1 The Model

There are two players, a sender and a receiver, and a state of nature $q \in [0,1]$ that is known to the sender but not the receiver. Let g denote the density of q. The sender sends an *intended* message $m(q) \in [0,1]$. The receiver receives a noisy version of the intended message, which we call the *received* message, $\widetilde{m} = m(q) + e$. e is distributed according to a continuous density f with support $[-\bar{e}, \bar{e}]$, where for some integer $N \geq 2$, $\bar{e} = \frac{1}{2N}$.

We specify f as follows. Let $x : [-\bar{e}, \bar{e}] \to [0,1]$ be given by

$$x(e) = \frac{1}{2}\left[1 + \frac{e}{\bar{e}}\right] \tag{1}$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1}(1 - x(e))^{b-1} \tag{2}$$

for constants $a > 0$ and $b > 0$. The receiver then takes an action $a(\widetilde{m}) \in [0,1]$. Both sender and receiver have utility over the state, q, and the receiver's action, a, of

$$U(q,a) = -\frac{1}{2}(q-a)^2 \tag{3}$$

Prior to the start of the game, the sender can specify an intended message function $m(q)$ that she will use. The receiver chooses an action based upon the function $m(q)$ and the received message $\widetilde{m}$, and we denote this function $a(\widetilde{m})$. We work backward and start with the optimal action, given $m(q)$ and $\widetilde{m}$.

Fix a message function m. Let $Q(\widetilde{m})$ denote the set of all states $q \in [0,1]$ such that for some noise $e \in [-\bar{e}, \bar{e}]$, $\widetilde{m} = m(q) + e$. In set builder notation,

$$Q(\widetilde{m}) = \{q \in [0,1] \mid m(q) - \bar{e} \leq \widetilde{m} \leq m(q) + \bar{e}\}. \tag{4}$$

Finally, define

$$q_+(\widetilde{m}) \equiv \sup Q(\widetilde{m}) \tag{5}$$

$$q_-(\widetilde{m}) \equiv \inf Q(\widetilde{m}) \tag{6}$$

$q_+(\widetilde{m})$ and $q_-(\widetilde{m})$ are the highest and lowest states that could possibly be associated with the received message $\widetilde{m}$. Define

$$I(\widetilde{m};\alpha,\beta,\gamma) \equiv \int\limits_{\alpha}^{\beta} q^{\gamma} f(\tilde{m}-m(q))g(q)dq \tag{7}$$

Suppose that the receiver receives the message $\widetilde{m} \in [-\bar{e}, 1+\bar{e}]$. $q|\widetilde{m}$ has support $[q_{-}(\widetilde{m}), q_{+}(\widetilde{m})]$ and density

$$g(q|\widetilde{m}) = \frac{f(\widetilde{m}-m(q))g(q)}{\int_0^1 f(\widetilde{m}-m(t))g(t)dt} \tag{8}$$

The receiver's problem is to

$$\max_{a(\widetilde{m})} \int\limits_{q_{-}(\widetilde{m})}^{q_{+}(\widetilde{m})} U(q,a)g(q|\widetilde{m})dq. \tag{9}$$

The receiver's optimal action is simply the expected value of the state, q, given the received message $\widetilde{m}$:

$$a(\widetilde{m}) = \int\limits_{q_{-}(\widetilde{m})}^{q_{+}(\widetilde{m})} qg(q|\widetilde{m})dq = \frac{I(\widetilde{m};q_{-}(\widetilde{m}),q_{+}(\widetilde{m}),1)}{I(\widetilde{m};q_{-}(\widetilde{m}),q_{+}(\widetilde{m}),0)} \tag{10}$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional C be given by

$$C[m] \equiv \int\limits_{0}^{1}\int\limits_{-\bar{e}}^{\bar{e}} (q-a(m(q)+e))^2 f(e)dedq, \tag{11}$$

where a is the receiver's optimal action from Equation (10). $C[m]$ is the expected loss for a given message function m. The integrand is the loss for a given state q and action $a(\widetilde{m})$. The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function m that minimizes $C[m]$:

$$\min_{m\in M} C[m] \tag{12}$$

where M is the space of weakly increasing piece-wise continuous functions on $[0,1]$. The change of variables $\widetilde{m} = m(q) + e$ and an application of Fubini's Theorem yield

$$C[m] = \int_{0}^{1} \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - a(\widetilde{m}))^2 f(\widetilde{m} - m(q)) g(q) d\widetilde{m} dq \tag{13}$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - a(\widetilde{m}))^2 f(\widetilde{m} - m(q)) g(q) dq d\widetilde{m}. \tag{14}$$

We consider the costs of identity ("$\mathcal{I}$") and discrete ("$\mathcal{D}$") message functions.

2 Identity Message

We consider the identity message function, $m(q) = q$, only for the case in which q is uniformly distributed.

$$a(\widetilde{m}) = \begin{cases} \overline{a}(\widetilde{m}) & \text{if } 1 - \overline{e} < \widetilde{m} \leq 1 + \overline{e} \\ a(\widetilde{m}) & \text{if } \overline{e} < \widetilde{m} \leq 1 - \overline{e} \\ \underline{a}(\widetilde{m}) & \text{if } -\overline{e} \leq \widetilde{m} \leq \overline{e} \end{cases} \tag{15}$$

where

$$\overline{a}(\widetilde{m}) = \frac{I(\widetilde{m}; \widetilde{m} - \bar{e}, 1, 1)}{I(\widetilde{m}; \widetilde{m} - \bar{e}, 1, 0)} \tag{16}$$

$$a(\widetilde{m}) = \frac{I(\widetilde{m}; \widetilde{m} - \bar{e}, \widetilde{m} + \bar{e}, 1)}{I(\widetilde{m}; \widetilde{m} - \bar{e}, \widetilde{m} + \bar{e}, 0)} \tag{17}$$

$$\underline{a}(\widetilde{m}) = \frac{I(\widetilde{m}; 0, \widetilde{m} + \bar{e}, 1)}{I(\widetilde{m}; 0, \widetilde{m} + \bar{e}, 0)} \tag{18}$$

Note that the normalizing constant cancels out when computing the conditional expectation. The cost of the identity message function is given by

$$C[m_{\mathcal{I}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - a(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m}. \tag{19}$$

where $q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\}$ and $q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\}$. Define

$$\overline{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\widetilde{m}-\bar{e}}^{1} (q - \overline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{20}$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\widetilde{m}-\bar{e}}^{\widetilde{m}+\bar{e}} (q - a(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{21}$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_{0}^{\widetilde{m}+\bar{e}} (q - \underline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{22}$$

so that $C[m_{\mathcal{I}}] = \overline{z} + z + \underline{z}$.

3 Discrete Message

Fix an integer $M \geq 1$ and define $K = M \times N$. Define $\bar{d} = \frac{1}{2K}$. Consider the partition

$$0 = x_0 < x_1 < \cdots < x_K < x_{K+1} = 1 \tag{23}$$

of $[0, 1]$. For each $i \in \{0, \ldots, K\}$, define $X_i = [x_i, x_{i+1})$. A discrete message with $K + 1$ messages is given by

$$m_{\mathcal{D}}(q) = \sum_{i=0}^{K} \tfrac{i}{K} \chi_{X_i}(q) \tag{24}$$

(where χ is the characteristic function). Let $k_+, k_- : [-\bar{e}, 1 + \bar{e}] \to \{0, \ldots, K\}$ be given by

$$k_+(\widetilde{m}) = \min\{\lfloor \widetilde{m} K \rfloor + M, K\} \tag{25}$$

$$k_-(\widetilde{m}) = \max\{0, \lfloor \widetilde{m} K \rfloor - M\} \tag{26}$$

Note that

$$q_+(\widetilde{m}) = x_{k_+(\widetilde{m})} \tag{27}$$

$$q_-(\widetilde{m}) = x_{k_-(\widetilde{m})} \tag{28}$$

There are

$$\frac{1 + 2\bar{e}}{2\bar{d}} = M + K \tag{29}$$

unique actions (corresponding to $M+K$ equispaced subintervals of $[-\bar{e}, 1+\bar{e}]$). For each $i \in \{-M, \ldots, K-1\}$, define

$$y_i = 2\bar{d}(i+M) - \bar{e} \tag{30}$$

$$Y_i = [y_i, y_{i+1}) \tag{31}$$

so that for each $y \in Y_i$,

$$k_+(y) = k_+(y_i) = \min\{i+M, K\} \tag{32}$$

$$k_-(y) = k_-(y_i) = \max\{0, i+1\} \tag{33}$$

Note that $y_{i+1} = y_i + 2\bar{d} = y_i + \frac{1}{K}$. Let

$$\alpha_{i,j} \equiv \int_{y_i}^{y_{i+1}} f\left(\widetilde{m} - \tfrac{j}{K}\right) d\widetilde{m} \tag{34}$$

$$= \int_{y_{i-1}+\frac{1}{K}}^{y_i+\frac{1}{K}} f\left(\left(\widetilde{m} + \tfrac{1}{K}\right) - \tfrac{j+1}{K}\right) d\widetilde{m} \tag{35}$$

$$= \int_{y_{i-1}}^{y_i} f\left(\widetilde{m} - \tfrac{j+1}{K}\right) d\widetilde{m} = \alpha_{i-1,j+1} \tag{36}$$

A is a Hankel matrix and therefore symmetric. To summarize,

$$\alpha_{i,j} = \alpha_{i-1,j+1} = \alpha_{j,i} \tag{37}$$

We can sum elements of a matrix Z two ways:

$$\sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)} z_{i,j} = \sum_{j=0}^{K} \sum_{i=j-M}^{j-1} z_{i,j} \tag{38}$$

The cost is given by

$$C[m_{\mathcal{D}}] = \int\limits_{-\bar{e}}^{1+\bar{e}} \left[\int\limits_{x_{k_-(\widetilde{m})}}^{x_{k_+(\widetilde{m})}} (q - a(\widetilde{m}))^2 f(\widetilde{m} - m_{\mathcal{D}}(q)) g(q) dq \right] d\widetilde{m}. \tag{39}$$

$$= \int\limits_{-\bar{e}}^{1+\bar{e}} \left[\sum_{j=k_-(\widetilde{m})}^{k_+(\widetilde{m})} \int\limits_{x_j}^{x_{j+1}} (q - a(\widetilde{m}))^2 f\left(\widetilde{m} - \tfrac{j}{K}\right) g(q) dq \right] d\widetilde{m} \tag{40}$$

$$= \sum_{i=-M}^{K-1} \int\limits_{y_i}^{y_{i+1}} \left[\sum_{j=k_-(y_i)}^{k_+(y_i)} \int\limits_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\widetilde{m} - \tfrac{j}{K}\right) g(q) dq \right] d\widetilde{m} \tag{41}$$

$$= \sum_{i=-M}^{K-1} \int\limits_{y_i}^{y_{i+1}} \left[\sum_{j=k_-(y_-)}^{k_+(y_i)} \int\limits_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\widetilde{m} - \tfrac{j}{K}\right) g(q) dq \right] d\widetilde{m} \tag{42}$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)} \left[\int\limits_{y_i}^{y_{i+1}} \int\limits_{x_j}^{x_{j+1}} (q - a_i)^2 f\left(\widetilde{m} - \tfrac{j}{K}\right) g(q) dq d\widetilde{m} \right] \tag{43}$$

$$= \sum_{i=-M}^{K-1} \sum_{j=k_-(y_i)}^{k_+(y_i)} \left[\alpha_{i,j} \int\limits_{x_j}^{x_{j+1}} (q - a_i)^2 g(q) dq \right] \tag{44}$$

Note that $C[m_{\mathcal{D}}]$ is the sum of $M(K+1)$ terms. The first-order condition for a_i is

$$0 = \frac{\partial}{\partial a_i} C[m_{\mathcal{D}}] = -2 \sum_{k=k_-(y_i)}^{k_+(y_i)} \alpha_{i,k} \int\limits_{x_k}^{x_{k+1}} (q - a_i(x)) g(q) dq \tag{45}$$

and $x = (x_0, \dots, x_{K+1})$. Now one may alternatively write $C[m_{\mathcal{D}}]$ as

$$C[m_{\mathcal{D}}] = \sum_{k=0}^{K} \sum_{i=k-M}^{k-1} \left[\alpha_{i,k} \int\limits_{x_k}^{x_{k+1}} (q - a_i(x))^2 g(q) dq \right] \tag{46}$$

so that the first-order condition for x_j is

$$0 = \frac{\partial}{\partial x_j} C[m_{\mathcal{D}}] = \sum_{i=j-1-M}^{j-2} \alpha_{i,j-1}(x_j - a_i(x))^2 g(x_j) - \sum_{i=j-M}^{j-1} \alpha_{i,j}(x_j - a_i(x))^2 g(x_j)$$
$$- 2 \sum_{i=-M}^{K-1} \sum_{k=k_-(y_i)}^{k_+(y_i)} \left[\alpha_{i,k} \int_{x_k}^{x_{k+1}} (q - a_i(x)) g(q) dq \right] \frac{\partial a_i}{\partial x_j} \tag{47}$$

3.1 Uniformly Distributed State

In what follows, suppose that $g(q) = \chi_{[0,1]}(q)$. We have

$$0 = \sum_{i=j-1-M}^{j-2} \alpha_{i,j-1}(x_j - a_i(x))^2 - \sum_{i=j-M}^{j-1} \alpha_{i,j}(x_j - a_i(x))^2 \tag{48}$$
$$= \sum_{i=j-1-M}^{j-2} \left(\alpha_{i,j-1}(x_j - a_i(x))^2 - \alpha_{i+1,j}(x_j - a_{i+1}(x))^2 \right) \tag{49}$$
$$= \sum_{i=j-1-M}^{j-2} \left(\alpha_{i,j-1} \left(x_j^2 - 2a_i(x)x_j - a_i(x)^2 \right) - \alpha_{i+1,j} \left(x_j^2 - 2a_{i+1}(x)x_j - a_{i+1}(x)^2 \right) \right) \tag{50}$$

Let the (i, j)-the element of the matrix B be given by

$$[B]_{i,j} = \begin{cases} \alpha_{j-1-M,j-1} & \text{if } j-1-M = i \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } j-1-M < i < j-2 \\ \alpha_{j-2,j-1} & \text{if } i = j-2 \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

so that

$$0 = B\left(x \circ x - 2a(x) \circ x - a(x) \circ a(x) \right) \tag{52}$$

then Equation 45 simplifies

$$0 = \left[\sum_{k=k_-(y_i)}^{k_+(y_i)} \alpha_{i,k} \left(x_{k+1}^2 - x_k^2 \right) \right] - 2a_i(x) \left[\sum_{k=k_-(y_i)}^{k_+(y_i)} \alpha_{i,k}(x_{k+1} - x_k) \right] \tag{53}$$

(where $i \in \{0, \ldots, K\}$). Let the (i,j)-the element of the matrix C be given by

$$[C]_{i,j} = \begin{cases} \alpha_{i,k_+(y_i)} & \text{if } j = k_+(y_i) \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{if } k_-(y_i) < j < k_+(y_i) \\ \alpha_{i,k_-(y_i)} & \text{if } j = k_-(y_i) \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

so that

$$0 = Cx \circ x + 2a(x) \circ Cx \tag{55}$$

We now wish to show that $B = C'$. x and $a(x)$ are jointly determined by the system

$$0 = Cx \circ x + 2a(x) \circ Cx \tag{56}$$

$$0 = B\left(x \circ x - 2a(x) \circ x - a(x) \circ a(x)\right) \tag{57}$$