
Question 5

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2

   b) 1. $p \to (q \land r)$ (Hypothesis 1)
      2. $\neg p \lor (q \land r)$ (Conditional identity, 1)
      3. $(\neg p \lor q) \land (\neg p \lor r)$ (Distributive law, 2)
      4. $\neg p \lor q$ (Simplification, 3)
      5. $p \to q$ (Conditional identity, 4)
      6. $\neg q$ (Hypothesis 2)
      7. $\neg p$ (Modus Tollens, 5, 6)

   e) 1. $p \lor q$ (Hypothesis 1)
      2. $\neg p \lor r$ (Hypothesis 2)
      3. $q \lor r$ (Resolution, 1, 2)
      4. $\neg q$ (Hypothesis 3)
      5. r (Disjunctive syllogism, 3, 4)

2. Exercise 1.12.3

   c) 1. $p \lor q$ (Hypothesis 1)
      2. $\neg(\neg p) \lor q$ (Double negation, 1)
      3. $(\neg p) \to q$ (Conditional identity, 2)
      4. $\neg p$ (Hypothesis 2)
      5. q (Modus ponens, 3, 4)

3. Exercise 1.12.5

   c) c: I will buy a new car
   h: I will buy a new house
   j: I will get a job

$$\begin{array}{l} (c \land h) \to j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

c	h	j	$(c \wedge h) \to j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

d) c: I will buy a new car
h: I will buy a new house
j: I will get a job

$$\begin{array}{l} (c \wedge h) \to j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

c	h	j	$(c \wedge h) \to j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

Answer: The argument is valid, as all of the hypotheses are true only for c=F, h=T, and j=F, and the conclusion for mentioned truth values of variables is also true. It can be also proved:

1. $\neg j$ (Hypothesis 2)
2. $(c \wedge h) \to j$ (Hypothesis 1)
3. $\neg(c \wedge h)$ (Modus tollens, 1, 2)
4. $\neg c \vee \neg h$ (De Morgan's law, 3)
5. $\neg h \vee \neg c$ (Commutative law, 4)
6. h (Hypothesis 3)
7. $\neg\neg h$ (Double negation law, 6)
8. $\neg c$ (Disjunctive syllogism, 6, 7)

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3

   b) To show that the argument is invalid, we need to show that hypotheses $\exists x(P(x) \vee Q(x))$ and $\exists x \neg Q(x)$ are true, and the conclusion $\exists x P(x)$ is false.

   For conclusion to be false, there has to be no P(a) and P(b) that are true, therefore we have:

	P	Q
a	F	
b	F	

   For hypothesis $\exists x \neg Q(x)$ to be true, Q(a) or Q(b) or both have to be false.

   However, for hypothesis $\exists x(P(x) \vee Q(x))$ to be true, as we already know that P(x) is always false, either Q(a) or Q(b) has to be true (either, because we know that one of them has to be false).

   Therefore, we have:

	P	Q
a	F	T
b	F	F

or

	P	Q
a	F	F
b	F	T

2. Exercise 1.13.5

   d) M(x): x missed the class
   D(x): x got a detention

   $\forall x(M(x) \to D(x))$
   Penelope is a particular student.
   $\neg M(Penelope)$
   $\therefore \neg D(Penelope)$

M(P)	D(P)	$\neg M(P)$	$\neg D(P)$	$M(P) \to D(P)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Clarification: P=Penelope

Answer: The argument isn't valid. If Penelope is the only student in class, D(Penelope) is true and M(Penelope) is false, the hypotheses are true, but the conclusion is false.

e) M(x): x missed the class

D(x): x got a detention

A(x): x got an A

$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$
Penelope is a particular student.
$\underline{A(Penelope)}$
$\therefore \neg D(Penelope)$

M(P)	D(P)	A(P)	$\neg A(P)$	$\neg D(P)$	((M(P) $\vee D(P)) \rightarrow \neg A(P))$
T	T	T	F	F	F
T	T	F	T	F	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	T	T
F	F	F	T	T	T

Clarification: P=Penelope

Answer: The argument is valid. The hypotheses are true only when M(P)=D(P)=F and A(P)=T, and for those truth values of variables the conclusion is true. The fact that argument is valid can also be proved:

1. $\forall x(M(x) \vee D(x) \rightarrow \neg A)$ (Hypothesis 1)
2. Penelope is a particular student. (Hypothesis 2)
3. $M(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$ (Universal installation, 1, 2)
4. $A(Penelope)$ (Hypothesis 3)
5. $\neg\neg A(Penelope)$ (Double negation law, 4)
6. $\neg(M(Penelope) \vee D(Penelope))$ (Modus Tollens, 3, 5)
7. $\neg M(Penelope) \wedge \neg D(Penelope))$ (De Morgan's law, 6)
8. $\neg D(Penelope) \wedge \neg M(Penelope))$ (Commutative law, 7)
9. $\neg D(Penelope)$ (Simplification, 8)

Question 6

1. Exercise 2.4.1

   d) Suppose that x and y are two odd integers (so $x = 2k_1 + 1$ and $y = 2k_2 + 1$, for some integers k_1 and k_2). We will prove that their product xy is an odd integer.

   xy = $(2\mathrm{k}_1 + 1)(2k_2 + 1) = 4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$

   Since k_1 and k_2 are integers, $(2k_1k_2+k_1+k_2)$ is also an integer (sum of all integers is also an integer).

   Since $xy = 2(2k_1k_2 + k_1 + k_2) + 1$, xy equals to two times an integer (and every integer multiplied by two is an even integer). For every even integer, if we add one to it, it becomes an odd integer, so the product of x and y is an odd integer. ■

2. Exercise 2.4.3

   b) Let x be a real number, for which $x \leq 3$ is true. We will prove that $12-7x+x^2 \geq 0$

   We can transform the standard form of quadratic function $12 - 7x + x^2$, to the factored form $(x - 3)(x - 4)$, by calculating zeros of the function.

   Then, $(x - 3)(x - 4) \geq 0$

   As $x \leq 3$, then the function $(x - 3)(x - 4)$ reaches its maximum value for x=3.

   Hence, its maximum value is
   $(x - 3)(x - 4) = (3 - 3)(3 - 4) = 0$

   Since the maximum value of the function is 0, the it is true that it is less or equal to 0, therefore, $12 - 7x + x^2 \geq 0$ is true. ■

Question 7

1. Exercise 2.5.1

   d) Let n be an even integer. We will prove that $n^2 - 2n + 7$ is odd integer.

   If n is an even integer then we can express it as n=2k, where k is an integer.

   Then, we can express $n^2 - 2n + 7$ as
   $(2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2k(k-2) + 7$.

   As 2k(k-2) is an even integer (because every integer multiplied by two is an even integer), when we add 7 to it, which is an odd integer (we can express it as 2n+1, because 7=2*3+1), we will obtain an odd integer (the sum of even integer and odd integer is always an odd integer). Therefore, $n^2 - 2n + 7$ is an odd integer. ■

2. Exercise 2.5.4

   a) Let x and y be real numbers that $x > y$. We will prove that $x^3 + xy^2 > x^2y + y^3$

   If $x > y$, then $x^2 > y^2$.

   Additionally, the square of every real numbers is greater or equal to 0, so $x^2 \geq 0$ and $y^2 \geq 0$. Therefore, $x^2 + y^2$ gives a non-negative number, which means that we can multiply both sides of the equation by $x^2 + y^2$ without changing the inequality sign.

   $x > y$
   $x(x^2 + y^2) > y(x^2 + y^2)$
   $x(x^2 + y^2) - y(x^2 + y^2) > 0$
   $(x - y)(x^2 + y^2) > 0$
   $(x^3 + xy^2 - x^2y - y^3) > 0$
   $x^3 + xy^2 > x^2y + y^3$

   Therefore, $x^3 + xy^2 > x^2y + y^3$ is true. ■

   b) Let's assume that for two real numbers x and y, it's not true that $x > 10$ or $y > 10$. We will prove that $x + y \leq 20$

   We can express our assumption as
   $\neg(x > 10 \lor y > 10)$

By applying De Morgan's law, we get
$\neg(x > 10 \vee y > 10) \equiv (\neg x > 10 \wedge \neg y > 10)) \equiv (x \leq 10 \wedge y \leq 10)$

The assumption is true, only when both $x \leq 10$ and $y \leq 10$ are true.

As the maximum value of x is 10 and the maximum value of y is 10, then the maximum value of x+y is 20. Since $x \leq 10$ and $y \leq 10$, the value of x+y can only get smaller than the maximum value of 20. Therefore, $x + y \leq 20$ is true. ■

3. Exercise 2.5.5

c) Let x be a non-zero real number. We assume that $\frac{1}{x}$ isn't irrational. We will prove that x is rational.

Since $\frac{1}{x}$ isn't irrational, then it is rational (every real number is either rational or irrational).

Since $\frac{1}{x}$ is rational, then $\frac{1}{x} = \frac{a}{b}$, where a and be are integers, and $b \neq 0$.

We can express $\frac{1}{x} = \frac{a}{b}$ as
$b = ax$

Since we know that $b \neq 0$, then $ax \neq 0$.

We also know, that $x \neq 0$. If $x \neq 0$ and $ax \neq 0$, then $a \neq 0$.

We can use our equation $b = ax$, to express x:
$b = ax$
$x = \frac{b}{a}$

Since $x = \frac{b}{a}$, $a \neq 0$, and a and b are both integers, x is rational (a number is rational, when we can express it as a fraction of two integers, where denominator of the fraction isn't 0). ■

Question 8

1. Exercise 2.6.6

   c) Theorem: The average of three real numbers is greater or equal to at least one of the numbers.

      Contradiction: The average of three real numbers is less than each of those numbers

      Proof:

      Assume that for three real numbers x, y, and z, its not true that $\frac{x+y+z}{3} \geq x \vee \frac{x+y+z}{3} \geq y \vee \frac{x+y+z}{3} \geq z$

      By applying De Morgan's law, we get
      $\neg(\frac{x+y+z}{3} \geq x \vee \frac{x+y+z}{3} \geq y \vee \frac{x+y+z}{3} < z) \equiv (\frac{x+y+z}{3} < x \wedge \frac{x+y+z}{3} < y \wedge \frac{x+y+z}{3} < z)$

      Since $(\frac{x+y+z}{3} < x \wedge \frac{x+y+z}{3} < y \wedge \frac{x+y+z}{3} < z)$, then it also must be true that $\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} < x + y + z$

      We can simplify the inequality
      $\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} < x + y + z$
      $\frac{3(x+y+z)}{3} < x + y + z$
      $x + y + z < x + y + z$

      We can divide both side of inequality by $x + y + z$, since we have two exact same numbers on the both sides of inequality (no matter if x+y+z is going to be positive number or negative number, because the direction of inequality sign won't change the truth value of the inequality).
      $x + y + z < x + y + z$
      $1 < 1$

      Since it not true, that $1 < 1$, then we've proven that the contradiction is false, and therefore the theorem that the average of three real numbers is greater or equal to at least one of the numbers is true.

   d) Theorem: There is no smallest integer.

      Contradiction: There is a smallest integer x

      Proof:

Suppose that x is the smallest integer.

Let's consider an integer $y = x - 1$ (y is an integer, as when we subtract an integer from an integer, we obtain an integer)

Since $x > x - 1$ and $y = x - 1$, then $x > y$.

We have proved that y is smaller integer than x. Hence, our contradiction that there exists a smallest integer is false, and therefore, the theorem that there is no smallest integer is true. ■

Question 9

1. Exercise 2.7.2

   b) If integers x and y have the same parity, then x+y is evern.

   Case 1: Both integers are even

   Let's express x as 2a, and y as 2b (since every integer multiplied by 2 is an even integer).

   Then, their sum is:
   $x + y = 2a + 2b = 2(a + b)$

   Since we know that (a+b) is an integer (because sum of two integers is also an integer) and x+y is equal to two times an integer (as said before, every integer multiplied by two is an even integer), then x+y is an even integer.

   Case 2: Both integers are odd

   Let's express x as (2a+1) and y as (2b+1) (since every integer multiplied by 2 is an even integer, and for every even integer, when we add 1 to it, it becomes an odd integer).

   Then, their sum is:
   $x + y = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b) + 2$

   Since 2(a+b) is an even integer, when we add 2 to it, which is also an even integer, we will obtain an even integer (as sum of two even integers gives and even integer). Therefore, x+y is an even integer.

   Answer: The sum of x and y is an even integer in both of the possible cases, therefore, the theorem is true.■