NYU Tandon Bridge Homework 6 #
Marta Bukowska

Question 5
Use the definition of θ in order to show the following:

a) $5n^3 + 2n^2 + 3n = \theta(n^3)$

Definition of θ: $f(n) = \theta(g(n))$, where f(n) and g(n) are two functions mapping Z^+ to R^+ if there exist positive real constants c_1 and c_2, and positive integer constant n_0, such that $c_2 g(n) \leq f(n) \leq c_1 g(n)$, for all $n \geq n_0$.

(We can also define θ using definitions of lower bound* and upper bound**: $f(n) = \theta(g(n))$ if $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$)

(*We can say that $f(n) = \Omega(g(n))$ if there exists positive real constant c_2 and positive integer constant n_0, such that $c_2 g(n) \leq f(n)$ for all $n \geq n_0$.)

(**We can say that $f(n) = O(g(n))$ if there exists positive real constant c_1 and positive integer constant n_0, such that $f(n) \leq c_1 g(n)$ for all $n \geq n_0$.)

Assumption:

$f(n) = 5n^3 + 2n^2 + 3n$

$g(n) = n^3$

Proof:

1. First, let's analyse the lower bound Ω.

   Let $c_2 = 5$ (we obtained that by dropping lower order terms in $5n^3 + 2n^2 + 3n$; as a result, we obtain $5n^3$, which can be expressed as $5g(n)$, hence $c_2 = 5$). The inequality $c_2 g(n) \leq f(n)$, can then be expressed as
   $5n^3 \leq 5n^3 + 2n^2 + 3n$

   Let's solve the inequality:
   $5n^3 \leq 5n^3 + 2n^2 + 3n$
   $0 \leq 2n^2 + 3n$
   $0 \leq n(2n + 3)$
   $n \in (-\infty, -\frac{3}{2}] \cup [0, \infty)$

   However, since our two functions are mapping Z^+ to R^+, the solution will be $n \geq 1$.

Therefore, we have proved that $5n^3+2n^2+3n = \Omega(n^3)$, since there exists positive real constant $c_2 = 5$ and positive integer constant $n_0 = 1$, such that $c_2n^3 \leq 5n^3 + 2n^2 + 3n$ for all $n \geq n_0$.

2. Let's analyse the upper bound O.

   Let $c_1 = 10$ (we obtain that by bringing every term in $5n^3+2n^2+3n$ to the power of 3; as a result, we obtain $5n^3 + 2n^3 + 3n^3 = 10n^3$, which can be expressed as $10g(n)$, hence $c_1 = 10$). The inequality $f(n) \leq c_1g(n)$, can then be expressed as $5n^3 + 2n^2 + 3n \leq 10n^3$

   Let's solve the inequality:
   $5n^3 + 2n^2 + 3n \leq 10n^3$
   $-5n^3 + 2n^2 + 3n \leq 0$
   $n(-5n^2 + 2n + 3) \leq 0$
   $n(5n + 3)(n - 1) \leq 0$
   $n \in [-\frac{3}{5}, 0] \cup [1, \infty)$

   However, since our two functions are mapping Z^+ to R^+, the solution will be $n \geq 1$.

   Therefore, we have proved that $5n^3+2n^2+3n = O(n^3)$, since there exists positive real constant $c_1 = 10$ and positive integer constant $n_0 = 1$, such that $5n^3 + 2n^2 + 3n \leq c_1n^3$ for all $n \geq n_0$.

Since there exist real constants $c_1 = 10$ and $c_2 = 5$, and positive integer constant $n_0 = 1$, such that $c_2n^3 \leq 5n^3 + 2n^2 + 3n \leq c_1n^3$, for all $n \geq n_0$, then it is true that $5n^3 + 2n^2 + 3n = \theta(n^3)$.

b) $\sqrt{7n^2 + 2n - 8} = \theta(n)$

Assumption:
$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = n$

Proof:

1. First, let's analyse the lower bound Ω.

   Let $c_2 = \sqrt{7}$ (we obtained that by dropping lower order terms in $\sqrt{7n^2 + 2n - 8}$; as a result, we obtain $\sqrt{7n^2}$, which can be expressed as $\sqrt{7}g(n)$, hence $c_2 = \sqrt{7}$). The inequality $c_2g(n) \leq f(n)$, can then be expressed as

$\sqrt{7}n \leq \sqrt{7n^2 + 2n - 8}$

Let's solve the inequality:
$\sqrt{7}n \leq \sqrt{7n^2 + 2n - 8}$
$\sqrt{7n^2} \leq \sqrt{7n^2 + 2n - 8}$
$7n^2 \leq 7n^2 + 2n - 8$
$0 \leq 2n - 8$
$8 \leq 2n$
$n \geq 4$

Therefore, we have proved that $\sqrt{7n^2 + 2n - 8} = \Omega(n)$, since there exists positive real constant $c_2 = \sqrt{7}$ and positive integer constant $n_0 = 4$, such that $c_2 n \leq \sqrt{7n^2 + 2n - 8}$ for all $n \geq n_0$.

2. Let's analyse the upper bound O.

   Let $c_1 = 3$ (we obtain that by bringing every term which includes n in $\sqrt{7n^2 + 2n - 8}$ to the power of 2; as a result, we obtain $\sqrt{7n^2 + 2n^2} = \sqrt{9n^2} = 3n$, which can be expressed as $3g(n)$, hence $c_1 = 3$). The inequality $f(n) \leq c_1 g(n)$, can then be expressed as
   $\sqrt{7n^2 + 2n - 8} \leq 3n$

   Let's solve the inequality:
   $\sqrt{7n^2 + 2n - 8} \leq 3n$
   $\sqrt{7n^2 + 2n - 8} \leq \sqrt{9n^2}$
   $7n^2 + 2n - 8 \leq 9n^2$
   $-2n^2 + 2n - 8 \leq 0$
   $-n^2 + n - 4 \leq 0$
   $n \in R$

   However, since our two functions are mapping Z^+ to R^+, the solution will be $n \geq 1$.
   Additionally, since $n \geq 1$, our inequality will be also true for every $n \geq 4$, which is the solution of inequality ($c_2 g(n) \leq f(n)$). Hence, we can also say that $n_0 = 4$

   Therefore, we have proved that $\sqrt{7n^2 + 2n - 8} = O(n)$, since there exists positive real constant $c_1 = 3$ and positive integer constant $n_0 = 4$, such that $\sqrt{7n^2 + 2n - 8} \leq c_1 n$ for all $n \geq n_0$.

Since there exist real constants $c_1 = 3$ and $c_2 = \sqrt{7}$, and positive integer constant $n_0 = 4$, such that $c_2 n \leq \sqrt{7n^2 + 2n - 8} \leq c_1 n$, for all $n \geq n_0$, then it is true that $\sqrt{7n^2 + 2n - 8} = \theta(n)$.