Computing in Tropical Geometry

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Overview of topics

- Lecture 1:
  Tropical polynomials, curves, dual subdivisions.
- Lecture 2:
  Intersections of tropical curves, fields with valuation, tropicalization, tropical varieties.
- Lecture 3:
  A tropical count of real bitangents to plane quartic curves.
- Practical Session 1:
  Tropical curves in `Oscar`.
- Practical Session 2:
  Tropical quartic curves and their bitangents in `polymake`.

References:

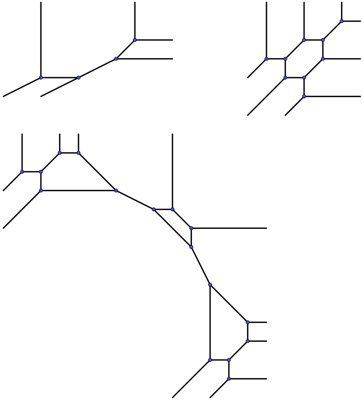
- Joswig, *Essentials of Tropical Combinatorics*, Graduate Studies in Mathematics, AMS, 2021.
- Maclagan and Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, AMS, 2015.

Degree

A tropical plane curve $T(F)$ has degree d if it is defined by a polynomial $F(X,Y) = \bigoplus_{u \in S} a_u X^{u_1} Y^{u_2}$ such that

$$\mathrm{Newt}(F) = \mathrm{conv}\big((0,0),(d,0),(0,d)\big).$$

$T(F)$ has d unbounded rays (with multiplicities) in each direction $(1,0)$, $(0,1)$ and $(-1,-1)$. If $T(F)$ is smooth, then these rays are exactly d.



Tropical plane curves

- Two general lines meet in one point.
- Two general points lie on a unique line.
- A general line and quadric meet in two points.
- Two general quadrics meet in four points.
- Five general points lie on a unique quadric.

Transversal intersection

Two tropical curves Γ_1 and Γ_2 intersect transversally if every point $\mathbf{p} \in \Gamma_1 \cap \Gamma_2$ lies in the relative interior of a unique edge e_1 of Γ_1 and a unique edge e_2 of Γ_2.

The multiplicity of $\mathbf{p}$ equals

$$(w_{e_1} w_{e_2})|u_1 v_2 - u_2 v_1|$$

where (u_1, u_2) and (v_1, v_2) are primitive direction vectors of e_1 and e_2, (and w_{e_1} and w_{e_2} their weights).

Stable intersection

Suppose that Γ_1 and Γ_2 do not intersect transversally. The stable intersection of Γ_1 and Γ_2 is the the limit as ϵ goes to zero of the intersection of Γ_1 and $\epsilon v + \Gamma_2$ for a generic vector v. This does not depend on the choice of v.

Multiplicities add up when points collide.

Bezout's Theorem

Theorem

Any two tropical curves of degrees c and d in $\mathbb{R}^2$ intersect in cd points.

Valuation

Let K be a field.
A valuation on K is a function $\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following axioms:

- $\mathrm{val}(a) = \infty$ if and only if $a = 0$;
- $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$; and
- $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in K$.

The ring

$$R = \{c \in K \,|\, \mathrm{val}(c) \geq 0\}$$

has a unique maximal ideal

$$\mathcal{M}_K = \{c \in K \,|\, \mathrm{val}(c) > 0\}.$$

The residue field is $\Bbbk = R/\mathcal{M}_K$.

Examples

Trivial valuation

$val(a) = 0$ for all $a \in K^*$.

p-adic valuation

$K = \mathbb{Q}$, $q = p^k a/b$, where $a, b \in \mathbb{Z}$ and p does not divide a or b.

$$\begin{array}{rrcl} \mathrm{val}_p : & \mathbb{Q}^* & \to & \mathbb{R} \\ & q = p^k a/b & \mapsto & k \end{array}$$

Puiseux series with complex coefficients $\mathbb{C}\{\{t\}\}$

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \cdots$$

where c_i are complex numbers and $a_1 < a_2 < a_3 < \ldots$ are rational numbers that have a common denominator.

$$\begin{array}{rrcl} \mathrm{val} : & \mathbb{C}\{\{t\}\}^* & \to & \mathbb{R} \\ & c(t) & \mapsto & a_1 \end{array}$$

Tropicalization

Given a polynomial $f(x_1, x_2) = \sum_{\mathbf{u} \in S} c_{\mathbf{u}} x_1^{u_1} x_2^{u_2}$ with coefficients in a valued field K, its tropicalization is

$$\mathrm{trop}(f)(X_1, X_2) = \bigoplus_{\mathbf{u} \in S} \mathrm{val}(c_{\mathbf{u}}) \odot X_1^{u_1} \odot X_2^{u_2}.$$

Theorem (Kapranov 90s)

Let K be an algebraically closed field with a nontrivial valuation, let $f = \sum_{\mathbf{u} \in S} c_{\mathbf{u}} x_1^{u_1} x_2^{u_2}$ be a polynomial in $K[x_1^{\pm}, x_2^{\pm}]$. The following subsets of $\mathbb{R}^2$ coincide:

- *the closure of $trop(V(f)) = \left\{ \left(val(y_1), val(y_2) \right) \mid (y_1, y_2) \in V(f) \subset (K^{\times})^2 \right\}$.*
- *tropical curve $T(trop(f))$.*

Smoothness

Tropically smooth implies smooth.

Proposition

Let $f = \sum_{\mathbf{u} \in S} c_{\mathbf{u}} x_1^{u_1} x_2^{u_2}$ be a polynomial in $K[x_1^{\pm}, x_2^{\pm}]$. Let $S(\mathrm{trop}(F))$ be the subdivision induced by the coefficients $\mathrm{val}(c_{\mathbf{u}})$ of $\mathrm{trop}(f)$ on $\mathrm{Newt}(\mathrm{trop}(f))$.

If $S(\mathrm{trop}(f))$ is a unimodular triangulations ($=$ $\mathrm{trop}(f)$ is tropically smooth), then $V(f) \subset (T^{\times})^n$ is a smooth curve.

Tropicalizing intersections

- Let f and g be polynomials in $K[x,y]$. If $\mathbf{x} \in V(f) \cap V(g)$, then $\mathrm{val}(\mathbf{x}) \in \mathrm{trop}(V(f)) \cap \mathrm{trop}(V(g))$.

- $K = \mathbb{C}\{\{t\}\}$, $f = x + y + 1$ and $g = x + 2y$.
  Then $V(f) \cap V(g) = (-2, 1)$ and $\mathrm{val}(-2, 1) = (0, 0)$.
  However $\mathrm{trop}(V(f)) \cap \mathrm{trop}(V(g)) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 \leq 0\}$.

- Let f and g be such that $\mathrm{trop}(V(f))$ and $\mathrm{trop}(V(g))$ intersect transversally at $\mathbf{w}$. Then there exists $x \in V(f) \cap V(g)$ such that $\mathrm{val}(\mathbf{x}) = \mathbf{w}$.

Tropical variety

Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $X = V(I)$ be its variety in the algebraic torus $(K^*)^n$. The tropicalization of X is

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)) = \bigcap_{f \in I} \mathrm{T}(\mathrm{trop}(f)) \subseteq \mathbb{R}^n.$$

Remark: It is not sufficient to take intersections over a generating set of I. But we can make the intersection finite by computing a tropical basis.

How to compute them?

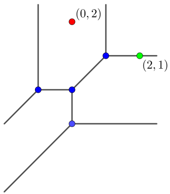
Given a polynomial f and a point $\mathbf{w}$ in $\mathbb{R}^n$, we can define the initial form of f with respect to $\mathbf{w}$ (taking into account the valuation).

$f = t^2 x^2 + t^2 y^2 + t\, xy + t\, x + t\, y + t^2 \ \in \mathbb{C}\{\{t\}\}[x^{\pm}, y^{\pm}]$.

$$\mathbf{w} = (0,2),\ \mathrm{in}_{\mathbf{w}}(f) = x \qquad \mathbf{w} = (2,1),\ \mathrm{in}_{\mathbf{w}}(f) = y + 1$$

For generic $\mathbf{w} \in \mathbb{R}^n$ we have $\mathrm{in}_{\mathbf{w}}(f)$ is a monomial.
Tropical Geometry cares about non-generic $\mathbf{w}$:

$$\{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(f) \neq \text{monomial}\}.$$



Remark: Taking the initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ of I gives a flat family over the valuation ring R of K. The general fiber is isomorphic to I and the special to $\mathrm{in}_{\mathbf{w}}(I)$.

What are tropical varieties?

Theorem (Fundamental Theorem of Tropical Geometry)

Let K be an algebraically closed field with a nontrivial valuation, let I be an ideal in $K[x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}]$, and let $X = V(I)$ its variety in $(K^*)^n$. The following three subsets of $\mathbb{R}^n$ coincide:

- *the closure of $\{(val(y_1), \ldots, val(y_n)) \mid (y_1, \ldots, y_n) \in X\}$.*
- $\{\mathbf{w} \in \mathbb{R}^n \mid in_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$.
- *tropical zeros of tropicalization of polynomials in the ideal I.*

Theorem (Structure Theorem)

Let X be an irreducible subvariety of dimension d in the algebraic torus $(K^*)^n$. The tropical variety $trop(X)$ is the support of a pure polyhedral complex of dimension n.

Thank you!