Pseudoscalar Quadrature Representation

for Real-Valued Signals

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October 29, 2025

10 Ways to i

The quantity i appears in many areas of mathematics and signal processing as a "rotation unit". Although often introduced only as a number with $i^2 = -1$, there are numerous realizations of i, each with a clear meaning:

1. **Algebraic extension**
   $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1),\ i := [x],\ i^2 = -1.$
   Practice: complex numbers = real polynomials modulo $x^2 + 1$.

2. **90° rotation as matrix**
   $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ J^2 = -I.$
   Practice: "multiply by i" = 90° rotation.

3. **Matrix exponential**
   $\left(I + \frac{\pi}{2n} J\right)^n \to e^{(\pi/2)J} = J.$
   Practice: many small rotations yield a 90° rotation.

4. **Eigenvalues of the 90° rotation**
   $\det(\lambda I - R(\pi/2)) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i.$
   Practice: complex eigenvalues reveal rotational structure.

5. **Euler on the unit circle**
   $e^{i\theta} = \cos\theta + i\sin\theta,\ \theta = \pi/2 \implies i.$
   Practice: rotation via multiplication.

6. **Oscillator differential equation**
   $u'' + u = 0 \Rightarrow z^2 + 1 = 0 \Rightarrow z = \pm i.$
   Practice: pure rotation in time = sinusoidal oscillation.

7. **Real power series**
   $e^{ix} = \sum_{k\geq 0} \frac{(ix)^k}{k!} = \cos x + i\sin x.$
   Practice: i arises from alternating signs.

8. **Geometric Algebra (pseudoscalar)**
   $I = e_1 e_2,\ I^2 = -1 \Rightarrow i \leftrightarrow I.$
   Practice: "imaginary" = oriented area element.

9. **Hilbert transform**
   $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\,\hat{f}(\xi),\ H^2 = -I.$
   Practice: 90° phase shift in DSP.

10. **Quaternions**
    In $\mathbb{H}$, any unit u with $u^2 = -1$ can play the role of i.
    Practice: the complex plane is just one 2D slice of quaternions.

Mini remark:
At $\theta = \pi/4$: $\sin\theta = \cos\theta \Rightarrow$ real = imaginary (balanced).
As $\theta \to \pi/2$: $e^{i\theta} \to i$: smooth 90° rotation.

Abstract

Instead of modeling the imaginary part of a "complex" signal using the pure number i (with $i^2 = -1$ implicitly), we propose using the pseudoscalar $\boldsymbol{I}$ of planar geometry. In 2D geometric algebra, $\boldsymbol{I}^2 = -1$ arises from oriented area rather than from numerical definition. Thus, the "imaginary part" becomes an oriented geometric quantity. We construct a window-based measure

$$\boxed{\mathcal{H} = H^\star + \boldsymbol{I}\,\kappa\,P^\dagger}$$

from a *normalized entropy* $H^\star = H_{\text{real}}/\log K \in [0,1]$ and a *pseudoscalar phase term* $P^\dagger \in [-1,1]$ that incorporates oddness, chirality, and statistical evidence. Decisions are made on the polar mapping $(H^\star, \kappa P^\dagger) \mapsto (r,\theta)$ rather than on an unscaled linear sum.

1 Geometric Background

In the Euclidean plane, let basis vectors e_1, e_2 satisfy $e_1^2 = e_2^2 = 1$ and $e_1 e_2 = -e_2 e_1$. The pseudoscalar $\boldsymbol{I} := e_1 e_2$ represents an oriented area, and thus

$$\boldsymbol{I}^2 = (e_1 e_2)^2 = -e_1^2 e_2^2 = -1.$$

The "-1" arises from orientation, not from arbitrary number definition [1–3].

2 Window-Based Decomposition and Normalization

For a real window $x = \{x_k\}_{k=0}^{W-1}$:

$$\begin{aligned} x_k^{\text{rev}} &= x_{W-1-k}, \\ x_k^{\text{even}} &= \frac{1}{2}(x_k + x_k^{\text{rev}}), \\ x_k^{\text{odd}} &= \frac{1}{2}(x_k - x_k^{\text{rev}}). \end{aligned}$$

With weights w_k:

$$E_{\text{tot}} = \sum_k w_k x_k^2, \quad E_{\text{odd}} = \sum_k w_k (x_k^{\text{odd}})^2,$$

Oddness ratio:

$$O = \frac{E_{\text{odd}}}{E_{\text{tot}} + \varepsilon} \in [0,1].$$

Normalized Shannon entropy:

$$H^\star = \frac{-\sum_i p_i \log p_i}{\log K} \in [0,1].$$

3 Chirality and Evidence Weighting

Quadrature component y via Hilbert transform:

$$C = \frac{\sum_k w_k(x_k \Delta y_k - y_k \Delta x_k)}{\sum_k w_k(x_k^2 + y_k^2) + \varepsilon} \in [-1, 1].$$

To avoid artifacts:

- Detrend and z-normalize per window,
- Taper (Hann or DPSS) with $\geq$50% overlap,
- Mirror padding $\geq 2W$,
- FIR Hilbert filter (129 taps).

Orientation evidence:

$$\hat{\theta}_k = \arg(x_k + iy_k), \quad R = \left| \frac{1}{W} \sum_k e^{i(\hat{\theta}_k - \bar{\theta})} \right|,$$

Rayleigh or V-test p_C. Evidence gate:

$$s_C = \sigma(-\beta \Phi^{-1}(1 - p_C)), \qquad P^\dagger = s_C \cdot (O \cdot C).$$

4 A "Complex" Quantity Without Numeric-i

Full measure:

$$\boxed{\mathcal{H} = H^\star + \boldsymbol{I}\, \kappa\, P^\dagger}$$

Polar form:

$$r = \sqrt{(H^\star)^2 + (\kappa P^\dagger)^2}, \qquad \theta = \operatorname{atan2}(\kappa P^\dagger, H^\star).$$

Transformations:

- Parity: $H^\star$ invariant, $P^\dagger \to -P^\dagger$.
- Scale invariance from normalized terms.

5 Applications

- **Audio:** $H^\star$ = loudness entropy, $P^\dagger$ = groove direction.
- **EEG:** $H^\star$ = activation entropy, $P^\dagger$ = lateralized oscillation.
- **Finance:** $H^\star$ = risk measure, $P^\dagger$ = trend chirality.
- **Processes:** $H^\star$ = disorder, $P^\dagger$ = directed causality.

6 Interpretation in Geometric Algebra

The expression $\mathcal{H} = H^\star + \boldsymbol{I}\, \kappa\, P^\dagger$ is formally equivalent to complex notation, yet $\boldsymbol{I}$ has geometric meaning: the oriented area element (parity-odd), while $H^\star$ is parity-even. The "imaginary" part becomes a real geometric object.

7 Conclusion

With $H^\star = H_{\text{real}}/\log K$, robust entropy and chirality estimators, evidence gating and FIR Hilbert processing, the model becomes scale-invariant, stable, and domain-independent. The pseudoscalar approach does not replace i, but gives it geometric-physical meaning.

References

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DOI: 10.5281/zenodo.17475177
Version: v1.0 (October 2025)

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