Sessão 6, TP4-6

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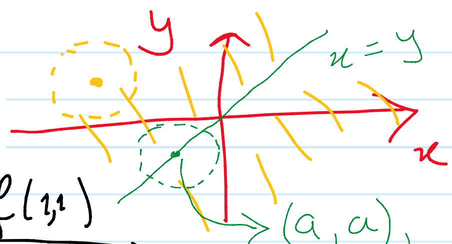
Exemplo do slide 28

Seja $f : \mathbb{R}^2 \to \mathbb{R}$ tal que $f(x,y) = \begin{cases} xy, & \text{se } x \neq y \\ x^3, & \text{se } x = y \end{cases}$. Mostre que $\frac{\partial f}{\partial x}(1,1) = 1$, $\frac{\partial f}{\partial y}(3,4) = 3$ e que $\frac{\partial f}{\partial y}(2,2)$ não existe.



$\to (a,a)$, $a \in \mathbb{R}$

$$\frac{\partial f}{\partial x}(1,1) = \lim_{h \to 0} \frac{f(1+h,1) - f(1,1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)\cdot 1 - 1^3}{h}$$

$$= \lim_{h \to 0} \frac{1+h}{h} = 1$$

uso obrigatório da definição!

$\frac{\partial f}{\partial y}(3,4) = ?$ Pode ser calculada usando regras de derivação.

$$(xy)'_y = x$$

logo, $\frac{\partial f}{\partial y}(3,4) = 3$

$\frac{\partial f}{\partial x}(3,4) = 4$

$(xy)'_x = y$

$$\frac{\partial f}{\partial y}(2,2) = \lim_{h \to 0} \frac{f(2,2+h) - f(2,2)}{h}$$

$$= \lim_{h \to 0} \frac{2(2+h) - 8}{h}$$

$$= \lim_{h \to 0} \frac{2h-4}{h}$$

$$= \lim_{h \to 0} \left(2 - \frac{4}{h}\right)$$ Não existe.

não existe $\to +\infty$, $\to -\infty$

Exemplo:

$f(x,y) = 2x^2y + x$, $D_f = \mathbb{R}^2$

$\frac{\partial f}{\partial x}(x,y) = 4xy + 1 \to$ Domínio $\mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = 2x^2 \to$ Domínio $\mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = 2x^2 \rightarrow$ Domínio $\mathbb{R}$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial x}(4xy+1) = 4y = \frac{\partial^2 f}{\partial x^2}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial y}(4xy+1) = 4x = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

$$\|$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial x}(2x^2) = 4x = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial y}(2x^2) = 0 = \frac{\partial^2 f}{\partial y^2}(x,y)$$

Derivadas de $2.^{\underline{a}}$ ordem:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

"Em geral", são iguais.

Exercício 10

$f(x,y) = \ln(x+y) - \ln(x-y)$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x+y}\cdot(x+y)'_x - \frac{1}{x-y}(x-y)'_x$$

$$= \frac{1}{x+y} - \frac{1}{x-y} = \frac{x-y-(x+y)}{(x+y)(x-y)}$$

$$= \frac{\cancel{x}-y-\cancel{x}-y}{x^2-y^2} = \frac{-2y}{x^2-y^2}, \quad (x,y) \in D_f$$

$$D_f = \{(x,y) \in \mathbb{R}^2 : x+y>0 \wedge x-y>0\}$$

$$= \{(x,y) \in \mathbb{R}^2 : y > -x \wedge y < x\}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{x+y} - \frac{1}{x-y}(-1)$$

$$= \frac{1}{x+y} + \frac{1}{x-y} = \frac{x-\cancel{y}+x+\cancel{y}}{x^2-y^2}$$

$$= \frac{1}{x+y} + \frac{1}{x-y} = \frac{x-\cancel{y}+x+\cancel{y}}{x^2-y^2}$$

$$= \frac{2x}{x^2-y^2}, \quad (x,y) \in D_f$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2-y^2}\right),$$

$$= \frac{0\cdot(x^2-y^2) - (-2y)\cdot(x^2-y^2)'_x}{(x^2-y^2)^2}$$

$$= \frac{2y\,(2x)}{(x^2-y^2)^2}$$

$$= \frac{4xy}{(x^2-y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \text{exercício}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{-2y}{x^2-y^2}\right)$$

$$= \frac{-2(x^2-y^2) - (-2y)(x^2-y^2)'_y}{(x^2-y^2)^2}$$

$$= \frac{-2(x^2-y^2) + 2y(-2y)}{(x^2-y^2)^2}$$

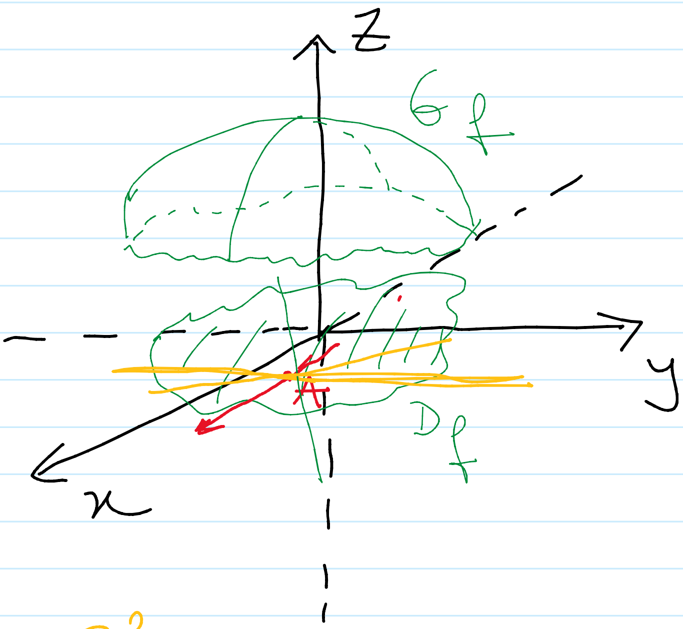
$$= \frac{-2x^2+2y^2-4y^2}{(x^2-y^2)^2}$$

$$= \frac{-2x^2-2y^2}{(x^2-y^2)^2}, \quad (x,y) \in D_f$$

$$= \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

porque f é $C^2(D_f)$.

Derivadas Direcionais:



Em $\mathbb{R}^2$:

$u = (u_1, u_2)$

$$\|u\| = 1 \Leftrightarrow \sqrt{u_1^2 + u_2^2} = 1$$

$$\Leftrightarrow u_1^2 + u_2^2 = 1$$

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$