1 Identity Message

Let $x : [-\bar{e}, \bar{e}] \to [0, 1]$ be given by

$$x(e) = \frac{1}{2}\left[1 + \frac{e}{\bar{e}}\right] \tag{1}$$

Define

$$I_{(\alpha,\beta,\gamma)}(\widetilde{m}) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - q)dq \tag{2}$$

and

$$A(\widetilde{m}) = \begin{cases} \overline{a}(\widetilde{m}) & \text{if } 1 - \bar{e} < \widetilde{m} \le 1 + \bar{e} \\ a(\widetilde{m}) & \text{if } \bar{e} < \widetilde{m} \le 1 - \bar{e} \\ \underline{a}(\widetilde{m}) & \text{if } -\bar{e} \le \widetilde{m} \le \bar{e} \end{cases} \tag{3}$$

where

$$\overline{a}(\widetilde{m}) = \frac{I_{(\widetilde{m}-\bar{e},1,1)}(\widetilde{m})}{I_{(\widetilde{m}-\bar{e},1,0)}(\widetilde{m})} \tag{4}$$

$$a(\widetilde{m}) = \frac{I_{(\widetilde{m}-\bar{e},\widetilde{m}+\bar{e},1)}(\widetilde{m})}{I_{(\widetilde{m}-\bar{e},\widetilde{m}+\bar{e},0)}(\widetilde{m})} \tag{5}$$

$$\underline{a}(\widetilde{m}) = \frac{I_{(0,\widetilde{m}+\bar{e},1)}(\widetilde{m})}{I_{(0,\widetilde{m}+\bar{e},0)}(\widetilde{m})} \tag{6}$$

Note that the normalizing constant cancels out when computing the conditional expectation. We need three sets of weights and knots.

The cost of the identity message function is given by

$$C[m_i] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\tilde{m} - q)dqd\widetilde{m}. \tag{7}$$

where

$$q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\} \tag{8}$$

$$q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\} \tag{9}$$

Define

$$\overline{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\widetilde{m}-\bar{e}}^{1} (q - \overline{a}(\widetilde{m}))^2 f(\tilde{m} - q)dqd\widetilde{m} \tag{10}$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\widetilde{m}-\bar{e}}^{\widetilde{m}+\bar{e}} (q - a(\widetilde{m}))^2 f(\tilde{m} - q)dqd\widetilde{m} \tag{11}$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_{0}^{\widetilde{m}+\bar{e}} (q - \underline{a}(\widetilde{m}))^2 f(\tilde{m} - q)dqd\widetilde{m} \tag{12}$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1}(1 - x(e))^{b-1} \tag{13}$$

for constants $a > 0$ and $b > 0$.

2 Discrete Message

A discrete message with $N+1$ message is given by

$$m_d(q) = \sum_{n=0}^{N} \frac{n}{N} \mathbf{1}_{\frac{n}{N+1} \leq q \frac{n+1}{N+1}}. \tag{14}$$

where $\bar{e} = 1/(2N)$.

$$q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\} \tag{15}$$

$$q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\} \tag{16}$$

We have

$$I_\gamma(k) = \int_{q_k}^{q_{k+1}} q^\gamma f(\widetilde{m} - m(q)) dq \tag{17}$$

The receiver knows which message was sent. The action function is

$$A(\widetilde{m}) = \tag{18}$$

The cost of the discrete message function is given by

$$C[m_d] = (N+1) \tag{19}$$