NYU Tandon Bridge
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Question 3

a) Exercise 4.1.3

b) $f(x) = \frac{1}{x^2-4}$
Answer: f(x) is not a function $R \to R$, because x $\notin R$ (since we can't divide by zero, f(x) is not well defined for $x \neq 2$ and $x \neq -2$).

c) $f(x) = \sqrt{x^2}$
Answer: f(x) is a function $R \to R$, because for every real number x, f(x) produces a real number such that $x \geq 0$. Therefore, the range of f(x) is a set of all non-negative real numbers.

b) Exercise 4.1.5

b) $A = \{2, 3, 4, 5\}$, $f : A \to Z, f(x) = x^2$
$f(2) = 2^2 = 4$
$f(3) = 3^2 = 9$
$f(4) = 4^2 = 16$
$f(5) = 5^2 = 25$
range f(x)= $\{4, 9, 16, 25\}$

d) $f : \{0, 1\}^5 \to Z$, $x \in \{0, 1\}^5$, $f(x)$ is the number of 1s that occur in x
When there are no 1s: $f(00000) = 0$
When there is one 1, for example: $f(00001) = 1$
When there are two 1s, for example: $f(00011) = 2$
When there are three 1s, for example: $f(00111) = 3$
When there are four 1s, for example: $f(01111) = 4$
When there are five 1s, for example: $f(11111) = 5$
Therefore:
range f(x)=$\{0, 1, 2, 3, 4, 5\}$

h) $A = \{1, 2, 3\}$, $f : A \times A \to Z \times Z, f(x, y) = (y, x)$
$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
range f(x)=$\{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$

i) $A = \{1, 2, 3\}$, $f : A \times A \to Z \times Z, f(x, y) = (x, y + 1)$
$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
$f(1, 1) = (1, 2)$
$f(1, 2) = (1, 3)$
$f(1, 3) = (1, 4)$

$f(2,1) = (2,2)$
$f(2,2) = (2,3)$
$f(2,3) = (2,4)$
$f(3,1) = (3,2)$
$f(3,2) = (3,3)$
$f(3,3) = (3,4)$
range f(x)=$\{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$

l) $A = \{1,2,3\}$, $f : P(A) \to P(A)$, $X \subseteq A$, $f(X) = X - \{1\}$
$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
range f(x)=$\{\emptyset, \{2\}, \{3\}, \{2,3\}$

Question 4

I a) Exercise 4.2.2

c) $h : Z \to Z$, $h(x) = x^3$
Answer: h(x) is one-to-one (each x corresponds to distinct value of h(x)), but not onto (there is no integer x such that $x^3 = 2$).

g) $Z \times Z \to Z \times Z$, $f(x, y) = (x + 1, 2y)$
Answer: f(x,y) is one-to-one (each pair of x and y corresponds to distinct value of f(x,y)), but no onto (for example, (2,3) will never be obtained, since 2y, where y is an integer, will never produce 3).

k) $f : Z^+ \times Z^+ \to Z^+$, $f(x, y) = 2^x + y$
Answer: f(x,y) is neither one-to-one (not every pair of x and y corresponds to distinct value of f(x,y), for example f(2,2)=f(1,4)=6) nor onto (since the minimum of f(x,y) is f(1,1)=3, we will never obtain f(x,y)=1 or f(x,y)=2).

b) Exercise 4.2.4

b) Answer: Function f is neither one-to-one (for example, f(001)=f(101)=101) nor onto (for example, function f will never produce 001).

c) Answer: Function f is both one-to-one (there are no two strings that produce the same output of the function) and onto (function f can take any value from its target)

d) Answer: Function f is one-to-one (there are no two strings that produce the same output of the function), but not onto (for example, function f will never produce 1000, which is in its target)

g) Answer: Function f is neither one-to-one (for example, f({1,2})=f({2})={2}) nor onto (for example, it will never be true that f(X)={1})

II a) $Z \to Z^+$, one-to-one but not onto

For example: $f(x) = 2x + 1$, for $x > 0$, and $f(x) = -2x + 2$, for $x \leq 0$

(f(x) is not onto, because there is no x for which f(x)=1; f(x) is one-to-one, because there are no two elements in the domain which obtain the same result of f(x))

b) $Z \to Z^+$, onto but not one-to-one

For example: f(x)=$|x|$ +1

(f(x) is not one-to-one, because f(-2)=f(2)=3, but it is onto, because the range of f(x) is equal to the set of positive integers Z^+, which is also the target of f(x))

c) $Z \to Z^+$, one-to-one and onto

For example: $f(x) = 2x + 1$, for $x \geq 0$, and $f(x) = -2x$, for $x < 0$

(f(x) is one-to-one, because it results in odd positive integers for $x \geq 0$, and even positive integers for $x < 0$, which means that there are no two elements in the domain that obtain the same result for f(x); f(x) is also onto, because the range of f(x) is equal to its target, which is a set of all positive integers Z^+)

d) $Z \to Z^+$, neither one-to-one nor onto

For example: $f(x) = 2x + 2$, for $x \geq 0$, and $f(x) = -2x$, for $x < 0$

(f(x) is not one-to-one, because f(1)=f(-2)=4; f(x) is also not onto, because there is no x for which f(x)=1)

Question 5

a) Exercise 4.3.2

c) Answer: Function f, which is both one-to-one and onto, has a well-defined inverse: $f^{-1} = \frac{x-3}{2}$
(Since, when we switch the roles of x and y in $y = 2x + 3$, and then solve for y, we obtain $y = \frac{x-3}{2}$)

d) Answer: Function f, which isn't one-to-one (for example, f({1})=f({2})=1), doesn't have a well-defined inverse.

g) Answer: Function f, which is both one-to-one and onto, has a well-defined inverse: $f^{-1} = f$
(Since the output of f^{-1} is obtained by reversing the bits of original function f)

i) Answer: Function f, which is both one-to-one and onto, has a well-defined inverse: $f^{-1}(x, y) = (x - 5, y + 2)$
(Let's say that g=x+5 and h=y-2 (so f(x,y)=(g,h)). Since, when we switch the roles of x and g in g=x+5, and solve for g, we obtain g=x-5. Then, when we switch the roles of y and h in h=y-2, and solve for h, we obtain h=y+2).

b) Exercise 4.4.8

c) $f \circ h = f(h(x)) = 2 * (x^2 + 1) + 3 = 2x^2 + 5$

d) $h \circ f = h(f(x)) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$

c) Exercise 4.4.2

b) $(f \circ h)(52) = f(h(52))$
$h(52) = \lceil \frac{52}{5} \rceil = 11$
$f(11) = 11^2 = 121$
$(f \circ h)(52) = 121$

c) $(g \circ h \circ f)(4) = g(h(f(4)))$
$f(4) = 4^2 = 16$
$h(16) = \lceil \frac{16}{5} \rceil = 4$
$g(4) = 2^4 = 16$
$(g \circ h \circ f)(4) = 16$

d) $h \circ f = h(f(x)) = \left\lceil \frac{x^2}{5} \right\rceil$

d) Exercise 4.4.6

c) $(h \circ f)(010) = h(f(010))$
$f(010) = 110$
$h(110) = 111$
$(h \circ f)(010) = 111$

d) We have 3 bits to disposition, and we know that the first bit and the last bit are both 1s. Therefore, only the middle bit can change - it can be either 0 or 1, therefore:

   range of $(h \circ f) = \{101, 111\}$

e) $g \circ f = g(f(x))$

   range of f(x)={100,101,110,111}

   range of g(f(x))={001,101,011,111}

Extra credit question

Question 5

e) Exercise 4.4.4

c) Let f(x) be a function that is not one-to-one. We will prove that $g \circ f$ can't be one-to-one.

Since f(x) is one-to-one there are at least two elements in the domain that otain the same value of f(x):

$f(x) = f(x') = y$

Let's consider $g \circ f$ for x and x':

$g \circ f = g(f(x))$

$g(f(x)) = g(y)$

$g(f(x')) = g(y)$

Since $g(f(x)) = g(y)$ and $g(f(x')) = g(y)$ then $g(f(x)) = g(f(x')) = g(y)$

We have proven there exist x and x' such that g(f(x))=g(f(x')) is true, which means that $g \circ f$ is not one-to-one (since there exist at least two elements in the domain which obtain the same value of g(f(x))). ■

d) Let g(x) be function that is not one-to-one, for example if $g(x) = x^2$ for all $x \in R$, and f(x) be function that is one-to-one, for example $f(x) = \sqrt{x}$ for $x \geq 0$. We will prove that it is possible that $g \circ f$ is one-to-one.

Let's consider $g \circ f$:

$g \circ f = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2$

Since the domain of g(x) is R, and the domain of f(x) are $x \geq 0$, then the domain of g(f(x)) is also $x \geq 0$.

For domain restricted to $x \geq 0$, function $g \circ f = (\sqrt{x})^2$ is one-to-one, as there are no two elements in the domain, that obtain the same value of g(f(x)). ■