j, n \in \mathbb{Z}^d \end{cases}. \tag{3.1}$$

Remark 3: One can also require that $\phi_F, \phi_M \in \mathcal{C}_c^J$ and that $\int x^\alpha \phi_M(x)\,dx = 0$ for all $|\alpha| \leqslant K$; here J, K are sufficiently large positive integers for the following calculations to hold. For the guarantee of smoothness and moment cancellations, see Theorem 1.61 and remark 1.62 in [9].

In what follows one is concerned with functions on $\mathbb{R}^{d\otimes l}$ is in the place of d - hence for example, d is replaced by $d \otimes l$ in 3.1.

Remark 4: By definition of $\Phi_n^{j,G}$ in 3.1, each $\Phi_n^{j,G}$ with $j \in \mathbb{N}_0, n \in \mathbb{Z}^{d\otimes l}$ can be written as $\Phi_{(k_1,\cdots,k_l)}^{j,G}$ with $k_i \in \mathbb{Z}^d$, and moreover, $\Phi_{(k_1,\cdots,k_l)}^{j,G} = \omega_{1,k_1}\cdots\omega_{l,k_l}$ with $\omega_{1,k}, \cdots, \omega_{l,k_l}$ being functions of only variables $x_1, \cdots, x_d; \cdots, \cdots, \cdots; x_{(l-1)d+1}, \cdots, x_{ld}$, respectively. By the assumption made in *Remark 3*, $\omega_{1,k_1}, \cdots, \omega_{l,k_l} \in \mathcal{C}_c^J$, for some sufficiently large J. The supports of ω_{i,k_i} have finite overlaps in k_i. Moreover

$$\|\omega_{i,k_i}\|_{L^\infty} \lesssim_d 2^{\frac{jd}{2}}. \tag{3.2}$$

An example of how the facts above are (frequently) used in the following analysis, is as follows,

$$\sum_{k_1} \|\omega_{1,k_1}\hat{f}_1\|_{L^2}^2 \lesssim_d \|\sum_{k_1}\omega_{1,k_1}\hat{f}_1\|_{L^2}^2 \leqslant \|f_1\|_{L^2}\|\sum_{k_1}\omega_{1,k_1}\|_{L^\infty} \lesssim_d 2^{jd/2}\|f_1\|_{L^2}.$$

Here the sufficient disjointness in supports of ω_{1,k_1}'s is used in the first inequality, and 3.2 is used in the last.

Let $m(\xi_1, \cdots, \xi_l)$ be a function on $\mathbb{R}^{d\otimes l}$ where $\xi_1, \cdots, \xi_l \in \mathbb{R}^d$. The following lemma is essentially given in [6].

Lemma 3. Let K be a positive integer. Assume that $m \in \mathcal{C}^{K+1}$ is a function on $\mathbb{R}^{d\otimes l}$ such that

$$\sup_{|\alpha|\leqslant K+1} \|\partial^\alpha m\|_{L^\infty} \leqslant C_0 < \infty. \tag{3.3}$$

Then one has,

$$|\langle \Phi_n^{j,G}, m\rangle| \lesssim_{C_0} 2^{-(K+1+d)j} \tag{3.4}$$

provided that ϕ_M has K vanishing moments.

Remark 5: See the appendix for a brief discussion about the proof of this lemma.

3.1 Main theorems

Theorem 4. Let $1 \leqslant q < 3$ and set $M_q = \lfloor \frac{18d}{3-q} \rfloor + 1$. Let $m(\xi_1, \xi_2, \xi_3)$ be a function in $L^q(\mathbb{R}^{3d}) \cap \mathcal{C}^{M_q}(\mathbb{R}^{3d})$ satisfying

$$\|\partial^\alpha m\|_{L^\infty} \leqslant C_0$$

for all $|\alpha| \leqslant M_q$. Then the trilinear operator T_m with the multiplier m satisfies

$$\|T_m\|_{L^2 \times L^2 \times L^2 \to L^{2/3}} \lesssim_{C_0, d, q} \|m\|_{L^q}^{q/3}. \tag{3.5}$$

Conversely, there is a function $m \in \bigcap_{q>3} L^q(\mathbb{R}^{3d}) \cap \mathcal{L}^\infty(\mathbb{R}^{3d})$ such that the associate operator T_m does not map $L^2 \times L^2 \times L^2 \to L^{2/3}$.

As a matter of fact, the first part of **Theorem 4** is a corollary of the following theorem:

Theorem 5. Let $l \geqslant 2$. Let $1 \leqslant q < \frac{2l}{l-1} =: A(l)$ and set $M_q = \lfloor \frac{6dl}{A(l)-q} \rfloor + 1$. Let $m(\xi_1, \cdots, \xi_l)$ be a function in $L^q(\mathbb{R}^{d \otimes l}) \cap \mathcal{C}^{M_q}(\mathbb{R}^{d \otimes l})$ satisfying

$$\|\partial^\alpha m\|_{L^\infty} \leqslant C_0 \tag{3.6}$$

for all $|\alpha| \leqslant M_q$. Then the l-multilinear operator T_m with the multiplier m satisfies

$$\|T_m\|_{L^2 \times \cdots \times L^2 \to L^{2/l}} \lesssim_{C_0, d, q} \|m\|_{L^q}^{q/A(l)}. \tag{3.7}$$

Remark 6: The dependence of expressions 3.5, 3.7 on C_0 can be taken to be, respectively, $C_0^{1-q/3}, C_0^{1-q/A(l)}$. This said dependence will be made clear in the body of the proof. The constants $18d$ and $6dl$ in the notions of M_q in **Theorem 4**, **Theorem 5**, respectively, are sufficient and not optimal.

4 Sufficiency for both theorems

It's sufficient to give a proof for **Theorem 5**. The proof will be based on induction on l.

Let j, G be as in 3.1 and $n = (k_1, \cdots, k_l) \in \mathbb{Z}^{d \otimes l}$. Set

$$b_n^{j,G} = \langle \Phi_n^{j,G}, m \rangle.$$

Then (see the appendix)

$$\|m\|_{L^q} \asymp_d \left\| \left(\sum_{(j,G)} \sum_{n \in \mathbb{Z}^{d \otimes l}} |b_n^{j,G} 2^{ljd/2} \chi_{Q_{jn}}|^2 \right)^{1/2} \right\|_{L^q}, \tag{4.1}$$

with Q_{jn} being a cube centered at $2^{-j}n$ with side-length 2^{1-j}. Let $\tilde{Q}_{jn} = (1/2)Q_{jn}$. Now fix j, G and refer to $b_n^{j,G}$ as simply b_n (or equivalently, $b_{(k_1, \cdots, k_l)}$). Then due to the pairwise disjoint of the cubes $\tilde{Q}_{jn}$ in n (for fixed j, G), one also has from 4.1 that

$$\begin{aligned}
\|m\|_{L^q} &\gtrsim 2^{ljd/2} \left\| \left(\sum_{n \in \mathbb{Z}^{d \otimes l}} |b_n|^2 \chi_{Q_{jn}} \right)^{1/2} \right\|_{L^q} \\
&\geqslant 2^{ljd/2} \left\| \left(\sum_{n \in \mathbb{Z}^{d \otimes l}} |b_n|^2 \chi_{\tilde{Q}_{jn}} \right)^{1/2} \right\|_{L^q} \\
&= 2^{ljd/2} \left\| \sum_{n \in \mathbb{Z}^{d \otimes l}} |b_n| \chi_{\tilde{Q}_{jn}} \right\|_{L^q} \\
&= 2^{ljd(1/2-1/q)} \left(\sum_{n \in \mathbb{Z}^{d \otimes l}} |b_n|^q \right)^{1/q}.
\end{aligned}$$

The disjointness of the cubes $\tilde{Q}_{jn}$ is used in the last two equalities above. Let $b = (b_n)_{n \in \mathbb{Z}^{d \otimes l}}$. Then the calculation above says,

$$\|b\|_{l^q} \lesssim 2^{ljd(1/q-1/2)} \|m\|_{L^q}. \tag{4.2}$$

Let $r \in \mathbb{N}_0$. Define,

$$U_r = \{(k_1, \cdots, k_l) \in \mathbb{Z}^{d\otimes l} : 2^{-r-1}\|b\|_{l^\infty} < |b_{(k_1,\cdots,k_l)}| \leqslant 2^{-r}\|b\|_{l^\infty}\}.$$

Let $\mathcal{C}$ denote the cardinality of U_r. Note that $\mathcal{C}$ is at most:

$$\mathcal{C} \lesssim 2^{rq}\|b\|_{l^q}^q\|b\|_{l^\infty}^{-q}. \tag{4.3}$$

Define,

$$m^r := \sum_{(k_1,\cdots,k_l)\in U_r} b_{(k_1,\cdots,k_l)}\omega_{1,k_1}\cdots\omega_{l,k_l} = \sum_{n\in U_r} b_n\omega_{1,k_1}\cdots\omega_{l,k_l},$$

where the meaning of ω_{i,k_i} is as described in *Remark 4*, and correspondingly,

$$T_{m^r}(f_1, \cdots, f_l)(x) := \int_{\mathbb{R}^{d\otimes l}} m^r(\xi_1, \cdots, \xi_l)\hat{f}_1(\xi_1)\cdots\hat{f}_l(\xi_l)e^{2\pi ix\cdot\sum_i\xi_i}\,d\xi_1\cdots d\xi_l.$$

In terms of Fourier transforms, T_{m^r} can be written as,

$$T_{m^r}(f_1, \cdots, f_l)(x) = \sum_{(k_1,\cdots,k_l)\in U_r} b_n\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)\cdots\mathcal{F}^{-1}(\omega_{l,k_l}\hat{f}_l).$$

The following lemma is key:

Lemma 8. For all $r \in \mathbb{N}$, let $\mathcal{C}$ be corresponding as above. Then

$$\|T_{m^r}(f_1, \cdots, f_l)\|_{L^{2/l}} \lesssim_d 2^{ljd/2}(2^{-r}\|b\|_{l^\infty})\cdot\mathcal{C}^{(l-1)/2l}\|f_1\|_{L^2}\cdots\|f_l\|_{L^2}. \tag{4.4}$$

Proof: One inducts on l. The base case $l = 2$ was proved in [7], for all $r \in \mathbb{N}_0$. Now suppose that 4.4 is true for l, one wants to deduce the same thing for $l + 1$. The following subsections are discussions of what one might encounter in the induction step.

4.1 A simple model

Consider $T(m^r)(f_1, \cdots, f_{l+1})$ where now all the quantities $m^r, U_r, \mathcal{C}$ are understood as defined at the $l + 1$ multilinear level, and that $n = (k_1, \cdots, k_{l+1})$. Let

$$U_r^i = \{k_i : (k_1, \cdots, k_{l+1}) \in U_r\}.$$

In other words, U_r^i is the "projection" of U_r onto the k_i-direction. Suppose that $|U_r^1|$ is smallest among all $|U_r^i|$'s; this holds no calculation significance, just for starter convenience.

For each k_1, let

$$A^{k_1} = \{(k_2, \cdots, k_{l+1}) : n = (k_1, \cdots, k_{l+1}) \in U_r\}$$

in the definition of $\mathcal{T}^{k_1,l}$. Let $\mathcal{A}^{k_1} = |A^{k_1}|$. Then,

$$\begin{aligned}\|T_{m^r}(f_1, \cdots, f_{l+1})\|_{L^{2/(l+1)}}^{2/(l+1)} &\leqslant \int_{\mathbb{R}^d}\sum_{k_1\in U_r^1}|\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^{2/(l+1)}\cdot\left|\sum_{(k_2,\cdots,k_{l+1}):n\in U_r} b_n\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)\right|^{2/(l+1)}dx\\ &= \int_{\mathbb{R}^d}\sum_{k_1\in U_r^{1,l}}\mathcal{F}^{-1}|(\omega_{1,k_1}\hat{f}_1)|^{2/(l+1)}\cdot\left|\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})\right|^{2/(l+1)}dx. \end{aligned}\tag{4.5}$$

where for each fixed k_1

$$\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1}) = \sum_{(k_2,\cdots,k_{l+1}):n\in U_r} b_n\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i).$$

Hence each operator $\mathcal{T}^{k_1,l}$ is simply a version of T_{m^r} at the l multilinear level. Let $r, r' \in \mathbb{N}$ be such that $1/r + 1/r' = 1$. Then from 4.5 and Hölder's inequality,

$$\begin{aligned}\|T_{m^r}(f_1, \cdots, f_{l+1})\|_{L^{2/(l+1)}}^{2/(l+1)} &\leqslant \int_{\mathbb{R}^d}\left(\sum_{k_1\in U_r^1}|\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^r\right)^{2/(r(l+1))}\left(\sum_{k_1\in U_r^1}|\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})|^{r'}\right)^{2/(r'(l+1))}dx\\ &\leqslant \left\|\left(\sum_{k_1\in U_r^1}|\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^r\right)^{1/r}\right\|_{L^2}^{2/(l+1)}\left\|\left(\sum_{k_1\in U_r^1}|\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})|^{r'}\right)^{1/r'}\right\|_{L^{2/l}}^{2/(l+1)}\\ &=: I_1 \times I_2. \end{aligned}\tag{4.6}$$

Applying Minkowski's integral inequality [3] to I_1 in 4.6 gives:

$$I_1 = \left(\int_{\mathbb{R}^d} \left(\sum_{k_1 \in U_r^1} |\mathcal{F}^{-1}(\omega_{1,k_1} \hat{f}_1)|^r \right)^{2/r} dx \right)^{1/(l+1)} \leqslant \left(\sum_{k_1 \in U_r^1} \left(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}(\omega_{1,k_1} \hat{f}_1)|^2 \, dx \right)^{r/2} \right)^{2/(r(l+1))}$$
$$= \left(\sum_{k_1 \in U_r^1} \|\mathcal{F}^{-1}(\omega_{1,k_1} \hat{f}_1)\|_{L^2}^r \right)^{2/(r(l+1))} = \left(\sum_{k_1 \in U_r^1} \|\omega_{1,k_1} \hat{f}_1\|_{L^2}^r \right)^{2/(r(l+1))} \tag{4.7}$$

by Plancherel's theorem. In order for Minkowski's inequality to apply, it must be that $2/r \geqslant 1$. Similarly for I_2 in 4.6 one has,

$$I_2 = \left(\int_{\mathbb{R}^d} \left(\sum_{k_1 \in U_r^1} |\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})|^{r'} \right)^{2/(lr')} dx \right)^{l/(l+1)}$$
$$\leqslant \left(\sum_{k_1 \in U_r^1} \left(\int_{\mathbb{R}^d} |\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})|^{2/l} \, dx \right)^{lr'/2} \right)^{2/(r'(l+1))} = \left(\sum_{k_1 \in U_r^1} \|\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})\|_{L^{2/l}}^{r'} \right)^{2/(r'(l+1))}. \tag{4.8}$$

It also requires here that $2/r' \geqslant 1$. Take $r = r' = 2$, then from 4.7, *Remark 4* and its 3.2,

$$I_1 \leqslant \left(\sum_{k_1 \in U_r^1} \|\omega_{1,k_1} \hat{f}_1\|_{L^2}^2 \right)^{1/(l+1)} \lesssim_d \left(\| \sum_{k_1 \in U_r^1} \omega_{1,k_1} \hat{f}_1\|_{L^2}^2 \right)^{1/(l+1)} \lesssim_d (2^{jd} \|f_1\|_{L^2}^2)^{1/(l+1)}. \tag{4.9}$$

From the induction hypothesis,

$$\|\mathcal{T}^{k_1,l}(f_2, \cdots, f_{l+1})\|_{L^{2/l}} \lesssim_d 2^{ljd/2} (2^{-r} \|b\|_{l^\infty}) \cdot (\mathcal{A}^{k_1})^{(l-1)/2l} \|f_2\|_{L^2} \cdots \|f_{l+1}\|_{L^2}. \tag{4.10}$$

That means 4.8 gives

$$I_2 \lesssim_{d,q} \left(\sum_{k_1 \in U_r^1} 2^{ljd} (2^{-r} \|b\|_{l^\infty})^2 \cdot [(\mathcal{A}^{k_1})^{(l-1)/2l}]^2 \|f_2\|_{L^2}^2 \cdots \|f_{l+1}\|_{L^2}^2 \right)^{1/(l+1)}. \tag{4.11}$$

Altogether 4.8, 4.11 gives,

$$\|T_{m^r}(f_1, \cdots, f_{l+1})\|_{L^{2/(l+1)}}^{2/(l+1)} \lesssim_d \left(\sum_{k_1 \in U_r^1} 2^{ljd} (2^{-r} \|b\|_{l^\infty})^2 \cdot [(\mathcal{A}^{k_1})^{(l-1)/2l}]^2 \|f_2\|_{L^2}^2 \cdots \|f_{l+1}\|_{L^2}^2 \right)^{1/(l+1)}$$
$$\times (2^{jd} \|f_1\|_{L^2}^2)^{1/(l+1)}. \tag{4.12}$$

If one is in an ideal situation, that means when $\mathcal{A}^{k_1} \approx \mathcal{C}^{l/(l+1)}$ and $|U_r^1| \approx \mathcal{C}^{1/(l+1)}$; that means, U_r most resembles a discrete cube in volume, then 4.12 concludes the induction step, because,

$$\|T_{m^r}\|_{L^2 \times \cdots L^2 \to L^{2/(l+1)}} \lesssim_d 2^{(l+1)jd/2} (2^{-r} \|b\|_{l^\infty}) \left(\mathcal{C}^{1/(l+1)} (\mathcal{C}^{l/(l+1)})^{(l-1)/l} \right)^{1/2}$$
$$= 2^{(l+1)jd/2} (2^{-r} \|b\|_{l^\infty}) \mathcal{C}^{l/2(l+1)}.$$

In a non-ideal situation, what can happen is either $\mathcal{A}^{k_1} > \mathcal{C}^{l/(l+1)}$ or that $\mathcal{A}^{k_1} \leqslant \mathcal{C}^{l/(l+1)}$ for all $k_1 \in U_r^1$, or a mixed case. In the first case,

$$\sum_{k_1 \in U_r^1} (\mathcal{A}^{k_1})^{(l-1)/l} \leqslant \mathcal{C} \mathcal{C}^{-1/(l+1)} = \mathcal{C}^{l/(l+1)}$$

as $\sum_{k_1 \in U_r^1} \mathcal{A}^{k_1} = \mathcal{C}$. This still gives the desired induction conclusion

$$\|T_{m^r}(f_1, \cdots, f_{l+1})\|_{L^2 \times \cdots L^2 \to L^{2/(l+1)}} \lesssim_d 2^{(l+1)jd/2} (2^{-r} \|b\|_{l^\infty}) \mathcal{C}^{l/2(l+1)}.$$

4.2 Second non-ideal case

One wants to modify 4.6 a bit

$$\|T_{m^r}(f_1,\cdots,f_{l+1})\|_{L^{2/(l+1)}}^{2/(l+1)} \leqslant \int_{\mathbb{R}^d}\Big(\sum_{k_2,\cdots,k_{l+1}}|\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)|^2\Big)^{1/(l+1)}\Big(\sum_{k_2,\cdots,k_{l+1}}|\sum_{k_1}b_n\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^2\Big)^{1/(l+1)}dx$$
$$\leqslant \Big\|\Big(\sum_{k_2,\cdots,k_{l+1}}|\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)|^2\Big)^{1/2}\Big\|_{L^{2/l}}^{2/(l+1)}\Big\|\Big(\sum_{k_2,\cdots,k_{l+1}}|\sum_{k_1}b_n\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^2\Big)^{1/2}\Big\|_{L^2}^{2/(l+1)} \tag{4.13}$$

where the range of k_1 is understood as,

$$\{k_1 : n=(k_1,\cdots,k_{l+1})\in U_r \text{ for some } (k_2,\cdots,k_{l+1})\}.$$

Similarly as in 4.8, one derives the following for the first factor in 4.13:

$$\Big(\int_{\mathbb{R}^d}\Big(\sum_{k_2,\cdots,k_{l+1}}|\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)|^2\Big)^{1/l}dx\Big)^{l/(l+1)} \leqslant \Big(\sum_{k_2,\cdots,k_{l+1}}\Big(\int_{\mathbb{R}^d}|\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)|^{2/l}dx\Big)^{l}\Big)^{1/(l+1)}$$
$$=\Big(\sum_{k_2,\cdots,k_{l+1}}\|\prod_{i\geqslant 2}\mathcal{F}^{-1}(\omega_{i,k_i}\hat{f}_i)\|_{L^{2/l}}^2\Big)^{1/(l+1)} = \Big(\sum_{k_2,\cdots,k_{l+1}}\prod_{i\geqslant 2}\|\omega_{i,k_i}\hat{f}_i\|_{L^2}^2\Big)^{1/(l+1)} \lesssim_d 2^{ljd/(l+1)}\prod_{i\geqslant 2}\|f_i\|_{L^2}^{2/(l+1)}. \tag{4.14}$$

The Hölder's inequality is used in the second to last step, and *Remark 4* and its 3.2 are used in the last.

As for the second factor in 4.13, one argues similarly as in 4.7

$$I_1 = \Big(\int_{\mathbb{R}^d}\Big(\sum_{k_2,\cdots,k_{l+1}}|\sum_{k_1}b_n\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^2\Big)dx\Big)^{1/(l+1)} \leqslant \Big(\sum_{k_2,\cdots,k_{l+1}}\Big(\int_{\mathbb{R}^d}|\sum_{k_1}b_n\mathcal{F}^{-1}(\omega_{1,k_1}\hat{f}_1)|^2dx\Big)\Big)^{1/(l+1)}$$
$$\leqslant (2^{-r}\|b\|_{l^\infty})^{2/(l+1)}\Big(\sum_{k_1\in U_r^1}\sum_{(k_2,\cdots,k_{l+1}):n\in U_r}\|\omega_{1,k_1}\hat{f}_1\|_{L^2}^2\Big)^{1/(l+1)} \lesssim_d 2^{jd/(l+1)}(2^{-r}\|b\|_{l^\infty})^{2/(l+1)}\|f_1\|_{L^2}^{2/(l+1)}\mathcal{C}^{l/(l+1)^2} \tag{4.15}$$

by the assumption about $\mathcal{A}^{k_1}$.

Then 4.13, 4.14, 4.15 gives the desired conclusion,

$$\|T_{m^r}(f_1,\cdots,f_{l+1})\|_{L^2\times\cdots L^2\to L^{2/(l+1)}} \lesssim_d 2^{(l+1)jd/2}(2^{-r}\|b\|_{l^\infty})\mathcal{C}^{l/2(l+1)}.$$

which concludes the induction step for the second non-ideal case.

4.3 For the mixed case

One simply splits the integral that defines $\|T_{m^r}(f_1,\cdots,f_{l+1})\|_{L^{2/(l+1)}}^{2/(l+1)}$ into two parts, one runs over $k_1\in U_r^1$ with each $\mathcal{A}^{k_1}\leqslant\mathcal{C}^{l/(l+1)}$, call this set $U_r^{1,s}$ and the other runs over $k_1\in U_r^1$ with each $\mathcal{A}^{k_1}>\mathcal{C}^{l/(l+1)}$, call this set $U_s^{1,g}$. Let $m^{r,s}$ denote the multiplier associated with the set $U_r^{1,s}$ and $m^{r,g}$ with $U_r^{1,g}$. As shown in the previous subsections,

$$\|T_{m^{r,s}}(f_1,\cdots,f_{l+1})\|_{L^2\times\cdots L^2\to L^{2/(l+1)}}, \|T_{m^{r,g}}(f_1,\cdots,f_{l+1})\|_{L^2\times\cdots L^2\to L^{2/(l+1)}}$$
$$\lesssim_d 2^{(l+1)jd/2}(2^{-r}\|b\|_{l^\infty})\mathcal{C}^{l/2(l+1)}$$

which implies the same for $\|T_{m^r}(f_1,\cdots,f_{l+1})\|_{L^2\times\cdots L^2\to L^{2/(l+1)}}$. The proof for the induction step here is complete; hence one is done with the lemma.

4.4 Conclusion for the sufficiency part

The **Lemma 8** is helpful, because the next step in this analysis is to make sure that the obtained dominance for $\|T_{m^r}(f_1, \cdots, f_l)\|_{L^{2/l}}$ in 4.4 is summable over r, j, G in that order. Putting 4.2, 4.3 into 4.4 gives:

$$\|T_{m^r}\|_{L^2\times\cdots L^2\to L^{2/l}} \lesssim_{d,q} 2^{jd/2(l+(l-1)/q-(l-1)/2)} 2^{r(q(l-1)/2l-1)} \|b\|_{l^\infty}^{1-q(l-1)/2l} \|m\|_{L^q}^{(l-1)/2l}. \tag{4.16}$$

Now 4.16 is summable in r as long as

$$(q(l-1)/2l) < 1$$

which gives the range $1 \leqslant q < A(l) = \frac{2l}{l-1}$. After summing 4.16 in terms of r, the dominance constant on the RHS is reduced to:

$$2^{jd/2(l+(l-1)/q-(l-1)/2)} \|b\|_{l^\infty}^{1-q(l-1)/2l} \|m\|_{L^q}^{(l-1)/2l},$$

which, after recalling from 3.4 that $\|b\|_{l^\infty} \lesssim_{C_0} 2^{-(K+1+d)j}$, where $K+1$ is the presupposed smoothness degree of m, becomes,

$$2^{jd/2(l+(l-1)/q-(l-1)/2)} (2^{-(K+1+d)j})^{1-q(l-1)/2l} \|m\|_{L^q}^{(l-1)/2l}. \tag{4.17}$$

Summing 4.17 in terms of j and requiring convergence gives,

$$(K+d+1)(1-q/A(l)) > (d/2)(l+(l-1)/q-(l-1)/2).$$

That means one can require $M_q := K+1 = \lfloor \frac{6dl}{A(l)-q} \rfloor + 1$, for instance. This completes the proof of **Theorem 5** and hence the sufficiency part of **Theorem 4**.

Remark 7: The dependence on the constant C_0 in 3.5, 3.7 is as follows. Note that C_0 was first invoked in **Lemma 3**, which was done in [6]. In that paper, the dependence of C_0 is found to be linear. In particular, the RHS of 3.4 can be taken to be $C_0 2^{-(K+1+d)j}$. This dominance is then put back in 4.17, which then gives the total dominance for $\|T_m\|_{L^2\times\cdots L^2\to L^{2/l}}$ as,

$$\|T_m\|_{L^2\times\cdots L^2\to L^{2/l}} \lesssim_{d,q} C_0^{1-q/A(l)} \|m\|_{L^q}^{1/A(l)}.$$

As discussed, the lack of duality theory prevents further extension from this approach to any other parts of the Banach range $1/p_1 + \cdots + 1/p_l = 1/r$ for the operator norm $\|T_m\|_{L^{p_1}\times\cdots L^{p_l}\to L^r}$.

5 Necessity for Theorem 4

Let ϕ be a Schwartz function on $\mathbb{R}$ whose Fourier transform has support in a symmetric interval I and let $\{a_j\}_{j\geqslant1}, \{b_j\}_{j\geqslant1}, \{c_j\}_{j\geqslant1}$ be two sequences of nonnegative numbers with only finitely many nonzero terms. This function ϕ is not related to the wavelet functions in the previous sections. Define f, g, h by

$$\hat{f}(\xi) = \sum_{j\geqslant1} a_j \hat{\phi}(\xi_1 - j) \prod_{r\geqslant2} \hat{\phi}(\xi_r - 1), \tag{5.1}$$

$$\hat{g}(\eta) = \sum_{j\geqslant1} b_j \hat{\phi}(\eta_1 - j) \prod_{r\geqslant2} \hat{\phi}(\eta_r - 1), \tag{5.2}$$

$$\hat{h}(\delta) = \sum_{j\geqslant1} c_j \hat{\phi}(\delta_1 - j) \prod_{r\geqslant2} \hat{\phi}(\delta_r - 1). \tag{5.3}$$

Then f, g, h are Schwartz functions whose L^2 norms are bounded by a constant multiple of $(\sum_{j\geqslant1} a_j^2)^{1/2}, (\sum_{j\geqslant1} b_j^2)^{1/2}, (\sum_{j\geqslant1} c_j^2)^{1/2}$, respectively.

Let $\{s_j(t)\}_{j\geqslant1}$ denote the sequence of Rademacher functions [3]. Let $\{v_j\}_{j\geqslant1}$ be a bounded sequence of nonnegative numbers. For $t \in [0,1]$, consider m_t by

$$m_t(\xi,\eta,\delta) = \sum_{j\geqslant1}\sum_{k\geqslant1}\sum_{l\geqslant1} v_{j+k+l} s_{j+k+l}(t) \psi(\xi_1 - j)\psi(\eta_1 - k)\psi(\delta_1 - l) \prod_{r\geqslant2} \psi(\xi_r - 1)\psi(\eta_1 - 1)\psi(\delta_1 - 1). \tag{5.4}$$

Here ψ is a smooth function on $\mathbb{R}$ supported in the interval $J = 10I$ and assuming value 1 in cJ, with c small enough so that $I \subset cJ$. Then from the definitions 5.1, 5.2, 5.3,

$$
\begin{aligned}
&T_{m_t}(f,g,h)(x) \\
&\quad = \sum_{j\geqslant 1}\sum_{k\geqslant 1}\sum_{l\geqslant 1} a_j b_k c_l v_{j+k+l} s_{j+k+l}(t)\phi(x_1)^3 e^{2\pi i x_1 (j+k+l)} \prod_{r\geqslant 2} e^{6\pi i x_r}\phi(x_r)^3 \\
&\quad = \sum_{l\geqslant 3} v_l s_l(t)\phi(x_1)^3 e^{2\pi i x_1 l} \sum_{j=1}^{l-1}\sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k} \prod_{r\geqslant 2} e^{6\pi i x_r}\phi(x_r)^3. \qquad (5.5)
\end{aligned}
$$

Utilizing 5.5, Khinchin's inequality [3] and Fubini's theorem, one then has

$$
\begin{aligned}
&\int_0^1 \|T_{m_t}(f,g,h)\|_{L^{2/3}}^{2/3}\,dt \\
&\quad = \left(\int_{\mathbb{R}} |\phi(y)|^2\,dy\right)^{n-1} \int_{\mathbb{R}}\int_0^1 \left|\sum_{l\geqslant 3} v_l s_l(t)\phi(x_1)^3 e^{2\pi i x_1 l} \sum_{j=1}^{l-1}\sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k}\right|^{2/3} dt dx_1 \\
&\approx_{C,d} \int_{\mathbb{R}} \left(\sum_{l\geqslant 3}\left(v_l|\phi(x_1)|^3 \sum_{j=1}^{l-1}\sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k}\right)^2\right)^{1/3} dx_1 \approx_{C,d} \left(\sum_{l\geqslant 3}\left(v_l \sum_{j=1}^{l-1}\sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k}\right)^2\right)^{1/3}.
\end{aligned}
\tag{5.6}
$$

Fix a positive integer $N \geqslant 2$ and set $a_j^N = b_j^N = c_j^N = 2^{-N/2}$ if $j = 2^N, \cdots, 2^{N+1}-1$ and zero otherwise. Observe that with this agreement, $a_j b_k c_{l-j-k}$ is only nonzero for a finite number of terms for each l. Moreover $\sum_{j\geqslant 1}\sum_{k\geqslant 1} a_j b_k c_{l-j-k} = 0$ if $l > 2^{N+3}$ or if $l < 2^{N+1}$. In effect,

$$
\sum_{j=1}^{l-1}\sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k} \geqslant \sum_{c2^N}^{C2^N}\sum_{c2^N}^{C2^N} 2^{-3N/2} \geqslant c2^{N/2} > 0. \tag{5.7}
$$

In 5.7 above, the constants C, c in the first instance can be found as follows. Independently of N, a_j, b_k, c_{l-j-k} are "activated" (nonzero) for ranges of j, k, l that have the same length. Hence within those ranges of j, k, one can fit in a "box" of sizes $[c2^N, C2^N] \times [c2^N, C2^N]$, for some appropriate c, C. The constant c in the second instance of 5.7 is unrelated to the previous one.

Define $v_l = (l-1)^{-1}(\log(l-1))^{1/2}$ and define f^N, g^N, h^N similarly to f, g, h, respectively, only with a_j, b_j, c_j being replaced by a_j^N, b_j^N, c_j^N, respectively. Then from 5.6, 5.7 and simple calculus,

$$
\begin{aligned}
\int_0^1 \|T_{m_t}(f,g,h)\|_{L^{2/3}}^{2/3}\,dt &\gtrsim_{C,d} 2^{N/3}\left(\sum_{c2^N\leqslant l\leqslant C2^N} (l-1)^{-2}\log(l-1)\right)^{1/3} \\
&\gtrsim_{C,d} 2^{N/3}(\log c2^N)^{1/3}\left(\sum_{c2^N\leqslant l\leqslant C2^N} (l-1)^{-2}\right)^{1/3} \\
&\gtrsim 2^{N/3} c N^{1/3}\left(\int_{c2^N}^{C2^N} y^{-2}\,dy\right)^{1/3} \gtrsim cN^{1/3}.
\end{aligned}
$$

Above, c, C denote positive numbers that possibly change from one instance to the next, with $C > c$. This means that for every $N \geqslant 2$ one can find $t_N \in [0,1]$ such that

$$
\|T_{m_{t_N}}(f,g,h)\|_{L^{2/3}} \geqslant CN^{1/2} \tag{5.8}
$$

with C being independent of N.

Define:

$$
m(\xi,\eta,\delta) = \sum_{j\geqslant 1}\sum_{k\geqslant 1}\sum_{l\geqslant 1} v_{j+k+l}\sigma_{j+k+l}\psi(\xi_1 - j)\psi(\eta_1 - k)\psi(\delta_1 - l)\prod_{r\geqslant 2}\psi(\xi_r - 1)\psi(\eta_1 - 1)\psi(\delta_1 - 1)
$$

with $\sigma_l = s_l(t_N)$ if $N \geqslant 2$ and $2^{N+1} \leqslant l \leqslant 2^{N+3}$. Then a quick calculation shows that,

$$
T_m(f^N, g^N, h^N)(x) = T_{m_{t_N}}(f^N, g^N, h^N)(x),
$$

which means, from 5.8,

$$\|T_m(f^N, g^N, h^N)\|_{L^{2/3}} \gtrsim N^{1/2}.$$

Hence T_m is unbounded on $L^2 \times L^2 \times L^2 \to L^{2/3}$. On the other hand with the definition, m is a smooth function with bounded derivatives and moreover,

$$\|m\|_{L^q} \approx_q \left(\sum_{l\geqslant 3} v_l^q (l-1)^2\right)^{1/q} = \left(\sum_{l\geqslant 3} (\log(l-1))^{q/2}(l-1)^{2-q}\right)^{1/q}$$

which, by comparison with the divergent series $\sum_{n\geqslant 3}(1/n)$, shows that $m \in \bigcap_{q>3} L^q$ and $m \notin L^q$ if $q \leqslant 3$.

Remark 8: It might be possible to construct systematically a counterexample multliplier $m \in \bigcap_{q>A(l)} L^q$ for every $l \geqslant 2$ for the sufficiency direction, to show that indeed $q = A(l)$ is the optimal Lebesgue exponent in the case of l-multilinear multiplier operator.

6 Appendix

The following utilizes the material in [9]. First one should see the definition of the function spaces $F^s_{pq}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < p \leqslant \infty, 0 < q \leqslant \infty$ [9]. The important point here is,

$$F^0_{q2} = L^q. \tag{6.1}$$

See the remark 1.65 in [9]. One also has the following fact, which is a rephrasing of parts of Theorem 1.64 in [9], using the notations already introduced in this paper:

Theorem A. Let $\Phi^{j,G}_n$ be as in 3.1 with sufficient smoothness (see *Remark 3*). Let $f \in F^s_{qp}(\mathbb{R}^d)$. Then f can be represented as,

$$f = \sum_{(j,G,n)} b^{j,G}_n \Phi^{j,G}_n.$$

Furthermore, this representation is unique and the map

$$I : f \mapsto \{b^{j,G}_n\}$$

is an isomorphic map of F^s_{qp} onto f^s_{qp}. The latter is a sequence space whose elements $\lambda = \{\lambda\}^{j,G}_n$ are given the following semi-norm:

$$\|\lambda\| = \left\| \left(\sum_{(j,G,n)} |2^{js}2^{jn}\lambda^{j,G}_n \chi_{Q_{jn}}|^p \right)^{1/q} \right\|_{L^q(\mathbb{R}^d)}. \tag{6.2}$$

Theorem A, 6.1, 6.2 imply 4.1.

6.1 Lemma 3

It was mentioned in [7] that the proof of **Lemma 3** is essentially done in Lemma 7 of [6]. Since the proof in [6] is quite involved, only its ideas will be presented here. This discussion aims to guarantee that although the version of the lemma stated in [7] was stated for $2d$, it's immaterial to change it to any ld.

First, a small point here is that, if one follows the argument there then because of the change in dimensions one should arrive at

$$|\langle \Phi^{j,G}_n, m\rangle| \lesssim_{C_0} 2^{-(K+1+ld/2)j}$$

for the conclusion. But clearly, $2^{-ld/2} \leqslant 2^{-d}$.

If the multiplier had any decay of the form,

$$|\partial^\alpha m(\vec{x})| \leqslant (1 + |\vec{x}|)^{M_1} \tag{6.3}$$

for $|\alpha| \leqslant K$, for some large K, M_1, then one can utilize this and the moment cancellation properties $\Phi_n^{j,G}$ $(j \neq 0)$ and apply the material in the Appendix B.2 of [4] to get the desired decay 3.4 of the wavelet transforms of m. The moment cancellation properties of $\Phi_n^{j,G}$ when $j \neq 0$ come from those of ϕ_M in its definition. There are no cancellation assumptions when $j = 0$, but then one can use the decay 6.1 and those of the wavelets to apply the material in Appendix B.1 of [4] instead.

Our m is assumed of no decay 6.3. Hence one can decompose m into parts,

$$m = \sum_i m_i$$

with m_i being defined as in [6]. Then $m_i = m_0(2^i \cdot)$. Then m_0 does possesses the decay 6.3 [6]. One can arrive at 3.4 for m_0. Now for $i < 0$, one has through change of variables,

$$\langle \Phi_n^{j,G}, m_0(2^i \cdot) \rangle = 2^{ic} \langle \Phi_n^{j+i,G}, m_0 \rangle = 2^{-c|i|} \langle \Phi_n^{j+i,G}, m_0 \rangle. \tag{6.4}$$

Here $c = c(d) = d/2$ for any dimension d of $\vec{x}$. Then one can take $j \geqslant 0$ in 6.4 large enough so that $k = j + i \in \mathbb{N}_0$. For $i > 0$, one uses,

$$\langle \Phi_n^{j,G}, m_0(2^i \cdot) \rangle = 2^{-c|i|} \langle \Phi_n^{j-i,G}, m_0 \rangle.$$

Putting all these back, one gets the desired conclusion for m. None of the tools mentioned here is particular to any dimension d. Hence the conclusion 3.4 holds for any ld.

References

[1] A. Bényi, R. Torres, *Almost orthogonality and a class of bounded bilinear pseudo-differential operators*, Math. Res. Lett. 11 (2004), 1–11.

[2] R. R. Coifman, Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) 28 (1978), 177–202.

[3] G. Folland, *Real Analysis: Modern Techniques and Their Applications, 2nd Ed.*, John Wiley and Sons, Inc, 1999.

[4] L. Grafakos, *Classical Fourier Analysis, 3rd Ed.*, GTM 249, Springer NY, 2014.

[5] L. Grafakos, *Modern Fourier Analysis, 3rd Ed.*, GTM 250, Springer NY, 2014.

[6] L. Grafakos and B. Park, *Characterization of multilinear multipliers in terms of Sobolev space regularity*, arXiv:2003.11739v1 [math.CA], 2020

[7] L. Grafakos, D. He and P. Honzík, *Rough bilinear singular integrals*, Adv. Math. 326 (2018), 54–78.

[8] L. Grafakos, D. He and L. Slavíková, *$L^2 \times L^2 \to L^1$ boundedness criteria*, arXiv:1802.09400v1 [math.CA], 2018.

[9] L. Grafakos, D. Miyachi and P. Tomita, *On multilinear Fourier multipliers of limited smoothness*, Can. Jour. Math. 65 (2013), no. 2, 299-330.

[10] H. Triebel, *Theory of function spaces, III*, Monographs in Mathematics, 100, Birkhäuser Verlag, Basel, 2006.

[11] H. Triebel, *Bases in function spaces, sampling, discrepancy, numerical integration*, EMS Tracts in Mathematics, 11, European Mathematical Society (EMS), Zürich, 2010.