Parking Functions with a Fixed Set of Lucky Cars

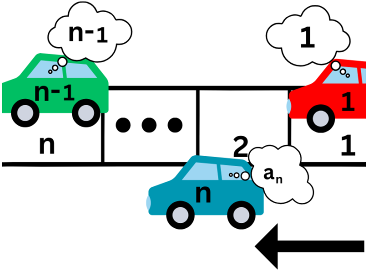
Lucy Martinez

Joint work with Pamela E. Harris

Outline

Background

Parking Functions



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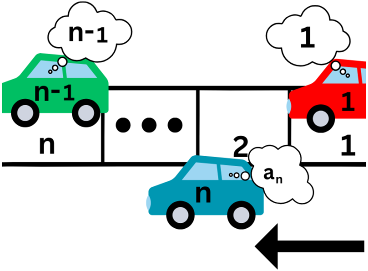
Parking Functions



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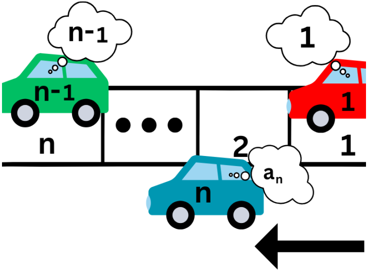

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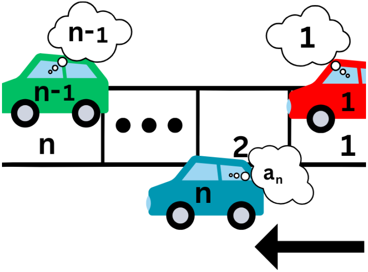

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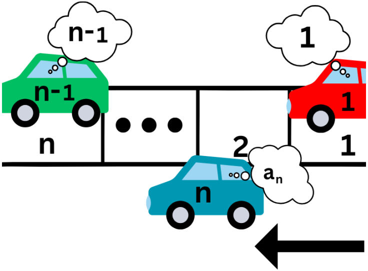

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- Each car i enters the street (one at a time) and has a preferred parking spot a_i.
- Parking Rule:
  - Car i tries to park in their favorite spot a_i.
  - If occupied: the car keeps driving and parks at the next available spot (if any).

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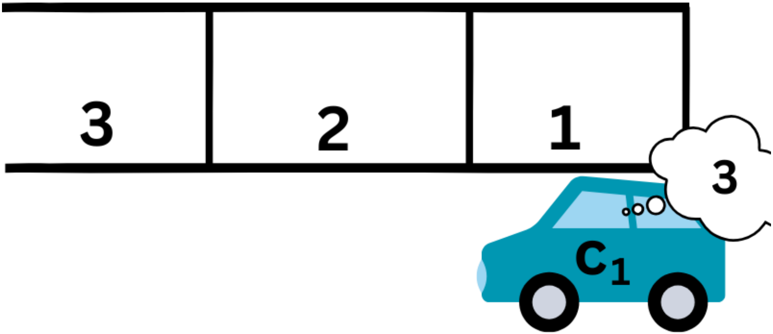

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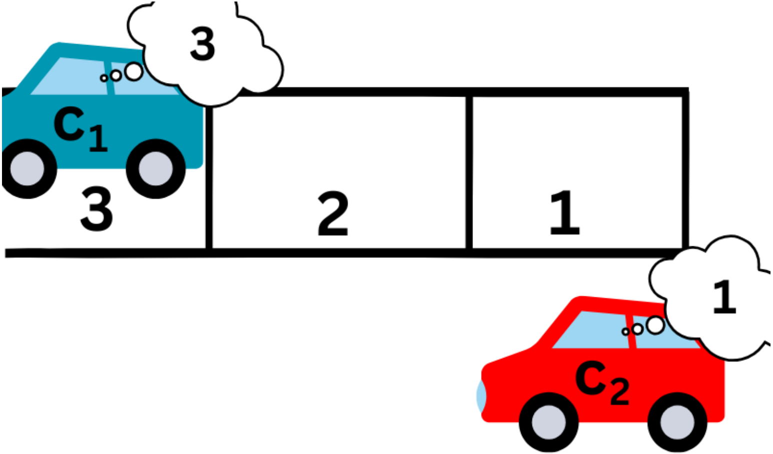

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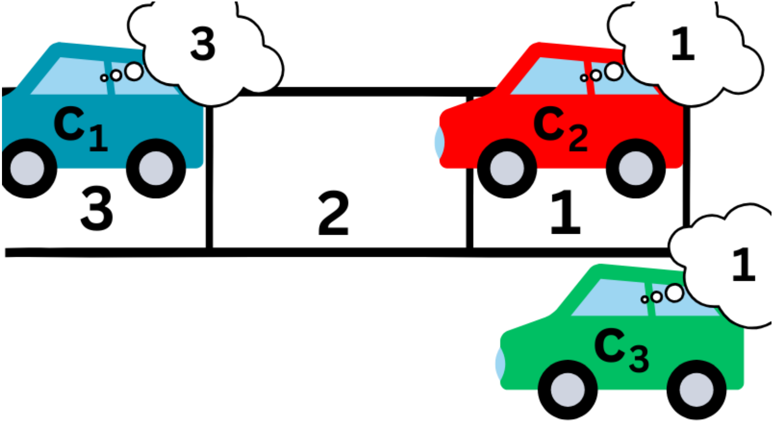

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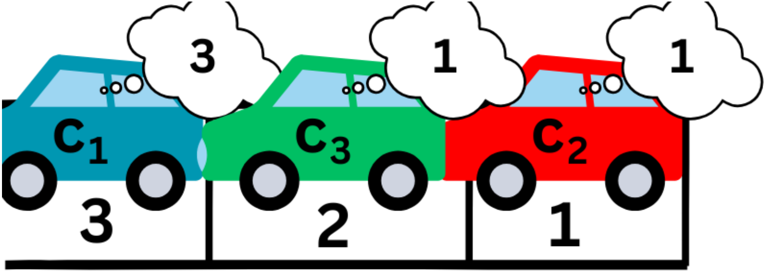

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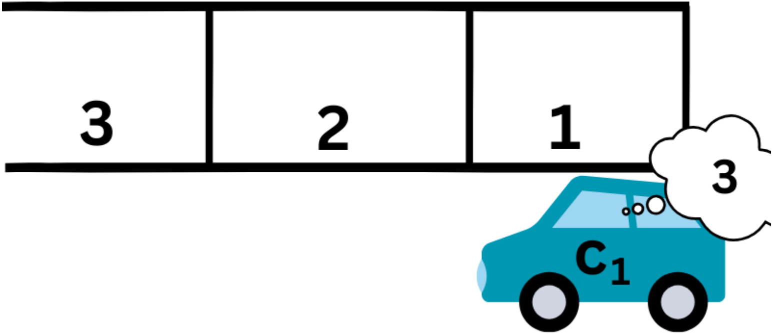

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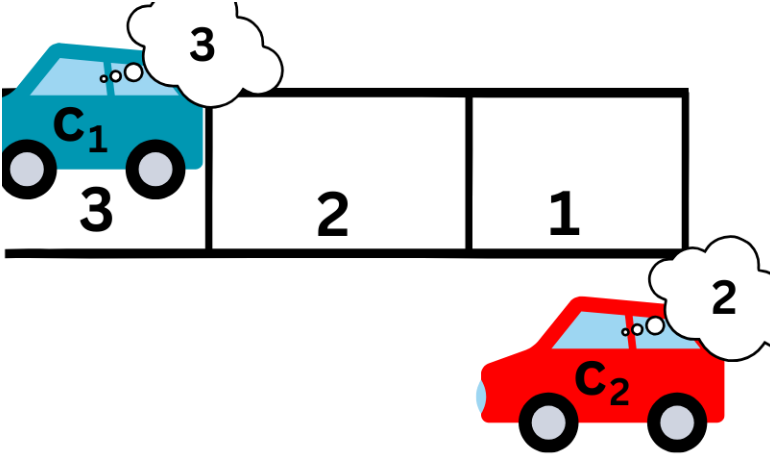

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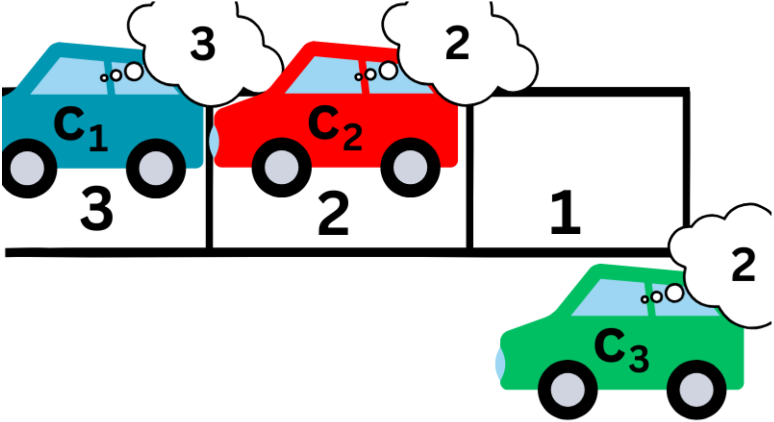

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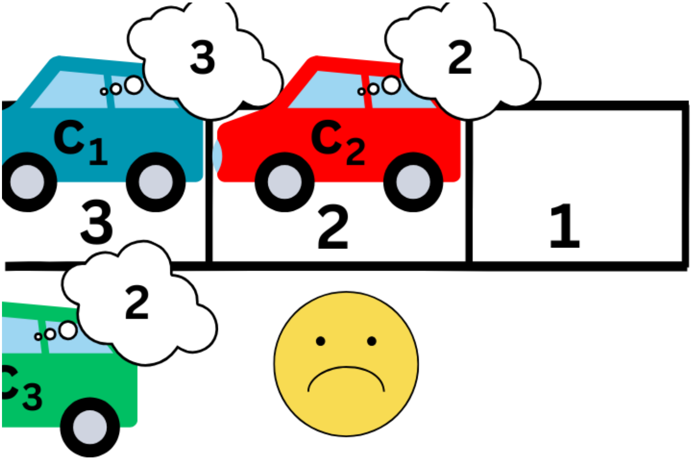

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- $n = 3$: $111, 211, 121, 112, 122, 221, 212, 113, 131, 311, 231, 123, 132, 213, 312, 321$

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- $n = 3$: $111, 211, 121, 112, 122, 221, 212, 113, 131, 311, 231, 123, 132, 213, 312, 321$

Theorem (Konheim and Weiss, 1966)

$|PF(n)| = (n + 1)^{(n-1)}$.

Outcomes of Parking Functions

Definition

If $\mathfrak{S}_n$ denotes the set of permutations of $[n]$ written in one-line notation, then the outcome of α is

$$\mathcal{O}(\alpha) = \pi_1\pi_2\cdots\pi_n,$$

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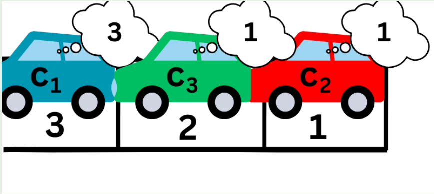
Example

From our previous example: If $\alpha = (3,1,1)$ then $\pi = 231$.



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- If $\alpha = (4, 1, 3, 2)$ then $\pi = 2431$
  - Observation: If the entries in a parking function are distinct, then all cars are lucky.

Previous Work

Theorem (Gessel and Seo)

The generating function for the number of parking functions based on the number of lucky cars is given by

$$L_n(q) = \sum_{\alpha \in PFn} q^{\mathsf{lucky}(\alpha)} = q \prod_{i=1}^{n-1} (i + (n-i+1)q).$$

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Note: This does not tell you which cars are lucky!

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- $\mathsf{Lucky}((1,1,3,2)) = \{1,3\}$
- $\mathsf{Lucky}(\alpha) = \{1,2,\ldots,n\}$ whenever $\alpha = (\pi_1, \pi_2, \ldots, \pi_n)$ and $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$.

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For any subset $I \subseteq [n]$, we define

$$\mathsf{LuckyPF}_n(I) := \{\alpha \in PF_n : \mathsf{Lucky}(\alpha) = I\},$$

which is the set of parking functions of length n with lucky cars being the cars in the set I and unlucky cars being the cars in the set $[n] \setminus I$.

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- We stress that lucky sets contain the lucky cars rather than the position at which the lucky cars park.

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  - Why? Car $3 = \pi_1$ parks in spot 1. Moreover, it was the first car to prefer spot 1 since it was empty when it entered the street.

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Question

For a fixed subset $I \subseteq [n]$ (containing 1), what permutations in $\mathfrak{S}_n$ arise as the outcome of parking functions with lucky set I?

Results

Lucky Sets

Theorem (Harris, M.)

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($\Leftarrow$) Assume $I = \{i_1 = 1, i_2, i_3, \ldots, i_k\}$, with $k \in [n]$.

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($\Leftarrow$) Assume $I = \{i_1 = 1, i_2, i_3, \ldots, i_k\}$, with $k \in [n]$.
Without loss of generality assume that $i_1 = 1 < i_2 < i_3 < \cdots < i_k$.

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Without loss of generality assume that $i_1 = 1 < i_2 < i_3 < \cdots < i_k$.
We want to construct α such that $\alpha \in PF_n$ and $\mathsf{Lucky}(\alpha) = I$. Let $\alpha = (a_1, a_2, \ldots, a_n)$ where

$$a_x = \begin{cases} x & \text{if } x \in I \\ i_\ell & \text{if } i_\ell < x < i_{\ell+1} \end{cases}.$$

□

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Car 2 must be in the lucky set.

Note: the descents in a permutation force certain cars to be lucky!

Descents

Definition

Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$, an index $1 < i \leq n$ is a *descent* of π if $\pi_{i-1} > \pi_i$, and the value π_i is called a *descent bottom* of π.

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Example

1. If $\pi = 1423$ then the descent bottoms are at $i = 1, 3$.
2. If $\pi = 45312$ then the descent bottoms are at $i = 1, 3, 4$.

Descent Bottoms and Lucky Sets

Lemma (Harris, M.)

Fix $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. *For any* $\alpha \in PF_n$ *with outcome* $\mathcal{O}(\alpha) = \pi$, *if* $i \in \mathrm{Des}(\pi)$, *then* $\pi_i \in \mathsf{Lucky}(\alpha)$. *In other words,*

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Remark: Descent bottoms are lucky cars but not all lucky cars have to be descent bottoms.

For example, if $I = \{1, 2, 3, 4\}$ and $\pi = 1234$. The only descent bottom is at $i = 1$.

Characterization

Theorem (Harris, M.)

Fix a lucky set $I \subseteq [n]$ of PF_n. Then $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is the outcome an $\alpha \in \mathsf{LuckyPF}_n(I)$ if and only if

1. $\pi_1 \in I$,
2. *if* $\pi_i \in I$ *and* $\pi_{i+1} \notin I$, *then* $\pi_i < \pi_{i+1}$, *and*
3. *if* $\pi_{i-1} > \pi_i$, *then* $\pi_i \in I$.

Note: Condition 2 follows from condition 3 via the converse.

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By the characterization theorem, $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 \in \mathcal{O}_5(\{1, 4\})$ if and only if

1. $\pi_1 = 1$ or $\pi_1 = 4$
2. if $\pi_i \in I$ and $\pi_{i+1} \notin I$, then $\pi_i < \pi_{i+1}$, and
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We know $\mathrm{DBottom}(\pi) \subseteq I$. There are $2 \cdot 4! = 48$ permutations of length $n = 5$ satisfying Condition (1), as $\pi = 1\sigma$ with $\sigma \in \mathfrak{S}_{\{2,3,4,5\}}$ or $\pi = 4\sigma$ with $\sigma \in \mathfrak{S}_{\{1,2,3,5\}}$.

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Thus $\mathcal{O}_5(\{1, 4\}) = \{12345, 12354, 41235, 45123\}$.

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Given a lucky set $I \subseteq [n]$, can we count the number of outcomes with that lucky set?

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$$|\mathcal{O}_n(I)| = \sum_{J \subseteq [n]} k! \binom{n-k}{j_2 - j_1 - 1, j_3 - j_2 - 1, \ldots, j_k - j_{k-1} - 1, n - j_k}$$

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Note: We have since found and proven a closed formula for this case and the general case!

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Theorem (Harris, M.)

Fix a lucky set $I = \{i_1, i_2, i_3, \ldots, i_k\} \subseteq [n]$ *of* $PF_n^{\uparrow}$ *and assume* $1 = i_1 < i_2 < \cdots < i_k \leq n$. *Then*

$$|\mathsf{LuckyPF}_n^{\uparrow}(I)| = \prod_{j=1}^{k} \mathrm{Cat}_{x_j}, \tag{1}$$

where $x_j = i_{j+1} - i_j - 1$ *for each* $j \in [k-1]$ *and* $x_k = n - i_k$.

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$1 \le a_2 \le 1$
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All tuples of the form $(1, a_2, a_3, a_4, a_5, 6, a_7, 8, a_9)$

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$$\underbrace{6 \leq a_7 \leq 6}_{\mathrm{Cat}_1} \qquad \underbrace{8 \leq a_9 \leq 8}_{\mathrm{Cat}_1 = \mathrm{Cat}_{9-8}}$$

$$\underbrace{\begin{matrix} 1 \leq a_2 \leq 1 \\ 1 \leq a_3 \leq 2 \\ 1 \leq a_4 \leq 3 \\ 1 \leq a_5 \leq 4 \end{matrix}}_{\mathrm{Cat}_4 = \mathrm{Cat}_{6-1-1}}$$

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$$\begin{array}{ccccccccc} \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$1 \leq a_2 \leq 1$, $1 \leq a_3 \leq 2$, $1 \leq a_4 \leq 3$, $1 \leq a_5 \leq 4$: $\underbrace{}_{\mathrm{Cat}_4 = \mathrm{Cat}_{6-1-1}}$

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All tuples of the form $(1, a_2, a_3, a_4, a_5, 6, a_7, 8, a_9)$
This implies that the number of weakly increasing parking functions with lucky set $I = \{1, 6, 8\}$ is $\mathrm{Cat}_4 \cdot \mathrm{Cat}_1 \cdot \mathrm{Cat}_1 = 14 \cdot 1 \cdot 1 = 14$.

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  What happens to all the previous results when there are more spots than cars?

Thank You!

https://marti310.github.io/research.html