I've done the math enough to know the dangers of our second guessing.
Doomed to crumble unless we grow, and strengthen our communication.

$\sim$ From "Schism," by Tool.

Uniform Noise, Quadratic Loss. Set $L(x) = \frac{1}{2}x^2$. Fix $\bar{\epsilon} \in (0,1)$, and $I \in C^1[0,1]$. As they will appear frequently, put $I_0(s,t) := \int_s^t \iota(r)dr$, and $I_1(s,t) := \int_s^t r\iota(r)dr$. Actions are maps from received messages to actions: $\tilde{m} \mapsto a(\tilde{m})$. Messages are maps from states to sent messages: $q \mapsto m(q)$. We consider strictly increasing, piecewise twice continuously differentiable action and message functions. Formally, define

$$\mathcal{A} := \{\alpha \in D^2[-\bar{\epsilon}, 1+\bar{\epsilon}] \mid \text{for all } q \in [-\bar{\epsilon}, 1+\bar{\epsilon}],\ \alpha'(q) > 0, \text{ and } \alpha(q) \in (0,1)\}, \text{ and} \tag{1}$$
$$\mathcal{M} := \{\mu \in D^2[0,1] \mid \text{for all } q \in [0,1],\ \mu(0) = 0, \mu(1) = 1, \mu'(q) > 0, \text{ and } \mu(q) \in [0,1]\}. \tag{2}$$

to be the sets of admissable action and message functions respectively. Note that the receiver's action has domain $[-\bar{\epsilon}, 1+\bar{\epsilon}]$. The sender's message has codomain $[0,1]$, but it is corrupted by noise. Although a *sent* message may be no smaller than zero, and no larger than one, a *received* message may be as small as $-\bar{\epsilon}$, or as large as $1+\bar{\epsilon}$. The receiver must choose an action in $[0,1]$ for messages less than zero or greater than one. The sender has ex-ante payoffs $S : \mathcal{A} \times \mathcal{M} \to \mathbb{R}$ given by

$$S(a,m) = \frac{1}{2\bar{\epsilon}} \int_{-\bar{\epsilon}}^{\bar{\epsilon}} I(q)L(a(m(q)+e) - q)g(q)de \tag{3}$$

Beliefs. Let $g(q \mid \tilde{m})$ denote the receiver's beliefs. Put $\tilde{m} = m(q) + e$, where $e \sim U[0,1]$. Define the upper and lower limits to be

$$\overline{q}(\tilde{m}) = m^{-1}(\min\{\tilde{m} + \bar{\epsilon}, 1\}), \text{ and} \tag{4}$$
$$\underline{q}(\tilde{m}) = m^{-1}(\max\{\tilde{m} - \bar{\epsilon}, 0\}) \tag{5}$$

respectively. Now $\tilde{m}|q \sim U[m(q) - \bar{\epsilon}, m(q) + \bar{\epsilon}]$. By Baye's Law,

$$g(q|\tilde{m}) = \frac{f(\tilde{m}|q)}{\int_0^1 f(\tilde{m}|t)g(t)dt} = \frac{\frac{1}{2\bar{\epsilon}}\mathbf{1}_{m(q)-\bar{\epsilon} \le \tilde{m} \le m(q)+\bar{\epsilon}}}{\int_0^1 \frac{1}{2\bar{\epsilon}}\mathbf{1}_{\underline{q}(\tilde{m}) \le t \le \overline{q}(\tilde{m})}dt} = \frac{\mathbf{1}_{\underline{q}(\tilde{m}) \le q \le \overline{q}(\tilde{m})}}{\overline{q}(\tilde{m}) - \underline{q}(\tilde{m})} \tag{6}$$

The sender has ex-ante payoffs $R : \mathcal{A} \times \mathcal{M} \to \mathbb{R}$ given by

$$R(a,m,\tilde{m}) = \int_0^1 I(q)L(a(\tilde{m}) - q)g(q|\tilde{m})dq. \tag{7}$$

Next, we change the order of integration, first integrating with respect to q, then integrating with respect to $\tilde{m}$. Proceeding with the change of variables, we obtain

$$J(a,m) = \int_{-\bar{\epsilon}}^{1+\bar{\epsilon}} \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} I(q)L(a(\tilde{m}) - q)dqd\tilde{m}. \tag{8}$$

We have omitted the density of the noise, as it is constant, and therefore does not affect the equilibrium message and action functions. The *sender and receiver's problem* is to choose $(a,m) \in \mathcal{A} \times \mathcal{M}$, which minimizes $J(a,m)$. Partition $[0,1]$ into the subsets $\{0\}$, $\mathcal{R}_1 := (0,\bar{\epsilon})$, $\mathcal{R}_2 := [\bar{\epsilon}, 1-\bar{\epsilon}]$, $\mathcal{R}_3 := (1-\bar{\epsilon}, 1)$, and $\{1\}$. We first show that there exists a unique solution to the sender and receiver's problem. We then show that steepness of the message function is increasing with the importance of getting the message right.

Lemma 1. *Given an action $a \in \mathcal{A}$, the sender's optimal message is.*

Proof. Suppose that $(a,m) \in \mathcal{A} \times \mathcal{M}$ is a solution to the sender and receiver's problem. We proceed by enumerating the conditions that must be met by a pair (a,m), and that (a,m) is unique among them. If the received message is $-\bar{\epsilon}$, then the sent message was 0 a.s., and hence the state of the world is also 0. Therefore, it is optimal for the receiver to play 0. If the received message is $1+\bar{\epsilon}$, then the sent message was 1 a.s., and hence the state of the world is also 1. Therefore, it is optimal for the receiver to play 1. We have established that $a(-\bar{\epsilon}) = 0$, and $a(1+\bar{\epsilon}) = 1$. (a,m) must satisfy the first-order condition with respect to m:

$$0 = \int_{-\bar{\epsilon}}^{\bar{\epsilon}} L'(a(m+e)-q)a'(m+e)de = L(a(m+\bar{\epsilon})-q) - L(a(m-\bar{\epsilon})-q). \tag{9}$$

Since L is even, there are two solutions: $a(m+\bar{\epsilon}) = a(m-\bar{\epsilon})$, and $a(m+\bar{\epsilon})-q = -(a(m-\bar{\epsilon})-q)$. The first implies that a is constant, and hence $a \notin \mathcal{A}$, which contradicts our assumption that $a \in \mathcal{A}$. Let $F : [0,1]^2 \to \mathbb{R}$ be given by

$$F(q,m) = a(m+\bar{\epsilon}) + a(m-\bar{\epsilon}) - 2q. \tag{10}$$

Note that the domain of the last argument is $[0,1]$, and *not* $\mathcal{M}$. Now

$$F_m(q,m) = a'(m+\bar{\epsilon}) + a'(m-\bar{\epsilon}) > 0. \tag{11}$$

Lastly, observe that $m' = -(F_m)^{-1}F_q > 0$. The first-order condition with respect to a, requires a change of coordinates. Figure 1 illustrates this change in the order of integration. (a,m) must satisfy the first-order condition with respect to a:

$$0 = \int_{\underline{q}(\tilde{m})}^{\overline{q}(\tilde{m})} I(q)L'(a-q)dq \;\Rightarrow\; a(\tilde{m}) = \frac{I_1(\overline{q}(\tilde{m}), \underline{q}(\tilde{m}))}{I_0(\overline{q}(\tilde{m}), \underline{q}(\tilde{m}))} \tag{12}$$

Observe that

$$2q = \frac{I_1(\overline{q}(m(q)+\bar{\epsilon}), \underline{q}(m(q)+\bar{\epsilon}))}{I_0(\overline{q}(m(q)+\bar{\epsilon}), \underline{q}(m(q)+\bar{\epsilon}))} - \frac{I_1(\overline{q}(m(q)-\bar{\epsilon}), \underline{q}(m(q)-\bar{\epsilon}))}{I_0(\overline{q}(m(q)-\bar{\epsilon}), \underline{q}(m(q)-\bar{\epsilon}))} \tag{13}$$

$$= \frac{I_1(m^{-1}(\min\{m(q)+2\bar{\epsilon},1\}), q)}{I_0(m^{-1}(\min\{m(q)+2\bar{\epsilon},1\}), q)} - \frac{I_1(q, m^{-1}(\max\{m(q)-2\bar{\epsilon},0\}))}{I_0(q, m^{-1}(\max\{m(q)-2\bar{\epsilon},0\}))} \tag{14}$$

Let $\mathcal{T} : C^2[0,1] \to C^2[0,1]$ be given by

$$[\mathcal{T}\ell](\tilde{m}) = \frac{1}{2}\left(\frac{I_1(\ell(\min\{\tilde{m}+2\bar{\epsilon},1\}), \ell(\tilde{m}))}{I_0(\ell(\min\{\tilde{m}+2\bar{\epsilon},1\}), \ell(\tilde{m}))} + \frac{I_1(\ell(\tilde{m}), \ell(\max\{\tilde{m}-2\bar{\epsilon},0\}))}{I_0(\ell(\tilde{m}), \ell(\max\{\tilde{m}-2\bar{\epsilon},0\}))}\right). \tag{15}$$

Example: $I \equiv 1$. $I_1(s,t)/I_0(s,t) = (s+t)/2$, which implies that

$$2\ell(\tilde{m}) = \ell(\min\{\tilde{m} + 2\bar{\epsilon}, 1\}) + \ell(\max\{\tilde{m} - 2\bar{\epsilon}, 0\}). \tag{16}$$

There are three cases:

1. $\tilde{m} \in (0, 2\bar{\epsilon})$: $\ell(x) = 2^{x/2\bar{\epsilon}}$.
2. $\tilde{m} \in (2\bar{\epsilon}, 1 - 2\bar{\epsilon})$: $\ell(x) = x$.
3. $\tilde{m} \in (1 - 2\bar{\epsilon}, 1)$:
$$\ell(x) = 1 + \frac{2\bar{\epsilon}}{\log(2)} \left(2^{-\frac{1}{2\bar{\epsilon}}} - 2^{-\frac{x}{2\bar{\epsilon}}}\right) \tag{17}$$

■

Lemma 2. *Given a message $m \in \mathcal{M}$, the receiver's optimal action is.*

Proposition 1. *There is a unique equilibrium $(a, m) \in \mathcal{A} \times \mathcal{M}$.*

Next, we show that. We assume that the zeros of I' are isolated: for each $q_0 \in [0, 1]$ satisfying $I'(q_0) = 0$, there is a neighborhood $\mathcal{U}$ of q such that for all $q \in \mathcal{U} \setminus \{q_0\}$, $I'(q) \neq 0$. If I is analytic, then its zeros are isolated.

Corollary 1. *The steepness of the message function is increasing with the importance.*

Proof. Let $(a, m) \in \mathcal{A} \times \mathcal{M}$ be the unique solution of the sender and receiver's problem. Choose $q \in (0, 1)$ at which $I'(q) \neq 0$ (so long as I is non-constant, there is at least one such q). Let F be as in the proof of Proposition **??**. Since $m' = -(F_m)^{-1} F_q$, $F_{qq} = 0$, $F_{mm} > 0$, and $F_{qm} = 0$, we have that

$$m'' = -(F_m)^{-2}((F_{qq} + F_{qm} m')F_m - F_q(F_{mq} + F_{mm} m')) = (F_m)^{-2} F_q F_{mm} m' > 0. \tag{18}$$

By the inverse function theorem, I inverts on $\mathcal{U}$. We conclude that,

$$\frac{dm'}{dI} = \frac{dm'}{dq}\frac{dq}{dI} > 0 \tag{19}$$

as desired. ■

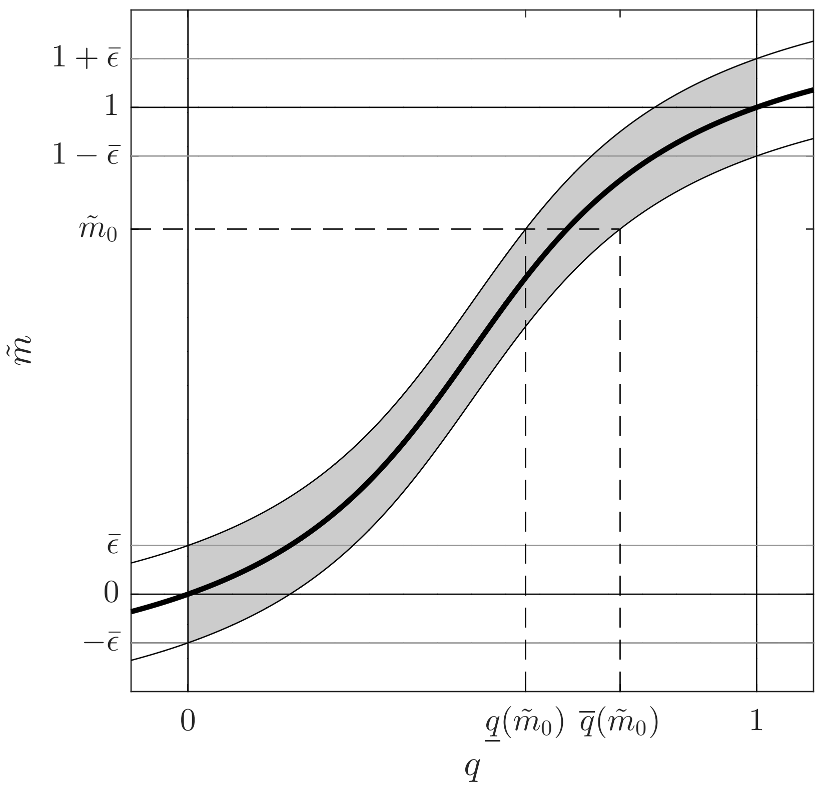

Figure 1: **The Region of Integration.** The solid line is $m(q)$, the upper dashed line is $m(q)+\bar{\epsilon}$, and the lower dashed line is $m(q)-\bar{\epsilon}$. The shaded area is the region of integration.