NYU Tandon Bridge
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Homework 8 #

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**Question 7**

a) Exercise 6.1.5

b) The first three cards are of the same rank and any suit. Hence, they can be chosen in C(13,1)*C(4,3) ways (since we are choosing 1 of 13 ranks, and then 3 out of 4 suits). Then, the remaining 2 cards both need to be of different ranks. Hence, we need to choose 2 out of remaining 12 ranks, and there are C(12,2) ways of doing that. Additionally, each of the remaining 2 cards can be of any of the 4 suits, so the number of possible suits combinations of 2 remaining cards is C(4,1)*C(4,1). Therefore, the number of ways of choosing "three of a kind" is:
$|E| = C(13,1) * C(4,3) * C(12,2) * C(4,1) * C(4,1) = 13 * 4 * 66 * 4 * 4 = 54,912$
The number of possibilities of choosing 5-card hand out of 52-card deck is:
$|S| = C(52,5) = 2,598,960$
Therefore, the probability of an event of choosing "three of a kind" is:
$p(E) = \frac{|E|}{|S|} = \frac{54{,}912}{2{,}598{,}960} \approx 0.0211285$

c) If all of the cards in 5-card hand are of the same suit, they they all must be of different ranks. Hence, since we are choosing 5 out of 13 ranks, and 1 out of 4 suits, the number of choosing 5-card hand in which all the cards are of the same suit is:
$|E| = C(13,5) * C(4,1) = 1,287 * 4 = 5,148$
The number of possibilities of choosing 5-card hand out of 52-card deck is:
$|S| = C(52,5) = 2,598,960$
Therefore, the probability of an event of choosing 5 cards of the same suit is:
$p(E) = \frac{|E|}{|S|} = \frac{5{,}148}{2{,}598{,}960} \approx 0.00198$

d) The first two cards are of the same rank and any suits. Hence, they can be chosen in C(13,1)*C(4,2) ways (since we are choosing 1 of 13 ranks, and then 2 out of 4 suits). The remaining 3 cards all need to be of different ranks. Hence, we need to choose 3 out of remaining 12 ranks, and there are C(12,3) ways of doing that. Additionally, each of the remaining 3 cards cam be of any of the 4 suits, so the number of possible suits combinations of 3 remaining cards is C(4,1)*C(4,1)*C(4,1). Therefore, the number of ways of choosing the 5-card hand that is "two of a kind" is:
$|E| = C(13,1)*C(4,2)*C(12,3)*C(4,1)*C(4,1)*C(4,1) = 13*6*220*4*4*4 = 1,098,240$
The number of possibilities of choosing 5-card hand out of 52-card deck is:
$|S| = C(52,5) = 2,598,960$
Therefore, the probability of an event of choosing "two of a kind" is:
$p(E) = \frac{|E|}{|S|} = \frac{1{,}098{,}240}{2{,}598{,}960} \approx 0.42257$

b) Exercise 6.2.4

a) A: there is at least one club in the 5-card hand

$\bar{A}$: there are no clubs in the 5-card hand

There are 13 clubs in the deck, so if we know that there are no clubs, we are choosing 5 out of 52-13=39 cards, hence:

$|\bar{A}| = C(39, 5) = 575,757$

The number of ways of choosing 5-card hand out of 52 card deck:

$|S| = C(52, 5) = 2,598,960$

Therefore, the probability of the event $\bar{A}$ is:

$p(\bar{A}) = \frac{|\bar{A}|}{|S|} = \frac{575{,}757}{2{,}598{,}960} \approx 0.22153$

Consequently, the probability of the event $A$, that there is at least one club is:

$p(A) = 1 - p(\bar{A}) \approx 0.77847$

b) B: The hand has at least two cards with the same rank

$\bar{B}$: There are no two cards with the same rank

If there are no two cards with the same rank, then every card must be of different rank. Hence, we are choosing 5 out of 13 ranks, and additionally, each card can be of any of the 4 suits. Therefore, the number of ways of choosing 5-card hand such that there are no two cards of the same rank is:

$|\bar{B}| = C(13, 5) * C(4, 1)^5 = 1,317,888$

The number of ways of choosing 5-card hand out of 52 card deck:

$|S| = C(52, 5) = 2,598,960$

Therefore, the probability of the event $\bar{B}$ is:

$p(\bar{B}) = \frac{|\bar{B}|}{|S|} = \frac{1{,}317{,}888}{2{,}598{,}960} \approx 0.507083$

Consequently, the probability of the event $B$, that there are no two cards with the same rank is:

$p(B) = 1 - p(\bar{B}) = 0.492917$

c) E:The hand has exactly one club or exactly one spade

$E_1$: The hand has exactly one club

$E_1$: The hand has exactly one spade

If the hand has exactly one spade, then the card that is the spade can be chosen in 13 ways since it can be of any of 13 ranks. The remaining 4 cards can be any of the 39 cards (52-13), which means that we are choosing 4 out of 39 cards. Hence, the number of ways of choosing a hand that has exactly one club is:

$|E_1| = 13 * C(39, 4) = 1,069,263$

The number of ways of choosing a hand that has exactly one spade is the same as the number of ways of choosing a hand that has exactly one club:

$|E_2| = 13 * C(39, 4) = 1,069,263$

However, $E_1$ and $E_2$ are not mutually exclusive events, therefore:

$p(E) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

The number of ways of choosing a hand that has one club and one spade is:

$|E_1 \cap E_2| = 13 * 13 * C(26, 3) = 439, 400$

The number of ways of choosing 5-card hand out of 52 card deck:

$|S| = C(52, 5) = 2, 598, 960$

Therefore, the probability of event E that the hand has exactly one club or exactly one spade is:

$p(E) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} = \frac{1{,}069{,}263 + 1{,}069{,}263 - 439{,}400}{2{,}598{,}960} \approx 0.65377$

d) E: The hand has at least one club or at least one spade

$\bar{E}$: The hand has no clubs and no spades

If the hand has no clubs and no spades, it means that we are choosing 5 out of 26 cards (since there are 13 clubs and 13 spades, and 52-13-13=26). Therefore, the number of ways of choosing a hand that has no clubs and no spades is:

$|\bar{E}| = C(26, 5) = 65, 780$

The number of ways of choosing 5-card hand out of 52 card deck:

$|S| = C(52, 5) = 2, 598, 960$

Therefore, the probability of the event E, that the hand has at least one club or at least one spade is:

$p(E) = 1 - p(\bar{E}) = 1 - \frac{|\bar{E}|}{|S|} = 1 - \frac{65{,}780}{2{,}598{,}960} \approx 0.97469$

## Question 8

a) Exercise 6.3.2

a) $|A| = 1 * 6! = 6!$ (because b can be expressed in only 1 way, and the remaining 6 letters can be expressed in 6! ways)

$|B| = 5! * (6+5+4+3+2+1)$ (because if b is the first letter, then c can take any of the 6 spots, and the remaining 5 letters can take any of the 5 spots, so they can be expressed in 5! ways; then, if b is the second letter, then c can take any of the 5 spots, and the remaining 5 letters can take any of the 5 spots, so they can be expressed in 5! ways, etc.; we use a product rule there)

$|C| = 5 * 4! = 5!$ (because 3 letters "def" can be placed among 7 spots in 5 ways, and the remaining 4 letters can be expressed in 4! ways)

$|S| = 7!$ (because there are 7! ways of placing 7 letters among 7 spots)

$p(A) = \frac{6!}{7!} = \frac{720}{5040} = \frac{1}{7}$
$p(B) = \frac{5!*(6+5+4+3+2+1)}{7!} = \frac{2520}{5040} = \frac{1}{2}$
$p(C) = \frac{5!}{7!} = \frac{120}{5040} = \frac{1}{42}$

b) $p(A|C) = \frac{p(A \cap C)}{p(C)}$

$A \cap C$: b is in the middle and "def" occur together

$|A \cap C| = 1 * 2 * 3! = 12$ (b can be expressed in only 1 way, "def" can be expressed in 2 ways (it can be before or after letter b), and the remaining 3 letters can be expressed in 3! ways)

$p(A \cap C) = \frac{12}{7!} = \frac{1}{420}$
$p(A|C) = \frac{p(A \cap C)}{p(C)} = \frac{1}{420} : \frac{1}{42} = \frac{1}{10}$

c) $p(B|C) = \frac{p(B \cap C)}{p(C)}$

$B \cap C$: c appears to the right of b and "def" occur together

$|B \cap C| = 3!(4+3+2+1)$ (because if b is the first letter, then c can take any of the 4 spots after the b (we treat "def" as one letter, since it can be expressed in only one way, and can only change the spots), and the remaining 3 letters can be expressed in 3! ways; if b is the second letter, then c can take any out of 3 spots after the b, and the remaining 3 letters can be expressed in 3! ways, etc; we use a product rule there)

$p(B \cap C) = \frac{3!(4+3+2+1)}{7!} = \frac{60}{5040} = \frac{1}{84}$
$p(B|C) = \frac{p(B \cap C)}{p(C)} = \frac{1}{84} : \frac{1}{42} = \frac{1}{2}$

d) $p(A|B) = \frac{p(A \cap B)}{p(B)}$

$A \cap B$: b is in the middle and c appears to the right of b

$|A \cap B| = 5! * 3 = 360$ (b can be expressed in only 1 way, so c can take any of the 3 spots after b, and the remaining 5 letters can be expressed in 5! ways)

$p(A \cap B) = \frac{360}{7!} = \frac{1}{14}$
$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1}{14} : \frac{1}{2} = \frac{1}{7}$

e) Events A and B are independent since $p(A \cap B) = \frac{1}{14} = \frac{1}{7} * \frac{1}{2} = p(A) * p(B)$. Events B and C are independent since $p(B \cap C) = \frac{1}{84} = \frac{1}{2} * \frac{1}{42} = p(A) * p(B)$. Events A and C aren't independent since $p(A \cap C) = \frac{1}{420} \neq \frac{1}{7} * \frac{1}{42} = p(A) * p(B)$

b) Exercise 6.3.6

b) A: first 5 flips come up heads
B: last 5 flips come up tails
$p(A \cap B) = p(A) * p(B) = (\frac{1}{3})^5 * (\frac{2}{3})^5 = \frac{1}{243} * \frac{32}{243} = \frac{32}{59049} \approx 0.00054$

c) A: first flip come up heads
B: rest of the flips come up tails
$p(A \cap B) = p(A) * p(B) = \frac{1}{3} * (\frac{2}{3})^9 = \frac{1}{3} * \frac{512}{19683} = \frac{512}{59049} \approx 0.00867$

c) Exercise 6.4.2

a) B: a dice is fair
$\bar{B}$: a dice isn't fair
A: we got 4,3,6,6,5,5 after rolling the randomly chosen dice 6 times
$p(B|A) = \frac{p(A|B)*p(B)}{p(A|B)*p(B)+p(A|\bar{B})*p(\bar{B})} =?$
$p(B) = \frac{1}{2}$
$p(\bar{B}) = \frac{1}{2}$
$p(A \cap B) = \frac{1}{2} * (\frac{1}{6})^6 = \frac{1}{93312}$ (since the probability of choosing fair dice is 0.5, and the probability of getting 4,3,6,6,5,5 is $(\frac{1}{6})^6$, as each value has the same probability of occurring)
$p(A \cap \bar{B}) = \frac{1}{2} * 0.15^4 * 0.25^2 = \frac{81}{5120000}$ (since the probability of choosing not fair dice is 0.5, the probability of getting 4,3,5,5 is $0.15^4$, and the probability of getting 6,6 is $0.25^2$)
$p(A|B) = \frac{p(A \cap B)}{p(B)} = (\frac{1}{6})^6$
$p(A|\bar{B}) = \frac{p(A \cap \bar{B})}{p(\bar{B})} = 0.15^4 * 0.25^2$
$p(B|A) = \frac{(\frac{1}{6})^6 * \frac{1}{2}}{(\frac{1}{6})^6 * \frac{1}{2} + 0.15^4 * 0.25^2 * \frac{1}{2}} \approx 0.4038$

**Question 9**

a) Exercise 6.5.2

  a) range of A $= \{0, 1, 2, 3, 4\}$ (since we can have 0, 1, 2, 3 or 4 aces in 5-card hand)
  b) distribution over A:
  $\{(0, \frac{C(48,5)}{C(52,5)}), (1, \frac{C(4,1)*C(48,4)}{C(52,5)}), (2, \frac{C(4,2)*C(48,3)}{C(52,5)}), (3, \frac{C(4,3)*C(48,2)}{C(52,5)}), (4, \frac{C(4,4)*C(48,1)}{C(52,5)})\}$

b) Exercise 6.6.1

  a) G: number of girls chosen
  range of G: $\{0, 1, 2\}$
  $p(G = 0) = \frac{C(3,2)}{C(10,2)}$ (if there are no girls, then we are choosing 2 representatives out of 3 boys and there are C(3,2) ways of doing that; $|S| = C(10, 2)$ since we are choosing 2 representatives out of 10 people)
  $p(G = 1) = \frac{C(3,1)*C(7,1)}{C(10,2)}$ (if there is only 1 girl, then the other representative is a boy, so we are choosing 1 out of 7 girls, and 1 out of 3 boys)
  $p(G = 2) = \frac{C(7,2)}{C(10,2)}$ (since there are only 2 girls, we are choosing 2 out of 7 girls)
  Therefore:
  $E[G] = 0 * \frac{C(3,2)}{C(10,2)} + 1 * \frac{C(3,1)*C(7,1)}{C(10,2)} + 2 * \frac{C(7,2)}{C(10,2)} = \frac{7}{5}$

c) Exercise 6.6.4

  a) X: the square of the number on the die
  range of X: $\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2\} = \{1, 4, 9, 16, 25, 36\}$
  it's a fair die, so the possibility of rolling out every number is $\frac{1}{6}$, therefore
  $E[X] = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{1}{6} * 91 = \frac{91}{6} \approx 15.16667$
  b) Y: the square of the number of heads
  range of Y: $\{0^2, 1^2, 2^2, 3^3\} = \{0, 1, 4, 9\}$
  $p(Y = 0) = \frac{1}{8}$ (since there is only one outcome (T,T,T) that the number of heads is 0, and $|S| = 8$)
  $p(Y = 1) = \frac{3}{8}$ (since there are 3 outcomes $\{(H, T, T), (T, H, T), (T, T, H)\}$ that the number of heads is 1, and $|S| = 8$)
  $p(Y = 4) = \frac{3}{8}$ (since there are 3 outcomes $\{(H, H, T), (H, T, H), (T, H, H)\}$ that the number of heads is 2, and $|S| = 8$)
  $p(Y = 9) = \frac{1}{8}$ (since there is only one outcome (H,H,H) that the number of heads is 0, and $|S| = 8$)
  Therefore:
  $E[Y] = \frac{1}{8} * 0 + \frac{3}{8} * 1 + \frac{3}{8} * 4 + \frac{1}{8} * 9 = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$

d) Exercise 6.7.4

a) $X_j$: if the j-th kid is given the right coat $X_j = 1$, otherwise $X_j = 0$

X: number of children who get his or her own coat

The probability of giving the coat to the right child is the same every time, so

$E[X_j] = 0 * \frac{9}{10} + 1 * \frac{1}{10} = \frac{1}{10}$

$E[X_j] = E[X_1] = E[X_2] = ... = E[X_{10}]$

Therefore, the expected number of kids who got their coat is:

$E[X] = E[X_1] + E[X_2] + ... + E[X_{10}] = 10E[X_j] = 10 * \frac{1}{10} = 1$

**Question 10**

a) Exercise 6.8.1

a) $b(k;n,p) = C(n,k) * p^k * q^{n-k}$
$k = 2$
$n = 100$
$p = \frac{1}{100}$
Therefore, the probability that out of 100 circuit boards made exactly 2 have defects is:
$b(2;100,\frac{1}{100}) = C(100,2) * (\frac{1}{100})^2 * (\frac{99}{100})^{98}$

b) E: out of 100 circuit boards made at least 2 have defects
$\bar{E}$: out of 100 circuit boards made at 0 or 1 has defects
The probability that out of 100 circuit boards made at 0 has defects:
$b(0;100,\frac{1}{100}) = C(100,0) * (\frac{1}{100})^0 * (\frac{99}{100})^{100} = (\frac{99}{100})^{100}$
The probability that out of 100 circuit boards made 1 has defects:
$b(1;100,\frac{1}{100}) = C(100,1) * (\frac{1}{100})^1 * (\frac{99}{100})^{99} = (\frac{99}{100})^{99}$
The probability that out of 100 circuit boards made at 0 or 1 has defects:
$p(\bar{E}) = (\frac{99}{100})^{100} + (\frac{99}{100})^{99}$
The probability that out of 100 circuit boards made at least 2 have defects:
$p(E) = 1 - p(\bar{E}) = 1 - (\frac{99}{100})^{100} - (\frac{99}{100})^{99}$

c) Expected number of circuit board with defects out of 1 made:
$E[D_j] = 0 * \frac{99}{100} + 1 * \frac{1}{100} = \frac{1}{100}$
Expected number of circuit board with defects out of 100 made:
$E[D] = E[D_1] + E[D_2] + ... + E[D_{100}] = 100 * E[D_j] = 100 * \frac{1}{100} = 1$

d) E: out of 50 batches at least 1 batch has defects
$\bar{E}$: out of 50 batches 0 has defects
The probability that out of 50 batches 0 has defects:
$p(\bar{E}) = b(0;50,\frac{1}{100}) = C(50,0) * (\frac{1}{100})^0 * (\frac{99}{100})^{50} = (\frac{99}{100})^{50}$
The probability that out of 50 batches at least 1 has defects:
$p(E) = 1 - (\frac{99}{100})^{50}$
Q1 Answer: The probability has changed. When we calculated the probability of at least 2 out of 100 circuit boards having defects, we obtained $1 - (\frac{99}{100})^{100} - (\frac{99}{100})^{99} \approx 0.26424$. Then, when we calculated the probability of at least 1 out of 50 batches having defects, we obtained $1 - (\frac{99}{100})^{50} \approx 0.39499$. It means that when producing circuit boards in batches, the probability of at least 1 out of 50 batches (or in other words at least 2 out of 100 circuit boards) having defects is greater, than the probability of at least 2 out of 100 circuit boards having defects when they are produced separately.
Expected number of batches with defects out of 1 made:

$E[D_i] = \frac{1}{100}$
Expected number of batches with defects out of 50 made:
$E[D] = E[D_1] + ... + E[D_{50}] = 50 * \frac{1}{100} = 0.5$
Q2 Answer: Expected number of batches with defects out of 50 batches made is 0.5 batch, which is equal to 1 board. The result is the same as the one calculated for expected number of boards with defects out of 100 boards made. Therefore, the expected number of boards with defect didn't change, and is still 1.

b) Exercise 6.8.3

b) E: the conclusion is incorrect (number of heads $\geq 4$)
$\bar{E}$: the conclusion is correct (number of heads $< 4$)
$p(\bar{E}) = p(k = 0) + p(k = 1) + p(k = 2) + p(k = 3)$
$p(k = 0) = C(10, 0) * 0.3^0 * 0.7^{10} = 0.7^{10}$
$p(k = 1) = C(10, 1) * 0.3^1 * 0.7^9 = 3 * 0.7^9$
$p(k = 2) = C(10, 2) * 0.3^2 * 0.7^8 = 4.05 * 0.7^8$
$p(k = 3) = C(10, 3) * 0.3^3 * 0.7^7 = 3.24 * 0.7^7$
$p(\bar{E}) = 0.7^{10} + 3 * 0.7^9 + 4.05 * 0.7^8 + 3.24 * 0.7^7 \approx 0.64961$
$p(E) = 1 - p(\bar{E}) \approx 1 - 0.64961 \approx 0.35039$
Answer: Probability of incorrect conclusion is approx. 0.35039