TTK4150 Nonlinear Systems and Control

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> Learning goals:
> * Hello

Contents

1 | SECOND-ORDER NONLINEAR TIME-INVARIANT SYSTEMS

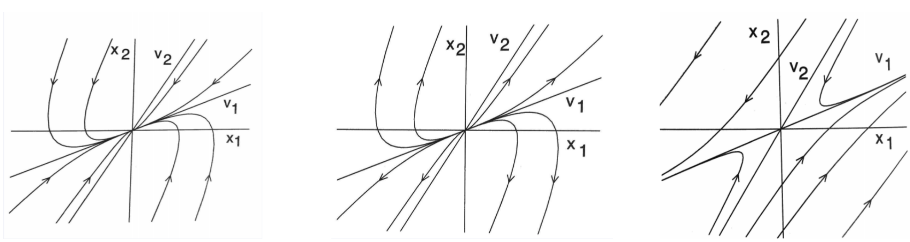
We first consider the system

$$\begin{aligned} \dot{x}_1 &= f_1\left(x_1, x_2\right) \\ \dot{x}_2 &= f_2\left(x_1, x_2\right) \end{aligned} \tag{1}$$

Phase-plane analysis: Determine the system behavior by constructing a phase portrait, i.e. plotting different IVP solutions in the phase space.

Local analysis:

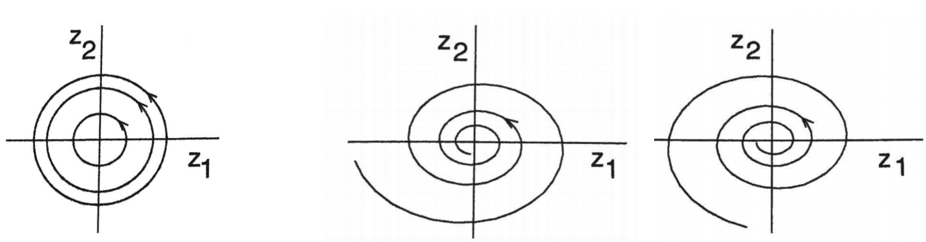
* Linearize about x^*.
* Find egeinvalues $\lambda(A)$.
* Classify equilibrium points for $f(x^*) = 0$. If λ is real, them we either get a stable node ($\lambda_2 < \lambda_1 < 0$), unstable node ($0 < \lambda_2 < \lambda_1$) or a saddle point ($\lambda_2 < 0 < \lambda_1$). In the complex case $\lambda_{1,2} = \alpha \pm \beta i$, then we either get a center focus ($\alpha = 0$), a stable focus ($\alpha < 0$) or an unstable focus ($\alpha > 0$).



(a) Stable node

(b) Unstable node

(c) Saddle point



(a) Center focus

(b) Stable focus

(c) Unstable focus

Topological equivalence: if the real part of the eigenvalues are nonzero, then the local phase-portrait corresponds to the phase portrait of the linearized system.

1.1 | Periodic orbits and limit cycles

Definition Periodic orbit: $\exists T > 0$ s.t. $x(t+T) = x(t) \quad \forall t \geq 0$.

Definition Limit cycle: non-trivial isolated periodic orbit.

Lemma 1 *Poincaré-Bendixson criterion:*
Let M be a closed bounded subset of the plane s.t.:
* *M contains no x^*, or it contains only one x^* with the property that the eigenvalues of the Jacobian matrix at x^* have positive real parts (unstable focus or unstable node).*
* *Every trajectory starting in M stays in M $\forall t > t_0$.*

Then M contains a periodic orbit of the system.

Lemma 2 *Bendixson negative criterion:*
If on a simply connected region D, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in D.

Corollary 3 *C is a periodic orbit $\implies \Sigma_i I = 1$ (sum of indeces of equilibrium points in C, where saddle points have index -1 and others have index 1)*

2 | FUNDAMENTAL PROPERTIES

Lipschitz: $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$
Either locally Lipschitz on $\mathbb{D}$ (L varies), Lipschitz in $\mathbb{D}$ or globally Lipschitz.

Theorem 4 *Local existence and uniqueness:*
If
* *$f(t,x)$ is piecewise continuous in t,*
* *$f(t,x)$ is Lipschitz $\forall x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\} \forall t \in [t_0, t_1]$,*

Then there exists a unique solution of the IVP $x(t)$ on $t \in [t_0, t_0 + \delta]$.

3 | LYAPUNOV STABILITY

3.1 | Stability of equilibrium points

Asymptotic stabilization problem: Find $\gamma(t, e)$ s.t. $e = 0$ is an asymptotically stable equilibrium point.
Regulation vs. trajectory tracking.

Definition Stability: $x = 0$ is stable iff $\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad$ s.t. $\quad \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$

Definition Asymptotic stability: $x = 0$ is (locally) asymptotically stable iff it is stable, and
$\exists r > 0 \quad$ s.t. $\quad \|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$

Definition Region of attraction: $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$. We denote R_A as the union of all the regions of attraction.

Definition Global asymptotic stability: $x = 0$ is GAS iff it is stable, and $\lim_{t\to\infty} x(t) = 0 \quad \forall x(0)$

Definition Exponential stability: $x = 0$ is exponentially stable iff
$\exists r, k, \lambda > 0$ s.t. $\|x(0)\| < r \Rightarrow \|x(t)\| \leq k\|x(0)\|e^{-\lambda t} \quad \forall t \geq 0$

Definition Global exponential stability: $x = 0$ is GES iff $\exists k, \lambda > 0$ s.t. $\forall x(0) \quad \|x(t)\| \leq k\|x(0)\|e^{-\lambda t} \quad \forall t \geq 0$

Remark It is useful to think in terms of stability + convergence to seperate the different stability properties.

3.2 | Lyapunov's indirect method

Theorem 5 *Lyapunov's indirect method:*
Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \quad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad C^1 \tag{2}$$

1. *Linearize about $x = 0$, $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}\big|_{x=0}$.*
2. *Find the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$.*
3. *Categorize the eigenvalues:*

* $\forall i \quad \mathrm{Re}\,(\lambda_i) < 0 \Rightarrow$ *asymptotically(exponentially) stable*
* $\exists i \quad \mathrm{Re}\,(\lambda_i) > 0 \Rightarrow$ *unstable*
* $\forall i \quad \mathrm{Re}\,(\lambda_i) \leq 0 \Rightarrow$ *inconclusive*

While Lyapunov's indirect method is simple to use, the results are only local and often inconclusive. Let's see if we can do better ey?

3.3 | Lyapunov's direct method

Definition Lyapunov function:
V is a Lyapunov function for $x = 0$ iff

* V is C^1
* $V(0) = 0, \quad V(x) > 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$
* $\dot{V}(0) = 0, \quad \dot{V}(x) \leq 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$

If $\dot{V}(x) < 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$ then V is a strict Lyapunov function for $x = 0$.

Theorem 6 *Lyapunov's stability theorem:*

* *If $\exists V(x)$ for $x = 0$, then $x = 0$ is stable.*
* *If $\exists$ strict $V(x)$ for $x = 0$, then $x = 0$ is asymptotically stable.*

Theorem 7 *Chetaev's instability theorem:*
If $\dot{V}(x) > 0$ in a set $U = \{x \in B_r | V(x) > 0\}$, then $x = 0$ is unstable.

Definition Radially unboundedness: $\quad \|x\| \to \infty \implies V(x) \to \infty$

Theorem 8 *If $\exists$ strict $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$ and V is radially unbounded, then $x = 0$ is GAS.*

Theorem 9 *If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ s.t.*

* *V is C^1*
* $k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a \quad \forall x \in \mathbb{D}$
* $\dot{V}(x) \leq -k_3\|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable. If these conditions hold for $\mathbb{D} = \mathbb{R}^n$, then $x = 0$ is GES.

Remark $\lambda_{min}(P)\|x\|^2 \leq x^\top Px \leq \lambda_{max}(P)\|x\|^2$

Remark How to deal with indeterminate signs in $\dot{V}$?

* Completion of squares: $x_1x_2 \leq \frac{1}{2}(x_1^2 + x_2^2)$
* Young's inequality: $x_1x_2 \leq \epsilon x_1^2 + \frac{1}{4\epsilon}x_2^2$
* Cauchy-Schwarz' inequality: $|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq \sqrt{\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)}\|x\|_2$

3.4 | The invariance principle

Definition Invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \in \mathbb{R}$

Definition Positively invariant set: $x(0) \in M \implies x(t) \in M \quad \forall t \geq 0$

Definition Level set: $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$

Theorem 10 *La Salle's theorem:*
If $\exists V : \mathbb{D} \to \mathbb{R}$ s.t.

* *V is C^1*
* *$\exists c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$ is bounded*
* *$\dot{V}(x) \leq 0 \quad \forall x \in \Omega_c$*

Let $E = \left\{x \in \Omega_c | \dot{V}(x) = 0\right\}$. Let M be the largest invariant set contained in E. Then $x(0) \in \Omega_c \Rightarrow x(t) \overset{t\to\infty}{\longrightarrow} M$.

Definition Region of attraction:
Let $x = 0$ be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz and $\mathbb{D} \subset \mathbb{R}^n$ contains the origin. Let $\phi(t, x_0)$ be the solution. Then the region of attraction is

$$R_A = \{x_0 \in \mathbb{D} \quad | \quad \phi(t, x_0) \text{ is defined } \forall t \geq 0 \text{ and } \phi(t, x_0) \to 0 \text{ as } t \to \infty\} \tag{3}$$

(I.e. all the points with a corresponding solution that converges to the origin).

Remark GAS iff $R_A = \mathbb{R}^n$.

Estimate of R_A: choose the largest set Ω_c in $\mathbb{D}$ which is bounded, and only the connected component of Ω_c that contains the origin. Then this subset is a subset of R_A.

3.5 | Stability analysis of time-variant systems

We now consider the system $\dot{x} = f(t, x)$.

Definition Stability: $\forall \varepsilon > 0, \quad \exists \delta(\varepsilon, t_0) > 0 \quad$ s.t. $\quad \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \geq 0$

Definition Uniform stability: stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

Definition Asymptotic stability: stable and $\exists c(t_0) > 0 \quad$ s.t. $\quad \|x(t_0)\| < c \Rightarrow x(t) \overset{t\to\infty}{\longrightarrow} 0$.

Definition Uniform asymptotic stability: asymptotically stable with $\delta(\varepsilon, t_0) = \delta(\varepsilon)$.

Definition Global uniform asymptotic stability: uniform stability with $\delta(\varepsilon) \overset{\varepsilon\to\infty}{\longrightarrow} \infty$ and $\forall c > 0 \quad \|x(t_0)\| < c \implies x(t) \overset{t\to\infty}{\longrightarrow} 0$ uniformly in t_0.

Definition Exponential stability: $\exists c, k, \lambda > 0 \quad$ s.t. $\quad \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} t \geq t_0 \geq 0 \, \|x(t_0)\| \leq c$. GES if $\forall c$.

Definition A continuous function $\alpha : [0, a) \to [0, \infty)$ is a class $\mathscr{K}$ function iff:
$\alpha(0) = 0$ and $\alpha(r)$ is strictly increasing, i.e. $\frac{\partial \alpha}{\partial r} > 0 \quad \forall r > 0$.

Definition If in addition $a \to \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then α is a class $\mathscr{K}_\infty$ function.

Definition A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is a class $\mathscr{K}\mathscr{L}$ function if for each fixed s

* $\beta(r, s)$ is a class $\mathscr{K}$ function w.r.t. r

and for each fixed r

* $\beta(r, s)$ is decreasing w.r.t. s,
* $\beta(r, s) \to 0$ as $s \to \infty$.

We can now define stability in terms of class $\mathscr{K}$ functions:

Definition Uniform stability: $\exists$ class $\mathscr{K}$ function α and $\exists c > 0$ s.t. $\|x(t)\| \leq \alpha(\|x(t_0)\|) \, \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$.

Definition Uniform asymptotic stability: $\exists$ class $\mathscr{K}\mathscr{L}$ function β and $\exists c > 0$ s.t. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \, \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$. GUAS if $\forall c$.

Definition $V(t, x)$ is positive definite iff

* $V(t, 0) = 0$
* $V(t, x) \geq W_1(x)$

$\forall t \geq 0, W_1(x) > 0$

Definition $V(t, x)$ is decrescent iff

* $V(t, 0) = 0$
* $V(t, x) \leq W_2(x)$

$\forall t \geq 0, W_2(x) > 0$

We can summarize the stability theorems for time-varying systems like this:

	Stable	Uniformly stable	UAS	GUAS
V	Pos. def.	Pos. def., decrescent	Pos. def., decrescent.	Pos. def., decrescent, radially unbounded
$\dot{V}$	Neg. semidef.	Neg. semidef.	Neg. def.	Neg. def.
	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{R}^n$

Estimate of R_A: $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathbb{D}, c < \min_{\|x\|=r} W_1(x) \implies \{x \in B_r : W_2(x) \leq c\}$ is a region of attraction, when the origin is UAS.

Lemma 11 *Barbalat's lemma:*

Let $\dot{f} : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$. If $\lim_{t\to\infty} f(t)$ exists and is finite, then $\dot{f} \to 0$ as $t \to \infty$.

Rephrased: if V is lower bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is uniformly bounded, then $\dot{V} \to 0$ as $t \to \infty$.

4 | INPUT-TO-STATE STABILITY

4.1 | Input-to-state stability

Now we consider the system $\Sigma : \dot{x} = f(t, x, u)$, where we consider $u(t)$ to be a disturbance/modelling error.

Definition Input-to-state stability (ISS): $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ s.t. $\|x(t, x_0, u)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_\infty)\}$

Remark This is really just an extension of GUAS that says that x is bounded the input as well. So naturally if Σ is ISS then it is also 0-GUAS.

Definition $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for Σ iff
* V is C^1.

$\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ s.t.
* $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$
* $\dot{V}(t, x) = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$

where $W_3 > 0$.
It can be shown that $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.
$\exists \mathrm{ISS-LF}$ for $\Sigma \Rightarrow \Sigma$ is ISS

Lemma 12 *if f is C^1 and globally Lipschitz in (x, u), then Σ is $0-\mathrm{GES} \Rightarrow \Sigma$ is ISS*

Theorem 13 *Consider the cascaded system $\Sigma_2 \longrightarrow \Sigma_1$, where $\Sigma_1 : \dot{x}_1 = f_1(t, x_1, x_2)$ and $\Sigma_2 : \dot{x}_2 = f_2(t, x_2)$. If Σ_2 is GUAS and Σ_1 is ISS, then the cascaded system is GUAS.*

4.2 | Input-output stability

We consider the system $y = Hu$.

Definition $\mathcal{L}_p^m$ space: $\quad u \in \mathcal{L}_p^m \quad 1 \leq p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|_p^p dt\right)^{\frac{1}{p}} < \infty$

Remark This makes $\mathcal{L}_2$ the space of all continuous, square-integrable functions, for instance.

Definition $\mathcal{L}_{pe}^m$ space: $u \in \mathcal{L}_{pe}^m \Leftrightarrow u_\tau \in \mathcal{L}_p^m \quad \forall \tau \in [0, \infty)$, where u_τ is the truncated version if u.

Definition $H : \mathcal{L}_{pe}^m \to \mathcal{L}_{pe}^q$ is $\mathcal{L}_p$ stable iff
* $\exists \alpha$ class $\mathcal{K} \quad \alpha : [0, \infty) \to [0, \infty)$
* $\exists$ constant $\beta \geq 0$

s.t. $\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \alpha\left(\|u_\tau\|_{\mathcal{L}_p}\right) + \beta \quad \forall u \in \mathcal{L}_{pe}^m$ and $\tau \in [0, \infty)$

Definition Finite-gain $\mathcal{L}_p$ stable: $\exists \gamma, \beta \geq 0$ s.t. $\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \gamma \|u_\tau\|_{\mathcal{L}_p} + \beta$

Definition Causal system: $(Hu)_\tau = (Hu_\tau)_\tau$
The two definitons above hold for non-truncated signals if the systems are causal.

Theorem 14 *Small-gain theorem:*
The feedback interconnection of H_1 and H_2 are finite-gain $\mathcal{L}_p$ stable iff $\gamma_1\gamma_2 < 1$.

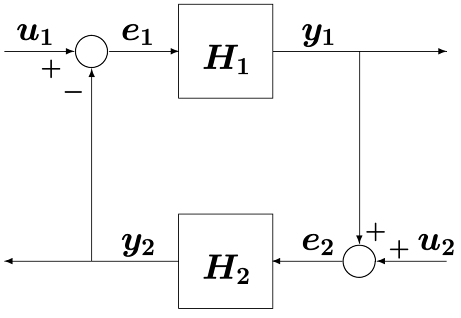


FIGURE 3 Feedback interconnection

5 | PASSIVITY

6 | NONLINEAR CONTROL

6.1 | Passivity-based control

6.2 | Feedback linearization

6.3 | Adaptive control

6.4 | Backstepping

A | LINEAR METHODS

Definition We define the p-norm as:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{4}$$

$$\|f\|_{\mathcal{L}_p} = \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}}, \quad p \in [1, \infty] \tag{5}$$

Theorem 15 *Schwarz' inequality:*

$$| < x, y > | \le \|x\| \cdot \|y\| \tag{6}$$

Definition $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then the Jacobian is defined as:

$$\frac{\partial f}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{7}$$

Which in the scalar case $m = 1$ is the gradient.