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**Question 5**

a) Prove that for any positive integer $n$, 3 divides $n^3 + 2n$ leaving no remainder.

PROOF:

By induction on n

P(n): $n^3 + 2n$

1) BASE CASE (for n=1)

P(1): $(1)^3 + 2 * (1) = 1 + 2 = 3 =$

Since 3 divides 3 leaving no remainder (3:3=1), we have proven that for n=1,P(n) is true.

2) INDUCTIVE STEP

Suppose that for any integer $k \geq 1$, $k^3 + 2k$ is divisible by 3 (so it can be expressed as $k^3 + 2k = 3m$, where $m$ is an integer). We will show that under this assumption $(k + 1)^3 + 2(k + 1)$ is also divisible by 3.

Starting form the left side of the equality to be proven:

$(k + 1)^3 + 2(k + 1) =$

$= (k^3 + 3k^2 + 3k + 1) + 2(k + 1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= (k^3 + 2k) + 3(k^2 + k + 1)$

$= 3m + 3(k^2 + k + 1)$ (by inductive hypothesis)

$= 3(m + k^2 + k + 1)$

As m and k are integers, and the sum of integers results in integer, $(m + k^2 + k + 1)$ is also an integer. Therefore, we have proven that $(k + 1)^3 + 2(k + 1)$ can be expressed as 3 times an integer, which means that it is divisible by 3.

Therefore, we have proven that for every $k \geq 1$, if $k^3 + 2k$ is divisible by 3 leaving no remainder (so it can be expressed an integer multiplied by 3), then it's also true that $(k + 1)^3 + 2(k + 1)$ is divisible by 3 leaving no remainder (it can also be expressed as integer multiplied by 3).

Hence, for every positive integer n, 3 divides $n^3 + 2n$ leaving no remainder. ■

b) Prove that any positive integer n ($n \geq 2$) can be written as a product of primes

PROOF:

By strong induction on n.

P(n): n such that ($n \geq 2$) can be written as a product of primes

1) BASE CASE (for n=2)

Since 2 is a prime number, then it's already a product of one prime number (2). Therefore, for n=2, P(n) is true.

2) INDUCTIVE STEP

Suppose that for every $k \geq 2$, any integer j such that $2 \leq j \leq k$, it's true that j can be expressed as a product of prime numbers. We will show that k+1 can also be expressed as product of prime numbers.

Let's consider two cases:

1) k+1 is prime
   If k+1 is prime, then it's already a product of one prime number (which is k+1).
2) k+1 isn't prime
   If k+1 isn't prime, it's composite, so it can be expressed as a product of two integers a and b, such that $a \geq 2$ and $b \geq 2$. Therefore, we have:
   $k+1 = a * b$
   Hence, a and b can be expressed as:
   $a = \frac{k+1}{b}$
   $b = \frac{k+1}{a}$
   Also, we know that $a \geq 2$ and $b \geq 2$, so
   $a = \frac{k+1}{b} < k+1$
   $a < k+1$
   $a \leq k$ (since the largest value that a can take without violating $a < k+1$ is k)
   and
   $b = \frac{k+1}{a} < k+1$
   $b < k+1$
   $b \leq k$ (since the largest value that b can take without violating $b < k+1$ is k)
   Therefore, $2 \leq a \leq k$ and $2 \leq b \leq k$. It means that the inductive hypothesis (for every $k \geq 2$, any integer j such that $2 \leq j \leq k$, it's true that j can be expressed as a product of prime numbers) can be applied.
   Hence, a and b can both be expressed as products of primes
   $a = prime_1 * prime_2 * ... * prime_x$
   $b = prime_1 * prime_2 * ... * prime_y$
   Consequently:
   $k+1 = a * b = prime_{a1} * prime_{a2} * ... * prime_{ax} * prime_{b1} * prime_{b2} * ... * prime_{bx}$
   Meaning that k+1 can be expressed as product of primes

Therefore, we have proven that no matter if k+1 is prime or not, it can always be expressed as a product of primes.

Hence, we have proven that for every $k \geq 2$, if P(j) then P(k+1). Therefore, any positive integer n, such that $n \geq 2$, can be written as product of primes. ■

**Question 6**

a) Exercise 7.4.1

$P(n) : \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

a) $P(3) : \sum_{j=1}^{3} j^2 = \frac{3(3+1)(2*3+1)}{6}$

$1^2 + 2^2 + 3^2 = \frac{3*4*7}{6}$

$1 + 4 + 9 = \frac{84}{6}$

$14 = 14$

$L = R$

Since the equation on the left-hand side is equal to the equation on the right-hand side, we have proven that for n=3, P(n) is true.

b) $P(k) : \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

c) $P(k+1) : \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k^2+3k+2)(2k+3)}{6} = \frac{2k^3+9k^2+13k+6}{6}$

d) In BASE CASE we should prove that P(1) is true

e) In INDUCTIVE STEP we should prove that for any positive integer k, P(k) implies P(k+1)

f) INDUCTIVE HYPOTHESIS is P(k)

g) PROOF:

By induction on n

1) BASE CASE (for n=1)

P(1): $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2+1)}{6}$

$1^2 = \frac{1*2*3}{6}$

$1 = \frac{6}{6}$

$1 = 1$

$L = R$

Since the equation on the left-hand side is equal to the equation on the right-hand side, we have proven that for n=1, P(n) is true.

2) INDUCTIVE STEP

Suppose that for every positive integer k, $P(k) : \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$ is true. We will show that $P(k+1) : \sum_{j=1}^{k+1} j^2 = \frac{2k^3+9k^2+13k+6}{6}$ is also true.

Starting from the left side of the equality to be proven:

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ (by inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}
\end{aligned}$$

$= \frac{k(2k^2+3k+1)+(6k^2+12k+6)}{6}$

$= \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$

$= \frac{2k^3+9k^2+13k+6}{6}$

We have proven that for any positive integer k, if P(k), then P(k+1). Hence, for every natural number n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. ■

b) Exercise 7.4.3

P(n): $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

PROOF:

By induction on n

1) BASE CASE (for n=1)

P(1): $\sum_{j=1}^{1} \frac{1}{j^2} \leq 2 - \frac{1}{1}$

$\frac{1}{1} \leq 2 - 1$

$1 \leq 1$

Since it's true that $1 \leq 1$, we have proven that for n=1, it's true that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

2) INDUCTIVE STEP

Suppose that for every integer $k \geq 1$, P(k): $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$ is true. We will show that under this assumption P(k+1): $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ is also true.

$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$

$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ (by inductive hypothesis)

$\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$ (by hint)

$= 2 - \frac{(k+1)}{k(k+1)} + \frac{1}{k(k+1)}$

$= 2 + \frac{(-k-1+1)}{k(k+1)}$

$= 2 - \frac{k}{k(k+1)}$

$= 2 - \frac{1}{k+1}$

Therefore, we have proven that for every integer $k \geq 1$, if P(k) then P(k+1). Hence, for every natural number n, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ ■

c) Exercise 7.5.1

P(n): 4 evelny divides $3^{2n} - 1$

PROOF:

By induction on n

1) BASE CASE (n=1)

P(1): $3^{2*1} - 1 = 9 - 1 = 8$

Since 4 evenly divides 8 ($8 : 4 = 2$), then we have proven that for n=1 P(n) is true.

2) INDUCTIVE STEP

Suppose that for any positive integer k, P(k): 4 evenly divides $3^{2k} - 1$ is true. We will show that under this assumption P(k+1): 4 evenly divides $3^{2(k+1)} - 1$ is also true.

Since 4 evenly divides $3^{2k} - 1$, then P(k) can be expressed as 4 times an in integer, so $3^{2k} - 1 = 4m$, where m is an integer.

Starting on the left side of the equality to be proven

$$
\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 3^{2k} * 3^2 - 1 \\
&= 9(4m + 1) - 1 \text{ (by inductive hypothesis } 3^{2k} = 4m - 1) \\
&= 36m + 9 - 1 \\
&= 36m + 8 \\
&= 4(9m + 2)
\end{aligned}
$$

Since m is an integer, then (9m+2) is also an integer. Hence, $3^{2(k+1)} - 1$ can be expressed as 4 times an integer, which means that $3^{2(k+1)} - 1$ is divisible by 4.

Therefore, we have proven that for every integer $k \geq 1$, if P(k) then P(k+1). Hence, for every natural number n, 4 evenly divides $3^{2n} - 1$ ■