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Graduate Research Plan Statement

Intellectual Merits. A partition of a positive integer n, is a sum of positive integers that equals the integer n. For example, $n = 4$ has partitions $1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 1, 3 + 1$, and 4 itself. Thus, there are 5 partitions of 4 (note that the order of the summands does not matter). The concept of a partition can be extended to vectors. Given a set of vectors, $v_1, \ldots, v_n$, we can partition a vector as a nonnegative integral sum $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$ where $a_1, \ldots, a_n \in \mathbb{Z}^+$. As an example, take the positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Let $v = 2\alpha_1 + \alpha_2$, then the vector partitions of v are $\alpha_1 + \alpha_1 + \alpha_2$ and $\alpha_1 + (\alpha_1 + \alpha_2)$. Thus, there are two partitions of the vector v as a sum of the positive roots of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

Vector partitions are studied in different fields of combinatorics, algebra, and number theory. One important use of vector partitions is in the volume formulas of flow polytopes [1] and Kostant's partition function [5]. My work is based on the q-analog of Kostant's partition function $\wp_q$ which is a polynomial-valued function in the variable q whose evaluation at $q = 1$ counts the number of ways to write the vector called the weight ξ as a sum of positive roots, Φ^+, of a classical simple Lie algebra of rank r, i.e.,

$$\wp_q(\xi) = a_1q + a_2q^2 + a_3q^3 + \ldots + a_iq^i \tag{1}$$

where a_i is the number of ways to write the weight ξ as a sum of exactly i positive roots.

In the combinatorial representation theory of finite-dimensional simple Lie algebras, Lusztig defined the q-analog of Kostant's weight multiplicity formula in [7] to give the q-multiplicity of the weight μ in the finite-dimensional irreducible representation with highest weight λ. First, we note the q-analog of Kostant's weight multiplicity formula is given by

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - (\mu + \rho)) \tag{2}$$

where W denotes the Weyl group of the Lie algebra $\mathfrak{g}$, ρ is half the sum of the positive roots, and $\ell(\sigma)$ is the length of $\sigma \in W$. Further, $\wp_q$ is the q-analog of Kostant's partition function as defined in (1). Thus, $m_q(\lambda, \mu)|_{q=1}$ gives the multiplicity of a dominant integral weight μ in a highest weight representation $L(\lambda)$ of $\mathfrak{g}$, [5]. A main challenge in using (2) for computations is that formulas for the partition function or its q-analog, $\wp_q$, do not exist in much generality. Moreover, for a Lie algebra of rank r, the number of terms appearing in this sum is factorial in the rank.

In summer of 2019, I attended the Mathematical Sciences Research Institute Undergraduate Program (MSRI-UP). My research at MSRI found closed formulas for the q-analog of Kostant's partition function for the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$, [4]. These formulas generalize the well-known results of De Loera and Sturmfels; evaluation at $q = 1$ recovers results in [6]. Our strategy was to construct a bijection that reduced the problem of counting partitions of a weight with a fixed number of parts to the problem of counting restricted partitions of an integer with parts drawn from a multiset of natural numbers. This connection provides a partial answer to a question of Sylvie Corteel and Jeremy Lovejoy posed in [3].

"We therefore ask whether the bijection [restricted colored integer partitions and Kostant's partition function] can be generalized to the Lie algebras in $\mathfrak{sl}_n(\mathbb{C})$ for n in the natural numbers." (3)

If such bijection is found, it would provide support on finding closed formulas for the q-analog of the partition function in $\mathfrak{sl}_n(\mathbb{C})$ as we did for $n = 4$. In addition, given previous work connecting Kostant's partition function to multiplex juggling sequences [2], my work further motivates the study of bijections between restricted colored integer partitions and magic multiplex juggling sequences. Such bijections would be the first of their kind in the combinatorics literature.

After submitting paper [4], we discovered an additional avenue of research:

"How do the polynomials arising from the q-analog of Kostant's weight multiplicity function change as μ varies under the action of the Weyl group elements of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$?" (4)

Some of my initial computations provide examples where the polynomials do in fact change as the Weyl group acts on the weight μ. Take $\lambda = \varpi_1 + 2\varpi_2$ and $\mu = \varpi_1$ where ϖ_1 and ϖ_2 are called the fundamental weights (linear combinations of the $\alpha's$) in the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ as described in [4]. Then, using the formulas for the q-analog of Kostant's weight multiplicity function from my prior work in [4], we find that the q-polynomials of $m_q(\lambda, \sigma(\mu))$ when $\sigma = 1, s_1, s_2s_1$ (three distinct elements of the Weyl group of $\mathfrak{sl}_4(\mathbb{C})$) is given by

$$m_q(\lambda,\mu) = q^4+q^3+q^2, \quad m_q(\lambda, s_1(\mu)) = q^5+q^4+q^3, \quad \text{and} \quad m_q(\lambda, s_2s_1(\mu)) = q^6+q^5+q^4.$$

My graduate research aims to address these two research problems. First, I want to generalize the bijection from $\mathfrak{sl}_4(\mathbb{C})$ to Lie algebras $\mathfrak{sl}_n(\mathbb{C})$ with $n \geq 5$. I will approach conjecture (3) using similar methods from [4]. This project will consist of writing code to test examples given any n. After proving the bijection for the general case, I would provide the scripts of the program for public use. Using the generalized bijection, I plan to construct closed formulas for q-analogs of Kostant's partition function for $\mathfrak{sl}_n(\mathbb{C})$.

Additionally, my work will investigate how the polynomials change in the q-analog of Kostant's weight multiplicity function if μ varies under the action of elements of the Weyl group. Note in the above example, although we have three different q-polynomials, their evaluation at $q = 1$ is 3, which recovers a known result in Lie theory: the multiplicity $m(\lambda,\mu) = m(\lambda,\sigma(\mu))$ for all σ elements of the Weyl group. However, we can see that although the q-polynomials are different, they are also related by a factor of q. Our examples for $\mathfrak{sl}_4(\mathbb{C})$ provide computational evidence supporting a need to answer question (4) as it is completely unknown. For this part of my research, I will first compute some examples in the Lie algebras $\mathfrak{sl}_n(\mathbb{C})$ for $n = 5$ and $n = 6$. To compute examples, I will begin by using the existing program from MSRI, and generalize it to expand to my graduate research.

Broader Impacts. Omitted

References

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