Taylor expansion:

$$m^{-1}(m(q)+\epsilon) = m^{-1}(m(q)) + \frac{\epsilon}{m'(m^{-1}(m(q)))} + \cdots = q + \frac{\epsilon}{m'(q)} + \cdots \tag{1}$$

Cost functional:

$$\begin{aligned} C[m] &= \int_0^1 \int_{-\bar\epsilon}^{\bar\epsilon} \left(q - m^{-1}(m(q)+\epsilon)\right)^2 \cdot \frac{d\epsilon}{2\bar\epsilon} \cdot g(q)dq &(2)\\ &\approx \int_0^1 \int_{-\bar\epsilon}^{\bar\epsilon} \left(q - \left(q + \frac{\epsilon}{m'(q)}\right)\right)^2 \cdot \frac{d\epsilon}{2\bar\epsilon} \cdot g(q)dq &(3)\\ &= \int_0^1 \int_{-\bar\epsilon}^{\bar\epsilon} \frac{\epsilon^2}{m'(q)^2} \cdot \frac{d\epsilon}{2\bar\epsilon} \cdot g(q)dq &(4)\\ &= \frac{\bar\epsilon^2}{3} \cdot \int_0^1 \frac{g(q)dq}{m'(q)^2} &(5) \end{aligned}$$

Euler-Lagrange:

$$0 = \frac{d}{dq}\left(\frac{g(q)}{m'(q)^3}\right) = \frac{g'(q)m'(q)^3 - 3g(q)m'(q)^2m''(q)}{m'(q)^6}. \tag{6}$$

Equivalently:

$$0 = g'(q)m'(q) - 3g(q)m''(q) \tag{7}$$

subject to $m(0)=0$ and $m(1)=1$. Solution:

$$\boxed{m(q) = \frac{1}{C}\int_0^q g(t)^{1/3}dt} \tag{8}$$

where

$$C = \int_0^1 g(t)^{1/3}dt \tag{9}$$

from which it immediately follows that

$$m'(q) = C^{-1}g(q)^{1/3}. \tag{10}$$

Example. $g(q) = q^3 \Rightarrow m(q) = q^2$. Put

$$m_n(q) = \sum_{k=0}^{n} \mathbf{1}_{q_{k-1}\le q<q_k}. \tag{11}$$

The optimal action is

$$a_k = A(q_k, q_{k+1}) \equiv \frac{\int_{q_k}^{q_{k+1}} tg(t)dt}{\int_{q_k}^{q_{k+1}} g(t)dt}. \tag{12}$$

The q_k obey the second-order difference equation

$$2q_k = A(q_{k-1}, q_k) + A(q_k, q_{k+1}) \tag{13}$$

subject to $q_0 = 0$ and $q_{n+1} = 1$.