1 Markowitz

Suppose the returns of $N \geq 1$ assets are normally distributed with mean $\mu \in \mathbb{R}^N$ and covariance $\Sigma \in \mathbb{R}^{N \times N}$. By definition, Σ is symmetric positive definite. Fix, for the moment, the desired return μ_P. For each portfolio $x \in \mathbb{R}^N$, we require that $\mu' x = \mu_P$ and $\mathbf{1}' x = w_P$. Let $\sigma_P^2(x)$ denote the variance of its returns: $\sigma_P^2(x) = x' \Sigma x$. The investor's problem is to choose the minimum-variance portfolio subject to it earning a return of μ_P.

$$
\begin{array}{rl}
 & \min_{x \in \mathbb{R}} \sigma_P^2(x) \\
\text{s.t.} & m' x = \mu \\
 & \mathbf{1}' x = 1
\end{array}
\tag{1}
$$

Let $a(\mu)$ and $b(\mu)$ denote the Lagrange multipliers of the constraints $m' x = \mu$ and $\mathbf{1}' x = 1$ respectively. The first-order condition is

$$
0 = \Sigma x - a(\mu) m - b(\mu) \mathbf{1} \tag{2}
$$

which implies that

$$
x^\circ = a(\mu) \Sigma^{-1} m + b(\mu) \Sigma^{-1} \mathbf{1} \tag{3}
$$

and hence

$$
\mu = a(\mu) \left(m' \Sigma^{-1} m \right) + b(\mu) \left(m' \Sigma^{-1} \mathbf{1} \right) \tag{4}
$$

$$
1 = a(\mu) \left(\mathbf{1}' \Sigma^{-1} m \right) + b(\mu) \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \tag{5}
$$

Define

$$
D^{-1} = \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \left(m' \Sigma^{-1} m \right) - \left(m' \Sigma^{-1} \mathbf{1} \right) \left(\mathbf{1}' \Sigma^{-1} m \right) \tag{6}
$$

so that

$$
a(\mu) = D \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \mu - D \left(m' \Sigma^{-1} \mathbf{1} \right) \tag{7}
$$

$$
b(\mu) = D \left(m' \Sigma^{-1} m \right) - D \left(\mathbf{1}' \Sigma^{-1} m \right) \mu \tag{8}
$$

Therefore,

$$
s_P(\mu) \equiv \sqrt{\sigma_P(x^\circ(\mu))} \tag{9}
$$

$$
= \sqrt{x^\circ(\mu)' \Sigma x^\circ(\mu)} \tag{10}
$$

The security market line is

$$\sigma_T(\mu) = r_F + \beta\mu \tag{11}$$

μ^* and β^* satisfy $s_P(\mu^*) = \sigma_T(\mu^*)$ and $s'_P(\mu^*) = \sigma'_T(\mu^*)$.

1.1 Alternative

The investor's problem is to choose the minimum-variance portfolio subject to it earning a return of μ_P.

$$\begin{aligned} & \max_{x\in\mathbb{R}} \tfrac{m'x - r_F}{\sqrt{\sigma_P^2(x)}} \\ \text{s.t.} \quad & \mathbf{1}'x = 1 \end{aligned} \tag{12}$$

1.2 Examples

1.2.1 Constant Risk

Suppose that $\Sigma = I$.