Set $L(x) = \frac{1}{2}x^2$ and $g(q) = \mathbf{1}_{0 \le q \le 1}$. Fix $\bar{\epsilon} > 0$, and $I \in C[0, 1]$, where $I > 0$ on $[0, 1]$. Action functions are maps from *received messages* to actions: $\tilde{m} \mapsto a(\tilde{m})$. Message functions are maps from states to *sent messages*: $q \mapsto m(q)$. Sent messages may be no smaller than $\underline{m}$ and no larger than $\overline{m}$, where $\overline{m} - \underline{m} > 2\bar{\epsilon}$. Sent and received messages are related by the identity $\tilde{m} = m(q) + e$, where e is uniformly distributed on $[-\bar{\epsilon}, \bar{\epsilon}]$. q is known to the sender, but not the receiver; e is known to neither the sender, nor the receiver.

Definition 1. *A map $a : [\underline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon}] \to \mathbb{R}$ is an action function if it is weakly increasing.*

Message functions are similarly defined, but with the additional requirement that they satisfy certain boundary conditions.

Definition 2. *A map $m : [0, 1] \to [\underline{m}, \overline{m}]$ is a message function if it is weakly increasing, and if $m(0) = \underline{m}$ and $m(1) = \overline{m}$.*

The boundary conditions $m(0) = \underline{m}$ and $m(1) = \overline{m}$ are made to quarantee the uniqueness of an equilibrium. Note that the receiver's action has domain $[\underline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon}]$. The sender's message has codomain $[\underline{m}, \overline{m}]$, but it is corrupted by noise. Although a *sent* message may be no smaller than $\underline{m}$, and no larger than $\overline{m}$, a *received* message may be as small as $\underline{m} - \bar{\epsilon}$, or as large as $\overline{m} + \bar{\epsilon}$. The receiver must choose an action in $\mathbb{R}$ for messages less than $\underline{m}$ or greater than $\overline{m}$. Since e is uniformly distributed on $[-\bar{\epsilon}, \bar{\epsilon}]$, $\tilde{m}$ has density

$$f(\tilde{m}) = \frac{1}{2\bar{\epsilon}} \mathbf{1}_{m(q) - \bar{\epsilon} \le \tilde{m} \le m(q) + \bar{\epsilon}}. \tag{1}$$

The sender's and receiver's payoffs are

$$S(a, m; q) = \int_{m(q)-\bar{\epsilon}}^{m(q)+\bar{\epsilon}} I(q) L(a(\tilde{m}) - q) f(\tilde{m}) d\tilde{m}, \text{ and} \tag{2}$$

$$R(a, m; \tilde{m}) = \int_0^1 I(q) L(a(\tilde{m}) - q) g(q|\tilde{m}) dq \tag{3}$$

respectively.

Proposition 1 (Continuous and Strictly Increasing Message Functions)**.** *There does not exist an equilibrium in which the message function is continuous and strictly increasing, and the receiver believes it to be so.*

Proof. By way of contradiction, let m be an equilibrium message function, and suppose that it is continuous and strictly increasing. Put $\ell = m^{-1}$. Since m is continuous and strictly increasing, so too is ℓ. Moreover, $\ell(\underline{m}) = 0$ and $\ell(\overline{m}) = 1$. Define

$$\overline{q}(\tilde{m}) = \ell(\min\{\tilde{m} + \bar{\epsilon}, \overline{m}\}), \text{ and} \tag{4}$$

$$\underline{q}(\tilde{m}) = \ell(\max\{\tilde{m} - \bar{\epsilon}, \underline{m}\}). \tag{5}$$

Given a received message $\tilde{m}$, $\overline{q}(\tilde{m})$ is the *largest* possible state that could have resulted in $\tilde{m}$, while $\underline{q}(\tilde{m})$ is the *smallest* possible state that could have resulted in $\tilde{m}$. For $\tilde{m} \in (\underline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon})$,

$\overline{q}(\tilde{m}) > \underline{q}(\tilde{m})$. Moreover, $\overline{q}(\underline{m}-\bar{\epsilon}) = \underline{q}(\underline{m}-\bar{\epsilon})$, and $\overline{q}(\underline{m}+\bar{\epsilon}) = \underline{q}(\underline{m}+\bar{\epsilon})$. As they will appear often, define

$$\underline{\mathcal{I}} := (\underline{m} - \bar{\epsilon}, \underline{m} + \bar{\epsilon}), \tag{6}$$

$$\mathcal{I} := (\underline{m} + \bar{\epsilon}, \overline{m} - \bar{\epsilon}), \text{ and} \tag{7}$$

$$\overline{\mathcal{I}} := (\overline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon}). \tag{8}$$

On $\underline{\mathcal{I}}$, $\underline{q}' = 0$ and $\overline{q}' > 0$; on $\mathcal{I}$, $\underline{q}' > 0$ and $\overline{q}' > 0$; on $\overline{\mathcal{I}}$, $\underline{q}' > 0$ and $\overline{q}' = 0$. If $\tilde{m} \in (\underline{m}-\bar{\epsilon}, \overline{m}+\bar{\epsilon})$, then the receiver has beliefs,

$$g(q|\tilde{m}) = \frac{f(\tilde{m}|q)g(q)}{f(\tilde{m})} = \frac{\frac{1}{2\bar{\epsilon}}\mathbf{1}_{m(q)-\bar{\epsilon}\leq\tilde{m}\leq m(q)+\bar{\epsilon}} \cdot \mathbf{1}_{0\leq q\leq 1}}{\int_{-\infty}^{\infty} \frac{1}{2\bar{\epsilon}}\mathbf{1}_{m(t)-\bar{\epsilon}\leq\tilde{m}\leq m(t)+\bar{\epsilon}} \cdot \mathbf{1}_{0\leq t\leq 1} dt} = \frac{\mathbf{1}_{\underline{q}(\tilde{m})\leq q\leq\overline{q}(\tilde{m})}}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})}. \tag{9}$$

If she receives the message $\underline{m} - \bar{\epsilon}$, then she believes that the state is 0 almost surely, while if she receives the message $\overline{m} + \bar{\epsilon}$, then she believes that the state is 1 almost surely. Formally, $g(q|\underline{m} - \bar{\epsilon}) = \delta(q - (\underline{m} - \bar{\epsilon}))$ and $g(q|\overline{m} + \bar{\epsilon}) = \delta(q - (\overline{m} + \bar{\epsilon}))$, where $\delta(\bullet)$ is the Dirac delta function. Next, we compute the receiver's best response action function. If $\tilde{m} \in (\underline{m}-\bar{\epsilon}, \overline{m}+\bar{\epsilon})$, then R is strictly convex in its first argument:

$$R_{aa}(a, m; \tilde{m}) = \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} \frac{I(q)L''(a(\tilde{m}) - q)dq}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})} = \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} \frac{I(q)dq}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})} > 0. \tag{10}$$

The solution of the first-order condition is the unique minimizer:

$$0 = R_a(\tilde{m}, a, m) = \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} \frac{I(q)L'(a(\tilde{m}) - q)dq}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})} = \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} \frac{I(q)(a(\tilde{m}) - q)dq}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})}, \tag{11}$$

which implies that the receiver plays

$$a(\tilde{m}) = \left[\int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} I(r)dr\right]^{-1} \left[\int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} rI(r)dr\right]. \tag{12}$$

If she receives the message $\tilde{m} - \bar{\epsilon}$, then she believes that the state is 0 almost surely, in which case she plays $a(\tilde{m} - \bar{\epsilon}) = 0$. If she receives the message $\tilde{m} + \bar{\epsilon}$, then she beliefs that the state is 1 almost surely, in which case she plays $a(\tilde{m} + \bar{\epsilon}) = 1$. Since $\underline{q}$ and $\overline{q}$ are continuous, so too is a. On $\underline{\mathcal{I}} \cup \mathcal{I} \cup \overline{\mathcal{I}}$,

$$a' = \left[\int_{\underline{q}}^{\overline{q}} I(r)dr\right]^{-2} \left[(\overline{q}'\overline{q}I(\overline{q}) - \underline{q}'\underline{q}I(\underline{q})) \int_{\underline{q}}^{\overline{q}} I(r)dr - (\overline{q}'I(\overline{q}) - \underline{q}'I(\underline{q})) \int_{\underline{q}}^{\overline{q}} rI(r)dr\right] \tag{13}$$

$$= \left[\int_{\underline{q}}^{\overline{q}} I(r)dr\right]^{-2} \left[\overline{q}'I(\overline{q}) \int_{\underline{q}}^{\overline{q}} (\overline{q} - r)dr + \underline{q}'I(\underline{q}) \int_{\underline{q}}^{\overline{q}} (r - \underline{q})dr\right] > 0. \tag{14}$$

a is strictly increasing on $\underline{\mathcal{I}} \cup \mathcal{I} \cup \overline{\mathcal{I}}$, and since it is continuous on $[\underline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon}]$, it is strictly increasing on $[\underline{m} - \bar{\epsilon}, \overline{m} + \bar{\epsilon}]$. Finally, we show that the sender has a profitable deviation. Define $\bar{a}(m) := \frac{1}{2}(a(m + \bar{\epsilon}) + a(m - \bar{\epsilon}))$. Fix $q \in [0, \bar{a}(\underline{m}))$ and define

$$r_1(m) := a(m + \bar{\epsilon}) - a(m - \bar{\epsilon}), \text{ and} \tag{15}$$

$$r_2(m) := \bar{a}(m) - q. \tag{16}$$

Note that $r_2(\underline{m}) > 0$. Since a is strictly increasing, r_1 is strictly positive, and r_2 is strictly increasing. Observe that

$$\begin{aligned} S_{m(q)}(a, m; q) &= L(a(m + \bar{\epsilon}) - q) - L(a(m - \bar{\epsilon}) - q) && (17) \\ &= \tfrac{1}{2}(a(m + \bar{\epsilon}) - q)^2 - \tfrac{1}{2}(a(m - \bar{\epsilon}) - q)^2 && (18) \\ &= \tfrac{1}{2}(a(m + \bar{\epsilon}) - a(m - \bar{\epsilon}))(a(m + \bar{\epsilon}) + a(m - \bar{\epsilon}) - 2q) && (19) \\ &= (a(m + \bar{\epsilon}) - a(m - \bar{\epsilon}))(\tfrac{1}{2}(a(m + \bar{\epsilon}) + a(m - \bar{\epsilon})) - q) && (20) \\ &= r_1(m) r_2(m). && (21) \end{aligned}$$

Since $r_2(\underline{m}) > 0$ and r_2 is strictly increasing, $r_2 > 0$ on $[\underline{m}, \overline{m}]$, and hence $S_{m(q)} > 0$ on $[\underline{m}, \overline{m}]$. The sender finds $m^*(q) = \underline{m}$ optimal. The message function obtained cannot be the equilibrium message function, since m is strictly increasing, and m^* is not. Therefore, m^* is a profitable deviation for the sender, contradicting the assumption that m is an equilibrium message function. ∎

The above proof suggests that a continuous and strictly increasing message function could be rectified by taking $\overline{m} \to \infty$ and $\underline{m} \to -\infty$. This is not the case. One obtains $\ell(q) = \frac{1}{2}$ for $q \in (0, 1)$, which is also constant.