Appendix: Numerical Solution with Non-Uniformly Distributed Errors

We look for solutions in the space of increasing piecewise-continuous functions on $[0,1]$, which we called M. The sender's problem is intractable. We therefore approximate the solution by a step-function. We have two choices. First, we can establish a necessary condition for optimality, namely, the Euler-Lagrange equation, and hope that a solution exists. We can then numerically approximate the solution to the Euler-Lagrange equation with $\ell_0(x) = x$:

$$a_n(m) = E[q|\ell_n(\max\{m-\bar{\epsilon},0\}) \leq q \leq \ell_n(\min\{m+\bar{\epsilon},1\})] \tag{1}$$

where the expectation is taken with respect to the PDF g. The integral can be written as

$$D_n(m) = (\ell_n(m) - a(m+\bar{\epsilon}))^2 f(\bar{\epsilon}) - (\ell_n(m) - a(m-\bar{\epsilon}))^2 f(-\bar{\epsilon}) \tag{2}$$

Define

$$A_n^k(m) = \int_{m-\bar{\epsilon}}^{m+\bar{\epsilon}} a_n(\tilde{m})^k f'(\tilde{m}-m) d\tilde{m} \tag{3}$$

We iterate according to

$$A_n^{k=0}(m)\ell_{n+1}^2(m) - 2A_n^{k=1}(m)\ell_{n+1}(m) + A_n^{k=2}(m) - D_n(m,\ell_n(m)) = 0 \tag{4}$$

Second, we can look for solutions in the space of step-functions on $[0,1]$. Such a step-function should *not* be confused with the step-function obtained as a solution. Step-functions are dense in M: for each $m \in M$ and $\epsilon > 0$, there is an $n \geq 0$ such that $||m - m_n||_\infty < 0$. Step-functions make for robust approximations, as they allow for discontinuities in the solution (a feature which we find in equilibrium). For $n \geq 1$, $m_n : [0,1] \to \{0, \alpha_1, \alpha_2, \ldots, a_{n-1}, 1\}$ be given by

$$m_n(q) = \sum_{j=0}^{n-1} \frac{j}{n} \cdot \mathbf{1}_{\alpha_j \leq q < \alpha j+1} \tag{5}$$

Now $m_n : \{0, \alpha_1, \alpha_2, \ldots, a_{n-1}, 1\} \to [0,1]$ will not be sufficient for our uses, since the receiver receives an element of the codomain of m_n *plus* noise. Let $\tilde{m} \mapsto \ell(\tilde{m}) \subset [0,1]$ be given by

$$\ell_n(p) = \left[\sum_{j=0}^{n-1} \frac{j}{n} \cdot \mathbf{1}_{\alpha_j \leq p < \alpha_{j+1}}, \sum_{j=0}^{n-1} \frac{j+1}{n} \cdot \mathbf{1}_{\alpha_j \leq p < \alpha_{j+1}}\right) \tag{6}$$

Now

$$q_{+}(\tilde{m}) = \sup \ell_n(\min\{\tilde{m} + \bar{\epsilon}, 1\}) = \sum_{j=0}^{n-1} \frac{j+1}{n} \cdot \mathbf{1}_{\alpha_j \leq \min\{\tilde{m}+\bar{\epsilon},1\} < \alpha_{j+1}}. \tag{7}$$

Similarly,

$$q_{-}(\tilde{m}) = \inf \ell_n(\max\{\tilde{m} + \bar{\epsilon}, 1\}) = \sum_{j=0}^{n-1} \frac{j}{n} \cdot \mathbf{1}_{\alpha_j \leq \max\{\tilde{m}-\bar{\epsilon},0\} < \alpha_{j+1}}. \tag{8}$$

The sender's problem is to choose a vector $a = (0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 1)$ to

$$\min_{a} \int_0^1 \int_{-\bar{\epsilon}}^{\bar{\epsilon}} \left(q - \frac{\overline{q}(m(q) + e + \bar{\epsilon}) + \underline{q}(m(q) + e - \bar{\epsilon})}{2} \right)^2 f_e(e) dedq \tag{9}$$

Fix $j \in \{0, 1, \ldots, n\}$. Note that m is constant on $[q_j, q_{j+1}]$. For $e \in [a, b]$, define

$$T_A(a, b) = \int_{q_j}^{q_{j+1}} q^2 g(q) dq \cdot \int_a^b f(e) de \tag{10}$$

$$T_B(a, b) = \int_{q_j}^{q_{j+1}} q g(q) dq \cdot \int_a^b A(m + e) f(e) de \tag{11}$$

$$T_C(a, b) = \int_{q_j}^{q_{j+1}} g(q) dq \cdot \int_a^b A(m + e)^2 f(e) de \tag{12}$$

so that

$$\int_{q_j}^{q_{j+1}} \int_a^b (q - A(m(q) - e))^2 f(e) deg(q) dq = T_A(a, b) - 2T_B(a, b) + T_C(a, b) \tag{13}$$