

Session 1.2: Introduction to INLA and R-INLA

Geospatial Analytics using R and R-INLA

MRC
Centre for Environment & Health



Medical
Research
Council

Imperial College
London



Learning objectives

After this lecture you should be able to

- Present the class of latent Gaussian models
- Present the Laplace approximation and the INLA approach
- Use the basic functions of the R-INLA package

The topics treated in this lecture are presented in Chapter 4 of Blangiardo and Cameletti (2015).

Outline

1. MCMC to INLA
2. Latent Gaussian models
3. The INLA approach
4. R-INLA package

MCMC to INLA

From MCMC to INLA

- Standard MCMC sampler are generally easy-ish to program and are in fact implemented in readily available software
- MCMC methods are flexible and able to deal with virtually any type of data and model, but they involve computationally- and time- intensive simulations to obtain the posterior distribution for the parameters. For this reason the complexity of the model and the database dimension often remain fundamental issues.

From MCMC to INLA

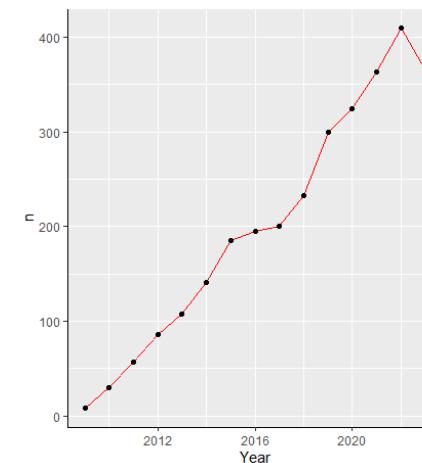
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- The INLA algorithm is designed for the class of *latent Gaussian models* and compared to MCMC it provides (as) accurate results in a shorter time.

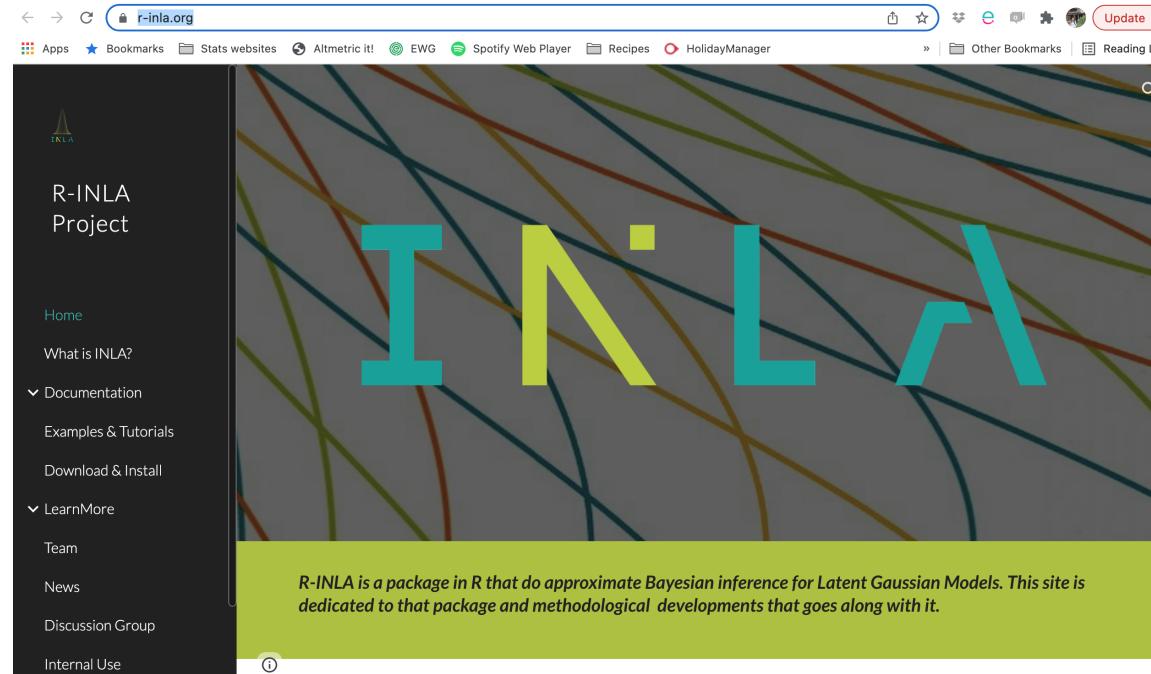
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- The INLA algorithm proposed by Rue, Martino, and Chopin (2009) is a *deterministic* algorithm for Bayesian inference and it represents an alternative to MCMC which is instead a simulation based algorithm.
- The INLA algorithm is designed for the class of *latent Gaussian models* and compared to MCMC it provides (as) accurate results in a shorter time.
- INLA has become very popular among statisticians and applied researchers and in the past few years the number of papers reporting usage and extensions of the INLA method has increased considerably.



INLA website and community

- The website contains source code, examples, papers and reports discussing the theory and applications of INLA.
- There is also a discussion forum where users can post queries and requests of help.
- Almost each year there is an INLA-related scientific meeting.



INLA website

Latent Gaussian models

Latent Gaussian models (LGMs)

- The general problem of (parametric) inference is posited by assuming a probability model for the observed data $\mathbf{y} = (y_1, \dots, y_n)$, as a function of some relevant parameters

$$\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\psi} \sim p(\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\psi}) = \prod_{i=1}^n p(y_i | \boldsymbol{\theta}, \boldsymbol{\psi})$$

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- Often (in fact for a surprisingly large range of models!), we can assume that the parameters are described by a **Gaussian Markov Random Field (GMRF)**

$$\boldsymbol{\theta} | \boldsymbol{\psi} \sim \text{Normal}(\mathbf{0}, \mathbf{Q}^{-1}(\boldsymbol{\psi}))$$

$$\theta_i \perp\!\!\!\perp \theta_j | \boldsymbol{\theta}_{-i,j} \iff Q_{ij}(\boldsymbol{\psi}) = 0$$

where

- The precision matrix \mathbf{Q} depends on some hyperparameters $\boldsymbol{\psi}$.
- The notation $-i, j$ indicates all the other elements of the parameters vector, excluding elements i and j
- The components of $\boldsymbol{\theta}$ are supposed to be *conditionally independent* with the consequence that \mathbf{Q} is a sparse precision matrix.
- This kind of models is often referred to as **Latent Gaussian models**.

LGMs as a general framework

- In general

$$\mathbf{y} \mid \boldsymbol{\theta}, \boldsymbol{\psi} \sim \prod_i p(y_i \mid \boldsymbol{\theta}, \boldsymbol{\psi}) \text{ (Data model)}$$

$$\boldsymbol{\theta} \mid \boldsymbol{\psi} \sim p(\boldsymbol{\theta} \mid \boldsymbol{\psi}) = \text{Normal}(0, \mathbf{Q}^{-1}(\boldsymbol{\psi})) \text{ (Latent Gaussian Field)}$$

$$\boldsymbol{\psi} \sim p(\boldsymbol{\psi}) \text{ (Hyperprior)}$$

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- The dimension of $\boldsymbol{\theta}$ can be very large (eg 10^2 - 10^5).
- Conversely, the dimension of $\boldsymbol{\psi}$ must be relatively small (less than 20 is recommended) to avoid an exponential increase in the computational costs of the model.

LGMs as a general framework

- A very general way of specifying the problem is specifying a distribution for y_i characterized by a parameter ϕ_i (usually the mean) defined as a function of a structured additive predictor η_i , defined on a suitable scale, such that $g(\phi_i) = \eta_i$ (e.g. logistic for binomial data):

$$\eta_i = \beta_0 + \sum_{m=1}^M \beta_m x_{mi} + \sum_{l=1}^L f_l(z_{li})$$

where

- β_0 is the intercept;
- $\beta = \{\beta_1, \dots, \beta_M\}$ quantify the effect of the covariates $x = (x_1, \dots, x_M)$ on the response;
- $f = \{f_1(\cdot), \dots, f_L(\cdot)\}$ is a set of functions defined in terms of some covariates $z = (z_1, \dots, z_L)$

and then assume

$$\theta = \{\beta_0, \beta, f\} \sim \text{Normal}(\mathbf{0}, \mathbf{Q}^{-1}(\psi)) = \text{GMRF}(\psi)$$

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- NB: This of course implies some form of Normally-distributed marginals for β_0 , β and f

LGMs as a general framework --- examples

Upon varying the form of the functions $f_l(\cdot)$, this formulation can accommodate a wide range of models (see Martins, Simpson, Lindgren, and Rue (2013) for a review)

- Standard regression
 - $f_l(\cdot) = \text{NULL}$

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- Spatial and spatio-temporal models
 - Areal data: $f_1(\cdot) \sim \text{CAR}$ (Spatially structured effects)
 $f_2(\cdot) \sim \text{Normal}(0, \sigma_{f_2}^2)$ (Unstructured residual)
 - Geostatistical data: $f(\cdot) \sim \text{Gaussian field}$
 - Temporal component: $f(\cdot) \sim \text{RW}$

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 - $f_l(\cdot) \sim \text{AR}(\phi, \sigma_\varepsilon^2)$
- Survival models, logGaussian Cox Processes, etc.

The INLA approach

Integrated Nested Laplace Approximation (INLA)

- The first *ingredient* of the INLA approach is the definition of conditional probability, which holds for any pair of variables (x, z) .

Technically, provided $p(z) > 0$

$$p(x \mid z) =: \frac{p(x, z)}{p(z)} \rightarrow p(x, z) = p(x \mid z)p(z)$$

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- In particular, a conditional version can be obtained further considering a third variable w as

$$p(z | w) = \frac{p(x, z | w)}{p(x | z, w)}$$

which is particularly relevant to the Bayesian case.

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- Main idea: approximate $\log f(x)$ using a quadratic function by means of a Taylor's series expansion around the mode $x^* = \operatorname{argmax}_x \log f(x)$

$$\begin{aligned}\log f(x) &\approx \log f(x^*) + (x - x^*) \frac{\partial \log f(x)}{\partial x} \Big|_{x=x^*} + \frac{(x - x^*)^2}{2} \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*} \\ &= \log f(x^*) + \frac{(x - x^*)^2}{2} \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*} \quad \left(\text{since } \frac{\partial \log f(x)}{\partial x} \Big|_{x=x^*} = 0 \right)\end{aligned}$$

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- Setting $\sigma^{2*} = -1 / \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*}$ we can re-write

$$\log f(x) \approx \log f(x^*) - \frac{1}{2\sigma^{2*}}(x - x^*)^2$$

or equivalently

$$\int f(x)dx = \int \exp[\log f(x)]dx \approx f(x^*) \int \exp\left[-\frac{(x - x^*)^2}{2\sigma^{2*}}\right] dx$$

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- Thus, under LA, $f(x) \approx \text{Normal}(x^*, \sigma^{2*})$.

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- Consequently, we can approximate $f(x)$ as

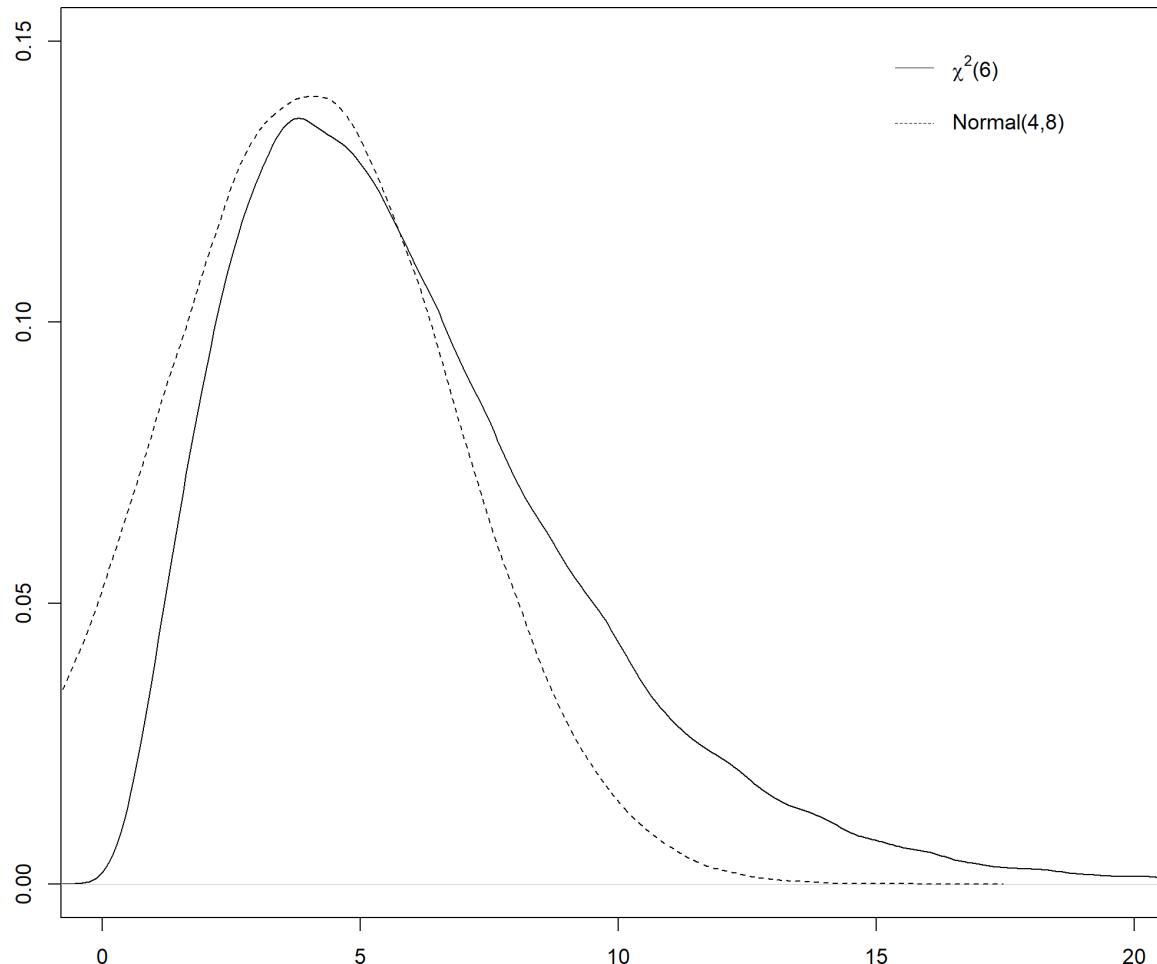
$$f(x) \approx \text{Normal}(k - 2, 2(k - 2))$$

Laplace approximation --- example

Fig 1

Fig 2

Fig 3

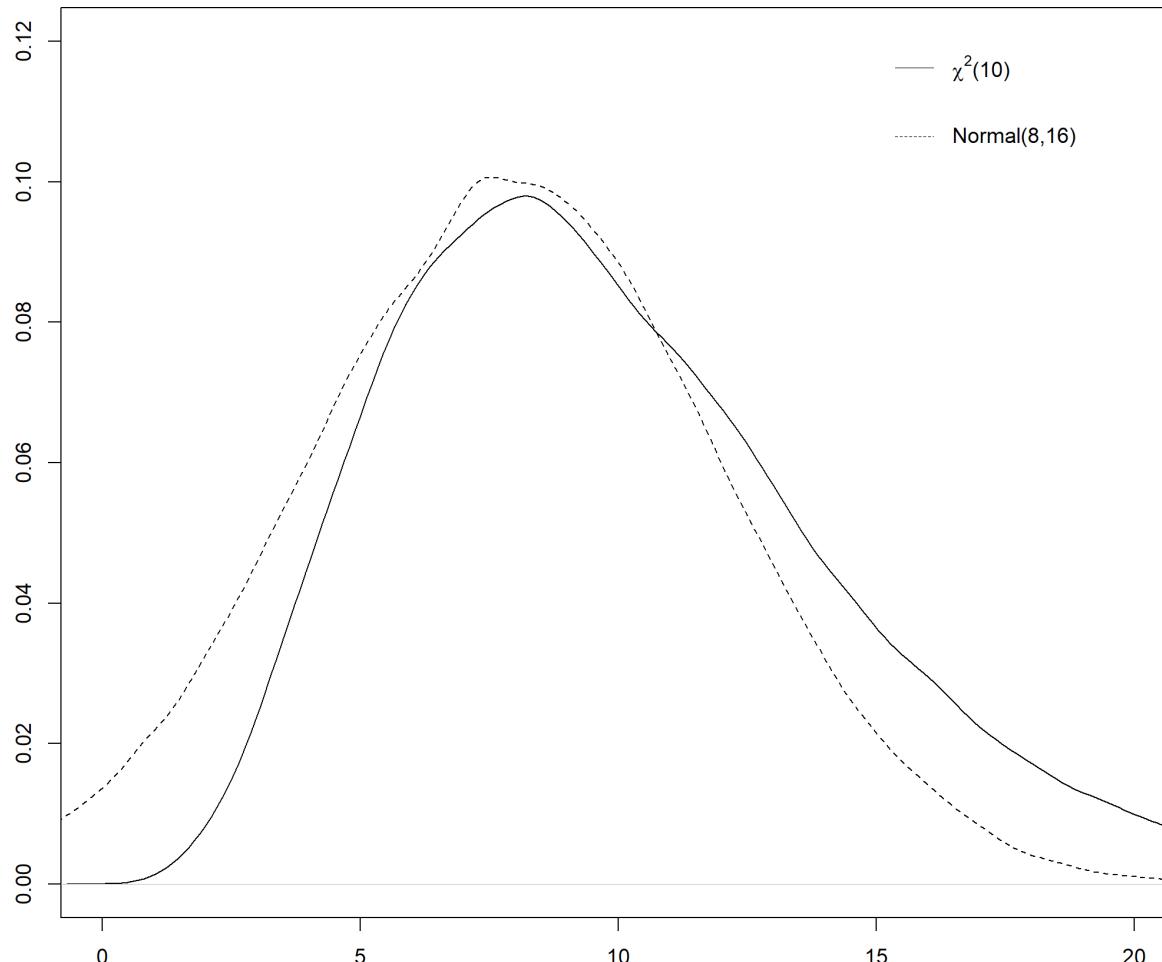


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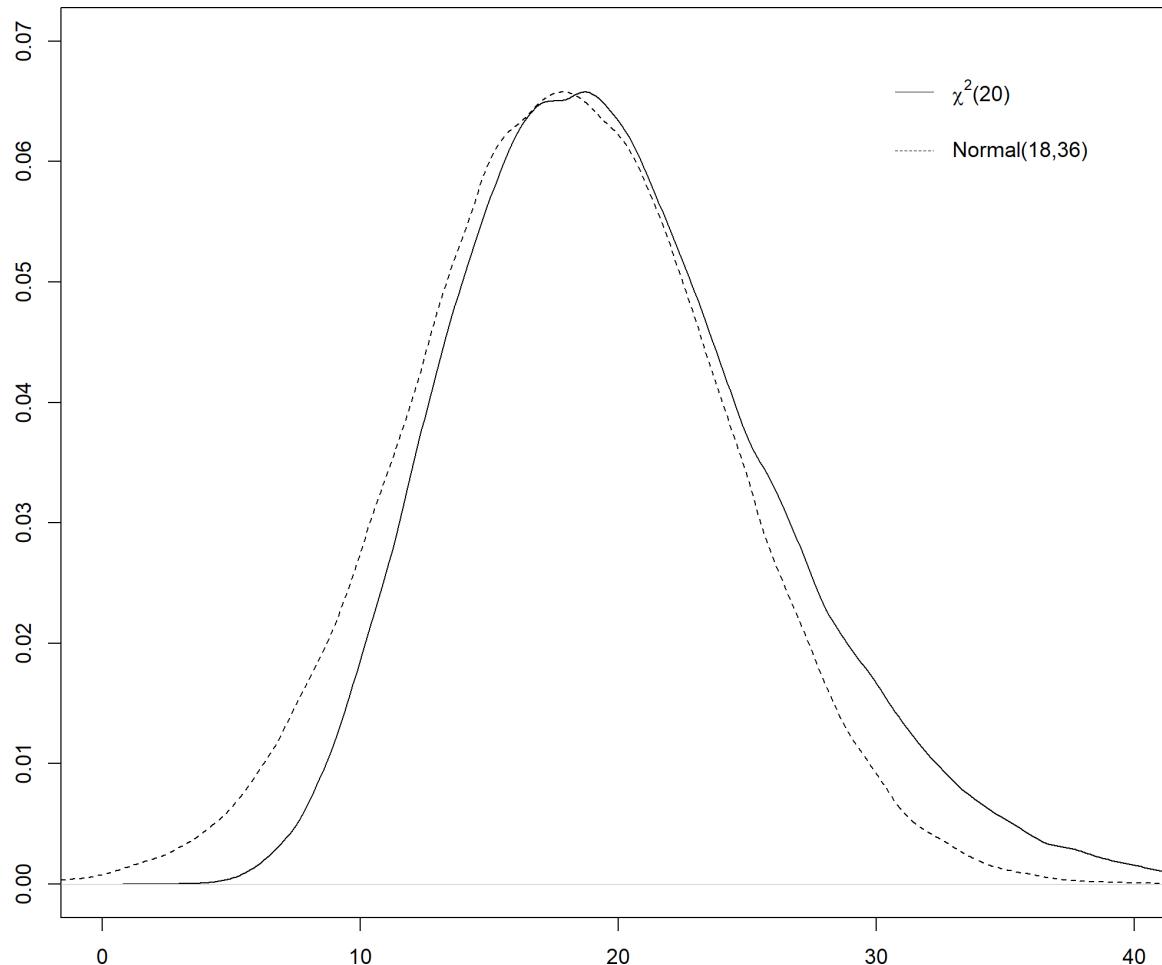


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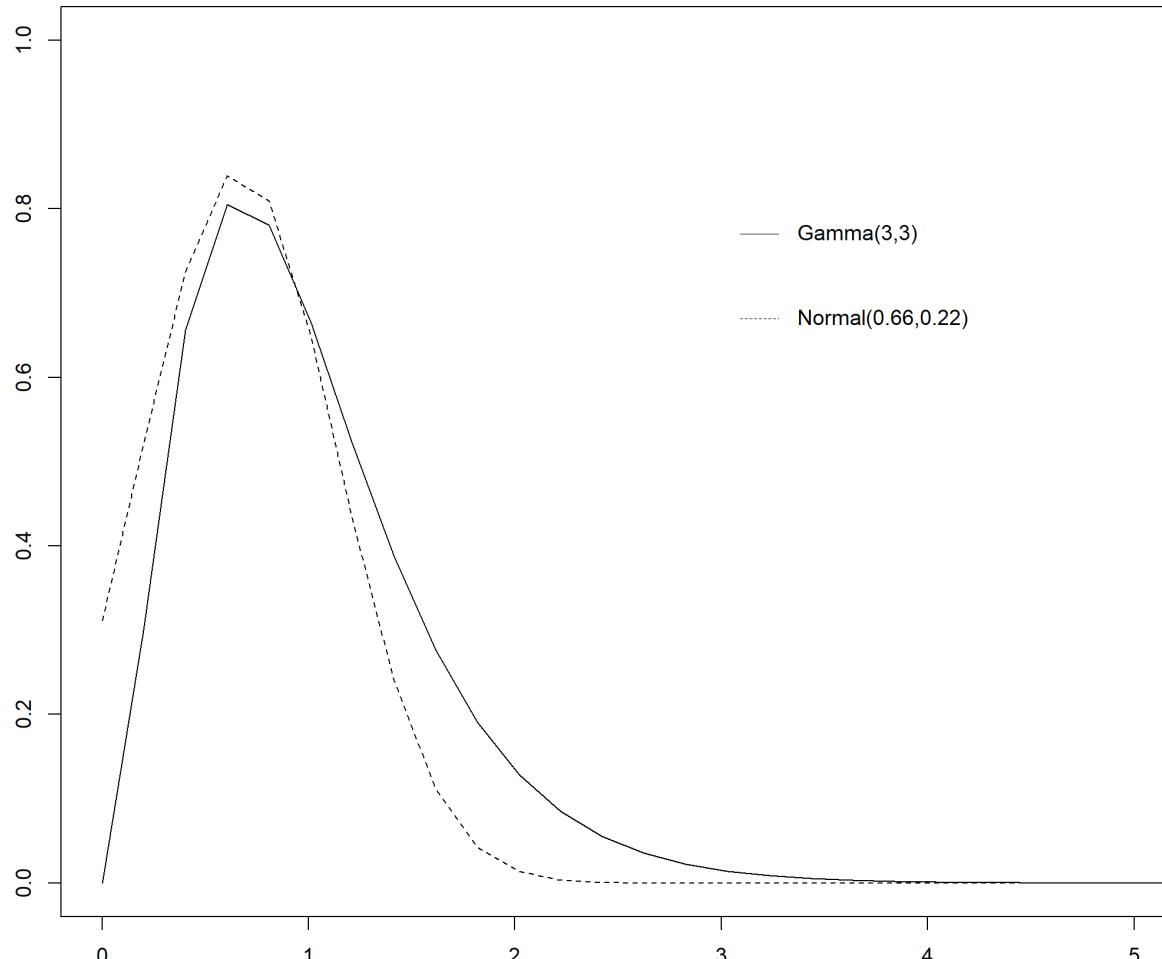
$$\text{Gamma}(a, b) \approx \text{Normal}\left(x^* = \frac{a-1}{b}, \sigma^{2*} = \frac{a-1}{b^2}\right)$$

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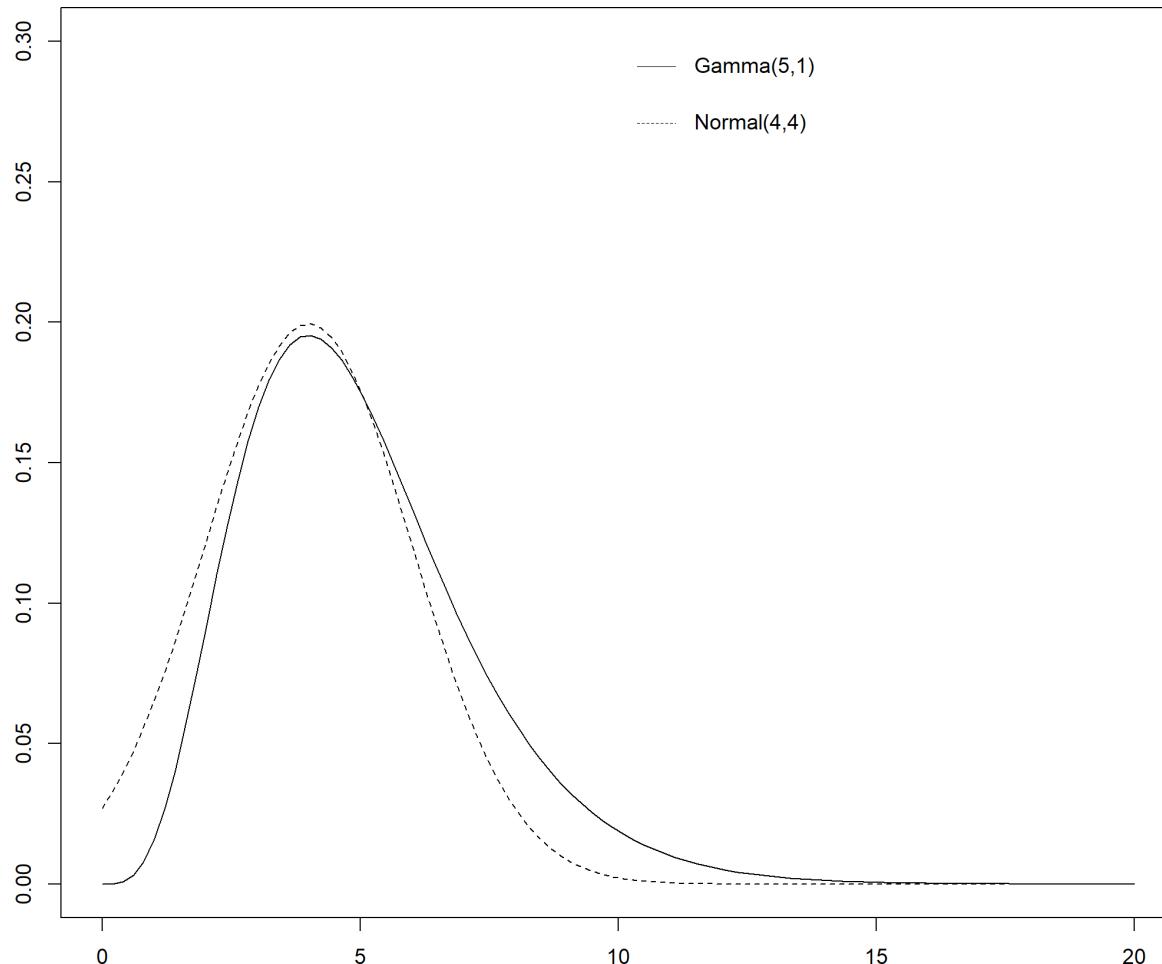


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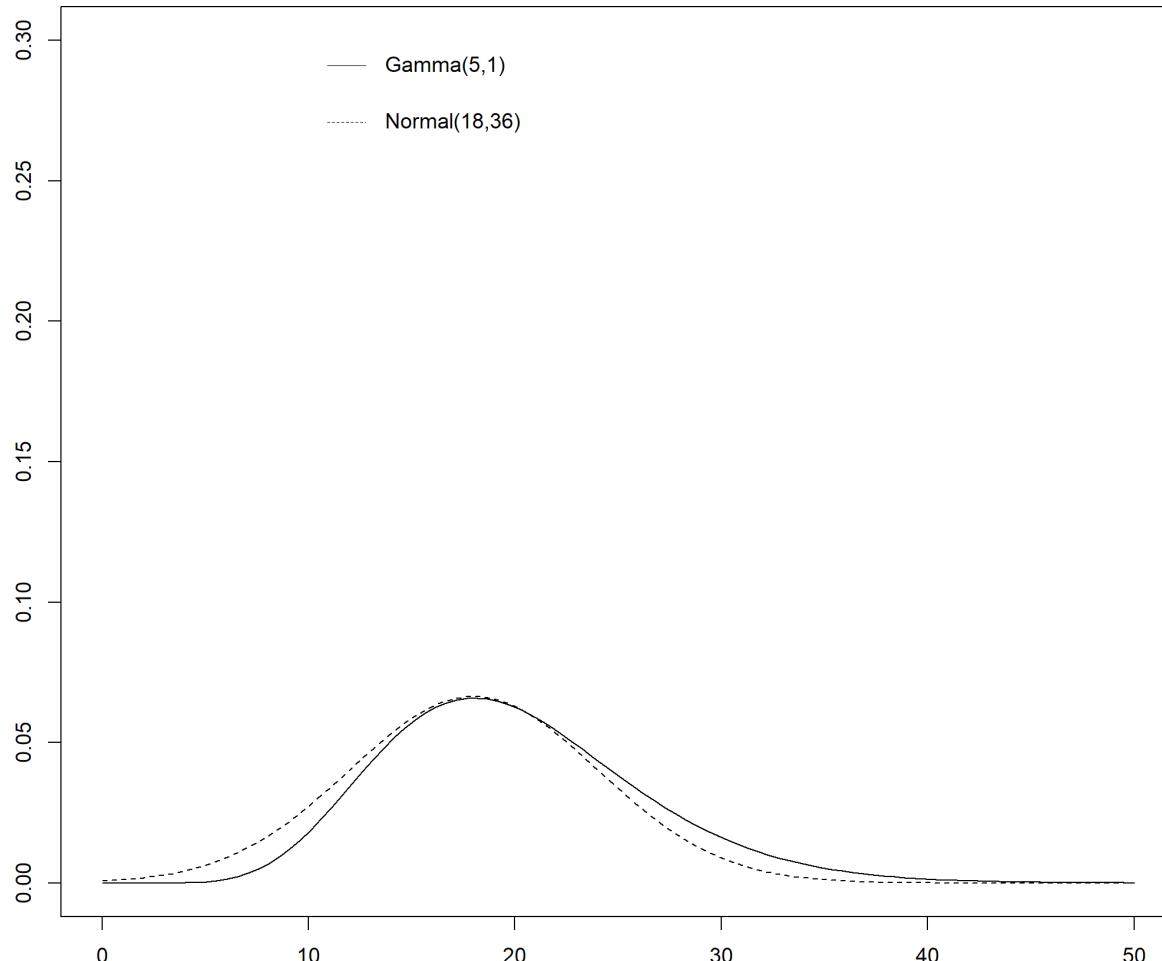


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Integrated Nested Laplace Approximation (INLA)

Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

$$p(\theta_i \mid \mathbf{y}) = \int p(\theta_i, \boldsymbol{\psi} \mid \mathbf{y}) d\boldsymbol{\psi} = \int \color{red}{p(\boldsymbol{\psi} \mid \mathbf{y})} \color{orange}{p(\theta_i \mid \boldsymbol{\psi}, \mathbf{y})} d\boldsymbol{\psi}$$

$$p(\psi_k \mid \mathbf{y}) = \int \color{red}{p(\boldsymbol{\psi} \mid \mathbf{y})} d\boldsymbol{\psi}_{-k}$$

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 2. $p(\theta_i | \boldsymbol{\psi}, \mathbf{y})$, which is needed to compute the marginal posterior for the parameters

Integrated Nested Laplace Approximation (INLA)

1. can be easily estimated as

$$p(\psi \mid \mathbf{y}) = \frac{p(\theta, \psi \mid \mathbf{y})}{p(\theta \mid \psi, \mathbf{y})} \text{ (Recall slide 17)}$$

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where

- $\tilde{p}(\theta \mid \psi, \mathbf{y})$ is the Gaussian approximation given by the Laplace method of $p(\theta \mid \psi, \mathbf{y})$
- $\theta = \theta^*(\psi)$ is its mode for a given ψ

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Full Laplace Approximation: alternatively, we can write $\theta = \{\theta_i, \theta_{-i}\}$, use the definition of conditional probability and again Laplace approximation to obtain $\tilde{p}(\theta_i | \psi, \mathbf{y})$ (very computationally expensive)

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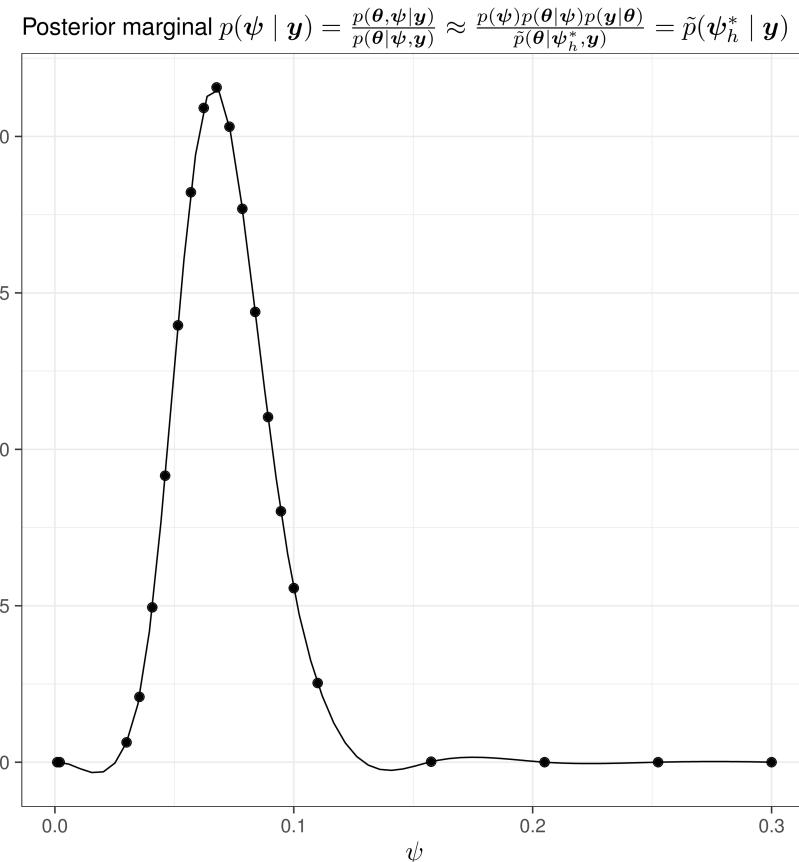
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The most efficient algorithm is the **Simplified Laplace Approximation**:

- Based on a Taylor's series expansion of the Laplace approximation $\tilde{p}(\theta_i | \psi, \mathbf{y})$.
- The accuracy of this approximation is sufficient in many applied cases and that the computing time is considerably shorter, it is the standard option.

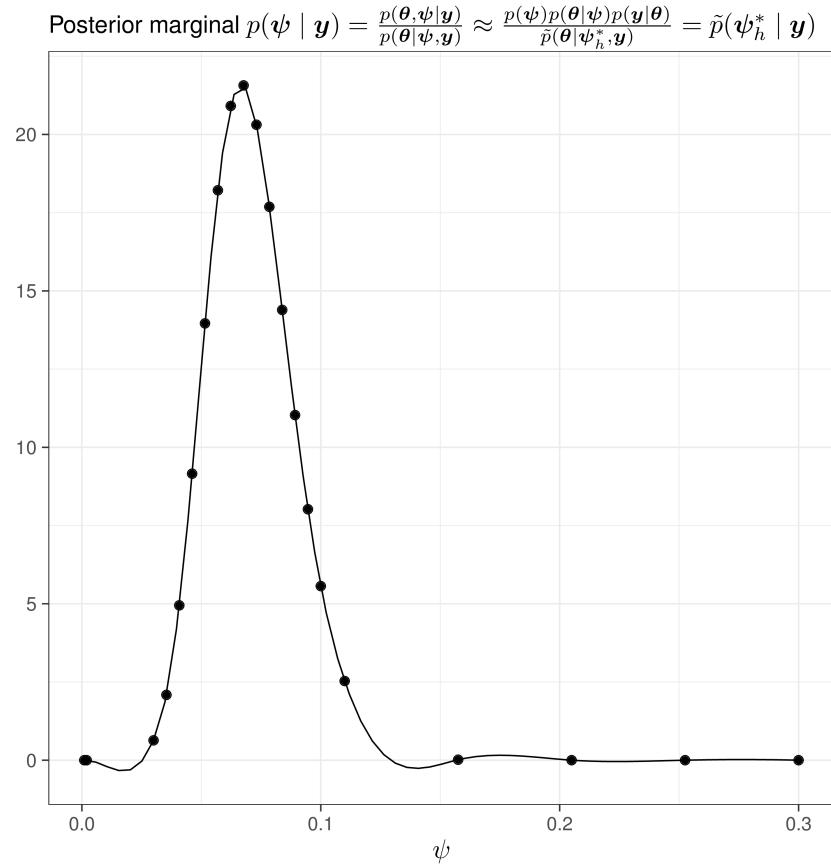
INLA – in a nutshell...

- I Select a grid of H points $\{\psi_h^*\}$ and area weights $\{\Delta_h\}$; interpolate the density to approximate to the posterior

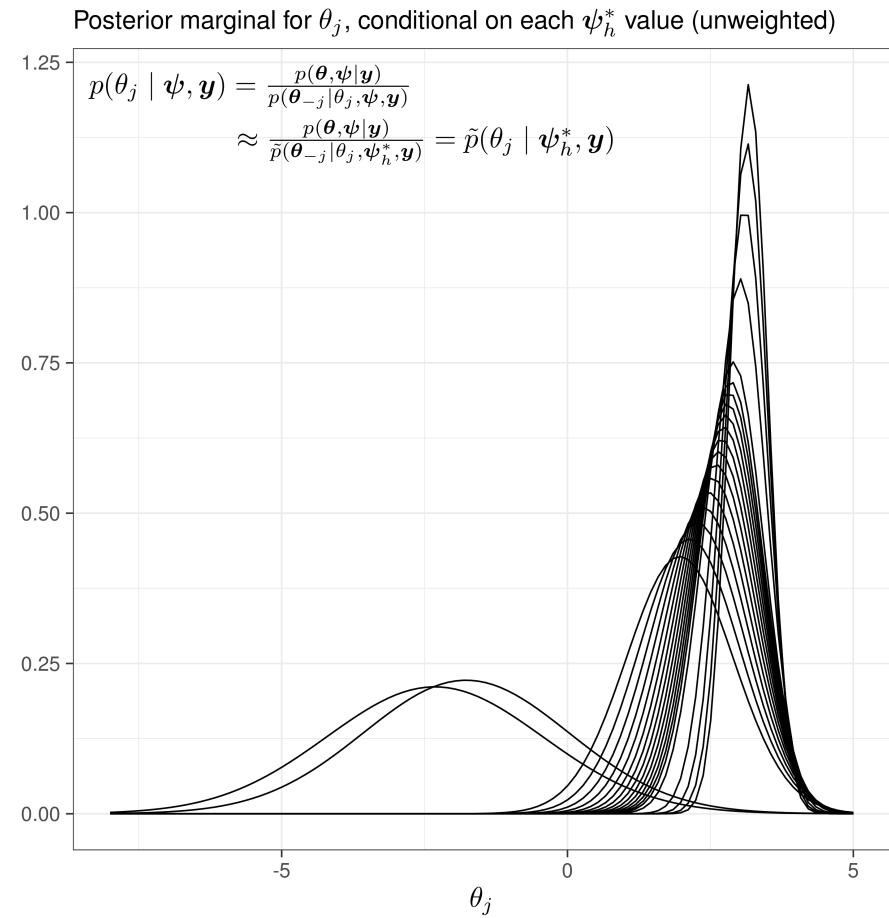


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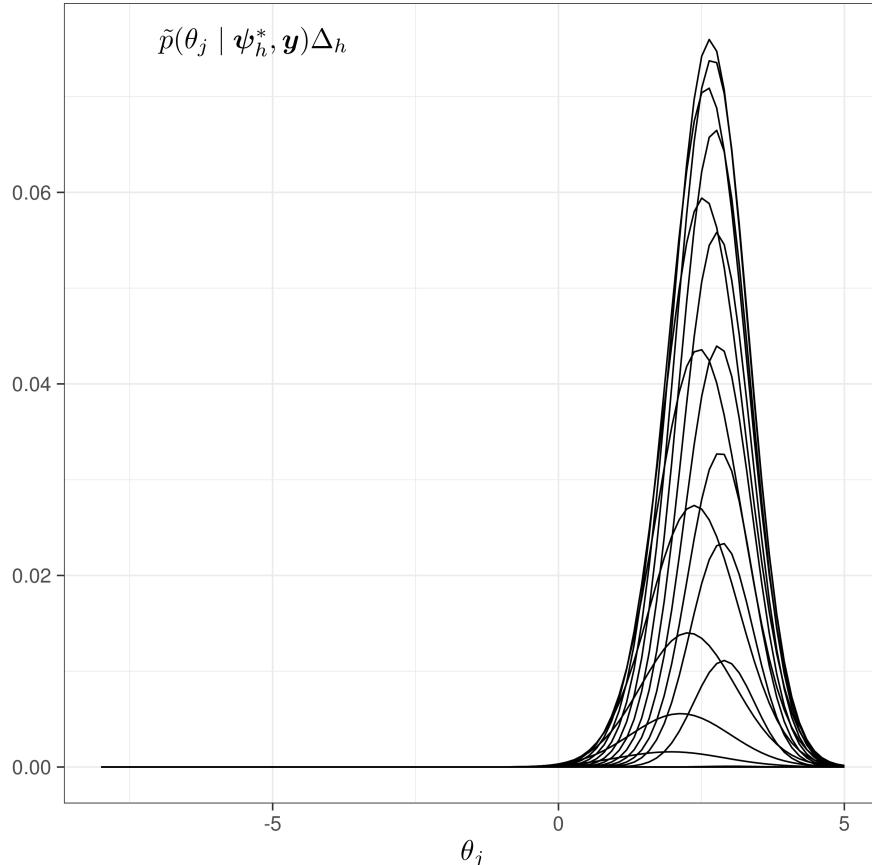
- 2 Approximates the conditional posterior of each θ_j , given ψ, \mathbf{y} on the H -dimensional grid



INLA – in a nutshell...

- 3 Weight the conditional marginal posteriors by the density associated with each ψ_h^*

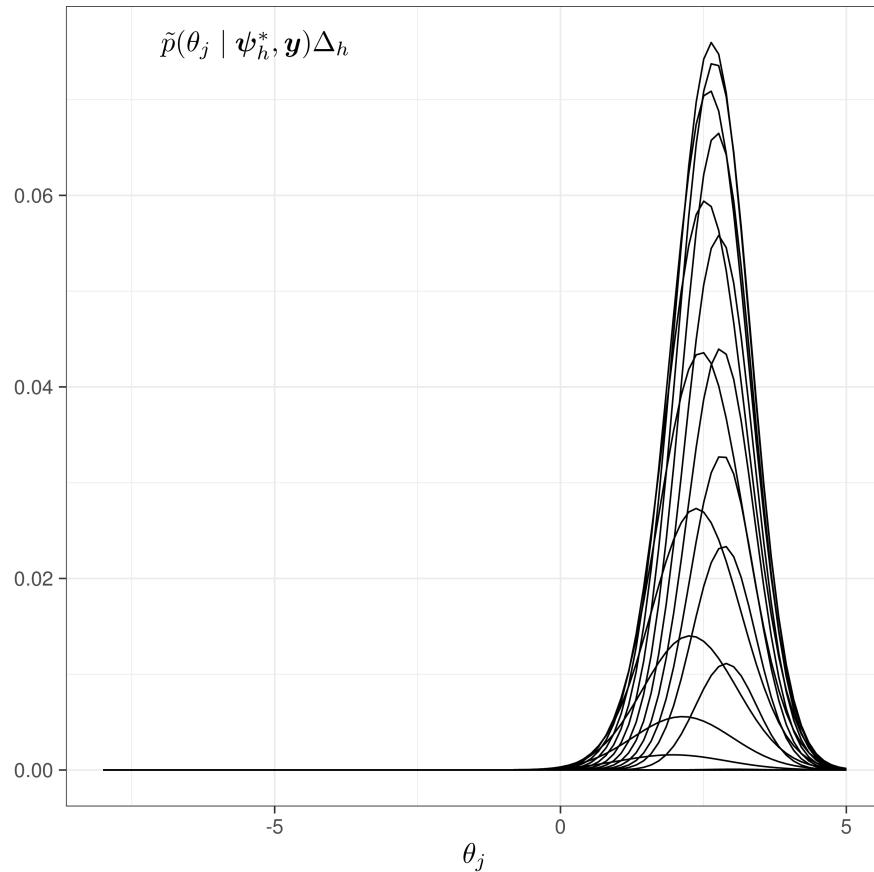
Posterior marginal for θ_j , conditional on each ψ_h^* value (weighted)



INLA – in a nutshell...

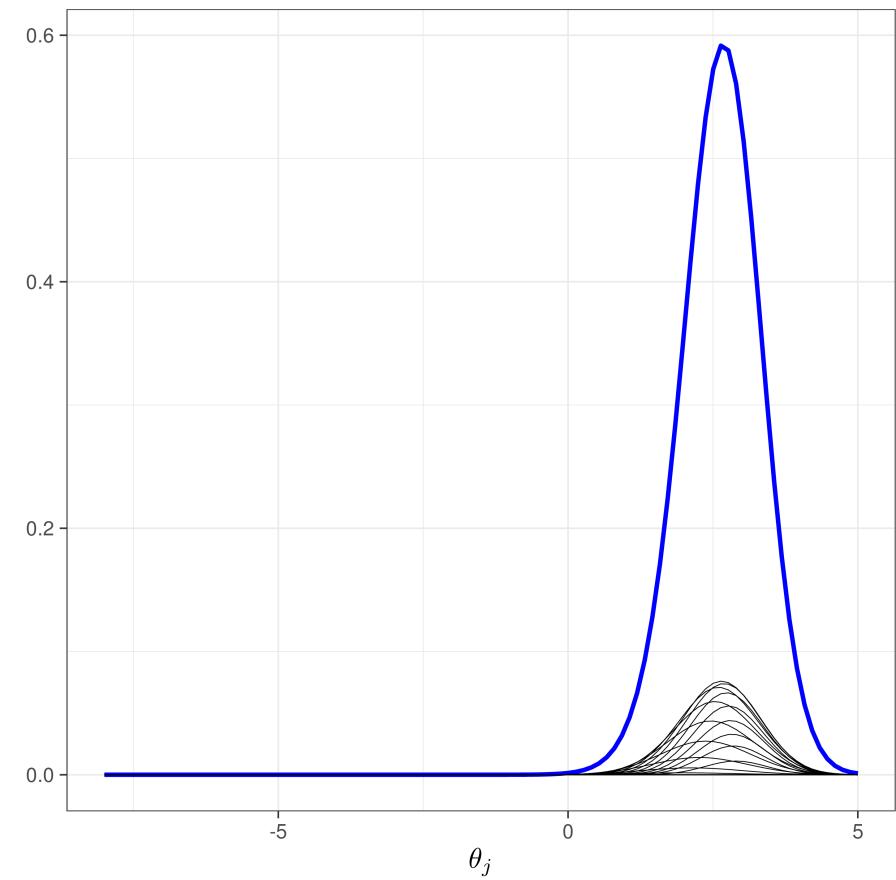
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Posterior marginal for θ_j , conditional on each ψ_h^* value (weighted)



4 (Numerically) sum over all the conditional densities to obtain the marginal posterior for θ_j

Posterior marginal for θ_j : $p(\theta_j | \mathbf{y}) \approx \sum_h \tilde{p}(\psi_h^* | \mathbf{y})\tilde{p}(\theta_j | \psi_h^*, \mathbf{y})\Delta_h$



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- Because the Laplace approximations are nested within one another
 - Since (1.) is needed to estimate (2.)
 - NB: Consequently the estimation of (1.) might not be good enough, but it can be refined (eg using a finer grid)
- Because the required marginal posterior distributions are obtained by (numerical) integration.

The R-INLA package

The INLA package for R

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1. The [GMRFLib](#) library

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- A standalone C program build upon the GMRFLib library (it performs the relevant computation and returns the results in a standardised way)

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2. The [inla](#) program

- A standalone C program build upon the GMRFLib library (it performs the relevant computation and returns the results in a standardised way)

NB: Because the package R-INLA relies on a standalone C program (and other reasons...), it is not available directly from CRAN.

R-INLA runs natively under Linux, Windows and Mac and it is possible to do multi-threading using OpenMP

The INLA package R - Installation

- From R, installation of the stable version is performed typing

```
install.packages("INLA", repos=cgetOption("repos"), INLA="https://inla.r-inla-download.org/R/stable"),  
dep=TRUE)
```

- Later, you can upgrade the package by typing

```
library(INLA)  
inla.upgrade()
```

- A test-version (which may contain unstable updates/new functions) can be obtained by typing

```
inla.upgrade(testing=TRUE)
```

- Type `inla.version()` to find out the installed version

Step by step guide to using R-INLA

1. The first thing to do is to **specify the model**

- For example, assume we have a generic model

$$y_i \stackrel{iid}{\sim} p(y_i | \theta_i)$$
$$\eta_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + f(z_i)$$

where

- $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ are observed covariates
- $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2) \sim \text{Normal}(0, \tau_1^{-1})$ are unstructured (*fixed*) effects
- \mathbf{z} is an **index**. This can be used to include structured (*random*), spatial, spatio-temporal effect, etc.
- $f \sim \text{Normal}(0, Q_f^{-1}(\tau_2))$ is a suitable function used to model the structured effects

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- $f \sim \text{Normal}(0, \mathbf{Q}_f^{-1}(\tau_2))$ is a suitable function used to model the structured effects
- As mentioned earlier, this formulation can actually be used to represent quite a wide class of models!

Step by step guide to using R-INLA

- The model is translated in R code using a **formula**
- This is sort of standard in R (you would do pretty much the same for calls to functions such as `lm` or `glm`)

```
formula = y ~ 1 + x1 + x2 + f(name=z, model="...", hyper=...)
```

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`formula = y ~ 1 + x1 + x2 + f(name=z, model="...", hyper=...)`
- The **f()** function can account for several structured nonlinear effects. We have for example:
 - **iid** specify independent random effects
 - **rw1, rw2, ar1** are smooth effect of covariates or time effects
 - **besag** models spatially structured effects (CAR)
 - **generic** is a user-defined precision matrix
- Type `inla.list.models("latent")` for the complete list and find descriptions at <https://www.r-inla.org/documentation>
- Some options can be specified for the **f()** term: for example **hyper** is used to specify the prior on the hyperparameters (more later).

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- Some options can be specified for the **f()** term: for example **hyper** is used to specify the prior on the hyperparameters (more later).
- It is possible to include in the formula several **f()** terms specifying them separately, e.g.

```
formula <- y ~ x1 + x2 + f(z1,...) + f(z2,...) + f(z3,...)
```

Step by step guide to using R-INLA

2. Call the function `inla` to fit the model, specifying the data and options (more on this later), e.g.

```
m = inla(formula, family="...", data=data.frame(y,x1,x2,z))
```

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- The data need to be included in a suitable `data.frame`
- The distribution of the data (i.e. the likelihood) is specified with the `family` option.
 - Type `'inla.list.models("likelihood")'` for the complete list of likelihood function (we have `poisson`, `binomial`, `gamma`, `beta`, `gaussian` and many others).

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Step by step guide to using R-INLA

The `control.xxx=list(...)` statements in the `inla` function control various part of the INLA program:

- `control.compute`: for computing measures of fit (eg DIC)
- `control.predictor`: for specifying the *Observation matrix A* which links the latent field to the data
- `control.family`: for changing the prior distribution of the likelihood hyperparameters
- `control.fixed`: for changing the prior distribution of the fixed effects
- `control.inla`: for changing the strategy to use for the approximations ('gaussian', 'simplified.laplace' (default) or 'laplace') or the grid exploration strategy
- and many others for expert use.

Step by step guide to using R-INLA

R returns an object `m` in the class `inla`, which has some methods available as for example `summary()` and `plot()`.

Some of the elements in an `inla` object returned by a call to `inla()` are:

Name	Description
<code>summary.fitted.values</code>	Summary statistics of fitted values.
<code>summary.fixed</code>	Summary statistics of fixed effects.
<code>summary.random</code>	Summary statistics of random effects.
<code>summary.linear.predictor</code>	Summary statistics of linear predictors.
<code>marginals.fitted.values</code>	Posterior marginals of fitted values.
<code>marginals.fixed</code>	Posterior marginals of fixed effects.
<code>marginals.random</code>	Posterior marginals of random effects.
<code>marginals.linear.predictor</code>	Posterior marginals of linear predictors.
<code>mlik</code>	Estimate of marginal likelihood.

Source: Gómez-Rubio (2020), Section 2.3.1

Toy example

We consider the `iris` dataset included in the `R` datasets (see `?iris`), regarding the measurements in centimeters of the variables *sepal length* and *width* and *petal length* and *width*, respectively, for 50 flowers from each of 3 species of iris. The species are Iris setosa, versicolor, and virginica. See Section 2.6 of the INLA book.

```
> summary(iris)
```

Sepal.Length	Sepal.Width	Petal.Length	Petal.Width	Species
Min. :4.300	Min. :2.000	Min. :1.000	Min. :0.100	setosa :50
1st Qu.:5.100	1st Qu.:2.800	1st Qu.:1.600	1st Qu.:0.300	versicolor:50
Median :5.800	Median :3.000	Median :4.350	Median :1.300	virginica :50
Mean :5.843	Mean :3.057	Mean :3.758	Mean :1.199	
3rd Qu.:6.400	3rd Qu.:3.300	3rd Qu.:5.100	3rd Qu.:1.800	
Max. :7.900	Max. :4.400	Max. :6.900	Max. :2.500	

We specify a simple regression model with Petal.length and Petal.width as dependent and independent variables, respectively:

$$\text{Petal.length}_i \sim \text{Normal}(\eta_i, 1/\sigma_e^2)$$
$$\eta_i = \beta_0 + \beta_1 \text{Petal.width}_i$$

Toy example: run INLA + exploring the fixed effects output

```
> library(INLA)
> formula <- Petal.Length ~ 1 + Petal.Width
> output <- inla(formula, family="gaussian", data=iris)
```

```
> output$summary.fixed
```

	mean	sd	0.025quant	0.5quant	0.975quant	mode	kld
(Intercept)	1.083565	0.07286615	0.9404841	1.083565	1.226646	1.083565	3.639717e-09
Petal.Width	2.229935	0.05132520	2.1291515	2.229935	2.330718	2.229935	3.639633e-09

- For each unstructured *fixed effect*, R-INLA reports a set of summary statistics from the posterior distribution.

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- For each unstructured *fixed effect*, R-INLA reports a set of summary statistics from the posterior distribution.
- The value of the Kullback-Leibler divergence *kld* describes the difference between the Gaussian approximation and the Simplified Laplace Approximation (SLA) to the marginal posterior densities:
 - Small values indicate that the posterior distribution is well approximated by a Normal distribution
 - If so, the more sophisticated SLA gives a *good* error rate and therefore there is no need to use the more computationally intensive *full* Laplace approximation.

Exploring the output: hyperparameters

```
> output$summary.hyperpar
```

	mean	sd	0.025quant	0.5quant	0.975quant	mode
Precision for the Gaussian observations	4.431766	0.5117049	3.486716	4.412179	5.490152	4.372817

- For each hyperparameter the summary statistics are reported to describe the posterior distribution.
- **NB:** INLA reports results on the **precision scale** (more on this later).

Manipulating the marginals from R-INLA: fixed effects

marginals1

marginals2

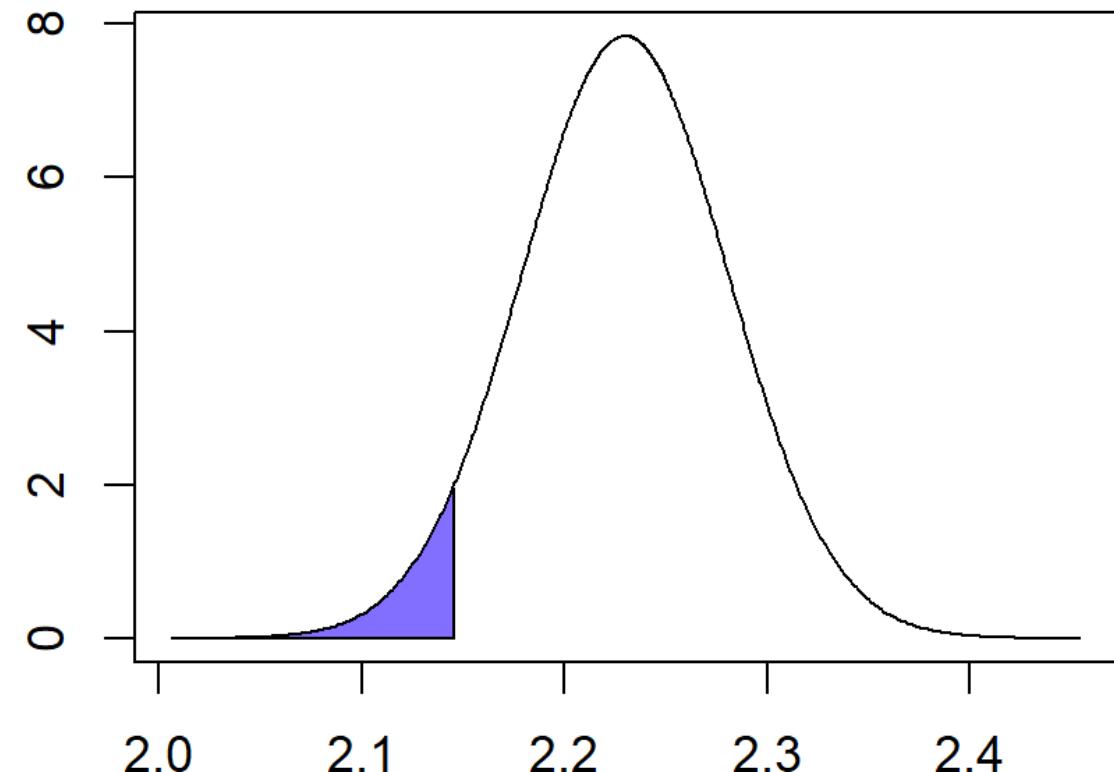
```
> names(output$marginals.fixed)
```

```
[1] "(Intercept)" "Petal.Width"
```

```
> beta1_post <- output$marginals.fixed[[2]]  
> marg <- inla.smarginal(beta1_post)  
> q <- inla.qmarginal(0.05, beta1_post)
```

```
> plot(marg, t="l",  
+       ylab="", xlab="",  
+       main=expression(paste("p(",beta[1], "| y"))  
> polygon(c(marg$x[marg$x <= q ], q),  
+           c(marg$y[marg$x <= q ], 0),  
+           col = "slateblue1", border = 1)
```

$$p(\beta_1 | y)$$



Manipulating the marginals from R-INLA: fixed effects

marginals1

marginals2

```
> inla.pmarginal(q,beta1_post)
```

```
[1] 0.05
```

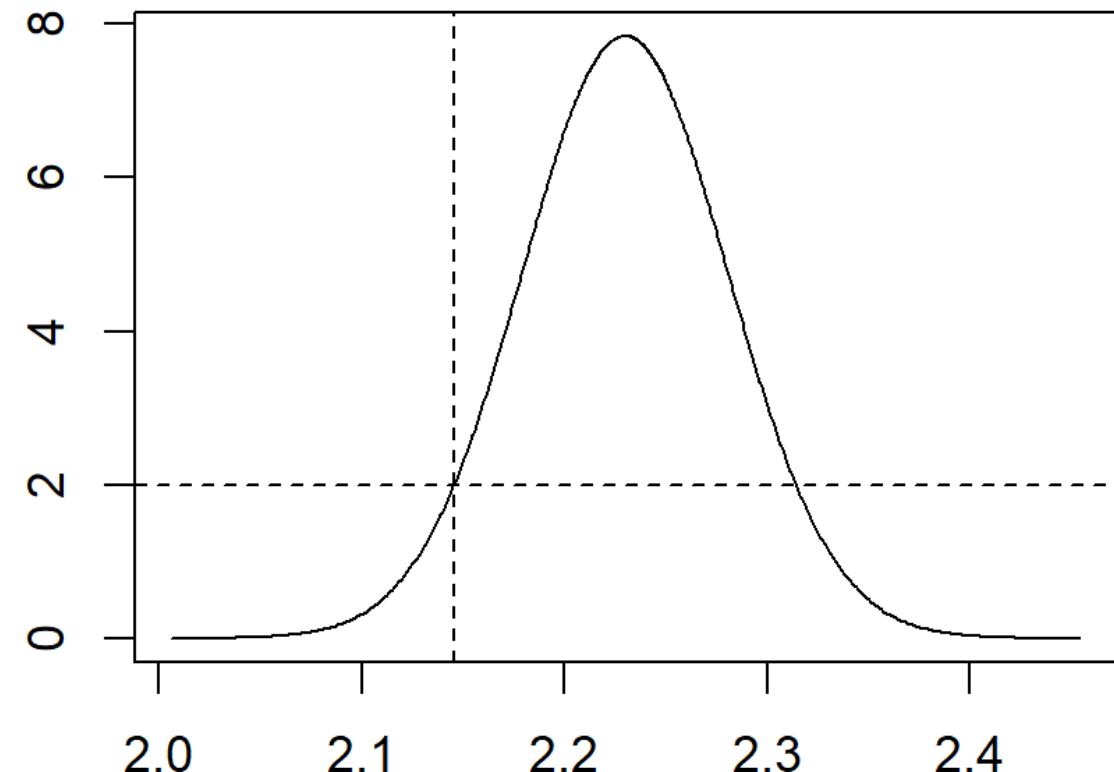
```
> d <-inla.dmarginal(q,beta1_post)  
> d
```

```
[1] 1.994373
```

```
> inla.rmarginal(4, beta1_post)
```

```
[1] 2.346620 2.178577 2.102739 2.316937
```

$$p(\beta_1 | y)$$



Manipulating the marginals from R-INLA: hyperparameters

INLA works with the precision by default

Precision Variance

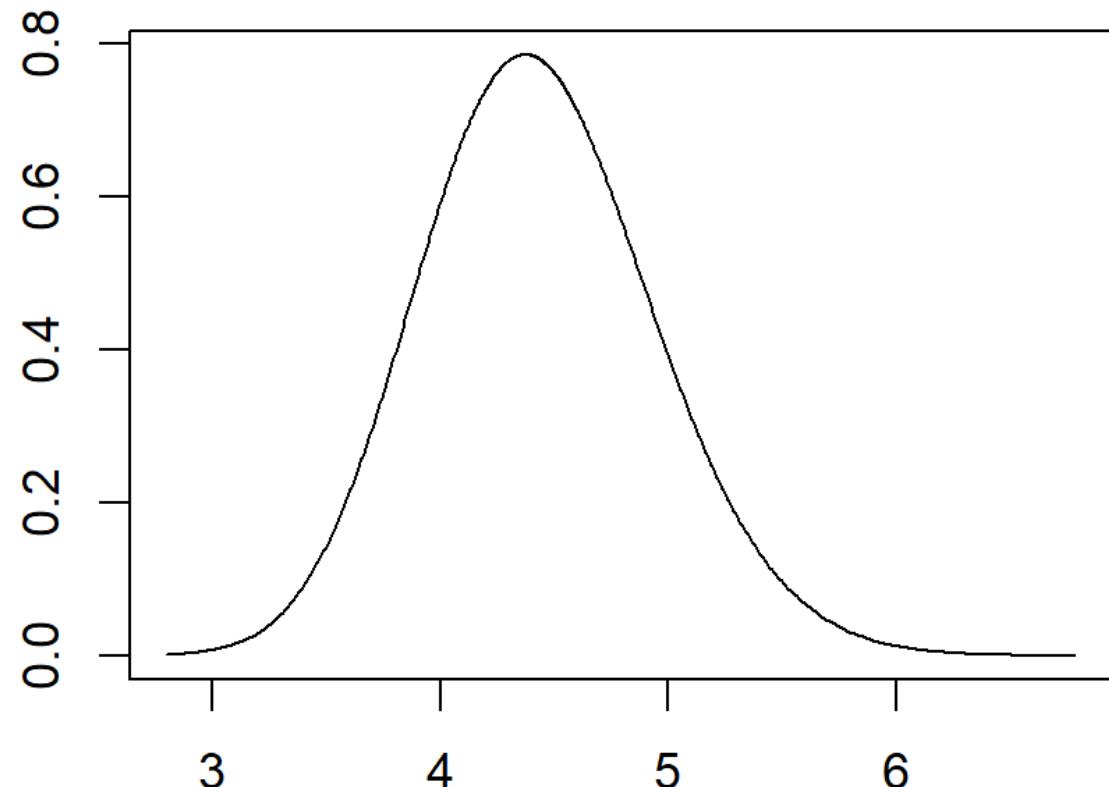
```
> names(output$marginals.hyperpar)
```

```
[1] "Precision for the Gaussian observations"
```

```
> prec_post <- output$marginals.hyperpar[[1]]
```

```
> plot(inla.smarginal(prec_post), t="1",
+       ylab="", xlab="",
+       main=expression(paste("p(", 1/sigma[e]^2, "
```

$$p(1/\sigma_e^2 | y)$$



Manipulating the marginals from R-INLA: hyperparameters

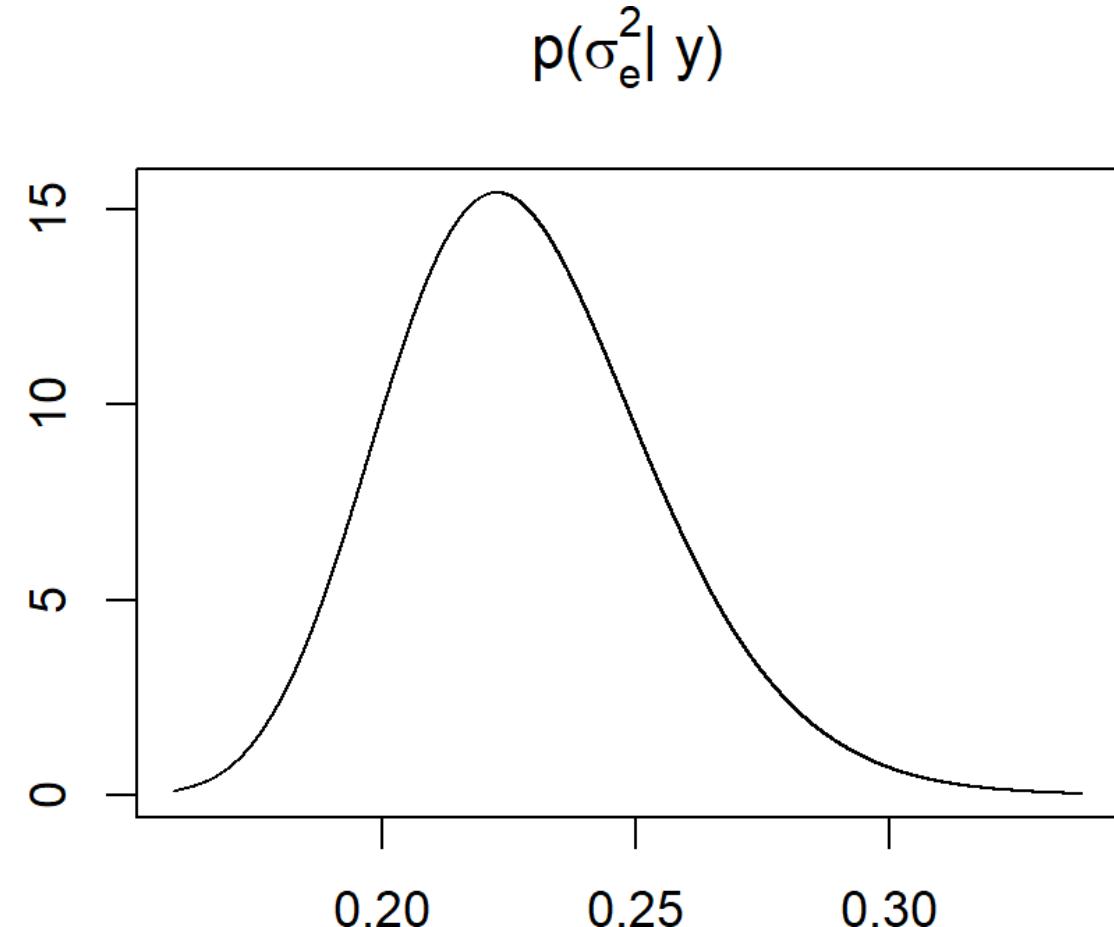
INLA works with the precision by default

Precision **Variance**

```
> var_post = inla.tmarginal(fun=function(x)
+           1/x, mar=prec_post)
> inla.emarginal(fun=function(x) 1/x,
+   marg=prec_post)
```

```
[1] 0.228669
```

```
> plot(inla.smarginal(var_post), t="l",
+       ylab="", xlab="",
+       main=expression(paste("p(", sigma[e]^2, " |
```



Summary

The INLA approach is not a rival/competitor/replacement to/of MCMC, just a better option for the class of LGMs.

- The basic idea behind the INLA procedure is simple
 - Repeatedly use Laplace approximation and take advantage of computational simplifications due to the structure of the model
 - Use numerical integration to compute the required posterior marginal distributions
- Complications are mostly computational and occur when
 - Extending to a large number of hyperparameters
 - Markedly non-Gaussian observations

References

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- Rue, H., S. Martino, and N. Chopin (2009). "Approximate Bayesian inference for latent Gaussian model by using integrated nested Laplace approximations (with discussion)". In: *J. R. Statist. Soc. B* 71, pp. 319-392.