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## 1 The Model

There are two players, a sender and a receiver, and a state of nature  $q \in [0, 1]$  that is known to the sender but not the receiver. Let  $g$  denote the density of  $q$ . The sender sends an *intended* message  $m(q) \in [0, 1]$ . The receiver receives a noisy version of the intended message, which we call the *received* message,  $\tilde{m} = m(q) + e$ .  $e$  is distributed according to a continuous density  $f$  with support  $[-\bar{e}, \bar{e}]$ , where for some integer  $N \geq 2$ ,  $\bar{e} = \frac{1}{2N}$ .

We specify  $f$  as follows. Let  $x : [-\bar{e}, \bar{e}] \rightarrow [0, 1]$  be given by

$$x(e) = \frac{1}{2} \left[ 1 + \frac{e}{\bar{e}} \right] \quad (1)$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1} (1 - x(e))^{b-1} \quad (2)$$

for constants  $a > 0$  and  $b > 0$ . The receiver then takes an action  $a(\tilde{m}) \in [0, 1]$ . Both sender and receiver have utility over the state,  $q$ , and the receiver's action,  $a$ , of

$$U(q, a) = -\frac{1}{2}(q - a)^2 \quad (3)$$

Prior to the start of the game, the sender can specify an intended message function  $m(q)$  that she will use. The receiver chooses an action based upon the function  $m(q)$  and the received message  $\tilde{m}$ , and we denote this function  $a(\tilde{m})$ . We work backward and start with

the optimal action, given  $m(q)$  and  $\tilde{m}$ .

Fix a message function  $m$ . Let  $Q(\tilde{m})$  denote the set of all states  $q \in [0, 1]$  such that for some noise  $e \in [-\bar{e}, \bar{e}]$ ,  $\tilde{m} = m(q) + e$ . In set builder notation,

$$Q(\tilde{m}) = \{q \in [0, 1] \mid m(q) - \bar{e} \leq \tilde{m} \leq m(q) + \bar{e}\}. \quad (4)$$

Finally, define

$$q_+(\tilde{m}) \equiv \sup Q(\tilde{m}) \quad (5)$$

$$q_-(\tilde{m}) \equiv \inf Q(\tilde{m}) \quad (6)$$

$q_+(\tilde{m})$  and  $q_-(\tilde{m})$  are the highest and lowest states that could possibly be associated with the received message  $\tilde{m}$ . Define

$$I(\tilde{m}; \alpha, \beta, \gamma) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - m(q)) g(q) dq \quad (7)$$

Suppose that the receiver receives the message  $\tilde{m} \in [-\bar{e}, 1 + \bar{e}]$ .  $q|\tilde{m}$  has support  $[q_-(\tilde{m}), q_+(\tilde{m})]$  and density

$$g(q|\tilde{m}) = \frac{f(\tilde{m} - m(q))g(q)}{\int_0^1 f(\tilde{m} - m(t))g(t)dt} \quad (8)$$

The receiver's problem is to

$$\max_{a(\tilde{m})} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} U(q, a) g(q|\tilde{m}) dq. \quad (9)$$

The receiver's optimal action is simply the expected value of the state,  $q$ , given the received message  $\tilde{m}$  :

$$a(\tilde{m}) = \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} q g(q|\tilde{m}) dq = \frac{I(\tilde{m}; q_-(\tilde{m}), q_+(\tilde{m}), 1)}{I(\tilde{m}; q_-(\tilde{m}), q_+(\tilde{m}), 0)} \quad (10)$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional  $C$  be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{e}}^{\bar{e}} (q - a(m(q) + e))^2 f(e) de dq, \quad (11)$$

where  $a$  is the receiver's optimal action from Equation (10).  $\mathcal{C}[m]$  is the expected loss for a given message function  $m$ . The integrand is the loss for a given state  $q$  and action  $a(\tilde{m})$ . The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function  $m$  that minimizes  $\mathcal{C}[m]$ :

$$\min_{m \in M} \mathcal{C}[m] \quad (12)$$

where  $M$  is the space of weakly increasing piece-wise continuous functions on  $[0, 1]$ . The change of variables  $\tilde{m} = m(q) + e$  and an application of Fubini's Theorem yield

$$\mathcal{C}[m] = \int_0^1 \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) d\tilde{m} dq \quad (13)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) dq d\tilde{m}. \quad (14)$$

We consider the costs of identity (“ $\mathcal{I}$ ”) and discrete (“ $\mathcal{D}$ ”) message functions.

## 2 Identity Message

We consider the identity message function,  $m(q) = q$ , only for the case in which  $q$  is uniformly distributed.

$$a(\tilde{m}) = \begin{cases} \bar{a}(\tilde{m}) & \text{if } 1 - \bar{e} < \tilde{m} \leq 1 + \bar{e} \\ a(\tilde{m}) & \text{if } \bar{e} < \tilde{m} \leq 1 - \bar{e} \\ \underline{a}(\tilde{m}) & \text{if } -\bar{e} \leq \tilde{m} \leq \bar{e} \end{cases} \quad (15)$$

where

$$\bar{a}(\tilde{m}) = \frac{I(\tilde{m}; \tilde{m} - \bar{e}, 1, 1)}{I(\tilde{m}; \tilde{m} - \bar{e}, 1, 0)} \quad (16)$$

$$a(\tilde{m}) = \frac{I(\tilde{m}; \tilde{m} - \bar{e}, \tilde{m} + \bar{e}, 1)}{I(\tilde{m}; \tilde{m} - \bar{e}, \tilde{m} + \bar{e}, 0)} \quad (17)$$

$$\underline{a}(\tilde{m}) = \frac{I(\tilde{m}; 0, \tilde{m} + \bar{e}, 1)}{I(\tilde{m}; 0, \tilde{m} + \bar{e}, 0)} \quad (18)$$

Note that the normalizing constant cancels out when computing the conditional expectation. The cost of the identity message function is given by

$$\mathcal{C}[m_{\mathcal{I}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m}. \quad (19)$$

where  $q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\}$  and  $q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\}$ . Define

$$\bar{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\tilde{m}-\bar{e}}^1 (q - \bar{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (20)$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\tilde{m}-\bar{e}}^{\tilde{m}+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (21)$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_0^{\tilde{m}+\bar{e}} (q - \underline{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (22)$$

so that  $\mathcal{C}[m_{\mathcal{I}}] = \bar{z} + z + \underline{z}$ .

### 3 Discrete Message

#### 3.1 Setup

Fix an integer  $M \geq 1$  and define  $K = M \times N$ . Define  $\bar{d} = \frac{1}{2K}$ . Consider the partition

$$0 = x_0 < x_1 < \cdots < x_K < x_{K+1} = 1 \quad (23)$$

of  $[0, 1]$ . Let  $x = (x_0, \dots, x_{K+1})$ . For each  $i \in \{0, \dots, K\}$ , define  $X_i = [x_i, x_{i+1})$ . A discrete message with  $K + 1$  messages is given by

$$m_{\mathcal{D}}(q) = \sum_{i=0}^K \frac{i}{K} \chi_{X_i}(q) \quad (24)$$

(where  $\chi$  is the characteristic function). Let  $k_- : [-\bar{e}, 1 + \bar{e}] \rightarrow \{0, \dots, K\}$  and  $k_+ : [-\bar{e}, 1 + \bar{e}] \rightarrow \{1, \dots, K + 1\}$  be given by

$$k_+(\tilde{m}) = \min\{\lfloor \tilde{m}K \rfloor + M + 1, K + 1\} \quad (25)$$

$$k_-(\tilde{m}) = \max\{0, \lfloor \tilde{m}K \rfloor - M\} \quad (26)$$

respectively. Note that

$$q_+(\tilde{m}) = x_{k_+(\tilde{m})} \quad (27)$$

$$q_-(\tilde{m}) = x_{k_-(\tilde{m})} \quad (28)$$

There are

$$\frac{1 + 2\bar{e}}{2\bar{d}} = M + K \quad (29)$$

unique actions (corresponding to  $M + K$  equispaced subintervals of  $[-\bar{e}, 1 + \bar{e}]$ ). For each  $i \in \{-M, \dots, K - 1\}$ , define

$$y_i = 2\bar{d}(i + M) - \bar{e} \quad (30)$$

$$Y_i = [y_i, y_{i+1}) \quad (31)$$

so that for each  $y \in Y_i$ ,

$$k_+(y) = k_+(y_i) = \min\{i + M + 1, K + 1\} \quad (32)$$

$$k_-(y) = k_-(y_i) = \max\{0, i + 1\} \quad (33)$$

### 3.2 Message Cost

Note that  $y_{i+1} = y_i + 2\bar{d}$ . For  $i \in \{-M, \dots, K - 1\}$  and  $j \in \{0, \dots, K + 1\}$ , let

$$\alpha_{i+1,j+1} \equiv \int_{y_{i+1}}^{y_{i+2}} f(\tilde{m} - 2\bar{d}(j + 1)) d\tilde{m} \quad (34)$$

$$= \int_{y_i + 2\bar{d}}^{y_{i+1} + 2\bar{d}} f((\tilde{m} - 2\bar{d}) - 2\bar{d}j) d\tilde{m} \quad (35)$$

$$= \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) d\tilde{m} = \alpha_{i,j} \quad (36)$$

Moreover,  $A$  is sparse:

$$\alpha_{i,j} = \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) d\tilde{m} \quad (37)$$

$$= \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) \mathbf{1}_{-\bar{e} \leq \tilde{m} - 2\bar{d}j \leq \bar{e}} d\tilde{m} \quad (38)$$

$$= \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) \mathbf{1}_{2\bar{d}j - \bar{e} \leq \tilde{m} \leq 2\bar{d}j + \bar{e}} d\tilde{m} \quad (39)$$

$$= \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) \mathbf{1}_{y_{j-M} \leq \tilde{m} \leq y_j} d\tilde{m} \quad (40)$$

$$= \mathbf{1}_{j-M \leq i < j} \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) d\tilde{m} \quad (41)$$

Note that when the error is uniformly distributed, The cost is given by

$$\mathcal{C}[m_{\mathcal{D}}] = \int_{-\bar{e}}^{1+\bar{e}} \left[ \int_{x_{k_-}(\tilde{m})}^{x_{k_+}(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - m_{\mathcal{D}}(q)) g(q) dq \right] d\tilde{m}. \quad (42)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \left[ \sum_{j=k_-}^{k_+(\tilde{m})-1} \int_{x_j}^{x_{j+1}} (q - a(\tilde{m}))^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (43)$$

$$= \sum_{i=-M}^{K-1} \int_{y_i}^{y_{i+1}} \left[ \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (44)$$

$$= \sum_{i=-M}^{K-1} \int_{y_i}^{y_{i+1}} \left[ \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (45)$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \left[ \int_{y_i}^{y_{i+1}} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq d\tilde{m} \right] \quad (46)$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \left[ \alpha_{i,j} \int_{x_j}^{x_{j+1}} (q - a_i)^2 g(q) dq \right] \quad (47)$$

$$= \sum_{j=0}^K \sum_{i=j-M}^{j-1} \left[ \alpha_{i,j} \int_{x_j}^{x_{j+1}} (q - a_i)^2 g(q) dq \right] \quad (48)$$

Equations 47 and 48 provide two different characterizations of  $\mathcal{C}[m_{\mathcal{D}}]$ .

### 3.3 First-Order Conditions

Define

$$\beta_{\gamma}(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} q^{\gamma} g(q) dq. \quad (49)$$

For  $i = -M, \dots, K-1$  and  $j = 1, \dots, K$ , the first-order conditions for  $a_i$  and  $x_j$  are

$$0 = \frac{\partial}{\partial a_i} \mathcal{C}[m_{\mathcal{D}}] = -2 \sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} \int_{x_k}^{x_{k+1}} (q - a_i) g(q) dq \quad (50)$$

$$0 = \frac{\partial}{\partial x_j} \mathcal{C}[m_{\mathcal{D}}] = \sum_{i=j-1-M}^{j-2} \alpha_{i,j-1} (x_j - a_i)^2 g(x_j) - \sum_{i=j-M}^{j-1} \alpha_{i,j} (x_j - a_i)^2 g(x_j) \quad (51)$$

respectively. If  $g(x_i) > 0$ , then for  $j = 1, \dots, K$ , we have

$$0 = \sum_{i=j-1-M}^{j-2} (x_j - a_i)^2 \alpha_{i,j-1} - \sum_{i=j-M}^{j-1} (x_j - a_i)^2 \alpha_{i,j} \quad (52)$$

$$= \sum_{i=j-1-M}^{j-2} ((x_j - a_i)^2 \alpha_{i,j-1} - (x_j - a_{i+1})^2 \alpha_{i+1,j}) \quad (53)$$

$$= \sum_{i=j-1-M}^{j-2} ((x_j^2 - 2a_i x_j - a_i^2) \alpha_{i,j-1} - (x_j^2 - 2a_{i+1} x_j - a_{i+1}^2) \alpha_{i+1,j}) \quad (54)$$

$$= x_j^2 \left[ \sum_{i=j-1-M}^{j-2} (\alpha_{i,j-1} - \alpha_{i+1,j}) \right] - 2x_j \left[ \sum_{i=j-1-M}^{j-2} (a_i \alpha_{i,j-1} - a_{i+1} \alpha_{i+1,j}) \right] \\ + \left[ \sum_{i=j-1-M}^{j-2} (a_i^2 \alpha_{i,j-1} - a_{i+1}^2 \alpha_{i+1,j}) \right] \quad (55)$$

Let the  $(i, j)$ -the element of the matrix  $B \in \mathbb{R}^{(M+K) \times K}$  be given by

$$[B]_{i,j} = \begin{cases} \alpha_{j-1-M,j-1} & \text{if } j-1-M = i \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } j-1-M < i < j-1 \\ -\alpha_{j-1,j} & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases} \quad (56)$$

so that

$$0 = \mathcal{F}(\tilde{x}, a) := (\tilde{x} \circ \tilde{x}) \circ (B'u) - (2\tilde{x}) \circ (B'a) + u \circ (B'(a \circ a)) \quad (57)$$

where  $\circ$  denotes Hadamard multiplication,  $u = (1, \dots, 1)' \in \mathbb{R}^K$ ,  $\tilde{x} = (x_1, \dots, x_K)' \in \mathbb{R}^K$ , and  $\mathcal{F} : \mathbb{R}^K \times \mathbb{R}^{M+K} \rightarrow \mathbb{R}^K$ .

### 3.4 Uniformly Distributed State

In what follows, suppose that  $g(q) = \chi_{[0,1]}(q)$ . Equation 50 becomes

$$0 = \left[ \sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} (x_{k+1}^2 - x_k^2) \right] - 2a_i \left[ \sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} (x_{k+1} - x_k) \right] \quad (58)$$

Let the  $(i, j)$ -the element of the matrix  $C \in \mathbb{R}^{(M+K) \times (K+2)}$  be given by

$$[C]_{i,j} = \begin{cases} -\alpha_{i,k_-(y_i)} & \text{if } k_-(y_i) = j \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } k_-(y_i) < j < k_+(y_i) \\ \alpha_{i,k_+(y_i)-1} & \text{if } j = k_+(y_i) \\ 0 & \text{otherwise} \end{cases} \quad (59)$$

It transpires that for  $i = -M, \dots, K-1$  and  $j = 1, \dots, K$ ,  $[B]_{i,j} = [C]_{i,j}$  ( $C$  is padded on both sides by an extra column):

$$C = \begin{bmatrix} \underline{c} & B & \bar{c} \end{bmatrix} \quad (60)$$

We have

$$0 = \mathcal{G}(x, a) := C(x \circ x) - 2a \circ (Cx) \quad (61)$$

where  $\mathcal{G} : \mathbb{R}^{K+2} \times \mathbb{R}^{M+K} \rightarrow \mathbb{R}^{M+K}$ . Hence,

$$a = \frac{1}{2} (C(x \circ x)) \oslash (Cx) \quad (62)$$

$$= \frac{1}{2} (B(\tilde{x} \circ \tilde{x} + \bar{c})) \oslash (B\tilde{x} + \bar{c}) \quad (63)$$

where  $\oslash$  denotes Hadamard division. To summarize,  $\mathcal{F}(x, a) = 0$  is a system of  $K$  equations and  $\mathcal{G}(x, a) = 0$  a system of  $K + M$  equations.  $x$  contains  $K$  unknowns (since  $x_0 = 0$  and  $x_{K+1} = 1$ ) and  $a$  contains  $K + M$  unknowns. In total, there are  $M + 2K$  equations and  $M + 2K$  unknowns.



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