Computing in Tropical Geometry

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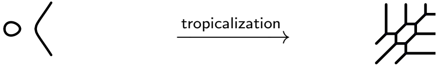
Computational geometry

Computational methods at the interface between

- Tropical Geometry
- Polyhedral Geometry
- Algebraic Geometry

Tropical degenerations of plane curves to planar graphs



- New computational approaches and combinatorial prospectives
- Triangulations, secondary fans, hyperplane arrangements, ...

polymake

OSCAR
SYMBOLIC TOOLS

Overview of topics

- Lecture 1:
  Tropical polynomials, curves, dual subdivisions.
- Lecture 2:
  Intersections of tropical curves, fields with valuation, tropicalization, tropical varieties.
- Lecture 3:
  A tropical count of real bitangents to plane quartic curves.
- Practical Session 1:
  Tropical curves in `Oscar`.
- Practical Session 2:
  Tropical quartic curves and their bitangents in `polymake`.

References:

- Joswig, *Essentials of Tropical Combinatorics*, Graduate Studies in Mathematics, AMS, 2021.
- Maclagan and Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, AMS, 2015.

Tropical polynomials

Tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, $x \oplus y = \min\{x, y\} \quad x \odot y = x + y$.
We abbreviate $\mathbb{T} = \mathbb{R} \cup \{\infty\}$.

Remark: $(\mathbb{R} \cup \{\infty\}, \oplus = \min, \odot)$ and $(\mathbb{R} \cup \{-\infty\}, \oplus = \max, \odot)$ are isomorphic.

A tropical polynomial $F : \mathbb{R}^n \to \mathbb{R}$ is the minimum of finitely many linear functions.

$$
\begin{aligned}
&F(X_1, X_2, \ldots, X_n) = a_u \odot X_1^{u_1} X_2^{u_2} \cdots X_n^{u_n} \oplus a_v \odot X_1^{v_1} X_2^{v_2} \cdots X_n^{v_n} \oplus \cdots \\
&= \min\{a_u + u_1 X_1 + u_2 X_2 + \cdots + u_n X_n,\ a_v + v_1 X_1 + v_2 X_2 + \cdots + v_n X_n,\ \cdots\}.
\end{aligned}
$$

Formally, a n-variate tropical polynomial F is a map from a finite set $S \subset \mathbb{Z}^n$ of exponents to $\mathbb{T}$, the set of coefficients.
Each exponent $\mathbf{u}$ is mapped to a coefficient $a_{\mathbf{u}}$.

Then F induces the evaluation function

$$
F(\mathbf{p}) = \bigoplus_{\mathbf{u} \in S} a_u \odot p_1^{u_1} \cdots p_n^{u_n} = \min\{a_u + \langle \mathbf{u}, \mathbf{p} \rangle \mid \mathbf{u} \in S\}.
$$

Tropical hypersurfaces

Given a tropical polynomial F, its hypersurface is

$$T(F) = \{\mathbf{x} \in \mathbb{R}^n \text{at which } \textit{the minimum is attained at least twice}\}.$$

Example: Cubic tropical polynomial in one variable

$$F(X) = a \odot X^3 \oplus b \odot X^2 \oplus c \odot X \oplus d = \min\{a + 3X, b + 2X, c + X, d\}$$

Suppose that $b - a \leq c - b \leq d - c$. Then

$$T(F) = \{b - a, c - b, d - c\}.$$

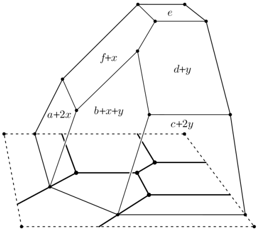
$$F(b - a) = \min\{3b - 2a, 3b - 2a, b - a + c, d\}$$

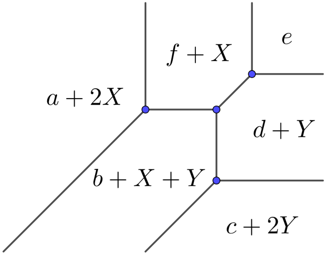
Tropical quadratic curves

$$F(X, Y) = a \odot X^2 \oplus b \odot XY \oplus c \odot Y^2 \oplus d \odot Y \oplus e \oplus f \odot X.$$

Each term defines an affine plane in $\mathbb{R}^3$.
The graph $F : \mathbb{R}^2 \to \mathbb{R}$ is the lower envelope of at most six planes in $\mathbb{R}^3$.

Suppose $b + f < a + d, \quad d + f < b + e, \quad b + d < c + f.$



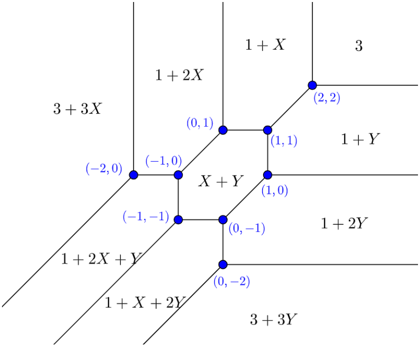


Picture from Maclagan–Sturmfels.

Tropical cubic curve

$$F(X,Y) = 3X^3 \oplus 1X^2Y \oplus 1XY^2 \oplus 3Y^3 \oplus 1X^2 \oplus XY \oplus 1Y^2 \oplus 1X \oplus 1Y \oplus 3$$

$$\min\{3+3X, 1+2X+Y, 1+X+2Y, 3+3Y, 1+2X, X+Y, 1+2Y, 1+X, 1+Y, 3\}$$

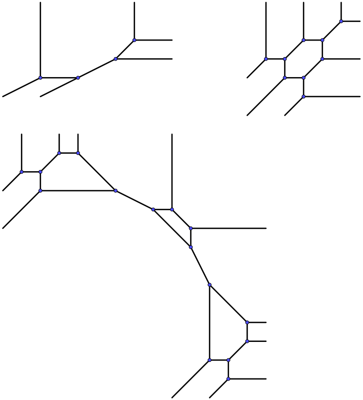


Tropical plane curves

$$F(X,Y) = \bigoplus_{\mathbf{u}\in S} a_{\mathbf{u}} X^{u_1} Y^{u_2},\ S \neq \emptyset,\ a_{\mathbf{u}} \in \mathbb{R}.$$

Proposition

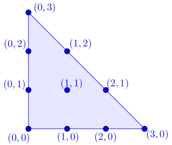
The tropical curve $T(F)$ is a finite graph embedded in $\mathbb{R}^2$. It has both bounded and unbounded edges, all slopes are rational, (and the graph satisfies a balancing condition around each vertex).



Newton polytopes and regular subdivisions

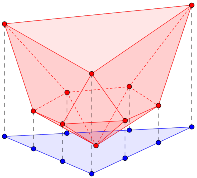
Given a tropical polynomial $F(X, Y) = \bigoplus_{\mathbf{u} \in S} a_{\mathbf{u}} X^{u_1} Y^{u_2}$, its Newton polytope is

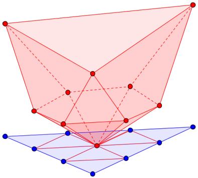
$$\mathrm{Newt}(F) = \mathrm{conv}(\mathbf{u} \; : \; a_{\mathbf{u}} \neq \infty).$$



$$F(X, Y) = 3X^3 \oplus 1X^2Y \oplus 1XY^2 \oplus 3Y^3 \oplus 1X^2 \oplus XY \oplus 1Y^2 \oplus 1X \oplus 1Y \oplus 3$$

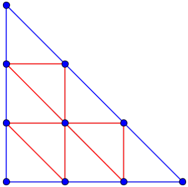
We consider the convex hull in $\mathbb{R}^3$ of the points $(\mathbf{u}, a_{\mathbf{u}})$. The projection of the lower faces on $\mathrm{Newt}(F)$ induces a subdivision of the Newton polytope.

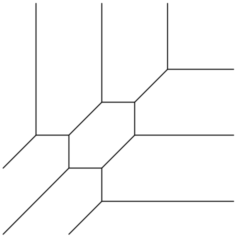




Regular subdivisions and tropical hypersurfaces

Tropical plane curves are dual to the regular subdivisions of their Newton polytopes induced by the coefficients.





Regular subdivisions and tropical hypersurfaces

Theorem

Let F be a n-variate tropical polynomial. Let $S(F)$ be the regular subdivision induced by the coefficients of F on the (lattice points) of $Newt(F)$.

There is a bijection between the k-dimensional cells of the tropical hypersurface $T(F)$ to the $(n-k)$-dimensional cells of $S(F)$ for $0 \leq k < n$.

The closure of the connected components of the complement of $T(F)$, called regions, correspond to the vertices of $S(F)$.

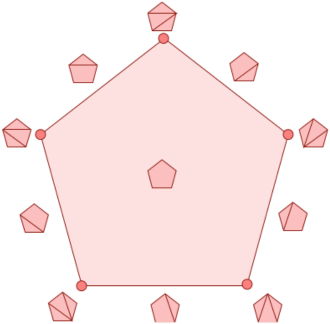
Regular subdivisions

- Let A be a point configuration in $\mathbb{R}^n$. A subdivision of A induced by a weight $\mathbf{w} \in \mathbb{R}^{|A|}$ is called regular.
- The set of weights inducing the same subdivision form a relatively open polyhderal cone in $\mathbb{R}^{|A|}$ called secondary cone.
- The collection of secondary cones of a A form the secondary fan of A.
- Triagulations, i.e., all cells in the subdivision are simplices, are maximal dimensional cones.

- Subdivisions and secondary cones are parameter spaces for tropical hypersurfaces.
- Subdivision $\rightarrow$ Combinatorial type
  Secondary cone $\rightarrow$ Metric properties
- They encode relevant geometry properties.
- Important computational and combinatorial tool for tropical methods.

Computing regular subdivisions

- Combinatorics and geometry play an important role in algorithms for fast enumeration and analysis of triangulations.

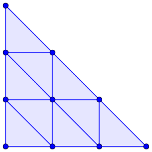


Picture from De Loera–Rambau–Santos.

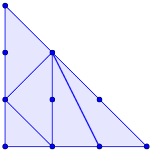
De Loera, Rambau, Santos, *Triangulations*, Structures for Algorithms and Application, Springer, 2010.
Gelfand, Kapranov, Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, 1994.

Smooth tropical curves

A tropical curve $T(F)$ is smooth if the regular subdivision induced by its coefficients is a unimodular triangulation, i.e., cells in the subdivision are simplices of minimal volume $\frac{1}{2}$.



Unimodular



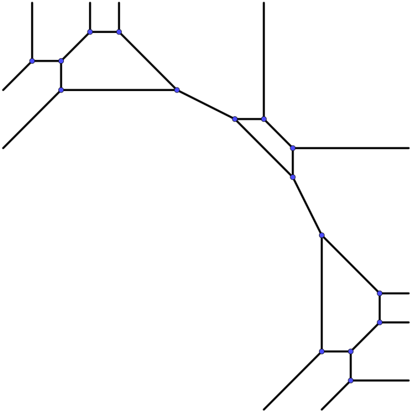
Not unimodular

The genus of $T(F)$ is the number of interior lattice points of $\mathrm{Newt}(F)$.

Planar graphs

Theorem

A smooth tropical plane curve $T(F)$ of genus g is a trivalent and connected planar graph with $2g - 2$ vertices and $3g - 3$ bounded edges.



Cycle of tropical plane cubics

Exercise 13, Section 1.3 of Maclagan–Sturmfels: show that the unique bounded region of a smooth cubic curve in the plane is an m-gon with $m \in \{3, 4, 5, 6, 7, 8, 9\}$.

- Vigeland '09: The boundary of the m-gon carries the group structure of the tropical elliptic curve.
- Katz-Markwig-Markwig '08: The lattice length of the cycle is the tropical j-invariant.

Thank you!