

Question 3

a) Exercise 8.2.2

b) $f(n) = n^3 + 3n^2 + 4$, $g(n) = n^3$

Theorem: $f = \theta(n^3)$ if $f = O(n^3)$ and $f = \Omega(n^3)$

Proof:
Let $c = 8$ and $n_0 = 1$. We will prove that $f = O(n^3)$, by showing that for any $n \geq 1$, $f(n) \leq 8g(n)$.

For $n \geq 1$, it is true that $n^2 \leq n^3$, so it is true that:
$n^3 + 3n^2 \leq n^3 + 3n^3$

Additionally, for $n \geq 1$, it is also true that $1 \leq n^3$, so it is also true that:
$4 \leq 4n^3$

Since $n^3 + 3n^2 \leq n^3 + 3n^3$ and $4 \leq 4n^3$ are true, putting the inequalities together, we get that for $n \geq 1$, it is true that:
$n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3$
$n^3 + 3n^2 + 4 \leq 8n^3$
$f(n) \leq 8g(n)$

Therefore, we have proven that $f(n) \leq 8g(n)$ for every $n \geq 1$. Hence, $f(n) = O(n^3)$.

Now, let $c = 1$ and $n_0 = 1$. We will prove that $f = \Omega(n^3)$, by showing that for any $n \geq 1$, $g(n) \leq f(n)$.

For $n \geq 1$, it is true that $n^3 \leq n^3 + 3n^2$, as $3n^2$ is always positive.

Additionally, when we add a positive number to any other number, we will obtain a greater number. Hence, $g(n) = n^3 \leq n^3 + 3n^2 + 4 = f(n)$

Therefore, we have proven that $g(n) \leq f(n)$ for every $n \geq 1$. Hence, $f(n) = \Omega(n^3)$.

Finally, as we have shown that $f = O(n^3)$ and $f = \Omega(n^3)$, we have proven that $f = \theta(n^3)$. ■

b) Exercise 8.3.5

   a) The first inner loop identifies numbers that are less than p, starting from the first number and moving towards the last number. The second outer loop identifies the numbers that are greater or equal to p, starting from the last number, and moving towards the first number. Then, as long as $i < j$, the two numbers, each identified by one inner loop, are swapped. The iteration of outer while loop ends when all of the numbers are checked, so when i exceeds j. Then, at the end of algorithm, the sorted sequence of numbers is returned - the numbers which value is less than the value of p are on the left, and the numbers which value is greater than or equal to p are on the right.
   b) The number of times that the lines $i := i + 1$ or $j := j - 1$ are executed on a sequence of length n, depends only on the length of the sequence, and can be described as n-1.
   c) The number of times the swap operation is executed depends on the input length and the numbers in the sequence. The maximum number of times that swap can be executed is when every element is swapped. Thus, if there are n elements, the number of swaps is $\frac{n}{2}$. The minimum number of times swap can be executed is when the sequence is already sorted in a way that numbers less than p are on the left and numbers greater than or equal to p are on the right. Then the number of swap operations is 0.
   d) Since no matter what the input is, two inner loops together iterate n-1 times. Thus, the asymptotic lower bound is $\Omega(n)$. It is not important to consider the worst-case input when determining an asymptotic lower bound because it does not directly influence it. The asymptotic lower bound indicates the minimum time complexity required by the algorithm, no matter what the specific input is.
   e) Since the number of operations (ignoring constant numbers of operations before and after the outer loop) is at most $n - 1 + \frac{n}{2} = \frac{3}{2}n - 1$, the upper bound is $O(n)$.

Question 4

a) Exercise 5.1.2

b) Let D be a set of digits, L - a set of letters, and S - a set of special characters.

Then: $|D| = 10$, $|L| = 26$, and $|S| = 4$.

Therefore, since every character in the password can be a digit, a letter or a special character, a single character can be expressed in 40 ways:

$|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$

We are supposed to take into considerations 3 types of passwords: one consisting of 7 characters (P_7), one consisting 8 characters (P_8), and one consisting of 9 characters (P_9). Hence:

$|P_7| = 40^7$ (there are 7 characters, each one can be expressed in 40 ways)
$|P_8| = 40^8$ (there are 8 characters, each one can be expressed in 40 ways)
$|P_9| = 40^9$ (there are 9 characters, each one can be expressed in 40 ways)

Therefore, total number of passwords is equal to $|P_7|+|P_8|+|P_9| = 40^7+40^8+40^9 = 2.6886144 * 10^{14}$

c) Let D be a set of digits, L - a set of letters, and S - a set of special characters.

Then: $|D| = 10$, $|L| = 26$, and $|S| = 4$.

The first character can only be a digit or a special character, therefore it can be expressed in 14 ways, since:

$|D \cup S| = |D| + |S| = 10 + 4 = 14$
Every character in the password other than the first character can be a digit, a letter or a special character, therefore, it can be expressed in 40 ways:

$|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$

We are supposed to take into considerations 3 types of passwords: one consisting of 7 characters (P_7), one consisting 8 characters (P_8), and one consisting of 9 characters (P_9). Hence:

$|P_7| = 14 * 40^6$ (there are 7 characters, first one can be expressed in 14 ways, the other 6 characters can be expressed in 40 ways)

$|P_8| = 14 * 40^7$ (there are 8 characters, first one can be expressed in 14 ways, the other 7 characters can be expressed in 40 ways)

$|P_9| = 14 * 40^8$ (there are 9 characters, first one can be expressed in 14 ways, the other 8 characters can be expressed in 40 ways)

Therefore, total number of passwords is equal to $|P_7| + |P_8| + |P_9| = 14 * 40^6 + 14 * 40^7 + 14 * 40^8 = 14(40^6 + 40^7 + 40^8) = 9.4101504 * 10^{13}$

b) Exercise 5.3.2

a) We have 10 spots to disposition. The first spot can be expressed in 3 ways, since it can be any element of the set $\{a, b, c\}$. Each of remaining 9 spots can be expressed in 2 ways, since no consecutive characters can be the same.

Therefore, the number strings over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same is:

$3 * 2^9 = 1536$

c) Exercise 5.3.3

b) We have 7 spots to fill

The 1st spot can be any digit, therefore it can be expressed in 10 ways (because there are 10 digits).

Spots 2-5 can be any letter, therefore each of them can be expressed in 26 ways (because there are 26 letters), and all of them can be expressed in 26^4 ways.

Spots 6 and 7 are digits, but no digit can appear more than once, hence 6th spot can be expressed in 9 ways (because there are 10 digits, and we already used 1 of them), and 7th spot - in 8 ways (because there are 10 digits and we already used 2 of them).

Therefore, the possible number of licence plate numbers is:

$10 * 26^4 * 9 * 8 = 329,022,720$

c) We have 7 spots to fill.

The 1st spot can be any digit, therefore it can be expressed in 10 ways (because there are 10 digits).

Spots 2-5 are letters, but no letter can appear more than once, therefore 2nd spot can be expressed in 26 ways (because there are 26 letters), 3rd spot - in 25 ways (because there are 26 letters, and we used 1 of them), 4th spot - in 24 ways (there are 26 letters, we already used 2 of them), and 5th spot - in 23 ways (there are 26 letters, we already used 3 of them).

Spots 6 and 7 are digits, but no digit can appear more than once, hence 6th spot can be expressed in 9 ways (there are 10 digits, and we already used 1 of them), and 7th spot - in 8 ways (there are 10 digits, and we already used 2 of them).

The possible number of licence plate numbers is:
$10 * 26 * 25 * 24 * 23 * 9 * 8 = 258,336,000$

d) Exercise 5.2.3

a) Let B^9 be a set of 9-bit binary strings and set E_{10} a set of 10-bit binary strings with an even number of 1s. We will prove that there is a bijection between B^9 and E_{10}.

There is a bijection between B^9 and E_{10} if there exists a function f, which maps elements of set B^9 to the elements of set E_{10}, that is both one-to-one and onto.

Let's prove that f(x) is both one-to-one and onto:

1. Proof that f(x) is one-to-one:

   Let f be a function which maps elements of set B^9 to the elements of set E_{10}. We will prove that f is one-to-one.

   By definition, function f(x) is one-to-one, if there are no two elements x_1 and x_2 in its domain (in our case B^9), such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

   Since by applying function f we add either 0 or 1 to the 9-bit binary string, as a result, we obtain a 10-bit binary string; therefore, it's impossible for $f(x_1)$ to equal $f(x_2)$ for any $x_1 \in B^9$ and $x_2 \in B^9$, such that $x_1 \neq x_2$, because we will never obtain the same 10-bit binary string, by adding 0 or 1 to two different 9-bit binary strings.

   Therefore, f(x) is one-to-one.

2. Proof that f(x) is onto:
   Let f be a function which maps elements of set B^9 to the elements of set E_{10}. We will prove that f is onto.

   By definition, function f(x) is onto, if for every $y \in E_{10}$ there is $x \in B^9$, such that $f(x) = y$.

   Since by applying function f, we add either 0 or 1 to the 9-bit binary string, and as a result, obtain a 10-bit binary string that has an even number of 1s, then:

   a) if the element of B^9 is a string with even number of 1s, we will always add 0 to it

b) if the element of B^9 is a string with odd number of 1s, we will always add 1 to it (since for every odd number, if we add 1 to it, it will become an even number (any odd number can be expressed as $2k+1$, where k is an even integer))

It shows that by adding a 0 or 1 to any 9-bit binary string, we can obtain a 10-bit binary string with an even number of 1s.

For every 10-bit binary string with even number of 1s y, there is a 9-bit binary string x, such that $f(x) = y$.

Therefore, f(x) is onto.

Since we have proved that function f(x), mapping elements of set B^9 to the elements of set E_{10}, is both one-to-one and onto, the it is true that there is a bijection between B^9 and E_{10}. ■

b) Since we have proved that there is a bijection between B^9 and E_{10}, by applying the bijection rule, we know that $|B^9| = |E_{10}|$

We can calculate the $|B^9|$:
$|B^9| = |\{1,0\}^9| = |\{1,0\}|^9 = 2^9$

Therefore:
$|E_{10}| = 2^9 = 512$

Question 5

a) Exercise 5.4.2

  a) The phone number has 7 digits. First 3 digits can be either 824 or 825, which means that they can be expressed in 2 ways. For remaining 4 digits, each digit can be expressed in 10 ways (as there are 10 digits), so all of them can be expressed in 10^4 ways.

     Therefore, the number of possible phone number combinations is:

     $2 * 10^4 = 20,000$

  b) The phone number has 7 digits. First 3 digits can be either 824 or 825, which means that they can be expressed in 2 ways. The 4th digit can be expressed in 10 ways (since there are 10 digits), the 5th in 9 ways (since there are 10 digits, and we already used one of them), 6th in 8 ways (we already used 2 digits), and 7th in 7 ways (we already used 3 digits).

     Therefore, the number of possible phone number combinations is:

     $2 * 10 * 9 * 8 * 7 = 10,080$

b) Exercise 5.5.3

  a) We have 10 bits, and each bit can be expressed in 2 ways (either 0 or 1). Therefore, the number of ways to express a 10-bit binary string is:
     $2^{10} = 1024$

  b) The first 3 bits are 001, therefore they all can be expressed in only 1 way. Each of the remaining 7 bits can be expressed in 2 ways (either 0 or 1), so the remaining 7 bits can all be expressed in 2^7 ways. Therefore, the number of ways to express a 10-bit binary string that starts with 001 is:
     $1 * 2^7 = 128$

  c) We need to consider two cases there:

     1) When the string starts with 001: The first 3 bits are 001, therefore they all can be expressed in only 1 way. Each of the remaining 7 bits can be expressed in 2 ways (either 0 or 1), so the remaining 7 bits can all be expressed in 2^7 ways. Therefore, the number of ways to express a 10-bit binary string that starts with 001 is:
        $1 * 2^7 = 128$

     2) When the string starts with 10: The first 2 bits are 10, therefore, together, they can be expressed in only 1 way. Each of the remaining 8 bits can be expressed in 2 ways (either 0 or 1), so the remaining 8 bits can all be expressed in 2^8 ways. Therefore, the number of ways to express a 10-bit binary string that starts with 10 is:
        $1 * 2^8 = 256$

Hence, the the number of ways to express a 10-bit binary string that starts with 001 or 10 is:

$2^7 + 2^8 = 128 + 256 = 384$

d) Each of the first 2 bits can be expressed in 2 ways (either 0 or 1), so together, they can be expressed in 2^2 ways. Since the first 2 bits and the last 2 bits are the same, then the last 2 bits can be expressed only in 1 way. Each of the remaining 6 bits in the middle can be expressed in 2 ways (either 0 or 1), so they all can be expressed in 2^6 ways. Therefore, the number of ways to express a 10-bit binary string, which 2 first bits are the same as the 2 last bits is:

$2^2 * 2^6 * 1 = 256$

e) We know that the string has 10-bits, and contains exactly 6 0s. The number of ways of placing 6 0s among 10 spots is:

$C(10, 6) = \frac{10!}{6!4!} = 210$

f) Since the string's first bit is 1, then it can be expressed in only 1 way. Among the remaining 9 there are placed 6 0s. The number of ways of placing 6 0s among 9 spots is: $C(9, 6)$. Therefore, the number of ways of expressing a 10-bit binary sting which starts with 1 and contains exactly 6 0s is:

$1 * C(9, 6) = \frac{9!}{6!3!} = 84$

g) There is exactly one 1 in the first half of the 10-bit binary string. The number of ways of placing one 1 among 5 spots is C(5,1). In the second half of the string, there are exactly three 1s. The number of ways of placing three 1s among 5 spots is C(5,3). Therefore, the number of ways of expressing a 10-bit string, which has one 1 in its first half, and three 1s in its second half is:

$C(5, 1) * C(5, 3) = \frac{5!}{1!4!} * \frac{5!}{3!2!} = 50$

c) Exercise 5.5.5

a) The choir director will choose 10 girls from the group of 35 girls and 10 boys from the group of 30 boys. The number of ways of choosing girls is C(35,10) and the number of ways of choosing boys is C(30,10). Therefore, the choir director can make this selection in the following number of ways:

$C(35, 10) * C(30, 10) = \frac{35!}{10!25!} * \frac{30!}{10!20!} = 183,579,396 * 30,045,015 = 5.515645707 * 10^{15}$

d) Exercise 5.5.8

c) There are 13 hearts and 13 diamonds in the deck. We are choosing 5 cards from 13 hearts and 13 diamonds, therefore we can say that we are choosing 5 cards form 26 cards (13+13). The number of ways of choosing 5 cards from 26 cards is:

$C(26, 5) = \frac{26!}{5!21!} = 65,780$

d) We are choosing 5 cards out of the deck, and we know that we have chosen 4 cards of the same rank.

There are 13 different ranks, hence there are 13 ways to choose 4 cards out of the 4 cards (since we are choosing 4 cards o the same rank, and there are only 4 suits). Therefore, the number of ways of choosing 4 cards of the same rank is 13 * C(4,4).

The number of ways of choosing one card that is of one of the remaining 12 ranks and any of the suits is 12 * C(4,1).

Hence, the number of five-card hands that have four cards of the same rank is:
$13 * C(4,4) * 12 * (C(4,1) = 13 * \frac{4!}{4!0!} * 12 * \frac{4!}{1!3!} = 13 * 1 * 12 * 4 = 624$

e) We are choosing 5 cards out of the deck, and we know that 2 of them are of the same rank, and 3 of the are of the same different rank.

Since there are 13 ranks, there are 13 ways of choosing 2 out of the 4 cards. Therefore, the number ways of choosing 2 cards of the same rank is: $13 * C(4,2)$.

Then, since there are 12 remaining ranks to choose from, there are 12 ways to choose 3 cards out of the 4 cards. Therefore, the number of ways of choosing 3 cards of the same rank, that are of different rank than the previously chosen 2 cards is $12 * C(4,3)$.

Hence, the number of choosing 5 cards out of the deck, where 2 of them are of the same rank, and 3 of the are of the same different rank is:
$13 * C(4,2) * 12 * C(4,3) = 13 * \frac{4!}{2!2!} * 12 * \frac{4!}{3!1!} = 13 * 6 * 12 * 4 = 3,744$

f) We are choosing 5 cards, so that there are no two cards of the same rank.

There are 13 ranks, and we will be choosing 5 of them (we are choosing 5 cards, and each of them must be of different rank). Hence, the number of ways of choosing 5 cards of different ranks is C(13,5). However, each chosen card can also be of any of the 4 suits. Considering the suits, there are 4^5 ways of choosing 5 cards. Therefore, by applying the product rule, the number of ways of choosing 5 cards, so that no two cards are of the same rank is:
$C(13,5) * 4^5 = \frac{13!}{5!8!} * 4^5 = 1,317,888$

e) Exercise 5.6.6

a) We need to elect a committee of 10 senate members with the same number of Democrats and Republicans. There are $C(44,5)$ ways of choosing 5 out of group of 44 Democrats (the order of choosing is not important). Additionally, there are $C(56,5)$ ways of choosing 5 out of group of 56 Republicans. Therefore, by applying the product rule, the number of ways of selecting a committee of 10 senate members with the same number of Democrats and Republicans is:
$C(44,5) * C(56,5) = \frac{44!}{5!39!} * \frac{56!}{5!51!} = 1,086,008 * 3,819,816 = 4.148350735 * 10^{12}$

b) We need to select a speaker and a vice speaker form each party. In this case the order isn't important. There are $P(44,2)$ ways of choosing a speaker and a vice speaker from the group of Democrats, and $P(56,2)$ ways of of choosing a speaker and a vice speaker from the group of Republicans. Therefore, by applying the product rule, the number of ways of choosing a speaker and a vice speaker form two parties is: $P(44,2) * P(56,6) = \frac{44!}{42!} * \frac{56!}{54!} = 43 * 44 * 55 * 56 = 5,827,360$

Question 6

a) Exercise 5.7.2

   a) The number of ways of choosing 5-card hands from a deck of 52 cards is $|S| = C(52,5)$. Club is one of 4 suits. There are 13 clubs in the deck. Hence, the number of ways of choosing 5-card hand that has no clubs is $|\bar{P}| = C(39,5)$ (we are choosing 5 out of the 39 cards, since there are no clubs, and 52-13=39). Therefore, the number of ways of choosing 5-card hands that have at east one club is: $|P| = |S| - |\bar{P}| = C(52,5) - C(39,5) = \frac{52!}{5!47!} - \frac{39!}{5!34!} = 2,598,960 - 575,757 = 2,023,203$

   b) The number of ways of choosing 5-card hands from a deck of 52 cards is $|S| = C(52,5)$. Suppose we are choosing 5-card hands that don't contain 2 cards of the same rank. Hence, every card has to be of different rank, and we will select 5 out of 13 ranks (C(13,5)). Additionally, every card can be chosen in 4 ways, since there are 4 suits. Therefore, the number of ways of choosing 5-card hands that don't contain 2 cards of the same rank is $|\bar{P}| = C(13,5) * 4^5$. Hence, the number of ways of choosing 5-card hands that have at least two cards with the same rank is: $|P| = |S| - |\bar{P}| = C(52,5) - C(13,5) * 4^5 = 1,281,072$

b) Exercise 5.8.4

   a) There are 20 comic books and 5 kids. We don't know how many comic books each kid gets, however, we do know that each comic book can be given to any of the 5 kids. Hence, there are 5 ways to give out each comic book, therefore, the number of ways to give out 20 comic books to 5 kids is: 5^{20}

   b) If each of 5 kids gets exactly 4 comic books, then the first kid chooses 4 out of 20 comic books, the second - 4 out of 16, the third 4 out of 12, the fourth 4 out of 8, and the fifth - 4 out of 4. Therefore, the number of ways of distributing the comic books so that 4 go to each of 5 kids is: $C(20,4)*C(16,4)*C(12,4)*C(8,4)*C(4,4)$

Question 7

a) Function f mapping elements of set X to elements of set Y is one-to-one if $x_1 \neq x_2$ implies that $f(x_1) = f(x_2)$. If set X consists of 5 elements and set Y consists of 4 elements, it means that an element from set X will be mapped to two elements of set Y or won't be mapped to any of the elements in set Y at all. Therefore, the number of one-to-one functions from a set with 5 elements to a set with 4 elements is 0.

b) Function f mapping elements of set X to elements of set Y is one-to-one if $x_1 \neq x_2$ implies that $f(x_1) = f(x_2)$. If set X consists of 5 elements and set Y consists of 5 elements, the number of one-to-one functions from a set X to Y is : $P(5,5) = 5! = 120$ (the order is important there, hence we use permutations, to determine the number of 5-permutations form a set with 5 elements))

c) Function f mapping elements of set X to elements of set Y is one-to-one if $x_1 \neq x_2$ implies that $f(x_1) = f(x_2)$. If set X consists of 5 elements and set Y consists of 6 elements, the number of one-to-one functions from a set X to Y is: $P(6,5) = \frac{6!}{1!} = 720$ (the order is important there, hence we use permutations, to determine the number of 5-permutations form a set with 6 elements))

d) Function f mapping elements of set X to elements of set Y is one-to-one if $x_1 \neq x_2$ implies that $f(x_1) = f(x_2)$. If set X consists of 5 elements and set Y consists of 7 elements, the number of one-to-one functions from a set X to Y is: $P(7,5) = \frac{7!}{2!} = 2520$ (the order is important there, hence we use permutations, to determine the number of 5-permutations form a set with 7 elements))