

# 1 The Model

There are two players, a sender and a receiver, and a state of nature  $q \in [0, 1]$  that is known to the sender but not the receiver. Let  $g$  denote the density of  $q$ . The sender sends an *intended* message  $m(q) \in [0, 1]$ . The receiver receives a noisy version of the intended message, which we call the *received* message,  $\tilde{m} = m + \epsilon$ .  $\epsilon$  is distributed according to a continuous density  $f$  with support  $[-\bar{\epsilon}, \bar{\epsilon}]$ , where  $\bar{\epsilon} = \frac{1}{2N} > 0$ . The receiver then takes an action  $A(\tilde{m}) \in [0, 1]$ .

Both sender and receiver have utility over the state,  $q$ , and the receiver's action,  $A$ , of

$$U(q, A) = -\frac{1}{2}(q - A)^2 I(q) \quad (1)$$

Prior to the start of the game, the sender can specify an intended message function  $m(q)$  that she will use. The receiver chooses an action based upon the function  $m(q)$  and the received message  $\tilde{m}$ , and we denote this function  $A(\tilde{m})$ . We work backward and start with the optimal action, given  $m(q)$  and  $\tilde{m}$ .

Fix a message function  $m$ . Let  $Q(\tilde{m})$  denote the set of all states  $q \in [0, 1]$  such that for some noise  $e \in [-\bar{\epsilon}, \bar{\epsilon}]$ ,  $\tilde{m} = m(q) + e$ . Put differently,

$$Q(\tilde{m}) = \{q \in [0, 1] | m(q) - \bar{\epsilon} \leq \tilde{m} \leq m(q) + \bar{\epsilon}\}. \quad (2)$$

Finally, let  $q_+(\tilde{m}) \equiv \sup Q(\tilde{m})$ ,  $q_-(\tilde{m}) \equiv \inf Q(\tilde{m})$ , and  $w(\tilde{m}) \equiv q_+(\tilde{m}) - q_-(\tilde{m})$ .  $q_+(\tilde{m})$  and  $q_-(\tilde{m})$  are the highest and lowest states that could possibly be associated with the received message  $\tilde{m}$ .  $w(\tilde{m}) \equiv q_+(\tilde{m}) - q_-(\tilde{m})$  is the distance between these two bounds. Except near the boundaries of the message space,  $w(\tilde{m})$  is equal to  $2\bar{\epsilon}$ .

Suppose that the receiver receives the message  $\tilde{m} \in [-\bar{\epsilon}, 1 + \bar{\epsilon}]$ .  $q|\tilde{m}$  has support  $[q_-(\tilde{m}), q_+(\tilde{m})]$ , and

$$g(q|\tilde{m}) = \frac{f(\tilde{m} - m(q))g(q)}{\int_0^1 f(\tilde{m} - m(t))g(t)dt} \quad (3)$$

The receiver's problem is to choose an action that maximizes her expected utility:

$$\max_{a(\tilde{m})} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} U(q, a) g(q|\tilde{m}) dq. \quad (4)$$

Because of Assumptions A1 to A3, the receiver's optimal action is simply the expected value of the state,  $q$ , given the received message  $\tilde{m}$ :

$$A(\tilde{m}) = \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} q g(q|\tilde{m}) dq. \quad (5)$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional  $C$  be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{\epsilon}}^{\bar{\epsilon}} (q - A(m(q) + e))^2 f(e) de dq, \quad (6)$$

where  $A$  is the receiver's optimal action from Equation (5).  $C[m]$  is the expected loss for a given message function  $m$ . The integrand is the loss for a given state  $q$  and action  $A(\tilde{m})$ . The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function  $m$  that minimizes  $C[m]$ :

$$\min_{m \in M} C[m] \quad (7)$$

where  $M$  is the space of weakly increasing piece-wise continuous functions on  $[0, 1]$ . The change of variables  $\tilde{m} = m(q) + e$  and an application of Fubini's Theorem yield

$$C[m] = \int_0^1 \int_{m(q)-\bar{e}}^{m(q)+\bar{e}} (q - A(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) d\tilde{m} dq \quad (8)$$

$$= \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - A(\tilde{m}))^2 f(\tilde{m} - m(q)) g(q) dq d\tilde{m}. \quad (9)$$

## 2 Identity Message

Define

$$I_{(\alpha, \beta, \gamma)}(\tilde{m}) \equiv \int_{\alpha}^{\beta} q^{\gamma} f(\tilde{m} - q) dq \quad (10)$$

and

$$A(\tilde{m}) = \begin{cases} \bar{a}(\tilde{m}) & \text{if } 1 - \bar{e} < \tilde{m} \leq 1 + \bar{e} \\ a(\tilde{m}) & \text{if } \bar{e} < \tilde{m} \leq 1 - \bar{e} \\ \underline{a}(\tilde{m}) & \text{if } -\bar{e} \leq \tilde{m} \leq \bar{e} \end{cases} \quad (11)$$

where

$$\bar{a}(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, 1, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, 1, 0)}(\tilde{m})} \quad (12)$$

$$a(\tilde{m}) = \frac{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(\tilde{m}-\bar{e}, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (13)$$

$$\underline{a}(\tilde{m}) = \frac{I_{(0, \tilde{m}+\bar{e}, 1)}(\tilde{m})}{I_{(0, \tilde{m}+\bar{e}, 0)}(\tilde{m})} \quad (14)$$

Note that the normalizing constant cancels out when computing the conditional expectation. We need three sets of weights and knots.

The cost of the identity message function is given by

$$C[m_i] = \int_{-\bar{e}}^{1+\bar{e}} \int_{q_-(\tilde{m})}^{q_+(\tilde{m})} (q - A(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m}. \quad (15)$$

where

$$q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\} \quad (16)$$

$$q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\} \quad (17)$$

Define

$$\bar{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\tilde{m}-\bar{e}}^1 (q - \bar{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (18)$$

$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\tilde{m}-\bar{e}}^{\tilde{m}+\bar{e}} (q - a(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (19)$$

$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_0^{\tilde{m}+\bar{e}} (q - \underline{a}(\tilde{m}))^2 f(\tilde{m} - q) dq d\tilde{m} \quad (20)$$

Let  $x : [-\bar{e}, \bar{e}] \rightarrow [0, 1]$  be given by

$$x(e) = \frac{1}{2} \left[ 1 + \frac{e}{\bar{e}} \right] \quad (21)$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1} (1 - x(e))^{b-1} \quad (22)$$

for constants  $a > 0$  and  $b > 0$ .

### 3 Discrete Message

A discrete message with  $N + 1$  message is given by

$$m_d(q) = \sum_{n=0}^N \frac{n}{N} \mathbf{1}_{\frac{n}{N+1} \leq q \leq \frac{n+1}{N+1}}. \quad (23)$$

where  $\bar{e} = 1/(2N)$ .

$$q_+(\tilde{m}) = \min\{\tilde{m} + \bar{e}, 1\} \quad (24)$$

$$q_-(\tilde{m}) = \max\{\tilde{m} - \bar{e}, 0\} \quad (25)$$

We have

$$I_\gamma(k) = \int_{q_k}^{q_{k+1}} q^\gamma f(\tilde{m} - m(q)) dq \quad (26)$$

The receiver knows which message was sent. The action function is

$$A(\tilde{m}) = \quad (27)$$

The cost of the discrete message function is given by

$$C[m_d] = \frac{1}{3} \left( \frac{\bar{e}}{1 + 2\bar{e}} \right)^2 \quad (28)$$

## 4 Smooth Message

Suppose that for some  $N \geq 2$ ,  $\bar{e} = 1/(2N)$ . For  $i \in \{0, \dots, M\}$  and  $j \in \{0, \dots, N\}$ , let

$$x_{i,j} = \frac{i + jM}{MN} \tag{29}$$

Unlike the case in which the receiver can perfectly infer the message sent. We have that

$$x \tag{30}$$