Session 9.2: Spatio-temporal modelling for area data

Imperial College London

Learning Objectives

At the end of this session you should be able to:

- Explain how to extend spatial or temporal models to spatio-temporal models
- Fit Bayesian space-time models using `R-INLA`

The topics covered in this lecture can be found in Chapter 7 of the book Spatial and **Spatio-Temporal Bayesian Models with** `R-INLA`.

Outline

Temporal dependence

Temporal dependence

- Similarly to spatial dependence, it is sometimes necessary to model temporal dependence of data or of parameters:
- the weekly or monthly number of cases of many diseases exhibit often a seasonal pattern as well as short term dependence
- the underlying daily level of atmospheric pollutants, e.g. PM $_{10}$, will show strong correlation over consecutive days because their lifetime lasts over several days
- To the contrary of spatial models, there is a natural order to any time series data which is used in specifying the models.

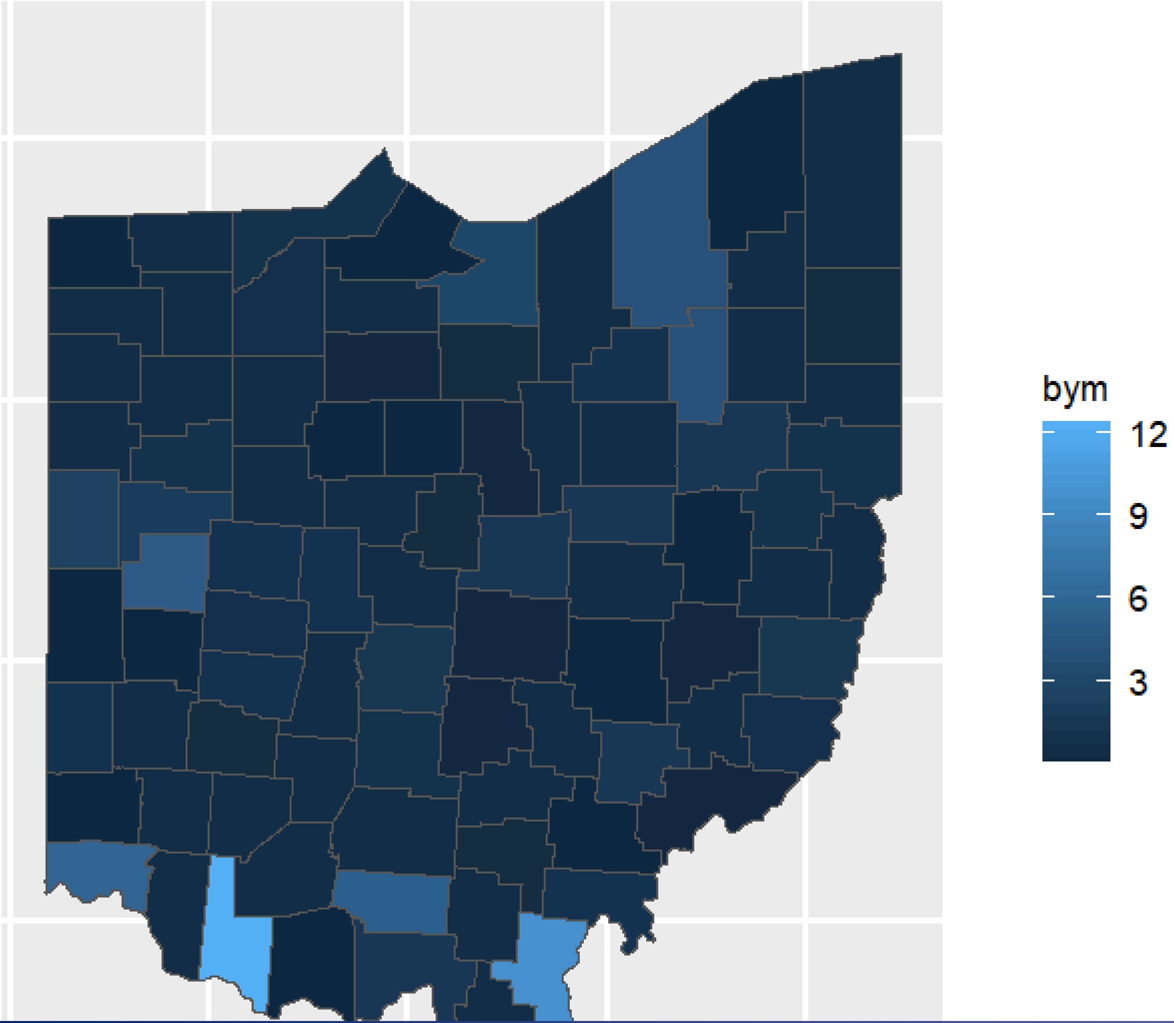
Spatial patterns

1. Data
   - Disease counts y_i in area i and stratum k, aggregated over a time period, $i = 1, \ldots, N$, $k = 1, \ldots, K$
   - Population counts n_{ik} in area i and stratum k, aggregated over a time period
   - Expected numbers $E_i = \sum_k n_{ik} r_k$, where r_k reference rate for stratum (age, gender,...)

2. Spatial smoothing using BYM model

$$
\begin{aligned}
y_i &\sim \text{Poisson}(E_i \rho_i); \quad i = 1, \ldots, N \\
\eta_i = \log \rho_i &= b_0 + b_i \\
\boldsymbol{b} &= \frac{1}{\sqrt{\tau_b}} (\sqrt{1-\phi}\,\boldsymbol{v}_* + \sqrt{\phi}\,\boldsymbol{u}_*)
\end{aligned}
$$

- Data on lung cancer in 88 counties of Ohio, 1968-1988
- Purely spatial analysis



Temporal trends

1. Data

- Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
- Population counts n_{tk} in time period t and stratum k, aggregated over space
- Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

Note that E_t makes sense if the time series covers a long period, e.g. years, otherwise it is common to assign $E_t = \sum_t y_t / T$

2. Temporal trends

$$\begin{aligned} y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\ \log \rho_t &= ??? \end{aligned}$$

Temporal trends

1. Data

- Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
- Population counts n_{tk} in time period t and stratum k, aggregated over space
- Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

Note that E_t makes sense if the time series covers a long period, e.g. years, otherwise it is common to assign $E_t = \sum_t y_t / T$

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \beta t \quad\quad \text{simple linear regression}
\end{aligned}
$$

Temporal trends

1. Data

- Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
- Population counts n_{tk} in time period t and stratum k, aggregated over space
- Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

Note that E_t makes sense if the time series covers a long period, e.g. years, otherwise it is common to assign $E_t = \sum_t y_t / T$

OR

2. Temporal trends

$$\begin{aligned} y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\ \log\rho_t &= b_0 + \psi_t \qquad \text{global temporal smoothing} \\ \psi_t &\sim \text{Normal}(0, \sigma^2_\psi) \end{aligned}$$

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

Note that E_t makes sense if the time series covers a long period, e.g. years, otherwise it is common to assign $E_t = \sum_t y_t / T$

OR

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \gamma_t \qquad \text{local temporal smoothing} \\
\gamma_t &\sim RW(\sigma^2_\gamma)
\end{aligned}
$$

Temporal trends

1. Data
   - Disease counts y_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)

Note that E_t makes sense if the time series covers a long period, e.g. years, otherwise it is common to assign $E_t = \sum_t y_t / T$

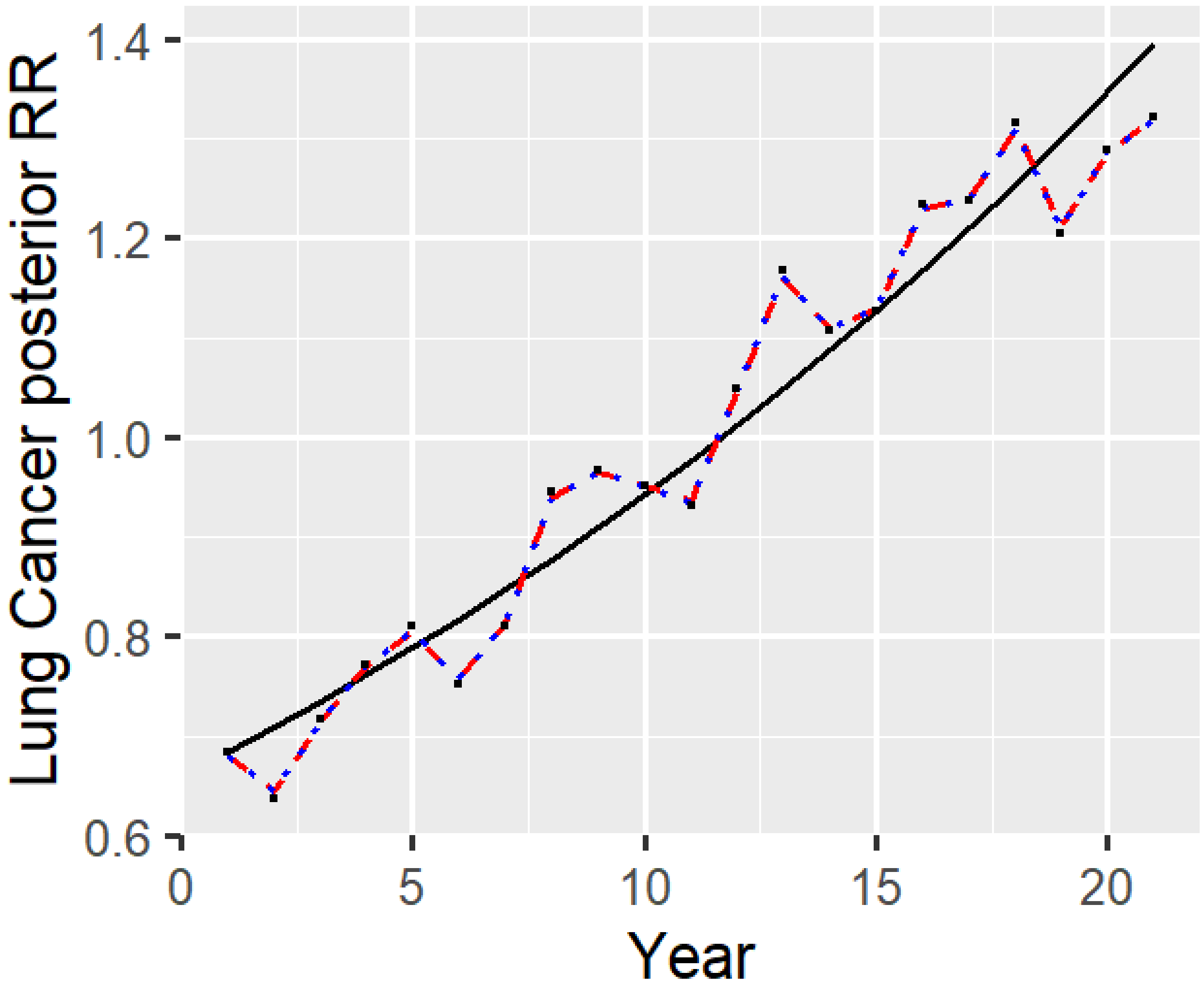
OR

2. Temporal trends

$$
\begin{aligned}
y_t &\sim \text{Poisson}(E_t \rho_t); \quad t = 1, \ldots, T \\
\log \rho_t &= b_0 + \gamma_t + \psi_t \qquad \text{global and local temporal smoothing} \\
\psi_t &\sim \text{Normal}(0, \sigma^2_\psi) \\
\gamma_t &\sim RW(\sigma^2_\gamma)
\end{aligned}
$$

Ohio Lung cancer data: modelled temporal trend

- Annual rates adjusted by gender and race
- Fitting different temporal trends (linear, local smoothing, local+global smoothing)



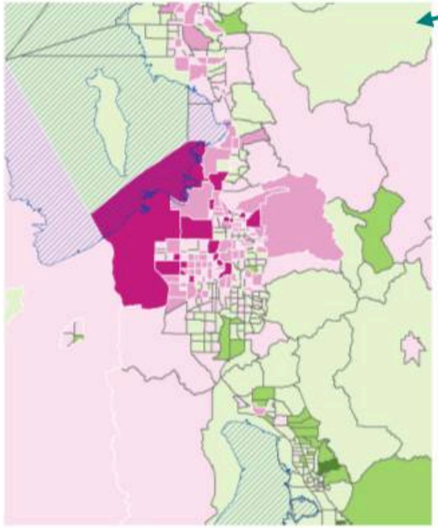
From space to space-time

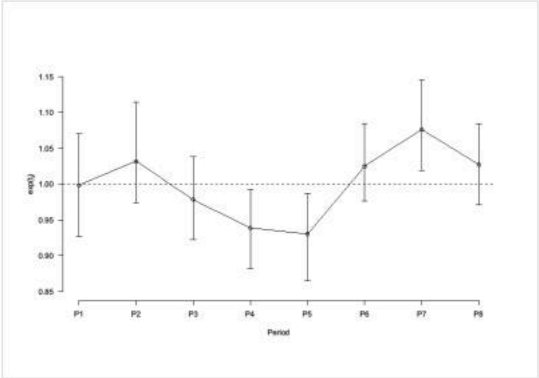
Disease mapping: Extending space to space-time

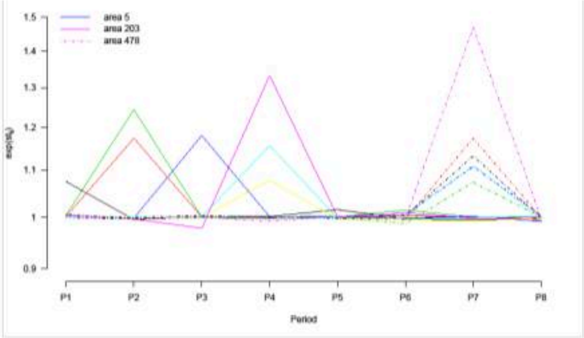
- Disease mapping is usually carried out on aggregated data over a time period
- Rather than suppressing the time dimension, it can be interesting to use models that combine the space and time dimension
- The stability (or not) of the spatial pattern can aid interpretation
- The specific space-time components of the model can potentially pinpoint unusual/emerging hazards
- Data:
  - y_{it}
  - E_{it}: the observed and expected number of cases in area i at time t calculated as $E_{it} = \sum_k n_{itk} r_k$, where r_k reference rate for stratum (age, gender,...)

Noise model: Poisson/Binomial

Latent structure: Space + Time + Interactions







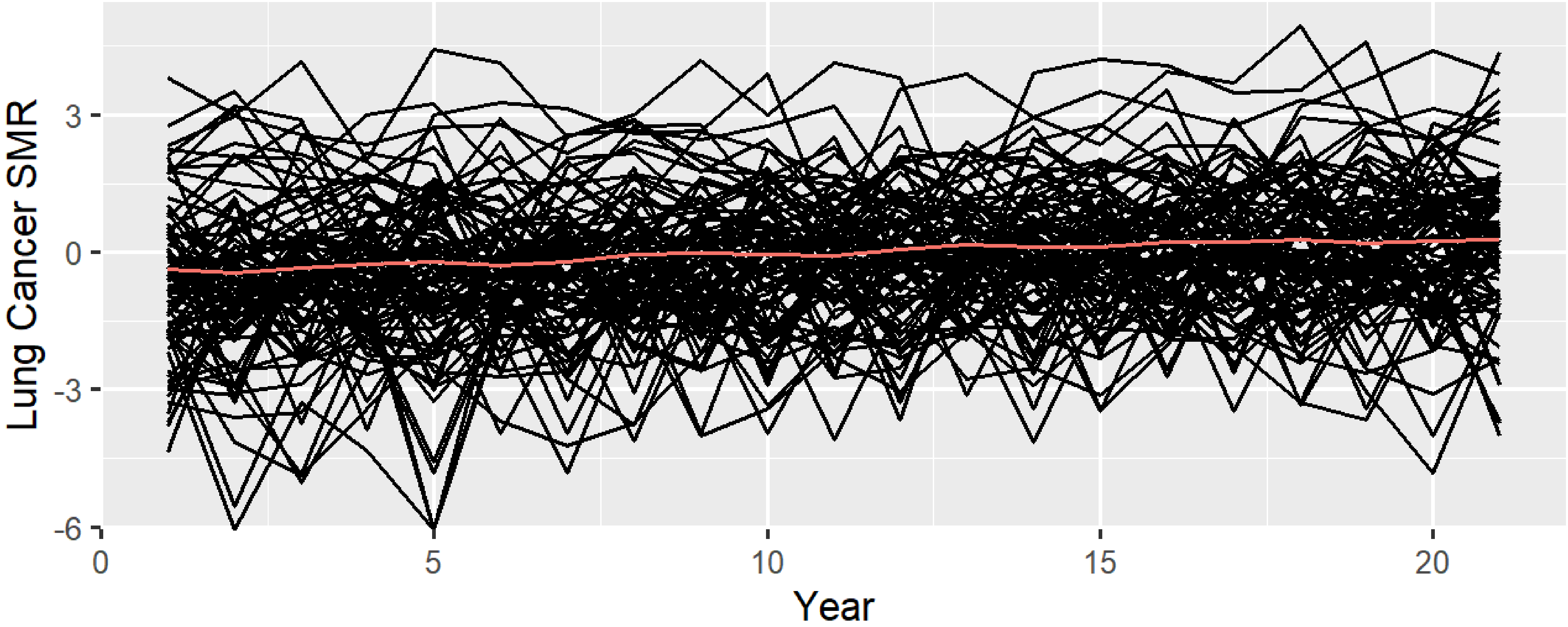
joint Bayesian estimation

Inference

Ohio Lung cancer - temporal SMRs

Log SMR time trends in each county

- Slightly increasing trend but lots of variation across the counties



Simple linear spatio-temporal model (Model 1)

$$y_{it} \sim \text{Poisson}(\text{E}_{it}\rho_{it})$$
$$\log \rho_{it} = b_0 + \beta \times t + b_i$$

where

- b_0 overall log RR in Ohio over the 21-year period
- b_i is the weighted average of spatially structured (ICAR) and unstructured random effects
- $\exp(\beta)$ is the change in the RR associated with a 1-year increase in time

```
> formula.mod1 = y ~ 1 + year + f(county,model="bym2",
+                        graph=Ohio.adj)
```

```
> mod1 = inla(data=ohio.data,formula=formula.mod1, E=E, family="poisson", control.compute=list(waic=TRUE)
```

Additional temporal random effects can be added...

Log-linear temporal model with structured temporal RE (Model 2)

$$y_{it} \sim \text{Poisson}(\text{E}_{it}\rho_{it})$$
$$\log\rho_{it} = b_0 + \beta \times t + b_i + \gamma_t$$

where

- b_0 overall log RR in Ohio over the 21-year period
- b_i is the weighted average of spatially structured (ICAR) and unstructured random effects
- $\exp(\beta)$ is the change in the RR associated with a 1-year increase in time
- Looking back at the purely temporal analysis it seems a random walk was able to explain the local changes in time $\gamma_t \sim \text{Normal}(\gamma_{t-1}, \sigma^2_\gamma)$

```
> year2<-ohio.data$year
> formula.mod2 = y ~ 1  +
+                  year + f(county,model="bym2",
+                  graph=Ohio.adj) +
+                  f(year2,model="rw1")
```

```
> mod2 = inla(data=ohio.data,formula=formula.mod2, E=E, family="poisson", control.compute=list(waic=TRUE)
```

Comparing model 1 and 2

- Fixed effects

```
> mod1$summary.fixed
```

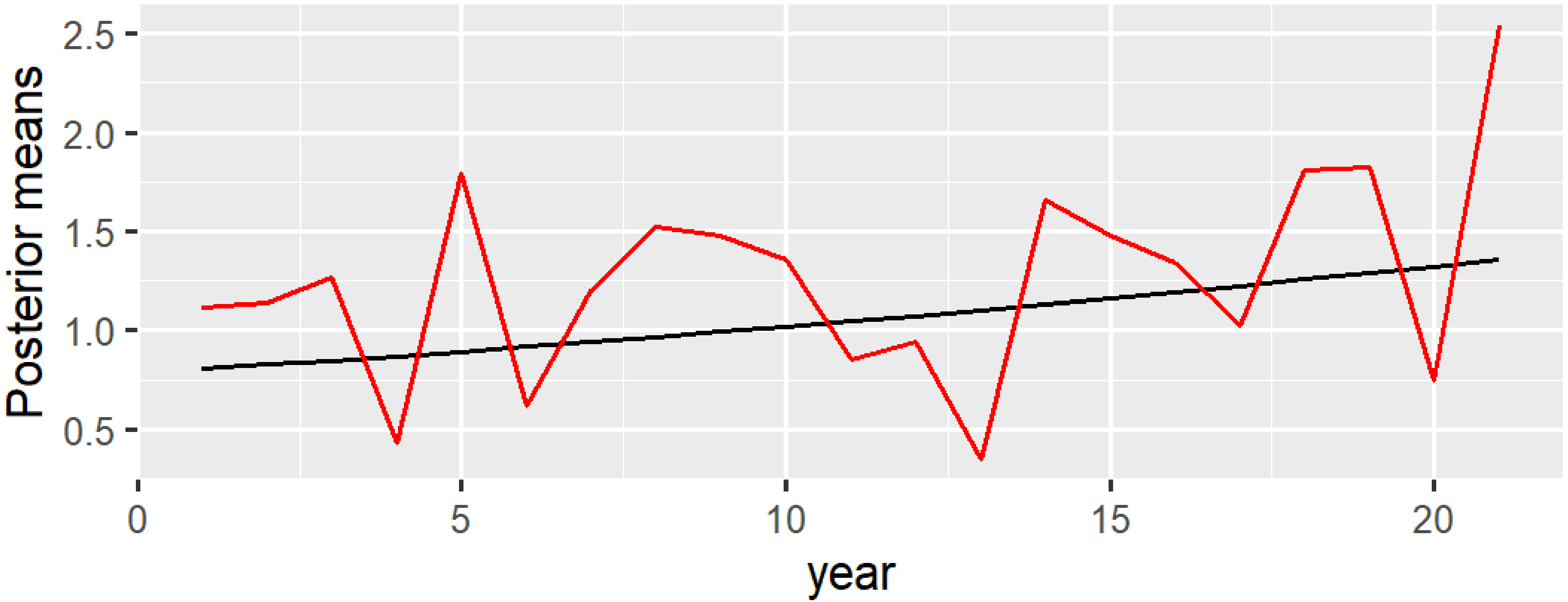
```
                   mean         sd  0.025quant    0.5quant  0.975quant        mode          kld
(Intercept) -0.98833298 0.11239029 -1.20942093 -0.98831791 -0.76733001 -0.98831792 1.473307e-08
year         0.02612109 0.00058628  0.02497146  0.02612109  0.02727073  0.02612109 5.512062e-11
```

```
> mod2$summary.fixed
```

```
                   mean        sd 0.025quant    0.5quant 0.975quant        mode          kld
(Intercept) -1.22479524 1.8557582  -4.899649 -1.22480739  2.4501320 -1.22480611 9.030611e-08
year         0.04083632 0.1683608  -0.292596  0.04083756  0.3742611  0.04083742 9.099647e-08
```

Comparing model 1 and 2

- Fitted values




```
> mod1$waic$waic
```

```
[1] 83781.91
```

```
> mod2$waic$waic
```

What if we remove the linear trend

- Exploring if a combination of random effects could be more flexible in capturing the temporal trend than a linear trend

$$
\begin{aligned}
y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\
\log\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t
\end{aligned}
$$

where

- b_0 overall log RR in Ohio over the 21-year period
- b_i is the weighted average of spatially structured (ICAR) and unstructured random effects
- The temporal pattern is now modelled with two random effects (global + local smoothing)

```
> formula.mod3 = y ~ 1  +
+                   f(county,model="bym2", graph=Ohio.adj) +
+                   f(year,model="rw1") +
+                   f(year2, model="iid")
```

```
> mod3 = inla(data=ohio.data,formula=formula.mod3, E=E, family="poisson", control.compute=list(waic=TRUE)
```

Comparing model 1, and 3

- Fixed effects

```
> mod1$summary.fixed
```

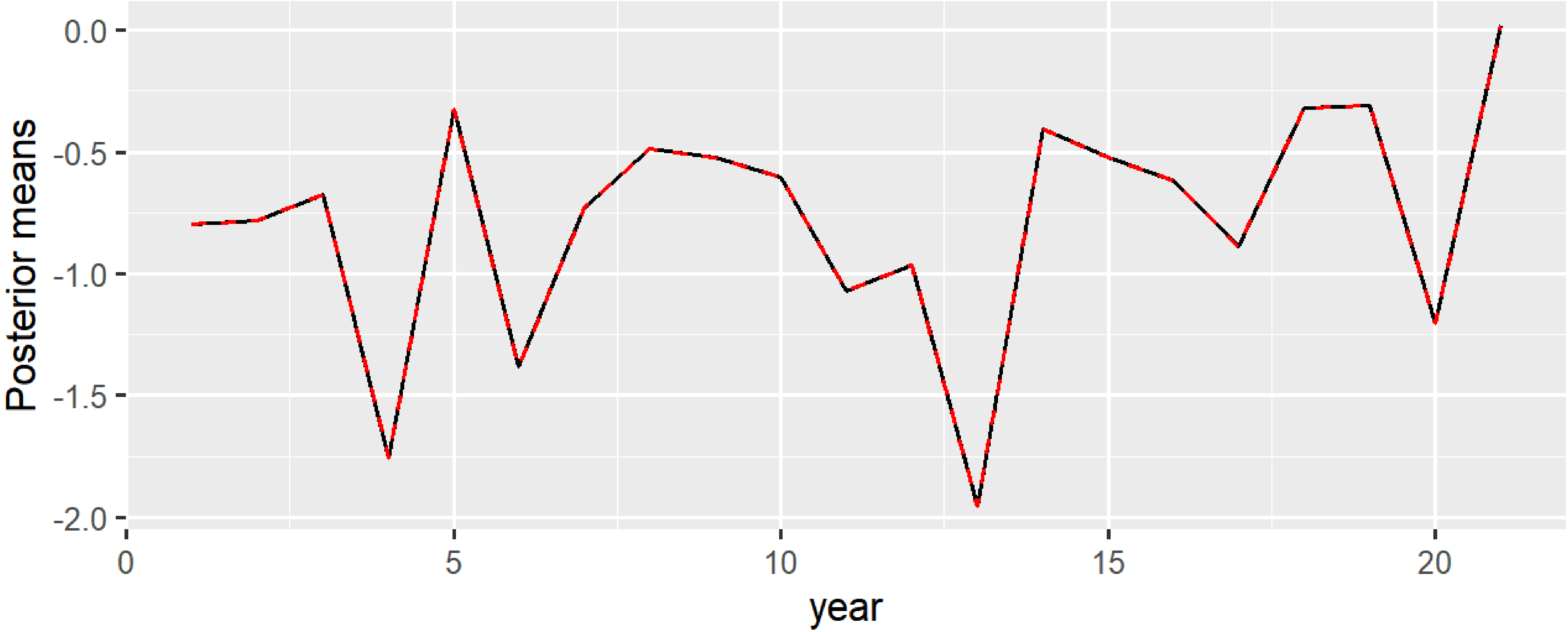
```
                   mean          sd  0.025quant    0.5quant  0.975quant        mode          kld
(Intercept) -0.98833298 0.11239029 -1.20942093 -0.98831791 -0.76733001 -0.98831792 1.473307e-08
year         0.02612109 0.00058628  0.02497146  0.02612109  0.02727073  0.02612109 5.512062e-11
```

```
> mod3$summary.fixed
```

```
                  mean        sd 0.025quant   0.5quant 0.975quant       mode          kld
(Intercept) -0.7759727 0.1580489  -1.086833 -0.7759593 -0.4651878 -0.7759591 1.532338e-08
```

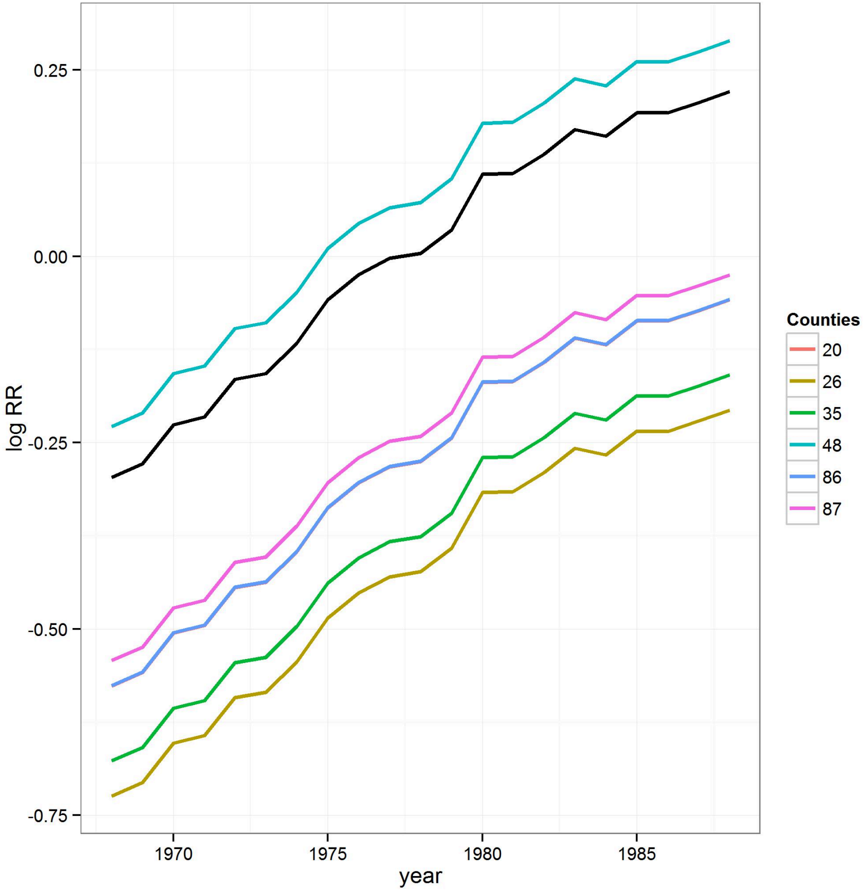
Comparing model 1, 2 and 3

- Fitted values



Do we need space-time interactions in spatio-temporal models?

- When we have temporal random effects, if we look at area specific trends this is what we would see:



Parallel lines as the temporal trend is assumed to be the same across all the spatial units

Space-time model with exchangeable interactions

- It is easy to allow for spatio-temporal interactions, which add flexibility to the model
- A general specification of the model will be

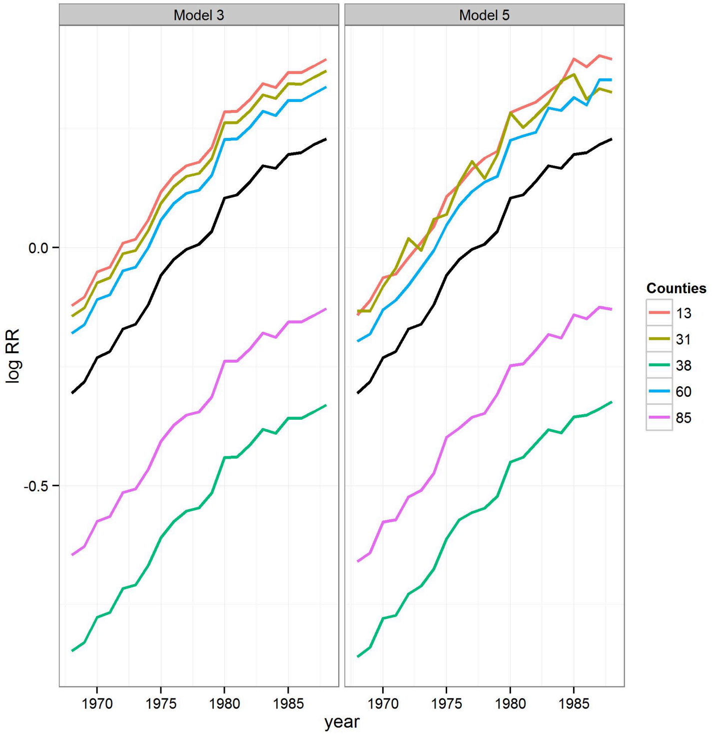
$$
\begin{aligned}
y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\
\log\rho_{it} &= b_0 + b_i + \ \gamma_t + \psi_t + \delta_{it} \\
b_i &= v_i + u_i \\
\gamma_t &\sim \text{RW}(1) \\
\psi_t &\sim \text{N}(0, \sigma^2_\psi) \\
\delta_{it} &\sim \text{Normal}(0, \sigma^2_\delta)
\end{aligned}
$$

The **space-time interaction parameter**, δ_{it}, is modelled as exchangeable random effects, that capture departure from the additive structure (given by the BYM2 in space and RW1+iid in time).

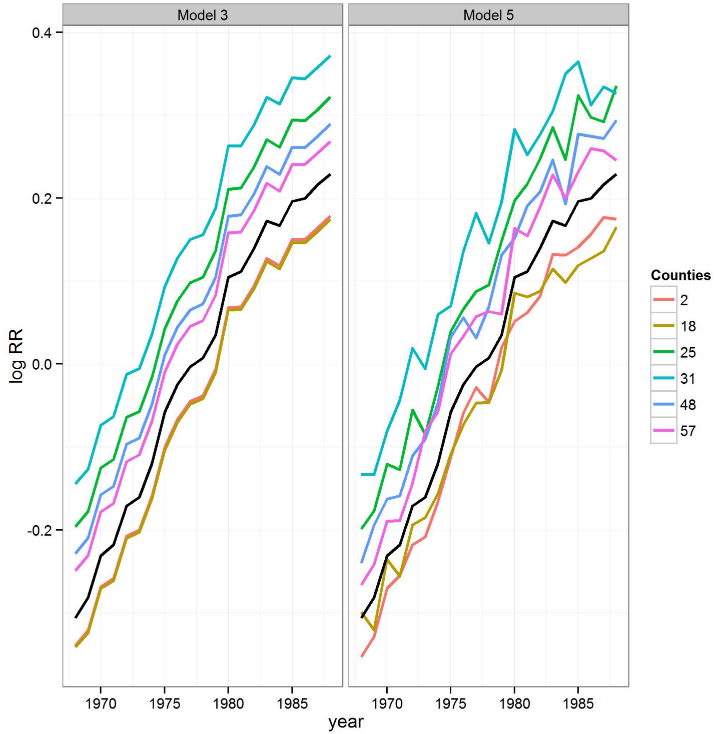
Interpretation of the interactions

- The interactions δ_{it} allow to highlight unusual temporal trends
- Rules based on the posterior probabilities $p(\delta_{it} > 0)$ for at least 1 time t

"usual" temporal trends



unusual temporal trends



INLA code for model with interaction

- In INLA is very easy to include the interaction in the `formula` environment
- We need to specify an index for the interaction, i.e. for each combination of area/time

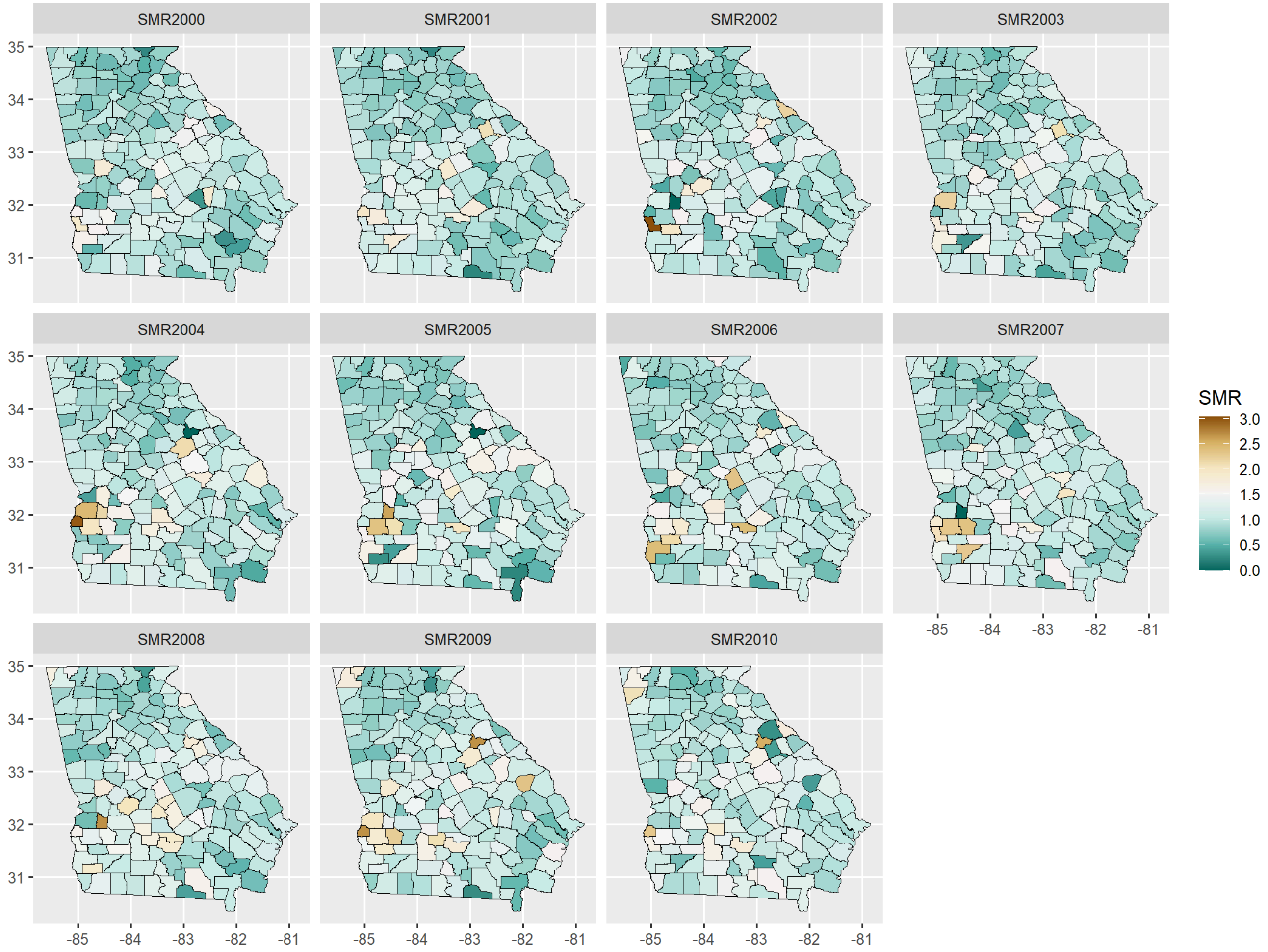
$$
\begin{aligned}
\log\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t + \delta_{it} \\
\gamma_t &\sim \mathrm{RW}(1) \\
\psi_t &\sim N(0, \sigma^2_\psi) \\
\delta_{it} &\sim \mathrm{Normal}(0, \sigma^2_\delta)
\end{aligned}
$$

```
> formula.intI = y ~ + f(county,model="bym2",
+                        graph=Ohio.adj) +
+                      f(year,model="rw1") +
+                      f(year2,model="iid") +
+                      f(area.year,model="iid")
```

Types of interactions

Another example: Birth weight in Georgia

- Count of babies weigthing less than 2500g in 159 counties in Georgia (US)
- Period 2000 – 2010



Different types of interactions

$$
\begin{aligned}
y_{it} &\sim \text{Poisson}(\text{E}_{it}\rho_{it}) \\
\log\rho_{it} &= b_0 + b_i + \gamma_t + \psi_t + \delta_{it} \\
b_i &= v_i + u_i \\
\gamma_t &\sim \text{RW}(1) \\
\psi_t &\sim N(0, \sigma^2_\psi)
\end{aligned}
$$

Characteristics of ST interaction

Interaction	**Parameters**	**Rank**
I	Unstructured in space and Unstructured in time	nT
II	Unstructured in space and RW in time	n(T-1) for RW1, n(T-2) for RW2
III	ICAR in space and Unstructured in time	(n-1)T
IV	ICAR in space and RW in time	(n-1)(T-1) for RW1, (n-1)(T-2) for RW2

How to model interactions

- Data in R format

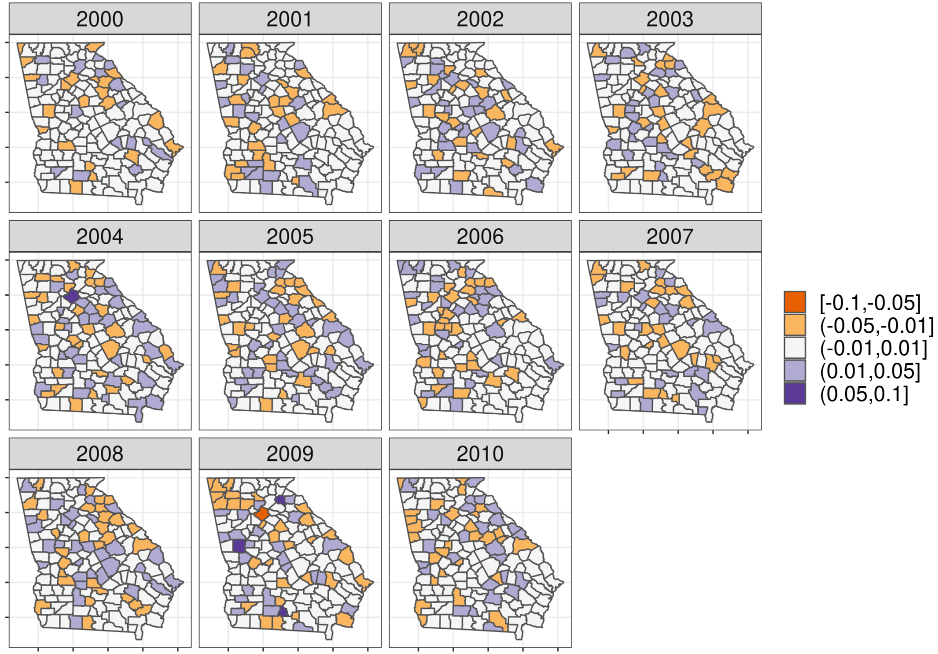
```
# A tibble: 1,749 × 8
   ID.area   Obs ID.year   Exp     y     E NAME    ID.area.year
     <int> <int>   <dbl> <dbl> <int> <dbl> <chr>          <int>
 1       1    20       1  25.8    20  25.8 Appling            1
 2       1    24       2  25.1    24  25.1 Appling            2
 3       1    25       3  22.8    25  22.8 Appling            3
 4       1    31       4  25.2    31  25.2 Appling            4
 5       1    24       5  24.9    24  24.9 Appling            5
 6       1    40       6  26.7    40  26.7 Appling            6
 7       1    29       7  26.0    29  26.0 Appling            7
 8       1    35       8  27.2    35  27.2 Appling            8
 9       1    26       9  25.1    26  25.1 Appling            9
10       1    25      10  24.5    25  24.5 Appling           10
# ℹ 1,739 more rows
```

- We need to make sure to have an index for area (`ID.area`), one for time (`ID.year`) and one for the interaction (`ID.area.year`)

Type I interaction in INLA

Let's adopt Type I interaction, which assumes that the two unstructured effects v_i and ψ_t interact

```
> Georgia.adj = "Georgia.graph"
> ID.year2 = data_INLA$ID.year
> formula.intI = y ~ + f(ID.area,model="bym2",
+                         graph=Georgia.adj) +
+                       f(ID.year,model="rw1") +
+                       f(ID.year2,model="iid") +
+                       f(ID.area.year,model="iid")
```



Kronecker product

- For the interactions of type II-IV we will need to use the **Kronecker product** to specify the dependencies. It is a matrix multiplication which returns a block matrix.

For instance

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$$

- There is a function in `R` which does exactly that: `kronecker(a,b)`
- We will not go through type II-IV of interactions, as they are pretty complex and also they can take a **very long** time to run
- If you are interested in knowing more how to set interactions using the Kronecker product in `INLA` see Goicoa, Adin, Ugarte, and Hodges (2018)

Type II interaction: in INLA

- Type II combines the structured temporal main effect γ_t and the unstructured spatial effect v_i.
- We use first create the structure matrix for the time component assuming a random walk of order 1

```
> #RW1
> D1 <- diff(diag(11),differences=1)
> Q.gammaRW1 <- t(D1)%*%D1
> #Let's look at it
> Q.gammaRW1[1:5,1:5]
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]    1   -1    0    0    0
[2,]   -1    2   -1    0    0
[3,]    0   -1    2   -1    0
[4,]    0    0   -1    2   -1
[5,]    0    0    0   -1    2
```

Type II interaction: in INLA

- Then we create the Kronecker product $v \otimes \gamma$ and note from the table in slide 35 that the rank deficiency is n (159 in our case)

```
> #Kronecker product (for RW1)
> R <- kronecker(diag(159),Q.gammaRW1)
> R[1:5,1:5]
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]    1   -1    0    0    0
[2,]   -1    2   -1    0    0
[3,]    0   -1    2   -1    0
[4,]    0    0   -1    2   -1
[5,]    0    0    0   -1    2
```

Type II interaction: in INLA

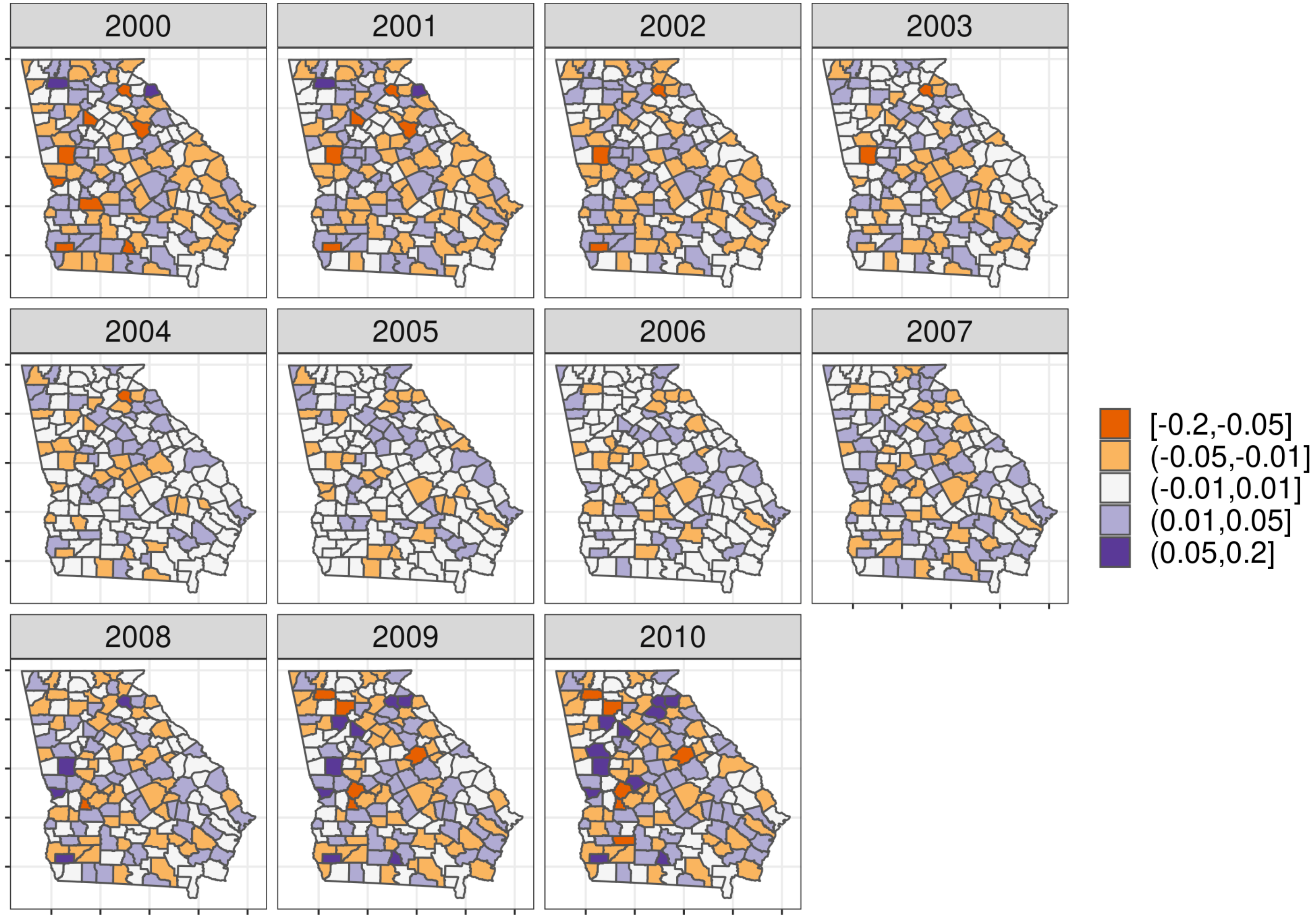
- Finally we need to impose some more constraints as the matrix have spatio-temporal interaction matrix has a rank deficiency
- We calculate the eigenvalues of the structure matrix of the interaction term and the extra constraints are those where the eigenvalues are zero

```
> eigenQ <- eigen(R)
> ids <- which(eigenQ$values < 0.00001)
> cMat <- t(eigenQ$vectors[,ids])
```

The `formula` environment needs to be changed to

```
> formula.intII<- y ~ f(ID.area,model="bym2",graph=Georgia.adj) +
+                     f(ID.year,model="rw1") +
+                     f(ID.year2,model="iid") +
+                     f(ID.area.year, model="generic0", Cmatrix=R, rankdef = nrow(cMat),
+                     extraconstr = list(A = cMat, e = rep(0, nrow(cMat))))
```

Type II interaction: in INLA



Type III interaction in INLA

- Type III combines the spatially structured main effect u_i and the unstructured temporal effect ψ_t.
- We use first create the structure matrix for the spatial component using the neighborhood graph

```
> g <- inla.read.graph(Georgia.adj)
> Q.xi <- matrix(0, g$n, g$n)
> for (i in 1:g$n){
+   Q.xi[i,i]=g$nnbs[[i]]
+   Q.xi[i,g$nbs[[i]]]=-1
+ }
> Q.xi[1:5,1:5]
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]    2   -1    0    0    0
[2,]   -1    4    0    0    0
[3,]    0    0    5   -1    0
[4,]    0    0   -1    4   -1
[5,]    0    0    0   -1    4
```

Type III interaction in INLA

- Then we create the Kronecker product $u \otimes \psi$ and note from the table in slide 35 that the rank deficiency is T (11 in our case)

```
> #Kronecker product (for RW1)
> R <- kronecker(Q.xi,diag(11))
> R[1:5,1:15]
```

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13] [,14] [,15]
[1,]    2    0    0    0    0    0    0    0    0     0     0    -1     0     0     0
[2,]    0    2    0    0    0    0    0    0    0     0     0     0    -1     0     0
[3,]    0    0    2    0    0    0    0    0    0     0     0     0     0    -1     0
[4,]    0    0    0    2    0    0    0    0    0     0     0     0     0     0    -1
[5,]    0    0    0    0    2    0    0    0    0     0     0     0     0     0     0
```

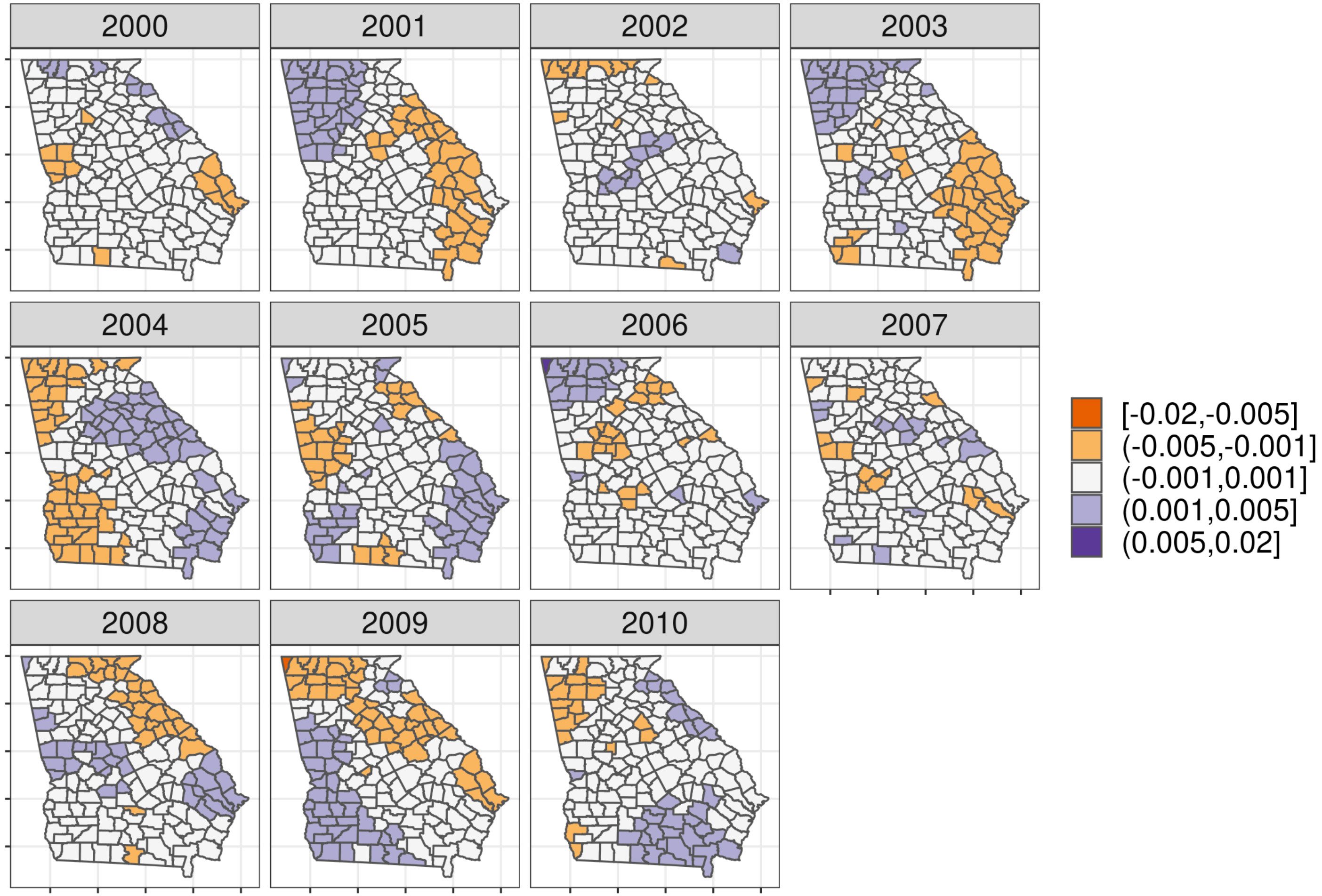
- Finally we need to impose some more constraints (on the eigenvalues equal to 0)

```
> eigenQ <- eigen(R)
> ids <- which(eigenQ$values < 0.00001)
> cMat <- t(eigenQ$vectors[,ids])
```

The `formula` environment becomes

```
> formula.intIII<- y ~ f(ID.area,model="bym2",graph=Georgia.adj) +
+                     f(ID.year,model="rw1") +
+                     f(ID.year2,model="iid") +
+                     f(ID.area.year, model="generic0", Cmatrix=R, rankdef = nrow(cMat),
+                      extraconstr = list(A = cMat, e = rep(0, nrow(cMat))))
```

Type III interaction: in INLA



Type IV interaction in INLA

- Type IV is the most complex type of interaction, assuming that the spatially and temporally structured effects u_i and γ_t interact
- We assume that the temporal dependency structure for each area is not independent from all the other areas anymore, but depends on the temporal pattern of the neighboring areas as well

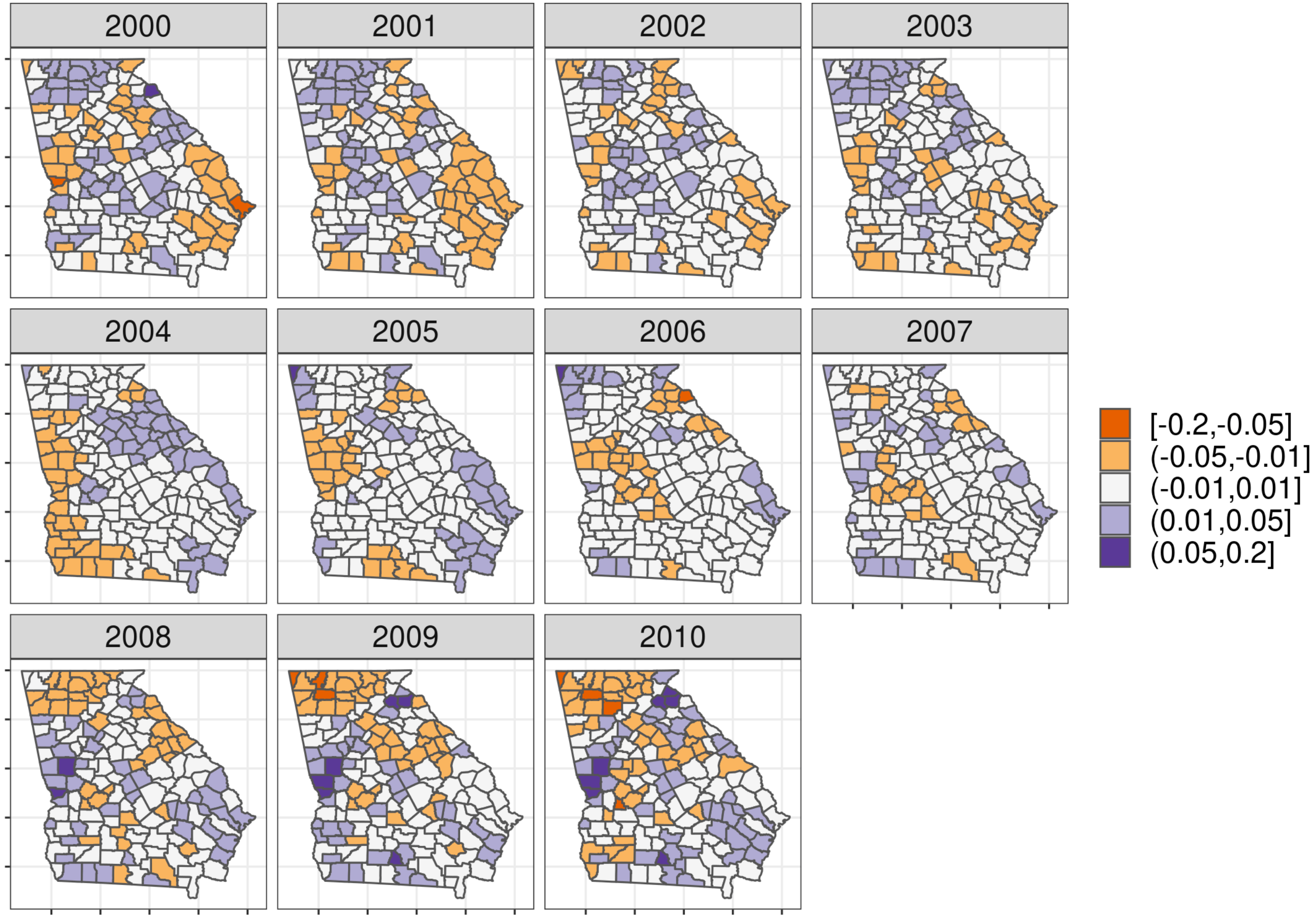
- We create the Kronecker product $u \otimes \gamma$ and note from the table in slide 35 that the rank deficiency is n+T-1 (169 in our case)

```
> #Kronecker product (for RW1)
> R <- kronecker(Q.xi,Q.gammaRW1)
> eigenQ <- eigen(R)
> ids <- which(eigenQ$values < 0.00001)
> cMat <- t(eigenQ$vectors[,ids])
```

The `formula` environment becomes

```
> formula.intIV<- y ~ f(ID.area,model="bym2",graph=Georgia.adj) +
+                     f(ID.year,model="rw1") +
+                     f(ID.year,model="iid") +
+                     f(ID.area.year, model="generic0", Cmatrix=R, rankdef = nrow(cMat),
+                     extraconstr = list(A = cMat, e = rep(0, nrow(cMat))))
```

Type IV interaction: in INLA



Model selection

It might be useful to employ indexes like DIC and WAIC to decide which model is the most appropriate for the problem at hand

In this example:

Model fitting for the different ST models

Interaction	**Parameters**	**Rank**	**DIC**	**WAIC**
I	Unstructured in space and Unstructured in time	nT	11570	11602
II	Unstructured in space and RW in time	n(T-1) for RW1, n(T-2) for RW2	11536	11569
III	ICAR in space and Unstructured in time	(n-1)T	11615	11659
IV	ICAR in space and RW in time	(n-1)(T-1) for RW1, (n-1)(T-2) for RW2	11570	11612

So we conclude that the model with the interaction of order II is the most appropriate for this data.

Summary

- Increase quality of datasets that are both spatially and temporally indexed
- Advanced methods to deal with this type of data
- Allow to interpret the stability (or not) of the spatial patterns

Summary

- Increase quality of datasets that are both spatially and temporally indexed
- Advanced methods to deal with this type of data
- Allow to interpret the stability (or not) of the spatial patterns
- Model selection tools are useful to choose which interaction to use
- Careful with interactions of order above I as they can take a **very long** time to run

References

Goicoa, T., A. Adin, M. Ugarte, et al. (2018). "In spatio-temporal disease mapping models, identifiability constraints affect PQL and INLA results". In: *Stochastic Environmental Research and Risk Assessment* 32.3, pp. 749-770.