1 Markowitz

Suppose the returns of $N \geq 1$ assets are normally distributed with mean $\mu \in \mathbb{R}^N$ and covariance $\Sigma \in \mathbb{R}^{N \times N}$. By definition, Σ is symmetric positive definite. Fix, for the moment, the desired return μ_P. For each portfolio $x \in \mathbb{R}^N$, we require that $\mu' x = \mu_P$ and $\mathbf{1}' x = w_P$. Let $\sigma_P^2(x)$ denote the variance of its returns: $\sigma_P^2(x) = x' \Sigma x$. The investor's problem is to choose the minimum-variance portfolio subject to it earning a return of μ_P.

$$\begin{array}{rl} & \min_{x \in \mathbb{R}} \sigma_P^2(x) \\ \text{s.t.} & m' x = \mu \\ & \mathbf{1}' x = 1 \end{array} \tag{1}$$

Let $a(\mu)$ and $b(\mu)$ denote the Lagrange multipliers of the constraints $m' x = \mu$ and $\mathbf{1}' x = 1$ respectively. The first-order condition is

$$0 = \Sigma x - a(\mu) m - b(\mu) \mathbf{1} \tag{2}$$

which implies that

$$x^\circ = a(\mu) \Sigma^{-1} m + b(\mu) \Sigma^{-1} \mathbf{1} \tag{3}$$

and hence

$$\mu = a(\mu) \left(m' \Sigma^{-1} m \right) + b(\mu) \left(m' \Sigma^{-1} \mathbf{1} \right) \tag{4}$$

$$1 = a(\mu) \left(\mathbf{1}' \Sigma^{-1} m \right) + b(\mu) \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \tag{5}$$

Define

$$D^{-1} = \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \left(m' \Sigma^{-1} m \right) - \left(m' \Sigma^{-1} \mathbf{1} \right) \left(\mathbf{1}' \Sigma^{-1} m \right) \tag{6}$$

so that

$$a(\mu) = D \left(\mathbf{1}' \Sigma^{-1} \mathbf{1} \right) \mu - D \left(m' \Sigma^{-1} \mathbf{1} \right) \tag{7}$$

$$b(\mu) = D \left(m' \Sigma^{-1} m \right) - D \left(\mathbf{1}' \Sigma^{-1} m \right) \mu \tag{8}$$

Therefore,

$$s_P(\mu) \equiv \sqrt{\sigma_P(x^\circ(\mu))} \tag{9}$$

$$= \sqrt{x^\circ(\mu)' \Sigma x^\circ(\mu)} \tag{10}$$

The security market line is

$$\sigma_T(\mu) = r_F + \beta\mu \tag{11}$$

μ^* and β^* satisfy $s_P(\mu^*) = \sigma_T(\mu^*)$ and $s'_P(\mu^*) = \sigma'_T(\mu^*)$.

1.1 Alternative

The investor's problem is to choose the minimum-variance portfolio subject to it earning a return of μ_P.

$$\begin{array}{ll} & \max_{x\in\mathbb{R}} \frac{m'x - r_F}{\sqrt{\sigma_P^2(x)}} \\ \text{s.t.} & \mathbf{1}'x = 1 \end{array} \tag{12}$$

1.2 Examples

1.2.1 Constant Risk

Suppose that $\Sigma = I$. Since $\mathbf{1}'\mathbf{1} = N$,

$$D^{-1} = (\mathbf{1}'\mathbf{1})(m'm) - (m'\mathbf{1})(\mathbf{1}'m) \tag{13}$$

$$= N^2 \left(\frac{m'm}{N} - \left(\frac{m'\mathbf{1}}{N} \right)^2 \right) \tag{14}$$

so that

$$a(\mu) = D(\mathbf{1}'\mathbf{1})\mu - D(m'\mathbf{1}) \tag{15}$$

$$b(\mu) = D(m'm) - D(\mathbf{1}'m)\mu \tag{16}$$

1.3 $N \to \infty$

$$\begin{array}{ll} & \min_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(i)x(j)\sigma(i,j)didj \\ \text{s.t.} & \int_{-\infty}^{\infty} \mu(i)x(i) = \mu_0 \\ & \int_{-\infty}^{\infty} x(i)di = 1 \end{array} \tag{17}$$