Session 9.1: Introduction to temporal modelling

Imperial College London

Learning objectives

After this lecture you should be able to

- Explain why time also matter and describe features of time series data
- Understand the difference between stationary and nonstationary temporal processes
- Describe basic temporal models
- Know the key functions to implement temporal models through the `R-INLA` package

Some of the topics covered in this lecture are presented in Chapter 8 of the book **Bayesian inference with INLA** by Virgilio Gómez-Rubio (link: https://becarioprecario.bitbucket.io/inla-gitbook/index.html).

Outline

1. Time series
2. Features of time series
3. Stationarity
4. Basic temporal models

Time series

Introduction

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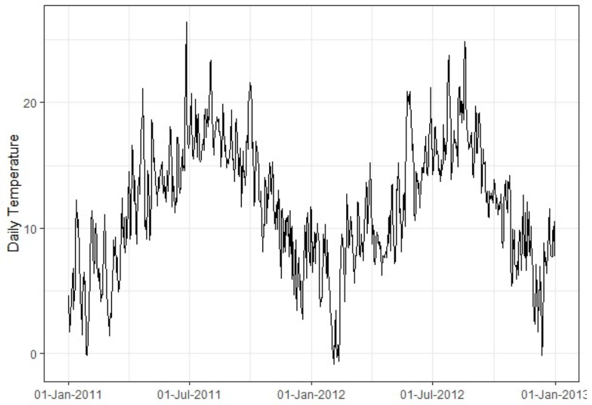
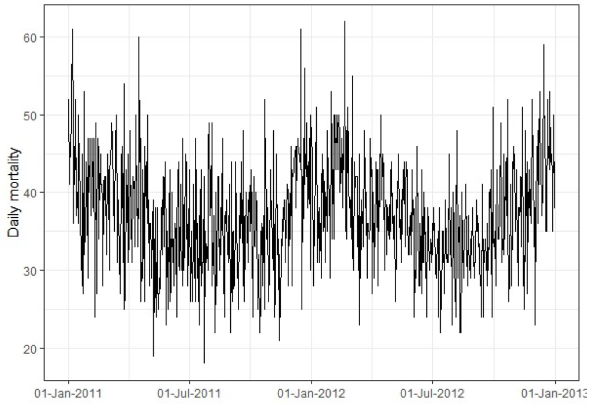
- Time also matter: the behaviour from one time point to the next is important and many data that we deal with in spatial analysis are actually both spatial and temporal in nature.
- Similarly to spatial dependence, it is sometime necessary to model temporal dependence on data and parameters.
- Unlike space, the temporal data hold a natural order.

Example of time series studies from environmental health studies

- In environmental epidemiology, time series have been widely used, notably for investigating the short-term associations between exposures such as air pollution or weather variables, and health outcomes such as cardiovascular and respiratory morbidity and mortality.
- Typically, for both exposure and outcome, data are available at regular time intervals (e.g. daily pollution levels and daily mortality counts) and the aim is to explore short-term associations between them.





Time series

- A time series is a set of observations taken sequentially in time.

- Depending on different applications, data may be collected hourly, daily, weekly, monthly, yearly, and so on.

- A time series that can be recorded continuously in time, is said to be continuous, while a time series that is taken only at specific time intervals is said to be discrete. We will work mainly with discrete time series data.

- We use notation such as: $\{Z_t : t \in \mathcal{D}_t\}$. Henceforth, we assume that $\mathcal{D}_t = \{0, 1, \ldots\}$ and we refer to $\{Z_t : t = 0, 1, \ldots\}$ as a time series.

- The natural (temporal) ordering in the time series creates an internal structure in the data, that shows, commonly, dependence in the observations, *such that values in the present depend upon observations available in the past*.

Autocorrelation or serial correlation

- Autocorrelation $\rightarrow$ the correlation of a variable with itself.

- Space: the correlation between the value of the variable at two different locations (or areas).

- Time series: the values of a variable at time t depends on the value of the same variable at time $t - h$, where h is the time-lag separation.

- Thus, autocorrelation is also sometimes called *lagged correlation* or *serial correlation*, which refers to the correlation between members of a series of numbers arranged in time.

Time series: mean, autocovariance and autocorrelation

A time series $\{Z_t\}$ has:

- Mean function: $\mu_t = E(Z_t)$
- Autocovariance function: $C(t,r) = Cov(Z_t, Z_r)$
- Autocorrelation function: $\rho(t,r) = \frac{C(t,r)}{\sqrt{C(t,t)C(r,r)}}$ where $\rho(t,r) \in [-1,1]$.

Notice that:

- The mean indicates the trend of the series
- The autocovariance function summarizes how the process co-varies across different time lags; we have $C(t,r) = C(r,t)$
- The variance is a special case of the autocovariance in which $C(t,t) = var(Z_t) = \sigma_t^2$

Features of time series

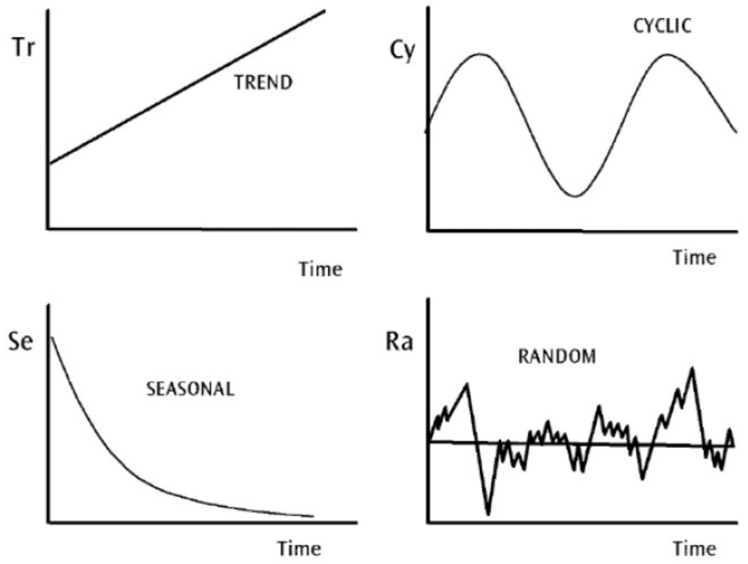
Components of time series

- Time series analysis typically presents challenges, as it exhibits patterns and irregular fluctuations.

Patterns can be specified as:

- Trend, that is the most common time series feature to account for and refers to long-term change in the mean level;
- Seasonal variation, which refers to periodic fluctuations which occur periodically within a year;
- Cyclic changes, which are recurrent rise and fall that are not of fixed period and are over a period longer than one year.

Irregular fluctuations are variations that are short in duration, following not regularity in the occurrence.



Stationarity

- In studying time series, a very important concept is given by stationary, that refers to the stability of the statistical properties of the process through time.
- Broadly speaking, a stationary process is one whose statistical properties do not change over time.
- There are two important forms of stationarity:
- strong stationarity;
- weak (or second order) stationarity.

Strong and weak stationarity

A time series is said to be strongly stationary if

- for any finite sequence of times $t_1, t_2, \ldots, t_n$ and any temporal lag h the probability distribution of the vector $(Z_{t_1}, \ldots, Z_{t_n})'$ is identical to the probability distribution of the vector $(Z_{t_1+h}, \ldots, Z_{t_n+h})'$.

- *In words: all aspects of the process's behavior are unaffected (unchanged) by a shift in time.*

A time series is said to be weak (or second order) stationary if

- $E(Z_t) = \mu$, i.e. the mean is constant for all t

- $var(Z_t) = \sigma^2$, i.e. the variance does not depend on t

- $Cov(Z_t, Z_r) = C(t - r)$, i.e. the autocovariance depends only on the on the elapsed time between t and r and not their actual location

- *In words: weak stationarity (only) concerns the shift-invariance of first and second moments of a process.*

Basic temporal Models

White noise process [1]

- A white noise process is a sequence of independent normally identically distributed random variables

- The term **noise** is due to the fact that there's no pattern, just random variation

- The Gaussian white noise is defined as:

$$W_t \overset{iid}{\sim} \mathrm{N}(0, \sigma^2_W)$$

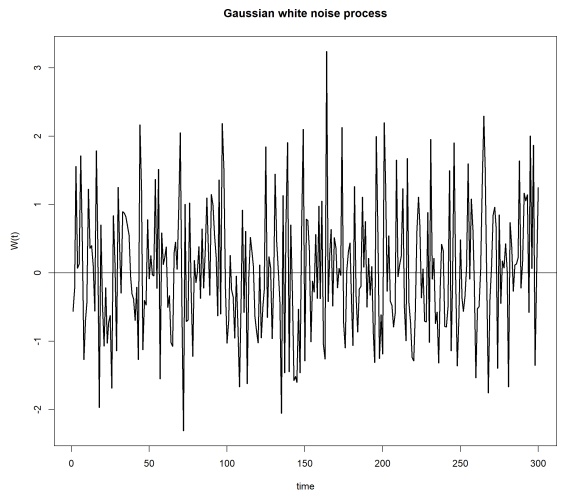
- This process is stationary:
  – $E[W_t] = 0$, i.e. the expectation is always constant and equal to zero
  – $var(W_t) = \sigma^2_W$, i.e. the variance is constant
  – $cov(W_t, W_r) = 0$ for $t \neq r$, i.e. the covariance is zero at all lags

- Note, iid stands for **independent and identically distributed**. An iid model assumes that observations on a phenomenon are taken under identically conditions and that each observation is taken independently of any other.

White noise process [2]

- Realization of a Gaussian white noise process $W_t \overset{iid}{\sim} \mathrm{N}(0, \sigma_W^2)$

```
> set.seed(123)  # set random number seed
> W = rnorm(300) # generate iid normal random va
> ts.plot(W, main="Gaussian white noise process",
+         xlab="time", ylab="W(t)",
+         col="black", lwd=2)
> abline(h=0)
```



- This figure shows that there are no discernible patterns and the distribution is completely random

- In `R-INLA` an iid Gaussian random effect is specified with the model `iid`

- To obtain details about the model `iid` we can type `inla.doc("iid")`

Random walk (RW) [1]

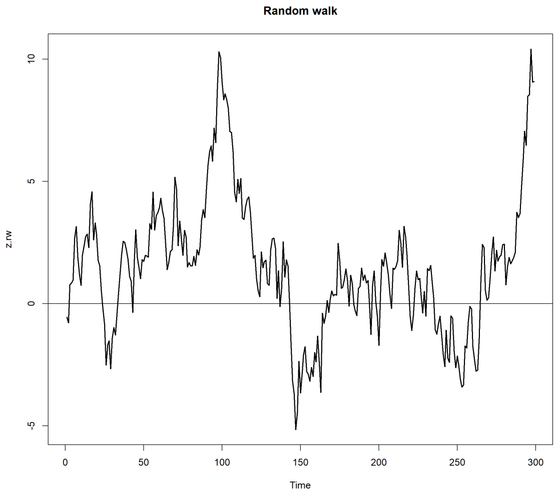
- The random walk (RW) describes how an observation directly depends upon one or more previous measurements plus a white noise process

- The random walk of order 1, RW1, is defined as:

$$Z_t = Z_{t-1} + W_t$$

  where W_t is a white noise process.
- Realization of a RW1

```
> set.seed(123) # set random number seed
> Z0 = 0 #Z0 is fixed
> T = 300
> W = c(Z0 + rnorm(T-1))
> z.rw = cumsum(W) # compute cumulative sum
> ts.plot(z.rw, main="Random walk",
+          lwd=2, col="black")
> abline(h=0)
```



- The RW1 only models the difference of levels on consecutive time points: $Z_t - Z_{t-1} = W_t$

Random walk (RW) [2]

- The RW is a non-stationary process (i.e. observations in a random walk are dependent on time)
- For a RW1, by recursively substitution, starting from $t = 1$, we have:

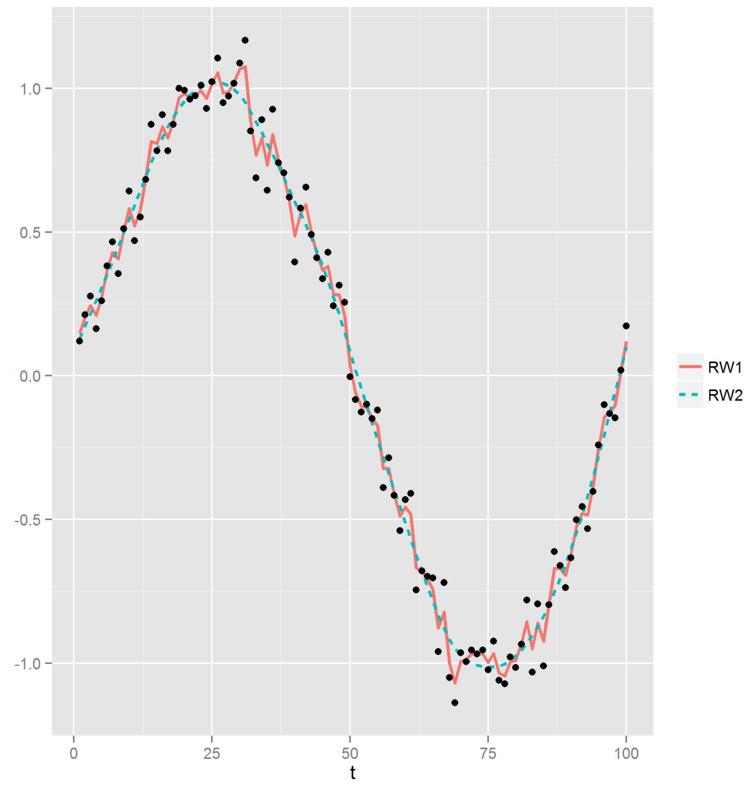
$$
\begin{aligned}
Z_1 &= Z_0 + W_1 \\
Z_2 &= Z_1 + W_2 = Z_0 + W_1 + W_2 \\
&\vdots \\
Z_t &= Z_0 + W_1 + \cdots + W_t \\
&= Z_0 + \sum_{j=1}^{t} W_j
\end{aligned}
$$

Hence, the first order moment (or the expected value) for this process is equal to:
$E(Z_t) = Z_0 + \sum_{j=1}^{t} E(W_j) = Z_0$, which is independent of t.

The variance is $var(Z_t) = var\left(\sum_{j=1}^{t} W_j \right) = \sum_{j=1}^{t} \sigma_W^2 = t\sigma_W^2$, which depends on t. Thus the random walk process $\{Z_t\}$ is not stationary.

- The random walk of order 2, RW2, is defined as: $Z_t = 2Z_{t-1} - Z_{t-2} + W_t$

- The RW2 only models a linear combination of levels on consecutive time points: $Z_t - 2Z_{t-1} + Z_{t-2} = W_t$



Parametrization RW1 in R-INLA

- The RW1 for the Gaussian vector $\boldsymbol{Z} = (Z_1, \ldots, Z_T)$ is constructed assuming independent increments:

$$\Delta Z_t = Z_t - Z_{t-1} \sim N(0, \tau^{-1})$$

- Hyperparameters: The precision parameter τ is represented as $\theta = log(\tau)$ and the prior is defined on θ

- Inclusion in the formula: `f(ID.time, model="rw1")`

Parametrization of RW2 in R-INLA

- The RW2 for the Gaussian vector $\boldsymbol{Z} = (Z_1, \dots, Z_T)$ is constructed assuming independent second order increments:

$$\Delta^2 Z_t = Z_t - 2Z_{t+1} + Z_{t+2} \sim N(0, \tau^{-1})$$

- Hyperparameters: the precision parameter τ is represented as $\theta = log(\tau)$ and the prior is defined on θ

- Inclusion in the formula: `f(ID.time, model="rw2")`

Autoregressive (AR) process [1]

- The autoregressive process of order (p), AR(p) is a time series model where the original data is expressed as a function of its previous values in time
- It is defined as:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + W_t$$

  where:
  - W_t is a Gaussian error term with mean zero and variance σ^2_W (i.e. a Gaussian white noise process)
  - $\{\phi_i : i = 1, \ldots, p\}$ is a sequence of unknown autoregressive parameters

- This class of models is called autoregressive because Z_t is regressed on past terms of the same process

- The simplest model is given by the AR1 (i.e. p=1) and is defined as:

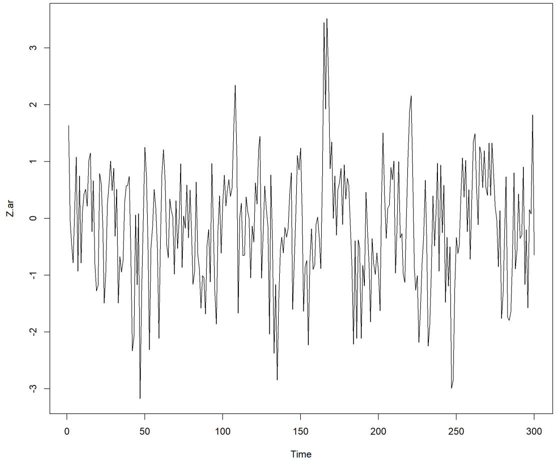
$$Z_t = \rho Z_{t-1} + W_t$$

  where $|\rho| < 1$ is the unknown temporal correlation term

Autoregressive (AR) process [2]

- Realization of an AR1 process

```
> set.seed(121)
> Z.ar = arima.sim(model=list(ar=.5), n=300)
> plot.ts(Z.ar)
```



- The AR1 process can be written as an infinite series of white noise random variables. Since $E(W_t) = 0$ and $var(W_t) = \sigma^2_W$, it follows that $E(Z_t) = 0$ and $var(Z_t) = \frac{\sigma^2_w}{1-\rho^2}$, which does not depend on t, thus the process is stationary. Note that if $\rho = 1$, the process is a random walk.

- In `R-INLA`, the AR1 model is implemented through the model specification `ar1`, while an AR model of arbitrary order is implemented through the specification `ar`.

- To obtain details about the specification of the AR1, and more in the general about the AR model, we can type `inla.doc("ar1")` and `inla.doc("ar")`.

To wrap up

A stationary time series doesn't exhibit trend or seasonality:

- Observations do not tend upwards or downwards
- Variance does not increase or decrease with time
- Observations do not tend to be large in some periods than others

References

- Broemeling D.L. (2019), Bayesian Analysis of Time Series, CRC Press
- Cressie N. and Wikle C.K. (2011), Statistics for spatio-temporal data, Wiley