A tropical count of real bitangents to plane quartic curves

joint work with Alheydis Geiger

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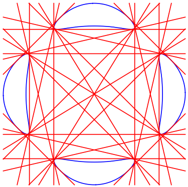
Quartic curves and their bitangents

A quartic curve in the projective plane is the zero set of a quartic polynomial

$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y \\ + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4.$$

- Plücker 1834: every smooth complex quartic curve admits 28 bitangent lines.
- Plücker 1839 and Zeuthen 1873: every smooth real quartic curve admits 28, 16, 8 or 4 real bitangent lines depending on its topology.

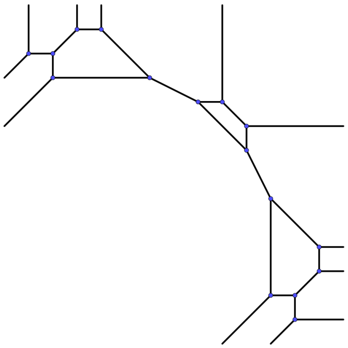


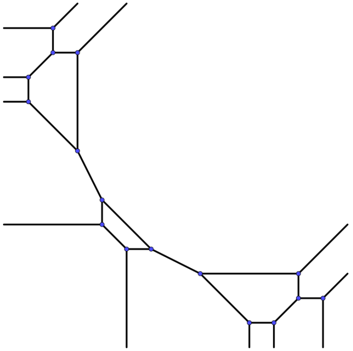
Question: What happens tropically?

Motivations:

- Mikhalkin '05: Tropical methods in enumerative geometry.
- Computational framework for further analysis.
- Counting bitangent lines over different fields.

From min to max

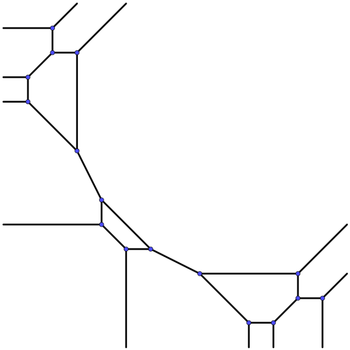




What happens tropically?

A tropical line Λ is bitangent to a smooth tropical plane quartic curve Γ if their intersection $\Lambda \cap \Gamma$ has two components with stable multiplicity 2, or one component with stable multiplicity 4.

- Baker et al. '16: Smooth tropical quartic curves have infinitely many tropical bitangents, grouped into 7 equivalence classes modulo continuous translations that preserve bitangency. [Superaboundance]



Lifts

$K = \mathbb{C}\{\{t\}\}, \mathbb{R}\{\{t\}\}$

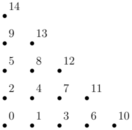
Let $V(f)$ a smooth plane quartic curve defined over K such that $\mathrm{Trop}(V(f)) = \Gamma$. A tropical bitangent Λ with tangency points P and P' lifts over K if there exists a bitangent ℓ to $V(f)$ defined over K with tangency points p and p' such that

$$\mathrm{Trop}(V(f)) = \Gamma, \quad \mathrm{Trop}(\ell) = \Lambda, \quad \mathrm{Trop}(p) = P, \quad \text{and} \quad \mathrm{Trop}(p') = P'.$$

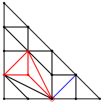
- Len–Markwig '20: Under some genericity assumptions, every bitangent class has 4 lifts over $\mathbb{C}\{\{t\}\}$. [$7 \cdot 4 = 28$]
- Cueto–Markwig '20: Under some genericity assumptions, every bitangent class has 0 or 4 real lifts. [Real bitangents are a multiple of 4]
- Geiger–P. '21: Computational tropical count of real bitangents.

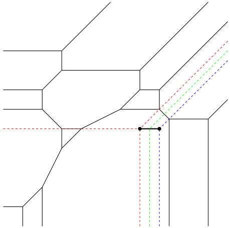
Unimodular regular triangulations

Smooth tropical quartic curves are dual to regular unimodular triangulations of the lattice points in $4\Delta_2$.

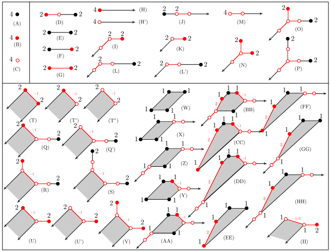


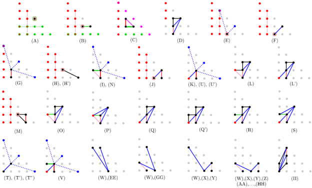
- Coefficients of the tropical polynomial are a height function defining a regular subdivision of the 15 lattice points in $4\Delta_2$.
- Coefficients are generic, then the dual subdivision is a triangulation $\mathcal{T}$.
- Triangulation is unimodular, then the tropical curve is smooth.
- Points in $\mathbb{R}^{15}$ inducing the same triangulation form a relatively open polyhedral cone, the secondary cone $\Sigma(\mathcal{T})$.





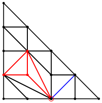
Shapes and real lifting conditions of bitangent classes [Cueto-Markwig '20]



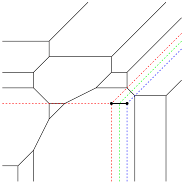


Shape	Lifting conditions
(A)	$(-s_{1v}s_{1,v+1})^i s_{0i}s_{22} > 0$ and $(-s_{u1}s_{u+1,1})^j s_{j0}s_{22} > 0$
(B)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $(-s_{21})^{j+1} s_{31}^j s_{1v}s_{1,v+1}s_{j0} > 0$
(C)	$(-s_{11}s_{12})^i s_{0i}s_{20} > 0$ and $(-s_{21}s_{12})^k s_{k,4-k}s_{20} > 0$ if j=2 $(-s_{11})^{i+1} s_{12}^i s_{21}s_{0i}s_{j0} > 0$ and $(-s_{21})^{k+1} s_{12}^k s_{11}s_{k,4-k}s_{j0} > 0$ if j=1,3
(H),(H')	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $-s_{21}s_{1v}s_{1,v+1}s_{40} > 0$
(M)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $s_{31}s_{1v}s_{1,v+1}s_{30} > 0$
(D)	$(-s_{10}s_{11})^i s_{0i}s_{22} > 0$
(E),(F),(J)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{20} > 0$
(G)	$(-s_{10}s_{11})^i s_{0i}s_{k,4-k} > 0$
(I),(N)	$-s_{10}s_{11}s_{01}s_{k,4-k} > 0$
(K),(T),(T'),(T''), (U),(U'),(V)	$s_{00}s_{k,4-k} > 0$
(L),(O),(P)	$-s_{10}s_{11}s_{01}s_{22} > 0$
(L'),(Q),(Q'), (R),(S)	$s_{00}s_{22} > 0$
rest	no conditions

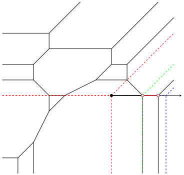
Example



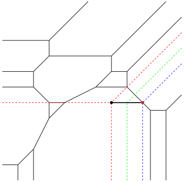
Dual triangulation



$(0, 5, 5, 9, 8, 5, 6.5, 9, 9, 4, 2, 7, 8, 7, 1)$
Shape (E)



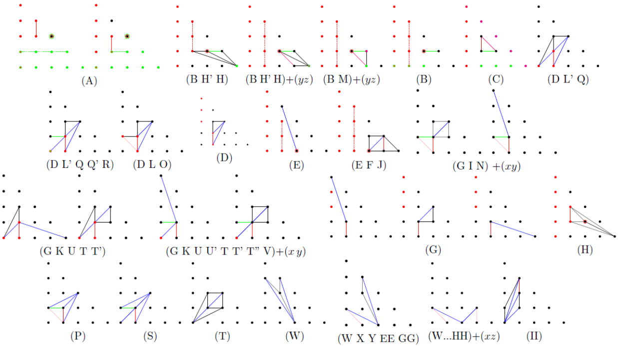
$(0, 5, 5, 9, 8, 5, 6, 9, 9, 4, 2, 7, 8, 7, 1)$
Shape (J)



$(0, 5, 5, 9, 8, 5, 5.5, 9, 9, 4, 1, 7, 8, 7, 1)$
Shape (F)

Deformation classes

Given a bitangent class, its deformation class is the collection of bitangnent classes to which it can deforms when continuously moving the coefficient to the tropical quartic curve in a path in the secondary cone.



Deformation classes only depend on the dual triangulation $\mathcal{T}$.

Real lifting conditions for deformation classes

Theorem (Geiger–P. '21)

The real lifting conditions only depend on the deformation class and not on the shapes. In other words, real lifting conditions of tropical bitangent classes only depend on the dual triangulation $\mathcal{T}$.

deformation class	Lifting conditions
(A)	$(-s_{1v}s_{1,v+1})^i s_{0i}s_{22} > 0$ and $(-s_{u1}s_{u+1,1})^j s_{j0}s_{22} > 0$
(B H' H), (B H' H)+(yz), (H)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $-s_{21}s_{1v}s_{1,v+1}s_{40} > 0$
(B M)+(yz)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $s_{31}s_{1v}s_{1,v+1}s_{30} > 0$
(B)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $(-s_{21})^{j+1}s_{31}^j s_{1v}s_{1,v+1}s_{j0} > 0$ with $j \in \{0, 1, 2\}$
(C)	$(-s_{11}s_{12})^i s_{0i}s_{20} > 0$ and $(-s_{21}s_{12})^k s_{k,4-k}s_{20} > 0$ if j=2 $(-s_{11})^{i+1}s_{12}^i s_{21}s_{0i}s_{j0} > 0$ and $(-s_{21})^{k+1}s_{12}^k s_{11}s_{k,4-k}s_{j0} > 0$ if j=1,3
(D)	$(-s_{10}s_{11})^i s_{0i}s_{22} > 0$ with $i \in \{2, 3, 4\}$
(D L O), (P)	$-s_{10}s_{11}s_{01}s_{22} > 0$
(D L' Q), (D L' Q Q' R), (S), (T)	$s_{00}s_{22} > 0$
(E), (E F J)	$(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{20} > 0$
(G)	$(-s_{10}s_{11})^i s_{0i}s_{k,4-k} > 0$ with $i \in \{2, 3, 4\}$
(G I N)+(xy)	$-s_{10}s_{11}s_{01}s_{k,4-k} > 0$
(G K U U' T T'), (G K U U' T- - T' T" V)+(xy)	$s_{00}s_{k,4-k} > 0$
rest	no conditions

Tropical count of real bitangents

Theorem (Geiger–P. '21)

Let Γ be a generic tropicalization of a smooth quartic plane curve defined over a real closed complete non-Archimedean valued field. Either 1, 2, 4 or 7 of its bitangent classes admit a lift to real bitangents. Every smooth quartic curve whose tropicalization is generic has either 4, 8, 16 or 28 totally real bitangents.

Proof.

Enumeration of deformation classes:

Input: Unimodular regular triangulation $\mathcal{T}$ of $4\Delta_2$.
Output: The list of all deformation classes in $\mathcal{T}$.

```
for each deformation class $M$ do
    for each $\sigma \in S_3$ do
        if the triangles of $\sigma(M)$ are contained in $\mathcal{T}$ then
            output $(\sigma(M), \sigma)$.
        end if
    end for
end for
```

Check of real lifting conditions:

Input: Unimodular regular triangulation $\mathcal{T}$ of $4\Delta_2$.
Output: The list of all possible numbers of real bitangents of an algebraic quartic curve with dual subdivision $\mathcal{T}$.

```
for each $v \in \{\pm 1\}^{15}$ starting with 1 do
    $n = 0$
    for each lifting condition $c$ of $\mathcal{T}$ do
        if $c(v)$ is true then
            $n + 4$.
        end if
    end for
    if the value of $n$ did not appear before then
        output $n$
    end if
end for
```

Run the two algorithms on the 1278 regular unimodular triangulations (modulo S_3-symmetry) of $4\Delta_2$ [Brodsky et al. '15] and on the 2^{15} sign vectors. □

Analyzing the data

Every generic combinatorial type of smooth tropical quartic curve admits a lift to a plane curve with 28 real bitangents.

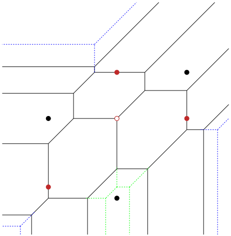
{4 8 16 28}	{4 8 28}	{4 16 28}	{8 16 28}	{4 28}	{8 28}	{16 28}
1200	15	26	18	6	3	2

Theorem (Caporaso–Sernesi '00)

The general plane quartic is uniquely determined by its 28 bitangent lines.

Theorem (Geiger-P. '21)

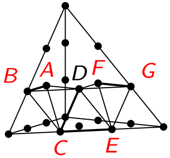
The combinatorial type of a tropical quartic curve is determined by its 7 deformation classes.



Tropical cubic surfaces

- 14 373 645 regular unimodular triangulations, parametrizing smooth tropical cubic surfaces.
- Enumerative question:
  how many tropical lines in a cubic surface?
- Answer: look at the triangulation.

- [P–Vigeland '21] Classification of 10 combinatorial types.
- [Joswig–P–Sturmfels '20] Enumeration and analysis.



- Combinatorics and computer algebra: software + database.

Thank you!