1 Markowitz

Suppose the returns of $N \geq 1$ assets are normally distributed with mean $\mu \in \mathbb{R}^N$ and covariance $\Sigma \in \mathbb{R}^{N \times N}$. By definition, Σ is symmetric positive definite. Fix, for the moment, the desired return μ. For each portfolio $x \in \mathbb{R}^N$, we require that $\mathbf{1}'x = 1$ and $\mu' x = \mu$. Let $\sigma_P^2(x)$ denote the variance of its returns: $\sigma_P^2(x) = x'\Sigma x$. The investor's problem is to choose the minimum-variance portfolio subject to it earning a return of μ.

$$\begin{aligned} & & \min\nolimits_{x \in \mathbb{R}} \tfrac{1}{2}\sigma_P^2(x) \\ \text{s.t.} \quad & a: & \mathbf{1}'x = 1 \\ & b: & m'x = \mu \end{aligned} \tag{1}$$

Let a and b denote the Lagrange multipliers of the constraints $m'x = \mu$ and $\mathbf{1}'x = 1$ respectively. The first-order condition is

$$0 = \Sigma x - a\mathbf{1} - bm \tag{2}$$

which implies that

$$x^\circ = a\Sigma^{-1}\mathbf{1} + b\Sigma^{-1}m \tag{3}$$

Let A be the 2×2 matrix with elements

$$[A]_{1,1} = \mathbf{1}'\Sigma^{-1}\mathbf{1} \tag{4}$$

$$[A]_{1,2} = \mathbf{1}'\Sigma^{-1}m \tag{5}$$

$$[A]_{2,1} = m'\Sigma^{-1}\mathbf{1} \tag{6}$$

$$[A]_{2,2} = m'\Sigma^{-1}m \tag{7}$$

Therefore,

$$s_P^2(\mu) = x^\circ(\mu)'\Sigma x^\circ(\mu) \tag{8}$$

The security market line is

$$\sigma_T(\mu) = r_F + \beta\mu \tag{9}$$

μ^* and β^* satisfy $s_P(\mu^*) = \sigma_T(\mu^*)$ and $s'_P(\mu^*) = \sigma'_T(\mu^*)$.

$$\sigma_P^2(x^\circ(\mu)) = (1, \mu) \begin{pmatrix} \mathbf{1}'\Sigma^{-1}\mathbf{1} & \mathbf{1}'\Sigma^{-1}m \\ m'\Sigma^{-1}\mathbf{1} & m'\Sigma^{-1}m \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mu \end{pmatrix} \tag{10}$$

Finally, we compute the mean of the tangency portfolio:

$$\max_{\mu} \frac{\mu - r_F}{\sqrt{\sigma_P^2(x^\circ(\mu))}} \tag{11}$$