TTK4150 Nonlinear Systems and Control

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Learning goals:

* A thorough knowledge of theory and methods for nonlinear dynamical systems.
* Know how to find the invariant sets of nonlinear dynamical systems, and know how to analyze the system behavior around these sets.
* Know the methods Phase plane analysis, Lyapunov stability analysis, Input-to-state stability analysis, Input-Output stability analysis, Passivity analysis, Lyapunov-based control, Energy-based control, Cascaded control, Passivity-based control, Input-Output linearization, and Backstepping control design.

Contents

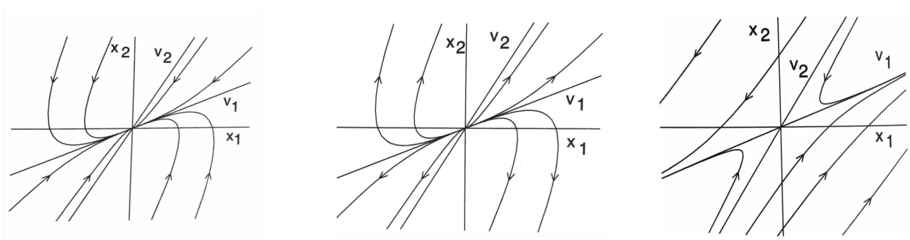
1 | SECOND-ORDER NONLINEAR TIME-INVARIANT SYSTEMS

We first consider the system

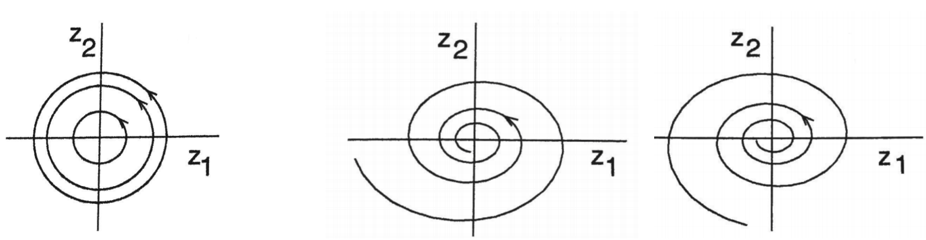
$$\begin{aligned} \dot{x}_1 &= f_1\left(x_1, x_2\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2\right) \end{aligned} \tag{1}$$

Phase-plane analysis: Determine the system behavior by constructing a phase portrait, i.e. plotting different IVP solutions in the phase space.

Local analysis:
* Linearize about x^*.
* Find egeinvalues $\lambda(A)$.
* Classify equilibrium points for $f(x^*) = 0$. If λ is real, them we either get a stable node ($\lambda_2 < \lambda_1 < 0$), unstable node ($0 < \lambda_2 < \lambda_1$) or a saddle point ($\lambda_2 < 0 < \lambda_1$). In the complex case $\lambda_{1,2} = \alpha \pm \beta i$, then we either get a center focus ($\alpha = 0$), a stable focus ($\alpha < 0$) or an unstable focus ($\alpha > 0$).



(a) Stable node (b) Unstable node (c) Saddle point



(a) Center focus (b) Stable focus (c) Unstable focus

Topological equivalence: if the real part of the eigenvalues are nonzero, then the local phase-portrait corresponds to the phase portrait of the linearized system.

1.1 | Periodic orbits and limit cycles

Definition Periodic orbit: $\exists T > 0$ s.t. $x(t+T) = x(t) \quad \forall t \geq 0$.

Definition Limit cycle: non-trivial isolated periodic orbit.

Lemma 1 *Poincaré-Bendixson criterion:*
Let M be a closed bounded subset of the plane s.t.:
$*$ *M contains no x^*, or it contains only one x^* with the property that the eigenvalues of the Jacobian matrix at x^* have positive real parts (unstable focus or unstable node).*
$*$ *Every trajectory starting in M stays in M $\forall t > t_0$.*
Then M contains a periodic orbit of the system.

Lemma 2 *Bendixson negative criterion:*
If on a simply connected region D, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in D.

Corollary 3 *C is a periodic orbit $\implies \Sigma_i I = 1$ (sum of indeces of equilibrium points in C, where saddle points have index -1 and others have index 1)*

2 | FUNDAMENTAL PROPERTIES

Lipschitz: $\|f(t,x) - f(t,y)\| \leq L\|x-y\|$
Either locally Lipschitz on $\mathbb{D}$ (L varies), Lipschitz in $\mathbb{D}$ or globally Lipschitz.

Theorem 4 *Local existence and uniqueness:*
If
$*$ *$f(t,x)$ is piecewise continuous in t,*
$*$ *$f(t,x)$ is Lipschitz $\forall x, y \in B = \{x \in \mathbb{R}^n \,|\, \|x - x_0\| \leq r\} \,\forall t \in [t_0, t_1]$,*
Then there exists a unique solution of the IVP $x(t)$ on $t \in [t_0, t_0 + \delta]$.

3 | LYAPUNOV STABILITY

3.1 | Stability of equilibrium points

Asymptotic stabilization problem: Find $\gamma(t,e)$ s.t. $e = 0$ is an asymptotically stable equilibrium point.
Regulation vs. trajectory tracking.

Definition Stability: $x = 0$ is stable iff $\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad$ s.t. $\quad \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$

Definition Asymptotic stability: $x = 0$ is (locally) asymptotically stable iff it is stable, and
$\exists r > 0 \quad$ s.t. $\quad \|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$

Definition Region of attraction: $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$. We denote R_A as the union of all the regions of attraction.

Definition Global asymptotic stability: $x = 0$ is GAS iff it is stable, and $\lim_{t\to\infty} x(t) = 0 \quad \forall x(0)$

Definition Exponential stability: $x = 0$ is exponentially stable iff
$\exists r, k, \lambda > 0$ s.t. $\|x(0)\| < r \Rightarrow \|x(t)\| \leq k\|x(0)\|e^{-\lambda t} \quad \forall t \geq 0$

Definition Global exponential stability: $x = 0$ is GES iff $\exists k, \lambda > 0$ s.t. $\forall x(0) \quad \|x(t)\| \leq k\|x(0)\|e^{-\lambda t} \quad \forall t \geq 0$

Remark It is useful to think in terms of stability + convergence to seperate the different stability properties.

3.2 | Lyapunov's indirect method

Theorem 5 *Lyapunov's indirect method:*
Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \quad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad C^1 \tag{2}$$

1. *Linearize about $x = 0$, $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}\big|_{x=0}$.*
2. *Find the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$.*
3. *Categorize the eigenvalues:*

* $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \Rightarrow$ *asymptotically(exponentially) stable*
* $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \Rightarrow$ *unstable*
* $\forall i \quad \mathrm{Re}(\lambda_i) \leq 0 \Rightarrow$ *inconclusive*

While Lyapunov's indirect method is simple to use, the results are only local and often inconclusive. Let's see if we can do better ey?

3.3 | Lyapunov's direct method

Definition Lyapunov function:
V is a Lyapunov function for $x = 0$ iff

* V is C^1
* $V(0) = 0, \quad V(x) > 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$
* $\dot{V}(0) = 0, \quad \dot{V}(x) \leq 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$

If $\dot{V}(x) < 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$ then V is a strict Lyapunov function for $x = 0$.

Theorem 6 *Lyapunov's stability theorem:*

* *If $\exists V(x)$ for $x = 0$, then $x = 0$ is stable.*
* *If $\exists$ strict $V(x)$ for $x = 0$, then $x = 0$ is asymptotically stable.*

Theorem 7 *Chetaev's instability theorem:*
If $\dot{V}(x) > 0$ in a set $U = \{x \in B_r | V(x) > 0\}$, then $x = 0$ is unstable.

Definition Radially unboundedness: $\quad \|x\| \to \infty \implies V(x) \to \infty$

Theorem 8 *If $\exists$ strict $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$ and V is radially unbounded, then $x = 0$ is GAS.*

Theorem 9 *If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ s.t.*
* *V is C^1*
* *$k_1 \|x\|^a \le V(x) \le k_2 \|x\|^a \quad \forall x \in \mathbb{D}$*
* *$\dot{V}(x) \le -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$*

then $x = 0$ is exponentially stable. If these conditions hold for $\mathbb{D} = \mathbb{R}^n$, then $x = 0$ is GES.

Remark $\lambda_{min}(P)\|x\|^2 \le x^\top P x \le \lambda_{max}(P)\|x\|^2$

Remark How to deal with indeterminate signs in $\dot{V}$?
* Completion of squares: $x_1 x_2 \le \frac{1}{2}(x_1^2 + x_2^2)$
* Young's inequality: $x_1 x_2 \le \epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2$
* Cauchy-Schwarz' inequality: $|a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \le \sqrt{\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)} \|x\|_2$

3.4 | The invariance principle

Definition Invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \in \mathbb{R}$

Definition Positively invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \ge 0$

Definition Level set: $\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\}$

Theorem 10 *La Salle's theorem:*
If $\exists V : \mathbb{D} \to \mathbb{R}$ s.t.
* *V is C^1*
* *$\exists c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \le c\} \subset \mathbb{D}$ is bounded*
* *$\dot{V}(x) \le 0 \quad \forall x \in \Omega_c$*

Let $E = \left\{x \in \Omega_c | \dot{V}(x) = 0\right\}$. Let M be the largest invariant set contained in E. Then $x(0) \in \Omega_c \Rightarrow x(t) \xrightarrow{t \to \infty} M$.

Definition Region of attraction:
Let $x = 0$ be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz and $\mathbb{D} \subset \mathbb{R}^n$ contains the origin. Let $\phi(t, x_0)$ be the solution. Then the region of attraction is

$$R_A = \{x_0 \in \mathbb{D} \quad | \quad \phi(t, x_0) \text{ is defined } \forall t \ge 0 \text{ and } \phi(t, x_0) \to 0 \text{ as } t \to \infty\} \tag{3}$$

(I.e. all the points with a corresponding solution that converges to the origin).

Remark GAS iff $R_A = \mathbb{R}^n$.

Estimate of R_A: choose the largest set Ω_c in $\mathbb{D}$ which is bounded, and only the connected component of Ω_c that contains the origin. Then this subset is a subset of R_A.

3.5 | Stability analysis of time-variant systems

We now consider the system $\dot{x} = f(t, x)$.

Definition Stability: $\forall \varepsilon > 0, \quad \exists \delta(\varepsilon, t_0) > 0 \quad \text{s.t.} \quad \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \geq 0$

Definition Uniform stability: stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

Definition Asymptotic stability: stable and $\exists c(t_0) > 0 \quad \text{s.t.} \quad \|x(t_0)\| < c \Rightarrow x(t) \overset{t\to\infty}{\longrightarrow} 0$.

Definition Uniform asymptotic stability: asymptotically stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

Definition Global uniform asymptotic stability: uniform stability with $\delta(\varepsilon) \overset{\varepsilon\to\infty}{\longrightarrow} \infty$ and $\forall c > 0 \quad \|x(t_0)\| < c \Rightarrow x(t) \overset{t\to\infty}{\longrightarrow} 0$ uniformly in t_0.

Definition Exponential stability: $\exists c, k, \lambda > 0 \quad \text{s.t.} \quad \|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)} t \geq t_0 \geq 0 \, \|x(t_0)\| \leq c$. GES if $\forall c$.

Definition A continuous function $\alpha : [0, a) \to [0, \infty)$ is a class $\mathscr{K}$ function iff:
$\alpha(0) = 0$ and $\alpha(r)$ is strictly increasing, i.e. $\frac{\partial\alpha}{\partial r} > 0 \quad \forall r > 0$.

Definition If in addition $a \to \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then α is a class $\mathscr{K}_\infty$ function.

Definition A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is a class $\mathscr{KL}$ function if for each fixed s

* $\beta(r, s)$ is a class $\mathscr{K}$ function w.r.t. r

and for each fixed r

* $\beta(r, s)$ is decreasing w.r.t. s,
* $\beta(r, s) \to 0$ as $s \to \infty$.

We can now define stability in terms of class $\mathscr{K}$ functions:

Definition Uniform stability: $\exists$ class $\mathscr{K}$ function α and $\exists c > 0$ s.t. $\|x(t)\| \leq \alpha(\|x(t_0)\|) \, \forall t \geq t_0 \geq 0, \quad \forall \, \|x(t_0)\| < c$.

Definition Uniform asymptotic stability: $\exists$ class $\mathscr{KL}$ function β and $\exists c > 0$ s.t. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \, \forall t \geq t_0 \geq 0, \quad \forall \, \|x(t_0)\| < c$. GUAS if $\forall c$.

Definition $V(t, x)$ is positive definite iff

* $V(t, 0) = 0$
* $V(t, x) \geq W_1(x)$

$\forall t \geq 0, W_1(x) > 0$

Definition $V(t, x)$ is decrescent iff

* $V(t, 0) = 0$
* $V(t, x) \leq W_2(x)$

$\forall t \geq 0, W_2(x) > 0$

We can summarize the stability theorems for time-varying systems like this:

	Stable	Uniformly stable	UAS	GUAS
V	Pos. def.	Pos. def., decrescent	Pos. def., decrescent.	Pos. def., decrescent, radially unbounded
$\dot{V}$	Neg. semidef.	Neg. semidef.	Neg. def.	Neg. def.
	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{R}^n$

Estimate of R_A: $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathbb{D}, c < \min_{\|x\|=r} W_1(x) \implies \{x \in B_r : W_2(x) \leq c\}$ is a region of attraction, when the origin is UAS.

Lemma 11 *Barbalat's lemma:*
Let $\dot{f} : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$. If $\lim_{t\to\infty} f(t)$ exists and is finite, then $\dot{f} \to 0$ as $t \to \infty$.
Rephrased: if V is lower bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is uniformly bounded, then $\dot{V} \to 0$ as $t \to \infty$.

4 | INPUT-TO-STATE STABILITY

4.1 | Input-to-state stability

Now we consider the system $\Sigma : \dot{x} = f(t, x, u)$, where we consider $u(t)$ to be a disturbance/modelling error.

Definition Input-to-state stability (ISS): $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ s.t. $\|x(t, x_0, u)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_\infty)\}$

Remark This is really just an extension of GUAS that says that x is bounded the input as well. So naturally if Σ is ISS then it is also 0-GUAS.

Definition $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for Σ iff
* V is C^1.

$\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ s.t.
* $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$
* $\dot{V}(t, x) = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$

where $W_3 > 0$.
It can be shown that $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.
$\exists \mathrm{ISS-LF}$ for $\Sigma \Rightarrow \Sigma$ is ISS

Lemma 12 *if f is C^1 and globally Lipschitz in (x, u), then Σ is $0-$GES $\Rightarrow \Sigma$ is ISS*

Theorem 13 *Consider the cascaded system $\Sigma_2 \longrightarrow \Sigma_1$, where $\Sigma_1 : \dot{x}_1 = f_1(t, x_1, x_2)$ and $\Sigma_2 : \dot{x}_2 = f_2(t, x_2)$.*
If Σ_2 is GUAS and Σ_1 is ISS, then the cascaded system is GUAS.

4.2 | Input-output stability

We consider the system $y = Hu$.

Definition $\mathscr{L}_p^m$ space: $u \in \mathscr{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_p^p dt\right)^{\frac{1}{p}} < \infty$

Remark This makes $\mathscr{L}_2$ the space of all continuous, square-integrable functions, for instance.

Definition $\mathscr{L}_{pe}^m$ space: $u \in \mathscr{L}_{pe}^m \Leftrightarrow u_\tau \in \mathscr{L}_p^m \quad \forall \tau \in [0, \infty)$, where u_τ is the truncated version if u.

Definition $H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is $\mathscr{L}_p$ stable iff
* $\exists \alpha$ class $\mathscr{K} \quad \alpha : [0, \infty) \to [0, \infty)$
* $\exists$ constant $\beta \ge 0$

s.t. $\|(Hu)_\tau\|_{\mathscr{L}_p} \le \alpha\left(\|u_\tau\|_{\mathscr{L}_p}\right) + \beta \quad \forall u \in \mathscr{L}_{pe}^m$ and $\tau \in [0, \infty)$

Definition Finite-gain $\mathscr{L}_p$ stable: $\exists \gamma, \beta \ge 0$ s.t. $\|(Hu)_\tau\|_{\mathscr{L}_p} \le \gamma \|u_\tau\|_{\mathscr{L}_p} + \beta$

Definition Causal system: $(Hu)_\tau = (Hu_\tau)_\tau$
The two definitons above hold for non-truncated signals if the systems are causal.

Theorem 14 *Small-gain theorem:*
The feedback interconnection of H_1 and H_2 are finite-gain $\mathscr{L}_p$ stable iff $\gamma_1\gamma_2 < 1$.

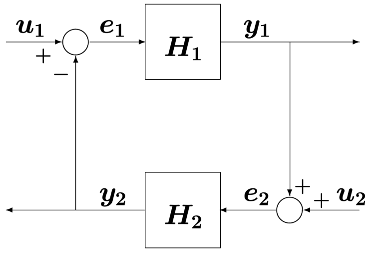


FIGURE 3 Feedback interconnection

5 | PASSIVITY

5.1 | Passivity for memoryless functions

Consider the memoryless function $y = h(t, u) \quad h : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^p$.

Definition The system is passive if $u^\top y \ge 0$ and lossless if $u^\top y = 0$.
The system is input-strictly passive if $u^\top y \ge u^\top \varphi(u)$, where $u^\top \varphi(u) > 0 \quad \forall u \ne 0$.
The system is output-strictly passive if $u^\top y \ge y^\top \rho(y)$, where $y^\top \rho(y) > 0 \quad \forall y \ne 0$.

5.2 | Passivity for dynamical systems

Now we extend this property for dynamical systems: $\Sigma: \quad \dot{x} = f(x,u), y = h(x,u)$
where $f: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is locally Lipschitz and $h: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ is continuous, and $f(0,0) = 0$ and $h(0,0) = 0$.

Definition The system Σ is passive iff $u^\top y \geq \dot{V} \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$
The system is lossless if $u^\top y = \dot{V}$.
The system is input-strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u), \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$.
The system is output-strictly passive if $u^T y \geq \dot{V} + y^T \rho(y), \quad y^T \rho(y) > 0 \quad \forall y \neq 0$.
The system is (state-) strictly passive if $u^T y \geq \dot{V} + \psi(x), \quad \psi(x) > 0$.

Remark Passivity really just generalizes the idea that the change of stored energy in the system should be less than the energy supplied to the system.

5.3 | Passivity and Lyapunov stability

Lemma 15 *If Σ is passive with a positive definite $V(x)$, then the origin of $f\dot{x} = (x,0)$ is stable.*

Lemma 16 *If Σ is output-strictly passive with $\rho(y) = \delta(y), \delta < 0$, then Σ is finite-gain $\mathscr{L}_2$ stable with $\gamma \leq \frac{1}{\delta}$.*

Definition Zero state observability: no solution of $\dot{x} = f(x,0)$ can stay identically in $S = \{x \in \mathbb{R}^n | h(x,0) = 0\}$ other than the trivial solution $x(t) = 0$.

Lemma 17 *The origin of $\dot{x} = f(x,0)$ is asymptotically stable if Σ is state-strictly passive, or output-strictly passive and zero state observable. If $V(x)$ is radially unbounded $\dot{x} = f(x,0)$ is GAS.*

Theorem 18 *If H_1 and H_2 is passive, then the feedback interconnection of H_1 and H_2 is passive with $V = V_1 + V_2$.*

Theorem 19 *If H_1 and H_2 satisfies $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i i = 1,2$ and $\varepsilon_1 + \delta_2 > 0, \varepsilon_2 + \delta_1 > 0$, then Σ is finite-gain $\mathscr{L}_2$ stable.*

Theorem 20 *If H_1 and H_2 are state-strictly passive,*
or H_1 and H_2 are output-strictly passive and zero state observable,
or H_1 is state-strictly passive and H_2 is output-strictly passive and zero state observable or opposite,
then Σ is 0-AS. If V_1 and V_2 are radially unbounded then Σ is 0-GAS.

6 | NONLINEAR CONTROL

6.1 | Lyapunov control design

* Propose a LFC $V(t,x)$, typically as desired energy in system.
* Choose $u = g(t,x)$ s.t. $\dot{V} < 0$ or $\dot{V} \leq 0$ with La Salle / Barbalat.

6.2 | Passivity-based control

Theorem 21 *For the LTI system* $y(s) = h(s)u(s)$ *with* $\mathrm{Re}\,(p_i) < 0, \forall i$ *we have:*

- *Passivity* $\Leftrightarrow \mathrm{Re}[h(j\omega)] \geq 0 \forall \omega$ *(note that if* $h(s)$ *has an integrator as well and* $\mathrm{Re}\,(z_i) < 0$ *this still holds)*
- *Input-strict passivity* $\Leftrightarrow \mathrm{Re}[h(j\omega)] \geq \delta > 0 \forall \omega$*, with* $\varphi(u) = \delta u$
- *Output-strict passivity* $\Leftrightarrow \exists \varepsilon > 0$ s.t. $\mathrm{Re}[h(j\omega)] \geq \varepsilon |h(j\omega)|^2 \forall \omega$*, with* $\rho(y) = \varepsilon y$

Remark This says that with non-negative real part the system is passive, with a strictly positive real part it is input-strictly passive and if it is lower bounded by the square of the magnitude it is output-strictly passive.

Remark Note that the previously discussed passivity theorems can be applied for passivity-based control.
Notably that means that if the controller and plant are passive, then the closed-loop system is passive.
Also if the controller is input- and output strictly passive and the plant is passive, then the system is finite gain $\mathscr{L}_2$ stable. The final passivity theorem can naturally also be applied.

Theorem 22 *Consider* H_1 *with* $\dot{x} = f(x, u)$ *locally Lipschitz and* $y = h(x)$ *continuous, both zero in the origin.*
Also consider $H_2 : \phi(y)$ *locally Lipschitz, memoryless and zero in origin.*
If H_1 *is passive with* $V > 0$*, radially unbounded and zero state observable, and* H_2 *satisifes* $y^\top \phi(y) > 0, y \neq 0$ *(passive, but not lossness) then the origin is GAS.*

Choice of y: for $\dot{x} = f(x) + G(x)u$, if $\exists$ LF $V(x)$ radially unbounded, let $y = \left[\frac{\partial V}{\partial x} G(x)\right]^T$ for the system to be passive.
Feedback passivation: choose $u = \alpha(x) + \beta(x)v$, $y = h(x)$ s.t. $\dot{x} = f(x) + G(x)u$ has desired passivity properties $v \mapsto y$.

6.3 | Feedback linearization

6.3.1 | Input-state linearization

Consider $\dot{x} = f(x) + G(x)u$. Find a state transformation $z = T(x)$ and input transformation $u = \alpha(x) + \beta(x)v$ s.t. the new system in z coordinates is linear and controllable. This is rearily possible to do, so we consider input-output linearization instead:

6.3.2 | Input-output linearization

We now consider the system $\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$.

Definition Lie derivative: $L_f h = \frac{\partial h}{\partial x} f, \quad L_f^2 h = \frac{\partial L_f h}{\partial x} f, \quad \ldots \quad L_f^i h = L_f \left(L_f^{i-1} h\right)$

Definition The system has relative degree ρ in a region $\mathbb{D}_0 \subset \mathbb{D} \subset \mathbb{R}^n$ if

$$\left.\begin{aligned} &L_g L_f^{i-1} h = 0, \quad 1 \leq i \leq \rho - 1 \\ &L_g L_f^{\rho-1} h \neq 0 \end{aligned}\right\} \forall x \in \mathbb{D}_0 \tag{4}$$

Remark (the number of differentiations of y before u appears)

Remark For linear systems we have $\rho = n - m$.

Remark If the relative degree is well defined in the region of interest $\mathbb{D}$, then the system can be input-output linearized.

Definition Zero dynamics: internal dynamics when output is kept at zero by the input i.e. $\psi = 0, \dot{\phi} = f_o(\phi, 0)$.

Definition Minimum-phase system: the zero-dynamics are asymptotically stable.

Input-output linearization:

1. **Find the relative degree ρ**

2. **Write the system in normal form**
Let $\psi_1 = y, \psi_2 = \dot{y}, \ldots$ The external dynamics are then $\dot{\psi_1} = \psi_2, \ldots, \dot{\psi}_\rho = L_f^\rho h + L_g L_f^{\rho-1} h \cdot u$.
Let $\phi_1, \ldots, \phi_{n-\rho}$ and $z = T(x) = \begin{bmatrix} \varphi^\top & \psi^\top \end{bmatrix}^\top$.
Choose φ s.t. T is a diffeomorphism, $L_g \varphi_i = 0$ and $\varphi_i(0) = 0$.
If the Jacobian $\frac{\partial T}{\partial x}\big|_{x_0}$ is nonsingular, then T is a diffeomorphism. Then the internal dynamics are $\dot{\psi} = \ldots$.
We can then finally write the system in normal form: $\dot{z} = \ldots$.

3. **Choose u to cancel nonlinearities**

$$u = \frac{1}{L_g L_f^{\rho-1} h} \left(-L_f^\rho h + v \right) \quad \Longrightarrow \quad \psi_\rho = v \tag{5}$$

4. **Analyze the zero-dynamics**

5. **Choose v to solve the control problem**
This is a special case of the system $\dot{\varphi} = f_0(\varphi, \psi), \quad \dot{\psi} = A\psi + Bv$.

Lemma 23 *If the system is minimum phase and $v = -K\psi$ is chosen s.t. $(A - BK)$ is Hurwitz, then the origin of the system is asymptotically stable.*

Lemma 24 *If $\dot{\phi} = f_0(\phi, \psi)$ is ISS, then the system is GAS.*

For tracking we have $v = -Ke + y_d^{(\rho)}$, but the same results apply, expect ϕ is now only bounded.

6.4 | Adaptive control

6.4.1 | MRAC for SISO systems

Consider the SISO system $\dot{x} = a_p x + c_p f(x) + b_p u$.

SISO MRAC:

1. **Specify desired closed-loop behaviour** by reference model: $\dot{x}_m = a_m x_m + b_m r(t)$
2. **Choose control law** s.t. plant output tracks reference model output when parameters are exactly known, by deriving error dynamics. Then replace parameters by estimates.
3. **Choose adaptation law** $(\dot{\hat{a}}_x, \dot{\hat{a}}_f, \dot{\hat{a}}_r)$ by first deriving new tracking error dynamics in terms of estimation errors, and then choosing adaptation law from suitable Lyapunov function e.g.

$$V(e, \tilde{a}) = \frac{1}{2}e^2 + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_f}\tilde{a}_f^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2 \tag{6}$$

By letting

$$\dot{\hat{a}}_x = -\gamma_x \text{sgn}(b_p) e x, \quad \dot{\hat{a}}_f = -\gamma_f \text{sgn}(b_p) e f, \quad \dot{\hat{a}}_r = -\gamma_r \text{sgn}(b_p) e r \tag{7}$$

the error will go to zero by Barbalat's lemma.

6.4.2 | Adaptive tracking control for a class of MIMO systems

Consider the system $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q)\dot{q} + g(q) = u$, with $M > 0$, $\dot{M} - 2C$ is skew symmetric, $z^T D z > 0 \quad \forall z \neq 0$. While the system is nonlinear it is linear in the unknown parameters a by the Regression matrix Y:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q)\dot{q} + g(q) = Y(q,\dot{q},\ddot{q})a \tag{8}$$

Adaptive MIMO tracking control:

1. Given the desired trajectory $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ bounded.
2. **Choose control law:**
We introduce the virtual reference velocity $s = \dot{e} + \lambda e = \dot{q} - \dot{q}_r$, where $\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$.

$$u = \hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{D}(q)\dot{q}_r + \hat{g}(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d) = Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\hat{a} - K_p e - K_d \dot{e} \tag{9}$$

From $V(s) = \frac{1}{2}s^\top M s$ and the cascade stability theorem we get GUAS with perfect tracking.
3. **Choose an adaptation law** such that tracking is achieved asymptotically. Again done by deriving error dynamics and analyzing Lyapunov function with Barbalat's lemma:

$$V(s, \tilde{a}) = \frac{1}{2}s^\top M s + \frac{1}{2}\tilde{a}^\top \Gamma^{-1}\tilde{a} \implies \dot{\hat{a}} = -\Gamma Y^\top s \tag{10}$$

6.5 | Backstepping

A | LINEAR METHODS

Definition We define the p-norm as:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{11}$$

$$\|f\|_{\mathscr{L}_p} = \left(\int_0^{\infty} |f(t)|^p dt\right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{12}$$

Theorem 25 *Schwarz' inequality:*

$$| < x, y > | \leq \|x\| \cdot \|y\| \tag{13}$$

Definition $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then the Jacobian is defined as:

$$\frac{\partial f}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{14}$$

Which in the scalar case $m = 1$ is the gradient.