

# Session 9.1: Introduction to temporal modelling

Bayesian modelling for Spatial and Spatio-temporal data, Imperial College London

# Learning objectives

After this lecture you should be able to

- Explain why time also matter and describe features of time series data
- Understand the difference between stationary and nonstationary temporal processes
- Describe basic temporal models
- Know the key functions to implement temporal models through the R-INLA package

Some of the topics treated in this lecture are presented in Chapter 8 of the book **Bayesian inference with INLA** by Virgilio Gómez-Rubio (link: <https://becarioprecario.bitbucket.io/inla-gitbook/index.html>).

# Outline

1. Time series
2. Features of time series
3. Stationarity
4. Basic temporal models

# Time series

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- Time also matter: the behaviour from one time point to the next is important and many data that we deal with in spatial analysis are actually both spatial and temporal in nature.
- Similarly to spatial dependence, it is sometime necessary to model temporal dependence on data and parameters.

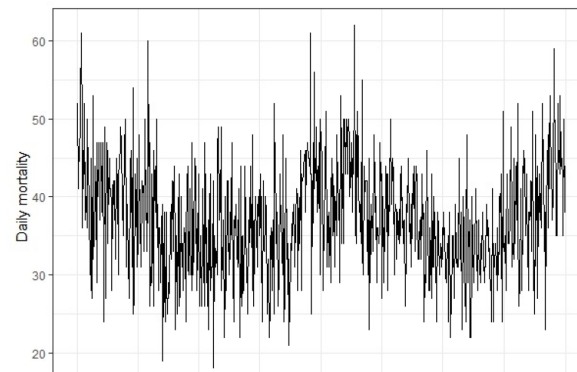
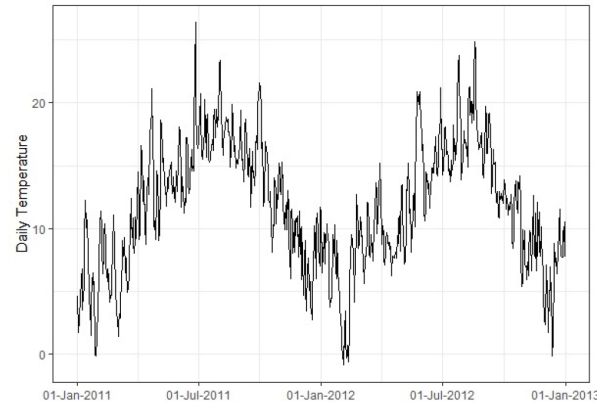
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- Time also matter: the behaviour from one time point to the next is important and many data that we deal with in spatial analysis are actually both spatial and temporal in nature.
- Similarly to spatial dependence, it is sometime necessary to model temporal dependence on data and parameters.
- Unlike space, the temporal data hold a **natural order**.



# Example of time series studies from environmental health studies

- In environmental epidemiology, time series have been widely used, notably for investigating the short-term associations between exposures such as air pollution or weather variables, and health outcomes such as cardiovascular and respiratory morbidity and mortality.
- Typically, for both exposure and outcome, data are available at regular time intervals (e.g. daily pollution levels and daily mortality counts) and the aim is to explore short-term associations between them.



# Time series

- A **time series** is a set of observations taken sequentially in time.
- Depending on different applications, data may be collected hourly, daily, weekly, monthly, yearly, and so on.
- A time series that can be recorded continuously in time, is said to be **continuous**, while a time series that is taken only at specific time intervals is said to be **discrete**. We will work mainly with discrete time series data.
- We use notation such as:  $\{Z_t : t \in \mathcal{D}_t\}$ . Henceforth, we assume that  $\mathcal{D}_t = \{0, 1, \dots\}$  and we refer to  $\{Z_t : t = 0, 1, \dots\}$  as a time series.
- The natural (temporal) ordering in the time series creates an internal structure in the data, that shows, commonly, dependence in the observations, *such that values in the present depend upon observations available in the past.*

# Autocorrelation or serial correlation

- Autocorrelation → the correlation of a variable with itself.
- **Space**: the correlation between the value of the variable at two different locations (or areas).
- **Time series**: the values of a variable at time  $t$  depends on the value of the same variable at time  $t - h$ , where  $h$  is the **time-lag separation**.
- Thus, autocorrelation is also sometimes called *lagged correlation* or *serial correlation*, which refers to the correlation between members of a series of numbers arranged in time.

# Time series: mean, autocovariance and autocorrelation

A time series  $\{Z_t\}$  has:

- **Mean function:**  $\mu_t = E(Z_t)$
- **Autocovariance function:**  $C(t, r) = Cov(Z_t, Z_r)$
- **Autocorrelation function:**  $\rho(t, r) = \frac{C(t, r)}{\sqrt{C(t, t)C(r, r)}}$  where  $\rho(t, r) \in [-1, 1]$ .

Notice that:

- The mean indicates the trend of the series
- The autocovariance function summarizes how the process co-varies across different time lags; we have  $C(t, r) = C(r, t)$
- The variance is a special case of the autocovariance in which  $C(t, t) = var(Z_t) = \sigma_t^2$

# Features of time series

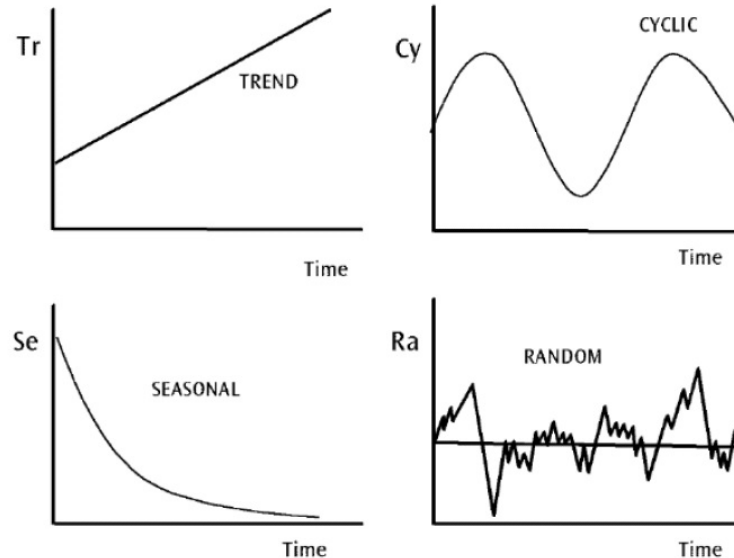
# Components of time series

- Time series analysis typically presents challenges, as it exhibits **patterns** and **irregular fluctuations**.

**Patterns** can be specified as:

- Trend**, that is the most common time series feature to account for and refers to long-term change in the mean level;
- Seasonal variation**, which refers to periodic fluctuations which occur periodically within a year;
- Cyclic changes**, which are recurrent rise and fall that are not of fixed period and are over a period longer than one year.

**Irregular fluctuations** are variations that are short in duration, following not regularity in the occurrence.



# Stationarity

- In studying time series, a very important concept is given by **stationary**, that refers to the stability of the statistical properties of the process through time.
- Broadly speaking, a stationary process is one whose statistical properties do not change over time.
- There are two important forms of stationarity:
  - **strong stationarity**;
  - **weak (or second order) stationarity**.

# Strong and weak stationarity

A time series is said to be **strongly stationary** if

- for any finite sequence of times  $t_1, t_2, \dots, t_n$  and any temporal lag  $h$  the probability distribution of the vector  $(Z_{t_1}, \dots, Z_{t_n})'$  is identical to the probability distribution of the vector  $(Z_{t_1+h}, \dots, Z_{t_n+h})'$ .
- *In words: all aspects of the process's behavior are unaffected (unchanged) by a shift in time.*

A time series is said to be **weak (or second order) stationary** if

- $E(Z_t) = \mu$ , i.e. the mean is constant for all  $t$
- $\text{var}(Z_t) = \sigma^2$ , i.e. the variance does not depend on  $t$
- $\text{Cov}(Z_t, Z_r) = C(t - r)$ , i.e. the autocovariance depends only on the elapsed time between  $t$  and  $r$  and not their actual location
- *In words: weak stationarity (only) concerns the shift-invariance of first and second moments of a process.*



# Basic temporal Models

# White noise process [1]

- A white noise process is a sequence of independent normally identically distributed random variables
- The term **noise** is due to the fact that there's no pattern, just random variation
- The **Gaussian white noise** is defined as:

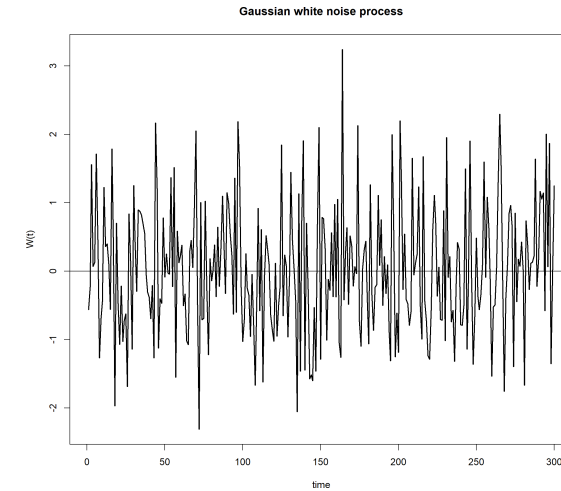
$$W_t \stackrel{iid}{\sim} N(0, \sigma_W^2)$$

- This process is stationary:
  - $E[W_t] = 0$ , i.e. the expectation is always constant and equal to zero
  - $var(W_t) = \sigma_W^2$ , i.e. the variance is constant
  - $cov(W_t, W_r) = 0$  for  $t \neq r$ , i.e. the covariance is zero at all lags
- Note, iid stands for **independent and identically distributed**. An iid model assumes that observations on a phenomenon are taken under identically conditions and that each observation is taken independently of any other.

# White noise process [2]

- Realization of a Gaussian white noise process  $W_t \stackrel{iid}{\sim} N(0, \sigma_W^2)$

```
> set.seed(123) # set random number seed
> W = rnorm(300) # generate iid normal random variates
> ts.plot(W, main="Gaussian white noise process",
+         xlab="time", ylab="W(t)",
+         col="black", lwd=2)
> abline(h=0)
```



- This figure shows that there are no discernible patterns and the distribution is completely random
- In R-INLA an iid Gaussian random effect is specified with the model `iid`
- To obtain details about the model `iid` we can type `inla.doc("iid")`

# Random walk (RW) [1]

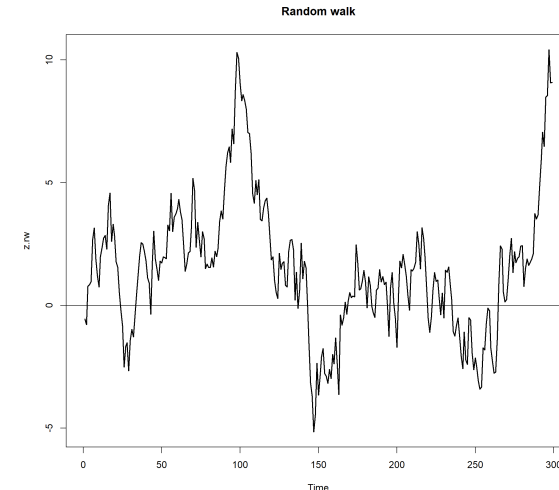
- The random walk (RW) describes how an observation directly depends upon one or more previous measurements plus a white noise process
- The **random walk of order 1, RW1**, is defined as:

$$Z_t = Z_{t-1} + W_t$$

where  $W_t$  is a white noise process.

- Realization of a RW1

```
> set.seed(123) # set random number seed
> Z0 = 0 #Z0 is fixed
> T = 300
> W = c(Z0 + rnorm(T-1))
> z.rw = cumsum(W) # compute cumulative sum
> ts.plot(z.rw, main="Random walk",
+         lwd=2, col="black")
> abline(h=0)
```



- The RW1 only models the difference of levels on consecutive time points:  $Z_t - Z_{t-1} = W_t$

# Random walk (RW) [2]

- The RW is a non-stationary process (i.e. observations in a random walk are dependent on time)
- For a RW1, by recursively substitution, starting from  $t = 1$ , we have:

$$\begin{aligned}Z_1 &= Z_0 + W_1 \\Z_2 &= Z_1 + W_2 = Z_0 + W_1 + W_2 \\&\vdots \\Z_t &= Z_0 + W_1 + \cdots + W_t \\&= Z_0 + \sum_{j=1}^t W_j\end{aligned}$$

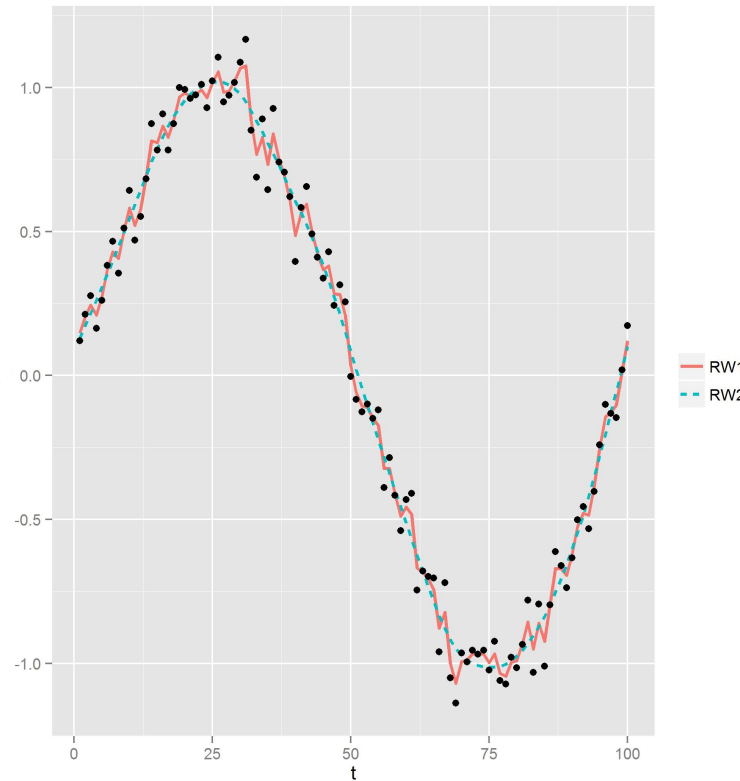
Hence, the first order moment (or the expected value) for this process is equal to:

$$E(Z_t) = Z_0 + \sum_{j=1}^t E(W_j) = Z_0, \text{ which is independent of } t.$$

The variance is  $var(Z_t) = var\left(\sum_{j=1}^t W_j\right) = \sum_{j=1}^t \sigma_W^2 = t\sigma_W^2$ , which depends on  $t$ . Thus the random walk process  $\{Z_t\}$  is not stationary.

# Random walk (RW) [3]

- The **random walk of order 2, RW2**, is defined as:  $Z_t = 2Z_{t-1} - Z_{t-2} + W_t$
- The RW2 only models a linear combination of levels on consecutive time points:  $Z_t - 2Z_{t-1} + Z_{t-2} = W_t$



# Parametrization RW1 in R-INLA

- The RW1 for the Gaussian vector  $\mathbf{Z} = (Z_1, \dots, Z_T)$  is constructed assuming independent increments:

$$\Delta Z_t = Z_t - Z_{t-1} \sim N(0, \tau^{-1})$$

- Hyperparameters: The precision parameter  $\tau$  is represented as  $\theta = \log(\tau)$  and the prior is defined on  $\theta$
- Inclusion in the formula: `f(ID.time, model="rw1")`

# Parametrization of RW2 in R-INLA

- The RW2 for the Gaussian vector  $\mathbf{Z} = (Z_1, \dots, Z_T)$  is constructed assuming independent second order increments:

$$\Delta^2 Z_t = Z_t - 2Z_{t+1} + Z_{t+2} \sim N(0, \tau^{-1})$$

- Hyperparameters: the precision parameter  $\tau$  is represented as  $\theta = \log(\tau)$  and the prior is defined on  $\theta$
- Inclusion in the formula: `f(ID.time, model="rw2")`



# Autoregressive (AR) process [1]

- The **autoregressive process of order (p), AR(p)** is a time series model where the original data is expressed as a function of its previous values in time
- It is defined as:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \dots + \phi_p Z_{t-p} + W_t$$

where:

- $W_t$  is a Gaussian error term with mean zero and variance  $\sigma_W^2$  (i.e. a Gaussian white noise process)
  - $\{\phi_i : i = 1, \dots, p\}$  is a sequence of unknown autoregressive parameters
- This class of models is called autoregressive because  $Z_t$  is regressed on past terms of the same process
  - The simplest model is given by the **AR1** (i.e.  $p=1$ ) and is defined as:

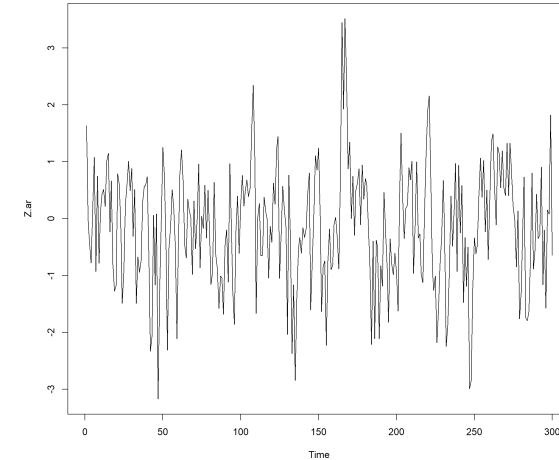
$$Z_t = \rho Z_{t-1} + W_t$$

where  $|\rho| < 1$  is the unknown temporal correlation term

# Autoregressive (AR) process [2]

- Realization of an AR1 process

```
> set.seed(121)
> Z.ar = arima.sim(model=list(ar=.5), n=300)
> plot.ts(Z.ar)
```



- The AR1 process can be written as an infinite series of white noise random variables. Since  $E(W_t) = 0$  and  $var(W_t) = \sigma_w^2$ , it follows that  $E(Z_t) = 0$  and  $var(Z_t) = \frac{\sigma_w^2}{1-\rho^2}$ , which does not depend on  $t$ , thus the process is stationary. Note that if  $\rho = 1$ , the process is a random walk.
- In R-INLA, the AR1 model is implemented through the model specification `ar1`, while an AR model of arbitrary order is implemented through the specification `ar`.
- To obtain details about the specification of the AR1, and more in the general about the AR model, we can type `inla.doc("ar1")` and `inla.doc("ar")`.

# To wrap up

A stationary time series **doesn't exhibit trend or seasonality**:

- Observations do not tend upwards or downwards
- Variance does not increase or decrease with time
- Observations do not tend to be large in some periods than others

# References

- Broemeling D.L. (2019), Bayesian Analysis of Time Series, CRC Press
- Cressie N. and Wikle C.K. (2011), Statistics for spatio-temporal data, Wiley