# TTK4150 Nonlinear Systems and Control

**Martin Brandt**

Learning goals:

∗ A thorough knowledge of theory and methods for nonlinear dynamical systems.

∗ Know how to find the invariant sets of nonlinear dynamical systems, and know how to analyze the system behavior around these sets.

∗ Know the methods Phase plane analysis, Lyapunov stability analysis, Input-to-state stability analysis, Input-Output stability analysis, Passivity analysis, Lyapunov-based control, Energy-based control, Cascaded control, Passivity-based control, Input-Output linearization, and Backstepping control design.

# Contents

# 1 | SECOND-ORDER NONLINEAR TIME-INVARIANT SYSTEMS

We first consider the system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$
(1)

**Phase-plane analysis:** Determine the system behavior by constructing a phase portrait, i.e. plotting different IVP solutions in the phase space.

> **Local analysis:**
> * Linearize about $x^*$.
> * Find eigenvalues $\lambda(A)$.
> * Classify equilibrium points for $f(x^*) = 0$. If $\lambda$ is real, them we either get a stable node ($\lambda_2 < \lambda_1 < 0$), unstable node ($0 < \lambda_2 < \lambda_1$) or a saddle point ($\lambda_2 < 0 < \lambda_1$). In the complex case $\lambda_{1,2} = \alpha \pm \beta i$, then we either get a center focus ($\alpha = 0$), a stable focus ($\alpha < 0$) or an unstable focus ($\alpha > 0$).
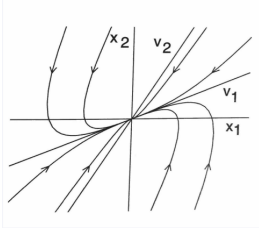
(a) Stable node
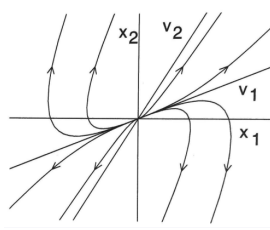
(b) Unstable node
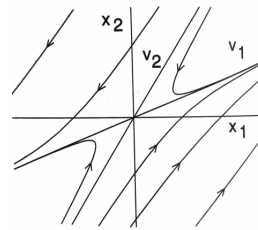
(c) Saddle point
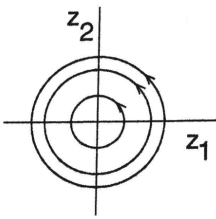
(a) Center focus
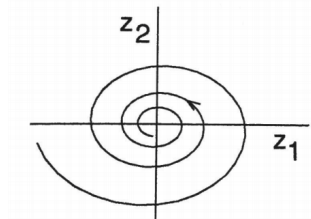
(b) Stable focus
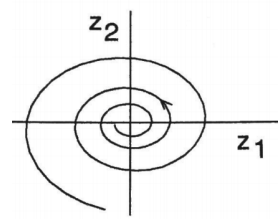
(c) Unstable focus

**Topological equivalence:** if the real part of the eigenvalues are nonzero, then the local phase-portrait corresponds to the phase portrait of the linearized system.

## 1.1 | Periodic orbits and limit cycles

**Definition** Periodic orbit: $\exists T > 0$ s.t. $x(t + T) = x(t) \quad \forall t \geq 0$.

**Definition** Limit cycle: non-trivial isolated periodic orbit.

> **Lemma 1** *Poincaré-Bendixson criterion:*
> *Let $M$ be a closed bounded subset of the plane s.t.*
> *∗ $M$ contains no $x^*$, or it contains only one $x^*$ with the property that the eigenvalues of the Jacobian matrix at $x^*$*
> *have positive real parts (unstable focus or unstable node).*
> *∗ Every trajectory starting in M stays in M $\forall t > t_0$.*
> *Then $M$ contains a periodic orbit of the system.*

**Lemma 2** *Bendixson negative criterion:*
*If on a simply connected region $D$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in $D$.*

**Corollary 3** *$C$ is a periodic orbit $\implies \Sigma_i I = 1$ (sum of indeces of equilibrium points in $C$, where saddle points have index -1 and others have index 1)*

## 2 | FUNDAMENTAL PROPERTIES

> **Lipschitz:** $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$
> Either locally Lipschitz on $\mathbb{D}$ (L varies), Lipschitz in $\mathbb{D}$ or globally Lipschitz.

**Theorem 4** *Local existence and uniqueness:*
*If*
*∗ $f(t, x)$ is piecewise continuous in $t$,*
*∗ $f(t, x)$ is Lipschitz $\forall x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\} \forall t \in [t_0, t_1]$,*
*Then there exists a unique solution of the IVP $x(t)$ on $t \in [t_0, t_0 + \delta]$.*

## 3 | LYAPUNOV STABILITY

### 3.1 | Stability of equilibrium points

**Asymptotic stabilization problem:** Find $\gamma(t, e)$ s.t. $e = 0$ is an asymptotically stable equilibrium point. Regulation vs. trajectory tracking.

**Definition** Stability: $x = 0$ is stable iff $\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad$ s.t. $\quad \|x(0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \forall t \geq 0$

**Definition** Asymptotic stability: $x = 0$ is (locally) asymptotically stable iff it is stable, and
$\exists r > 0 \quad$ s.t. $\quad \|x(0)\| < r \implies \lim_{t \to \infty} x(t) = 0$

**Definition** Region of attraction: $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$. We denote $R_A$ as the union of all the regions of attraction.

**Definition** Global asymptotic stability: $x = 0$ is GAS iff it is stable, and $\lim_{t \to \infty} x(t) = 0 \quad \forall x(0)$

**Definition** Exponential stability: $x = 0$ is exponentially stable iff

$\exists r, k, \lambda > 0$ s.t. $\|x(0)\| < r \implies \|x(t)\| \le k\|x(0)\|e^{-\lambda t} \quad \forall t \ge 0$

**Definition** Global exponential stability: $x = 0$ is GES iff $\exists k, \lambda > 0$ s.t. $\forall x(0) \quad \|x(t)\| \le k\|x(0)\|e^{-\lambda t} \quad \forall t \ge 0$

**Remark** It is useful to think in terms of stability + convergence to seperate the different stability properties.

## 3.2 | Lyapunov's indirect method

**Theorem 5** *Lyapunov's indirect method:*

*Let $x = 0$ be an equilibrium point for*

$$\dot{x} = f(x), \quad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \quad is \quad C^1 \tag{2}$$

1. *Linearize about $x = 0$, $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}\big|_{x=0}$.*
2. *Find the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$.*
3. *Categorize the eigenvalues:*
* $\forall i \quad \text{Re}(\lambda_i) < 0 \implies$ *asymptotically(exponentially) stable*
* $\exists i \quad \text{Re}(\lambda_i) > 0 \implies$ *unstable*
* $\forall i \quad \text{Re}(\lambda_i) \le 0 \implies$ *inconclusive*

While Lyapunov's indirect method is simple to use, the results are only local and often inconclusive. Let's see if we can do better ey?

## 3.3 | Lyapunov's direct method

**Definition** Lyapunov function:

$V$ is a Lyapunov function for $x = 0$ iff
* $V$ is $C^1$
* $V(0) = 0, \quad V(x) > 0 \quad$ in $\quad \mathbb{D}\backslash\{0\}$
* $\dot{V}(0) = 0, \quad \dot{V}(x) \le 0 \quad$ in $\quad \mathbb{D}\backslash\{0\}$
If $\dot{V}(x) < 0 \quad$ in $\quad \mathbb{D}\backslash\{0\}$ then $V$ is a strict Lyapunov function for $x = 0$.

**Theorem 6** *Lyapunov's stability theorem:*
* *If $\exists V(x)$ for $x = 0$, then $x = 0$ is stable.*
* *If $\exists$ strict $V(x)$ for $x = 0$, then $x = 0$ is asymptotically stable.*

**Theorem 7** *Chetaev's instability theorem:*
*If $\dot{V}(x) > 0$ in a set $U = \{x \in B_r \,|\, V(x) > 0\}$, then $x = 0$ is unstable.*

**Definition** Radially unboundedness: $\|x\| \to \infty \implies V(x) \to \infty$

**Theorem 8** *If $\exists$ strict $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$ and $V$ is radially unbounded, then $x = 0$ is GAS.*

**Theorem 9** *If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ s.t.*

* $V$ is $C^1$
* $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$
* $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

*then $x = 0$ is exponentially stable. If these conditions hold for $\mathbb{D} = \mathbb{R}^n$, then $x = 0$ is GES.*

**Remark** $\lambda_{min}(P)\|x\|^2 \leq x^\top P x \leq \lambda_{max}(P)\|x\|^2$

**Remark** How to deal with indeterminate signs in $\dot{V}$?

* Completion of squares: $x_1 x_2 \leq \frac{1}{2}(x_1^2 + x_2^2)$
* Young's inequality: $x_1 x_2 \leq \epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2$
* Cauchy-Schwarz' inequality: $|a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \leq \sqrt{\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)} \|x\|_2$

## 3.4  |  The invariance principle

**Definition** Invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \in \mathbb{R}$

**Definition** Positively invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \geq 0$

**Definition** Level set: $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$

> **Theorem 10** *La Salle's theorem:*
> *If $\exists V : \mathbb{D} \to \mathbb{R}$ s.t.*
> * $V$ is $C^1$
> * $\exists c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$ is bounded
> * $\dot{V}(x) \leq 0 \quad \forall x \in \Omega_c$
> *Let $E = \{x \in \Omega_c | \dot{V}(x) = 0\}$. Let $M$ be the largest invariant set contained in $E$. Then $x(0) \in \Omega_c \Rightarrow x(t) \xrightarrow{t \to \infty} M$.*

**Definition** Region of attraction:
Let $x = 0$ be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz and $\mathbb{D} \subset \mathbb{R}^n$ contains the origin. Let $\phi(t, x_0)$ be the solution. Then the region of attraction is

$$R_A = \{x_0 \in \mathbb{D} \quad | \quad \phi(t, x_0) \text{ is defined } \forall t \geq 0 \text{ and } \phi(t, x_0) \to 0 \text{ as } t \to \infty\} \tag{3}$$

(I.e. all the points with a corresponding solution that converges to the origin).

**Remark** GAS iff $R_A = \mathbb{R}^n$.

**Estimate of $R_A$:** choose the largest set $\Omega_c$ in $\mathbb{D}$ which is bounded, and only the connected component of $\Omega_c$ that contains the origin. Then this subset is a subset of $R_A$.

## 3.5 | Stability analysis of time-variant systems

We now consider the system $\dot{x} = f(t, x)$.

**Definition** Stability: $\forall \varepsilon > 0, \quad \exists \delta(\varepsilon, t_0) > 0 \quad$ s.t. $\quad \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \geq 0$

**Definition** Uniform stability: stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

**Definition** Asymptotic stability: stable and $\exists c(t_0) > 0 \quad$ s.t. $\quad \|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t \to \infty} 0$.

**Definition** Uniform asymptotic stability: asymptotically stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

**Definition** Global uniform asymptotic stability: uniform stability with $\delta(\varepsilon) \xrightarrow{\varepsilon \to \infty} \infty$ and $\forall c > 0 \quad \|x(t_0)\| < c \Rightarrow$ $x(t) \xrightarrow{t \to \infty} 0$ uniformly in $t_0$.

**Definition** Exponential stability: $\exists c, k, \lambda > 0 \quad$ s.t. $\quad \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} t \geq t_0 \geq 0 \|x(t_0)\| \leq c$. GES if $\forall c$.

**Definition** A continuous function $\alpha : [0, a) \to [0, \infty)$ is a class $\mathscr{K}$ function iff:
$\alpha(0) = 0$ and $\alpha(r)$ is strictly increasing, i.e. $\frac{\partial \alpha}{\partial r} > 0 \quad \forall r > 0$.

**Definition** If in addition $a \to \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then $\alpha$ is a class $\mathscr{K}_\infty$ function.

**Definition** A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is a class $\mathscr{K}\mathscr{L}$ function if for each fixed $s$
∗ $\beta(r, s)$ is a class $\mathscr{K}$ function w.r.t. $r$
and for each fixed $r$
∗ $\beta(r, s)$ is decreasing w.r.t. $s$,
∗ $\beta(r, s) \to 0$ as $s \to \infty$.

We can now define stability in terms of class $\mathscr{K}$ functions:

**Definition** Uniform stability: $\exists$ class $\mathscr{K}$ function $\alpha$ and $\exists c > 0$ s.t. $\|x(t)\| \leq \alpha(\|x(t_0)\|) \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$.

**Definition** Uniform asymptotic stability: $\exists$ class $\mathscr{K}\mathscr{L}$ function $\beta$ and $\exists c > 0$ s.t. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$. GUAS if $\forall c$.

**Definition** $V(t, x)$ is positive definite iff
∗ $V(t, 0) = 0$
∗ $V(t, x) \geq W_1(x)$
$\forall t \geq 0, W_1(x) > 0$

**Definition** $V(t, x)$ is decrescent iff
∗ $V(t, 0) = 0$
∗ $V(t, x) \leq W_2(x)$
$\forall t \geq 0, W_2(x) > 0$

We can summarize the stability theorems for time-varying systems like this:

| | Stable | Uniformly stable | UAS | GUAS |
|---|---|---|---|---|
| $V$ | Pos. def. | Pos. def., decrescent | Pos. def., decrescent. | Pos. def., decrescent, radially unbounded |
| $\dot{V}$ | Neg. semidef. | Neg. semidef. | Neg. def. | Neg. def. |
| | $\forall x \in \mathbb{D}$ | $\forall x \in \mathbb{D}$ | $\forall x \in \mathbb{D}$ | $\forall x \in \mathbb{R}^n$ |

**Estimate of** $R_A$: $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathbb{D}, c < \min_{\|x\|=r} W_1(x) \implies \{x \in B_r : W_2(x) \leq c\}$ is a region of attraction, when the origin is UAS.

---

**Lemma 11** *Barbalat's lemma:*

*Let $\dot{f} : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$. If $\lim_{t\to\infty} f(t)$ exists and is finite, then $\dot{f} \to 0$ as $t \to \infty$.*

*Rephrased: if $V$ is lower bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is uniformly bounded, then $\dot{V} \to 0$ as $t \to \infty$.*

---

# 4 | INPUT-TO-STATE STABILITY

## 4.1 | Input-to-state stability

Now we consider the system $\Sigma : \dot{x} = f(t, x, u)$, where we consider $u(t)$ to be a disturbance/modelling error.

**Definition** Input-to-state stability (ISS): $\exists \beta \in \mathscr{KL}, \gamma \in \mathscr{K}$ s.t. $\|x(t, x_0, u)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_\infty)\}$

**Remark** This is really just an extension of GUAS that says that $x$ is bounded the input as well. So naturally if $\Sigma$ is ISS then it is also 0-GUAS.

---

**Definition** $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for $\Sigma$ iff
* $V$ is $C^1$.
$\exists \alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\rho \in \mathscr{K}$ s.t.
* $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$
* $\dot{V}(t, x) = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$
where $W_3 > 0$.
It can be shown that $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.
$\exists \text{ISS} - \text{LF for } \Sigma \implies \Sigma$ is ISS

---

**Lemma 12** *if $f$ is $C^1$ and globally Lipschitz in $(x, u)$, then $\Sigma$ is $0 - \text{GES} \implies \Sigma$ is ISS*

**Theorem 13** *Consider the cascaded system $\Sigma_2 \longrightarrow \Sigma_1$, where $\Sigma_1 : \dot{x}_1 = f_1(t, x_1, x_2)$ and $\Sigma_2 : \dot{x}_2 = f_2(t, x_2)$. If $\Sigma_2$ is GUAS and $\Sigma_1$ is ISS, then the cascaded system is GUAS.*

## 4.2 | Input-output stability

We consider the system $y = Hu$.

**Definition** $\mathscr{L}_p^m$ space: $\quad u \in \mathscr{L}_p^m \quad 1 \leq p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left( \int_0^\infty \|u(t)\|_p^p dt \right)^{\frac{1}{p}} < \infty$

**Remark** This makes $\mathscr{L}_2$ the space of all continuous, square-integrable functions, for instance.

**Definition** $\mathscr{L}_{pe}^m$ space: $u \in \mathscr{L}_{pe}^m \Leftrightarrow u_\tau \in \mathscr{L}_p^m \quad \forall \tau \in [0, \infty)$, where $u_\tau$ is the truncated version if $u$.

> **Definition** $H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is $\mathscr{L}_p$ stable iff
> * $\exists \alpha$ class $\mathscr{K} \quad \alpha : [0, \infty) \to [0, \infty)$
> * $\exists$ constant $\beta \geq 0$
> s.t. $\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \alpha \left( \|u_\tau\|_{\mathscr{L}_p} \right) + \beta \quad \forall u \in \mathscr{L}_{pe}^m$ and $\tau \in [0, \infty)$

**Definition** Finite-gain $\mathscr{L}_p$ stable: $\exists \gamma, \beta \geq 0$ s.t. $\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \gamma \|u_\tau\|_{\mathscr{L}_p} + \beta$

**Definition** Causal system: $(Hu)_\tau = (Hu_\tau)_\tau$
The two definitons above hold for non-truncated signals if the systems are causal.

**Theorem 14** *Small-gain theorem:*
*The feedback interconnection of $H_1$ and $H_2$ are finite-gain $\mathscr{L}_p$ stable iff $\gamma_1 \gamma_2 < 1$.*
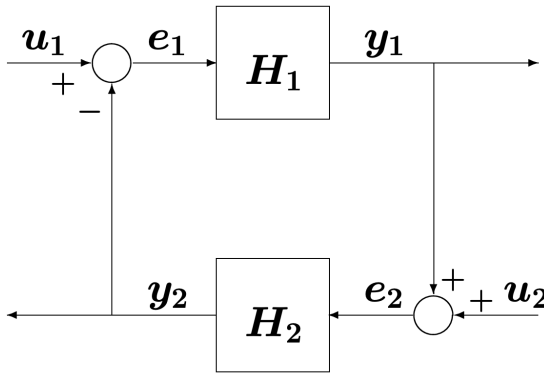


**FIGURE 3** Feedback interconnection

# 5 | PASSIVITY

## 5.1 | Passivity for memoryless functions

Consider the memoryless function $y = h(t, u)$    $h : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^p$.

**Definition**  The system is passive if $u^\top y \geq 0$ and lossless if $u^\top y = 0$.
The system is input-strictly passive if $u^\top y \geq u^\top \varphi(u)$, where $u^\top \varphi(u) > 0$    $\forall u \neq 0$.
The system is output-strictly passive if $u^\top y \geq y^\top \rho(y)$, where $y^\top \rho(y) > 0$    $\forall y \neq 0$.

## 5.2 | Passivity for dynamical systems

Now we extend this property for dynamical systems: $\Sigma$ :    $\dot{x} = f(x, u), y = h(x, u)$
where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz and $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous, and $f(0, 0) = 0$ and $h(0, 0) = 0$.

> **Definition**  The system $\Sigma$ is passive iff $u^\top y \geq \dot{V}$    $\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p$
> The system is lossless if $u^\top y = \dot{V}$.
> The system is input-strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u)$,    $u^T \varphi(u) > 0$    $\forall u \neq 0$.
> The system is output-strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$,    $y^T \rho(y) > 0$    $\forall y \neq 0$.
> The system is (state-) strictly passive if $u^T y \geq \dot{V} + \psi(x)$,    $\psi(x) > 0$.

**Remark**  Passivity really just generalizes the idea that the change of stored energy in the system should be less than the energy supplied to the system.

## 5.3 | Passivity and Lyapunov stability

**Lemma 15**  *If $\Sigma$ is passive with a positive definite $V(x)$, then the origin of $\dot{x} = f(x, 0)$ is stable.*

**Lemma 16**  *If $\Sigma$ is output-strictly passive with $\rho(y) = \delta(y), \delta < 0$, then $\Sigma$ is finite-gain $\mathscr{L}_2$ stable with $\gamma \leq \frac{1}{\delta}$.*

**Definition**  Zero state observability: no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{x \in \mathbb{R}^n | h(x, 0) = 0\}$ other than the trivial solution $x(t) = 0$.

**Lemma 17**  *The origin of $\dot{x} = f(x, 0)$ is asymptotically stable if $\Sigma$ is state-strictly passive, or output-strictly passive and zero state observable. If $V(x)$ is radially unbounded $\dot{x} = f(x, 0)$ is GAS.*

**Theorem 18**  *If $H_1$ and $H_2$ is passive, then the feedback interconnection of $H_1$ and $H_2$ is passive with $V = V_1 + V_2$.*

**Theorem 19**  *If $H_1$ and $H_2$ satisfies $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i i = 1, 2$ and $\varepsilon_1 + \delta_2 > 0, \varepsilon_2 + \delta_1 > 0$, then $\Sigma$ is finite-gain $\mathscr{L}_2$ stable.*

**Theorem 20**  *If $H_1$ and $H_2$ are state-strictly passive,*
*or $H_1$ and $H_2$ are output-strictly passive and zero state observable,*
*or $H_1$ is state-strictly passive and $H_2$ is output-strictly passive and zero state observable or opposite,*
*then $\Sigma$ is 0-AS. If $V_1$ and $V_2$ are radially unbounded then $\Sigma$ is 0-GAS.*

# 6 | NONLINEAR CONTROL

## 6.1 | Lyapunov control design

* Propose a LFC $V(t, x)$, typically as desired energy in system.
* Choose $u = g(t, x)$ s.t. $\dot{V} < 0$ or $\dot{V} \leq 0$ with La Salle / Barbalat.

## 6.2 | Passivity-based control

**Theorem 21**  *For the LTI system $y(s) = h(s)u(s)$ with $\text{Re}\,(p_i) < 0, \forall i$ we have:*

* *Passivity $\Leftrightarrow \text{Re}[h(j\omega)] \geq 0 \forall \omega$ (note that if $h(s)$ has an integrator as well and $\text{Re}\,(z_i) < 0$ this still holds)*
* *Input-strict passivity $\Leftrightarrow \text{Re}[h(j\omega)] \geq \delta > 0 \forall \omega$, with $\varphi(u) = \delta u$*
* *Output-strict passivity $\Leftrightarrow \exists \varepsilon > 0$ s.t. $\text{Re}[h(j\omega)] \geq \varepsilon |h(j\omega)|^2 \forall \omega$, with $\rho(y) = \varepsilon y$*

**Remark**  This says that with non-negative real part the system is passive, with a strictly positive real part it is input-strictly passive and if it is lower bounded by the square of the magnitude it is output-strictly passive.

**Remark**  Note that the previously discussed passivity theorems can be applied for passivity-based control.
Notably that means that if the controller and plant are passive, then the closed-loop system is passive.
Also if the controller is input- and output strictly passive and the plant is passive, then the system is finite gain $\mathscr{L}_2$ stable. The final passivity theorem can naturally also be applied.

**Theorem 22**  *Consider $H_1$ with $\dot{x} = f(x, u)$ locally Lipschitz and $y = h(x)$ continuous, both zero in the origin.*
*Also consider $H_2 : \phi(y)$ locally Lipschitz, memoryless and zero in origin.*
*If $H_1$ is passive with $V > 0$, radially unbounded and zero state observable, and $H_2$ satisifes $y^\top \phi(y) > 0, y \neq 0$*
*(passive, but not lossness) then the origin is GAS.*

**Choice of y:** for $\dot{x} = f(x) + G(x)u$, if $\exists$ LF $V(x)$ radially unbounded, let $y = \left[ \frac{\partial V}{\partial x} G(x) \right]^T$ for the system to be passive.
**Feedback passivation:** choose $u = \alpha(x) + \beta(x)v, y = h(x)$ s.t. $\dot{x} = f(x) + G(x)u$ has desired passivity properties $v \mapsto y$.

## 6.3 | Feedback linearization

### 6.3.1 | Input-state linearization

Consider $\dot{x} = f(x) + G(x)u$. Find a state transformation $z = T(x)$ and input transformation $u = \alpha(x) + \beta(x)v$ s.t. the new system in $z$ coordinates is linear and controllable. This is rearily possible to do, so we consider input-output linearization instead:

### 6.3.2 | Input-output linearization

We now consider the system $\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$ .

**Definition** Lie derivative: $L_f h = \frac{\partial h}{\partial x} f, \quad L_f^2 h = \frac{\partial L_f h}{\partial x} f, \quad \ldots \quad L_f^i h = L_f \left( L_f^{i-1} h \right)$

**Definition** The system has relative degree $\rho$ in a region $\mathbb{D}_0 \subset \mathbb{D} \subset \mathbb{R}^n$ if

$$\left. \begin{array}{l} L_g L_f^{i-1} h = 0, \quad 1 \le i \le \rho - 1 \\ L_g L_f^{\rho-1} h \ne 0 \end{array} \right\} \forall x \in \mathbb{D}_0 \tag{4}$$

**Remark** (the number of differentiations of $y$ before $u$ appears)

**Remark** For linear systems we have $\rho = n - m$.

**Remark** If the relative degree is well defined in the region of interest $\mathbb{D}$, then the system can be input-output linearized.

**Definition** Zero dynamics: internal dynamics when output is kept at zero by the input i.e. $\psi = 0, \dot{\phi} = f_o(\phi, 0)$.

**Definition** Minimum-phase system: the zero-dynamics are asymptotically stable.

---

**Input-output linearization:**

1. **Find the relative degree $\rho$**
2. **Write the system in normal form**
Let $\psi_1 = y, \psi_2 = \dot{y}, \ldots$ The external dynamics are then $\dot{\psi}_1 = \psi_2, \ldots, \dot{\psi}_\rho = L_f^\rho h + L_g L_f^{\rho-1} h \cdot u$.
Let $\phi_1, \ldots, \phi_{n-\rho}$ and $z = T(x) = [\ \varphi^\top \quad \psi^\top\ ]^\top$.
Choose $\varphi$ s.t. $T$ is a diffeomorphism, $L_g \varphi_i = 0$ and $\varphi_i(0) = 0$.
If the Jacobian $\frac{\partial T}{\partial x}\big|_{x_0}$ is nonsingular, then $T$ is a diffeomorphism. Then the internal dynamics are $\dot{\psi} = \ldots$.
We can then finally write the system in normal form: $\dot{z} = \ldots$.
3. **Choose u to cancel nonlinearities**

$$u = \frac{1}{L_g L_f^{\rho-1} h} \left( -L_f^\rho h + v \right) \quad \Longrightarrow \quad \psi_\rho = v \tag{5}$$

4. **Analyze the zero-dynamics**
5. **Choose $v$ to solve the control problem**
This is a special case of the system $\dot{\varphi} = f_0(\varphi, \psi), \quad \dot{\psi} = A\psi + Bv$.

**Lemma 23** *If the system is minimum phase and $v = -K\psi$ is chosen s.t. $(A - BK)$ is Hurwitz, then the origin of the system is asymptotically stable.*

**Lemma 24** *If $\dot{\phi} = f_0(\phi, \psi)$ is ISS, then the system is GAS.*

For tracking we have $v = -Ke + y_d^{(\rho)}$, but the same results apply, expect $\phi$ is now only bounded.

## 6.4 | Adaptive control

### 6.4.1 | MRAC for SISO systems

Consider the SISO system $\dot{x} = a_p x + c_p f(x) + b_p u$.

---

**SISO MRAC:**

1. **Specify desired closed-loop behaviour** by reference model: $\dot{x}_m = a_m x_m + b_m r(t)$
2. **Choose control law** s.t. plant output tracks reference model output when parameters are exactly known, by deriving error dynamics. Then replace parameters by estimates.
3. **Choose adaptation law** $(\dot{\hat{a}}_x, \dot{\hat{a}}_f, \dot{\hat{a}}_r)$ by first deriving new tracking error dynamics in terms of estimation errors, and then choosing adaptation law from suitable Lyapunov function e.g.

$$V(e, \tilde{a}) = \frac{1}{2} e^2 + \frac{|b_p|}{2\gamma_x} \tilde{a}_x^2 + \frac{|b_p|}{2\gamma_f} \tilde{a}_f^2 + \frac{|b_p|}{2\gamma_r} \tilde{a}_r^2 \tag{6}$$

By letting

$$\dot{\hat{a}}_x = -\gamma_x \mathrm{sgn}(b_p) ex, \quad \dot{\hat{a}}_f = -\gamma_f \mathrm{sgn}(b_p) ef, \quad \dot{\hat{a}}_r = -\gamma_r \mathrm{sgn}(b_p) er \tag{7}$$

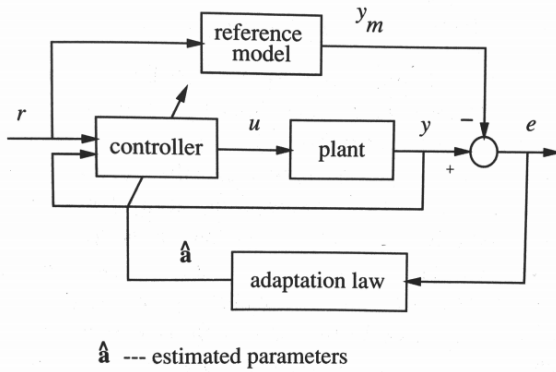the error will go to zero by Barbalat's lemma.

---



**FIGURE 4** Model reference adaptive control loop

### 6.4.2 | Adaptive tracking control for a class of MIMO systems

Consider the system $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(q)\dot{q} + g(q) = u$, with $M > 0$, $\dot{M} - 2C$ is skew symmetric, $z^T D z > 0 \quad \forall z \neq 0$. While the system is nonlinear it is linear in the unknown parameters $a$ by the Regression matrix $Y$:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(q)\dot{q} + g(q) = Y(q, \dot{q}, \ddot{q})a \tag{8}$$

**Adaptive MIMO tracking control:**

**1.** Given the desired trajectory $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ bounded.

**2. Choose control law:**

We introduce the virtual reference velocity $s = \dot{e} + \lambda e = \dot{q} - \dot{q}_r$, where $\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$.

$$u = \hat{M}(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{D}(q)\dot{q}_r + \hat{g}(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d) = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\,\hat{a} - K_p e - K_d \dot{e} \quad (9)$$

From $V(s) = \frac{1}{2}s^\top M s$ and the cascade stability theorem we get GUAS with perfect tracking.

**3. Choose an adaptation law** such that tracking is achieved asymptotically. Again done by deriving error dynamics and analyzing Lyapunov function with Barbalat's lemma:

$$V(s, \tilde{a}) = \frac{1}{2}s^\top M s + \frac{1}{2}\tilde{a}^\top \Gamma^{-1} \tilde{a} \implies \dot{\hat{a}} = -\Gamma Y^\top s \quad (10)$$

## 6.5 | Backstepping

We consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) + g\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) \cdot x_{n+1}, \quad \dot{x}_{n+1} = u \quad (11)$$

Which we can identify as the cascaded system $\Sigma_1 : \dot{\eta} = f(\eta) + g(\eta)\xi, \quad \Sigma_2 : \dot{\xi} = u$.

We can regard $\xi$ as the virtual control input of $\Sigma_1$.

**Integrator backstepping:**

**1. Find stabilizing function for $\Sigma_1$**

$\xi = \varphi(\eta), \quad \varphi(0) = 0$ s.t. $\xi = 0$ is asymptotically stable by some $V_1(\eta)$.

**2. Design actual control input $u$**

**a.** Introduce error variable $z = \xi - \varphi(\eta)$.

**b.** Derive dynamics in $(\eta, z)$ coordinates.

**c.** Choose LFC $V_2(\eta, z) = V_1(\eta) + \frac{1}{2}x^2$.

**d.** Find $u$ that asymptotically stabilizes $(\eta, z) = (0, 0)$. Generally we get:

$$u = -\frac{\partial V}{\partial \eta}g(\eta) + \dot{\varphi} - kz \quad (12)$$

If $V_2$ is radially unbounded in $\eta$ and $\mathbb{D} = \mathbb{R}^n$ the results are global.

**Remark** Note that this is a recursive process, and you may therefore do several steps of finding virtual control inputs before finally finding $u$. Also note that the method works fine even if $\Sigma_2$ is more exotic.

# A | LINEAR METHODS

**Definition** We define the p-norm as:

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{13}$$

$$\|f\|_{\mathscr{L}_p} = \left( \int_0^\infty |f(t)|^p \, dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{14}$$

**Theorem 25** *Schwarz' inequality:*

$$|<x, y>| \leq \|x\| \cdot \|y\| \tag{15}$$

**Definition** $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then the Jacobian is defined as:

$$\frac{\partial f}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{16}$$

Which in the scalar case $m = 1$ is the gradient.