1 The Model

There are two players, a sender and a receiver, and a state of nature $q \in [0,1]$ that is known to the sender but not the receiver. Let g denote the density of q. The sender sends an *intended* message $m(q) \in [0,1]$. The receiver receives a noisy version of the intended message, which we call the *received* message, $\widetilde{m} = m + \epsilon$. ϵ is distributed according to a continuous density f with support $[-\bar{\epsilon}, \bar{\epsilon}]$, where $\bar{\epsilon} = \frac{1}{2N} > 0$. The receiver then takes an action $A(\widetilde{m}) \in [0,1]$.

Both sender and receiver have utility over the state, q, and the receiver's action, A, of

$$U(q, A) = -\frac{1}{2}(q - A)^2 I(q) \tag{1}$$

Prior to the start of the game, the sender can specify an intended message function $m(q)$ that she will use. The receiver chooses an action based upon the function $m(q)$ and the received message $\widetilde{m}$, and we denote this function $A(\widetilde{m})$. We work backward and start with the optimal action, given $m(q)$ and $\widetilde{m}$.

Fix a message function m. Let $Q(\widetilde{m})$ denote the set of all states $q \in [0,1]$ such that for some noise $e \in [-\bar{\epsilon}, \bar{\epsilon}]$, $\widetilde{m} = m(q) + e$. Put differently,

$$Q(\widetilde{m}) = \{q \in [0,1] | m(q) - \bar{\epsilon} \leq \widetilde{m} \leq m(q) + \bar{\epsilon}\}. \tag{2}$$

Finally, let $q_+(\widetilde{m}) \equiv \sup Q(\widetilde{m})$, $q_-(\widetilde{m}) \equiv \inf Q(\widetilde{m})$, and $w(\widetilde{m}) \equiv q_+(\widetilde{m}) - q_-(\widetilde{m})$. $q_+(\widetilde{m})$ and $q_-(\widetilde{m})$ are the highest and lowest states that could possibly be associated with the received message $\widetilde{m}$. $w(\widetilde{m}) \equiv q_+(\widetilde{m}) - q_-(\widetilde{m})$ is the distance between these two bounds. Except near the boundaries of the message space, $w(\widetilde{m})$ is equal to $2\bar{\epsilon}$.

Suppose that the receiver receives the message $\widetilde{m} \in [-\bar{\epsilon}, 1 + \bar{\epsilon}]$. $q|\widetilde{m}$ has support $[q_-(\widetilde{m}), q_+(\widetilde{m})]$, and

$$g(q|\widetilde{m}) = \frac{f(\widetilde{m} - m(q))g(q)}{\int_0^1 f(\widetilde{m} - m(t))g(t)dt} \tag{3}$$

The receiver's problem is to choose an action that maximizes her expected utility:

$$\max_{a(\widetilde{m})} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} U(q, a)g(q|\widetilde{m})dq. \tag{4}$$

Because of Assumptions A1 to A3, the receiver's optimal action is simply the expected value of the state, q, given the received message $\widetilde{m}$:

$$A(\widetilde{m}) = \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} qg(q|\widetilde{m})dq. \tag{5}$$

It will be helpful to refer to the *cost* of a message function. Let the cost functional C be given by

$$C[m] \equiv \int_0^1 \int_{-\bar{\epsilon}}^{\bar{\epsilon}} (q - A(m(q) + e))^2 f(e)dedq, \tag{6}$$

where A is the receiver's optimal action from Equation (5). $C[m]$ is the expected loss for a given message function m. The integrand is the loss for a given state q and action $A(\widetilde{m})$. The interior integral integrates over the possible exogenous errors, to generate the expected loss given the state. The exterior integral integrates over possible states. Therefore, the *sender's problem* is to choose a message function m that minimizes $C[m]$:

$$\min_{m \in M} C[m] \tag{7}$$

where M is the space of weakly increasing piece-wise continuous functions on $[0, 1]$. The change of variables $\widetilde{m} = m(q) + e$ and an application of Fubini's Theorem yield

$$C[m] = \int_0^1 \int_{m(q)-\bar{\epsilon}}^{m(q)+\bar{\epsilon}} (q - A(\widetilde{m}))^2 f(\widetilde{m} - m(q))g(q)d\widetilde{m}dq \tag{8}$$

$$= \int_{-\bar{\epsilon}}^{1+\bar{\epsilon}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\widetilde{m} - m(q))g(q)dqd\widetilde{m}. \tag{9}$$

2 Identity Message

Define

$$I_{(\alpha,\beta,\gamma)}(\widetilde{m}) \equiv \int_\alpha^\beta q^\gamma f(\tilde{m} - q)dq \tag{10}$$

and

$$A(\widetilde{m}) = \begin{cases} \overline{a}(\widetilde{m}) & \text{if } 1 - \overline{e} < \widetilde{m} \leq 1 + \overline{e} \\ a(\widetilde{m}) & \text{if } \overline{e} < \widetilde{m} \leq 1 - \overline{e} \\ \underline{a}(\widetilde{m}) & \text{if } -\overline{e} \leq \widetilde{m} \leq \overline{e} \end{cases} \tag{11}$$

where

$$\overline{a}(\widetilde{m}) = \frac{I_{(\widetilde{m}-\overline{e},1,1)}(\widetilde{m})}{I_{(\widetilde{m}-\overline{e},1,0)}(\widetilde{m})} \tag{12}$$

$$a(\widetilde{m}) = \frac{I_{(\widetilde{m}-\overline{e},\widetilde{m}+\overline{e},1)}(\widetilde{m})}{I_{(\widetilde{m}-\overline{e},\widetilde{m}+\overline{e},0)}(\widetilde{m})} \tag{13}$$

$$\underline{a}(\widetilde{m}) = \frac{I_{(0,\widetilde{m}+\overline{e},1)}(\widetilde{m})}{I_{(0,\widetilde{m}+\overline{e},0)}(\widetilde{m})} \tag{14}$$

Note that the normalizing constant cancels out when computing the conditional expectation. We need three sets of weights and knots.

The cost of the identity message function is given by

$$C[m_i] = \int_{-\overline{e}}^{1+\overline{e}} \int_{q_-(\widetilde{m})}^{q_+(\widetilde{m})} (q - A(\widetilde{m}))^2 f(\widetilde{m} - q)dqd\widetilde{m}. \tag{15}$$

where

$$q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\} \tag{16}$$
$$q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\} \tag{17}$$

Define

$$\overline{z} = \int_{1-\bar{e}}^{1+\bar{e}} \int_{\widetilde{m}-\bar{e}}^{1} (q - \overline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{18}$$
$$z = \int_{\bar{e}}^{1-\bar{e}} \int_{\widetilde{m}-\bar{e}}^{\widetilde{m}+\bar{e}} (q - a(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{19}$$
$$\underline{z} = \int_{-\bar{e}}^{\bar{e}} \int_{0}^{\widetilde{m}+\bar{e}} (q - \underline{a}(\widetilde{m}))^2 f(\widetilde{m} - q) dq d\widetilde{m} \tag{20}$$

Let $x : [-\bar{e}, \bar{e}] \to [0, 1]$ be given by

$$x(e) = \frac{1}{2}\left[1 + \frac{e}{\bar{e}}\right] \tag{21}$$

Consider, as an example, the Beta PDF:

$$f(e) = x(e)^{a-1}(1 - x(e))^{b-1} \tag{22}$$

for constants $a > 0$ and $b > 0$.

3 Discrete Message

A discrete message with $N + 1$ message is given by

$$m_d(q) = \sum_{n=0}^{N} \frac{n}{N} \mathbf{1}_{\frac{n}{N+1} \le q \frac{n+1}{N+1}}. \tag{23}$$

where $\bar{e} = 1/(2N)$.

$$q_+(\widetilde{m}) = \min\{\widetilde{m} + \bar{e}, 1\} \tag{24}$$
$$q_-(\widetilde{m}) = \max\{\widetilde{m} - \bar{e}, 0\} \tag{25}$$

We have

$$I_\gamma(k) = \int_{q_k}^{q_{k+1}} q^\gamma f(\widetilde{m} - m(q)) dq \tag{26}$$

The receiver knows which message was sent. The action function is

$$A(\widetilde{m}) = \tag{27}$$

The cost of the discrete message function is given by

$$C[m_d] = \frac{1}{3}\left(\frac{\bar{e}}{1 + 2\bar{e}}\right)^2 \tag{28}$$

4 Smooth Message

Suppose that for some $n \geq 2$, $\bar{e} = 1/(2n)$. A continuous message function solves. We approximate m with the step function

$$m_d(q) = \sum_{n=0}^{N} \frac{n}{N} \mathbf{1}_{\frac{n}{N+1} \leq q \frac{n+1}{N+1}}. \tag{29}$$

Unlike the case in which the receiver can perfectly infer the message sent.