MIRROR, MIRROR: A RANDOM WALK WITH A TWIST

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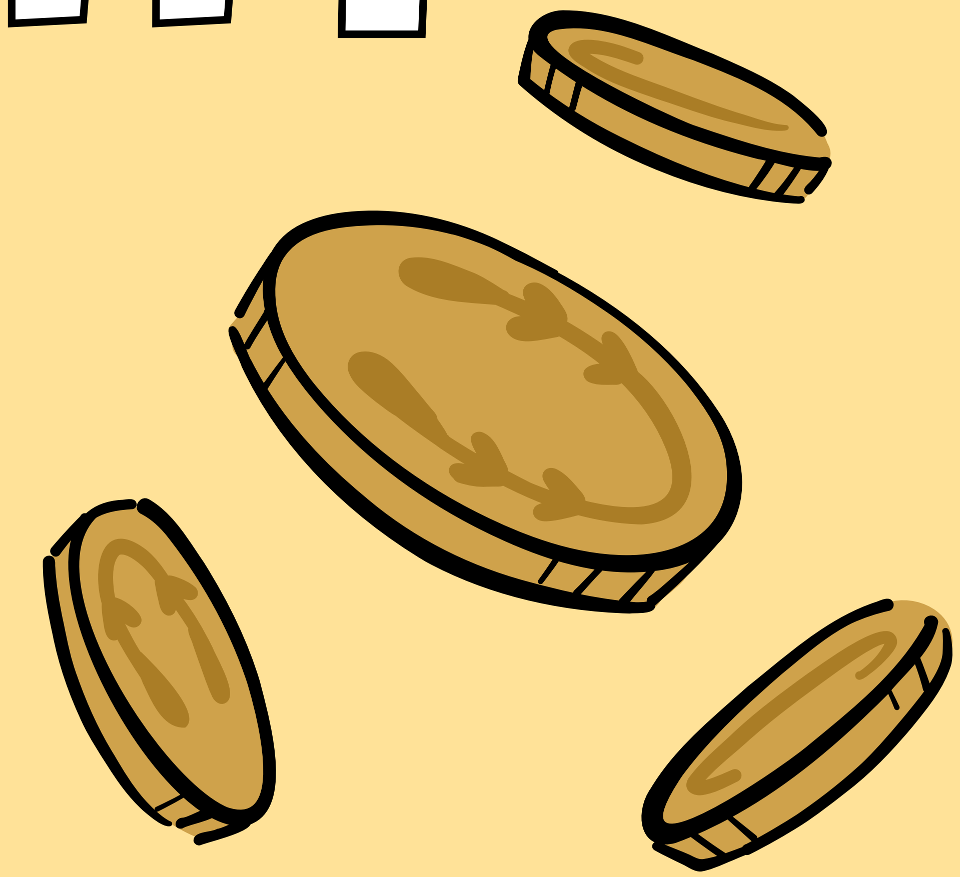
ABOUT ME

- Grade: 22
- Math: Games and puzzles

$$\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{N}}$$

PROBABILITY



PROBABILITY

Concerns events and numerical descriptions of how likely they are to occur

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The probability of an event is a number between zero and one

TYPES OF PROBABILITY

Empirical/Experimental

Estimates by using experience and observation

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Empirical/Experimental

Estimates by using experience and observation

Theoretical

Calculates the likeliness of an event happening based on reasoning and mathematics

EXAMPLE 1

Empirical Probability: You toss a coin 8 times and you record the following <u>outcomes</u>

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Experiment	1	2	3	4	5	6	7	8
Outcome	heads	heads	tails	heads	tails	tails	heads	heads

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Empirical Probability

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What is the empirical probability of getting tails?

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Empirical Probability

Experiment	1	2	3	4	5	6	7	8
Outcome	heads	heads	tails	heads	tails	tails	heads	heads

What is the empirical probability of getting tails? 3/8 = 37.5%

EXAMPLE 2

Theoretical Probability:

$$\frac{\text{number of desired outcomes}}{\text{total number of all outcomes}}$$

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Probability of getting tails? P(tails) =

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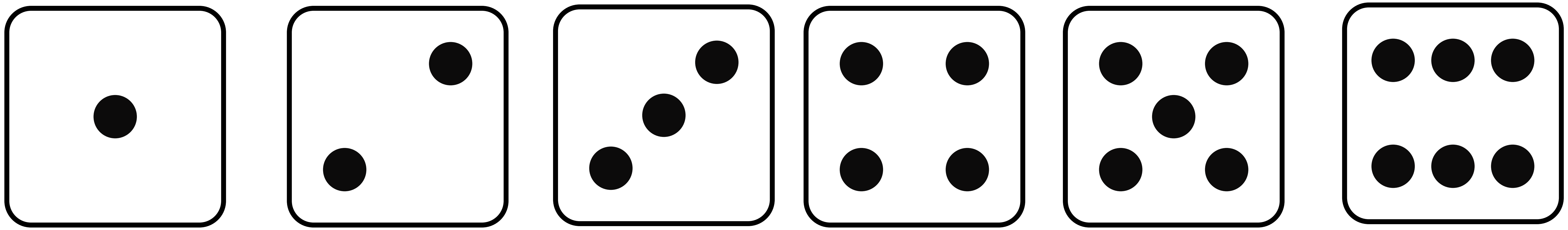
Theoretical Probability:

$$\frac{\text{number of desired outcomes}}{\text{total number of all outcomes}}$$

Probability of getting tails? P(tails) = $\frac{1}{2}$ = 50%

EXAMPLE 3

Example: Consider a six-sided die.



What is the probability of getting a 4? How likely do we get a 4 if you throw the dice?

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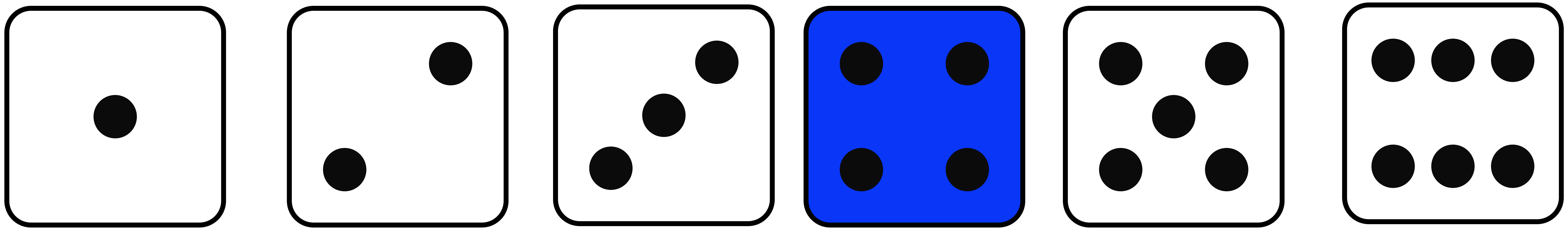
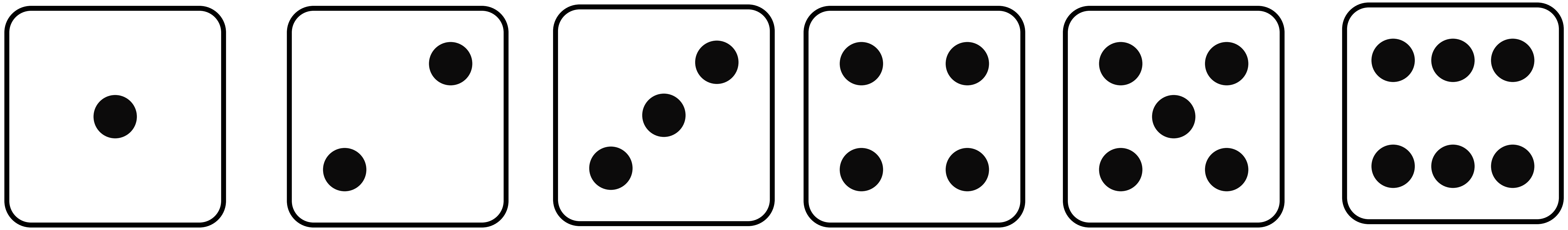


What is the probability of getting a 4? How likely do we get a 4 if you throw the dice? Answer: $\frac{1}{6}$

EXAMPLE 4

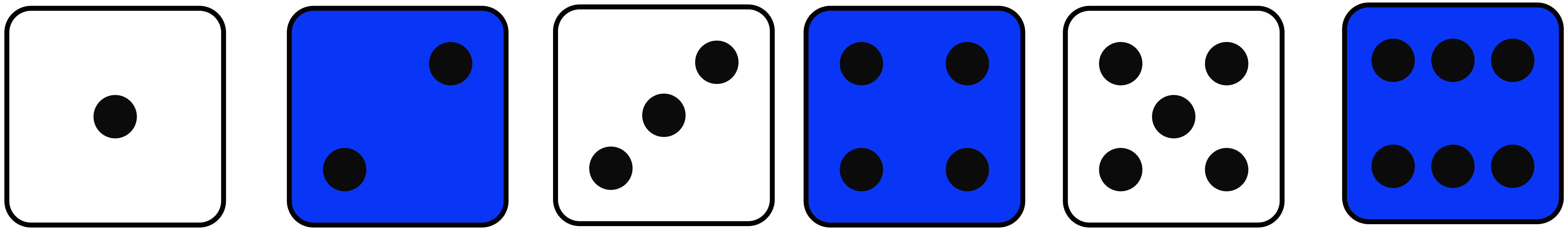
Example: Consider a six-sided die.



What is the probability of getting an even number?

EXAMPLE 4

Example: Consider a six-sided die.



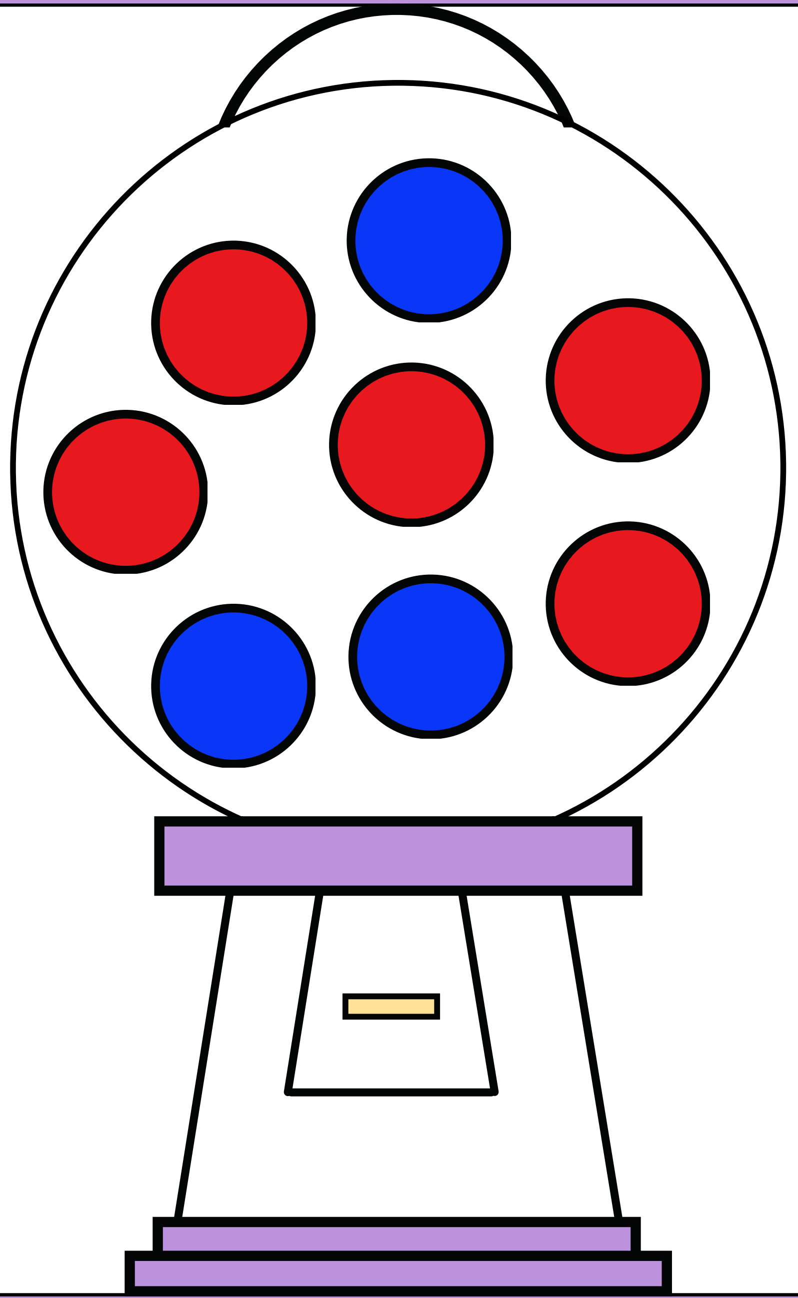
What is the probability of getting an even number? Answer:

$$\frac{3}{6} = \frac{1}{2} = 50\%$$

EXAMPLE 5

How likely is it you will get a blue gumball?

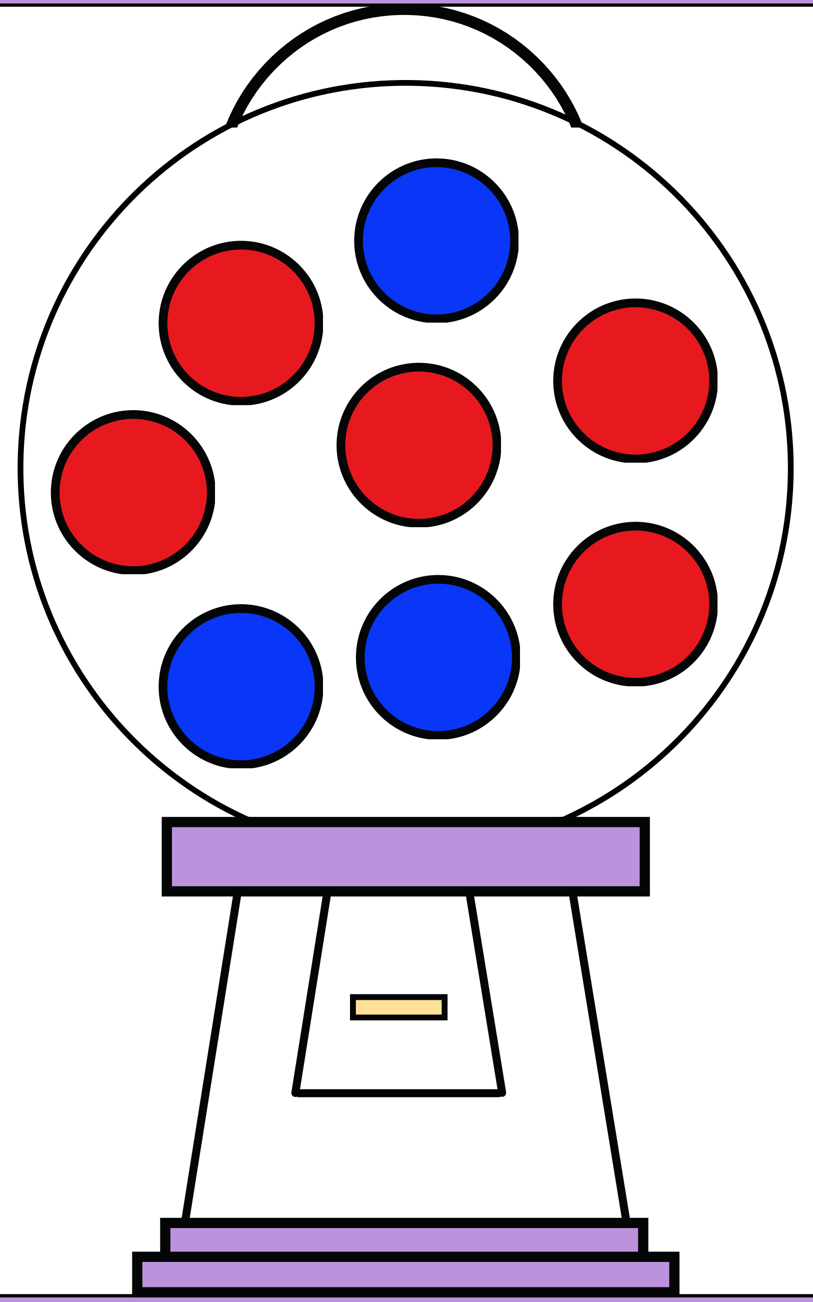
How likely is it you will get a red gumball?



EXAMPLE 5

How likely is it you will get a blue gumball? Answer: $\frac{3}{8}$

How likely is it you will get a red gumball? Answer: $\frac{5}{8}$



WALKS

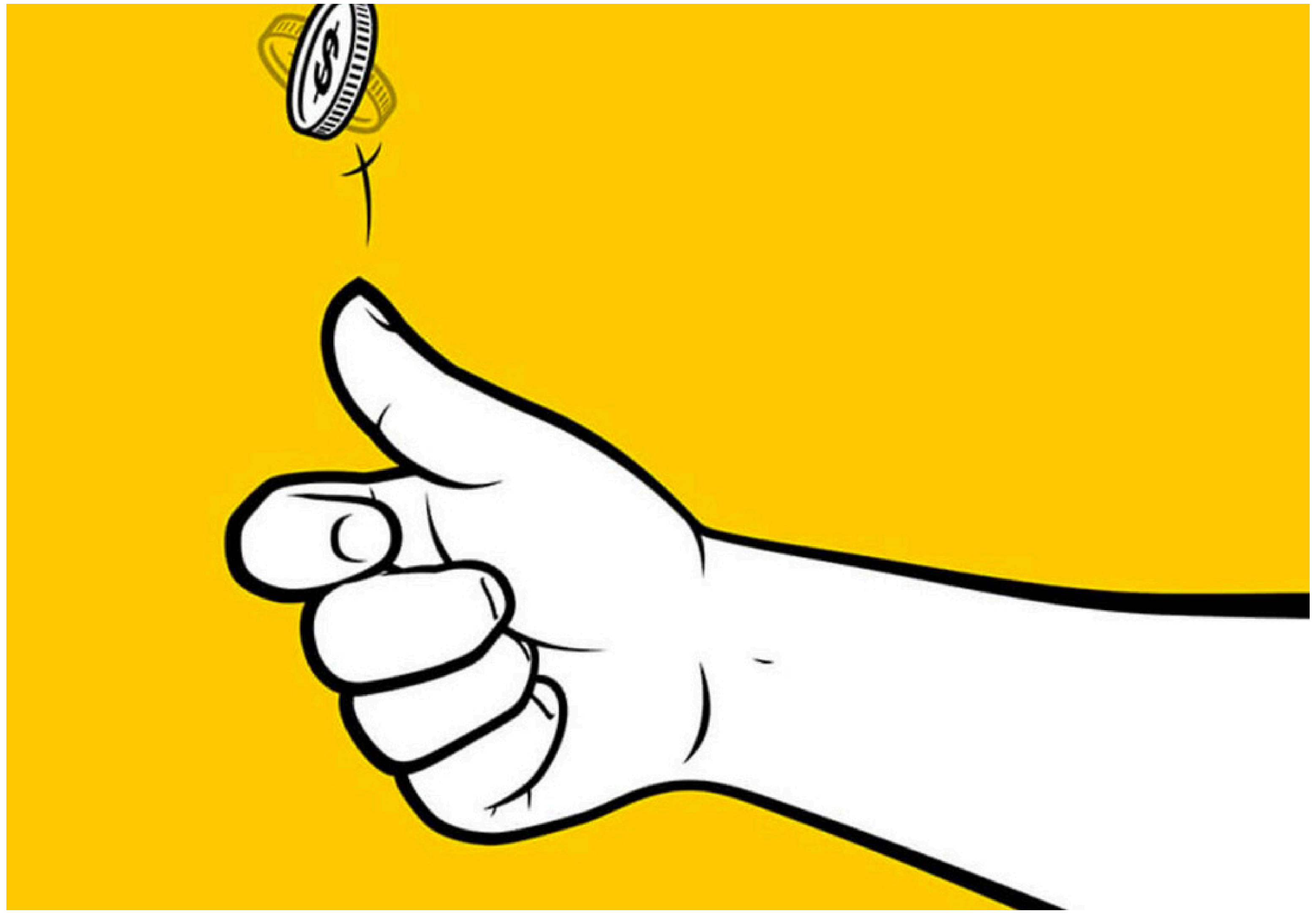
ACTIVITY 1

Flip a coin:

Heads: take one step forward

Tails: take one step back

Record the location



ACTIVITY 1

Share your thoughts: If you take a walk with coin flips as your guide like our volunteer just did, what would you wonder about?

What are you curious about? Write it down!

ACTIVITY 2

Let's investigate

Starting at zero on the number line:

  if heads: step forward one unit

  if tails: step back one unit

Where will you arrive on the number line after six flips?

ACTIVITY 2

	Flip 1	Flip 2	Flip 3	Flip 4
	heads	tails	tails	heads
Location				
0				

ACTIVITY 2

Let's all share our thoughts

ACTIVITY 2

Let's all share our thoughts

Now let's all do the activity!

OBSERVATIONS

After first step: Is this what you expected?

OBSERVATIONS

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How did you get to your location?

OBSERVATIONS

After first step: Is this what you expected?

How did you get to your location?

Final step: Single columns

MORE OBSERVATIONS

What do we notice in the bar graph we have created?

RANDOM WALKS!

MORE OBSERVATIONS

Question: What were some of your noticings and wonderings about the human bar graph we created? Let's record them on our handout and then share with someone!

LET'S ANALYZE

Let's predict what the human bar graph would like if we only did 2-flips, 3-flips, 4-flips, or 5-flips with the coin.

Be specific: Where could people arrive after 4-flips?
What fraction or percentage of the people would arrive in each column of the bar graph?

OUTCOMES

		TH		
TT		HT		HH
-2	-1	0	1	2

OUTCOMES (2)

		TTH		HHT		
		THT		HTH		
TTT		HTT		THH		HHH
-3	-2	-1	0	1	2	3

				TTHH				
				HTTH				
		TTTH		HHTT		HHHT		
		TTHT		HTHT		HHTH		
		THTT		THTH		HTHH		
TTTT		HTTT		THHT		THHH		HHHH
-4	-3	-2	-1	0	1	2	3	4

		HHTTT	TTHHH		
		HTHTT	THTHH		
		HTTHT	THHTH		
		HTTTH	THHHT		
		THHTT	HTTHH		
	TTTTH	THTHT	HTHTH	HHHHT	
	TTTHT	THTTH	HTHHT	HHHTH	
	TTHTT	TTHHT	HHTTH	HHTHH	
	THTTT	TTHTH	HHTHT	HTHHH	
TTTTT	HTTTT	TTTHH	HHHTT	THHHH	HHHHH

QUESTIONS

Is this a coincidence?

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How often will I get heads or tails?

QUESTIONS

Is this a coincidence?

How often will I get heads or tails?

What happens to the total possible outcomes when flipping a coin as the number of flips grows by 1? How many more outcomes do we get from 2-flips to 3-flips?

RESULTS

Number of possible outcomes doubles as the number of flips grows by one:

- 2 flips: 4 outcomes
- 3 flips: 8 outcomes
- 4 flips: 16 outcomes

QUESTIONS

Where will I arrive after a certain number of flips?

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What do you notice about the number of ways to arrive at a location in a N-flips compared to the number of ways of arriving at the two adjacent locations in (N-1)-flips?

QUESTIONS

Where will I arrive after a certain number of flips?

What do you notice about the number of ways to arrive at a location in a N-flips compared to the number of ways of arriving at the two adjacent locations in (N-1)-flips?
Look at the number of ways to arrive at 'zero' and the number of ways of arriving at 'two' in 4-flips and compare to the number of ways of arriving at 'one' in 5-flips

RESULTS

There are 6 ways to arrive at zero in 4-flips

There are 4 ways to arrive at one in 4-flips

There are 10 ways to arrive at one in 5-flips

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This is not a coincidence!

				TTHH				
				HTTH				
		TTTH		HHTT		HHHT		
		TTHT		HTHT		HHTH		
		THTT		THTH		HTHH		
TTTT		HTTT		THHT		THHH		HHHH
-4	-3	-2	-1	0	1	2	3	4

		HHTTT	TTHHH		
		HTHTT	THTHH		
		HTTHT	THHTH		
		HTTTH	THHHT		
		THHTT	HTTHH		
	TTTTH	THTHT	HTHTH	HHHHT	
	TTTHT	THTTH	HTHHT	HHHTH	
	TTHTT	TTHHT	HHTTH	HHTHH	
	THTTT	TTHTH	HHTHT	HTHHH	
TTTTT	HTTTT	TTTHH	HHHTT	THHHH	HHHHH

MORE QUESTIONS

What if we used dice and made a walk like so:

Move forward by one if you get 1,2,3, or 4

Move back by one if you get 5 or 6

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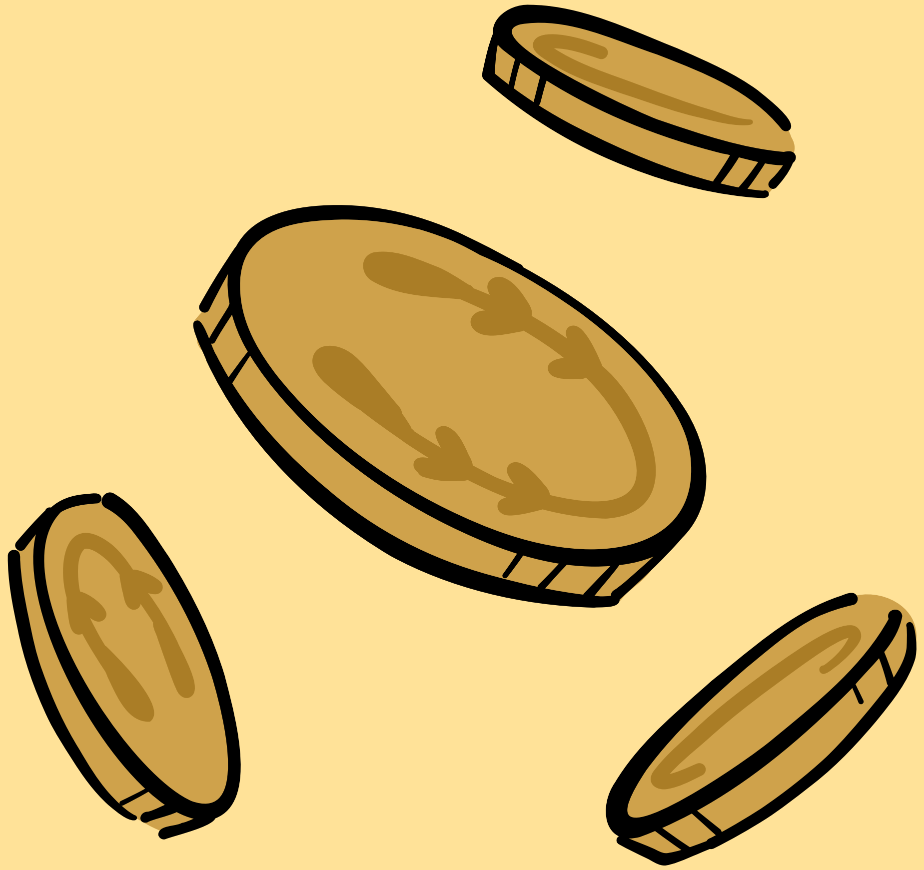
How would the final positions change?

MY WORK



WHAT DO I DO?

Use a die with 3 sides: one, two and three on the sides

if it lands on ‘one’, move forward by one step

if it lands on ‘two’, move back by one step

if it lands on ‘three’ move to the ‘mirror’ side

WHAT DO I DO?

Use a die with 3 sides: one, two and three on the sides

if it lands on ‘one’, move forward by one step

move from x to $x+1$

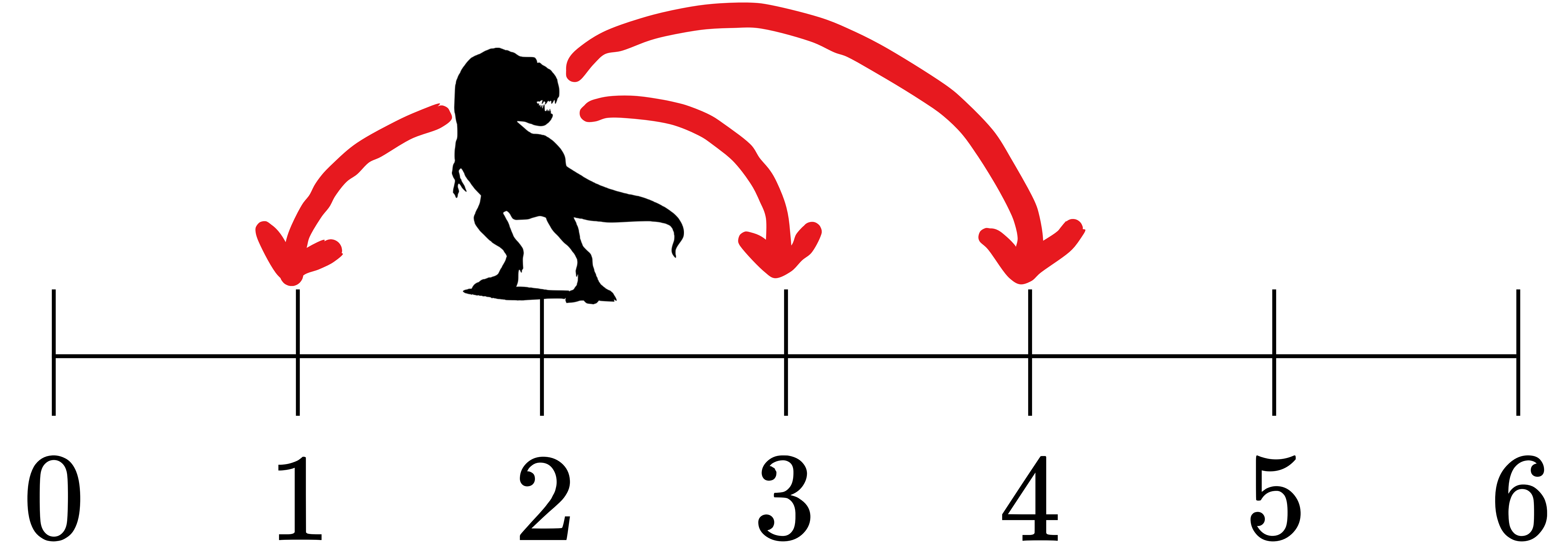
if it lands on ‘two’, move back by one step

move from x to $x-1$

if it lands on ‘three’ move to the ‘mirror’ side

move from x to $N-x$

RANDOM WALK



MY WORK

The kind of questions I studied:

- Probability of getting all the way to the right

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The kind of questions I studied:

- Probability of getting all the way to the right
- How long (on average) does it take to get to either of the ends?
- End behavior

QUESTIONS?

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GAMBLER'S RUIN PROBLEM

APPLICATIONS

- Physics: Used to model particle movement as a stochastic process

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- Hydrology: Sediment transport processes in rivers such as scouring and deposition
- Probability: Random walks
- Other fields...

SCENARIO

Consider a gambler who starts with $5 and plays the following game:

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The gambler keeps playing until they run out of money or wins a desirable amount. For now, let's say \$7

QUESTION

If the gambler starts with \$5, what is the probability of the gambler exiting the game with \$7?

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If the gambler starts with \$5, what is the probability of the gambler exiting the game with \$7?

This scenario is known as the gambler's ruin problem, first posed by Blaise Pascal in 1656

EXAMPLE

Gambler starts with \$5 and the goal is \$7:

Start	\$5		\$5	
First Round	\$6		\$4	
Second Round	\$5	\$7	\$3	\$5

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- Each step has two different outcomes: lose \$1 or win \$1
  - Ex: If the gambler is at \$4, in the previous step the gambler could have been at \$5 or \$3
- The gambler either loses \$1 or wins \$1 with probability $\frac{1}{2}$

PROBABILITY OF WINNING

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$$P(5) = \frac{1}{2}P(4) + \frac{1}{2}P(6)$$

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PROBABILITY OF WINNING

What do we actually know?

- We know $P(0)=0$
  - If the gambler starts with \$0 then the gambler ran out of money so the probability of winning \$7 is 0
- We know $P(7)=1$
  - If the gambler starts with \$7 then the gambler has reached the goal so the probability of winning \$7 is 1

MORE GENERALLY...

We can generalize this scenario by replacing any starting amount by a variable "A" and the goal amount by "N":

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$$P(A) = \frac{1}{2}P(A-1) + \frac{1}{2}P(A+1)$$

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If the gambler starts with \$A, what is the probability of the gambler exiting the game with \$N?

The formula is

$$P(A) = \frac{A}{N}$$

EXAMPLE

If the gambler starts with \$5, what is the probability of the gambler exiting the game with \$7?

$$P(5) = \frac{5}{7} \approx 71\%$$

RESEARCH QUESTION

If a particle starts at some point A on a line of length N. At each step:

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- The particle moves from A to A-1 or
- The particle moves from A to A+1 or
- The particle moves from A to N-A

RESEARCH QUESTION

If a particle starts at some point A on a line of length N. At each step:

- The particle moves from A to A-1 or
- The particle moves from A to A+1 or
- The particle moves from A to N-A

What is the probability that the particle reaches the end of the line?