Chapter 15
The Cumulative Distribution Function

1 Definition and Examples

Cumulative distribution functions play a central role in probability theory. As we will see in this chapter, they can be used to create new probability distributions from old ones. Modeling different phenomena such as wind speed or the time for the next earthquake require different models. This is why it is critical to be able to create new probability distributions.

Cumulative distribution functions are especially useful when dealing with continuous random variables. The following definition applies to any random variable, but we will only study the continuous case.

- Let X be a random variable. The function

$$F(x) = P(X \leq x)$$

is called **the cumulative distribution function** or just **distribution function** of X.

Assume X is a continuous random variable with density f and support (a, b). Then, the distribution function F of X is

$$F(x) = P(X \leq x) = \int_a^x f(t)dt,$$

for x in (a, b). By the Fundamental Theorem of Calculus if f is continuous at x, then F is differentiable at x and

$$F'(x) = f(x).$$

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Hence, the distribution function determines the density and therefore the distribution of a continuous random variable.

Example 1 Let U be a uniform random on (0, 1). That is, the density of U is $f(u) = 1$ for u in (0, 1) and $f(u) = 0$ elsewhere.

We now compute F.

If $u \le 0$, then $F(u) = P(U \le u) = 0$ (i.e., U is always positive). If $u \ge 1$, then $F(u) = P(U \le u) = 1$ (i.e., U is always smaller than 1). If $0 < u < 1$, then

$$F(u) = \int_0^u f(x)dx = \int_0^u dx = u.$$

Summarizing the computations above we get

$$F(u) = 0 \text{ if } u \le 0$$

$$F(u) = u \text{ if } 0 < u < 1$$

$$F(u) = 1 \text{ if } u \ge 1$$

There are three features of the graph in Fig. 15.1 that are true for all distribution functions. We now state these without proof. Let F be the distribution function of a random variable X. Then, we have the following three properties:

- $\lim_{x \to -\infty} F(x) = 0$.
- F is an increasing function. That is, if $x_1 < x_2$ then $F(x_1) \le F(x_2)$.
- $\lim_{x \to +\infty} F(x) = 1$.

Example 2 Let T be an exponential random variable with rate λ. What is its distribution function?

The density of T is $f(t) = \lambda e^{-\lambda t}$ for $t > 0$. Thus, $F(t) = 0$ for $t \le 0$. For $t > 0$,

$$F(t) = \int_0^t f(x)dx = -e^{-\lambda x}\Big]_0^t = 1 - e^{-\lambda t}.$$

Summarizing,

$$F(t) = 0 \text{ for } t \le 0$$

$$F(t) = 1 - e^{-\lambda t} \text{ for } t > 0$$

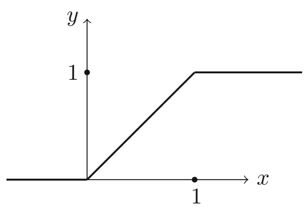

Fig. 15.1 This is the graph of the c.d.f. of a uniform random variable

2 Transformations of Random Variables

At this point we know relatively few different continuous distributions: uniform, exponential, and normal are the main distributions we have seen. In this section we will see a general method to obtain many more distributions from the known ones. We start with an example.

Example 3 Let U be a uniform random variable on $(0, 1)$. Define $X = U^2$. What is the probability density of X?

Since the support of U is $(0, 1)$ so is the support of $X = U^2$. Let F_X and F_U be the distribution functions of X and U, respectively.

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(U^2 \le x) \\ &= P(U \le \sqrt{x}) \\ &= F_U(\sqrt{x}). \end{aligned}$$

Recall that if the probability density f_X is continuous at x, then the distribution function F_X is differentiable at x and

$$\frac{d}{dx} F_X(x) = f_X(x).$$

Hence, for x in $(0, 1)$

$$\frac{d}{dx} F_X(x) = \frac{d}{dx} F_U(\sqrt{x}),$$

and therefore by the chain rule,

$$f_X(x) = \frac{1}{2\sqrt{x}} f_U(\sqrt{x}).$$

Since U is uniform on $(0, 1)$, $f_U(u) = 1$ for u in $(0, 1)$. Hence, $f_U(\sqrt{x}) = 1$ for x in $(0, 1)$. Thus,

$$f_X(x) = \frac{1}{2\sqrt{x}} \text{ for } x \in (0, 1).$$

- The preceding example gives a general method to compute the probability density of the transformed random variable. We first find the support of the transformed random variable. Then, we compute the distribution function of the transformed random variable. Assuming the distribution function is regular enough (which will always be the case for us) the probability density of the transformed variable is the derivative of the distribution function.

Example 4 (The Chi-Squared Distribution) Let Z be a standard normal random variable. What is the density of $Y = Z^2$?

The support of Z is $(-\infty, +\infty)$. Hence, the support of $Y = Z^2$ is $(0, +\infty)$.

Let $y > 0$,

$$\begin{aligned} F_Y(y) =& P(Y \leq y) \\ =& P(Z^2 \leq y) \\ =& P(-\sqrt{y} \leq Z \leq \sqrt{y}). \end{aligned}$$

Recall that the standard normal is centered around 0. Hence,

$$P(-\sqrt{y} \leq Z \leq \sqrt{y}) = 2P(0 \leq Z \leq \sqrt{y}).$$

Since

$$P(Z \leq \sqrt{y}) = \frac{1}{2} + P(0 \leq Z \leq \sqrt{y}),$$

$$\begin{aligned} F_Y(y) =& 2\left(P(Z \leq \sqrt{y}) - \frac{1}{2}\right) \\ =& 2F_Z(\sqrt{y}) - 1. \end{aligned}$$

We now take the derivative with respect to y, by the chain rule

$$f_Y(y) = \frac{1}{\sqrt{y}} f_Z(y).$$

Since the density of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

the probability density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} \text{ for } y > 0.$$

This is the density of the so-called Chi-Squared distribution with one degree of freedom.

Example 5 Let T be exponentially distributed with mean 1. What is the distribution of $X = \sqrt{T}$?

Since the support of T is $(0, +\infty)$, so is the support of X. Let $x > 0$,

$$\begin{aligned} F_X(x) =& P(X \leq x) \\ =& P(\sqrt{T} \leq x) \\ =& P(T \leq x^2) \\ =& F_T(x^2). \end{aligned}$$

By the chain rule, the density of X is

$$f_X(x) = \frac{d}{dx} F_T(x^2) = 2xe^{-x^2} \text{ for } x > 0.$$

Next we finally prove a property of normal random variables that we have already used many times.

Example 6 Let X be a normal random variable with mean μ and standard deviation σ. Show that $Y = \frac{X-\mu}{\sigma}$ is a standard normal random variable.

The support of X is $(-\infty, +\infty)$ and therefore so is the support of Y. We compute the distribution function of Y.

$$\begin{aligned} F_Y(y) =& P(Y \leq y) \\ =& P(\frac{X-\mu}{\sigma} \leq y) \\ =& P(X \leq \mu + \sigma y) \\ =& F_X(\mu + \sigma y). \end{aligned}$$

Therefore,

$$f_Y(y) = \frac{d}{dy} F_X(\mu + \sigma y) = f_X(\mu + \sigma y) \times \sigma.$$

Recall that the density of X is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Thus,

$$f_Y(y) = f_X(\mu + \sigma y) \times \sigma$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

This proves that Y is a standard normal random variable.

Problems

1. Graph the distribution function of a uniform random variable on $[-1,2]$.

2. Let X be a random variable with distribution function $F(x) = x^2$ for x in (0,1).

(a) What is $P(X < 1/3) =$?
(b) What is the expected value of X?

3. Consider a standard normal random variable Z. Use a normal table to sketch the graph of the distribution function of Z.

4. Let U be a uniform random variable on (0, 1). Define $X = \sqrt{U}$. What is the probability density of X?

5. Let T be exponentially distributed with mean 1. What is the probability density of $T^{1/3}$?

6. Let Z be a standard normal distribution. Find the probability density of $X = e^Z$. (X is called a **lognormal random variable**).

7. Let U be uniform on (0, 1). Find the probability density of $Y = \ln(1 - U)$.

8. Let T be exponentially distributed with rate λ. Find the probability density of $T^{1/a}$ where $a > 0$. ($T^{1/a}$ is called a **Weibull random variable** with parameters a and λ).

9. Let X be a continuous random variable, let $a > 0$ and b be two real numbers, and let $Y = aX + b$.

(a) Show that

$$f_Y(y) = \frac{1}{a} f_X(\frac{y-b}{a}).$$

(b) Show that if X is normally distributed, then so is Y. With what parameters?
(c) If X is exponentially distributed, is $Y = aX + b$ also exponentially distributed?

3 Sample Maximum and Minimum

Another important application of cumulative distribution functions is the computation of the distribution of the maximum and minimum of a random sample.

Example 7 Assume that T_1 and T_2 are two independent exponentially distributed random variables with rates λ_1 and λ_2, respectively. Let T be the minimum of T_1 and T_2, what is the distribution of T?

First observe that since the support of T_1 and T_2 is $(0, +\infty)$ so is the support of T. Let F be the distribution function of T. Let $t > 0$,

$$F(t) = P(T \le t) = P(\min(T_1, T_2) \le t).$$

In order to achieve $\min(T_1, T_2) \le t$ there are three possibilities, $T_1 < t$ and $T_2 > t$ or $T_1 > t$ and $T_2 < t$ or $T_1 < t$ and $T_2 < t$. This is why it is better to compute the probability of the complement. That is, $P(\min(T_1, T_2) > t)$. Observe that $\min(T_1, T_2) > t$ if and only if $T_1 > t$ and $T_2 > t$. Thus, since we are assuming that T_1 and T_2 are independent,

$$\begin{aligned} F(t) =& 1 - P(T_1 > t)P(T_2 > t) \\ =& 1 - (1 - F_1(t))(1 - F_2(t)), \end{aligned}$$

where F_1 and F_2 are the distribution functions of T_1 and T_2, respectively. Using the distribution function given in Example 2,

$$F(t) = 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} = 1 - e^{-(\lambda_1 + \lambda_2)t} \text{ for } t > 0.$$

Note that F is the distribution function of an exponential random variable with rate $\lambda_1 + \lambda_2$. Therefore, the minimum of two independent exponential random variables is also exponentially distributed and its rate is the sum of the two rates.

Next we look at the maximum of three uniform random variables.

Example 8 Let U_1, U_2, and U_3 be three independent random variables uniformly distributed on [0,1]. Let M be the maximum of U_1, U_2, U_3. What is the density of M?

The maximum of three numbers in $(0, 1)$ is also in $(0, 1)$. Hence, the support of M is $(0, 1)$. We first compute the distribution function of M and then differentiate to get the density. Let F_M be the distribution function of M. Then,

$$F_M(x) = P(M \le x) = P(\max(U_1, U_2, U_3) \le x).$$

Note that $\max(U_1, U_2, U_3) \le x$ if and only if $U_1 \le x$, $U_2 \le x$, and $U_3 \le x$. Thus, due to the independence of the U_i,

$$F_M(x) = P(U_1 \le x)P(U_2 \le x)P(U_3 \le x).$$

Since U_1, U_2, and U_3 all have the same distribution they have the same distribution function. Recall from Example 1 that $P(U \leq x) = x$ for $0 < x < 1$. Hence,

$$F_M(x) = x^3 \text{ for } 0 < x < 1.$$

Thus, the density of M that we denote by f_M is

$$f_M(x) = \frac{d}{dx} F_M(x) = 3x^2 \text{ for } x \text{ in } (0, 1).$$

Observe that the maximum of uniform random variables is not uniform!

Problems

10. Assume that waiting times for buses 5 and 8 are exponentially distributed with means 10 and 20 min, respectively. I can take either bus so I will take the first bus that comes.

(a) Compute the probability that I will have to wait at least 15 min.
(b) What is the mean time I will have to wait?

11. Consider a circuit with two components in parallel. Assume that both components have independent exponential lifetimes with means 1 and 2 years, respectively.

(a) What is the probability that the circuit lasts more than 3 years?
(b) What is the expected lifetime of the circuit?

12. Assume that T_1 and T_2 are two independent exponentially distributed random variables with rates λ_1 and λ_2, respectively. Let M be the maximum of T_1 and T_2, what is the density of M?

13. Let $U_1, U_2, \ldots, U_n$ be n i.i.d. uniform random variables on $(0, 1)$.

(a) Find the probability density of the maximum of the U_i.
(b) Find the probability density of the minimum of the U_i.

14. Let $X_1, X_2, \ldots, X_n$ be independent random variables with distribution functions $F_1, F_2, \ldots, F_n$, respectively. Let F_{max} and F_{min} be the distribution functions of the random variables $\max(X_1, X_2, \ldots, X_n)$ and $\min(X_1, X_2, \ldots, X_n)$, respectively.

(a) Show that $F_{max} = F_1 F_2 \ldots F_n$.
(b) Show that $F_{min} = 1 - (1 - F_1)(1 - F_2) \ldots (1 - F_n)$.

4 Simulations

Consider a continuous random variable X. Assume that the distribution function F_X of X is strictly increasing and continuous so that the inverse function F_X^{-1} is well defined. Let U be a uniform random variable on (0,1). Since F_X is an increasing function, $F_X^{-1}(U) \leq x$ if and only if $F_X(F_X^{-1}(U)) \leq F_X(x)$. Hence,

$$P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x)).$$

Since $F_X(x)$ is always in [0, 1] (why?), by Example 1 above

$$P(U \leq F_X(x)) = F_U(F_X(x)) = F_X(x).$$

Hence,

$$P(F_X^{-1}(U) \leq x) = F_X(x).$$

That is, the distribution function of $F_X^{-1}(U)$ is the same as the distribution function of X. This shows the following.

- Let X be a continuous random variable with a strictly increasing distribution function F_X. Let U be a uniform random variable on (0, 1). Then $F_X^{-1}(U)$ has the same distribution as X. That is, to simulate X it is enough to simulate a uniform random variable U and then compute $F^{-1}(U)$.

Random generators are computer programs that simulate uniform random variables. The remark above gives a recipe to go from a uniform to any distribution.

Example 9 A random generator gives us the following 10 random numbers,

$$0.38, 0.1, 0.6, 0.89, 0.96, 0.89, 0.01, 0.41, 0.86, 0.13.$$

Simulate 10 independent exponential random variables with rate 1.

By Example 2 we know that the distribution function F of an exponential random variable with rate 1 is

$$F(x) = 1 - e^{-x}.$$

We compute F^{-1}. If $y = 1 - e^{-x}$, then

$$x = -\ln(1 - y).$$

Thus, $F^{-1}(x) = -\ln(1 - x)$. We now compute $F^{-1}(x)$ for

$$x = 0.38, 0.1, 0.6, 0.89, 0.96, 0.89, 0.01, 0.41, 0.86, 0.13.$$

We get the following ten observations for ten independent exponential rate 1 random variables, 4.78, 1.05, 0.92, 2.21, 3.22, 2.21,4.6, 5.28, 1.97, 1.39.

Example 10 How do we simulate a standard normal distribution? In this case the distribution function is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

This is not an expression that is easy to use. Instead we use the normal table. For instance, if we want $F^{-1}(0.38)$ we are looking for z such that

$$P(Z \leq z) = 0.38.$$

Hence, $P(0 \leq Z \leq -z) = 0.12$. We read in the table $-z = 0.31$, that is, $z = -0.31$. Using the 10 random numbers from Example 6 we get the following ten observations for a standard normal distribution, -0.31, -1.28, 0.25, 1.22, 1.75, 1.22, -2.33, -0.23, 1.08, -1.13.

Example 11 Simulate a normal distribution X with mean $\mu = 5$ and variance $\sigma^2 = 4$. We know that if Z is a standard normal distribution, then $\mu + \sigma Z$ is a normal distribution with mean μ and variance σ^2. We can use the simulation of Z in Example 6 to get simulations of X. For instance, if $Z = -0.31$, then $X = 5 + (-0.31) \times 2 = 4.38$. Here are the 10 observations for a normal distribution X with mean $\mu = 5$ and variance $\sigma^2 = 4$. We get, 4.38, 2.44, 5.5, 7.44, 8.5, 7.44, 0.34, 4.54, 7.16, 2.74.

Problems

15. Using the random numbers 0.8147 0.9058 0.1270 0.9134 0.6324 0.0975 0.2785 0.5469 0.9575 0.9649, simulate 10 observations of a normal distribution with mean 3 and standard deviation 2.

16. Using the random numbers 0.8147 0.9058 0.1270 0.9134 0.6324 0.0975 0.2785 0.5469 0.9575 0.9649, simulate 10 observations of an exponential distribution with mean 2.