Differential Equations and Planetary Systems

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The main goal of this study is to investigate the properties of the forward Euler and velocity Verlet algorithms and model the Solar system. The mathematical framework for calculating acceleration is developed, for different forms of the gravitaional force, and various ways of testing the algortihms and verifying their accuracies are discussed before the modeling of different planetary cases are conducted. Both two-body, three-body and N-body systems are modelled. In addition, the perihelion precession of Mercury, in a two-body system with the Sun, is investigated with a general relativistic alteration to the gravitational force. The algorithms are timed and their respective optimal timesteps for the highest degree of numerical accuracy are determined. The optimal timesteps for the Euler and Verlet algorithms are found to be $\Delta t = 0.004861$ yr and $\Delta t = 0.003397$ yr respectively. The gravitational force is varied from an inverse square law to an inverse cube law and the effect on the Earth's orbit is described. The three-body system of the Sun, Earth and Jupiter is solved for several cases where the mass of Jupiter is altered in order to observe the effects on the orbits of the Sun and the Earth. Finally, the precession of the perihelion of Mercury is calculated over a period of a century and is found to be 1640 arcseconds. The velocity Verlet method is found to be more precise than the forward Euler method. However, it is also found to be considerably slower.

I. INTRODUCTION

In this project we want to create a simulation of the solar system by developing a code in which we will be using the so-called Verlet algorithm for solving coupled ordinary differential equations. By use of data from NASA[2], giving the position in x, y and z direction as well as their corresponding velocities in units of AU per day, we want to extract initial conditions in order to solve our coupled ordinary differential equations, and object orient our code in order to numerically model the solar system. 1 AU is the average distance between the Sun and Earth, 1.5×10^{11} m.

Before looking at a case including all the planets in our solar system, also Pluto, we will look at a system involving only the Earth and the Sun in two dimensions. In this case we will initialise the position by saying that x = 1 AU and y = 0 AU, solve the differential equations using Euler's method and the velocity Verlet method, and assume that the Sun is a fixed point and that Earth has a circular orbit. We need to perform different tests of the algorithm in order to create a plot of the Sun-Earth system and test the stability of our algorithm to make sure our code runs correctly before implementing the other planets. We also want to compare and look at the differences between the Euler algorithm and the Verlet algorithm. After this we include the other planets in the system, and convert to a centre of mass coordinate system. For these calculations we use only the Verlet algorithm. Looking at the orbit of Mercury is also of interest because this has a peculiar shape. We will run the simulation for a Sun-Mercury system as well in order to study this.

II. METHOD

Discretize Equations

We first want to compute the motion of the the Earth in the Sun-Earth system using Euler's method and the velocity Verlet method for solving ordinary differential equations. We assume the motion of Earth to be in the x-y plane, and we use Newton's second law of motion,

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_{Earth}}, \tag{1}$$

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_{Earth}}, \tag{2}$$

where $F_{G,x}$ and $F_{G,y}$ are the x and y components of the gravitational force respectively. We will in this paper use units convenient for astrophysics where mass is measured in units of solar masses, i.e. $M_\odot = 1$ and distances in units of AU so that $G = 4\pi^2$. Assuming the Earth has a circular orbit around the Sun the gravitational force can be defined as

$$F_G = \frac{M_{Earth}v^2}{r} = \frac{GM_\odot M_{Earth}}{r^2}, \tag{3}$$

which is simplified using the astrophysical unit system so that

$$v^2 r = GM_\odot = 4\pi^2 AU^3/yr^2. \tag{4}$$

This expression is used in order to determine what the initial velocity for a circular orbit of radius 1 AU is required to be by solving for v and inserting $r = 1$ AU

$$v_{\text{circ}} = 2\pi. \tag{5}$$

This will be used to test the special two body circular orbit solution of the planetary system.

Next in the process we want to discretize the equations obtained from Newton's law, Eq.(1) and Eq.(2), and implementing these in the differential equation algorithms. If we look at Eq.(3) on vector form, so that we get

$$\mathbf{F_G} = \frac{GM_{\odot}M_{Earth}}{|\mathbf{r}^2|}\hat{\mathbf{r}}, \tag{6}$$

and using $|\mathbf{r}| = \sqrt{x^2 + y^2}$ and the astrophysical units we get that the acceleration is

$$\frac{d^2x}{dt^2} = a_x = \frac{4\pi^2 x}{\sqrt{(x^2+y^2)^3}}, \tag{7}$$

$$\frac{d^2y}{dt^2} = a_y = \frac{4\pi^2 y}{\sqrt{(x^2+y^2)^3}}. \tag{8}$$

We then discretize this by use of Taylor expansion and insert into the Forward Euler and Velocity Verlet methods. These algorithms are presented below

Algorithm 1 Forward Euler

$a_x = \frac{4\pi^2 x_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $a_y = \frac{4\pi^2 y_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $v_{x,i+1} = v_{x,i} + a_x\Delta t$ $v_{y,i+1} = v_{x,i} + a_y\Delta t$ $x_{i+1} = x_i + v_{x,i}\Delta t + a_x\Delta t^2$ $y_{i+1} = y_i + v_{y,i}\Delta t + a_y\Delta t^2$

Algorithm 2 Velocity Verlet

$a_{x,i} = \frac{4\pi^2 x_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $a_{y,i} = \frac{4\pi^2 y_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $x_{i+1} = x_i + v_{x,i}\Delta t + \frac{a_{x,i}}{2}\Delta t^2$ $y_{i+1} = y_i + v_{y,i}\Delta t + \frac{a_{y,i}}{2}\Delta t^2$ $a_{x,i+1} = \frac{4\pi^2 x_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $a_{y,i+1} = \frac{4\pi^2 y_i}{\sqrt{(x_i^2+y_i^2)^3}}$ $v_{x,i+1} = v_{x,i} + \frac{a_{x,i+1}+a_{x,i}}{2}\Delta t$ $v_{y,i+1} = v_{y,i} + \frac{a_{y,i+1}+a_{y,i}}{2}\Delta t$

Test of Algorithm

Next, we want to test the algorithms, find out which initial velocity value gives a circular orbit, and test the stability of the algorithm as function of different time steps Δt. This we want to do in order to create a plot of the Sun-Earth system and make sure our code runs correctly before implementing the other planets.

We also want to check that both the kinetic and the potential energies in the circular orbit are conserved. As a planetary system can be taken to be frictionless and gravity is a conservative force, the mechanical energy of the system is conserved. In a system where the Earth moves in a circular orbit and the Sun is in the origin, the potential energy of the Sun is zero. Thus the potential energy of the system is the potential energy of the Earth. This can be expressed as

$$E_p = -\frac{GM_{\odot}M_{\text{Earth}}}{r}.$$

A circular orbit means that the radius is constant and so the potential energy is also constant. Because mechanical energy is the sum of potential and kinetic energy and both mechanical and potential energy is conserved the kinetic energy must also be conserved.

We also want to perform a timing of both the forward Euler and the velocity Verlet algorithm for equal final times, in order to investigate which algorithm is most efficient. The number of FLOPs involved is 4N FLOPs for the Euler method and 10N FLOPs for the velocity Verlet method.

Conservation of Angular Momentum

We can show that the angular momentum is conserved by use of Kepler's second law. Angular momentum is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v},$$

where $\mathbf{r}$ is the radial position vector of the object, m the mass of the object and $\mathbf{v}$ is the velocity of the object. For a circular orbit with conserved energy it is obvious that the angular momentum is conserved. Because the energy is conserved and the radius is constant, $\mathbf{r}$ and $\mathbf{v}$ are always normal to each other and the speed of the object must be constant. Thus

$$L_{\text{circ}} = rmv = \text{const.}$$

For an elliptical orbit both r and v are variable. The tangential velocity $\mathbf{v_T}$ can be expressed in terms of the angular velocity $\dot{\theta} = d\theta/dt$ as

$$\mathbf{v}_T = r\dot{\theta}.$$

As the cross product in the definition of the angular momentum only takes into account the tangential velocity the angular momentum is

$$L = mr^2\dot{\theta}.$$

Kepler's second law states that

$$\dot{A} = \frac{dA}{dt} = \frac{r^2\dot{\theta}}{2} = \text{const},$$

where A is the area swept over in a time interval dt, r is radius and $d\theta$ is the angular change over dt. Solving this for $r^2\dot{\theta}$ yields

$$r^2\dot{\theta} = 2\dot{A} = \text{const}.$$

so that the angular momentum for an elliptical orbit becomes

$$L = 2m\dot{A} = \text{const}.$$

Alternative Gravitational Forces

We replace our inverse-square force, Eq.(3), with

$$F_G = \frac{GM_\odot M_{Earth}}{r^\beta}, \tag{9}$$

where $\beta \in [2, 3]$. This we want to do in order to observe how planetary orbits would behave if gravity had a different form. The first question to ask is whether total mechanical energy and angular momentum is still conserved with an altered form of gravity. We investigate the case where $\beta = 3$. In this case there is still no friction in the planetary system, so the question becomes whether or not the altered gravitational force is conservative. One way of determining if a force is conservative is to verify that the work done over a closed path is equal to zero. In order to be thorough we do this for two different paths, one where we send the Earth in a circle around the Sun (i.e. constant r) and one where we let Earth move from 1 AU away from the Sun to 2 AU and back again. First the circular orbit

$$W_{\text{circ}} = \frac{GM_\odot m}{r^3}\int_0^{2\pi} \sin\theta \, d\theta = \frac{GM_\odot m}{r^3}\big(-\cos 2\pi + \cos 0\big) = 0, \tag{10}$$

and then the straight line in and out

$$\begin{aligned} W_{\text{straight}} &= GM_\odot m \left[\int_1^2 \frac{1}{r^3}\, dr + \int_2^1 \frac{1}{r^3}\, dr\right] \\ &= -\frac{GM_\odot m}{2}\big(\frac{1}{4} - 1 + 1 - \frac{1}{4}\big) = 0. \end{aligned} \tag{11}$$

For both paths we see that the net work done on the Earth by the altered gravitational force is zero. Thus we know that also the altered inverse cube law form of gravity is conservative, and therefore mechanical energy must be conserved. The next question is about whether the angular momentum is conserved. In order to show this we will look at the change in angular momentum with time, i.e. the derivative of $\mathbf{L}$ with regards to time

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = \frac{d}{dt}\big(\mathbf{r} \times m\mathbf{v}\big) &= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a} \\ &= \mathbf{r} \times \mathbf{F_G} = 0, \end{aligned} \tag{12}$$

where we have used that the cross product of parallel vectors is zero and the $\mathbf{F_G} = m\mathbf{a}$ Newton's second law. This means that the angular momentum is constant. We will attempt to verify that our program conserve these quantities by running simulations and checking the values for mechanical energy and angular momentum at the beginning and end of the simulations.

Escape Velocity

A physical quantity that is useful in order to verify how accurately our model mimics reality is the escape velocity. The escape velocity is defined as the required velocity an object in a gravitational field must have in order to be able to escape the area of influence of said gravitational field. Mathematically it is defined as the velocity of an object with kinetic energy equal to its potential energy

$$E_K - E_P = 0. \tag{13}$$

The expressions for kinetic and potential energy of an object in a gravitational field are well known and the equation is expanded to

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{r} = 0, \tag{14}$$

which is solved for v yielding

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}. \tag{15}$$

In this it has been assumed that $M >> m$ so that the mutual gravitational attraction can be neglected. Inserting values for the Sun-Earth system using astrophysical units gives

$$v_{\text{esc}} = 2\sqrt{2}\pi, \tag{16}$$

for objects in the Sun's gravitational field at a distance of 1 AU.

The Three-body Problem

In reality the Earth will not orbit unaffected around the Sun, the other planets in the solar system will also have a gravitational field which will impact the Earth's orbit. Thus we initially want to include one more planet in the Sun-Earth system, the most massive planet in the solar system - Jupiter. This we want to do in order to investigate how much Jupiter's attraction alters the Earth's orbit, plot the positions of Earth and Jupiter by use of the velocity Verlet algorithm, and study the stability of this algorithm. We now have a three-body problem, with the Sun still at a fixed center point, and Jupiter with a mass of $M_{Jupiter} = 1.9 \cdot 10^{27}$ kg $= 9.5 \cdot 10^{-4}\ M_{\odot}$. We also assume here that the orbits lie in the xy-plane.

When including Jupiter in the calculations, we need to add the gravitational force from Jupiter. This force can easily be found by altering Eq.(3), so that Jupiter is taken into account instead of the Sun,

$$F_{Earth-Jupiter} = \frac{GM_{Jupiter}M_{Earth}}{r^2_{Earth-Jupiter}}, \tag{17}$$

where $M_{Jupiter}$ is the mass of Jupiter and $r_{Earth-Jupiter}$ is the distance between Earth and Jupiter. In addition, after the first plot is created, we want to increase $M_{Jupiter}$ by a factor of 10 and 1000 in order to again study the stability of the Verlet solver.

Up until this point, the Sun's position has been fixed at the origin of the system, but as for the Earth, in reality the Sun's position is also affected by the gravitation from the other planets in the solar system. Instead of using the Sun's position as the system's center point, we now want to use the center-of-mass position of the three-body system as the origin. By giving the Sun an initial velocity, this point will be fixed as the total momentum of the system then becomes zero. This we want to do in order to compare the results of this case with the results from the previous case where the Sun is positioned at a fixed center point.

We demand that the total momentum of the system is zero,

$$\sum_i M_i \mathbf{v}_i = 0. \tag{18}$$

Here index i refers to the i'th body in the system. We decompose the momentum and solve for the Solar velocity

$$v_{x,y,Sun} = -\frac{1}{M_{sun}}\left(\sum M_i \mathbf{v}_{x,y,i}\right). \tag{19}$$

Above we see the expression for the initial Solar velocity in such a way that the total momentum of the system is zero.

Complete Solar System

After the code successfully run for the three-body problem we want to include all the planets in the solar system presented in Table I, except for Mercury, so that we later can plot and compare with the results with the previous systems. The reason we do not include Mercury is because we want to look at this later in a system alone. We will use initial positions and velocities for the planets from NASA's JPL's Horizons system presented in Table II and Table III respectively. These data can be found at the NASA website[2]. The positions and velocities are time specific and the time used is the Julian day 2459140.5 corresponding to October 18. 2020 00:00 TBD.

Comparing the prediction for the perihelion precession with the observed data was a test of the general theory of relativity. After one orbit around the Sun, Mercury will not end up at the same point where it started, leading to the orbit looking like an ellipse rotating in space after several orbits. The perihelion of the ellipse slowly precesses around the Sun. We now want to study this special orbit, so we need to add a general relativistic correction to the Newtonian gravitational force,

$$F_G = \frac{GM_{Sun}M_{Mercury}}{r^2}\Big[1 + \frac{3l^2}{r^2c^2}\Big], \tag{20}$$

where $M_{Mercury}$ is the mass of Mercury, c is the speed of light in vacuum, r is the distance between Mercury and the Sun, and $l = |\mathbf{r} \times \mathbf{v}|$ is the magnitude of Mercury's orbital angular momentum per unit mass.

In order to see this precession more clearly, we want to run the simulation and create a plot of the orbit of Mercury over 100 years in a Sun-Mercury system. By use of

$$\tan(\theta_p) = \frac{y_p}{x_p}, \tag{21}$$

we can find the value of the perihelion angle θ_p, where x_p is the x position of the perihelion, i.e. where Mercury is at its closest to the Sun, and y_p is the y position at the same point. We set that Mercury's initial position is at the perihelion at $x_0 = 0.3075$ AU and $y_0 = 0$, and that its initial velocity is 12.44 AU/yr. We also assume that the Sun is positioned at a fixed center point of the system. We can find θ_p by looking at the first and last orbit and find the angle between them by looking at the x and y positions. By setting $y_0 = y_p = 0$, we get that θ_p for the first orbit is zero, so we only need to find θ_p by using the last x and y position for the last orbit in order to find the final θ_p.

III. RESULTS

When running the simulation for 1.1 year with a time step $\Delta t = 1.1 \cdot 10^{-5}$ for the Sun-Earth system, assuming that the Sun is positioned at a fixed center point and that Earth is the only planet in the system, we find that the Forward Euler method uses 9.565 ms and the velocity Verlet method uses 15.363 ms. Next, we find that the optimal initial velocity in y-direction for Earth so that it orbits circularly, by use of $\Delta t = 10^{-3}$ and running for a period of 1.1 year, is approximately 6.2828 AU/yr, which is consistent with the theoretical value of 2π. This is presented in Figure 1.

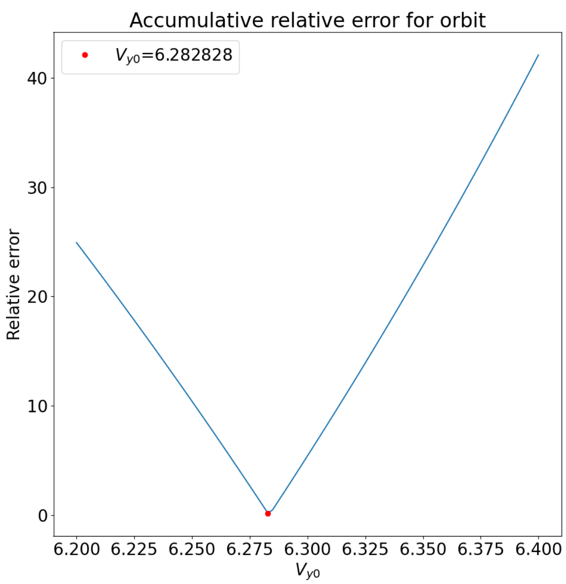

FIG. 1. Optimal initial velocity in y-direction $v_{y0} = 6.2828$ AU/yr, in order for Earth to get a circular orbit around the Sun in the Sun-Earth system. The simulation is run for 1.1 year with $\Delta t = 10^{-3}$.

Running the calculations for a time period of 1.1 year, using the optimal v_{y0} and evaluating $\Delta t \in [10^{-2}, 5 \cdot 10^{-4}]$ with 1001 points, we find two different optimal time steps Δt for this Sun-Earth system, one by use of the Forward Euler method and one by use of the velocity Verlet method. We find that the optimal t when using the Forward Euler method is approximately 0.0048 years, with a relative error of $1.3 \cdot 10^{-4}$, presented in Figure 2, and approximately 0.0033 years, with a relative error of $6 \cdot 10^{-5}$, when using the velocity Verlet method, presented in Figure 3.

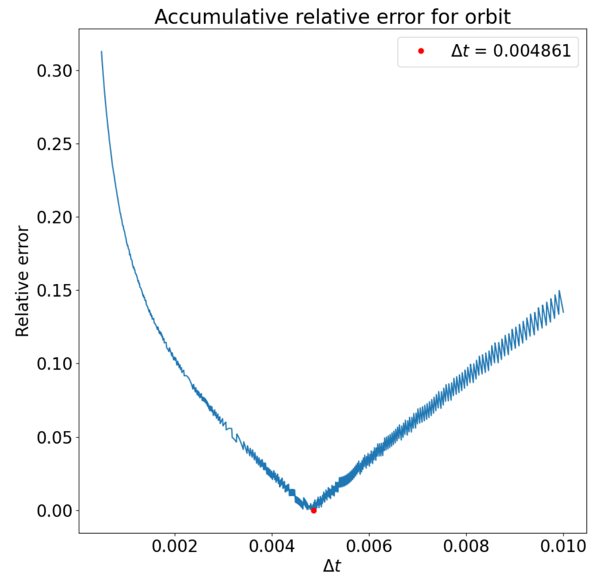

FIG. 2. Optimal time step $\Delta t = 0.0048$ yr in the Sun-Earth system when using the Forward Euler method.

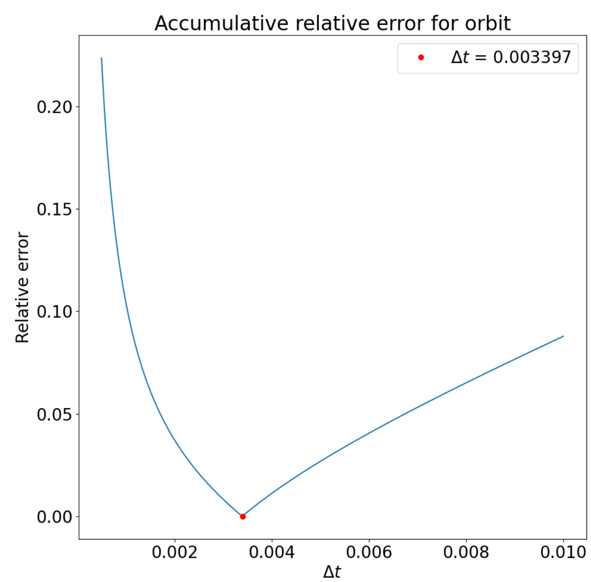

FIG. 3. Optimal time step $\Delta t = 0.0033$ yr in the Sun-Earth system when using the velocity Verlet method.

By running the simulation for 1.1 year with the optimal initial velocity and the optimal time step Δt for the Verlet method previously presented, we obtain the plot shown in Figure 4, visualizing the Earth's circular orbit around the Sun in the Sun-Earth system.

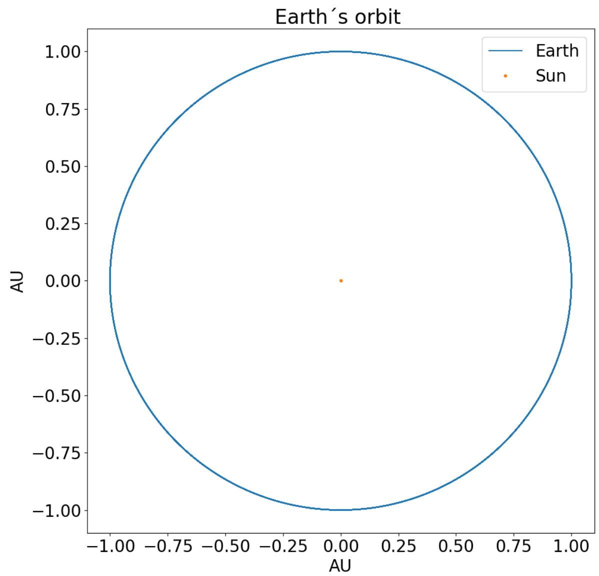

FIG. 4. Earth's circular orbit around the Sun fixed at the center point of the system.

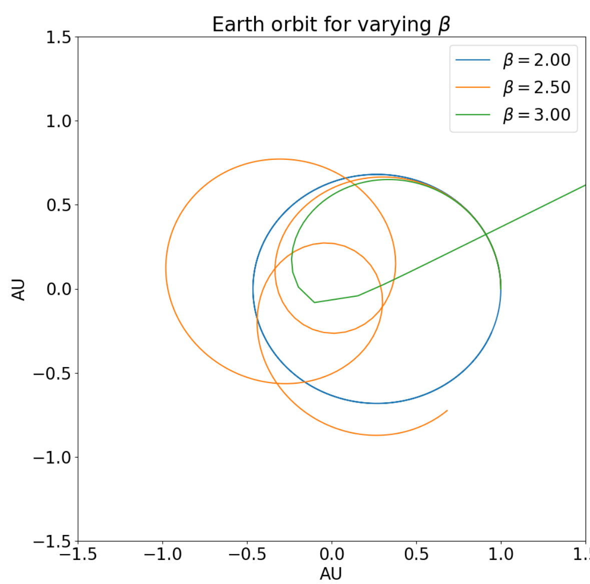

FIG. 5. When the value of β varies, we observe that the orbit of the Earth is changed for different values of β.

Varying the value of β in (9), over a time period of 1.1 year with $v_{y0} = 5$ and using the optimal Δt for the Verlet method, we observe that the orbit of the Earth is changed for different values of β, visualized in Figure 5.

When adding Jupiter to the system, no longer assuming the Sun to be fixed at the center point of the system, we obtain the plots presented in Figures 6, 7, 8 and 9. The center-of-mass position of the three-body system is now the origin. The Sun-Earth-Jupiter system is run for 15 years with the optimal Δt for the velocity Verlet method. In Figure 6 we observe that neither the Sun or the Earth are affected much by the mass of Jupiter. In Figures 7, 8 and 9, where the mass of Jupiter is increased by a factor of ten for each plot, we observe that both the Sun and the Earth is more and more affected as Jupiter's mass increases.

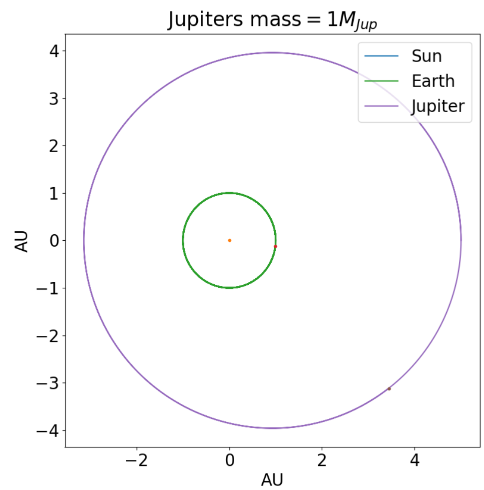

FIG. 6. Earth and Jupiter orbiting the Sun for a time period of 15 years, with the center-of-mass position of the three-body system in the origin.

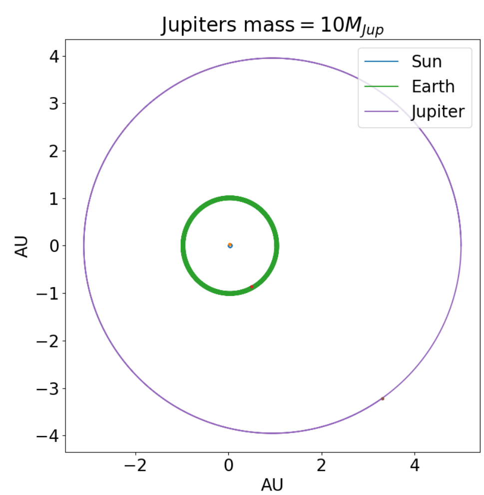

FIG. 7. Earth and Jupiter orbiting the Sun for a time period of 15 years, with the center-of-mass position of the three-body system in the origin. Jupiter has a mass ten times the original mass.

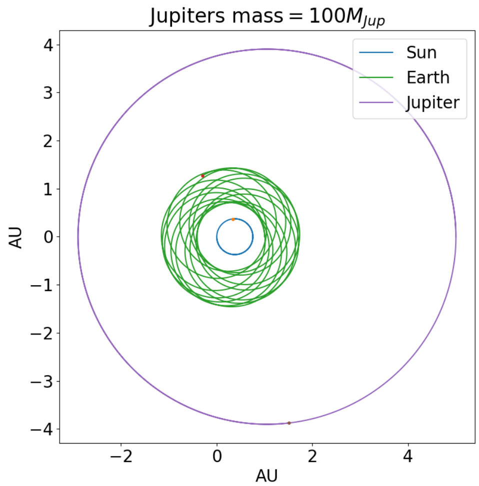

FIG. 8. Earth and Jupiter orbiting the Sun for a time period of 15 years, with the center-of-mass position of the three-body system in the origin. Jupiter has a mass one hundred times the original mass.

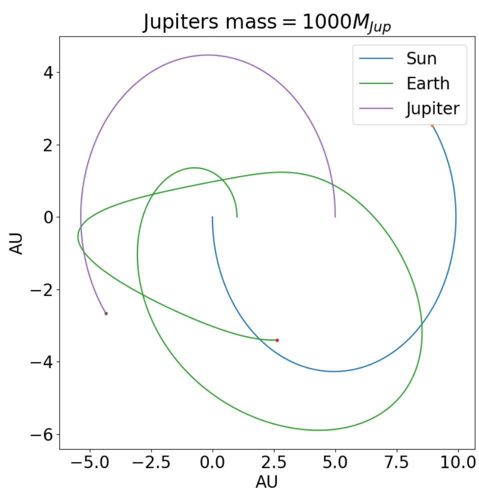

FIG. 9. Earth and Jupiter orbiting the Sun for a time period of 15 years, with the center-of-mass position of the three-body system in the origin. Jupiter has a mass one thousand times the original mass.

When looking at the whole solar system including all the planets except for Mercury, we run the simulation for 250 years, with a Δt equal to the optimal time step for the Verlet method, and using the initial positions and velocities presented in Table II and Table III we obtain the plot presented in Figure 10. We use (19) to find the initial velocity of the Sun in this system. The bottom plot in this figure shows a close-up on the inner planets in the system.

When studying the system only containing the Sun and Mercury, running the simulation for a time period of 100 years with a $\Delta t = 3 \cdot 10^{-4}$, again fixing the Sun to the center of the system and assuming Mercury has an initial position at the perihelion where $x_0 = 0.3075$ AU and $y_0 = 0$, and an initial velocity of 12.44 AU/yr in positive y-direction, we get the orbits visualized in Figure 11. In the top plot in this figure we observe the first and last orbit of Mercury during this time period. In the bottom plot we observe that the vector pointing from the center to the perihelion point in the first and last orbit has an angle θ_p between them. This angle is found to be 0.4575 degrees = 1640 arcsec.

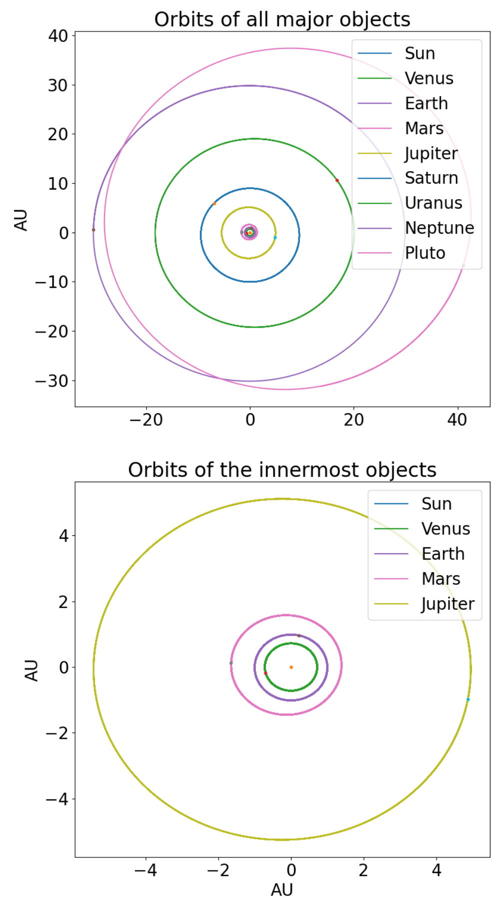

FIG. 10. All the planets in the solar system orbiting the Sun for a time period of 250 years, where the center-of-mass is the origin of the system. The bottom plot shows a close-up on the inner planets in the system.

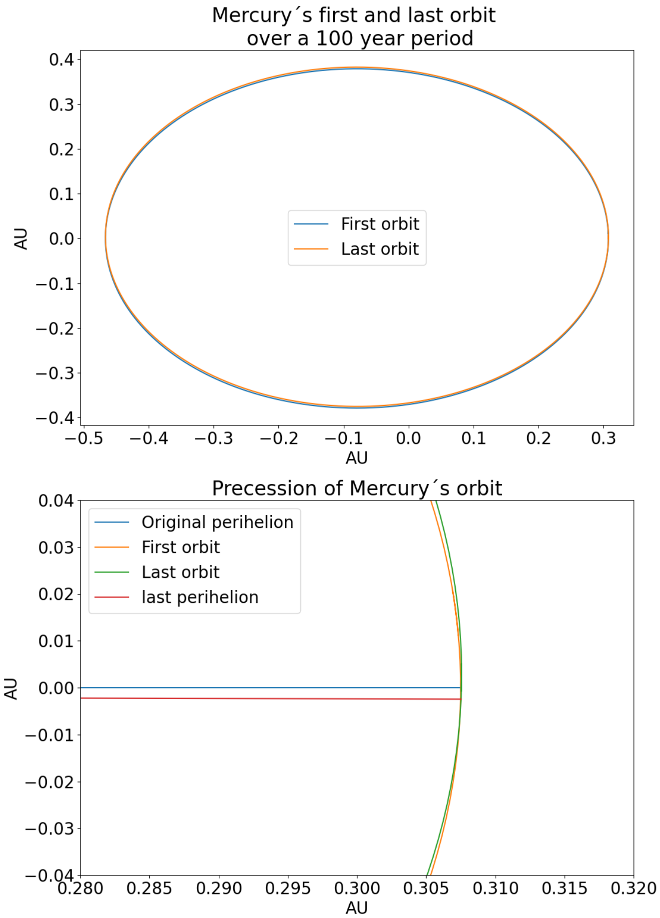

FIG. 11. Mercury orbiting the Sun fixed at a position in the center of the system. The bottom plot visualizes how the perihelion point changes after a time period of 100 years, with an angle θ_p between them.

All these results can also be found in the Github repository [1].

IV. DISCUSSION AND CONCLUSION

When looking at the run time for the Forward Euler method and the velocity Verlet method we find that the Verlet method is slower than the Euler method. For further analysis, to be able to compare this to the number of FLOPs for each method, we would have to look at the time evolution of each method as a function of number of time steps. When comparing these two methods and looking at the optimal Δt for each method in Figure 2 and Figure 3, we observe that the Forward Euler method reaches its optimal precision area for a larger Δt than the Verlet method. However, the Verlet method's relative error for the optimal Δt is twice as good as the relative error of the Euler method with its Δt. This indicates that the velocity Verlet method is a more precise method, which is exactly what we would expect.

When looking at the first system, the Sun-Earth system with the Sun fixed at the center, we observe that the orbit of Earth is changed as β varies, as shown in Figure 5. When looking at Eq.(9) we observe that F_G will decrease as β increases. We would expect this to cause the orbit to be more elliptical for a higher β value, but as observed in Figure 5, this is not the case. We observe that as β increases, Earth is drawn closer and closer to the Sun. The reason for this can possibly be miscalculations.

Moving on to the Jupiter-Earth-Sun system, where the center-of-mass is in the origin of the system and no longer assuming that the Sun is at a fixed center point, we observe from Figures 6, 7, 8, and 9 that as the mass of Jupiter increases, the more the Earth and the Sun is affected and drawn away from their original orbits. We observe from Eq.(17) that F_G increases as $M_{Jupiter}$ increases meaning that Jupiter will have a greater influence on both the Earth and the Sun.

When looking at the full solar system including all the planets except for Mercury in Figure 10, we observe that the Verlet method seems stable, as none of the planetary orbits have any significant deviations.

When studying the abnormal orbit of Mercury visualized in Figure 11, we observe that over a period of 100 years the perihelion precession is approximately 0.4575 degrees which corresponds to 1640 arcseconds. This number is significantly larger than our expectation of 43 arcseconds precession per century. The reason for this could possibly be a consequence of numerical error connected to the selection of a time step, Δt. The precession analysis of Mercury's orbit is dependant on high positional resolution which is dependant on small numerical errors. We have shown in Figure 3 that the Verlet method get an increase in numerical error for Δt smaller than $3.397 \cdot 10^{-3}$ years. Another possible solution to this problem can be an error following from the determination of the position of the perihelion of each orbit.

By achieving a circular orbit for the Earth in the Sun-Earth system with a total relative error of $6 \cdot 10^{-5}$, we validate the Verlet method. So, in conclusion according to our results, the velocity Verlet method is a more precise and in this case more suitable method than the Forward Euler method.

V. APPENDIX

Body	Mass in kg	Mass in $M_\odot$	Distance to Sun in AU
Sun	1.989×10^{30}	1	N/A
Mercury	3.3×10^{23}	1.7×10^{-7}	0.39
Venus	4.9×10^{24}	2.5×10^{-6}	0.72
Earth	6.0×10^{24}	3.0×10^{-6}	1
Mars	6.6×10^{23}	3.3×10^{-7}	1.52
Jupiter	1.9×10^{27}	9.6×10^{-4}	5.2
Saturn	5.5×10^{26}	2.8×10^{-4}	9.54
Uranus	8.8×10^{25}	4.4×10^{-5}	19.19
Neptune	1.0×10^{26}	$5.0\times^{-5}$	30.06
Pluto	1.3×10^{22}	6.5×10^{-9}	39.53

TABLE I. Data for the different bodies in the Solar system. Mass in kg, mass in solar masses and the average distance to the Sun is presented for each of the planets plus the Sun itself and the dwarf planet Pluto.

Body	x-position in AU	y-position in AU
Sun	$-6.122473398487174 \times 10^{-3}$	$6.410482460526987 \times 10^{-3}$
Mercury	$3.532243431882026 \times 10^{-1}$	$-3.883410607572125 \times 10^{-2}$
Venus	$-2.593074555306261 \times 10^{-1}$	$6.786561172635504 \times 10^{-1}$
Earth	$8.984239561537140 \times 10^{-1}$	$4.241593607051067 \times 10^{-1}$
Mars	1.298826322121149	$5.662069485224839 \times 10^{-1}$
Jupiter	2.569528464099691	-4.420315510127981
Saturn	5.152970564937138	-8.557980521297571
Uranus	$1.552991486743390 \times 10$	$1.225108114470249 \times 10$
Neptune	$2.941338926533959 \times 10$	-5.458868379880593
Pluto	$1.383356606211680 \times 10$	$-3.119798652278261 \times 10$

TABLE II. Position vectors from center of mass at time JD 2459140.5

Body	v_x in AU/day	v_y in AU/day
Mercury	$-1.934765136590339 \times 10^{-3}$	$2.916727934340103 \times 10^{-2}$
Venus	$-1.900391839216583 \times 10^{-2}$	$-7.247860847576755 \times 10^{-3}$
Earth	$-7.505790752250796 \times 10^{-3}$	$1.555524120490925 \times 10^{-2}$
Mars	$-4.988664931466998 \times 10^{-3}$	$1.404997171659795 \times 10^{-2}$
Jupiter	$6.432012793032114 \times 10^{-3}$	$4.150995088200494 \times 10^{-3}$
Saturn	$4.467883275314033 \times 10^{-3}$	$2.863553376831038 \times 10^{-3}$
Uranus	$-2.465040979168495 \times 10^{-3}$	$2.904702477181378 \times 10^{-3}$
Neptune	$5.522227196905099 \times 10^{-4}$	$3.105503404723992 \times -3$
Pluto	$2.942753009796773 \times 10^{-3}$	$5.927341579097886 \times 10^{-4}$

TABLE III. Velocity at time JD 2459140.5

[1] Wegger et al. Github FYS4150. Project3. https://github.com/martewegger/fys4150/tree/master/project3, 2020.
[2] Jet Propulsion Laboratory. Horizons web-interface. https://ssd.jpl.nasa.gov/horizons.cgi?fbclid=IwAR3JVLjO5LMFGInrhmHK79Q278OutQv6cVoTlwtluprmrS1-6xzSa2yny9g, 2020.