Recall that for some positive integer $N \in \mathbb{Z}_+$, $\bar{\epsilon}_N = 1/2N$. For some constant $c > 0$, the cost of the discrete message function m_d is

$$C[m_d](N) = \frac{1}{3}\left(\frac{2\bar{\epsilon}_N}{1+2\bar{\epsilon}_N}\right)^2 + cN = \frac{1}{12}\left(\frac{1}{1+N}\right)^2 + cN. \tag{1}$$

The sender's problem is to

$$\min_{N \in \mathbb{Z}_+} C[m_d](N). \tag{2}$$

Consider the nearby problem:

$$\min_{N \in \mathbb{R}_+} C[m_d](N). \tag{3}$$

As a function of real numbers $N \in \mathbb{R}_+$, $C[m_d]$ is convex. The problem in Equation 3 has solution

$$N^* = \frac{1}{\sqrt{6c}} - 1 \tag{4}$$

and hence the problem in Equation 2 has solution $\lfloor N^* \rceil$ (where $\lfloor \cdot \rceil$ denotes the nearest integer)