There are two players, sender and receiver, and a state of nature $q \sim U[0,1]$ which is known to the sender but not to the receiver. Let g denote the density of q. The sender and receiver have preferences over actions $U(q,A) = -(A-q)^2$. Let $\mathcal{M}$ denote the set of all non-decreasing functions from $[0,1]$ to $[0,1]$. Let $m \in \mathcal{M}$ be a message function. Let $Q(\widetilde{m})$ denote the set of all $q \in [0,1]$ such that for some $e \in [-\bar{\epsilon}, \bar{\epsilon}]$, $\widetilde{m} = m(q) + e$. Finally, put $q_+(\widetilde{m}) \equiv \sup Q(\widetilde{m})$, and $q_-(\widetilde{m}) \equiv \inf Q(\widetilde{m})$. Suppose that the receiver receives the message $\widetilde{m} \in [-\bar{\epsilon}, 1+\bar{\epsilon}]$. $q|\widetilde{m}$ has support $[q_-(\widetilde{m}), q_+(\widetilde{m})]$, and

$$g(q|\widetilde{m}) = \frac{f_e(\widetilde{m} - m(q))\mathbf{1}_{0 \leq q \leq 1}}{\int_0^1 f_e(\widetilde{m} - m(t))\mathbf{1}_{0 \leq t \leq 1} dt}. \tag{1}$$

Her optimal action is

$$A(\widetilde{m}) \equiv \underset{a}{\operatorname{argmin}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q-a)^2 g(q|\widetilde{m}) = \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} q g(q|\widetilde{m}) \tag{2}$$

As it will appear often, let the cost functional $C : \mathcal{M} \to \mathbb{R}$ be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{\epsilon}}^{\bar{\epsilon}} (q - A(m(q)+e))^2 f_\epsilon(e) de dq. \tag{3}$$

The *sender's problem* is to choose a message $m \in \mathcal{M}$ that minimizes C. A change of variables and an application of Fubini's Theorem yeild

$$C[m] = \int_0^1 \int_{m(q)-\bar{\epsilon}}^{m(q)+\bar{\epsilon}} (q - A(\widetilde{m}))^2 f_\epsilon(\widetilde{m} - m(q)) d\widetilde{m} dq \tag{4}$$

$$= \int_{-\bar{\epsilon}}^{1+\bar{\epsilon}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f_\epsilon(\widetilde{m} - m(q)) dq d\widetilde{m}. \tag{5}$$

Finally, note that $\int_{-\bar{\epsilon}}^{1+\bar{\epsilon}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} dq d\widetilde{m} = \int_0^1 \int_{m(q)-\bar{\epsilon}}^{m(q)+\bar{\epsilon}} d\widetilde{m} dq = 2\bar{\epsilon}$.

Proposition 1. *If the error is uniform, then the discrete message function is optimal.*

Proof. $f_e(e) = \frac{1}{2\bar{\epsilon}}\mathbf{1}_{-\bar{\epsilon}\leq e\leq\bar{\epsilon}}$, $g(q|\widetilde{m}) = \frac{1}{q_+(\widetilde{m})-q_-(\widetilde{m})}\mathbf{1}_{q_-(\widetilde{m})\leq q\leq q_+(\widetilde{m})}$, $A(\widetilde{m}) = \frac{q_+(\widetilde{m})+q_-(\widetilde{m})}{2}$, and

$$C[m] = \frac{1}{2\bar{\epsilon}}\int_{-\bar{\epsilon}}^{1+\bar{\epsilon}}\left[\frac{2}{3}\left(\frac{q_+(\widetilde{m})-q_-(\widetilde{m})}{2}\right)^3\right]d\widetilde{m}. \tag{6}$$

Under m_d, $q_+(\widetilde{m})-q_-(\widetilde{m}) = \frac{2\bar{\epsilon}}{2\bar{\epsilon}+1}$, and hence $C[m_d] = \frac{1}{3}\left(\frac{\bar{\epsilon}}{1+2\bar{\epsilon}}\right)^2$. Let $m \in \mathcal{M}$.

$$\begin{aligned}
C[m] &= \frac{1+2\bar{\epsilon}}{2\bar{\epsilon}}\int_{-\bar{\epsilon}}^{1+\bar{\epsilon}}\left[\frac{2}{3}\left(\frac{q_+(\widetilde{m})-q_-(\widetilde{m})}{2}\right)^3\right]\cdot\frac{d\widetilde{m}}{1+2\bar{\epsilon}} && (7)\\
&\geq \frac{1+2\bar{\epsilon}}{2\bar{\epsilon}}\cdot\frac{2}{3}\left(\int_{-\bar{\epsilon}}^{1+\bar{\epsilon}}\left(\frac{q_+(\widetilde{m})-q_-(\widetilde{m})}{2}\right)\cdot\frac{d\widetilde{m}}{1+2\bar{\epsilon}}\right)^3 && (8)\\
&= \frac{1+2\bar{\epsilon}}{3\bar{\epsilon}}\left(\frac{\bar{\epsilon}}{1+2\bar{\epsilon}}\right)^3 && (9)\\
&= C[m_d], && (10)
\end{aligned}$$

where the second line follows by Jensen's inequality. ■

Proposition 2. *If the error is not uniform and if its variance is sufficiently small* ($<$ $C[m_d]/(1+2\bar{\epsilon})$)*, then the optimal message function is not discrete.*

Proof. Put $m_i(q) \equiv q$. Under m_i, we have that $q_-(\widetilde{m}) = \widetilde{m}-\bar{\epsilon}$, $q_+(\widetilde{m}) = \widetilde{m}+\bar{\epsilon}$, and $g(q|\widetilde{m}) = f_e(\widetilde{m}-m(q))$. Put $\delta \equiv \widetilde{m}-m_i(q)$. $A(\widetilde{m}) = \widetilde{m} - \int_{-\bar{\epsilon}}^{\bar{\epsilon}}\delta f_e(\delta)d\delta$, and

$$C[m_i] = \int_{-\bar{\epsilon}}^{1+\bar{\epsilon}}\int_{-\bar{\epsilon}}^{\bar{\epsilon}}\left(\delta - \int_{-\bar{\epsilon}}^{\bar{\epsilon}}\delta' f_e(\delta')d\delta'\right)^2 f_\epsilon(\delta)d\delta d\widetilde{m} \leq C[m_d] \tag{11}$$

from which the result obtains. ■