

# Session 1.1: Introduction to INLA and R-INLA

VIBASS, University of Valencia

20 July 2022

# Learning objectives

After this lecture you should be able to

- Present the class of latent Gaussian models
- Present the Laplace approximation and the INLA approach
- Use the basic functions of the R-INLA package

The topics treated in this lecture are presented in Chapter 4 of the book **Spatial and Spatio-Temporal Bayesian models with R-INLA**.

# Outline

1. MCMC and INLA
2. Latent Gaussian models
3. The INLA approach
4. R-INLA package

# MCMC and INLA

# From MCMC to INLA

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- The INLA algorithm proposed by Rue, Martino, and Chopin (2009) is a *deterministic* algorithm for Bayesian inference and it represents an alternative to MCMC which is instead a simulation based algorithm.

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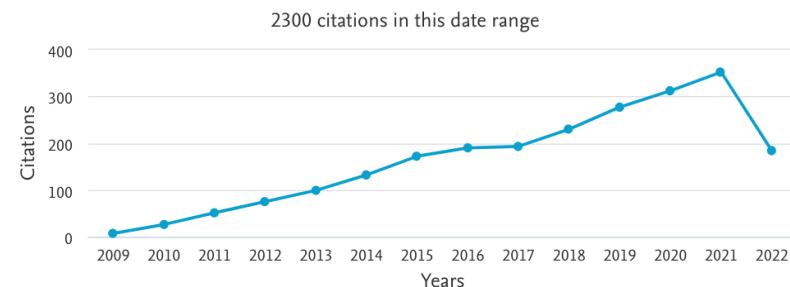
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- The INLA algorithm is designed for the class of *latent Gaussian models* and compared to MCMC it provides (as) accurate results in a shorter time.

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- The INLA algorithm proposed by Rue, Martino, and Chopin (2009) is a *deterministic* algorithm for Bayesian inference and it represents an alternative to MCMC which is instead a simulation based algorithm.
- The INLA algorithm is designed for the class of *latent Gaussian models* and compared to MCMC it provides (as) accurate results in a shorter time.
- INLA has become very popular among statisticians and applied researchers and in the past few years the number of papers reporting usage and extensions of the INLA method has increased considerably.

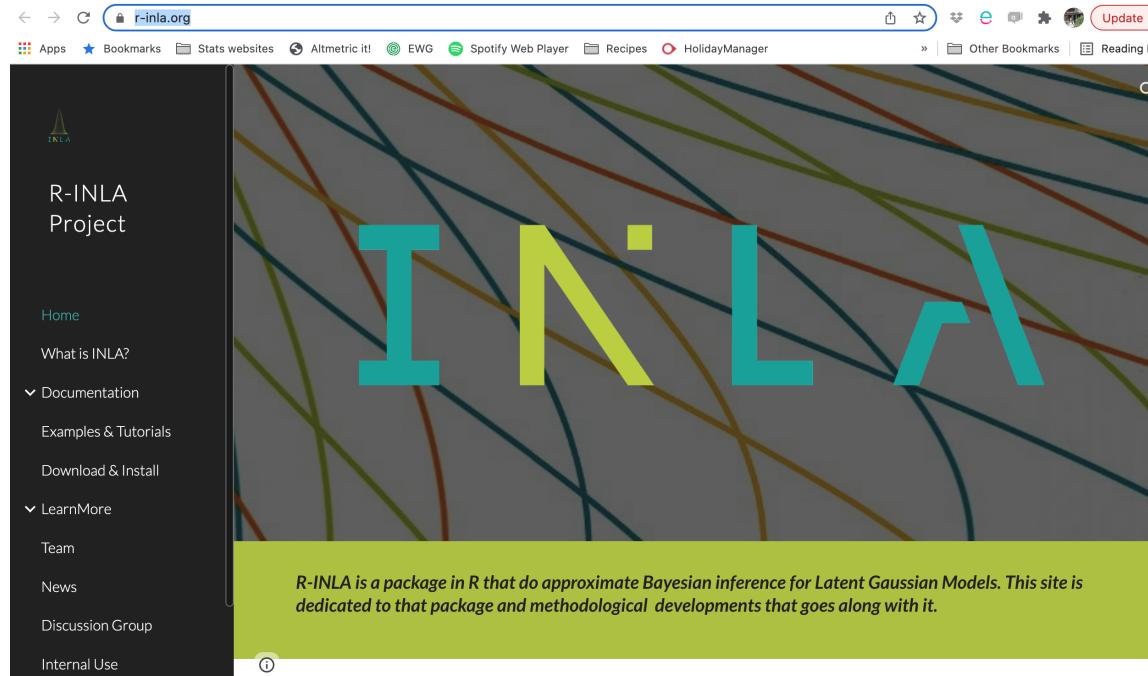
Date range: 2009  to 2022

Include all citations  
 Exclude self citations  
 Exclude citations from books



# INLA website and community

- The website contains source code, examples, papers and reports discussing the theory and applications of INLA.
- There is also a discussion forum where users can post queries and requests of help.
- Almost each year there is an INLA-related scientific meeting.



INLA website

# Latent Gaussian models

# Latent Gaussian models (LGMs)

- The general problem of (parametric) inference is posited by assuming a probability model for the observed data  $\mathbf{y} = (y_1, \dots, y_n)$ , as a function of some relevant parameters

$$\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\psi} \sim p(\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\psi}) = \prod_{i=1}^n p(y_i | \boldsymbol{\theta}, \boldsymbol{\psi})$$

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- Often (in fact for a surprisingly large range of models!), we can assume that the parameters are described by a **Gaussian Markov Random Field (GMRF)**

$$\boldsymbol{\theta} | \boldsymbol{\psi} \sim \text{Normal}(\mathbf{0}, \mathbf{Q}^{-1}(\boldsymbol{\psi}))$$

$$\theta_i \perp\!\!\!\perp \theta_j | \boldsymbol{\theta}_{-i,j} \iff Q_{ij}(\boldsymbol{\psi}) = 0$$

where

- The precision matrix  $\mathbf{Q}$  depends on some hyperparameters  $\boldsymbol{\psi}$ .
- The notation  $-i, j$  indicates all the other elements of the parameters vector, excluding elements  $i$  and  $j$
- The components of  $\boldsymbol{\theta}$  are supposed to be *conditionally independent* with the consequence that  $\mathbf{Q}$  is a sparse precision matrix.
- This kind of models is often referred to as **Latent Gaussian models**.

# LGMs as a general framework

- In general

$$\mathbf{y} \mid \boldsymbol{\theta}, \boldsymbol{\psi} \sim \prod_i p(y_i \mid \boldsymbol{\theta}, \boldsymbol{\psi}) \text{ (Data model)}$$

$$\boldsymbol{\theta} \mid \boldsymbol{\psi} \sim p(\boldsymbol{\theta} \mid \boldsymbol{\psi}) = \text{Normal}(0, \mathbf{Q}^{-1}(\boldsymbol{\psi})) \text{ (Latent Gaussian Field)}$$

$$\boldsymbol{\psi} \sim p(\boldsymbol{\psi}) \text{ (Hyperprior)}$$

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- The dimension of  $\boldsymbol{\theta}$  can be very large (eg  $10^2$ - $10^5$ ).
- Conversely, the dimension of  $\boldsymbol{\psi}$  must be relatively small (less than 20 is recommended) to avoid an exponential increase in the computational costs of the model.

# LGMs as a general framework

- A very general way of specifying the problem is specifying a distribution for  $y_i$  characterized by a parameter  $\phi_i$  (usually the mean) defined as a function of a structured additive predictor  $\eta_i$ , defined on a suitable scale, such that  $g(\phi_i) = \eta_i$  (e.g. logistic for binomial data):

$$\eta_i = \beta_0 + \sum_{m=1}^M \beta_m x_{mi} + \sum_{l=1}^L f_l(z_{li})$$

where

- $\beta_0$  is the intercept;
- $\beta = \{\beta_1, \dots, \beta_M\}$  quantify the effect of the covariates  $x = (x_1, \dots, x_M)$  on the response;
- $f = \{f_1(\cdot), \dots, f_L(\cdot)\}$  is a set of functions defined in terms of some covariates  $z = (z_1, \dots, z_L)$

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$$\theta = \{\beta_0, \beta, f\} \sim \text{Normal}(\mathbf{0}, \mathbf{Q}^{-1}(\psi)) = \text{GMRF}(\psi)$$

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- NB: This of course implies some form of Normally-distributed marginals for  $\beta_0$ ,  $\beta$  and  $f$

## LGMs as a general framework --- examples

Upon varying the form of the functions  $f_l(\cdot)$ , this formulation can accommodate a wide range of models (see Martins, Simpson, Lindgren, and Rue (2013) for a review)

- Standard regression
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- Spatial and spatio-temporal models
  - Areal data:  $f_1(\cdot) \sim \text{CAR}$  (Spatially structured effects)  
 $f_2(\cdot) \sim \text{Normal}(0, \sigma_{f_2}^2)$  (Unstructured residual)
  - Geostatistical data:  $f(\cdot) \sim \text{Gaussian field}$
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- Survival models, logGaussian Cox Processes, etc.

# The INLA approach

# Integrated Nested Laplace Approximation (INLA)

- The first *ingredient* of the INLA approach is the definition of conditional probability, which holds for any pair of variables  $(x, z)$ .

Technically, provided  $p(z) > 0$

$$p(x \mid z) =: \frac{p(x, z)}{p(z)} \rightarrow p(x, z) = p(x \mid z)p(z)$$

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- In particular, a conditional version can be obtained further considering a third variable  $w$  as

$$p(z | w) = \frac{p(x, z | w)}{p(x | z, w)}$$

which is particularly relevant to the Bayesian case.

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- Main idea: approximate  $\log f(x)$  using a quadratic function by means of a Taylor's series expansion around the mode  $x^* = \operatorname{argmax}_x \log f(x)$

$$\begin{aligned}\log f(x) &\approx \log f(x^*) + (x - x^*) \frac{\partial \log f(x)}{\partial x} \Big|_{x=x^*} + \frac{(x - x^*)^2}{2} \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*} \\ &= \log f(x^*) + \frac{(x - x^*)^2}{2} \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*} \quad \left( \text{since } \frac{\partial \log f(x)}{\partial x} \Big|_{x=x^*} = 0 \right)\end{aligned}$$

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- Setting  $\sigma^{2*} = -1 / \frac{\partial^2 \log f(x)}{\partial x^2} \Big|_{x=x^*}$  we can re-write

$$\log f(x) \approx \log f(x^*) - \frac{1}{2\sigma^{2*}} (x - x^*)^2$$

or equivalently

$$\int f(x) dx = \int \exp[\log f(x)] dx \approx f(x^*) \int \exp \left[ -\frac{(x - x^*)^2}{2\sigma^{2*}} \right] dx$$

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- Thus, under LA,  $f(x) \approx \text{Normal}(x^*, \sigma^{2*})$ .

## Laplace approximation --- example

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- Then
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- Consequently, we can approximate  $f(x)$  as

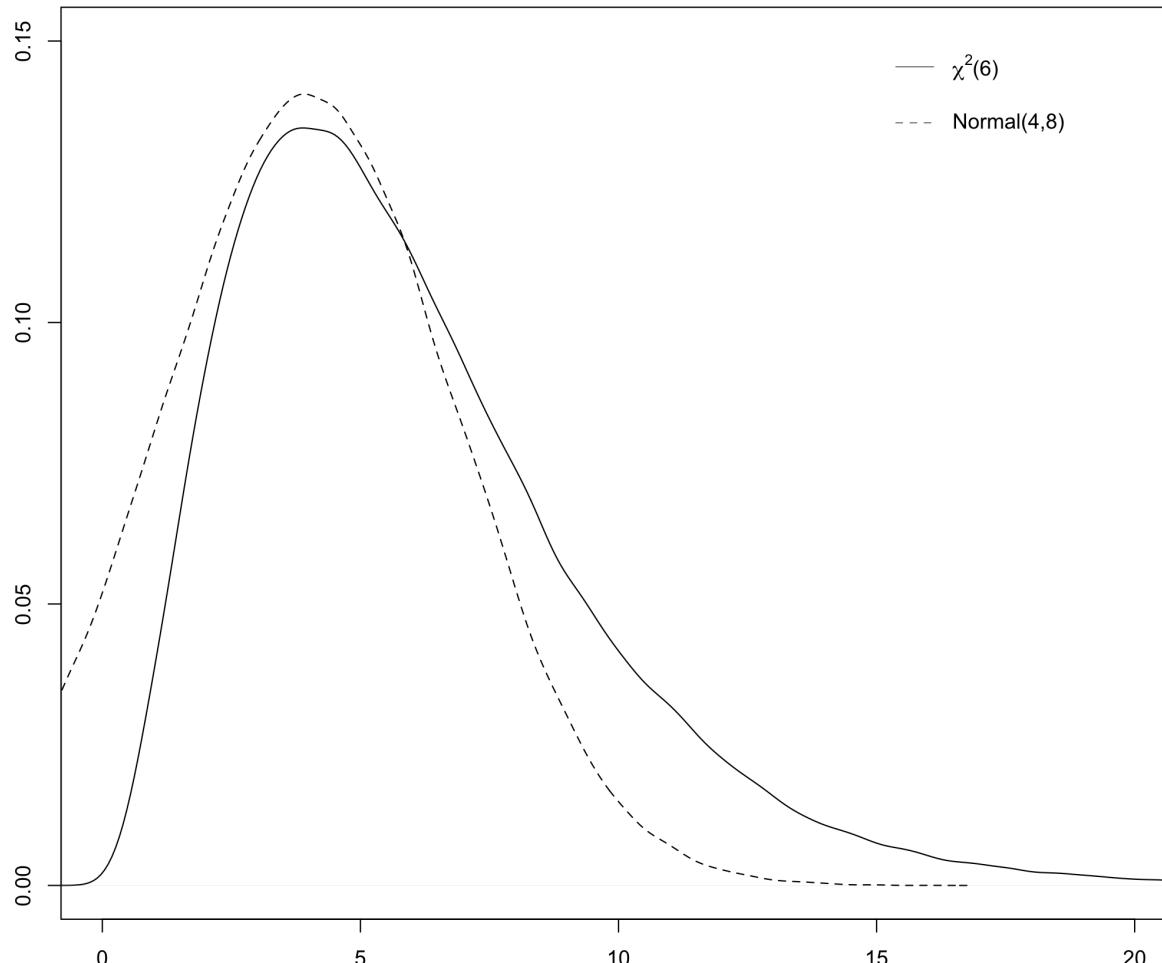
$$f(x) \approx \text{Normal}(k - 2, 2(k - 2))$$

# Laplace approximation --- example

Fig 1

Fig 2

Fig 3

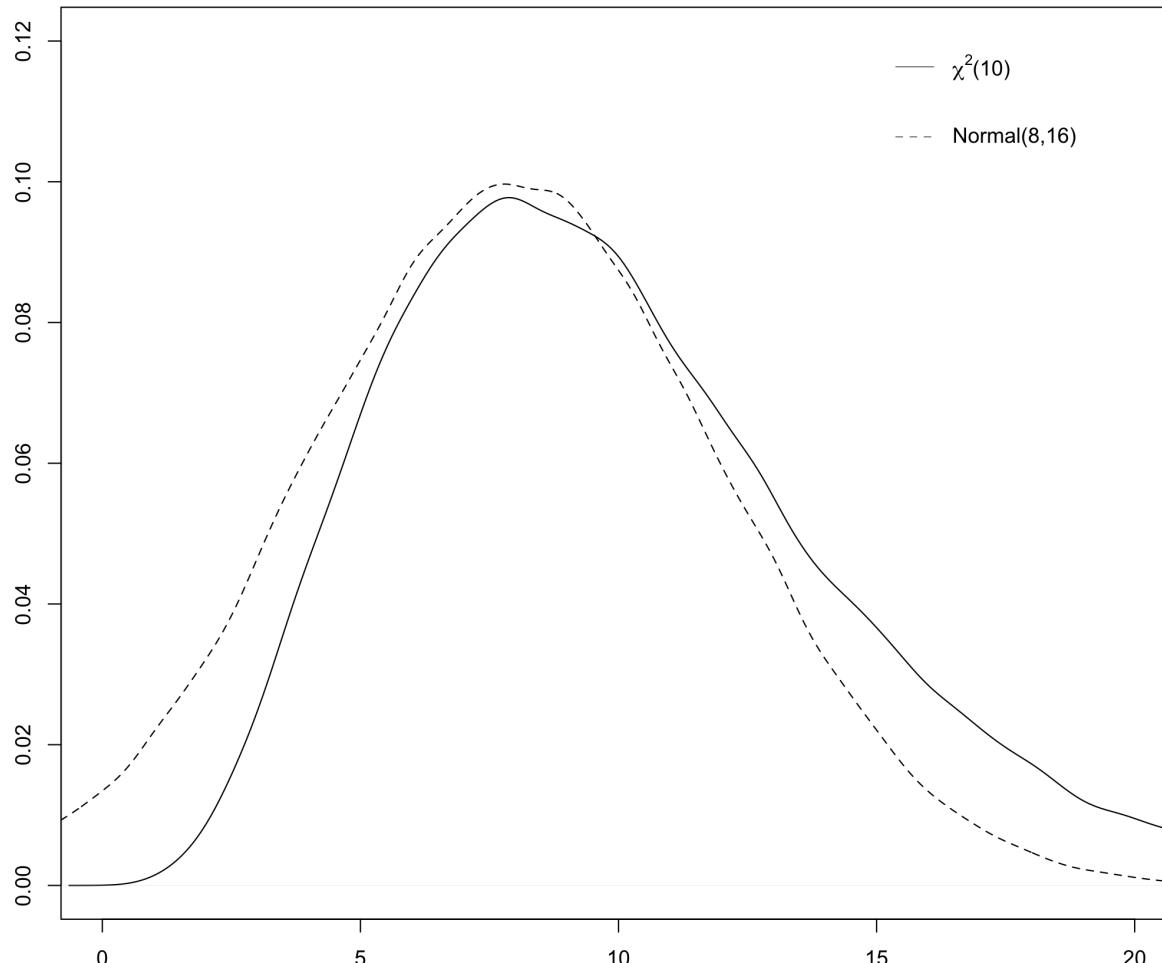


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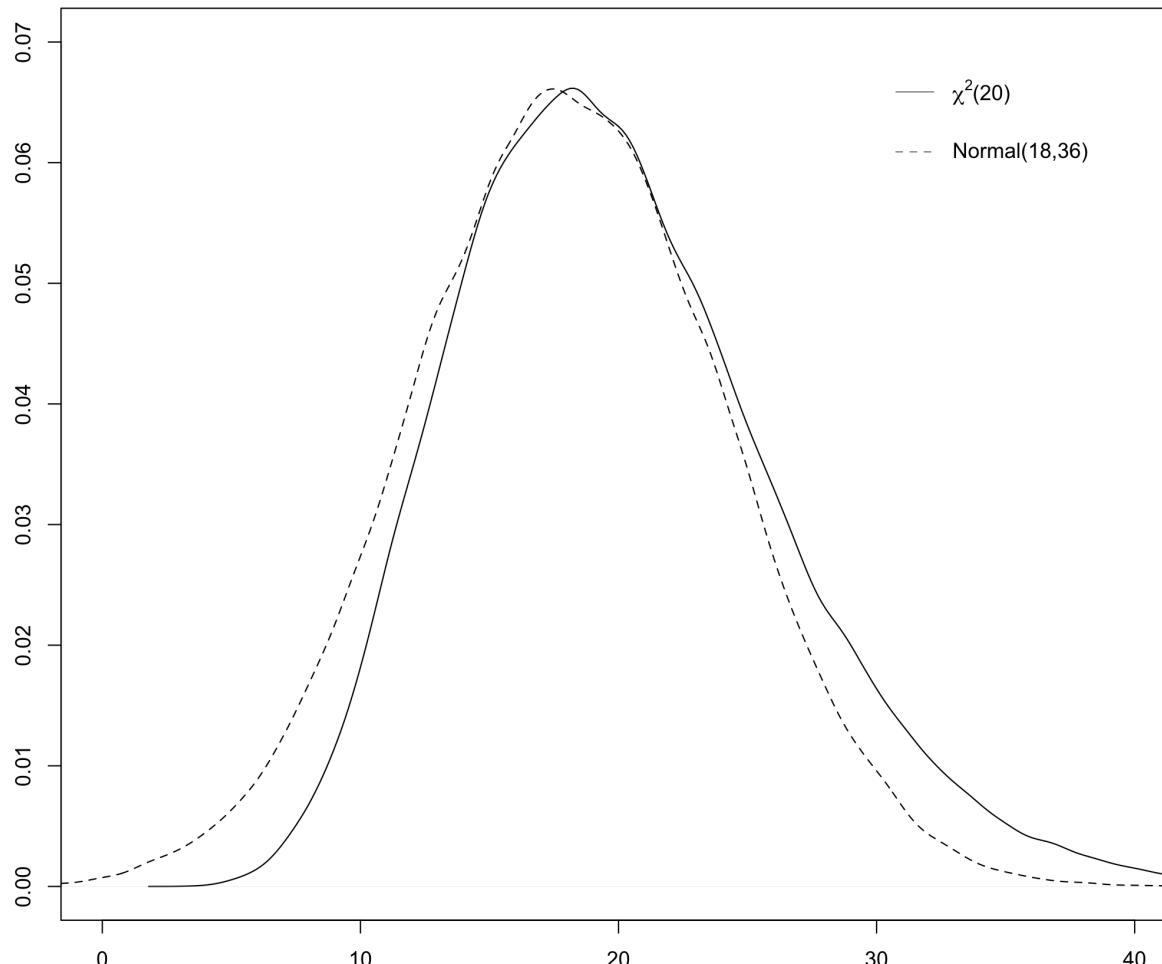


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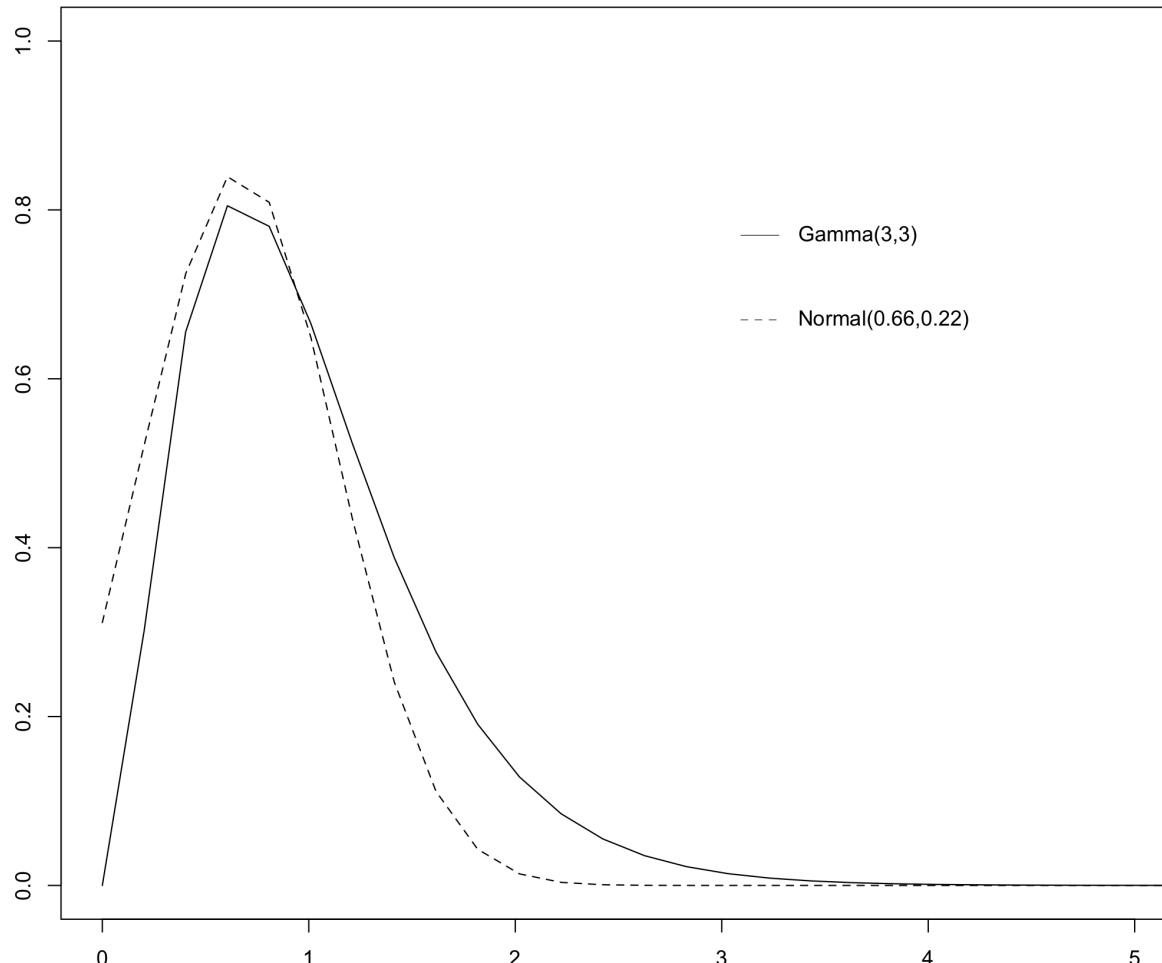
$$\text{Gamma}(a, b) \approx \text{Normal}\left(x^* = \frac{a-1}{b}, \sigma^{2*} = \frac{a-1}{b^2}\right)$$

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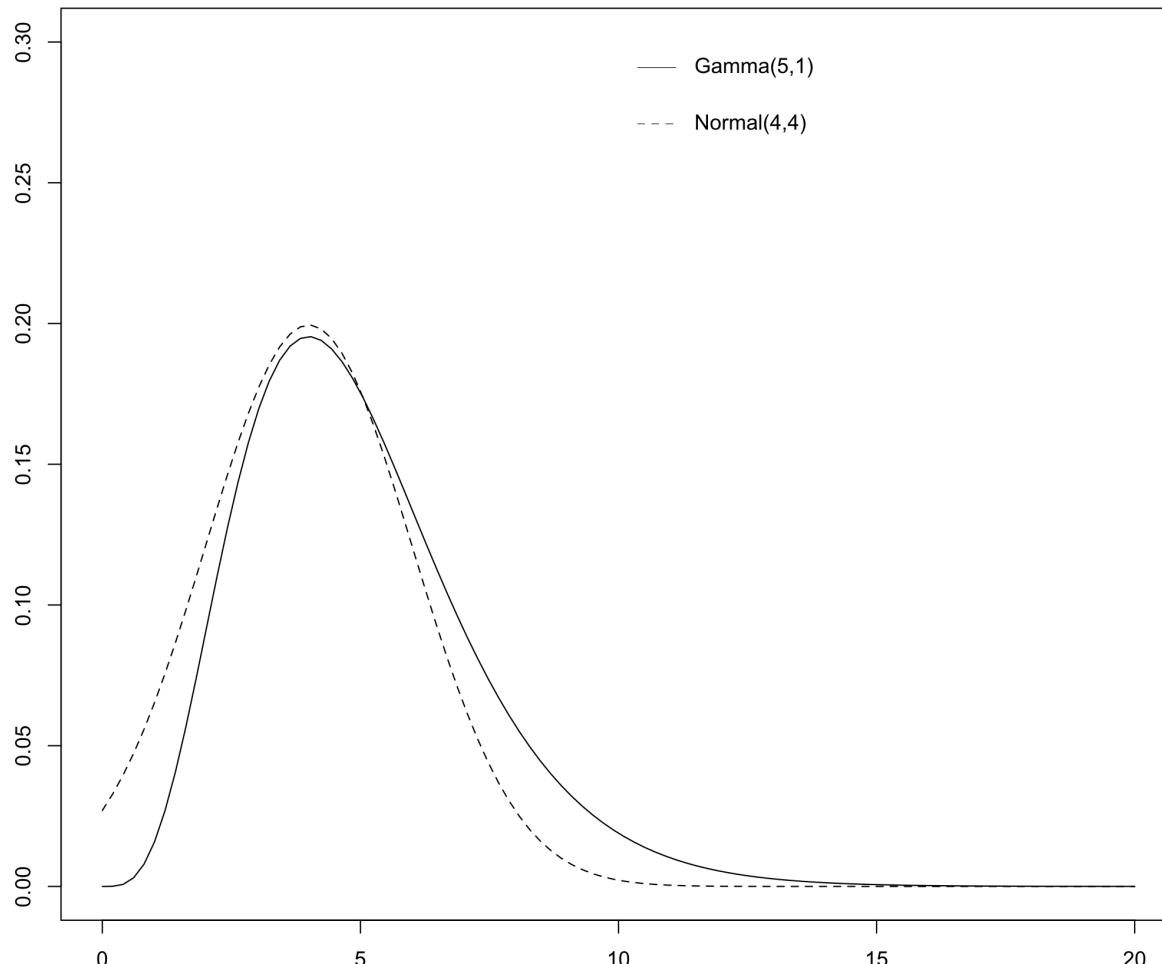


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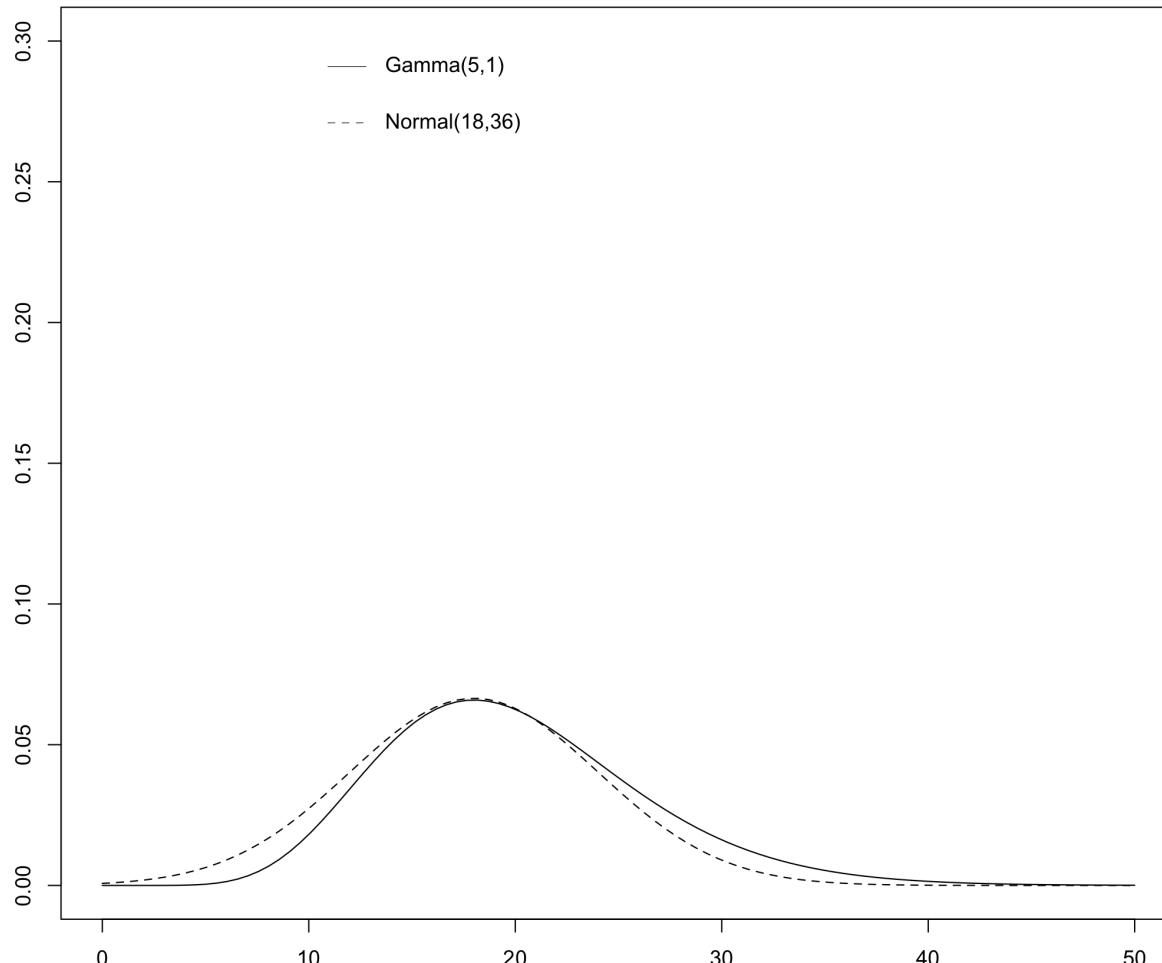


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# Integrated Nested Laplace Approximation (INLA)

## Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

$$p(\theta_i \mid \mathbf{y}) = \int p(\theta_i, \boldsymbol{\psi} \mid \mathbf{y}) d\boldsymbol{\psi} = \int \color{red}{p(\boldsymbol{\psi} \mid \mathbf{y})} \color{orange}{p(\theta_i \mid \boldsymbol{\psi}, \mathbf{y})} d\boldsymbol{\psi}$$

$$p(\psi_k \mid \mathbf{y}) = \int \color{red}{p(\boldsymbol{\psi} \mid \mathbf{y})} d\boldsymbol{\psi}_{-k}$$

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$$p(\psi_k \mid \mathbf{y}) = \int \color{red}{p(\boldsymbol{\psi} \mid \mathbf{y})} d\boldsymbol{\psi}_{-k}$$

- Thus we need to estimate:
  1.  $p(\boldsymbol{\psi} \mid \mathbf{y})$ , from which also all the relevant marginals  $p(\psi_k \mid \mathbf{y})$  can be obtained;

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## Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

$$p(\theta_i | \mathbf{y}) = \int p(\theta_i, \boldsymbol{\psi} | \mathbf{y}) d\boldsymbol{\psi} = \int \color{red}{p(\boldsymbol{\psi} | \mathbf{y})} \color{orange}{p(\theta_i | \boldsymbol{\psi}, \mathbf{y})} d\boldsymbol{\psi}$$

$$p(\psi_k | \mathbf{y}) = \int \color{red}{p(\boldsymbol{\psi} | \mathbf{y})} d\boldsymbol{\psi}_{-k}$$

- Thus we need to estimate:
  1.  $p(\boldsymbol{\psi} | \mathbf{y})$ , from which also all the relevant marginals  $p(\psi_k | \mathbf{y})$  can be obtained;
  2.  $p(\theta_i | \boldsymbol{\psi}, \mathbf{y})$ , which is needed to compute the marginal posterior for the parameters

# Integrated Nested Laplace Approximation (INLA)

1. can be easily estimated as

$$p(\psi \mid \mathbf{y}) = \frac{p(\theta, \psi \mid \mathbf{y})}{p(\theta \mid \psi, \mathbf{y})} \text{ (Recall slide 14)}$$

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where

- $\tilde{p}(\theta \mid \psi, \mathbf{y})$  is the Gaussian approximation given by the Laplace method of  $p(\theta \mid \psi, \mathbf{y})$
- $\theta = \theta^*(\psi)$  is its mode for a given  $\psi$

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- Because  $(\boldsymbol{\theta}_{-i} \mid \boldsymbol{\theta}_i, \boldsymbol{\psi}, \mathbf{y})$  are reasonably Normal, the approximation works generally well.
- However, this strategy can be computationally expensive as  $\tilde{p}(\boldsymbol{\theta}_{-i} \mid \boldsymbol{\theta}_i, \boldsymbol{\psi}, \mathbf{y})$  must be recomputed for each value of  $\boldsymbol{\theta}$  and  $\boldsymbol{\psi}$ .

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- The most efficient algorithm is the Simplified Laplace Approximation:
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  - This is usually corrected by including a mixing term (e.g. spline) to increase the fit to the required distribution.
  - The accuracy of this approximation is sufficient in many applied cases and that the computing time is considerably shorter, it is the standard option.

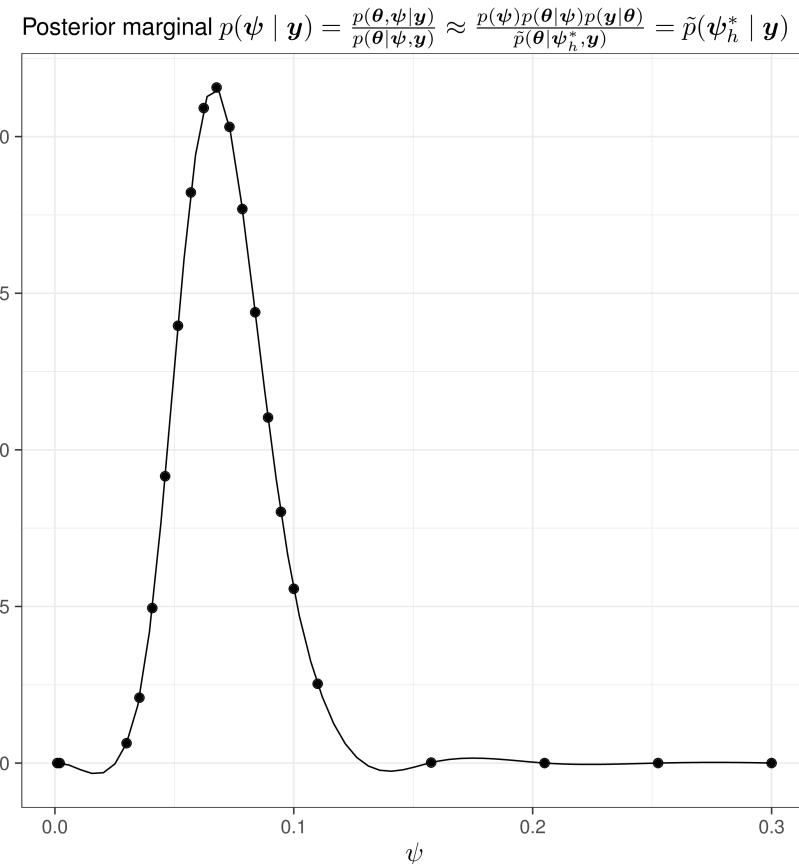
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  - The accuracy of this approximation is sufficient in many applied cases and that the computing time is considerably shorter, it is the standard option.
- This is the algorithm implemented by default by R-INLA, but this choice can be modified.
  - If extra precision is required, it is possible to run the full Laplace approximation --- of course at the expense of running time!

<!-- --> <!-- -->

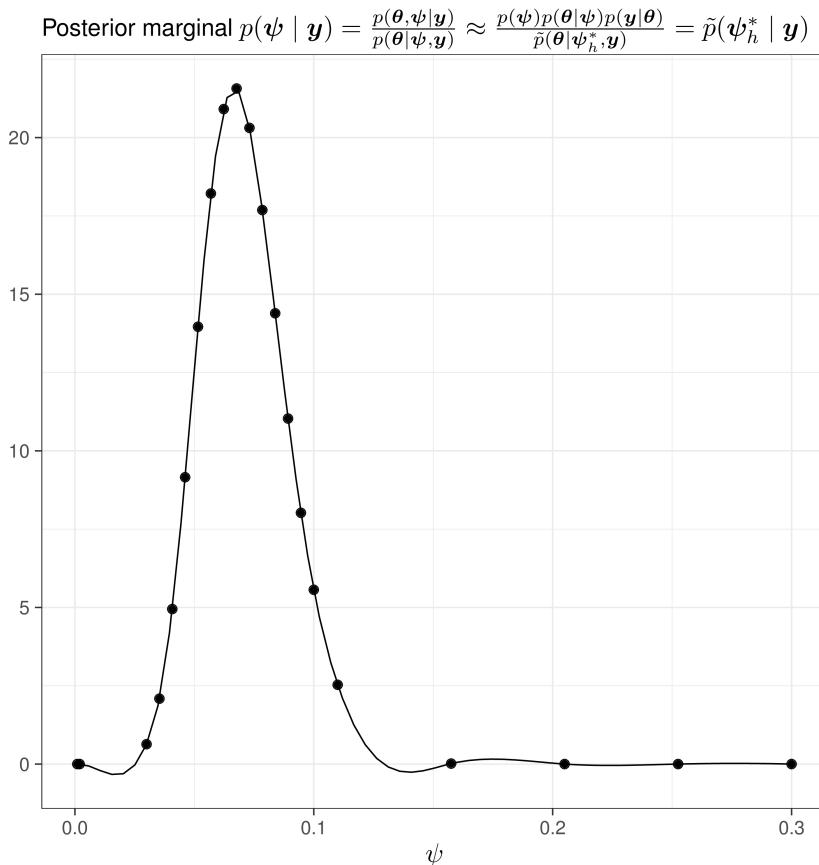
# INLA – in a nutshell...

- I Select a grid of  $H$  points  $\{\psi_h^*\}$  and area weights  $\{\Delta_h\}$ ; interpolate the density to approximate to the posterior

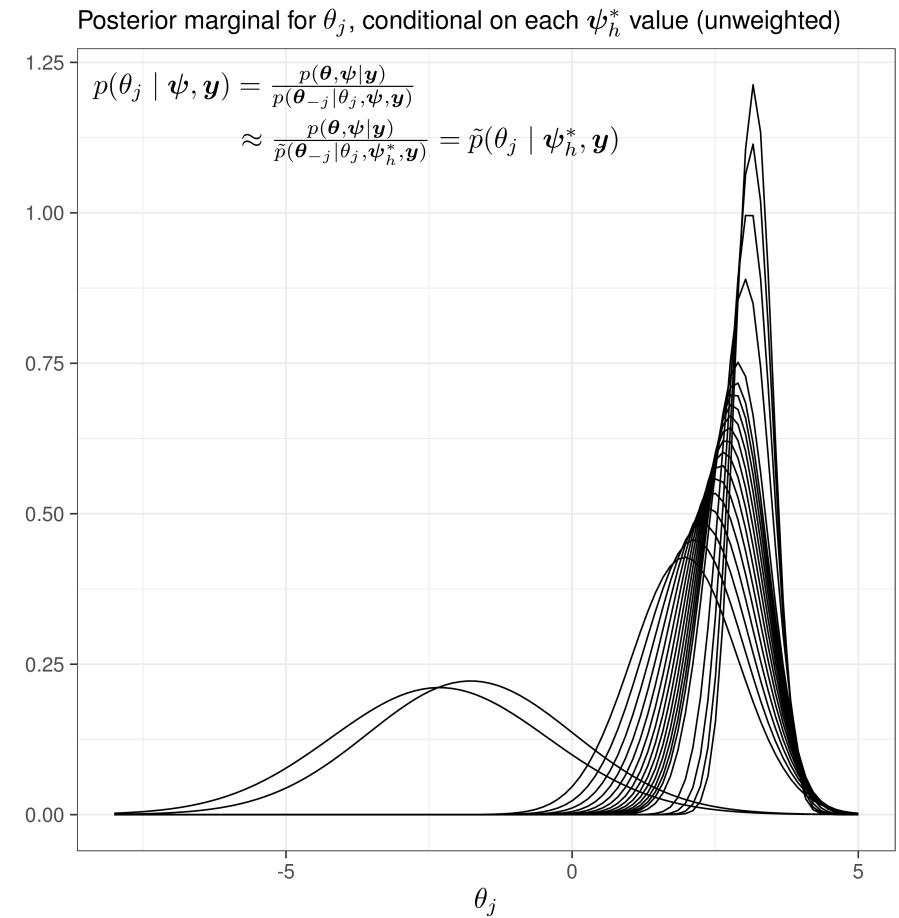


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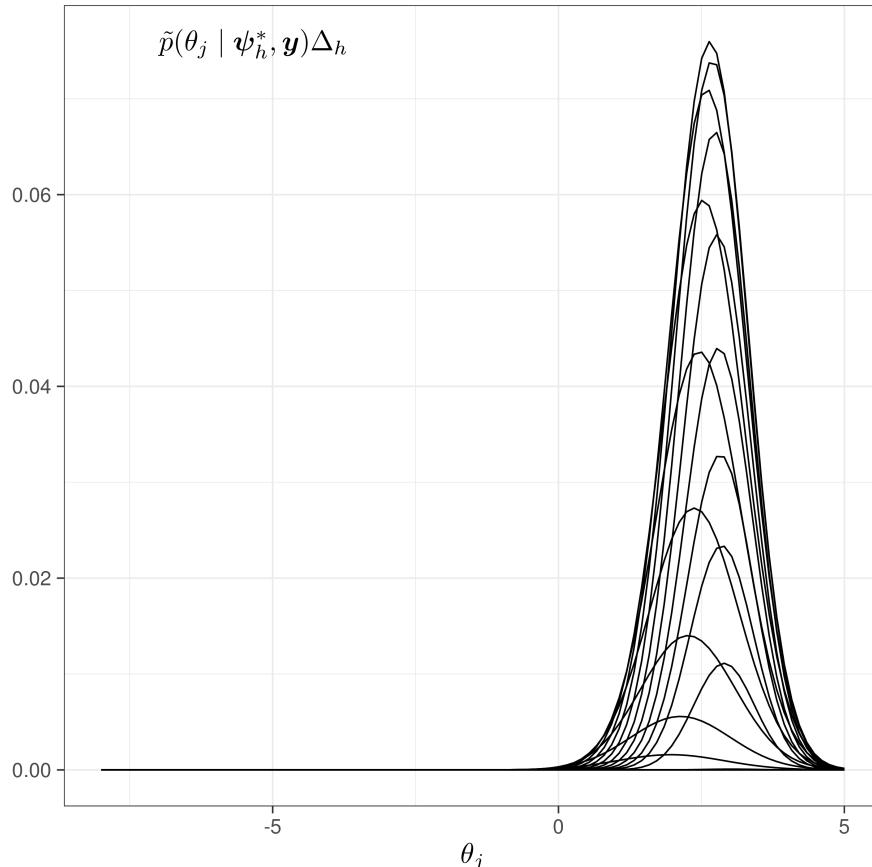
- 2 Approximates the conditional posterior of each  $\theta_j$ , given  $\psi, \mathbf{y}$  on the  $H$ -dimensional grid



# INLA – in a nutshell...

- 3 Weight the conditional marginal posteriors by the density associated with each  $\psi_h^*$

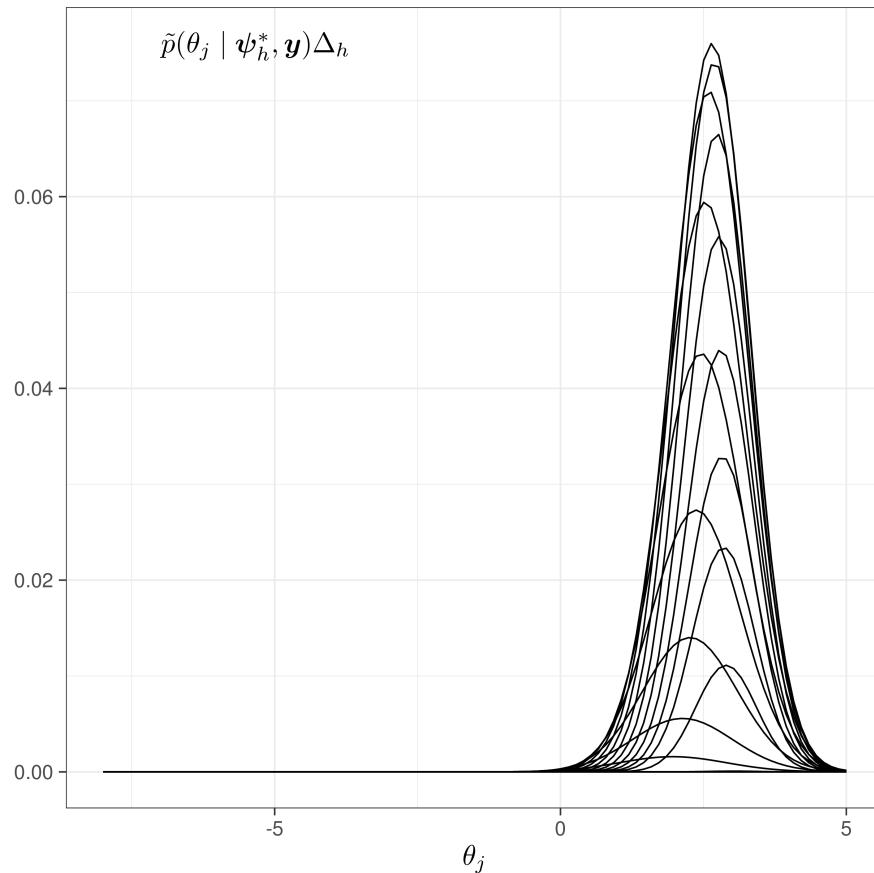
Posterior marginal for  $\theta_j$ , conditional on each  $\psi_h^*$  value (weighted)



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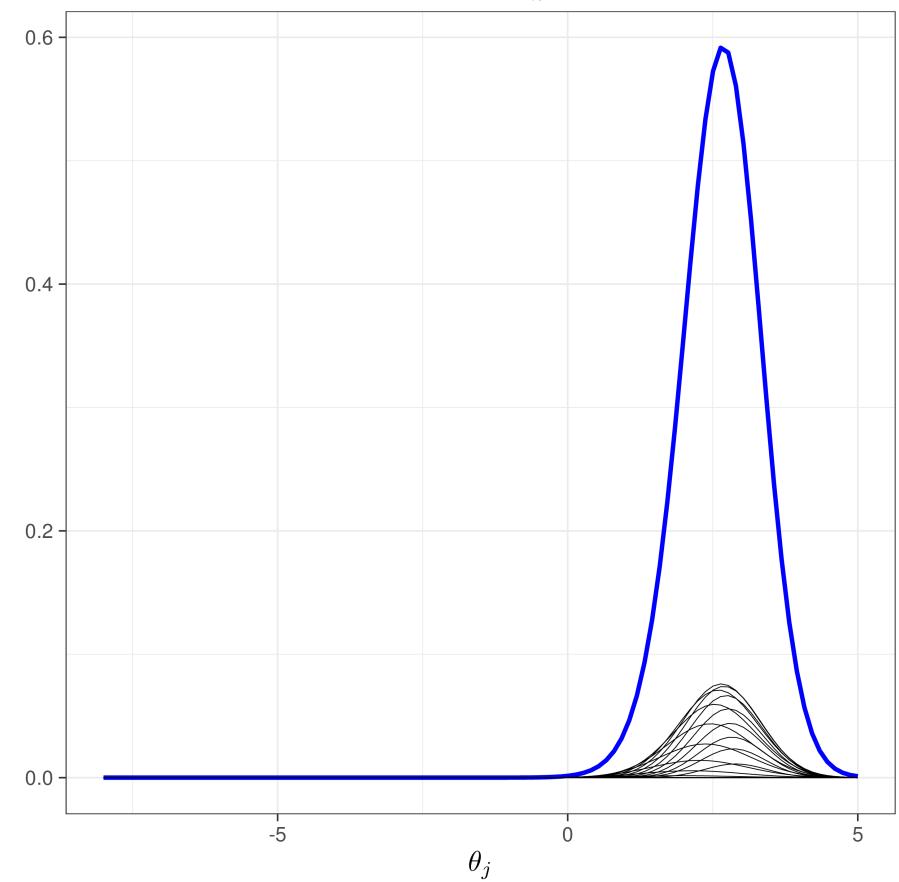
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4 (Numerically) sum over all the conditional densities to obtain the marginal posterior for  $\theta_j$

Posterior marginal for  $\theta_j$  :  $p(\theta_j | \mathbf{y}) \approx \sum_h \tilde{p}(\psi_h^* | \mathbf{y}) \tilde{p}(\theta_j | \psi_h^*, \mathbf{y}) \Delta_h$



# Integrated Nested Laplace Approximation (INLA)

So, it's all in the name...

## Integrated Nested **Laplace Approximation**

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  - NB: Consequently the estimation of (1.) might not be good enough, but it can be refined (eg using a finer grid)
- Because the required marginal posterior distributions are obtained by (numerical) integration.

# The R-INLA package

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- A standalone C program build upon the GMRFLib library (it performs the relevant computation and returns the results in a standardised way)

NB: Because the package R-INLA relies on a standalone C program (and other reasons...), it is not available directly from CRAN.

R-INLA runs natively under Linux, Windows and Mac and it is possible to do multi-threading using OpenMP

# The INLA package R - Installation

- From R, installation of the stable version is performed typing

```
install.packages("INLA", repos="http://www.math.ntnu.no/inla/R/stable")
```

- Later, you can upgrade the package by typing

```
library(INLA)
```

```
inla.upgrade()
```

- A test-version (which may contain unstable updates/new functions) can be obtained by typing

```
inla.upgrade(testing=TRUE)
```

- Type `inla.version()` to find out the installed version

# Step by step guide to using R-INLA

## 1. The first thing to do is to **specify the model**

- For example, assume we have a generic model

$$\begin{aligned}y_i &\stackrel{iid}{\sim} p(y_i \mid \theta_i) \\ \eta_i &= \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + f(z_i)\end{aligned}$$

where

- $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$  are observed covariates
- $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2) \sim \text{Normal}(0, \boldsymbol{\tau}_1^{-1})$  are unstructured (*fixed*) effects
- $\mathbf{z}$  is an **index**. This can be used to include structured (*random*), spatial, spatio-temporal effect, etc.
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- As mentioned earlier, this formulation can actually be used to represent quite a wide class of models!

# Step by step guide to using R-INLA

- The model is translated in R code using a **formula**
- This is sort of standard in R (you would do pretty much the same for calls to functions such as `lm` or `glm`)

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formula = y ~ 1 + x1 + x2 + f(name=z, model="...", hyper=...)
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- The **f()** function can account for several structured nonlinear effects. We have for example:
  - **iid** specify independent random effects
  - **rw1, rw2, ar1** are smooth effect of covariates or time effects
  - **besag** models spatially structured effects (CAR)
  - **generic** is a user-defined precision matrix
- Type `inla.list.models("latent")` for the complete list and find descriptions at <https://www.r-inla.org/documentation>
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- It is possible to include in the formula several **f()** terms specifying them separately, e.g.

```
formula <- y ~ x1 + x2 + f(z1, ...) + f(z2, ...) + f(z3, ...)
```

# Step by step guide to using R-INLA

2. Call the function `inla` to fit the model, specifying the data and options (more on this later), e.g.

```
m = inla(formula, family="...", data=data.frame(y,x1,x2,z))
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# Step by step guide to using R-INLA

The `control.xxx=list(...)` statements in the `inla` function control various part of the INLA program:

- `control.compute`: for computing measures of fit (eg DIC)
- `control.predictor`: for specifying the *Observation matrix A* which links the latent field to the data
- `control.family`: for changing the prior distribution of the likelihood hyperparameters
- `control.fixed`: for changing the prior distribution of the fixed effects
- `control.inla`: for changing the strategy to use for the approximations ('gaussian', 'simplified.laplace' (default) or 'laplace') or the grid exploration strategy
- and many others for expert use.

# Step by step guide to using R-INLA

R returns an object `m` in the class `inla`, which has some methods available as for example `summary()` and `plot()`.

```
> names(m)[1:38]
```

```
[1] "names.fixed"                      "summary.fixed"
[3] "marginals.fixed"                  "summary.lincomb"
[5] "marginals.lincomb"                 "size.lincomb"
[7] "summary.lincomb.derived"          "marginals.lincomb.derived"
[9] "size.lincomb.derived"              "mlik"
[11] "cpo"                                "gcpo"
[13] "po"                                 "waic"
[15] "model.random"                      "summary.random"
[17] "marginals.random"                  "size.random"
[19] "summary.linear.predictor"          "marginals.linear.predictor"
[21] "summary.fitted.values"             "marginals.fitted.values"
[23] "size.linear.predictor"             "summary.hyperpar"
[25] "marginals.hyperpar"                "internal.summary.hyperpar"
[27] "internal.marginals.hyperpar"       "offset.linear.predictor"
[29] "model.spde2.blc"                  "summary.spde2.blc"
[31] "marginals.spde2.blc"               "size.spde2.blc"
[33] "model.spde3.blc"                  "summary.spde3.blc"
[35] "marginals.spde3.blc"               "size.spde3.blc"
[37] "logfile"                            "misc"
```

## Toy example

We consider the `iris` dataset included in the `R` datasets (see `?iris`), regarding the measurements in centimeters of the variables *sepal length* and *width* and *petal length* and *width*, respectively, for 50 flowers from each of 3 species of iris. The species are Iris setosa, versicolor, and virginica. See Section 2.6 of the INLA book.

```
> summary(iris)
```

```
Sepal.Length      Sepal.Width       Petal.Length      Petal.Width  
Min.   :4.300    Min.   :2.000    Min.   :1.000    Min.   :0.100  
1st Qu.:5.100   1st Qu.:2.800   1st Qu.:1.600   1st Qu.:0.300  
Median  :5.800   Median :3.000    Median :4.350    Median :1.300  
Mean    :5.843   Mean   :3.057    Mean   :3.758    Mean   :1.199  
3rd Qu.:6.400   3rd Qu.:3.300   3rd Qu.:5.100   3rd Qu.:1.800  
Max.    :7.900   Max.   :4.400    Max.   :6.900    Max.   :2.500  
  
Species  
setosa    :50  
versicolor:50  
virginica:50
```

We specify a simple regression model with *Petal.length* and *Petal.width* as dependent and independent variables, respectively:

$$\begin{aligned}\text{Petal.length}_i &\sim \text{Normal}(\eta_i, 1/\sigma_e^2) \\ \eta_i &= \beta_0 + \beta_1 \text{Petal.width}_i\end{aligned}$$

## Toy example: run INLA + summary

```
> formula <- Petal.Length ~ 1 + Petal.Width  
> output <- inla(formula, family="gaussian", data=iris)  
> summary(output)
```

Call:

```
c("inla.core(formula = formula, family = family, contrasts = contrasts,  
", " data = data, quantiles = quantiles, E = E, offset = offset, ", "  
scale = scale, weights = weights, Ntrials = Ntrials, strata = strata,  
", " lp.scale = lp.scale, link.covariates = link.covariates, verbose =  
verbose, ", " lincomb = lincomb, selection = selection, control.compute  
= control.compute, ", " control.predictor = control.predictor,  
control.family = control.family, ", " control.inla = control.inla,  
control.fixed = control.fixed, ", " control.mode = control.mode,  
control.expert = control.expert, ", " control.hazard = control.hazard,  
control.lincomb = control.lincomb, ", " control.update =  
control.update, control.lp.scale = control.lp.scale, ", "  
control.pardiso = control.pardiso, only.hyperparam = only.hyperparam,  
", " inla.call = inla.call, inla.arg = inla.arg, num.threads =  
num.threads, ", " blas.num.threads = blas.num.threads, keep = keep,  
working.directory = working.directory, ", " silent = silent, inla.mode  
= inla.mode, safe = FALSE, debug = debug, ", " .parent.frame =  
.parent.frame)")
```

Time used:

Pre = 3.99, Running = 0.295, Post = 0.0246, Total = 4.31

Fixed effects

# Exploring the output: fixed effects

```
> output$summary.fixed
```

	mean	sd	0.025quant	0.5quant	0.975quant	mode
(Intercept)	1.083565	0.07292053	0.9403638	1.083565	1.226767	NA
Petal.Width	2.229935	0.05136350	2.1290668	2.229935	2.330802	NA
	kld					
(Intercept)	9.793727e-10					
Petal.Width	9.793732e-10					

- For each unstructured *fixed effect*, R-INLA reports a set of summary statistics from the posterior distribution.

# Exploring the output: fixed effects

```
> output$summary.fixed
```

	mean	sd	0.025quant	0.5quant	0.975quant	mode
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Petal.Width	2.229935	0.05136350	2.1290668	2.229935	2.330802	NA
	kld					
(Intercept)	9.793727e-10					
Petal.Width	9.793732e-10					

- For each unstructured *fixed effect*, R-INLA reports a set of summary statistics from the posterior distribution.
- The value of the Kullback-Leibler divergence kld describes the difference between the Gaussian approximation and the Simplified Laplace Approximation (SLA) to the marginal posterior densities:
  - Small values indicate that the posterior distribution is well approximated by a Normal distribution
  - If so, the more sophisticated SLA gives a *good* error rate and therefore there is no need to use the more computationally intensive *full* Laplace approximation.

# Exploring the output: hyperparameters

```
> output$summary.hyperpar
```

	mean	sd	0.025quant	0.5quant
Precision for the Gaussian observations	4.432669	0.5115441	3.50196	4.413411
	0.975quant	mode		
Precision for the Gaussian observations	5.484833	NA		

- For each hyperparameter the summary statistics are reported to describe the posterior distribution.
- **NB:** INLA reports results on the **precision** scale (more on this later).

## Exploring the output: effective parameters and replicates

- The expected number of effective parameters is basically the number of **independent** parameters included in the model:
- In this case there are 2 actual parameters  $\beta_0$  and  $\beta_1$ .
- In a hierarchical model with structured exchangeable (ie correlated) effects (see next lecture) information is shared across parameters and the expected number is lower.

## Exploring the output: effective parameters and replicates

- The expected number of effective parameters is basically the number of **independent** parameters included in the model:
- In this case there are 2 actual parameters  $\beta_0$  and  $\beta_1$ .
- In a hierarchical model with structured exchangeable (ie correlated) effects (see next lecture) information is shared across parameters and the expected number is lower.
- The number of equivalent replicates indicates the available information (in terms of sample size) per effective parameter
- Example: there are 150 data points and on average 2.011 parameters; so each is estimated using on average  $150/2.011 \approx 74.59$  data points
- Low values (with respect to the overall sample size) are indicative of poor fit.

# Manipulating the marginals from R-INLA: fixed effects

marginals1

marginals2

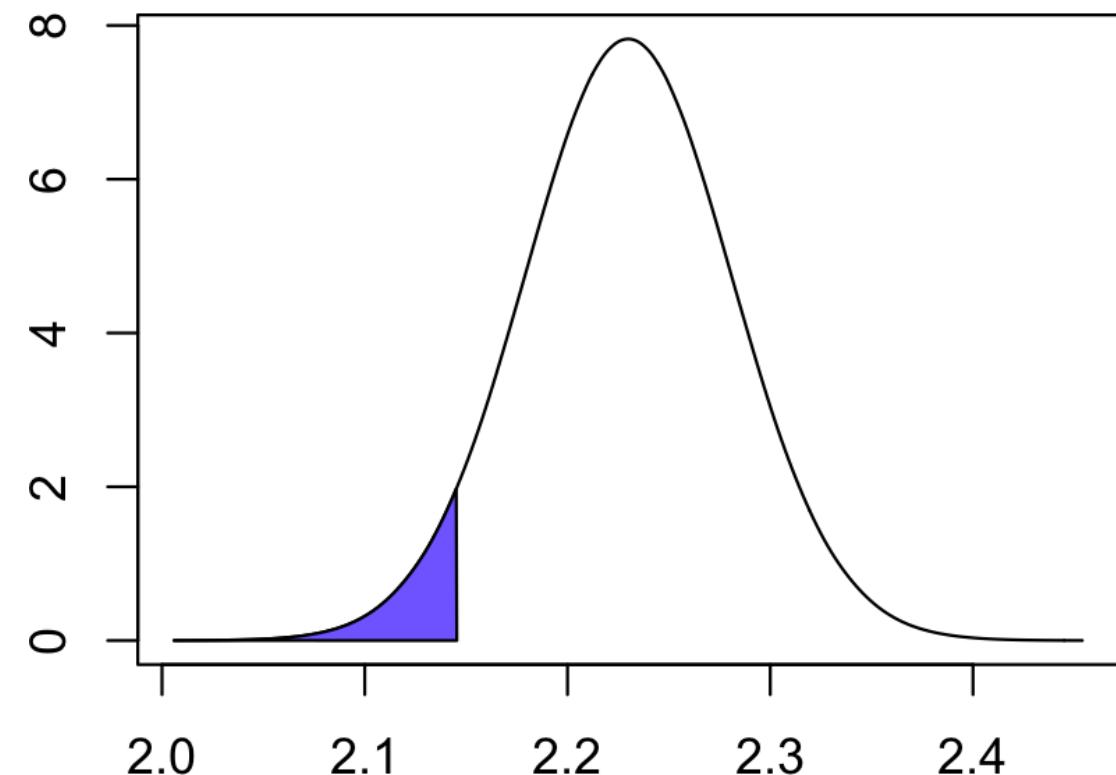
```
> names(output$marginals.fixed)
```

```
[1] "(Intercept)" "Petal.Width"
```

```
> beta1_post <- output$marginals.fixed[[2]]  
> marg <- inla.smarginal(beta1_post)  
> q <- inla.qmarginal(0.05, beta1_post)
```

```
> plot(marg, t="l",  
+       ylab="", xlab="",  
+       main=expression(paste("p(",beta[1], "| y"))  
> polygon(c(marg$x[marg$x <= q ], q),  
+           c(marg$y[marg$x <= q ], 0),  
+           col = "slateblue1", border = 1)
```

$p(\beta_1 | y)$



# Manipulating the marginals from R-INLA: fixed effects

marginals1

marginals2

```
> inla.pmarginal(q,beta1_post)
```

```
[1] 0.05
```

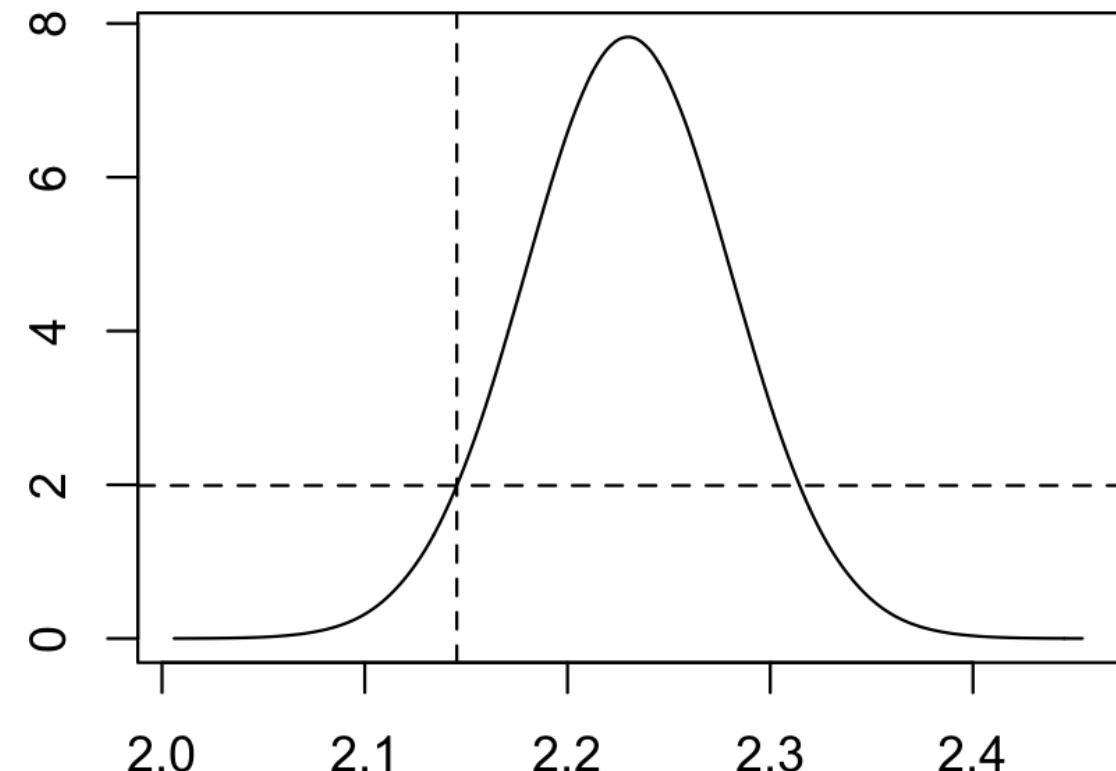
```
> d <-inla.dmarginal(q,beta1_post)  
> d
```

```
[1] 1.990405
```

```
> inla.rmarginal(4, beta1_post)
```

```
[1] 2.260884 2.257382 2.217607 2.208797
```

$$p(\beta_1 | y)$$



# Manipulating the marginals from R-INLA: hyperparameters

INLA works with the precision by default

Precision      Variance

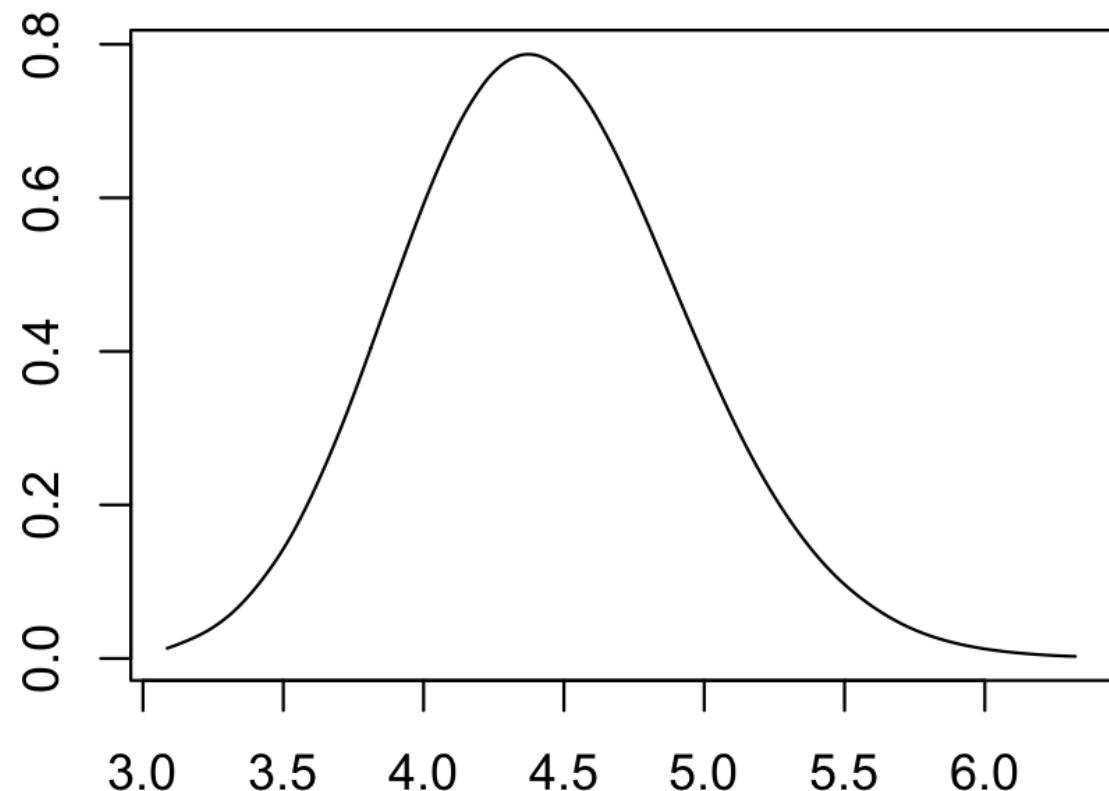
```
> names(output$marginals.hyperpar)
```

```
[1] "Precision for the Gaussian observations"
```

```
> prec_post <- output$marginals.hyperpar[[1]]
```

```
> plot(inla.smarginal(prec_post), t="1",
+       ylab="", xlab="",
+       main=expression(paste("p(", 1/sigma[e]^2, "
```

$$p(1/\sigma_e^2 | y)$$



# Manipulating the marginals from R-INLA: hyperparameters

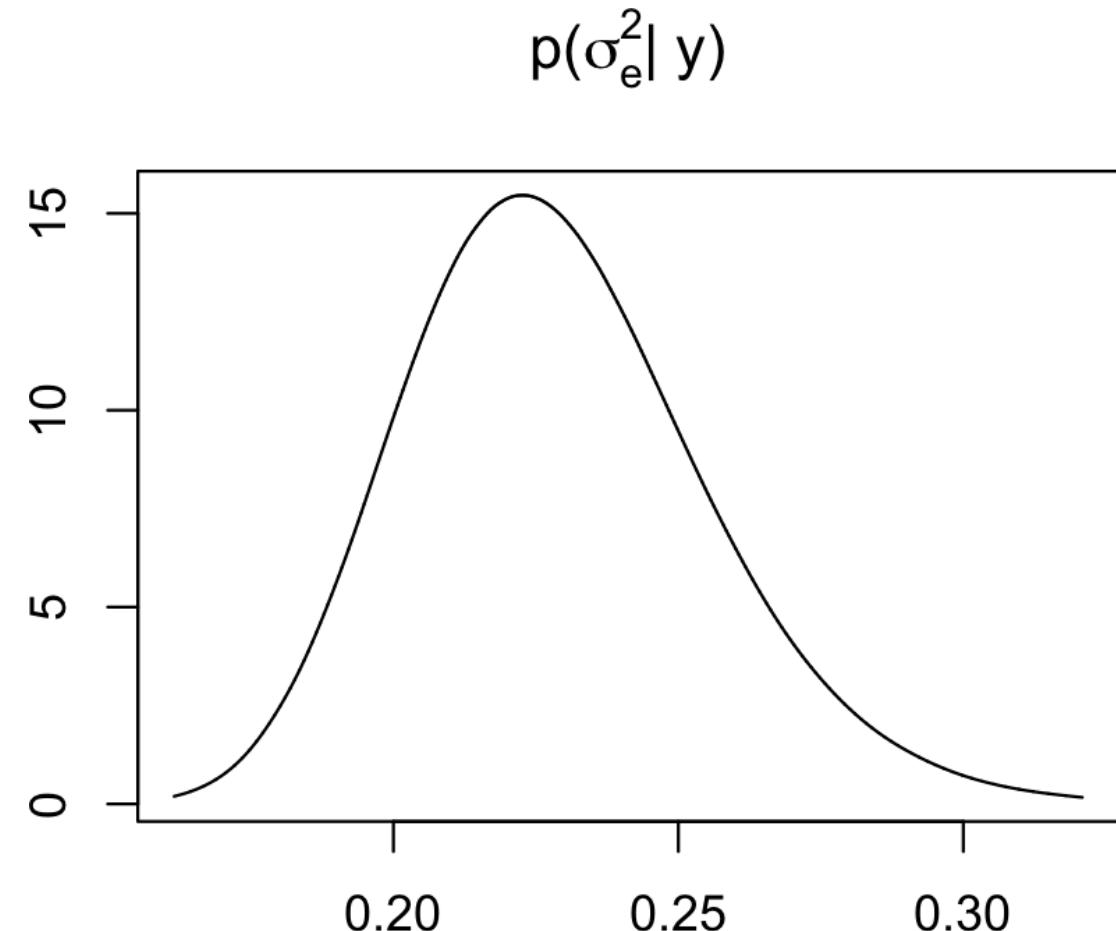
INLA works with the precision by default

Precision      Variance

```
> var_post = inla.tmarginal(fun=function(x)
+           1/x, mar=prec_post)
> inla.emarginal(fun=function(x) 1/x,
+   marg=prec_post)
```

```
[1] 0.228523
```

```
> plot(inla.smarginal(var_post), t="l",
+       ylab="", xlab="",
+       main=expression(paste("p(", sigma[e]^2, " |
```



# Summary

The INLA approach is not a rival/competitor/replacement to/of MCMC, just a better option for the class of LGMs.

- The basic idea behind the INLA procedure is simple
  - Repeatedly use Laplace approximation and take advantage of computational simplifications due to the structure of the model
  - Use numerical integration to compute the required posterior marginal distributions
- Complications are mostly computational and occur when
  - Extending to a large number of hyperparameters
  - Markedly non-Gaussian observations

# References

- Martins, T. G., D. Simpson, F. Lindgren, et al. (2013). "Bayesian computing with INLA: New features". In: *Computational Statistics & Data Analysis* 67, pp. 68-83.
- Rue, H., S. Martino, and N. Chopin (2009). "Approximate Bayesian inference for latent Gaussian model by using integrated nested Laplace approximations (with discussion)". In: *J. R. Statist. Soc. B* 71, pp. 319-392.