1 discrete messages

Assume throughout existence and uniqueness of solutions.

1.1 one if by land, two if by sea...

For each $x \in [0,1]$, let the message $m_x : [0,1] \to \{m_L, m_H\}$ be given by

$$m_x(q) = \begin{cases} m_L & \text{if } q < x \\ m_H & \text{otherwise} \end{cases}$$

and the action $A_x : \{m_L, m_H\} \to \mathbb{R}$ by

$$A(m) = \begin{cases} \operatorname{argmin}_a \int_0^x L(a-q)I(q)g(q)dq & \text{if } m = m_L \\ \operatorname{argmin}_a \int_x^1 L(a-q)I(q)g(q)dq & \text{otherwise} \end{cases}$$

where L, I and g are defined in the paper. The receiver's first-order conditions for $a_L := A(m_L)$ and $a_H := A(m_H)$ are

$$0 = \int_0^x L'(a_L(x)-q)I(q)g(q)dq;$$
$$0 = \int_x^1 L'(a_H(x)-q)I(q)g(q)dq.$$

The sender chooses x to minimize total loss:

$$\min_x \int_0^x L(a_L(x)-q)I(q)g(q)dq + \int_x^1 L(a_H(x)-q)I(q)g(q)dq.$$

The sender's first-order condition for x is

$$0 = \int_0^x L'(a_L(x)-q)a_L'(x)I(q)g(q)dq + L(a_L(x)-x)I(x)g(x)$$
$$+ \int_x^1 L'(a_H(x)-q)a_H'(x)I(q)g(q)dq - L(a_H(x)-x)I(x)g(x)$$

Using the receiver's first-order conditions, we obtain

$$L(a_L(x)-x) = L(a_H(x)-x).$$

$L(|\bullet|)$ is even and $L' > 0$, so there are two solutions: $a_L(x) = a_H(x)$ and $x = (a_L(x) + a_H(x))/2$. Assume (correctly, I think) that the second solution is the unique minimizer of the total loss. a_L, a_H and x obey

$$0 = \int_0^x L'(a_L - q)I(q)g(q)dq$$

$$0 = \int_x^1 L'(a_H - q)I(q)g(q)dq$$

and

$$x = \frac{a_L + a_H}{2}.$$

There are three equations and three unknowns.

1.1.1 example (in progress)

Let $L(x) = \frac{1}{2}x^2$, $I(x) = b + (1 - 2b)x$, and $g(x) = 1$.

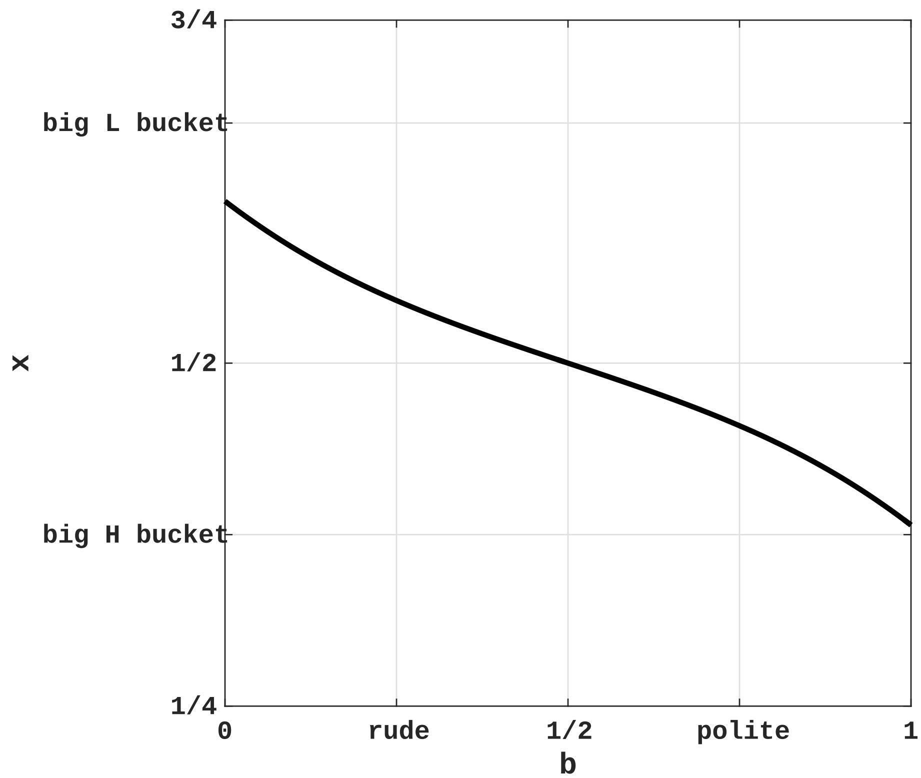

Figure 1: $L(x) = .5x^2$, $g(x) = 1$ and $n = 2$.

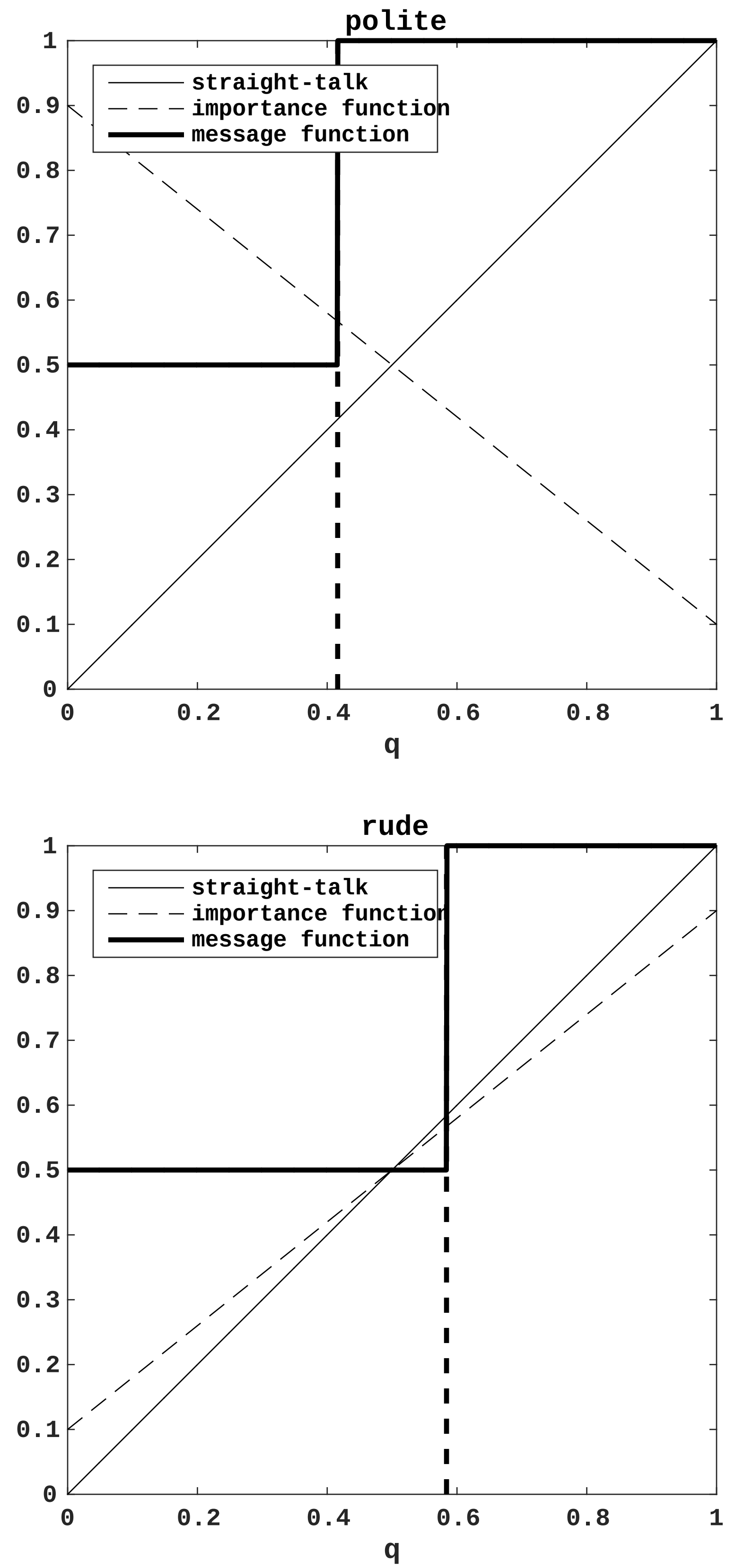

Figure 2: $L(q) = .5q^2$, $I_{rude}(q) = .1 + .8q$, $I_{polite}(q) = .9 - .8q$, $g(q) = 1$ and $n = 2$.

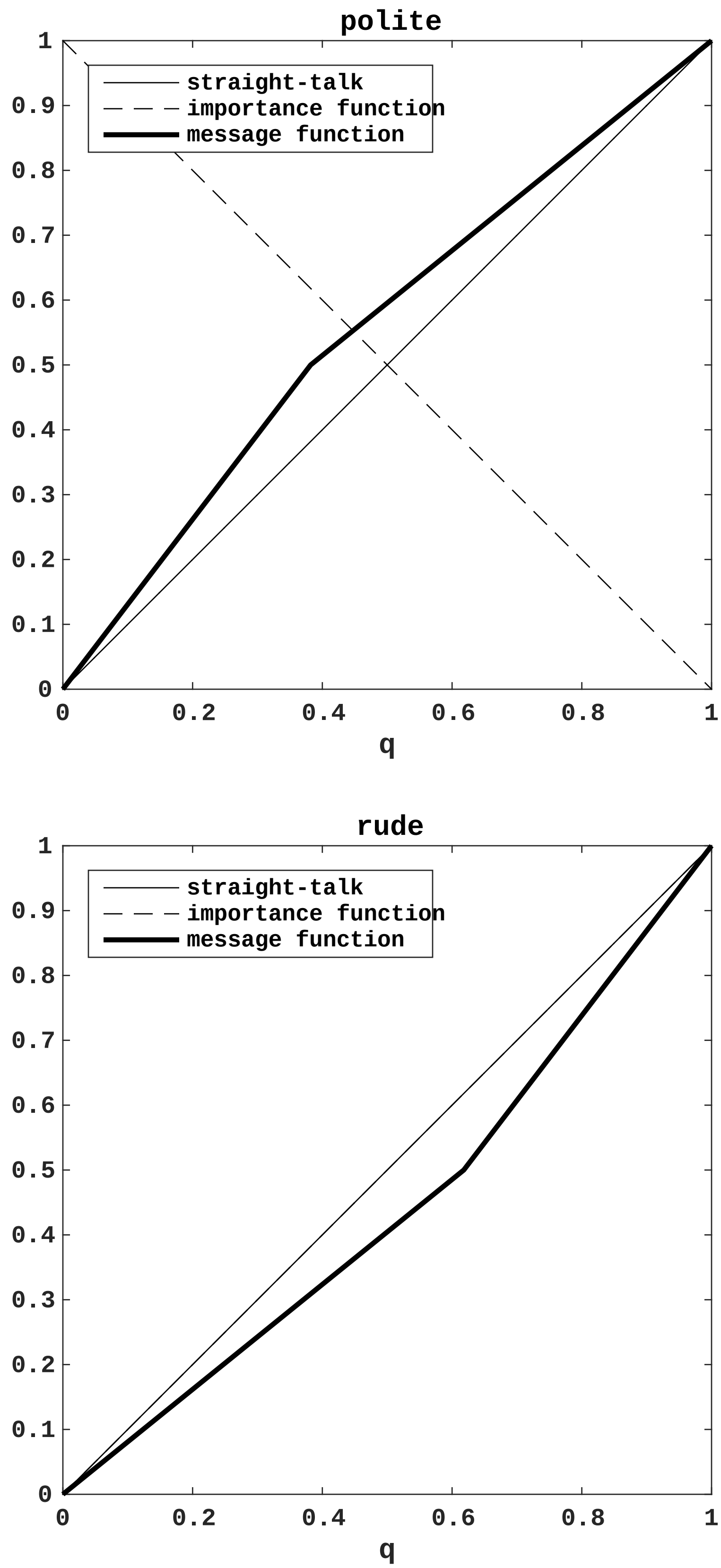

Figure 3: $L(q) = .5q^2$, $I_{rude}(q) = .1 + .8q$, $I_{polite}(q) = .9 - .8q$, $g(q) = 1$ and $n = 2$.

1.2 ...three if the British are drinking tea

Let there be $n \in \mathbb{N}$ messages: $m_1, m_2, \ldots, m_n$. For each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ with $0 = x_0 < x_1 < \cdots < x_n = 1$, let the message $m_P : [0,1] \to \{m_1, m_2, \ldots, m_n\}$ be given by

$$m_P(q) = \sum_{t=1}^{n} m_t \chi_{[x_{t-1}, x_t)}(q) + m_n \chi_{\{1\}}(q)$$

and the action $A_P : \{m_1, m_2, \ldots, m_n\} \to [0,1]$ by given by

$$A_P(m) = \sum_{t=1}^{n} \operatorname*{argmin}_{a_t} \chi_{\{m_t\}}(m) \int_{x_{t-1}}^{x_t} L(a_t - q) I(q) g(q) dq$$

where χ is the indicator function and L, I and g are defined in the paper. Fix $t \in \{1, 2, \ldots, n\}$. The receiver's first-order condition for $a_t := A(m_t)$ is

$$0 = \int_{x_{t-1}}^{x_t} L'(a_t(x_t, x_{t-1}) - q) I(q) g(q) dq.$$

The sender chooses P to minimize total loss:

$$\min_P \sum_{t=1}^{n} \int_{x_{t-1}}^{x_t} L(a_t(x_t, x_{t-1}) - q) I(q) g(q) dq.$$

The sender's first-order condition for x_t is

$$\begin{aligned} 0 = &\int_{x_{t-1}}^{x_t} L'(a_t - q) \frac{\partial a_t}{\partial x_t} I(q) g(q) dq + L(a_t - x_t) I(x_t) g(x_t) \\ &+ \int_{x_t}^{x_{t+1}} L'(a_{t+1} - q) \frac{\partial a_{t+1}}{\partial x_t} I(q) g(q) dq - L(a_{t+1} - x_t) I(x_t) g(x_t) \end{aligned}$$

so that

$$L(a_t - x_t) = L(a_{t+1} - x_t).$$

Again, there are two solutions: $a_t = a_{t+1}$ and $x_t = (a_t + a_{t+1})/2$. Assume that the second solution is the unique minimizer of the total loss. $a_1, a_2, \ldots, a_n$ and $x_1, x_2, \ldots, x_{n-1}$ obey

$$0 = \int_{x_{t-1}}^{x_t} L'(a_t - q) I(q)(q) dq$$

for $t \in \{1, 2, \ldots, n\}$ and

$$x_t = \frac{a_t + a_{t+1}}{2}$$

for $t \in \{1, 2, \ldots, n-1\}$. There are $2n + 1$ equations and $2n + 1$ unknowns.

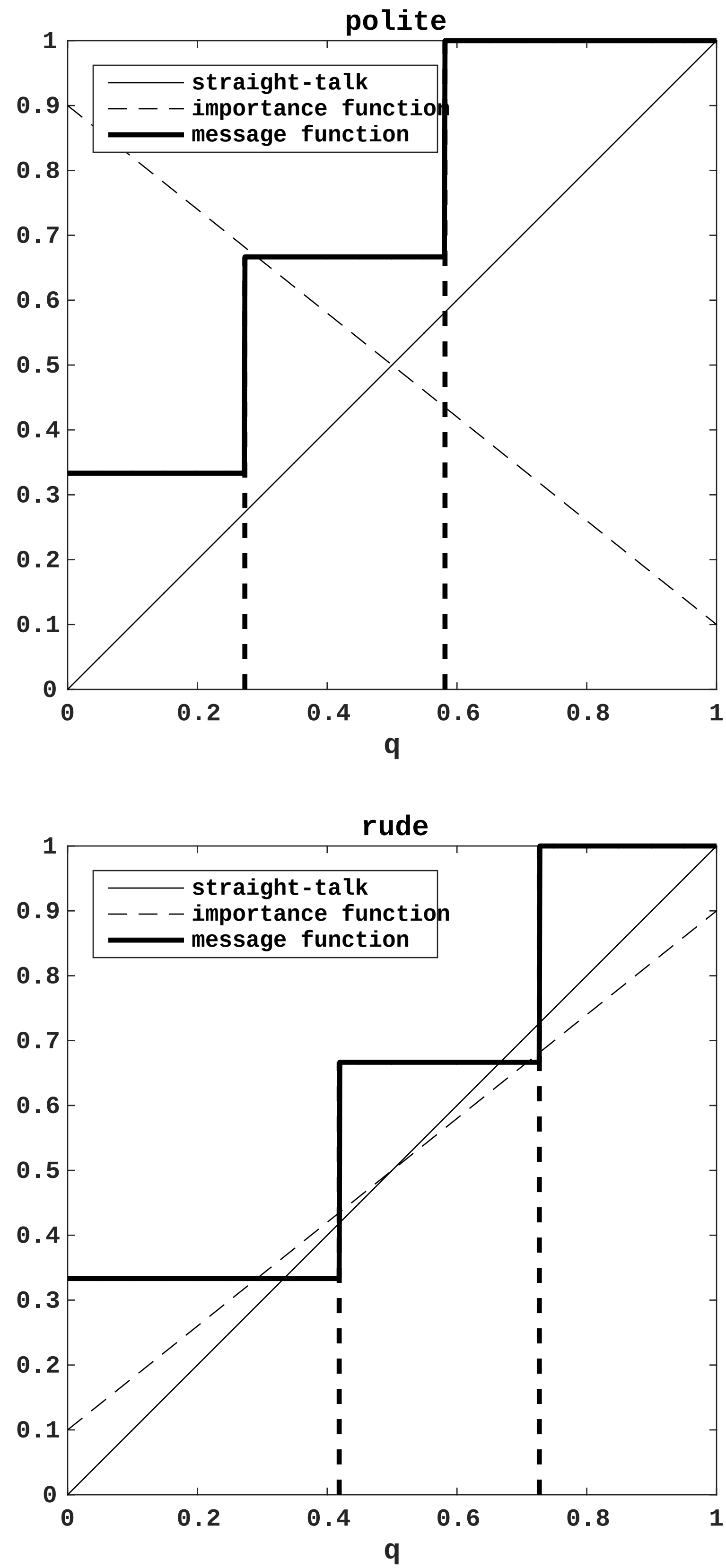

Figure 4: $L(q) = .5q^2$, $I_{rude}(q) = .1 + .8q$, $I_{polite}(q) = .9 - .8q$, $g(q) = 1$ and $n = 3$.

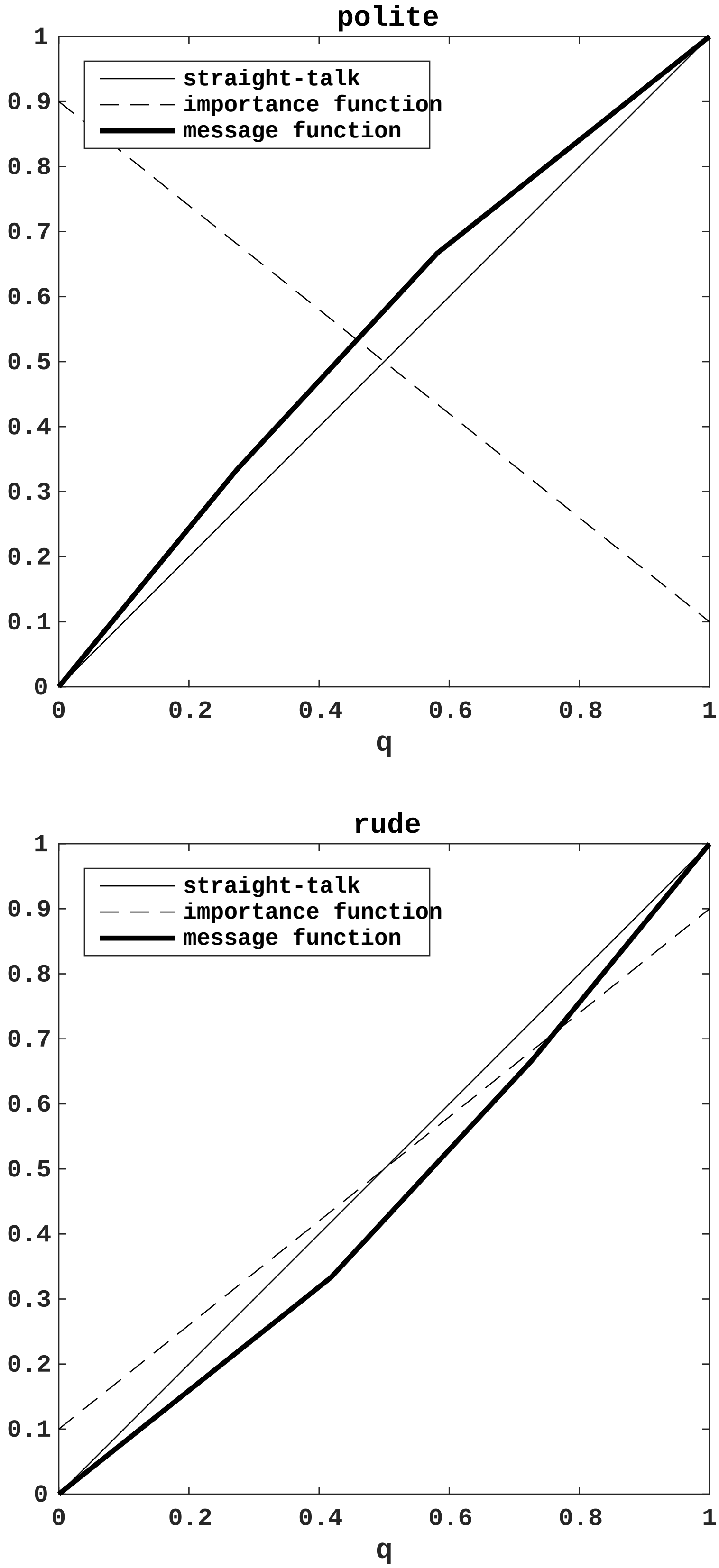

Figure 5: $L(q) = .5q^2$, $I_{rude}(q) = .1 + .8q$, $I_{polite}(q) = .9 - .8q$, $g(q) = 1$ and $n = 3$.