

1 Identity Message

Let $x : [-\bar{e}, \bar{e}] \rightarrow [0, 1]$ be given by

$$x(e) = \frac{1}{2} \left[1 + \frac{e}{\bar{e}} \right] \quad (1)$$

Define

$$I_{(\alpha, \beta, \gamma)}(\tilde{m}) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - q) dq \quad (2)$$

and

$$A(\tilde{m}) = \begin{cases} \bar{a}(\tilde{m}) & \text{if } 1 - \bar{e} < \tilde{m} \leq 1 + \bar{e} \\ a(\tilde{m}) & \text{if } \bar{e} < \tilde{m} \leq 1 - \bar{e} \\ \underline{a}(\tilde{m}) & \text{if } -\bar{e} \leq \tilde{m} \leq \bar{e} \end{cases} \quad (3)$$

where

$$\bar{a}(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, 1, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, 1, 0)}(\tilde{m})} \quad (4)$$

$$a(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (5)$$

$$\underline{a}(\tilde{m}) = \frac{I_{(0, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(0, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (6)$$

Note that the normalizing constant cancels out when computing the conditional expectation. We need three sets of weights and knots.

The cost of the identity message function is given by

$$C[m_i] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - A(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m}. \quad (7)$$

where

$$q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\} \quad (8)$$

$$q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\} \quad (9)$$

Define

$$\bar{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\tilde{m}-\bar{e}}^1 (q - \bar{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (10)$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\tilde{m}-\bar{e}}^{\tilde{m}+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (11)$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_0^{\tilde{m}+\bar{e}} (q - \underline{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (12)$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1} (1 - x(e))^{b-1} \quad (13)$$

for constants $a > 0$ and $b > 0$.

2 Discrete Message

A discrete message with $N + 1$ message is given by

$$m_d(q) = \sum_{n=0}^N \frac{n}{N} \mathbf{1}_{\frac{n}{N+1} \leq q \leq \frac{n+1}{N+1}}. \quad (14)$$

where $\bar{e} = 1/(2N)$.

$$q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\} \quad (15)$$

$$q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\} \quad (16)$$

We have

$$I_\gamma(k) = \int_{q_k}^{q_{k+1}} q^\gamma f(\tilde{m} - m(q)) dq \quad (17)$$

The receiver knows which message was sent. The action function is

$$A(\tilde{m}) = \quad (18)$$

The cost of the discrete message function is given by

$$C[m_d] = (N + 1) \quad (19)$$