Session 3.2: Spatio-temporal models for area data

Geospatial Analytics using R and R-INLA

MRC Centre for Environment & Health

UKRI Medical Research Council

Imperial College London

AIMS African Institute for Mathematical Sciences RWANDA

Learning Objectives

At the end of this session you should be able to:

- Explain how to extend spatial or temporal models to spatio-temporal models
- Fit Bayesian space-time models using `R-INLA`

The topics covered in this lecture can be found in Chapter 7 of the book Spatial and **Spatio-Temporal Bayesian Models with** `R-INLA`.

Outline

1. Temporal dependence
2. From space to space-time
3. Type of interactions

Temporal dependence

Temporal dependence

- Similarly to spatial dependence, it is sometimes necessary to model temporal dependence of data or of parameters:
- the weekly or monthly number of cases of many diseases exhibit often a seasonal pattern as well as short term dependence
- the underlying daily level of atmospheric pollutants, e.g. PM $_{10}$, will show strong correlation over consecutive days because their lifetime lasts over several days
- To the contrary of spatial models, there is a natural order to any time series data which is used in specifying the models.

Spatial patterns

1. Data
    - Disease counts y_i in area i and stratum k, aggregated over a time period, $i = 1, \ldots, N$, $k = 1, \ldots, K$
    - Population counts n_{ik} in area i and stratum k, aggregated over a time period
    - Expected numbers $E_i = \sum_k n_{ik} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Spatial smoothing using BYM model

$$
\begin{aligned}
y_i &\sim \text{Poisson}(E_i \rho_i); \quad i = 1, \ldots, N \\
\text{log}\rho_i &= b_0 + v_i + u_i \\
v_i &\sim \text{Normal}(0, \sigma_v^2) \\
\boldsymbol{u} &\sim \text{ICAR}(\boldsymbol{W}, \sigma_u^2) \rightarrow u_i | u_{j,\ j \neq i} \sim \text{Normal}\left(\frac{\sum_j w_{ij} u_j}{\sum_j w_{ij}}, \sigma_u^2 / n_i\right)
\end{aligned}
$$

with $w_{ij} = 1$ if areas i and j are neighbours, 0 otherwise

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\text{log}\rho_t &= ???
\end{aligned}
$$

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \beta t \quad \text{simple linear regression}
\end{aligned}
$$

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \beta t \qquad \text{simple linear regression} \\
&= b_0 + \psi_t \qquad \text{global temporal smoothing} \\
\psi_t &\sim \text{Normal}(0, \sigma^2_\psi)
\end{aligned}
$$

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \beta t \quad \text{simple linear regression} \\
&= b_0 + \psi_t \quad \text{global temporal smoothing} \\
\psi_t &\sim \text{Normal}(0, \sigma^2_\psi) \\
&= b_0 + \gamma_t \quad \text{local temporal smoothing} \\
\gamma_t &\sim \text{distribution ?}
\end{aligned}
$$

Temporal trends

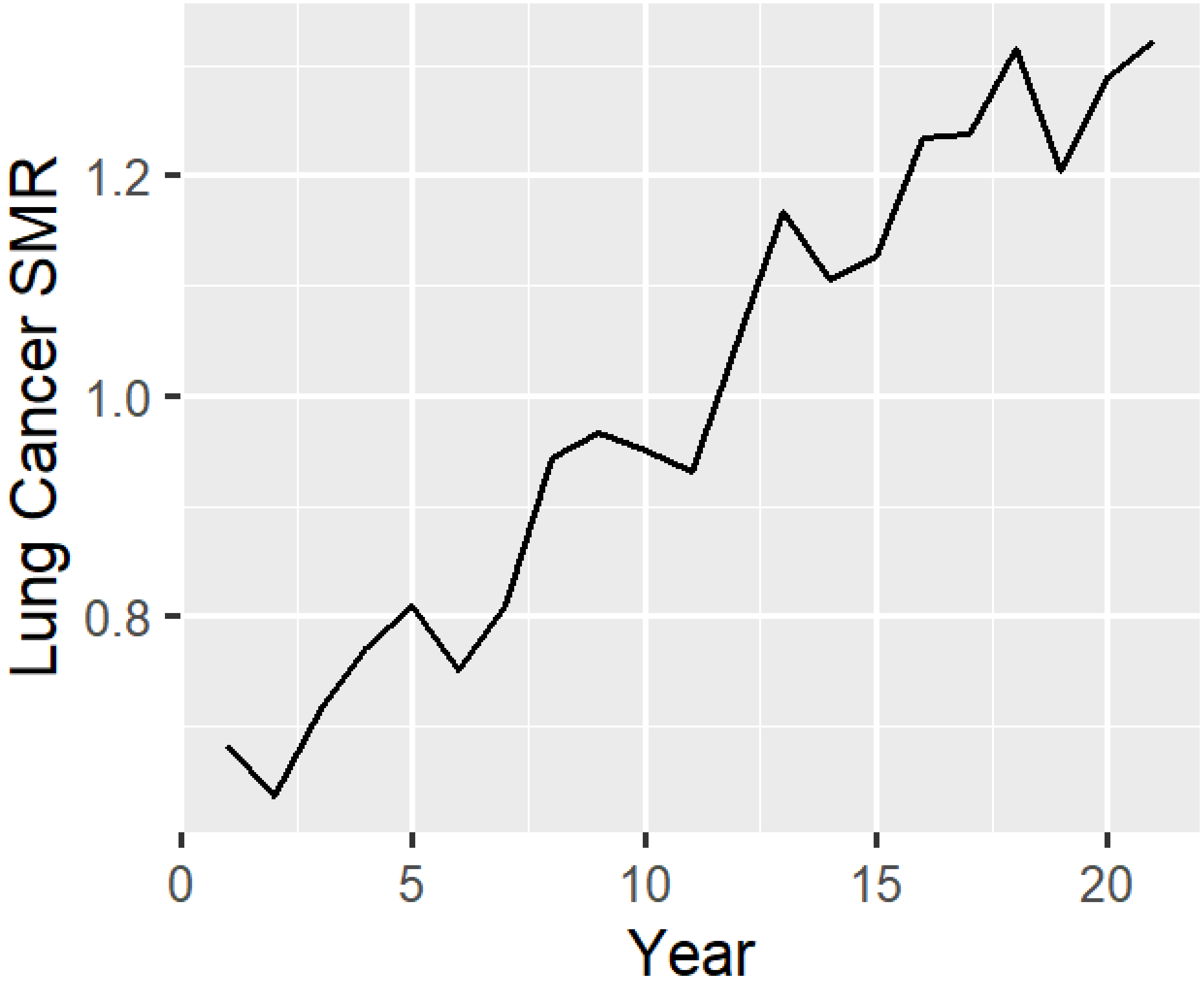
1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \dots, T$ (equally-spaced time intervals), $k = 1, \dots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \dots, T \\
\log \rho_t &= b_0 + \beta t \quad \text{simple linear regression} \\
&= b_0 + \psi_t \quad \text{global temporal smoothing} \\
\psi_t &\sim \text{Normal}(0, \sigma^2_\psi) \\
&= b_0 + \gamma_t \quad \text{local temporal smoothing} \\
\gamma_t &\sim \text{distribution ?} \\
&= b_0 + \gamma_t + \psi_t \quad \text{global and local temporal smoothing}
\end{aligned}
$$

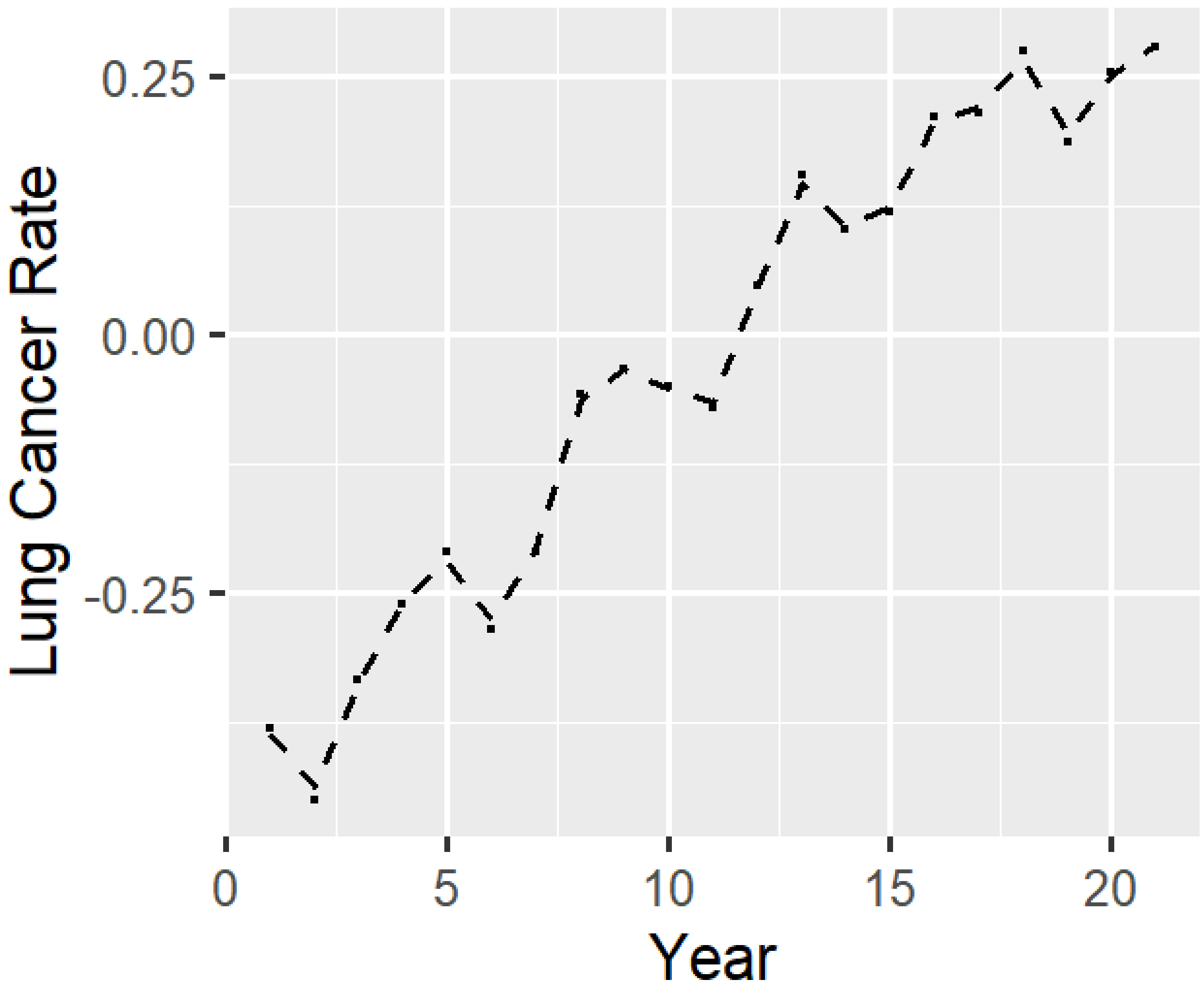
Ohio Lung cancer data

- Data on lung cancer in 88 counties of Ohio, 1968-1988
- Annual rates adjusted by gender and race

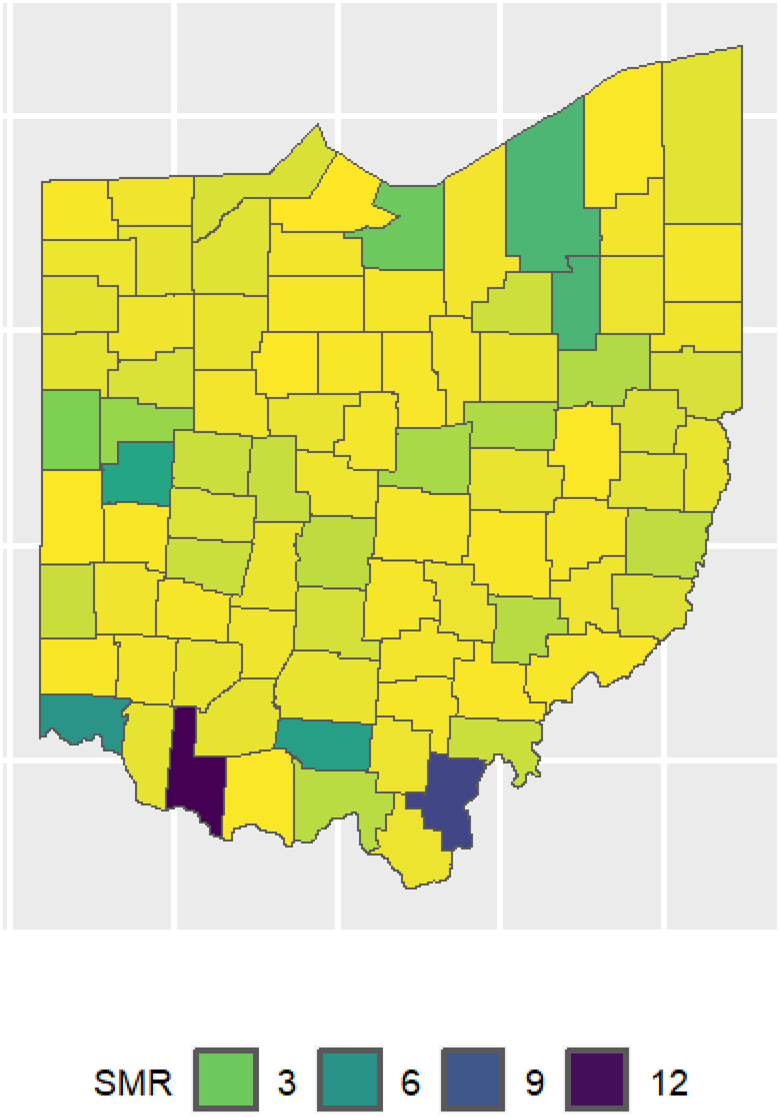


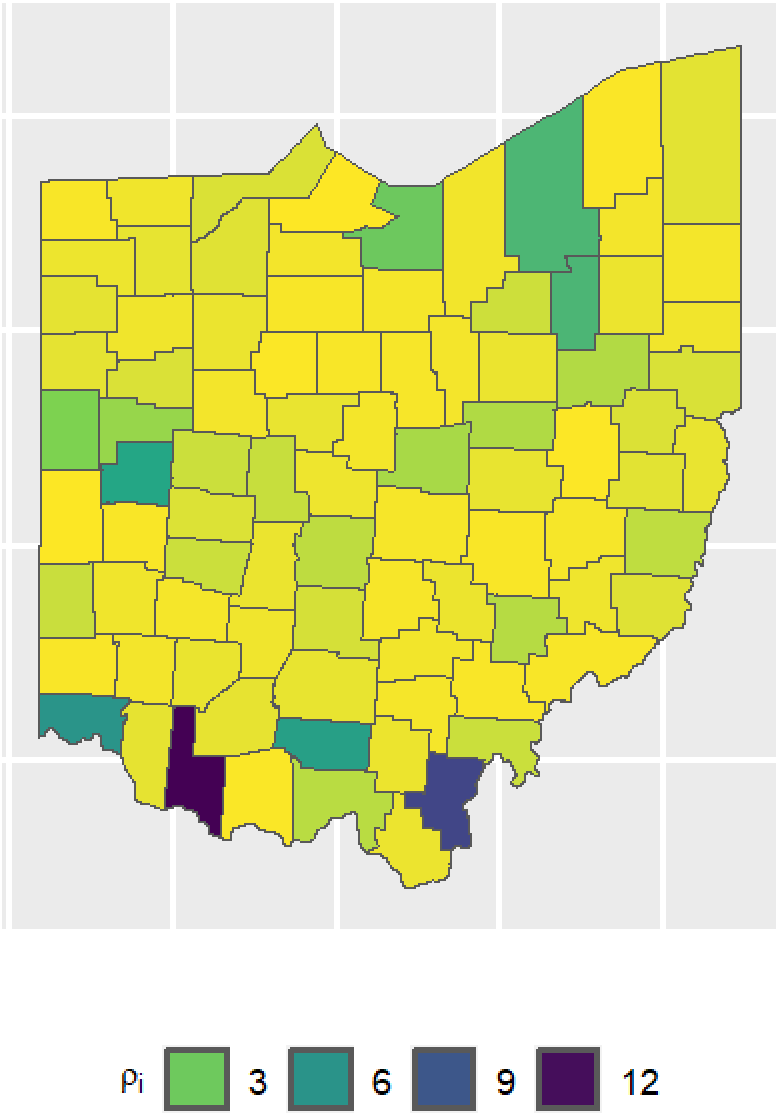
Ohio Lung cancer data: modelled temporal trend

- Smoothed temporal trend using random walk models (posterior mean) seen in the previous lecture



Ohio Lung cancer spatial pattern (no temporal dimension)



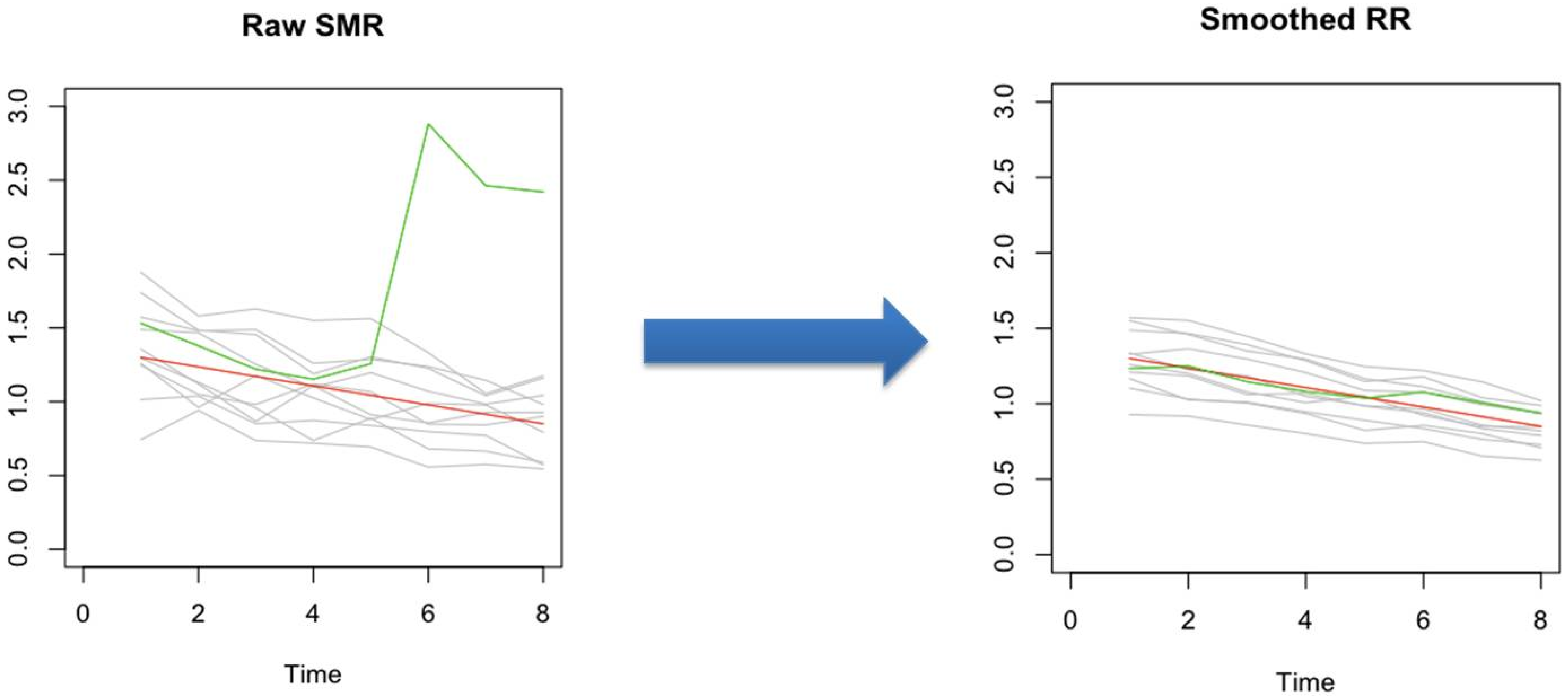


From space to space-time

Disease mapping: Extending space to space-time

- Disease mapping is usually carried out on aggregated data over a time period
- Rather than suppressing the time dimension, it can be interesting to use models that combine the space and time dimension
- The stability (or not) of the spatial pattern can aid interpretation
- The specific space-time components of the model can potentially pinpoint unusual/emerging hazards
- Data:
  - y_{it} and
  - E_{it}: the observed and expected number of cases in area i at time t calculated as $E_{it} = \sum_k n_{itk} r_k$, where r_k reference rate for stratum (age, gender,...)

Schematic representation I



Linear spatio-temporal model

- A simple model assumes a linear effect of time:

$$\log\rho_{it} = b_0 + u_i + v_i + \beta \times t$$

- Main spatial effect $u_i + v_i$
- Main linear trend β (global time effect)

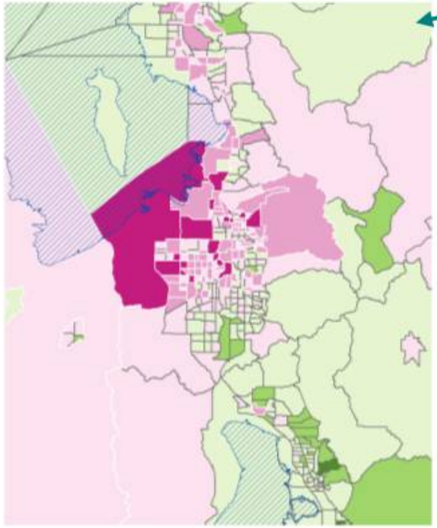
- A differential effect δ_i can be added

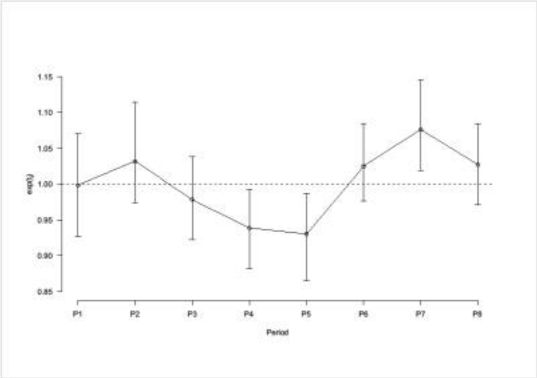
$$\log\rho_{it} = b_0 + u_i + v_i + (\beta + \delta_i) \times t$$

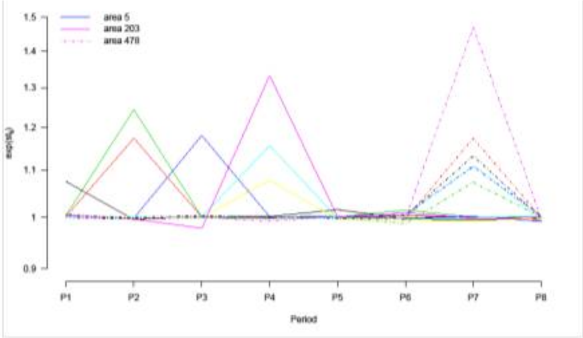
- If $\delta_i < 0$ then the area-specific trend is less steep than the mean trend, whilst $\delta_i > 0$ implies that the area-specific trend is steeper than the mean trend.

Noise model: Poisson/Binomial

Latent structure: Space + Time + Interactions





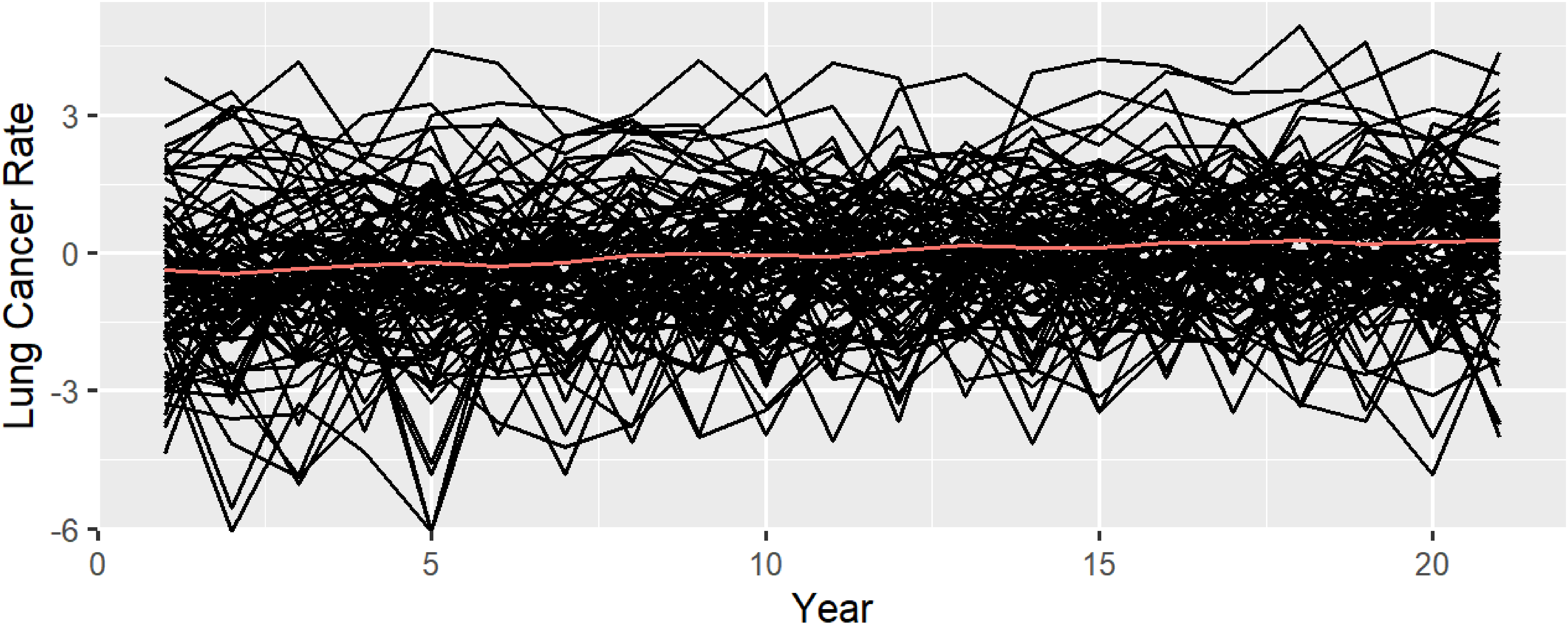


joint Bayesian estimation → Inference

Ohio Lung cancer - temporal SMRs

Log SMR time trends in each county

- Slightly increasing trend but lots of variation across the counties



Simple linear spatio-temporal model (Model 1)

$$y_{it} \sim \text{Poisson}(\text{E}_{it}\rho_{it})$$
$$\text{log}\rho_{it} = b_0 + \beta * t + u_i + v_i$$

where

- b_0 overall log RR in Ohio over the 21-year period
- $v_i \sim \text{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{u} \sim \text{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- $\exp(\beta)$ is the change in the RR associated with a 1-year increase in time

```
> formula.mod1 = y ~ 1 + f(county,model="bym",
+                      graph=Ohio.adj) + year
```

Log-linear temporal model with unstructured temporal RE (Model 2)

$$y_{it} \sim \text{Poisson}(\text{E}_{it}\rho_{it})$$
$$\text{log}\rho_{it} = b_0 + \beta * t + u_i + v_i + \psi_t$$

where

- b_0 overall log RR in Ohio over the 21-year period
- $v_i \sim \text{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{u} \sim \text{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- exp(β) is the change in the RR associated with a 1-year increase in time
- $\psi_t \sim \text{Normal}(0, \sigma_\psi^2)$ temporally unstructured RE

```
> year2 = ohio.data$year
> formula.mod2 = y ~ 1 + f(county,model="bym",
+                     graph=Ohio.adj) +
+                     f(year2,model="iid") +
+                     year
```

Ohio lung cancer - comparison models 1 and 2

```
> mod1 = inla(data=ohio.data,formula=formula.mod1, E=E, family="poisson")
> mod2 = inla(data=ohio.data,formula=formula.mod2, E=E, family="poisson")
> # Posterior mean and  95% CI for Model 1
> mod1$summary.fixed
```

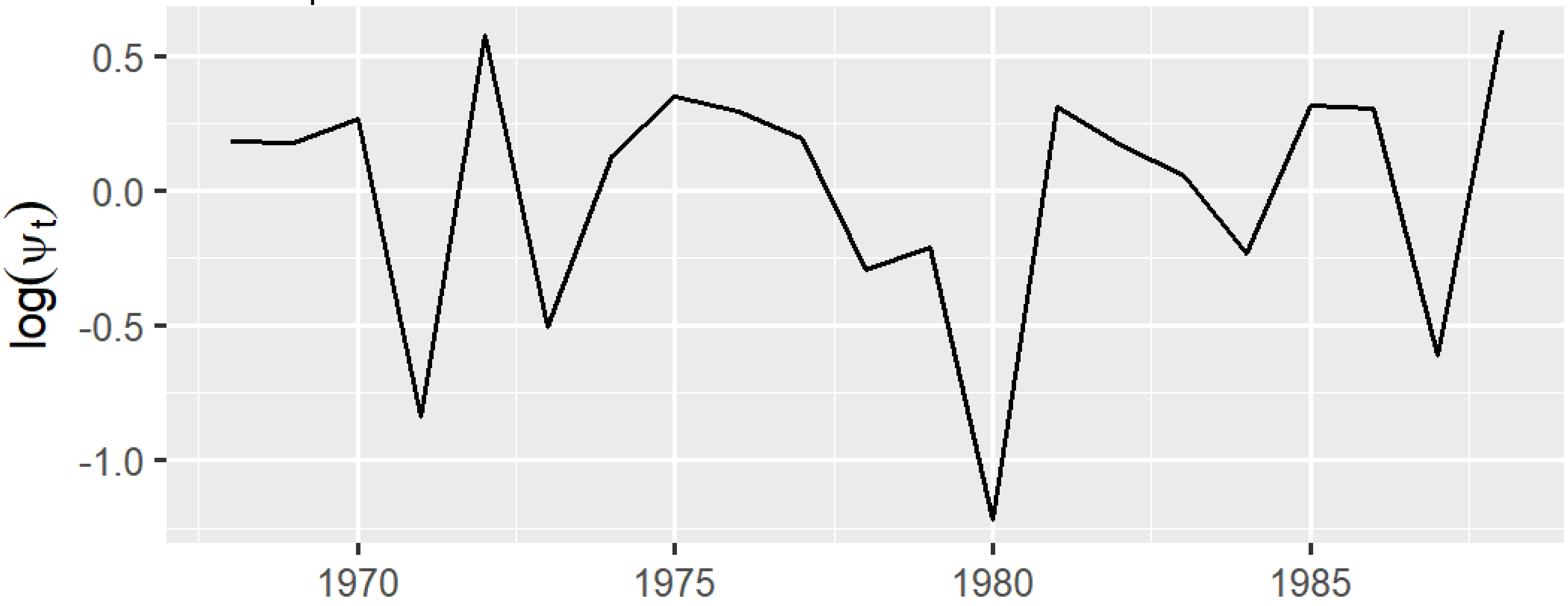
```
                  mean           sd  0.025quant   0.5quant  0.975quant       mode          kld
(Intercept) -0.9883759 0.1193437858 -1.22290645 -0.9883608 -0.75393070 -0.9883607 8.189035e-09
year         0.0261213 0.0005862816  0.02497166  0.0261213  0.02727094  0.0261213 5.512032e-11
```

```
> #Posterior mean and 95% CI for Model 2
> mod2$summary.fixed
```

```
                   mean         sd  0.025quant    0.5quant  0.975quant        mode          kld
(Intercept) -1.00052112 0.24889074 -1.49122140 -1.00051278 -0.50986801 -1.00051276 4.510179e-08
year         0.02042513 0.01706443 -0.01336415  0.02042531  0.05421336  0.02042531 8.986997e-08
```

Ohio lung cancer - comparison models 1 and 2

Let's look at the temporal random effect in model 2



Some temporal structure which suggests that the linear term is not capturing the temporal trend well

Simple additive space-time structure version 1 (Model 3)

This model assumes that the space time variations can be captured by the superimposition of a BYM spatial model and a structured time trend

$$\begin{aligned} y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\ \log\rho_{it} &= b_0 + b_i + \gamma_t \end{aligned}$$

- $b_i = v_i + u_i$
- $v_i \sim \text{Normal}(0, \sigma^2_v)$ spatially unstructured RE
- $\mathbf{u} \sim \text{ICAR}(\mathbf{W}, \sigma^2_u)$ spatially structured RE
- $\gamma_t \sim \text{RW}(1)$ temporally structured RE with variance parameter σ^2_γ
- Assuming a BYM2 specification, there are 3 hyperparameters
  - τ_b, the marginal precision of b_i
  - ϕ, the proportion of spatial variability explained by the spatially structured component
  - σ^2_γ, the conditional variance of the RW(1) modelling the time trend

```
> formula.mod3 = y ~ 1 + f(county,model="bym",
+                      graph=Ohio.adj) +
+                      f(year,model="rw1")
```

```
> mod3 = inla(data=ohio.data,formula=formula.mod3, E=E, family="poisson")
```

Simple additive space-time structure version 2 (Model 4)

$$y_{it} \sim \text{Poisson}(\text{E}_{it}\rho_{it})$$
$$\log\rho_{it} = b_0 + u_i + v_i + \psi_t + \gamma_t$$

where

- b_0 overall log RR in Ohio over the 21-year period
- $v_i \sim \text{Normal}(0, \sigma^2_v)$ spatially unstructured RE
- $\mathbf{u} \sim \text{ICAR}(\mathbf{W}, \sigma^2_u)$ spatially structured RE
- $\gamma_t \sim \text{RW}(1)$ temporally structured RE with variance parameter σ^2_γ
- $\psi_t \sim \text{Normal}(0, \sigma^2_\psi)$ temporally unstructured RE

```
> formula.mod4 = y ~ 1 + f(county,model="bym",
+                      graph=Ohio.adj) +
+                      f(year,model="rw1") +
+                      f(year2,model="iid")
```

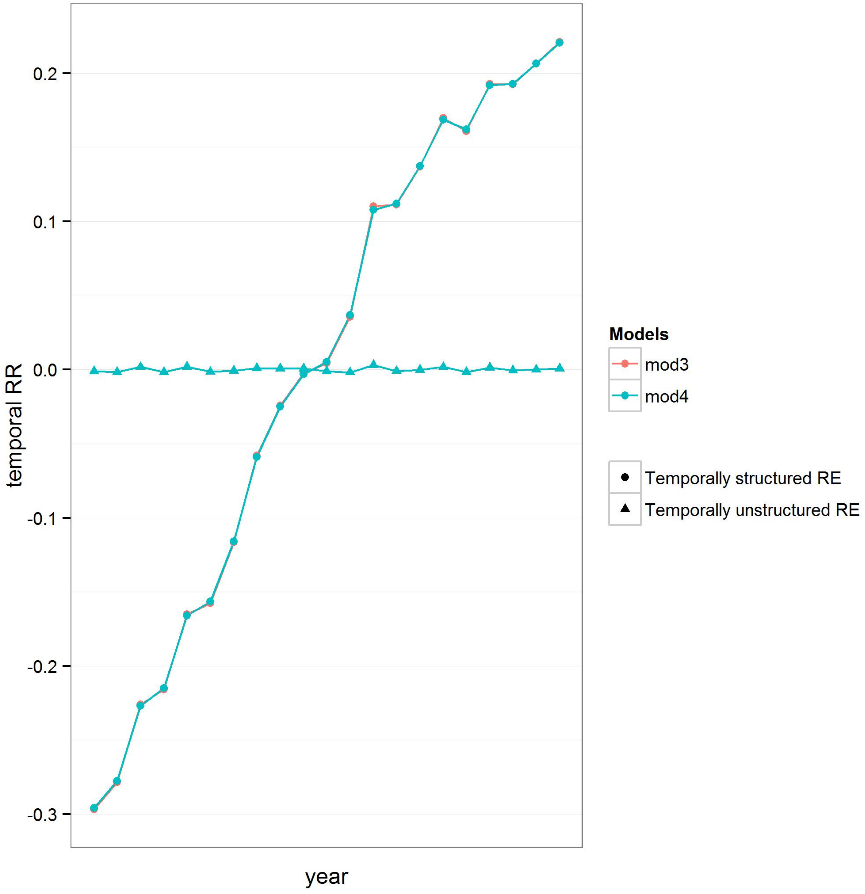
```
> mod4 = inla(data=ohio.data,formula=formula.mod4, E=E, family="poisson")
```

Ohio lung cancer - comparison models 3 and 4

Posterior means of the temporal RE

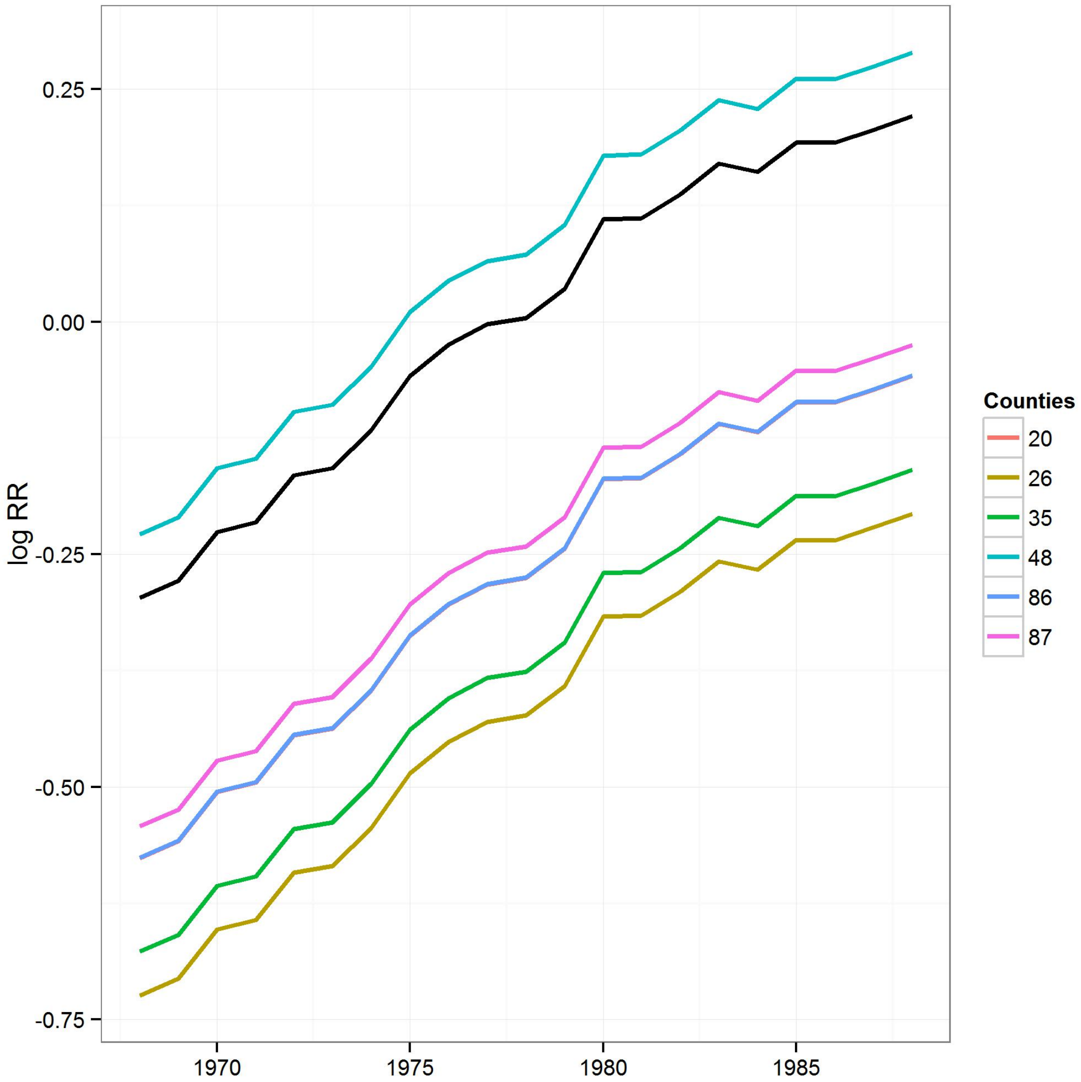
Model 3 = γ_t

Model 4 = $\psi_t + \gamma_t$



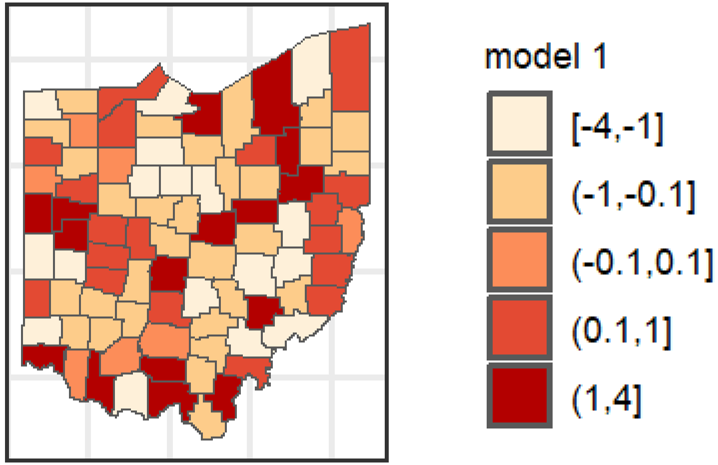
Ohio lung cancer - area-specific temporal RR

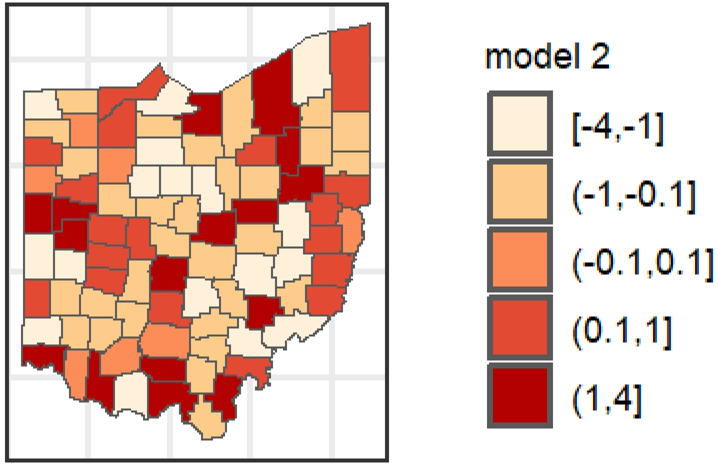
Posterior means of the temporal RE γ_t (model 3)

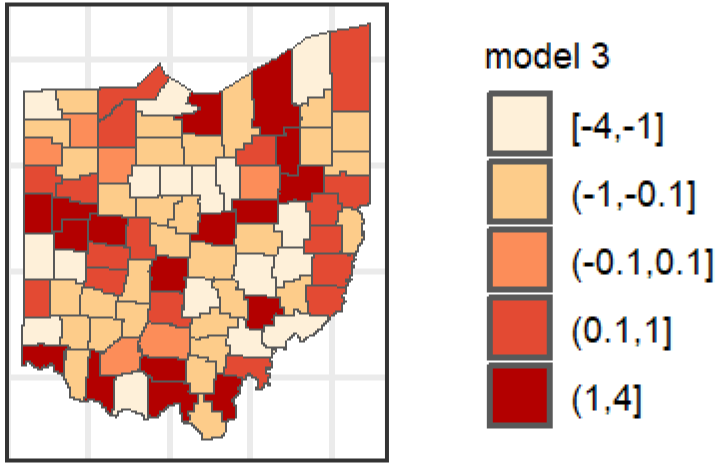


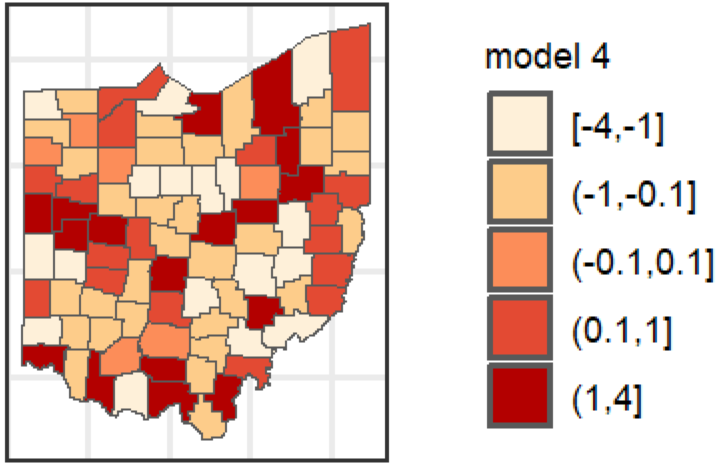
Ohio lung cancer - spatial patterns (all models)

Posterior means of the spatial random effects for the 4 different models









Similar patterns with the 4 models

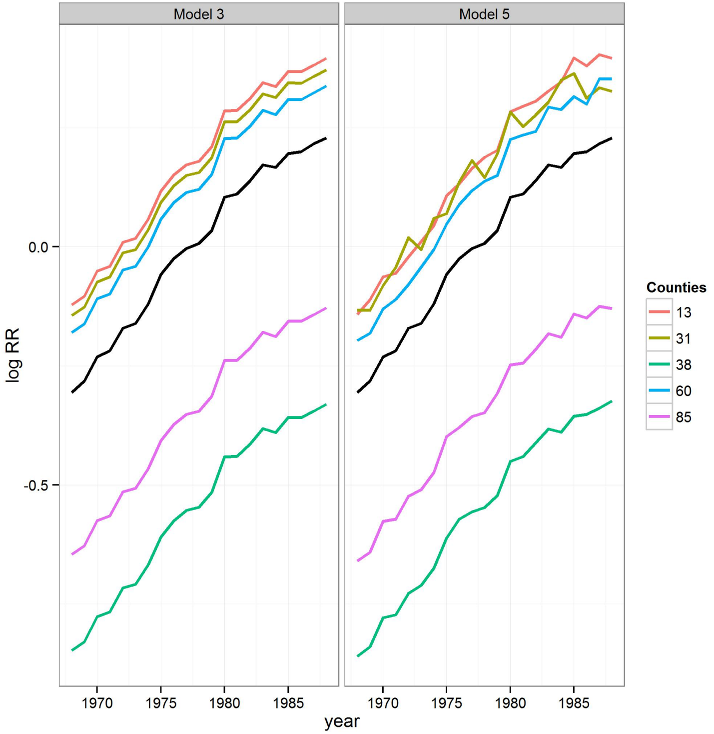
Space-time model with exchangeable interactions (Model 5)

$$
\begin{aligned}
y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\
\log\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t + \delta_{it} \\
b_i &= v_i + u_i \\
\gamma_t &\sim \text{RW}(1) \\
\psi_t &\sim \text{N}(0, \sigma^2_\psi) \\
\delta_{it} &\sim \text{Normal}(0, \sigma^2_\delta)
\end{aligned}
$$

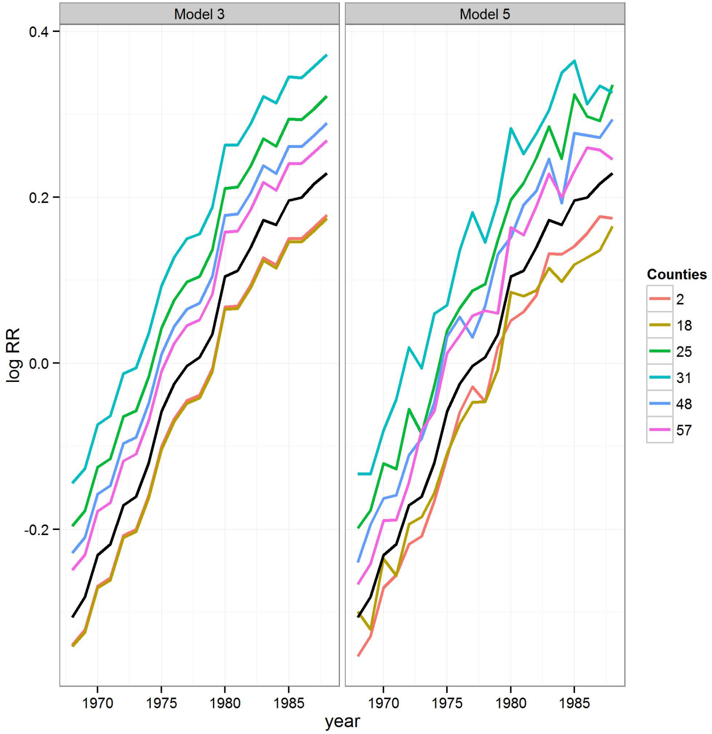
The simple additive model 4 can be extended by including **space-time interactions parameters**, δ_{it}, modelled as exchangeable random effects, that capture departure from the additive structure

- The interactions δ_{it} allow to highlight unusual temporal trends
- Rules based on the posterior probabilities $p(\delta_{it} > 0)$ for at least 1 time t
- Ohio lung cancer: 6 counties with unusual temporal trends

"usual" temporal trends



unusual temporal trends



INLA code for model with interaction (Model 5)

- In INLA is very easy to include the interaction in the `formula` environment
- We need to specify an index for the interaction, i.e. for each combination of area/time

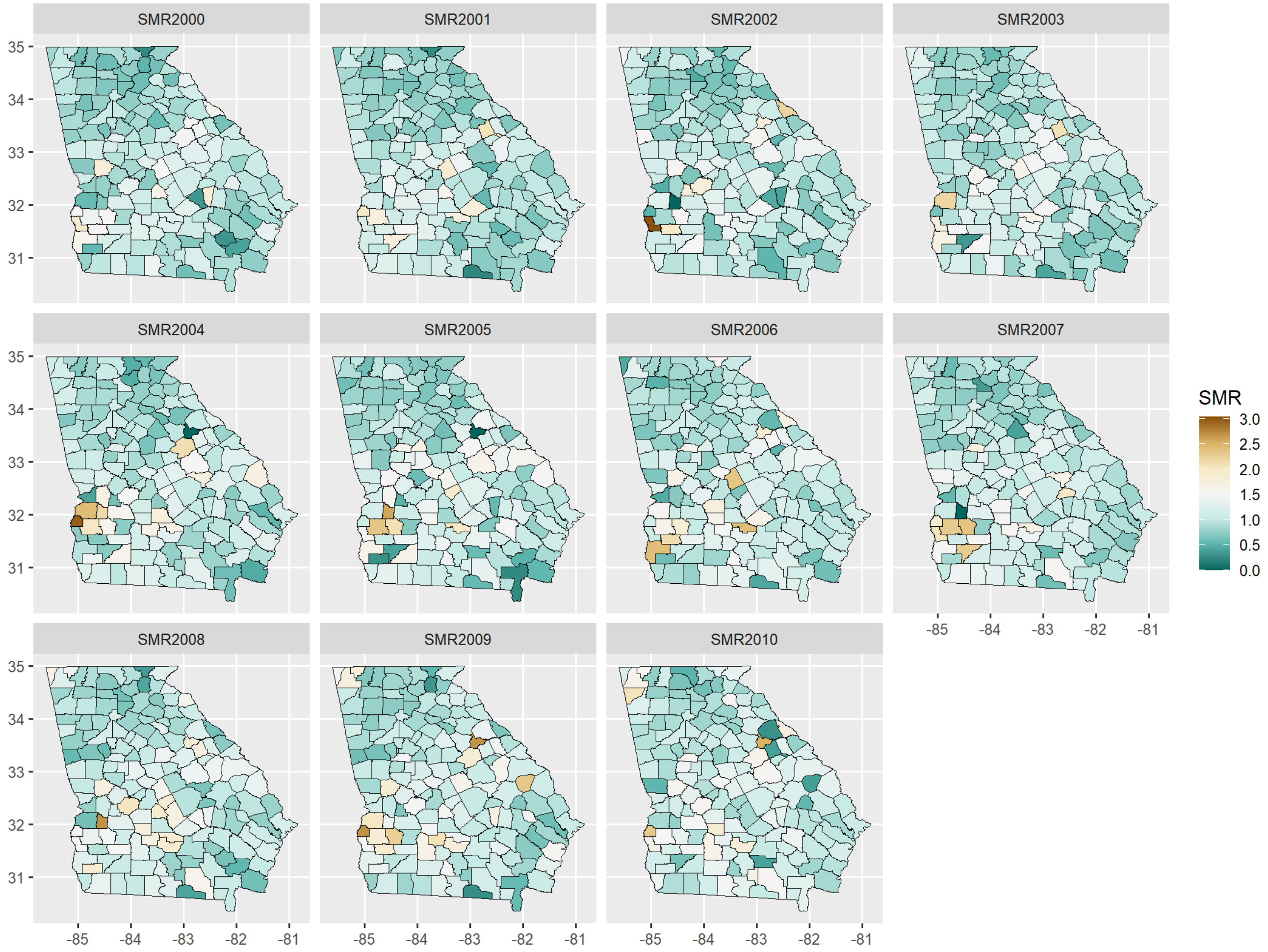
$$
\begin{aligned}
\log\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t + \delta_{it} \\
b_i &= v_i + u_i \\
\gamma_t &\sim \mathrm{RW}(1) \\
\psi_t &\sim N(0, \sigma^2_\psi) \\
\delta_{it} &\sim \mathrm{Normal}(0, \sigma^2_\delta)
\end{aligned}
$$

```
> formula.intI = y ~ + f(county,model="bym",
+                        graph=Ohio.adj) +
+                       f(year,model="rw1") +
+                       f(year2,model="iid") +
+                       f(area.year,model="iid")
```

Types of interactions

Another example: Birth weight in Georgia

- Count of babies weigthing less than 2500g in 159 counties in Georgia (US)
- Period 2000 – 2010



Different types of interactions

$$
\begin{aligned}
y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\
\text{log}\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t + \delta_{it} \\
b_i &= v_i + u_i \\
\gamma_t &\sim \text{RW}(1) \\
\psi_t &\sim N(0, \sigma^2_\psi)
\end{aligned}
$$

Characteristics of ST interaction

Interaction	**Parameters**	**Rank**
I	v and ψ	nT
II	v and γ	n(T-1) for RW1, n(T-2) for RW2
III	u and ψ	(n-1)T
IV	u and γ	(n-1)(T-1) for RW1, (n-1)(T-2) for RW2

- We will learn how to develop Type I interaction in INLA

How to model interactions

- Data in R format

```
# A tibble: 1,749 × 8
   ID.area   Obs ID.year   Exp     y     E NAME    ID.area.year
     <int> <int>   <dbl> <dbl> <int> <dbl> <chr>          <int>
 1       1    20       1  25.8    20  25.8 Appling            1
 2       1    24       2  25.1    24  25.1 Appling            2
 3       1    25       3  22.8    25  22.8 Appling            3
 4       1    31       4  25.2    31  25.2 Appling            4
 5       1    24       5  24.9    24  24.9 Appling            5
 6       1    40       6  26.7    40  26.7 Appling            6
 7       1    29       7  26.0    29  26.0 Appling            7
 8       1    35       8  27.2    35  27.2 Appling            8
 9       1    26       9  25.1    26  25.1 Appling            9
10       1    25      10  24.5    25  24.5 Appling           10
# ℹ 1,739 more rows
```

- We need to make sure to have an index for area (`ID.area`), one for time (`ID.year`) and one for the interaction (`ID.area.year`)

Type I interaction in INLA

Let's adopt Type I interaction, which assumes that the two unstructured effects v_i and ψ_t interact

```
> Georgia.adj = "Georgia.graph"
> ID.year2 = data_INLA$ID.year
> formula.intI = y ~ + f(ID.area,model="bym2",
+                         graph=Georgia.adj) +
+                       f(ID.year,model="rw1") +
+                       f(ID.year2,model="iid") +
+                       f(ID.area.year,model="iid")
```

Kronecker product

- For the interactions of type II-IV we will need to use the **Kronecker product** to specify the dependencies. It is a matrix multiplication which returns a block matrix.

For instance

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$$

- There is a function in `R` which does exactly that: `kronecker(a,b)`
- We will not go through type II-IV of interactions, as they are pretty complex and also they can take a **very long** time to run
- If you are interested in knowing more how to set interactions using the Kronecker product in `INLA` see Goicoa, Adin, Ugarte, and Hodges (2018)

Summary

- Increase quality of datasets that are both spatially and temporally indexed
- Advanced methods to deal with this type of data
- Allow to interpret the stability (or not) of the spatial patterns

References

Goicoa, T., A. Adin, M. Ugarte, et al. (2018). "In spatio-temporal disease mapping models, identifiability constraints affect PQL and INLA results". In: *Stochastic Environmental Research and Risk Assessment* 32.3, pp. 749-770.