Session 2.1: Bayesian inference

Imperial College London

Learning Objectives

After this session you should be able to:

- Understand how Bayes Theorem can be applied to random variables
- Describe what conjugacy means
- Obtain posterior distribution for the Beta-Binomial and Gamma-Poisson families

The topics treated in this lecture are presented in Chapter 3 of the book Blangiardo and Cameletti (2015) and in Chapter 2.3, 3.1-3.4 and 5.2-5.3 of the book Johnson, Ott, and Dogucu (2022).

Outline

1. Bayes Theorem for random variables
2. A quick recap
3. What is conjugacy?
4. Some conjugacy models: Beta-Binomial
5. Some conjugacy models: Gamma-Poisson

Bayes Theorem for random variables

An example

- A clinical trial is carried out to assess the efficacy of a preventive treatment for migraine
- A group of 20 patients are offered the new drug and they have to report if they have a migraine episode in the next week after taking the medication.
- Our aim is to estimate the probability of success of this new drug (θ)
- Our data here is Y: the number of patients reporting no migraine episodes.

  Note that Y is a **random variable** (look at recording 3 for a more detailed description of this concept)

Prior probability model

As a first step we need to assign a prior on θ.

As a simplification let's assume that θ is discrete and can only get the values 0.1,0.4,0.8 with the following probability function, which specifies the prior probability of each possible θ value:

θ	$p(\theta)$
0.1	0.2
0.4	0.5
0.8	0.3

Note that this prior reflects some sort of information from a previous study on a similar compound and put 50% probability on the event that 40% of the patients will report a reduction in migraine.

The Binomial data model

- Our random variable Y can take any discrete value between 0 and 20 (total number of patients)
- It will depend on the probability θ
- For our formal Bayesian analysis, we must model this dependence of Y on θ through a **conditional probability model**
- We make two assumptions about the trials: (1) the outcome of any one patient doesn't influence the outcome of another; and (2) the probability of success does not change among patients
- We can use the Binomial model

$Y \sim \text{Binomial}(n, \theta)$

with conditional probability function

$$p(y \mid \theta) = \frac{n!}{y!(n-y)!}\theta^y(1-\theta)^{n-y}$$

- Mean of this distribution is $E(Y) = n\theta$
- Variance is $V(Y) = n\theta(1-\theta)$

Check out recording 4 for a recap on the Binomial distribution

The Binomial data model

This model allows to calculate ANY conditional probability. For instance, conditioning on $\theta = 0.4$ let's see the difference in the probability of getting 12 or 15 successes

1. $$p(y = 12 \mid \theta = 0.4) = \frac{20!}{12!8!} 0.4^{12}(1 - 0.4)^8 = 0.035$$

2. $$p(y = 15 \mid \theta = 0.4) = \frac{20!}{15!5!} 0.4^{15}(1 - 0.4)^5 = 0.0013$$

--

Note you can get these results in R using

```
> #1
> dbinom(12,20,0.4)
```

```
[1] 0.03549744
```

```
> #2
> dbinom(15,20,0.4)
```

```
[1] 0.001294494
```

Binomial likelihood function

- The Binomial provides a theoretical model of the data we might observe.
- In the end we observe 10 successes out of the 20 patients $(y = 10)$.
- The next step in our Bayesian analysis is to determine how compatible this particular data is with the various possible θ

We need to evaluate the *likelihood* of getting 10 successes in the trial under each possible value of θ

Similarly to last week with events, the likelihood function follows from evaluating the conditional probability function $p(y = 10 \mid \theta)$ for all the possible θ values:

1. $$p(y = 10 \mid \theta = 0.1) = \frac{20!}{10!10!} 0.1^{10}(1 - 0.1)^{10} = 0.00000644$$
2. $$p(y = 10 \mid \theta = 0.4) = \frac{20!}{10!10!} 0.4^{10}(1 - 0.4)^{10} = 0.117$$
3. $$p(y = 10 \mid \theta = 0.8) = \frac{20!}{10!10!} 0.8^{10}(1 - 0.8)^{10} = 0.002$$

Likelihood and probability function

Let's recall the fundamental difference between probability function and likelihood function

When θ is known, the **conditional probability function** $p(\cdot \mid \theta)$ allows us to compare the probabilities of different values of the data Y occurring with θ:

$$p(y_1 \mid \theta) \text{ compared to } p(y_2 \mid \theta)$$

When Y is known, the **likelihood function** $L(\cdot \mid y) = p(y \mid \cdot)$ allows us to compare the relative likelihood of the data y under different possible values of θ:

$$L(\theta_1 \mid y) = p(y \mid \theta_1) \text{ compared to } L(\theta_2 \mid y) = p(y \mid \theta_2)$$

So $L(\cdot \mid y)$ provides the tool we need to evaluate the relative compatibility of data $Y = y$ with various θ values.

Normalising constant

- Now we have a prior for θ and a likelihood and as Bayesian we want to **balance** these two pieces of information to obtain the posterior.
- Something is missing...

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The normalising constant!

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Law of total probability:

$$P(A) = \sum_j P(A \mid B_j)P(B_j) \text{ (for events)}$$

$$P(Y = y) = \sum_j p(Y \mid \theta_j)p(\theta_j) \text{ (for discrete probability functions)}$$

So we get

$$P(Y = 10) = 0.00000644 \times 0.2 + 0.117 \times 0.5 + 0.002 \times 0.3 = 0.059$$

Interpretation: across all the possible θ there is only around 6% chance that there are 10 successes.

Posterior distribition

Finally we are ready to apply Bayes theorem and get the posterior distribution

$$p(\theta \mid y = 10) = \frac{p(\theta)L(\theta \mid y = 10)}{p(y = 10)} \text{ for } \theta \in \{0.1, 0.4, 0.8\}$$

which gives us:

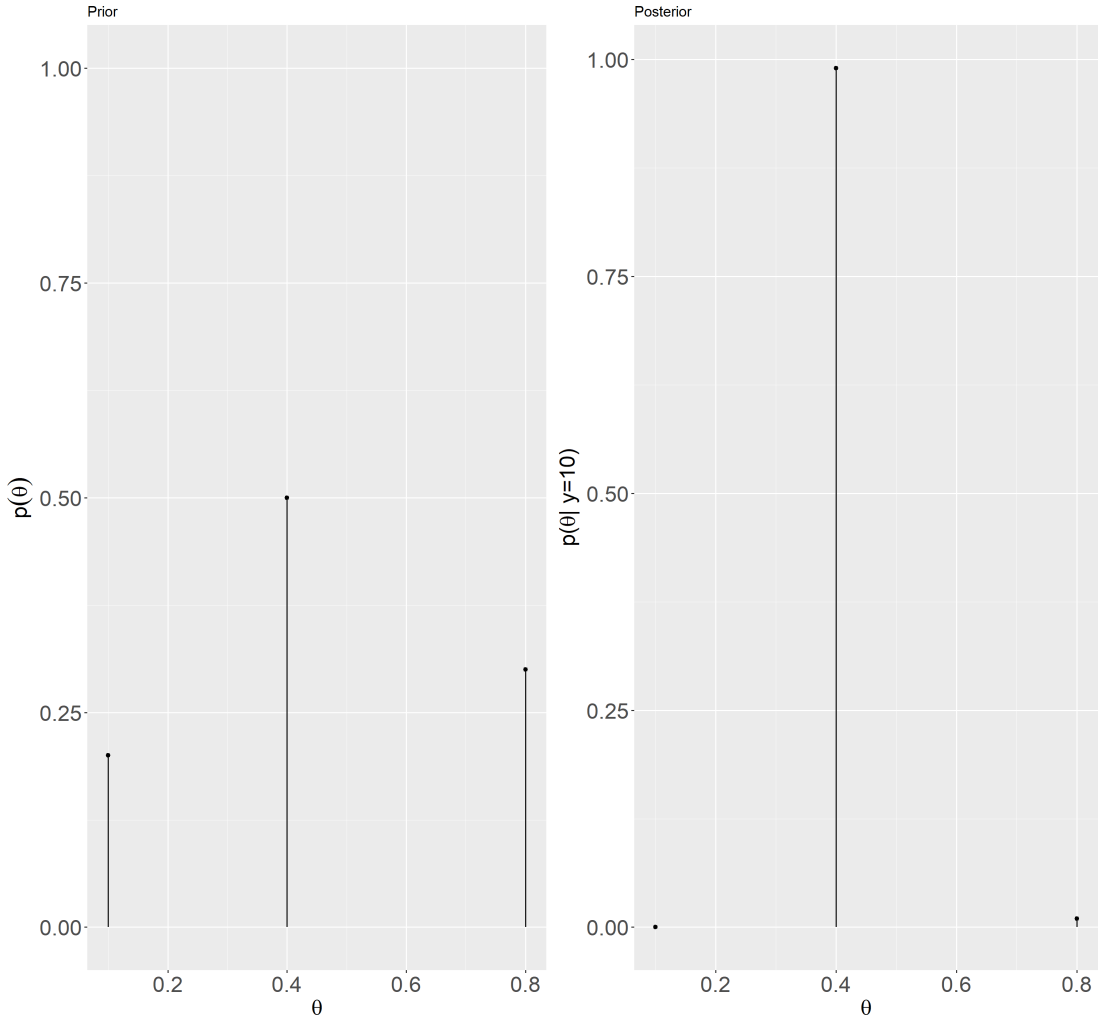
$$p(\theta = 0.1 \mid y = 10) = \frac{0.2 \times 0.0000064}{0.059} = 0$$

$$p(\theta = 0.4 \mid y = 10) = \frac{0.5 \times 0.117}{0.059} = 0.99$$

$$p(\theta = 0.8 \mid y = 10) = \frac{0.3 \times 0.002}{0.059} = 0.01$$

Comparing prior and posterior

θ	$p(\theta)$	$p(\theta \mid y = 10)$
0.1	0.2	0.00
0.4	0.5	0.99
0.8	0.3	0.01



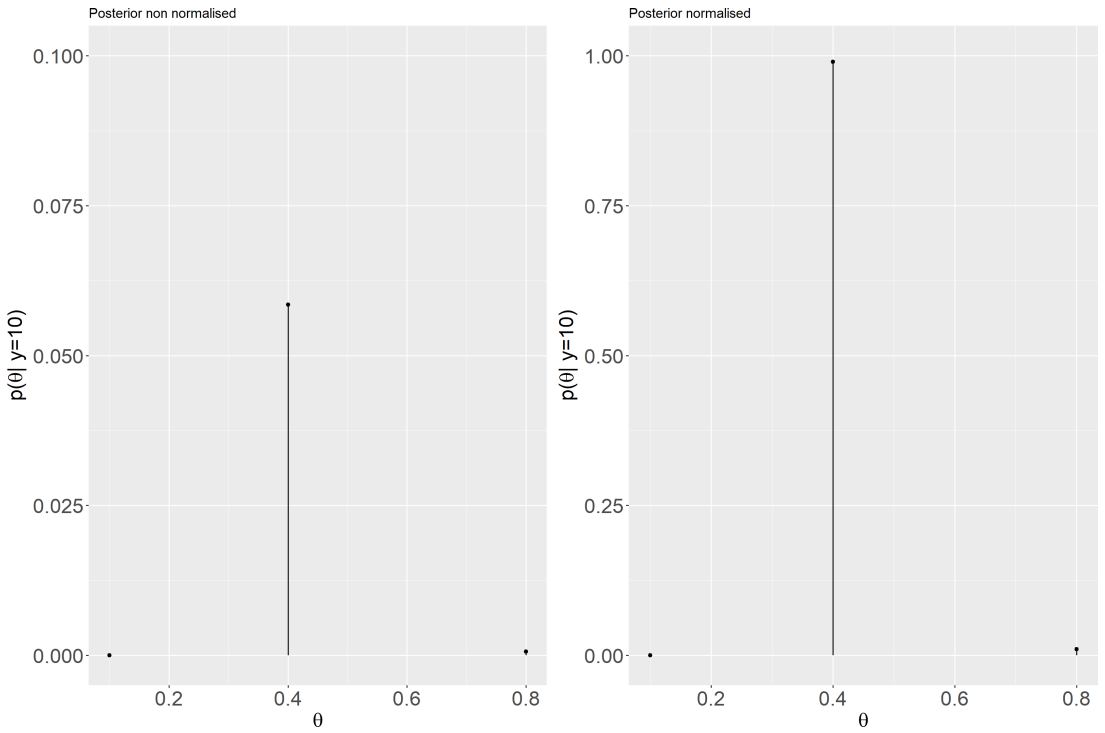
Posterior shortcut

- Moving forward we can actually forget about calculating the normalising constant
- Note that on slide 12 the $p(y = 10)$ appears at the denominator of all the $p(\theta \mid Y)$

  It normalises the posterior probabilities so they sum to 1
- So we can simply acknowledge that $p(y = 10) = c$ and replace the posterior as:

$$p(\theta \mid y = 10) \propto p(\theta) \times L(\theta \mid y)$$

- The proportionality means that if we compare the normalised and unnormalised posterior they preserve their relative relationship



A quick recap

So a quick general recap: Bayesian inference

Makes fundamental distinction between

- Observable quantities y, i.e.~the data
- Unknown quantities θ
- θ can be statistical parameters, missing data, mismeasured data...

  $\rightarrow$ parameters are treated as random variables

  $\rightarrow$ in the Bayesian framework, we make probability statements about model parameters

So a quick general recap: Bayesian inference

Makes fundamental distinction between

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Note that in the Frequentist framework, parameters are fixed non-random quantities and the probability statements concern the data

Bayesian inference

- As with any statistical analysis, we start building a model which specifies $p(Y = y \mid \theta)$
- This is the data distribution, which relates all variables into a **full probability model**
- The choice of data distribution depends on the nature of the data:

e.g. are we analysing continuous or discrete data, are the data symmetric or skewed, etc.

- As we observe the data, we can use descriptive tools (e.g. plots) to visualise the data and choose the best likelihood

Bayesian inference

From a Bayesian point of view

- θ is unknown so should have a probability distribution reflecting our uncertainty about it before seeing the data

$\rightarrow$ need to specify a prior distribution $p(\theta)$

- y is known so we should condition on it

  $\rightarrow$ use Bayes theorem to obtain conditional probability distribution for unobserved quantities of interest given the data:

$$p(\theta \mid y) = \frac{p(\theta)\, p(y \mid \theta)}{\int p(\theta)\, p(y \mid \theta)\, d\theta} \propto p(\theta)\, p(y \mid \theta)$$

This is the posterior distribution

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This is the posterior distribution

- The prior distribution $p(\theta)$, expresses our uncertainty about θ before seeing the data
- The posterior distribution $p(\theta \mid y)$, expresses our uncertainty about θ after seeing the data

The posterior distribution

Posterior distribution forms basis for all inference --- can be summarised to provide

- point and interval estimates of Quantities of Interest (QOI), e.g. treatment effect, small area estimates, ...
- point and interval estimates of any function of the parameters
- probability that QOI (e.g. treatment effect) exceeds a critical threshold
- prediction of QOI in a new unit
- prior information for future experiments, trials, surveys, ...
- inputs for decision making
- ...

What is conjugacy?

How to select a prior

- Selecting the prior is crucial for a Bayesian analysis

  - There is no right way to select a prior
  - The choices often depend on the objective of the study and the nature of the data

- There are other criteria to consider when choosing a prior model:

Computational ease

Especially if we don't have access to computing power, it is helpful if the posterior model is easy to build.

Interpretability

The posterior balance (is a compromise) the data and the prior. A posterior model is interpretable, and thus more useful, when you can look at its formulation and identify the contribution of the data relative to that of the prior.

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We introduce now a type of models which satisfy both properties

Conjugate prior

Let the prior model for a parameter θ with $p(\theta)$ and the model of data Y conditioned on θ have likelihood function $L(\theta \mid y)$.

If the resulting posterior model $p(\theta \mid y) \propto p(\theta) \times p(y \mid \theta)$ is of the same family as the prior, then we say it is a **conjugate prior**.

Some conjugacy models: Beta-Binomial

Beta-Binomial model for proportions: example

- We consider an early investigation of a new drug
- Experience with similar compounds has suggested that response rates between 0.2 and 0.6 could be feasible
- We interpret this as a distribution with mean = 0.4 and standard deviation 0.1
- A Beta(9.2,13.8) distribution has these properties (Check recording 7 to see how to go from the mean and sd to the a and b parameters of a Beta distribution)
- Suppose we now treat $n = 20$ volunteers with the compound and observe $y = 15$ positive responses

Identifying the different model components

- Assuming patients are independent, with common unknown response rate θ, leads to a binomial data distribution

$$p(y \mid n, \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y} \propto \theta^y (1-\theta)^{n-y}$$

- θ needs a continuous prior distribution:

$$\theta \sim \text{Beta}(a, b)$$
$$p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}$$

Combining prior and data

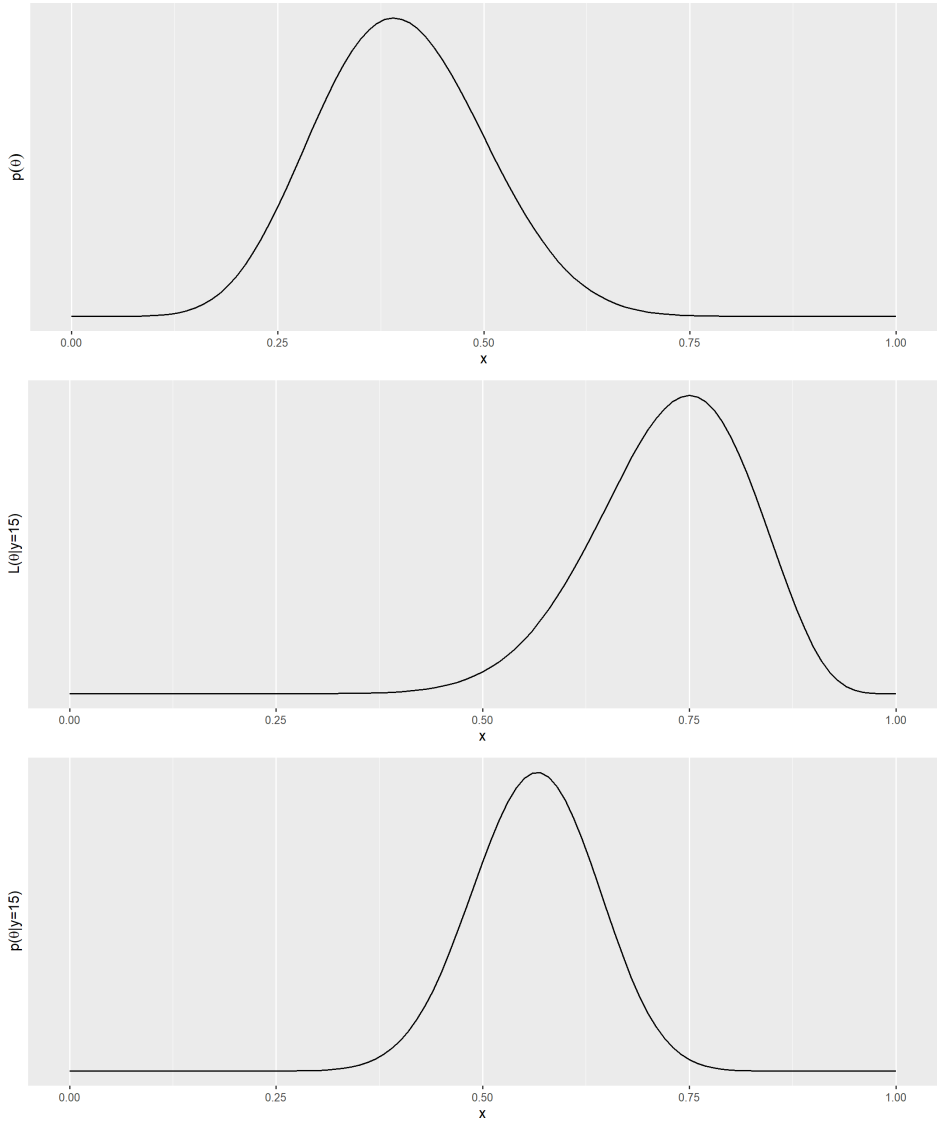
Combining the Binomial data and the Beta prior gives the following posterior distribution

$$\begin{aligned} p(\theta \mid y, n) &\propto p(y \mid \theta, n) p(\theta) \\ &\propto \theta^{y} (1-\theta)^{n-y} \theta^{a-1} (1-\theta)^{b-1} \\ &= \theta^{y+a-1} (1-\theta)^{n-y+b-1} \end{aligned}$$

The posterior is still a Beta distribution (with different parameters):

$$p(\theta \mid y, n) \propto \text{Beta}(y + a,\ n - y + b)$$

Comparing prior, likelihood and posterior



Gamma-Poisson model for count data: example

- For a recap on the Poisson distribution see recording 5

- In epidemiology we are often interested in estimating the rate or relative risk rather than the mean for Poisson data:

- Suppose we observe $y = 7$ cases of leukaemia in one region;
- The expected number of cases is $E = 4$

- **Data distribution**: Poisson with mean $\theta = \lambda \times E$, where λ is the unknown incidence ratio:

$$p(y \mid \lambda, E) = \frac{(\lambda E)^y e^{-\lambda E}}{y!}$$

- **Prior**: $\text{Gamma}(a, b)$ on the the risk λ:

$$p(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$$

Check recording 8 for a recap on the Gamma distribution

Combining likelihood and prior

This implies the following posterior

$$
\begin{aligned}
p(\lambda \mid y) &\propto p(\lambda)\, p(y \mid \lambda) \\
&= \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b(\lambda)} e^{-(\lambda E)} \frac{(\lambda E)^y}{y!} \\
&\propto \lambda^{a+y-1} e^{-(b+E)\lambda} \\
&= \text{Gamma}(a+y,\, b+E)
\end{aligned}
$$

Combining likelihood and prior

This implies the following posterior

$$
\begin{aligned}
p(\lambda \mid y) &\propto p(\lambda)\, p(y \mid \lambda) \\
&= \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b(\lambda)} e^{-(\lambda E)} \frac{(\lambda E)^y}{y!} \\
&\propto \lambda^{a+y-1} e^{-(b+E)\lambda} \\
&= \text{Gamma}(a+y,\, b+E)
\end{aligned}
$$

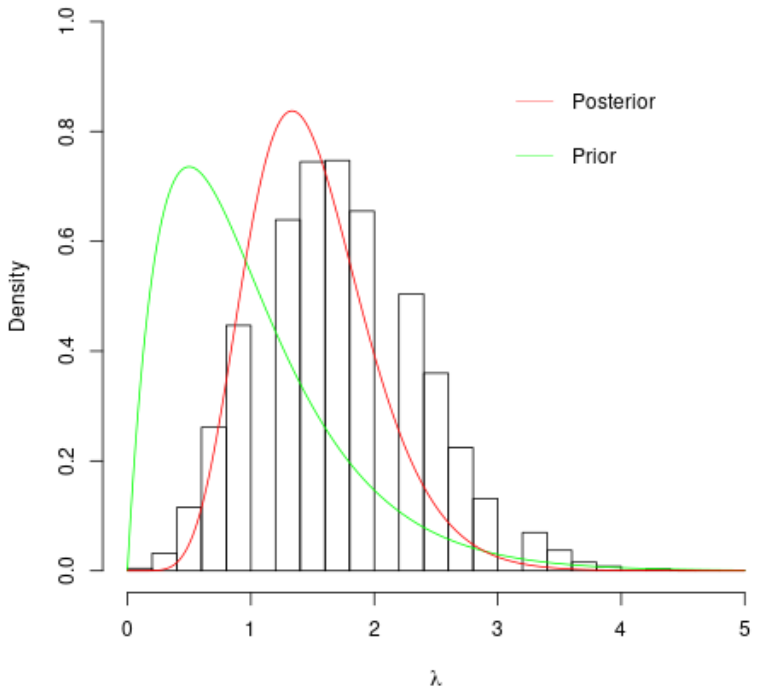
The posterior is another (different) Gamma distribution

$$
E(\theta \mid y) = \frac{a+y}{b+E}
$$

So posterior mean depends on the prior (a, b) and on the data (y, E)

Prior, likelihood, posterior for the Leukaemia example

- Assuming a prior $\lambda \sim \text{Gamma}(2, 2)$
- Considering the data $y = 7, E = 4$
- We obtain a posterior $\lambda \mid y \sim \text{Gamma}(9, 6)$ centered around 1.5



The posterior becomes a compromise between the prior and the data

References

Blangiardo, M. and M. Cameletti (2015). *Spatial and spatio-temporal Bayesian models with R-INLA*. John Wiley & Sons.

Johnson, A. A., M. Q. Ott, and M. Dogucu (2022). *Bayes Rules!: An Introduction to Applied Bayesian Modeling*. CRC Press.