

*I've done the math enough to know the dangers of our second guessing
Doom to crumble unless we grow and strengthen our communication.*

~ Tool

1 The Model

There are two players, a sender and a receiver, and a state of nature $q \in [0, 1]$ that is known to the sender but not the receiver. Let g denote the density of q . The sender sends an *intended* message $m(q) \in [0, 1]$. The receiver receives a noisy version of the intended message, which we call the *received* message, $\tilde{m} = m(q) + e$. e is distributed according to a continuous density f with support $[-\bar{e}, \bar{e}]$, where for some integer $N \geq 2$, $\bar{e} = \frac{1}{2N}$.

We specify f as follows. Let $x : [-\bar{e}, \bar{e}] \rightarrow [0, 1]$ be given by

$$x(e) = \frac{1}{2} \left[1 + \frac{e}{\bar{e}} \right] \quad (1)$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1} (1 - x(e))^{b-1} \quad (2)$$

for constants $a > 0$ and $b > 0$. The receiver then takes an action $a(\tilde{m}) \in [0, 1]$. Both sender and receiver have utility over the state, q , and the receiver's action, a , of

$$U(q, a) = -\frac{1}{2}(q - a)^2 \quad (3)$$

Prior to the start of the game, the sender can specify an intended message function $m(q)$ that she will use. The receiver chooses an action based upon the function $m(q)$ and the received message \tilde{m} , and we denote this function $a(\tilde{m})$. We work backward and start with the optimal action, given $m(q)$ and \tilde{m} .

Fix a message function m . Let $Q(\tilde{m})$ denote the set of all states $q \in [0, 1]$ such that for some noise $e \in [-\bar{e}, \bar{e}]$, $\tilde{m} = m(q) + e$. In set builder notation,

$$Q(\tilde{m}) = \{q \in [0, 1] \mid m(q) - \bar{e} \leq \tilde{m} \leq m(q) + \bar{e}\}. \quad (4)$$

Finally, define

$$q_+(\tilde{m}) \equiv \sup Q(\tilde{m}) \quad (5)$$

$$q_-(\tilde{m}) \equiv \inf Q(\tilde{m}) \quad (6)$$

$q_+(\tilde{m})$ and $q_-(\tilde{m})$ are the highest and lowest states that could possibly be associated with the received message \tilde{m} . Define

$$I_{(\alpha, \beta, \gamma)}(\tilde{m}) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - m(q)) dq \quad (7)$$

Suppose that the receiver receives the message $\tilde{m} \in [-\bar{e}, 1 + \bar{e}]$. $q|\tilde{m}$ has support $[q_-(\tilde{m}), q_+(\tilde{m})]$ and density

$$g(q|\tilde{m}) = \frac{f(\tilde{m} - m(q))g(q)}{\int_0^1 f(\tilde{m} - m(t))g(t)dt} \quad (8)$$

The receiver's problem is to

$$\max_{a(\tilde{m})} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} U(q, a)g(q|\tilde{m})dq. \quad (9)$$

The receiver's optimal action is simply the expected value of the state, q , given the received message \tilde{m} :

$$a(\tilde{m}) = \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} qg(q|\tilde{m})dq. \quad (10)$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional C be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{e}}^{\bar{e}} (q - a(m(q) + e))^2 f(e) dedq, \quad (11)$$

where a is the receiver's optimal action from Equation (10). $C[m]$ is the expected loss for a given message function m . The integrand is the loss for a given state q and action $a(\tilde{m})$. The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function m that minimizes $C[m]$:

$$\min_{m \in M} C[m] \quad (12)$$

where M is the space of weakly increasing piece-wise continuous functions on $[0, 1]$. The change of variables $\tilde{m} = m(q) + e$ and an application of Fubini's Theorem yield

$$C[m] = \int_0^1 \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q))g(q)d\tilde{m}dq \quad (13)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - m(q))g(q)dq d\tilde{m}. \quad (14)$$

We consider the costs of identity and discrete message functions.

2 Identity Message

We consider the identity message function, $m(q) = q$, only for the case in which q is uniformly distributed.

$$a(\tilde{m}) = \begin{cases} \bar{a}(\tilde{m}) & \text{if } 1 - \bar{e} < \tilde{m} \leq 1 + \bar{e} \\ a(\tilde{m}) & \text{if } \bar{e} < \tilde{m} \leq 1 - \bar{e} \\ \underline{a}(\tilde{m}) & \text{if } -\bar{e} \leq \tilde{m} \leq \bar{e} \end{cases} \quad (15)$$

where

$$\bar{a}(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, 1, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, 1, 0)}(\tilde{m})} \quad (16)$$

$$a(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (17)$$

$$\underline{a}(\tilde{m}) = \frac{I_{(0, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(0, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (18)$$

Note that the normalizing constant cancels out when computing the conditional expectation. The cost of the identity message function is given by

$$C[m_{\mathcal{I}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m}. \quad (19)$$

where $q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\}$ and $q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\}$. Define

$$\bar{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\tilde{m}-\bar{e}}^1 (q - \bar{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (20)$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\tilde{m}-\bar{e}}^{\tilde{m}+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (21)$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_0^{\tilde{m}+\bar{e}} (q - \underline{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (22)$$

so that $C[m_{\mathcal{I}}] = \bar{z} + z + \underline{z}$.

3 Discrete Message

Fix an integer $M \geq 1$ and define $K = M \times N$. Consider the partition

$$0 = x_0 < x_1 < \dots < x_K < x_{K+1} = 1 \quad (23)$$

of $[0, 1]$. For each $i \in \{0, \dots, K\}$, define $X_i = [x_i, x_{i+1})$. A discrete message with $K + 1$ messages is given by

$$m_{\mathcal{D}}(q) = \frac{1}{K} \sum_{i=0}^K \chi_{X_i}(q). \quad (24)$$

Each $\tilde{m} \in [-\bar{e}, 1 + \bar{e}]$ maps to M potential bins. The action function is

$$a(\tilde{m}) = \quad (25)$$

Let $j_+, j_- : [-\bar{e}, 1 + \bar{e}] \rightarrow \{0, \dots, K\}$ be given by

$$j_+(\tilde{m}) = \min\{\lfloor \tilde{m}K \rfloor + M, K\} \quad (26)$$

$$j_-(\tilde{m}) = \max\{0, \lfloor \tilde{m}K \rfloor - M\} \quad (27)$$

Note that

$$q_+(\tilde{m}) = x_{j_+(\tilde{m})} \quad (28)$$

$$q_-(\tilde{m}) = x_{j_-(\tilde{m})} \quad (29)$$

The cost is given by

$$C[m_{\mathcal{D}}] = \int_{-\bar{e}}^{1+\bar{e}} \int_{x_{j_-(\tilde{m})}}^{x_{j_+(\tilde{m})}} (q - a(\tilde{m}))^2 f(\tilde{m} - m_{\mathcal{D}}(q)) g(q) dq d\tilde{m}. \quad (30)$$

$$= \sum_{j=0}^{j=K} \int_{x_j}^{x_j} (q - a(\tilde{m}))^2 f(\tilde{m} - m_{\mathcal{D}}(q)) g(q) dq d\tilde{m}. \quad (31)$$

$$(32)$$

It is sufficient to choose the partition in Equation 23. The first-order conditions are

$$x \quad (33)$$