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1 The Model

There are two players, a sender and a receiver, and a state of nature $q \in [0, 1]$ that is known to the sender but not the receiver. Let g denote the density of q . The sender sends an *intended* message $m(q) \in [0, 1]$. The receiver receives a noisy version of the intended message, which we call the *received* message, $\tilde{m} = m(q) + e$. e is distributed according to a continuous density f with support $[-\bar{e}, \bar{e}]$, where for some integer $N \geq 2$, $\bar{e} = \frac{1}{2N}$.

We specify f as follows. Let $x : [-\bar{e}, \bar{e}] \rightarrow [0, 1]$ be given by

$$x(e) = \frac{1}{2} \left[1 + \frac{e}{\bar{e}} \right] \quad (1)$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1} (1 - x(e))^{b-1} \quad (2)$$

for constants $a > 0$ and $b > 0$. The receiver then takes an action $a(\tilde{m}) \in [0, 1]$. Both sender and receiver have utility over the state, q , and the receiver's action, a , of

$$U(q, a) = -\frac{1}{2}(q - a)^2 \quad (3)$$

Prior to the start of the game, the sender can specify an intended message function $m(q)$ that she will use. The receiver chooses an action based upon the function $m(q)$ and the received message \tilde{m} , and we denote this function $a(\tilde{m})$. We work backward and start with

the optimal action, given $m(q)$ and \tilde{m} .

Fix a message function m . Let $Q(\tilde{m})$ denote the set of all states $q \in [0, 1]$ such that for some noise $e \in [-\bar{e}, \bar{e}]$, $\tilde{m} = m(q) + e$. In set builder notation,

$$Q(\tilde{m}) = \{q \in [0, 1] \mid m(q) - \bar{e} \leq \tilde{m} \leq m(q) + \bar{e}\}. \quad (4)$$

Finally, define

$$q_+(\tilde{m}) \equiv \sup Q(\tilde{m}) \quad (5)$$

$$q_-(\tilde{m}) \equiv \inf Q(\tilde{m}) \quad (6)$$

$q_+(\tilde{m})$ and $q_-(\tilde{m})$ are the highest and lowest states that could possibly be associated with the received message \tilde{m} . Define

$$I(\tilde{m}; \alpha, \beta, \gamma) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - m(q)) g(q) dq \quad (7)$$

Suppose that the receiver receives the message $\tilde{m} \in [-\bar{e}, 1 + \bar{e}]$. $q|\tilde{m}$ has support $[q_-(\tilde{m}), q_+(\tilde{m})]$ and density

$$g(q|\tilde{m}) = \frac{f(\tilde{m} - m(q))g(q)}{\int_0^1 f(\tilde{m} - m(t))g(t)dt} \quad (8)$$

The receiver's problem is to

$$\max_{a(\tilde{m})} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} U(q, a) g(q|\tilde{m}) dq. \quad (9)$$

The receiver's optimal action is simply the expected value of the state, q , given the received message \tilde{m} :

$$a(\tilde{m}) = \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} q g(q|\tilde{m}) dq = \frac{I(\tilde{m}; q_-(\tilde{m}), q_+(\tilde{m}), 1)}{I(\tilde{m}; q_-(\tilde{m}), q_+(\tilde{m}), 0)} \quad (10)$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional C be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{e}}^{\bar{e}} (q - a(m(q) + e))^2 f(e) de dq, \quad (11)$$

where a is the receiver's optimal action from Equation (10). $\mathcal{C}[m]$ is the expected loss for a given message function m . The integrand is the loss for a given state q and action $a(\tilde{m})$. The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function m that minimizes $\mathcal{C}[m]$:

$$\min_{m \in M} \mathcal{C}[m] \quad (12)$$

where M is the space of weakly increasing piece-wise continuous functions on $[0, 1]$. The change of variables $\tilde{m} = m(q) + e$ and an application of Fubini's Theorem yield

$$\mathcal{C}[m] = \int_0^1 \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) d\tilde{m} dq \quad (13)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) dq d\tilde{m}. \quad (14)$$

We consider the costs of identity (“ \mathcal{I} ”) and discrete (“ \mathcal{D} ”) message functions.

2 Identity Message

We consider the identity message function, $m(q) = q$, only for the case in which q is uniformly distributed.

$$a(\tilde{m}) = \begin{cases} \bar{a}(\tilde{m}) & \text{if } 1 - \bar{e} < \tilde{m} \leq 1 + \bar{e} \\ a(\tilde{m}) & \text{if } \bar{e} < \tilde{m} \leq 1 - \bar{e} \\ \underline{a}(\tilde{m}) & \text{if } -\bar{e} \leq \tilde{m} \leq \bar{e} \end{cases} \quad (15)$$

where

$$\bar{a}(\tilde{m}) = \frac{I(\tilde{m}; \tilde{m} - \bar{e}, 1, 1)}{I(\tilde{m}; \tilde{m} - \bar{e}, 1, 0)} \quad (16)$$

$$a(\tilde{m}) = \frac{I(\tilde{m}; \tilde{m} - \bar{e}, \tilde{m} + \bar{e}, 1)}{I(\tilde{m}; \tilde{m} - \bar{e}, \tilde{m} + \bar{e}, 0)} \quad (17)$$

$$\underline{a}(\tilde{m}) = \frac{I(\tilde{m}; 0, \tilde{m} + \bar{e}, 1)}{I(\tilde{m}; 0, \tilde{m} + \bar{e}, 0)} \quad (18)$$

Note that the normalizing constant cancels out when computing the conditional expectation. The cost of the identity message function is given by

$$\mathcal{C}[m_{\mathcal{I}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m}. \quad (19)$$

where $q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\}$ and $q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\}$. Define

$$\bar{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\tilde{m}-\bar{e}}^1 (q - \bar{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (20)$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\tilde{m}-\bar{e}}^{\tilde{m}+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (21)$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_0^{\tilde{m}+\bar{e}} (q - \underline{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (22)$$

so that $\mathcal{C}[m_{\mathcal{I}}] = \bar{z} + z + \underline{z}$.

3 Discrete Message

3.1 Setup

Fix an integer $M \geq 1$ and define $K = M \times N$. Define $\bar{d} = \frac{1}{2K}$. Consider the partition

$$0 = x_0 < x_1 < \cdots < x_K < x_{K+1} = 1 \quad (23)$$

of $[0, 1]$. Let $x = (x_0, \dots, x_{K+1})$. For each $i \in \{0, \dots, K\}$, define $X_i = [x_i, x_{i+1})$. A discrete message with $K + 1$ messages is given by

$$m_{\mathcal{D}}(q) = \sum_{i=0}^K \frac{i}{K} \chi_{X_i}(q) \quad (24)$$

(where χ is the characteristic function). Let $k_- : [-\bar{e}, 1 + \bar{e}] \rightarrow \{0, \dots, K\}$ and $k_+ : [-\bar{e}, 1 + \bar{e}] \rightarrow \{1, \dots, K + 1\}$ be given by

$$k_+(\tilde{m}) = \min\{\lfloor \tilde{m}K \rfloor + M + 1, K + 1\} \quad (25)$$

$$k_-(\tilde{m}) = \max\{0, \lfloor \tilde{m}K \rfloor - M\} \quad (26)$$

respectively. Note that

$$q_+(\tilde{m}) = x_{k_+(\tilde{m})} \quad (27)$$

$$q_-(\tilde{m}) = x_{k_-(\tilde{m})} \quad (28)$$

There are

$$\frac{1 + 2\bar{e}}{2\bar{d}} = M + K \quad (29)$$

unique actions (corresponding to $M + K$ equispaced subintervals of $[-\bar{e}, 1 + \bar{e}]$). For each $i \in \{-M, \dots, K - 1\}$, define

$$y_i = 2\bar{d}(i + M) - \bar{e} \quad (30)$$

$$Y_i = [y_i, y_{i+1}) \quad (31)$$

so that for each $y \in Y_i$,

$$k_+(y) = k_+(y_i) = \min\{i + M + 1, K + 1\} \quad (32)$$

$$k_-(y) = k_-(y_i) = \max\{0, i + 1\} \quad (33)$$

3.2 Message Cost

Note that $y_{i+1} = y_i + 2\bar{d}$. For $i \in \{-M, \dots, K - 1\}$ and $j \in \{0, \dots, K + 1\}$, let

$$\alpha_{i,j} \equiv \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) d\tilde{m} \quad (34)$$

$$= \int_{y_{i-1} + 2\bar{d}}^{y_i + 2\bar{d}} f((\tilde{m} + 2\bar{d}) - 2\bar{d}(j + 1)) d\tilde{m} \quad (35)$$

$$= \int_{y_{i-1}}^{y_i} f(\tilde{m} - 2\bar{d}(j + 1)) d\tilde{m} = \alpha_{i-1,j+1} \quad (36)$$

A is a Hankel matrix and therefore symmetric. To summarize,

$$\alpha_{i,j} = \alpha_{i-1,j+1} = \alpha_{j,i} \quad (37)$$

Note that when the error is uniformly distributed, The cost is given by

$$\mathcal{C}[m_{\mathcal{D}}] = \int_{-\bar{e}}^{1+\bar{e}} \left[\int_{x_{k_-}(\tilde{m})}^{x_{k_+}(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - m_{\mathcal{D}}(q)) g(q) dq \right] d\tilde{m}. \quad (38)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \left[\sum_{j=k_-(\tilde{m})}^{k_+(\tilde{m})-1} \int_{x_j}^{x_{j+1}} (q - a(\tilde{m}))^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (39)$$

$$= \sum_{i=-M}^{K-1} \int_{y_i}^{y_{i+1}} \left[\sum_{j=k_-(y_i)}^{k_+(y_i)-1} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (40)$$

$$= \sum_{i=-M}^{K-1} \int_{y_i}^{y_{i+1}} \left[\sum_{j=k_-(y_i)}^{k_+(y_i)-1} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq \right] d\tilde{m} \quad (41)$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \left[\int_{y_i}^{y_{i+1}} \int_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\tilde{m} - \frac{j}{K}\right) g(q) dq d\tilde{m} \right] \quad (42)$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)-1} \left[\alpha_{i,j} \int_{x_j}^{x_{j+1}} (q - a_i)^2 g(q) dq \right] \quad (43)$$

$$= \sum_{j=0}^K \sum_{i=j-M}^{j-1} \left[\alpha_{i,j} \int_{x_j}^{x_{j+1}} (q - a_i)^2 g(q) dq \right] \quad (44)$$

Equations 43 and 44 provide two different characterizations of $\mathcal{C}[m_{\mathcal{D}}]$.

3.3 First-Order Conditions

Define

$$\beta_{\gamma}(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} q^{\gamma} g(q) dq. \quad (45)$$

For $i = -M, \dots, K-1$ and $j = 1, \dots, K$, the first-order conditions for a_i and x_j are

$$0 = \frac{\partial}{\partial a_i} \mathcal{C}[m_{\mathcal{D}}] = -2 \sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} \int_{x_k}^{x_{k+1}} (q - a_i) g(q) dq \quad (46)$$

$$0 = \frac{\partial}{\partial x_j} \mathcal{C}[m_{\mathcal{D}}] = \sum_{i=j-1-M}^{j-2} \alpha_{i,j-1} (x_j - a_i)^2 g(x_j) - \sum_{i=j-M}^{j-1} \alpha_{i,j} (x_j - a_i)^2 g(x_j) \quad (47)$$

respectively. If $g(x_i) > 0$, then for $j = 1, \dots, K$, we have

$$0 = \sum_{i=j-1-M}^{j-2} (x_j - a_i)^2 \alpha_{i,j-1} - \sum_{i=j-M}^{j-1} (x_j - a_i)^2 \alpha_{i,j} \quad (48)$$

$$= \sum_{i=j-1-M}^{j-2} ((x_j - a_i)^2 \alpha_{i,j-1} - (x_j - a_{i+1})^2 \alpha_{i+1,j}) \quad (49)$$

$$= \sum_{i=j-1-M}^{j-2} ((x_j^2 - 2a_i x_j - a_i^2) \alpha_{i,j-1} - (x_j^2 - 2a_{i+1} x_j - a_{i+1}^2) \alpha_{i+1,j}) \quad (50)$$

$$= x_j^2 \left[\sum_{i=j-1-M}^{j-2} (\alpha_{i,j-1} - \alpha_{i+1,j}) \right] - 2x_j \left[\sum_{i=j-1-M}^{j-2} (a_i \alpha_{i,j-1} - a_{i+1} \alpha_{i+1,j}) \right] \\ + \left[\sum_{i=j-1-M}^{j-2} (a_i^2 \alpha_{i,j-1} - a_{i+1}^2 \alpha_{i+1,j}) \right] \quad (51)$$

Let the (i, j) -the element of the matrix $B \in \mathbb{R}^{(M+K) \times K}$ be given by

$$[B]_{i,j} = \begin{cases} \alpha_{j-1-M,j-1} & \text{if } j-1-M = i \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } j-1-M < i < j-1 \\ -\alpha_{j-1,j} & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases} \quad (52)$$

so that

$$0 = \mathcal{F}(\tilde{x}, a) := (\tilde{x} \circ \tilde{x}) \circ (B' u) - (2\tilde{x}) \circ (B' a) + u \circ (B' (a \circ a)) \quad (53)$$

where \circ denotes the Hadamard product, $u = (1, \dots, 1)' \in \mathbb{R}^K$, $\tilde{x} = (x_1, \dots, x_K)' \in \mathbb{R}^K$, and $\mathcal{F} : \mathbb{R}^K \times \mathbb{R}^{M+K} \rightarrow \mathbb{R}^K$.

3.4 Uniformly Distributed State

In what follows, suppose that $g(q) = \chi_{[0,1]}(q)$. Equation 46 becomes

$$0 = \left[\sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} (x_{k+1}^2 - x_k^2) \right] - 2a_i \left[\sum_{k=k_-(y_i)}^{k_+(y_i)-1} \alpha_{i,k} (x_{k+1} - x_k) \right] \quad (54)$$

Let the (i, j) -the element of the matrix $C \in \mathbb{R}^{(M+K) \times (K+2)}$ be given by

$$[C]_{i,j} = \begin{cases} -\alpha_{i,k_-(y_i)} & \text{if } k_-(y_i) = j \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } k_-(y_i) < j < k_+(y_i) \\ \alpha_{i,k_+(y_i)-1} & \text{if } j = k_+(y_i) \\ 0 & \text{otherwise} \end{cases} \quad (55)$$

It transpires that for $i = -M, \dots, K-1$ and $j = 1, \dots, K$, $[B]_{i,j} = [C]_{i,j}$ (C is padded on both sides by an extra column). We have

$$0 = \mathcal{G}(x, a) := C(x \circ x) - 2a \circ (Cx) \quad (56)$$

where $\mathcal{G} : \mathbb{R}^{K+2} \times \mathbb{R}^{M+K} \rightarrow \mathbb{R}^{M+K}$.

To summarize, $\mathcal{F}(x, a) = 0$ is a system of K equations and $\mathcal{G}(x, a) = 0$ a system of $K+M$ equations. x contains K unknowns (since $x_0 = 0$ and $x_{K+1} = 1$) and a contains $K+M$ unknowns. In total, there are $M+2K$ equations and $M+2K$ unknowns.

3.5 Uniformly Distributed Error

In what follows, suppose that $f(e) = N\mathbf{1}_{-\bar{e} \leq e \leq \bar{e}}$. Equation 34 becomes

$$\alpha_{i,j} = \int_{y_i}^{y_{i+1}} f(\tilde{m} - 2\bar{d}j) d\tilde{m} \quad (57)$$

$$= \int_{y_i}^{y_{i+1}} N\mathbf{1}_{-\bar{e} \leq \tilde{m} - 2\bar{d}j \leq \bar{e}} d\tilde{m} \quad (58)$$

$$= \int_{y_i}^{y_{i+1}} N\mathbf{1}_{2\bar{d}j - \bar{e} \leq \tilde{m} \leq 2\bar{d}j + \bar{e}} d\tilde{m} \quad (59)$$

$$= \int_{y_i}^{y_{i+1}} N\mathbf{1}_{y_j - M \leq \tilde{m} \leq y_j} d\tilde{m} \quad (60)$$

$$= \frac{1}{M} \mathbf{1}_{j-M \leq i < j} \quad (61)$$