Sessão 20 (aula 28); TP4-6

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Sumário: Resolução de exercícios envolvendo transformadas de Laplace (e inversa).

6. Usando transformadas de Laplace mostre que

$$t^m * t^n = \frac{m!\,n!}{(m+n+1)!} t^{m+n+1} \qquad (m, n \in \mathbb{N}_0).$$

Formulario...

F6_TL_19-20

Observações:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$$

$$t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$t^m * t^n = \int_0^t \tau^m (t-\tau)^n \, d\tau$$

$$t^m * t^n = \mathcal{L}^{-1}\left\{\frac{m!}{s^{m+1}}\right\} * \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{m!}{s^{m+1}} \cdot \frac{n!}{s^{n+1}}\right\}$$

$$= \frac{m!\,n!}{(m+n+1)!}\mathcal{L}^{-1}\left\{\frac{1\,(m+n+1)!}{s^{m+n+2}}\right\}$$

$$= \frac{m!\,n!}{(m+n+1)!} t^{m+n+1}, \quad m, n \in \mathbb{N}_0$$

8. Resolva cada um dos seguintes problemas de Cauchy usando transformadas de Laplace.

(a) $3x' - x = \cos t\,, \quad x(0) = -1;$

$x = x(t)$

$X(s) = \mathcal{L}\{x\}$.

$$\mathcal{L}\{3x' - x\} = \mathcal{L}\{\cos(t)\}$$

$$3\mathcal{L}\{x'\} - \mathcal{L}\{x\} = \frac{s}{s^2+1}, \quad s>0$$

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$$3(\underbrace{s\mathcal{L}\{x\}}_{X(s)} - \underbrace{x(0)}_{-1}) - \mathcal{L}\{x\} = \frac{s}{s^2+1}$$

$$3s\,X(s) + 3 - X(s) = \frac{s}{s^2+1}$$

$$(3s-1)\,X(s) = \frac{s}{s^2+1} - 3$$

$$X(s) = \frac{s - 3(s^2+1)}{(s^2+1)(3s-1)}$$

$$X(s) = \frac{-3s^2+s-3}{(s^2+1)(3s-1)}$$

$$\therefore \quad x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{-3s^2+s-3}{(s^2+1)(3s-1)}\right\}$$

C.A.

$$\frac{-3s^2+s-3}{\underbrace{(s^2+1)}_{\text{raízes complexas conjugadas}}\underbrace{(3s-1)}_{\text{raiz real}}} = \frac{A}{3s-1} + \frac{Bs+C}{s^2+1}, \quad A, B, C \in \mathbb{R}$$

$$-3s^2+s-3 = A(s^2+1) + (Bs+C)(3s-1)$$

complexos conjugados

$$-3s^2+s-3 = A(s^2+1) + (Bs+C)(3s-1)$$
$$= As^2 + A + 3Bs^2 - Bs + 3Cs - C$$
$$= (A+3B)s^2 + (-B+3C)s + (A-C)$$

$$\begin{cases} A+3B=-3 \\ -B+3C=1 \\ A-C=-3 \end{cases} \qquad \begin{cases} A+3B=-3 \\ -B+3C=1 \\ 0-C-3B=0 \end{cases} \qquad \begin{cases} -B-9B=1 \\ C=-3B \end{cases}$$

$$\begin{cases} A=-3+\frac{3}{10} \\ B=-\frac{1}{10} \\ C=\frac{3}{10} \end{cases} \qquad \begin{cases} A=\frac{-30+3}{10} \\ B=-\frac{1}{10} \\ C=\frac{3}{10} \end{cases} \qquad \begin{cases} A=-\frac{27}{10} \\ B=-\frac{1}{10} \\ C=\frac{3}{10} \end{cases}$$

$$x(t) = \mathcal{L}^{-1}\left\{ \frac{-27/10}{3s-1} + \frac{-1/10\, s + \frac{3}{10}}{s^2+1} \right\}$$

$$= -\frac{27}{10}\mathcal{L}^{-1}\left\{\frac{1}{3s-1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s-3}{s^2+1}\right\}$$

$$= -\frac{27}{3\cdot 10}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{1}{3}}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

linha 2 linha 4 linha 3

$$x(t) = -\frac{9}{10}e^{\frac{1}{3}t} - \frac{1}{10}\cos(t) + \frac{3}{10}\operatorname{sen}(t), \quad t \geq 0$$

(c) $y'' + 2y' + 3y = 3t$, $y(0) = 0$, $y'(0) = 1$;

$\mathcal{L}\{y''+2y'+3y\} = \mathcal{L}\{3t\}$, considerando que
$$Y(s) = \mathcal{L}\{y(t)\}(s).$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 3\cdot\frac{1}{s^2}, \quad s>0$$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - \cancel{s}\underbrace{y'(0)}_{1} + 2(sY(s) - \underbrace{y(0)}_{0}) + 3Y(s) = \frac{3}{s^2}$$

$$s^2Y(s) - 1 + 2sY(s) + 3Y(s) = \frac{3}{s^2}$$

$$(s^2+2s+3)Y(s) = \frac{3}{s^2} + 1$$

$$Y(s) = \frac{3+s^2}{s^2(s^2+2s+3)}$$

Logo, $y(t) = \mathcal{L}^{-1}\left\{\frac{s^2+3}{s^2(s^2+2s+3)}\right\}$

C.A.
$s^2+2s+3=0 \Leftrightarrow s = \frac{-2\pm\sqrt{4-12}}{2}$ soluções complexas

$s^2+2s+3 = (s+1)^2+2$

$$\frac{s^2+3}{s^2(s^2+2s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+3}, \quad A,B,C,D \in \mathbb{R}$$

$$s^2+3 = As(s^2+2s+3) + B(s^2+2s+3) + (Cs+D)s^2$$

$\vdots$

$$\begin{cases} A = -\frac{2}{3} \\ B = 1 \\ C = 2/3 \\ D = 4/3 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2(s^2+2s+3)}\right\} = -\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+3}\right\}$$

$$= -\frac{2}{3}\cdot 1 + t + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{s+\overbrace{2}^{1+1}}{(s+1)^2+2}\right\}$$

$$= -\frac{2}{3} + t + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}$$

$$= -\frac{2}{3} + t + \frac{2}{3}e^{-t}\mathcal{L}\left\{\frac{s}{s^2+2}\right\} + \frac{2}{3\sqrt{2}}e^{-t}\mathcal{L}^{-1}\left\{\frac{1\cdot\sqrt{2}}{s^2+2}\right\}$$

$$= -\frac{2}{3} + t + \frac{2}{3}e^{-t}\cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t)$$

Nota:

$\mathcal{L}\{e^{\lambda t}f(t)\}(s) = F(s-\lambda),\ \lambda\in\mathbb{R}$

$e^{\lambda t}\underbrace{f(t)}_{\mathcal{L}^{-1}\{F(s)\}} = \mathcal{L}^{-1}\{F(s-\lambda)\}(t)$

Logo,

$$y(t) = -\frac{2}{3} + t + \frac{2}{3}e^{-t}\cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t),\quad t\geq 0$$

(e) $y'' + y' = \frac{e^{-t}}{2}$, $y(0) = 0 = y'(0)$.

$$\mathcal{L}\{y''+y'\} = \mathcal{L}\left\{\frac{e^{-t}}{2}\right\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + sY(s) - \underbrace{y(0)}_{=0} = \frac{1}{2}\cdot\frac{1}{s+1}$$

$$(s^2+s)Y(s) = \frac{1}{2(s+1)}$$

$$Y(s) = \frac{1}{2(s+1)(s^2+s)}$$

$$Y(s) = \frac{1}{2s(s+1)^2} = \frac{1}{2}\cdot\frac{1}{s(s+1)^2}$$

c.A.

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2},\quad A,B,C\in\mathbb{R}$$

$\vdots$

$$\begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$

$$y(t) = \frac{1}{2}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right)$$

$$= \frac{1}{2}\left(1 - e^{-t} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right)$$

$$= \frac{1}{2}\left(1 - e^{-t} - e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right)$$

$$= \frac{1}{2}\left(1 - e^{-t} - e^{-t}\,t\right) \quad , t \geq 0$$

9. Resolva o seguinte problema de valores iniciais recorrendo às transformadas de Laplace:

$$y'' + y = t^2 + 1\,, \qquad y(\pi) = \pi^2\,, \quad y'(\pi) = 2\pi.$$

(Sugestão: Efetuar a substituição definida por $x = t - \pi$).

$y = y(t)$ $\qquad y(\pi) = \pi^2$

$\qquad\qquad\quad y'(\pi) = 2\pi$

$x = t - \pi \iff t = x + \pi$

$y = y(x+\pi)$

$$y''(x+\pi) + y(x+\pi) = (x+\pi)^2 + 1$$

$z(x) = y(x+\pi)$

$z'(x) = y'(x+\pi)$

$z''(x) = y''(x+\pi)$

$$\begin{cases} z''(x) + z(x) = (x+\pi)^2 + 1 \\ z(0) = \pi^2 \qquad z'(0) = 2\pi \end{cases}$$

Aplicando transformadas de Laplace:

$$s^2 Z(s) - s\,\overset{=\pi^2}{z(0)} - \overset{=2\pi}{z'(0)} + Z(s) = \mathcal{L}\left\{\underbrace{(x+\pi)^2 + 1}_{x^2 + 2\pi x + \pi^2 + 1}\right\}$$

$$(s^2+1)\,Z(s) = \ldots$$

$\vdots$

$z(x) = \ldots$

$y(t) = z(t-\pi)$