1 The Model

There are two players, a sender and a receiver, and a state of nature $q \in [0, 1]$ that is known to the sender but not the receiver. Let g denote the density of q. The sender sends an *intended* message $m(q) \in [0, 1]$. The receiver receives a noisy version of the intended message, which we call the *received* message, $\widetilde{m} = m + \epsilon$. ϵ is distributed according to a continuous density f with support $[-\overline{\epsilon}, \overline{\epsilon}]$, where for some integer $N \geq 2$, $\overline{e} = \frac{1}{2N}$.

We specify f as follows. Let $x : [-\overline{e}, \overline{e}] \to [0, 1]$ be given by

$$x(e) = \frac{1}{2}\left[1 + \frac{e}{\overline{e}}\right] \tag{1}$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1}(1 - x(e))^{b-1} \tag{2}$$

for constants $a > 0$ and $b > 0$. The receiver then takes an action $A(\widetilde{m}) \in [0, 1]$. Both sender and receiver have utility over the state, q, and the receiver's action, A, of

$$U(q, A) = -\frac{1}{2}(q - A)^2 I(q) \tag{3}$$

Prior to the start of the game, the sender can specify an intended message function $m(q)$ that she will use. The receiver chooses an action based upon the function $m(q)$ and the received message $\widetilde{m}$, and we denote this function $A(\widetilde{m})$. We work backward and start with the optimal action, given $m(q)$ and $\widetilde{m}$.

Fix a message function m. Let $Q(\widetilde{m})$ denote the set of all states $q \in [0, 1]$ such that for some noise $e \in [-\bar{e}, \bar{e}]$, $\widetilde{m} = m(q) + e$. Put differently,

$$Q(\widetilde{m}) = \{q \in [0, 1] | m(q) - \bar{e} \leq \widetilde{m} \leq m(q) + \bar{e}\}. \tag{4}$$

Finally, let $q_+(\widetilde{m}) \equiv \sup Q(\widetilde{m})$, $q_-(\widetilde{m}) \equiv \inf Q(\widetilde{m})$, and $w(\widetilde{m}) \equiv q_+(\widetilde{m}) - q_-(\widetilde{m})$. $q_+(\widetilde{m})$ and $q_-(\widetilde{m})$ are the highest and lowest states that could possibly be associated with the received message $\widetilde{m}$. $w(\widetilde{m}) \equiv q_+(\widetilde{m}) - q_-(\widetilde{m})$ is the distance between these two bounds. Except near the boundaries of the message space, $w(\widetilde{m})$ is equal to $2\bar{e}$. We have

$$I_\gamma(k) = \int_{q_k}^{q_{k+1}} q^\gamma f(\widetilde{m} - m(q))g(q)dq \tag{5}$$

$$I_{(\alpha,\beta,\gamma)}(\widetilde{m}) \equiv \int_\alpha^\beta q^\gamma f(\tilde{m} - q)dq \tag{6}$$

Suppose that the receiver receives the message $\widetilde{m} \in [-\bar{e}, 1 + \bar{e}]$. $q|\widetilde{m}$ has support $[q_-(\widetilde{m}), q_+(\widetilde{m})]$, and

$$g(q|\widetilde{m}) = \frac{f(\widetilde{m} - m(q))g(q)}{\int_0^1 f(\widetilde{m} - m(t))g(t)dt} \tag{7}$$

The receiver's problem is to choose an action that maximizes her expected utility:

$$\max_{a(\widetilde{m})} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} U(q,a) g(q|\widetilde{m}) dq. \tag{8}$$

Because of Assumptions A1 to A3, the receiver's optimal action is simply the expected value of the state, q, given the received message $\widetilde{m}$:

$$A(\widetilde{m}) = \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} q g(q|\widetilde{m}) dq. \tag{9}$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional C be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{e}}^{\bar{e}} (q - A(m(q)+e))^2 f(e) de dq, \tag{10}$$

where A is the receiver's optimal action from Equation (9). $C[m]$ is the expected loss for a given message function m. The integrand is the loss for a given state q and action $A(\widetilde{m})$. The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function m that minimizes $C[m]$:

$$\min_{m \in M} C[m] \tag{11}$$

where M is the space of weakly increasing piece-wise continuous functions on $[0,1]$. The change of variables $\widetilde{m} = m(q) + e$ and an application of Fubini's Theorem yield

$$\begin{aligned} C[m] &= \int_0^1 \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - A(\widetilde{m}))^2 f(\widetilde{m} - m(q)) g(q) d\widetilde{m} dq & (12) \\ &= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\widetilde{m} - m(q)) g(q) dq d\widetilde{m}. & (13) \end{aligned}$$

We consider the costs of identity and discrete message functions.

2 Identity Message

We consider the identity message function, $m(q) = q$, only for the case in which q is uniformly distributed.

$$A(\widetilde{m}) = \begin{cases} \overline{a}(\widetilde{m}) & \text{if } 1 - \bar{e} < \widetilde{m} \le 1 + \bar{e} \\ a(\widetilde{m}) & \text{if } \bar{e} < \widetilde{m} \le 1 - \bar{e} \\ \underline{a}(\widetilde{m}) & \text{if } -\bar{e} \le \widetilde{m} \le \bar{e} \end{cases} \tag{14}$$

where

$$\overline{a}(\widetilde{m}) = \frac{I_{(\widetilde{m}-\bar{e},1,1)}(\widetilde{m})}{I_{(\widetilde{m}-\bar{e},1,0)}(\widetilde{m})} \tag{15}$$

$$a(\widetilde{m}) = \frac{I_{(\widetilde{m}-\bar{e},\widetilde{m}+\bar{e},1)}(\widetilde{m})}{I_{(\widetilde{m}-\bar{e},\widetilde{m}+\bar{e},0)}(\widetilde{m})} \tag{16}$$

$$\underline{a}(\widetilde{m}) = \frac{I_{(0,\widetilde{m}+\bar{e},1)}(\widetilde{m})}{I_{(0,\widetilde{m}+\bar{e},0)}(\widetilde{m})} \tag{17}$$

Note that the normalizing constant cancels out when computing the conditional expectation. The cost of the identity message function is given by

$$C[m_{\mathcal{I}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m}. \tag{18}$$

where $q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\}$ and $q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\}$. Define

$$\overline{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\widetilde{m}-\bar{e}}^{1} (q - \overline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{19}$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\widetilde{m}-\bar{e}}^{\widetilde{m}+\bar{e}} (q - a(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{20}$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_{0}^{\widetilde{m}+\bar{e}} (q - \underline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{21}$$

so that $C[m_{\mathcal{I}}] = \overline{z} + z + \underline{z}$.

3 Discrete Message

Fix an integer $M \geq 1$ and define $K = M \times N$. Consider the partition

$$0 = x_0 < x_1 < \cdots < x_K < x_{K+1} = 1 \tag{22}$$

of $[0, 1]$. For each $i \in \{0, \ldots, K\}$, define $X_i = [x_i, x_{i+1})$. A discrete message with $K + 1$ messages is given by

$$m_{\mathcal{D}}(q) = \frac{1}{K} \sum_{i=0}^{K} \chi_{X_i}(q). \tag{23}$$

Note that $q_+(\widetilde{m}), q_-(\widetilde{m}) \in \{x_0, x_1, \cdots, x_{K-1}, x_K\}$ for each $\widetilde{m} \in [-\bar{e}, 1+\bar{e}]$. The action function is

$$A(\widetilde{m}) = \tag{24}$$

The cost is given by

$$C[m_{\mathcal{D}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\widetilde{m} - m_{\mathcal{D}}(q)) g(q) dq d\widetilde{m}. \tag{25}$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{x_{j(\widetilde{m})}}^{x_{j(\widetilde{m})+N}} (q - A(\widetilde{m}))^2 f(\widetilde{m} - j/K) g(q) dq d\widetilde{m}. \tag{26}$$

$$\tag{27}$$

Each $\widetilde{m} \in [-\bar{e}, 1 + \bar{e}]$ maps to M potential bins.