

MARTA BRZEZIŃSKA

BEYOND THE TEN-FOLD WAY: NOVEL TOPOLOGICAL
PHASES IN LOW-DIMENSIONAL SYSTEMS

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presented by
MARTA BRZEZIŃSKA
Master of Science in Physics,
Wrocław University of Science
and Technology, Poland
born on 22 August 1991

supervised by
Prof. Titus Neupert,
Prof. Arkadiusz Wójs,
Dr. Paweł Potasz

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ABSTRACT

This thesis covers selected recent developments in the field of topological aspects of condensed matter physics. In particular, we focus on three directions which can be seen as extensions of well-established classification of free-fermionic gapped states: i) investigating a realization of topological states in the systems defined in non-integer spatial dimensions, ii) the role of crystal symmetries and how they affect the distinction between topologically trivial and non-trivial states, and iii) non-Hermitian Hamiltonians arising from a minimal modelling of gains and losses exhibiting observable phenomena without Hermitian counterparts. In all cases, we propose material candidates or experimental setups to support our theoretical findings.

ABSTRAKT

Przedłożona rozprawa doktorska obejmuje wybrane najnowsze osiągnięcia w dziedzinie topologicznych aspektów fizyki materii skondensowanej. W szczególności uwaga zostanie poświęcona trzem kierunkom badań, które można postrzegać jako rozszerzenie ugruntowanej klasyfikacji układów nieoddziałujących fermionów: i) próba realizacji stanów topologicznych w układach scharakteryzowanych przez liczbę wymiarów przestrzennych będących liczbą niecałkowitą, ii) rola symetrii krystalicznych i ich wpływ na rozróżnienie pomiędzy stanami topologicznymi oraz trywialnymi, a także iii) hamiltonianami niehermitowskimi będącymi efektywnym opisem układów otwartych i wykazującymi obserwowalne zjawiska bez odpowiedników w modelach hermitowskich. We wszystkich omawianych zagadnieniach zostaje przedyskutowana możliwa realizacja eksperymentalna w celu poparcia wyników teoretycznych.

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ABBREVIATIONS

1D	one-dimensional
2D	two-dimensional
BZ	Brillouin zone
CB	conduction band
NN	nearest-neighbour
NNN	next-nearest-neighbour
QH	quantum Hall
SOC	spin-orbit coupling
SPT phase	symmetry protected topological phase
TI	topological insulator
VB	valence band

INTRODUCTION

Classification is a theme that lies at the heart of condensed matter physics. For a long time, it was believed that Ginzburg-Landau theory [1] of the phase transitions based on the symmetry-breaking paradigm provides a complete list of phases of matter. It states that the continuous transition between phases is described by a local order parameter, which vanishes in the high-symmetry (disordered) phase and becomes non-zero in the low-symmetry (ordered) phase. For instance, magnetization serves as an order parameter at the ferromagnetic-paramagnetic transition; crystals break continuous translational symmetry and hence are characterized by a discrete space group. Another example is superconductivity, where the non-zero gap Δ indicates the superconducting state.

At the beginning of 1970s, Berezinskii [2, 3], Kosterlitz and Thouless [4, 5] investigated a two-dimensional classical magnet with $U(1)$ (XY model) in which the phase transition falls beyond the Ginzburg-Landau paradigm. At the zero-temperature, the system is ferromagnetic as the spins are perfectly arranged. At finite (but small) temperature, the system still has long-range correlations but a local spin structures called vortices may be created. Vortices are said to be topological excitations as the only way to destroy them is to annihilate vortex with an antivortex. With an increase of the temperature, more pairs of vortex-antivortex pairs are created and they become less bounded, hence destroying the long-range correlations. This is the Kosterlitz-Thouless (KT) phase transition in which no continuous symmetry is spontaneously broken (which is in an agreement with Mermin-Wagner theorem), yet the system undergoes a transition to the phase with short-range correlations. For their developments in the field of topological matter, Kosterlitz and Thouless (together with Haldane) obtained the Nobel Prize in 2016.

An even more pronounced example is a discovery of the integer quantum Hall effect (IQHE) [6], and subsequently, the fractional version of this phenomenon (FQHE) [7]. In a two-dimensional electron gas at low temperature exposed to a strong magnetic field, applying the voltage on the two sides of a sample results in a current generated in the perpendicular direction. As a function of the magnetic field, one observes the perfectly flat plateaus in the transverse Hall conductivity, while the longitudinal conductivity vanishes. Hall conductivity σ_{xy} takes quantized values being multiplies ν (ν is an integer in case of IQHE, while a fraction in case of

FQHE) of elementary constants e^2/h (where e is the electron charge and h is the Planck constant) and has been measured to the accuracy of the order 10^{-9} [8]. For the discovery of IQHE, Klaus von Klitzing got the Nobel Prize in 1985 and now the quantum of conductance serves as a universal constant. The origin of this quantization is universal, in a sense that it is observed regardless of microscopic details of a sample such as disorder. In a finite geometry, the systems exhibits robust edge currents, which are chiral, that is they flow in one fixed direction. These are two different manifestations of the topological properties of the systems.

IQH states can be understood from the perspective of the Landau levels formed in a strong magnetic field. A remarkable idea given by Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) [9] was to relate the number of gapless edge modes with the topological invariant computed for the bulk. This became the first observation of the *bulk-boundary correspondence*. Later on, the theoretical proposals of realization of IQH states on the honeycomb lattice in the absence of magnetic field [10] or topological superconductors [11] ignited the experimental search for topological materials. On the other hand, understanding FQHE requires taking into account the electron-electron interactions. The exotic properties emerge from the collective behavior of electrons and give rise to the concept of topological order as the low-energy effective theory can be described in terms of topological quantum field theories (such as Chern-Simons or BF). Apart from FQHE, another paradigmatic example of topological order is chiral spin liquid developed as an attempt to understand high-temperature superconductivity [12, 13]. Topologically ordered states may posses unique features such as fractionalized excitations (carrying the fractions of elemental charge) with the anyonic exchange statistic (not bosonic nor fermionic) or long-range entanglement pattern [14]. In addition, they exhibit robust ground state degeneracy depending on the manifold on which they are defined. These properties are very promising for a fault-tolerant quantum computing [15].

As no symmetry breaking occurs, in order to classify topological phases a notion of equivalence classes has to be introduced. To do so, we will investigate the systems at zero temperature with a spectral gap separating the ground state from the first excited state. States are said to be topologically equivalent if they can be connected by unitary transformations that act infinitely slowly (adiabatically). The physical constrains of unitaries are that at every point of evolution they have to preserve the energy gap and they have to involve only local degrees of freedom. In addition, one may take into account the symmetries to enlarge the number of possible unitary transformations and investigate symmetry-protected (or symmetry-enriched) topological states.

Not all the states that are topological possess intrinsic topological order: some of them may be short-range entangled and they can be understood from the single-particle physics perspective (a more detailed explanation will be given in the following chapter). Interestingly, a lot of theoretical developments were followed-up by experimental efforts.

ORGANIZATION OF THIS THESIS

In this thesis, we study non-interacting fermionic systems which cannot be completely classified by means of the so-called ten-fold way, that is the classification based on a dimensionality and the presence or absence of internal symmetries: time-reversal, particle-hole and chiral.

Chapter 2 serves as a brief introduction to topological band theory, with an emphasis on the basic definitions, topological invariants and illustrative examples of toy models.

In Chapter 3, a realization of topological states in systems with non-integer spatial dimension is discussed. Detecting topological properties without the translational invariants requires employing real-space methods.

Chapter 4 is devoted to the states protected by the crystalline symmetries. However, they fall beyond a sharp distinction what is topological or trivial. Obstructed atomic limits are recent refinements and can be understood from the perspective of Wannier functions.

Topological states in non-Hermitian systems are investigated in Chapter 5. Recently, a studies of non-Hermitian Hamiltonians emerged as an effective modeling of open systems in which the energy or particle number is not preserved.

Finally, Chapter 6 provides a summary and indicates further directions. In addition, some lengthy derivations are given in Appendix A.

TOPOLOGICAL BAND THEORY

In this chapter, we introduce more formally the concept of topology in the band theory. We provide the definitions of topological invariants, precise the bulk-boundary correspondence, and discuss the details of the classification for gapped free-fermion models.

2.1 CONVENTIONAL BAND THEORY

We introduce the formalism for translational invariant lattice models described within the tight-binding formalism for non-interacting particles. This imposes only quadratic terms in creation and annihilation operators in the second-quantized Hamiltonian.

It is often convenient to work in the reciprocal space. This can be done by performing the Fourier transform, which allows to diagonalize only a single block corresponding to all degrees of freedom in a unit cell.

with vectors satisfying $a_i \cdot b_j = 2\pi\delta_{ij}$, where δ_{ij} is the Kronecker delta function.

Hence, in order to change

2.2 SYMMETRIES AND CLASSIFICATION OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

In quantum mechanics, the symmetries are constructed according to the Wigner's theorem: they are the operators that preserve transition probabilities and are therefore implemented via unitaries or antiunitaries which commute with the Hamiltonian.

A fundamental example of antiunitary operator is the time-reversal, which takes $t \rightarrow -t$. Acting on Hamiltonian H :

$$\mathcal{T}H\mathcal{T}^{-1} = +H \quad (2.1)$$

As all antiunitary operators, \mathcal{T} can be decomposed into a product of a unitary operator T and complex conjugate \mathcal{K} , $\mathcal{T} = T\mathcal{K}$. Applying \mathcal{T} twice will lead to the rise of the phase factor, hence $\mathcal{T}^2 = \pm 1$. This is true for all antiunitary symmetries. For spin-1/2 systems, TRS systems follow the Kramers' theorem which tells that all states comes in time-reversal symmetric pairs $\mathcal{T}|n\rangle = |n\rangle$.

		\mathcal{T}	\mathcal{P}	\mathcal{C}	$d = 1$	$d = 2$	$d = 3$
standard	A (unitary)	0	0	0	-	\mathbb{Z}	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	-1	0	0	-	\mathbb{Z}_2	\mathbb{Z}_2
chiral	AIII (chiral unitary)	0	0	1	\mathbb{Z}	-	\mathbb{Z}
	BDI (chiral orthogonal)	+1	+1	1	\mathbb{Z}	-	-
	CII (chiral symplectic)	-1	-1	1	\mathbb{Z}	-	\mathbb{Z}_2
BdG	D	0	+1	0	\mathbb{Z}_2	\mathbb{Z}	-
	C	0	-1	0	-	\mathbb{Z}	-
	DIII	-1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
	CI	+1	-1	1	-	-	\mathbb{Z}

TABLE 2.1: Table of symmetry classes of non-interacting Hamiltonians taken from Refs. [17] and [18]

Another (antiunitary) symmetry is the particle-hole \mathcal{P} , defined as:

$$\mathcal{P}H\mathcal{P}^{-1} = -H \quad (2.2)$$

Finally, we can define chiral symmetry being a product of \mathcal{T} and \mathcal{P} , $\mathcal{C} = \mathcal{T}\mathcal{P}$ ¹

$$\mathcal{C}H\mathcal{C}^{-1} = -H \quad (2.3)$$

\mathcal{C} may be present in the system, even if \mathcal{T} and \mathcal{P} are absent. Although chiral symmetry is not a symmetry in a strict sense, but guarantees that non-zero eigenvalues of the Hamiltonian come in pairs, hence there is a spectral symmetry.

Dates back to the idea of Élie Cartan's classification of symmetric spaces in differential geometry. Later on, Altland and Zirnbauer applied this classification in the context of random matrix theory [16]

All possible combination of internal symmetries in $d = 1, 2, 3$ spatial dimensions are tabulated in Tab. 2.1. There are ten distinguished classes.

This was first proposed in Ref. [19], with further mathematical improvements from K-theory (such is pattern called Bott periodicity; with a period two for the complex classes AI, AIII and a period eight for remaining eight real classes) given by Kitaev [18] to finally provide the complete classification [17].

For higher dimensions, topological invariants repeat themselves and show the Bott periodicity with period 2 for classes A and AIII, and period 8 for the remaining ones.

¹ Note that \mathcal{P} and \mathcal{C} anticommute with the single-particle Hamiltonian

2.3 TOPOLOGICAL INVARIANTS

Berry connection of Bloch states:

$$\mathbf{A} = i \langle u_n(\mathbf{k}) | \nabla | u_m(\mathbf{k}) \rangle \quad (2.4)$$

$\Delta \equiv \Delta_{\mathbf{k}}$ Berry curvature:

$$F = \nabla \times \mathbf{A} \quad (2.5)$$

Gauss-Bonnet theorem for a two-dimensional manifold M without a boundary:

$$\int_M K dA = 2\pi(2 - 2g) \quad (2.6)$$

with K being the Gauss curvature and g - genus.

2.3.1 Chern number

For two-dimensions, it is given by the (first) Chern number:

$$C = \frac{1}{2\pi} \int_{BZ} F \quad (2.7)$$

(note, some conventions incorporate imaginary unit i explicitly in the prefactor).

2.3.2 Z_2 topological index

Some systems can have one trivial and one topological phase, which is characterized by a two-valued index ν being 0 or 1, respectively. Z_2 invariant can be computed easily in the presence of inversion symmetry using Fu-Kane formula [20]:

$$\delta_i = \prod_{m=1}^N \zeta_{2m}(\Gamma_i); \quad (-1)^\nu = \prod_i \delta_i \quad (2.8)$$

where Γ_i are time-reversal invariant momenta in the BZ, $\zeta_{2m}(\Gamma_i)$ is every second occupied Kramers pair. In case of 3D systems, two types of TI can be observed, defined whether they have a strong or weak indices. Non-zero strong index means that the system cannot be seen as a set of stacked 2D layers, it's intrinsically three-dimensional.

Because for TRS systems C always vanishes, if S_z (spin component pointing out of 2D plane) is conserved, one may define the spin Chern number [21] based on the fact that spins up and down have independent Chern numbers:

$$C_s = \frac{C_\uparrow - C_\downarrow}{2} \quad (2.9)$$

and is related to ν as $\nu = C_s \bmod 2$
Pfaffian formalism

2.4 TOPICAL EXAMPLES AND BRIEF REVIEW ON EXPERIMENTAL PROGRESS

2.4.1 *Su-Schrieffer-Heeger chain*

SSH model, firstly proposed to describe polyacetylene, describes spinless fermions in 1D chain [22]

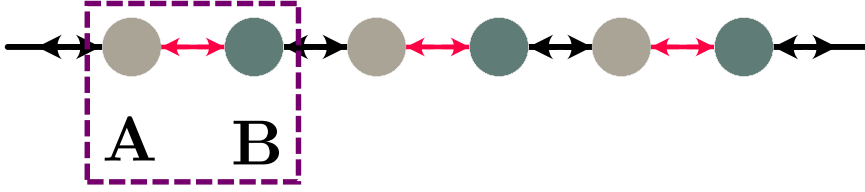


FIGURE 2.1: A schematic of SSH chain. There are two atomic sites A and B per unit cell and alternating hoppings with strength t (within unit cell) and t' (between neighbouring unit cells). Depending on t and t' , the model exhibits two phases: if $t' > t$, the system is in a topological phase with gapless edge modes in an open geometry; conversely, if $t' < t$ the system is in a trivial phase with $t' = 0$ as a fully dimerized case.

Bloch Hamiltonian reads:

$$H(k) = \begin{pmatrix} 0 & t + t'e^{ik} \\ t + t'e^{-ik} & 0 \end{pmatrix} \quad (2.10)$$

The system has the chiral symmetry realized by σ_z and is in BDI class, hence obeys \mathbb{Z} classification.

2.4.2 *Integer quantum Hall states and Chern insulators*

IQH states can be seen as the most robust topological states without intrinsic topological order as they don't need any symmetry to persist. In particular, TR is broken. Actually, in the classification based on the entanglement (Wen vs. Kitaev)

Haldane model is a first example of a Chern insulator, where TR symmetry is broken because of complex NNN hopping. Experimental realizations include fermionic ultracold atoms [23] or classical wave systems [24].

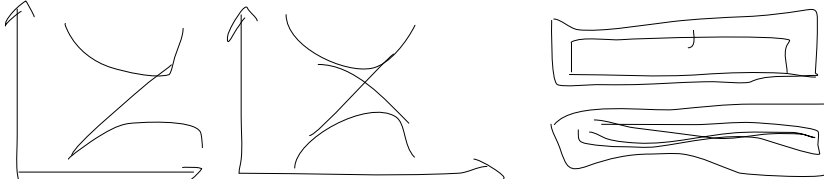


FIGURE 2.2: Comparison between the band structures for Chern insulators with $C = +1$ and the quantum spin Hall system seen as two copies of CI with opposite chiralities

2.4.3 Quantum spin Hall systems

First by Kane and Mele by taking two Haldane models of Chern insulators of opposite chirality. The resulting model has now restored time-reversal and can be described by Z_2 index. It was proposed to describe graphene, but due to negligible spin-orbit coupling it's not realizable experimentally under realistic conditions.

In the same time, QSHE was proposed to be realized in strained zinc-blende semiconductors (such as GaAs) [25]. In 2006, Bernevig, Hughes and Zhang gave an idea that CdTe/ HgTe/ CdTe quantum wells may exhibit the band inversion at a critical thickness. Due to strong spin-orbit coupling in HgTe, CB has a p -like character and VB is composed of s -orbitals instead of normal ordering which is observed in CdTe: s -orbital character of conduction band and p -orbital of valence band. Hence, by changing the thickness of the HgTe layer, it is possible to manipulate which effect will contribute more. It was confirmed experimentally year later by observing quantized resistance $h^2/2e$ (factor 2 comes from the spin contribution) in a thick quantum well.

In Ref. [26]

TOPOLOGICAL STATES IN SELF-SIMILAR LATTICES

Underlying geometry in quantum lattice models plays an important role in their electronic properties. For instance, frustration in spin models on triangular or kagome lattices arises due to inability of defining unique ground state which minimizes the total energy.

Having a

Interestingly, the authors in pointed out that topological states can be realized in amorphous lattice models [27]. BHZ on fractals [28]

Experimental realizations: CO molecules on a Cu (111) surface [29], assembled molecules [30], focused ion beam epitaxy [31], in metamaterials that mimic the physics of quantum models (amorphous Chern insulators [32]).

Prediction of peculiar transport properties: [33, 34]

3.1 REAL-SPACE METHODS FOR COMPUTING TOPOLOGICAL INVARIANTS

Disordered or amorphous systems, due to lack of translational invariance, require defining topological invariants in a real space. Non-commutative

Bott index

Algorithm for computing the Bott index [35]

Chern number

$$C = 12\pi i \sum_{j \in A} \sum_{k \in B} \sum_{l \in C} \left(P_{jk} P_{kl} P_{lj} - P_{jl} P_{lk} P_{kj} \right), \quad (3.1)$$

where P is the projector operator onto occupied states and i, j, l label the lattice sites.

Fractal lattices comprise of many interesting features: they are aperiodic, but scale invariant. Also, there is no sharp notion between bulk and edge.

Here, we are interested in the lattice regularization of two fractals, Sierpiński carpet (SC) and triangle (or gasket) (SG). This approach is relevant for potential experimental realization as it introduces the distance between nearest-neighbouring sites (lattice constant) to be a natural cutoff.

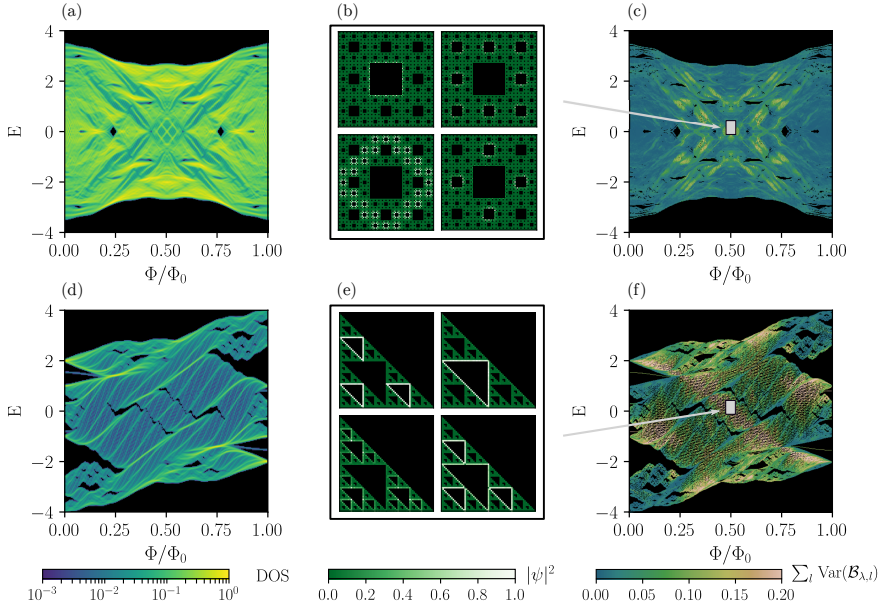


FIGURE 3.1: Density of states, localization of selected eigenstates and edge-locality marker.

The reason why we investigate these lattices is motivated by their distinct Hausdorff dimensions ($d_H = \ln A / \ln L$, where A is the area and L the linear size) and connectivity properties. Firstly, $d_H = 1.892 \dots$ for SC and $d_H = 1.585 \dots$ for SG.

We consider tight-binding model of spinless electrons exposed to a magnetic field. The Hamiltonian reads

$$H = -t \sum_{\langle i,j \rangle} e^{iA_{ij}} c_i^\dagger c_j + \text{h.c.}, \quad (3.2)$$

where we set $t = 1$. Introducing a finite field leads to lifting the macroscopic degeneracy

In Fig. 3.1 we observe Hofstadter's butterfly [36]

One of the difficulties is to compute topological invariants in that systems as they do not exhibit translational invariance. One may therefore employ real-space methods. Other methods (for example, the Bott index) may be numerically insufficient. Here, we used the real-space expression for the Chern number:

To compute the Chern number as a function of the Fermi level, we use the formula Eq. (3.1).

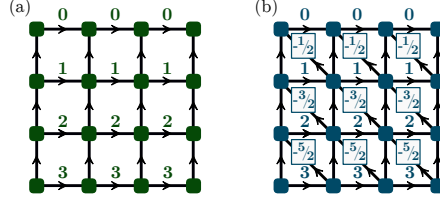


FIGURE 3.2: Phase distribution on 4×4 (a) square and (b) triangle lattices with open boundary conditions. A_{ij} phase between site i and j is equal to the number shown above the bond in 2π units. A phase acquired with the respect to the direction pointed by arrows has a positive sign.

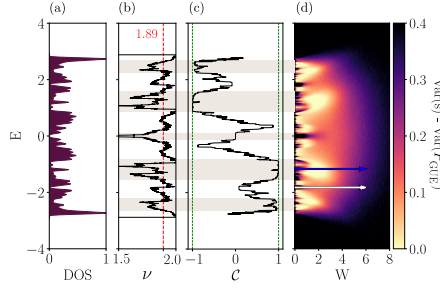


FIGURE 3.3: Density of states at $\Phi/\Phi_0 = 1/4$, scaling of density of states, the Chern number and disorder-induced phase transition.

3.2 EDGE STATES

Edge-locality marker:

$$\mathcal{B}_{\lambda,l} = \sum_{i \in \mathcal{E}_l} |\psi_{\lambda,i}|^2, \quad (3.3)$$

Previous studies for fractal lattices [37, 38] suggested that the presence of a magnetic field leads to an increase in the degree of delocalization of eigenstates. This can be confirmed by calculating inverse participation ratio:

$$I_\psi = \frac{\sum_i |\psi_i|^4}{(\sum_i |\psi_i|^2)^2}. \quad (3.4)$$

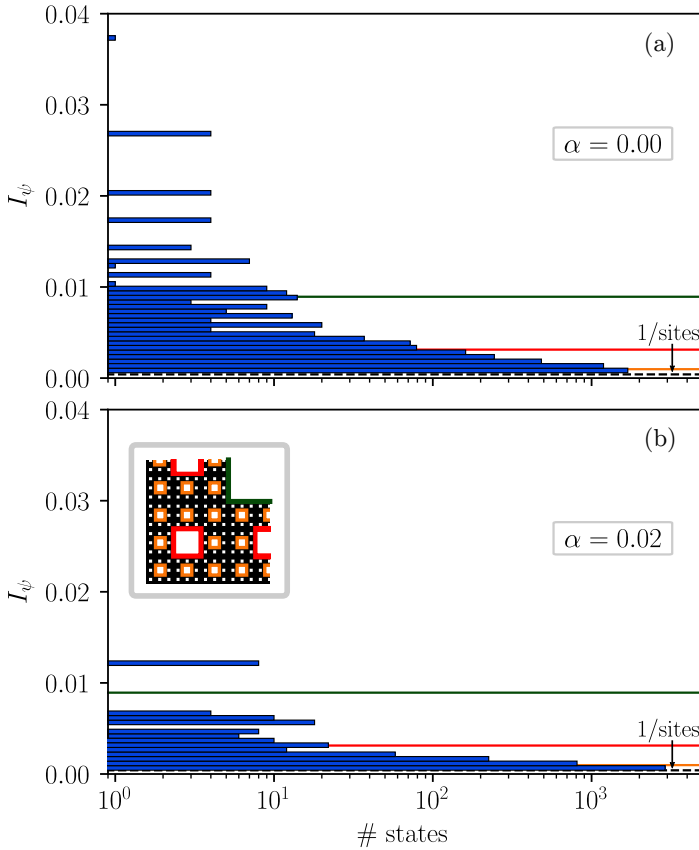


FIGURE 3.4: Distribution of IPR for SC at (a) $\alpha = 0$ and (b) $\alpha = 0.02$. Inset: a closeup of carpet with internal edges of different hierarchies indicated by distinct colors.

At zero flux $\alpha = 0$, i.e., in absence of a magnetic field, the distribution of IPRs is peaked close to the inverse of the number of sites belonging to the edges of the second-smallest squares or triangles. As magnetic field is introduced, exemplified here with $\alpha = 0.02$, the distribution of IPRs shifts to smaller, i.e., more delocalized values. This effect is more pronounced for the SC compared to the SG.

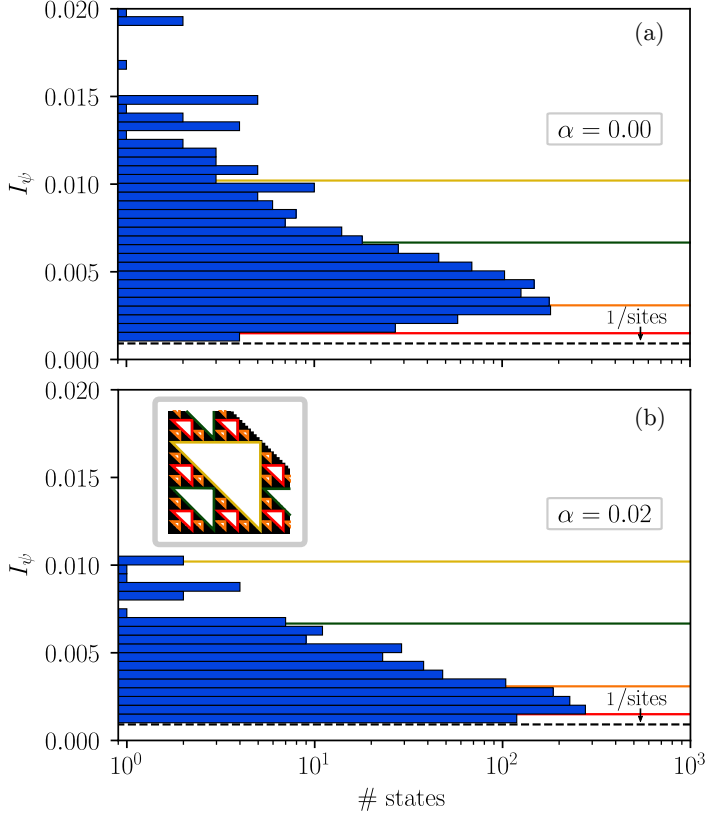


FIGURE 3.5: Statistics of IPR for SG at (a) $\alpha = 0$ and (b) $\alpha = 0.02$. Inset: a closeup of gasket with highlighted edges of different hierarchies.

3.3 EFFECT OF DISORDER

As an ultimate probe, we study potential topological phase transition with the level spacing statistics. Depending whether states are extended or localized, they follow Wigner-Dyson or Poisson distribution, respectively. Such approach was successfully applied to disordered topological and Chern insulators [39, 40]

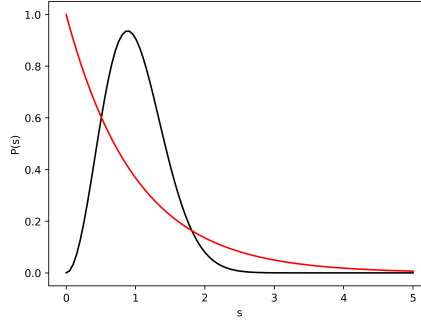


FIGURE 3.6: Comparison of the level spacing distribution

However, authors in Ref. [41] claim that a power-law like level statistics is a generic feature of fractals.

To do so, we add the on-site disorder term $\sum_i V_i c_i^\dagger c_i$ in Eq. (3.2), where V_i is drawn from a uniform distribution $[-W/2, W/2]$. Having computed the level spacings, we calculate their variance and average over 500 disorder realizations for each disorder strength W .

It is worth mention that follow-up works investigated the Hall conductance in Sierpiński carpet. It was found that the edge states corresponding to non-zero σ_{xy} are always present for a finite field strength and stable as one approaches the thermodynamic limit [42]. Interestingly, the Hall conductivity is not proportional to the Chern number KatnelsonFractal2020. Models realizing spinless chiral $p-$ and $p + ip-$ wave superconductors on SC and SG were discussed in [43]

OBSTRUCTED ATOMIC LIMITS AND FRACTIONAL CORNER CHARGES

4.1 CRYSTAL SYMMETRIES

Gapless boundary modes in topological crystalline insulators are protected by spatial symmetries which act non-locally (i.e. mirror symmetry $M_z : x \rightarrow -x$ Prediction of 3D TCI with two-dimensional surface states in the semiconducting SnTe protected by the mirror symmetry [44, 45].

Extending the classification schemes to crystal symmetries [46–50].

Crystal symmetries give rise to a generalization of TIs called higher-order topological insulators, where non-trivial d -dimensional bulk is accompanied by boundary modes in less than $(d - 1)$ -dimensions [51–53]. For instance, 3D system may exhibit 1D hinge modes/

A general framework of topological quantum chemistry [54]

4.2 MODERN THEORY OF POLARIZATION: WANNIER FUNCTIONS

Those systems can be understood using the Wannier functions. Given a set of occupied Bloch states, one may construct the Wannier function as their Fourier transform:

$$|\mathbf{R}n\rangle = \frac{V}{(2\pi)^3} \int_{BZ} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{R}} \sum_{m=1}^J U_{mn}(\mathbf{k}) |\psi_{m\mathbf{k}}\rangle \quad (4.1)$$

In general, Wannier functions are not uniquely defined, only the sum over the Wannier centers in a given unit cell. Most often, the rotation matrix U_{nm} is determined in such a way the minimizes the spread in a real space (= the sum of the mean squares) of the Wannier function. For more detailed discussion, we refer to the review by Marziari *et al.* [55].

Obstructed atomic limits admit the Wannier function representation, (with Wannier functions being exponentially localized and symmetry-preserving) which is in contrast to strong topological phases. However, the Wannier centers do not coincide with atomic positions (as in the case of trivial atomic limit), but they are rather localized on other symmetric points in the unit cell called the Wyckoff positions.

Note that the charge fractionalization mechanism is different that in, for example, spin liquids. Here it's due to cutting Wannier functions in a finite sample.



FIGURE 4.1: In the presence of crystal symmetries, a distinction between topological and trivial phases is less pronounced. Atomic limit corresponds to the situation where Wannier charge centers are located on atomic positions. In case of obstructed atomic limits, Wannier functions are exponentially localized on the Wyckoff positions. Fragile phases cannot be represented in terms of Wannier functions, but a strong index vanishes. Strong topological phases (such as TI or CI) do not admit Wannier representation.

4.3 BULK INDICES

The key goal was to construct the bulk indices for all layer groups to be sure that boundary effects are related to the nontrivial bulk rather than suitable edge termination.

4.3.1 *Symmetry indicators*

Fu-Kane formula presented in Chapter 2 was actually the first symmetry indicator. In a similar manner, one may construct the quantities which will count

4.3.2 *Wilson loops*

Given the projector onto occupied states $P := \sum_{E < E_F} |\psi\rangle \langle \psi|$, the Wilson loop is defined as a product of P along closed path γ in k -space:

$$W_\gamma = \prod_{\mathbf{k}} P(\mathbf{k}) \quad (4.2)$$

and it is gauge invariant.

4.4 MATERIAL CANDIDATES

Predicted also in atomically thin carbon allotrope called graphdiyne [56, 57].

As potential experimental realization, we propose group-V honeycomb monolayers of bismuth, antimony and arsenic. They share the very same

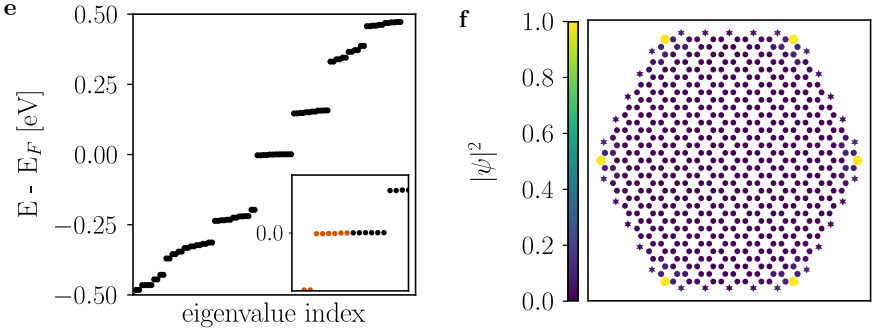


FIGURE 4.2: Low-energy spectrum of a finite armchair-terminated flake of the 3cOAL. The inset presents the energies around the Fermi level, with filled states in orange. (f) The electronic densities of the cornerstates with color scale proportional to the normalized square modulus of the eigenstates $|\psi|^2$ (normalized with respect to the largest $|\psi|^2$). The tellurium atoms used for edge passivation are shown as stars.

crystal structure. With non-zero buckling, these systems preserve C_3 and \mathcal{I} . When $d_z = 0$, TCI phase is exhibited.

In Fig. 4.2 we show

In quantum mechanics, the condition that the observable must be a Hermitian operator has a deep physical reasoning - the corresponding expectation value has to be a real-valued number. However, this strong assumption can be relaxed - it is possible to have a non-Hermitian operator with real spectrum. This observation gave rise to the concept of PT-symmetric Hamiltonians, where the real spectrum is guaranteed by the product of parity and time-reversal symmetries [58].

Another motivation comes from the open systems. Instead of a full treatment with Lindblad formalism, for instance, nH Hamiltonians can effectively capture the coupling of the system with its environment, where the non-Hermiticity models the gains and losses.

Such system exhibits interesting phenomena without Hermitian counterpart: exceptional points, the skin effect and, as a consequence, the breakdown of bulk-boundary correspondence.

Novel features of nH systems are seen at the level of 2×2 matrices. Consider a matrix M : $M =$

$$\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \quad (5.1)$$

If $\alpha \neq 1$, M is not diagonalizable, and only admits the Jordan block form. NH matrices have distinct left- and right- eigenvectors. Therefore, a remedy for some problems may be to consider quantities of interests within the biorthogonal quantum mechanics. For instance, the norm is then given by the inner product between left and right eigenvectors. This attempt allowed to restore BB correspondence in some models. Another way is to consider the singular value decomposition (SVD) instead of eigenvalue problem. However, the interpretation of the singular values is not physical (in contrast to the eigendecomposition, where the eigenvalues are the energies) [59].

In non-Hermitian case, the topology is already manifested in single-band systems (in contrast to Hermitian systems where at least two bands are needed). Also, the winding number for 1D systems is defined through the eigenvalues, not the eigenstates.

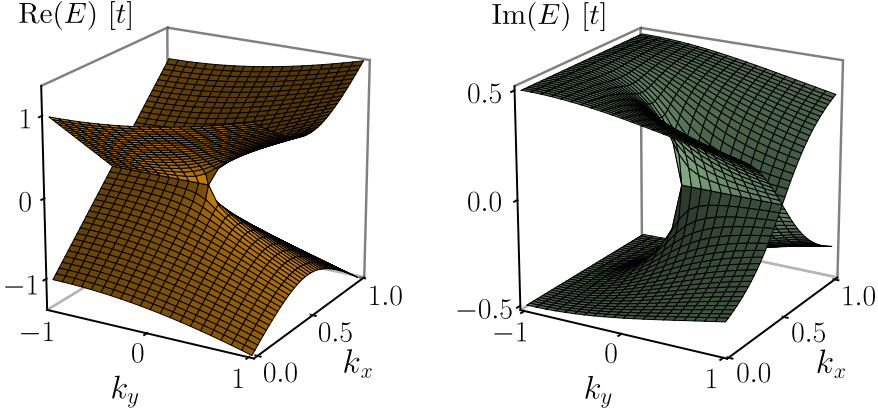


FIGURE 5.1: Real (left) and imaginary (right) part of the spectrum of the Hamiltonian defined by Eq.(5.3). Gapless region of real part of the spectrum corresponds to gapped imaginary spectrum and vice versa.

5.1 EXCEPTIONAL POINTS

Let us remind the concept of Weyl points. Consider the Hamiltonian in 3D

$$H = \mathbf{k} \cdot \boldsymbol{\sigma} = k_x \sigma_x + k_y \sigma_y + k_z \sigma_z. \quad (5.2)$$

This model exhibits a robust generic degeneracy. As all σ_i are used, adding other term proportional to σ_i only shifts the touching point. Now compare the following model in 2D:

$$H = k_x \sigma_x + k_y \sigma_y + ir \sigma_y \quad (5.3)$$

Non-zero r gives rise to the degeneracies in non-Hermitian band structure called exceptional points. Exceptional points can be seen as equivalents of Weyl nodes as they appear in generic points in k -space and one has to get them closer to annihilate them (for example by adding very large mass term).

5.2 BREAKDOWN OF BULK-BOUNDARY CORRESPONDENCE

As Hermitian conjugate is not longer equal to complex conjugate and transpose, different type of symmetries appear. Classification, firstly by [60], extended recently in Refs. [61–63]

5.3 SKIN EFFECT

nH Hamiltonians are sensitive to the boundary conditions. Eigenstates localization properties may change dramatically. All states for the system in an open geometry may be exponentially localized on the one edge, which is dubbed the skin effect (note: this has nothing in common with a typical skin effect, where the electrons in a conductor prefer to flow far from the middle due to electron-electron repulsion).

Previously, it was known that the skin effect can be induced by having unbalanced directed hoppings. However, this breaks the reciprocity, defined as

$$H(k) = H^T(-k) \quad (5.4)$$

5.3.1 *Reciprocal skin effect*

Here, we show that in two- or higher-dimensions it is possible to have the skin effect with the condition defined by Eq. (5.4).

SUMMARY

It is of great interest to provide a systematic classification for the systems with the scale symmetry instead of translational invariance.



APPENDIX

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CURRICULUM VITAE

PERSONAL DATA

Name	Marta Agnieszka Brzezińska
Date of Birth	August 22, 1991
Place of Birth	Legnica, Poland

EDUCATION

since 2015	Ph. D. in Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Technology, Poland Supervisors: Prof. Titus Neupert, Prof. Arkadiusz Wójs
2013 – 2015	M. Sc. in Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Technology, Poland Specialization: Physical Foundations of Computer Science Thesis: Entanglement entropy in d-dimensional interacting many-body systems Supervisor: Prof. Arkadiusz Wójs Grade: 5.5 (excellent)
2010 – 2013	B. Sc. in Physics, Faculty of Fundamental Problems of Technology, Wrocław University of Technology, Poland Grade: 5.0 (very good)

SCHOLARSHIPS

Sep 2019 – Aug 2020	The Iwanowska Programme for Doctoral Students from Polish National Agency for Academic Exchange
Sep 2018 – Aug 2019	Swiss Government Excellence Scholarship from Federal Commission for Scholarships for Foreign Students

PUBLICATIONS

Articles in peer-reviewed journals:

1. Schindler, F., Brzezińska, M., Benalcazar, W. A., Iraola, M., Bouhon, A., Tsirkin, S. S., Vergniory, M. G. & Neupert, T. Fractional corner charges in spin-orbit coupled crystals. *Phys. Rev. Research* **1**, 033074 (3 2019).
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