

# TTK4150 Nonlinear Systems and Control

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Learning goals:

- \* A thorough knowledge of theory and methods for nonlinear dynamical systems.
- \* Know how to find the invariant sets of nonlinear dynamical systems, and know how to analyze the system behavior around these sets.
- \* Know the methods Phase plane analysis, Lyapunov stability analysis, Input-to-state stability analysis, Input-Output stability analysis, Passivity analysis, Lyapunov-based control, Energy-based control, Cascaded control, Passivity-based control, Input-Output linearization, and Backstepping control design.

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# 1 | SECOND-ORDER NONLINEAR TIME-INVARIANT SYSTEMS

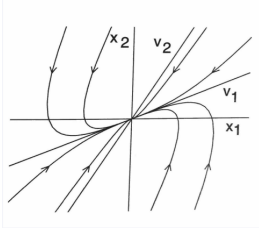
We first consider the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\tag{1}$$

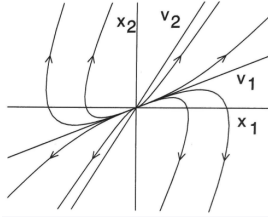
**Phase-plane analysis:** Determine the system behavior by constructing a **phase portrait**, i.e. plotting different IVP solutions in the phase space.

## Local analysis:

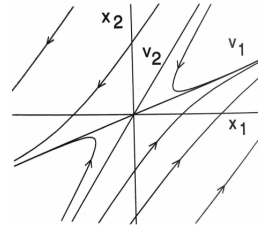
- \* Linearize about  $x^*$ .
- \* Find eigenvalues  $\lambda(A)$ .
- \* Classify equilibrium points for  $f(x^*) = 0$ . If  $\lambda$  is real, then we either get a stable node ( $\lambda_2 < \lambda_1 < 0$ ), unstable node ( $0 < \lambda_2 < \lambda_1$ ) or a saddle point ( $\lambda_2 < 0 < \lambda_1$ ). In the complex case  $\lambda_{1,2} = \alpha \pm \beta i$ , then we either get a center focus ( $\alpha = 0$ ), a stable focus ( $\alpha < 0$ ) or an unstable focus ( $\alpha > 0$ ).



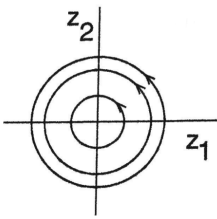
(a) Stable node



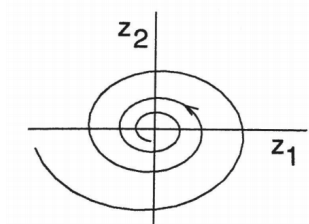
(b) Unstable node



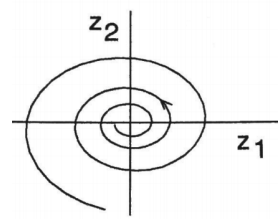
(c) Saddle point



(a) Center focus



(b) Stable focus



(c) Unstable focus

**Topological equivalence:** if the real part of the eigenvalues are nonzero, then the local phase-portrait corresponds to the phase portrait of the linearized system.

## 1.1 | Periodic orbits and limit cycles

**Definition** Periodic orbit:  $\exists T > 0$  s.t.  $x(t + T) = x(t) \quad \forall t \geq 0$ .

**Definition** Limit cycle: non-trivial isolated periodic orbit.

**Lemma 1** Poincaré-Bendixson criterion:

Let  $M$  be a closed bounded subset of the plane s.t.:

\*  $M$  contains no  $x^*$ , or it contains only one  $x^*$  with the property that the eigenvalues of the Jacobian matrix at  $x^*$  have positive real parts (unstable focus or unstable node).

\* Every trajectory starting in  $M$  stays in  $M \forall t > t_0$ .

Then  $M$  contains a periodic orbit of the system.

**Lemma 2** Bendixson negative criterion:

If on a simply connected region  $D$ ,  $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$  is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in  $D$ .

**Corollary 3**  $C$  is a periodic orbit  $\implies \sum_i I = 1$  (sum of indeces of equilibrium points in  $C$ , where saddle points have index -1 and others have index 1)

## 2 | FUNDAMENTAL PROPERTIES

**Lipschitz:**  $\|f(t, x) - f(t, y)\| \leq L \|x - y\|$

Either locally Lipschitz on  $\mathbb{D}$  ( $L$  varies), Lipschitz in  $\mathbb{D}$  or globally Lipschitz.

**Theorem 4** Local existence and uniqueness:

If

\*  $f(t, x)$  is piecewise continuous in  $t$ ,

\*  $f(t, x)$  is Lipschitz  $\forall x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\} \forall t \in [t_0, t_1]$ ,

Then there exists a unique solution of the IVP  $x(t)$  on  $t \in [t_0, t_0 + \delta]$ .

## 3 | LYAPUNOV STABILITY

### 3.1 | Stability of equilibrium points

**Asymptotic stabilization problem:** Find  $\gamma(t, e)$  s.t.  $e = 0$  is an asymptotically stable equilibrium point.

Regulation vs. trajectory tracking.

**Definition** Stability:  $x = 0$  is stable iff  $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$  s.t.  $\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \forall t \geq 0$

**Definition** Asymptotic stability:  $x = 0$  is (locally) asymptotically stable iff it is stable, and

$\exists r > 0$  s.t.  $\|x(0)\| < r \implies \lim_{t \rightarrow \infty} x(t) = 0$

**Definition** Region of attraction:  $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ . We denote  $R_A$  as the union of all the regions of attraction.

**Definition** Global asymptotic stability:  $x = 0$  is GAS iff it is stable, and  $\lim_{t \rightarrow \infty} x(t) = 0 \quad \forall x(0)$

**Definition** Exponential stability:  $x = 0$  is exponentially stable iff

$$\exists r, k, \lambda > 0 \text{ s.t. } \|x(0)\| < r \Rightarrow \|x(t)\| \leq k \|x(0)\| e^{-\lambda t} \quad \forall t \geq 0$$

**Definition** Global exponential stability:  $x = 0$  is GES iff  $\exists k, \lambda > 0$  s.t.  $\forall x(0) \quad \|x(t)\| \leq k \|x(0)\| e^{-\lambda t} \quad \forall t \geq 0$

**Remark** It is useful to think in terms of stability + convergence to separate the different stability properties.

### 3.2 | Lyapunov's indirect method

**Theorem 5** Lyapunov's indirect method:

Let  $x = 0$  be an equilibrium point for

$$\dot{x} = f(x), \quad f : \mathbb{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{is } C^1 \quad (2)$$

1. Linearize about  $x = 0$ ,  $\dot{x} = Ax$ , where  $A = \frac{\partial f}{\partial x} \Big|_{x=0}$ .
2. Find the eigenvalues  $\lambda_1(A), \dots, \lambda_n(A)$ .
3. Categorize the eigenvalues:
  - \*  $\forall i \quad \operatorname{Re}(\lambda_i) < 0 \Rightarrow \text{asymptotically(exponentially) stable}$
  - \*  $\exists i \quad \operatorname{Re}(\lambda_i) > 0 \Rightarrow \text{unstable}$
  - \*  $\forall i \quad \operatorname{Re}(\lambda_i) \leq 0 \Rightarrow \text{inconclusive}$

While Lyapunov's indirect method is simple to use, the results are only local and often inconclusive. Let's see if we can do better ey?

### 3.3 | Lyapunov's direct method

**Definition** Lyapunov function:

$V$  is a Lyapunov function for  $x = 0$  iff

- \*  $V$  is  $C^1$
  - \*  $V(0) = 0, \quad V(x) > 0 \quad \text{in } \mathbb{D} \setminus \{0\}$
  - \*  $\dot{V}(0) = 0, \quad \dot{V}(x) \leq 0 \quad \text{in } \mathbb{D} \setminus \{0\}$
- If  $\dot{V}(x) < 0 \quad \text{in } \mathbb{D} \setminus \{0\}$  then  $V$  is a strict Lyapunov function for  $x = 0$ .

**Theorem 6** Lyapunov's stability theorem:

- \* If  $\exists V(x)$  for  $x = 0$ , then  $x = 0$  is stable.
- \* If  $\exists$  strict  $V(x)$  for  $x = 0$ , then  $x = 0$  is asymptotically stable.

**Theorem 7** Chetaev's instability theorem:

If  $\dot{V}(x) > 0$  in a set  $U = \{x \in B_r \mid V(x) > 0\}$ , then  $x = 0$  is unstable.

**Definition** Radially unboundedness:  $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$

**Theorem 8** If  $\exists$  strict  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $x = 0$  and  $V$  is radially unbounded, then  $x = 0$  is GAS.

**Theorem 9** If there exist a function  $V : \mathbb{D} \rightarrow \mathbb{R}$  and constants  $a, k_1, k_2, k_3 > 0$  s.t.

\*  $V$  is  $C^1$

\*  $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

\*  $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then  $x = 0$  is exponentially stable. If these conditions hold for  $\mathbb{D} = \mathbb{R}^n$ , then  $x = 0$  is GES.

**Remark**  $\lambda_{\min}(P) \|x\|^2 \leq x^\top P x \leq \lambda_{\max}(P) \|x\|^2$

**Remark** How to deal with indeterminate signs in  $\dot{V}$ ?

\* Completion of squares:  $x_1 x_2 \leq \frac{1}{2}(x_1^2 + x_2^2)$

\* Young's inequality:  $x_1 x_2 \leq \epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2$

\* Cauchy-Schwarz' inequality:  $|a_1 x_1 + a_2 x_2 + \dots + a_n x_n| \leq \sqrt{(a_1^2 + a_2^2 + \dots + a_n^2)} \|x\|_2$

### 3.4 | The invariance principle

**Definition** Invariant set:  $x(0) \in M \implies x(t) \in M \quad \forall t \in \mathbb{R}$

**Definition** Positively invariant set:  $x(0) \in M \implies x(t) \in M \quad \forall t \geq 0$

**Definition** Level set:  $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$

**Theorem 10** La Salle's theorem:

If  $\exists V : \mathbb{D} \rightarrow \mathbb{R}$  s.t.

\*  $V$  is  $C^1$

\*  $\exists c > 0$  such that  $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$  is bounded

\*  $\dot{V}(x) \leq 0 \quad \forall x \in \Omega_c$

Let  $E = \{x \in \Omega_c | \dot{V}(x) = 0\}$ . Let  $M$  be the largest invariant set contained in  $E$ . Then  $x(0) \in \Omega_c \implies x(t) \xrightarrow{t \rightarrow \infty} M$ .

**Definition** Region of attraction:

Let  $x = 0$  be an asymptotically stable equilibrium point of the system  $\dot{x} = f(x)$ , where  $f : \mathbb{D} \rightarrow \mathbb{R}^n$  is locally Lipschitz and  $\mathbb{D} \subset \mathbb{R}^n$  contains the origin. Let  $\phi(t, x_0)$  be the solution. Then the region of attraction is

$$R_A = \{x_0 \in \mathbb{D} \mid \phi(t, x_0) \text{ is defined } \forall t \geq 0 \text{ and } \phi(t, x_0) \rightarrow 0 \text{ as } t \rightarrow \infty\} \quad (3)$$

(I.e. all the points with a corresponding solution that converges to the origin).

**Remark** GAS iff  $R_A = \mathbb{R}^n$ .

**Estimate of  $R_A$ :** choose the largest set  $\Omega_c$  in  $\mathbb{D}$  which is bounded, and only the connected component of  $\Omega_c$  that contains the origin. Then this subset is a subset of  $R_A$ .

### 3.5 | Stability analysis of time-variant systems

We now consider the system  $\dot{x} = f(t, x)$ .

**Definition** Stability:  $\forall \varepsilon > 0, \exists \delta(\varepsilon, t_0) > 0$  s.t.  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \geq 0$

**Definition** Uniform stability: stable with  $\delta(\varepsilon, t_0) = \delta(\varepsilon)$ .

**Definition** Asymptotic stability: stable and  $\exists c(t_0) > 0$  s.t.  $\|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t \rightarrow \infty} 0$ .

**Definition** Uniform asymptotic stability: asymptotically stable with  $\delta(\varepsilon, t_0) = \delta(\varepsilon)$ .

**Definition** Global uniform asymptotic stability: uniform stability with  $\delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow \infty} \infty$  and  $\forall c > 0 \quad \|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t \rightarrow \infty} 0$  uniformly in  $t_0$ .

**Definition** Exponential stability:  $\exists c, k, \lambda > 0$  s.t.  $\|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} \quad t \geq t_0 \geq 0 \quad \|x(t_0)\| \leq c$ . GES if  $\forall c$ .

**Definition** A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  is a **class  $\mathcal{K}$  function** iff:

$\alpha(0) = 0$  and  $\alpha(r)$  is strictly increasing, i.e.  $\frac{\partial \alpha}{\partial r} > 0 \quad \forall r > 0$ .

**Definition** If in addition  $a \rightarrow \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , then  $\alpha$  is a **class  $\mathcal{K}_\infty$  function**.

**Definition** A continuous function  $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$  is a **class  $\mathcal{KL}$  function** if for each fixed  $s$

\*  $\beta(r, s)$  is a class  $\mathcal{K}$  function w.r.t.  $r$

and for each fixed  $r$

\*  $\beta(r, s)$  is decreasing w.r.t.  $s$ ,

\*  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

We can now define stability in terms of class  $\mathcal{K}$  functions:

**Definition** Uniform stability:  $\exists$  class  $\mathcal{K}$  function  $\alpha$  and  $\exists c > 0$  s.t.  $\|x(t)\| \leq \alpha(\|x(t_0)\|) \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$ .

**Definition** Uniform asymptotic stability:  $\exists$  class  $\mathcal{KL}$  function  $\beta$  and  $\exists c > 0$  s.t.  $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$ . GUAS if  $\forall c$ .

**Definition**  $V(t, x)$  is **positive definite** iff

\*  $V(t, 0) = 0$

\*  $V(t, x) \geq W_1(x)$

$\forall t \geq 0, W_1(x) > 0$

**Definition**  $V(t, x)$  is **decreascent** iff

\*  $V(t, 0) = 0$

\*  $V(t, x) \leq W_2(x)$

$\forall t \geq 0, W_2(x) > 0$

We can summarize the stability theorems for time-varying systems like this:

	Stable	Uniformly stable	UAS	GUAS
$V$	Pos. def.	Pos. def., decreascent	Pos. def., decreascent.	Pos. def., decreascent, radially unbounded
$\dot{V}$	Neg. semidef.	Neg. semidef.	Neg. def.	Neg. def.
	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{R}^n$

**Estimate of  $R_A$ :**  $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathbb{D}, c < \min_{\|x\|=r} W_1(x) \implies \{x \in B_r : W_2(x) \leq c\}$  is a region of attraction, when the origin is UAS.

**Lemma 11** *Barbalat's lemma:*

Let  $\dot{f} : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly continuous on  $[0, \infty)$ . If  $\lim_{t \rightarrow \infty} f(t)$  exists and is finite, then  $\dot{f} \rightarrow 0$  as  $t \rightarrow \infty$ .

Rephrased: if  $V$  is lower bounded,  $\dot{V} \leq 0$  and  $\ddot{V}$  is uniformly bounded, then  $\dot{V} \rightarrow 0$  as  $t \rightarrow \infty$ .

## 4 | INPUT-TO-STATE STABILITY

### 4.1 | Input-to-state stability

Now we consider the system  $\Sigma : \dot{x} = f(t, x, u)$ , where we consider  $u(t)$  to be a disturbance/modelling error.

**Definition** **Input-to-state stability (ISS):**  $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}$  s.t.  $\|x(t, x_0, u)\| \leq \max \{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_\infty)\}$

**Remark** This is really just an extension of GUAS that says that  $x$  is bounded the input as well. So naturally if  $\Sigma$  is ISS then it is also 0-GUAS.

**Definition**  $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$  is an **ISS-LF** for  $\Sigma$  iff

- $V$  is  $C^1$ .
- $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\rho \in \mathcal{K}$  s.t.
- $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$
- $\dot{V}(t, x) = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$

where  $W_3 > 0$ .

It can be shown that  $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$ .

$\exists \text{ISS-LF for } \Sigma \implies \Sigma \text{ is ISS}$

**Lemma 12** if  $f$  is  $C^1$  and globally Lipschitz in  $(x, u)$ , then  $\Sigma$  is 0 – GES  $\implies \Sigma$  is ISS

**Theorem 13** Consider the cascaded system  $\Sigma_2 \longrightarrow \Sigma_1$ , where  $\Sigma_1 : \dot{x}_1 = f_1(t, x_1, x_2)$  and  $\Sigma_2 : \dot{x}_2 = f_2(t, x_2)$ . If  $\Sigma_2$  is GUAS and  $\Sigma_1$  is ISS, then the cascaded system is GUAS.

### 4.2 | Input-output stability

We consider the system  $y = Hu$ .



**Definition**  $\mathcal{L}_p^m$  space:  $u \in \mathcal{L}_p^m \quad 1 \leq p < \infty \Leftrightarrow \|u\|_{\mathcal{L}_p} = \left( \int_0^\infty \|u(t)\|_p^p dt \right)^{\frac{1}{p}} < \infty$

**Remark** This makes  $\mathcal{L}_2$  the space of all continuous, square-integrable functions, for instance.

**Definition**  $\mathcal{L}_{pe}^m$  space:  $u \in \mathcal{L}_{pe}^m \Leftrightarrow u_\tau \in \mathcal{L}_p^m \quad \forall \tau \in [0, \infty)$ , where  $u_\tau$  is the truncated version of  $u$ .

**Definition**  $H : \mathcal{L}_{pe}^m \rightarrow \mathcal{L}_{pe}^q$  is  $\mathcal{L}_p$  stable iff

\*  $\exists \alpha$  class  $\mathcal{K} \quad \alpha : [0, \infty) \rightarrow [0, \infty)$

\*  $\exists$  constant  $\beta \geq 0$

s.t.  $\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \alpha(\|u_\tau\|_{\mathcal{L}_p}) + \beta \quad \forall u \in \mathcal{L}_{pe}^m$  and  $\tau \in [0, \infty)$

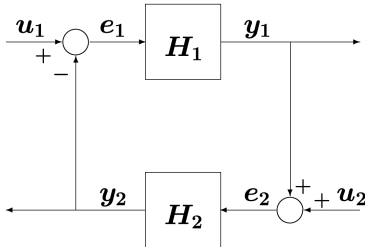
**Definition** Finite-gain  $\mathcal{L}_p$  stable:  $\exists \gamma, \beta \geq 0$  s.t.  $\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \gamma \|u_\tau\|_{\mathcal{L}_p} + \beta$

**Definition** Causal system:  $(Hu)_\tau = (Hu_\tau)_\tau$

The two definitions above hold for non-truncated signals if the systems are causal.

**Theorem 14** Small-gain theorem:

The feedback interconnection of  $H_1$  and  $H_2$  are finite-gain  $\mathcal{L}_p$  stable iff  $\gamma_1 \gamma_2 < 1$ .



**FIGURE 3** Feedback interconnection

## 5 | PASSIVITY

### 5.1 | Passivity for memoryless functions

Consider the memoryless function  $y = h(t, u) \quad h : [0, \infty) \times \mathbb{R}^P \rightarrow \mathbb{R}^P$ .

**Definition** The system is **passive** if  $u^\top y \geq 0$  and lossless if  $u^\top y = 0$ .

The system is **input-strictly passive** if  $u^\top y \geq u^\top \varphi(u)$ , where  $u^\top \varphi(u) > 0 \quad \forall u \neq 0$ .

The system is **output-strictly passive** if  $u^\top y \geq y^\top \rho(y)$ , where  $y^\top \rho(y) > 0 \quad \forall y \neq 0$ .

## 5.2 | Passivity for dynamical systems

Now we extend this property for dynamical systems:  $\Sigma : \dot{x} = f(x, u), y = h(x, u)$

where  $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is locally Lipschitz and  $h : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^p$  is continuous, and  $f(0, 0) = 0$  and  $h(0, 0) = 0$ .

**Definition** The system  $\Sigma$  is **passive** iff  $u^T y \geq \dot{V} \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p$

The system is **lossless** if  $u^T y = \dot{V}$ .

The system is **input-strictly passive** if  $u^T y \geq \dot{V} + u^T \varphi(u), \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$ .

The system is **output-strictly passive** if  $u^T y \geq \dot{V} + y^T \rho(y), \quad y^T \rho(y) > 0 \quad \forall y \neq 0$ .

The system is **(state-) strictly passive** if  $u^T y \geq \dot{V} + \psi(x), \quad \psi(x) > 0$ .

**Remark** Passivity really just generalizes the idea that the change of stored energy in the system should be less than the energy supplied to the system.

## 5.3 | Passivity and Lyapunov stability

**Lemma 15** If  $\Sigma$  is passive with a positive definite  $V(x)$ , then the origin of  $\dot{x} = f(x, 0)$  is stable.

**Lemma 16** If  $\Sigma$  is output-strictly passive with  $\rho(y) = \delta(y), \delta < 0$ , then  $\Sigma$  is finite-gain  $\mathcal{L}_2$  stable with  $\gamma \leq \frac{1}{\delta}$ .

**Definition** Zero state observability: no solution of  $\dot{x} = f(x, 0)$  can stay identically in  $S = \{x \in \mathbb{R}^n | h(x, 0) = 0\}$  other than the trivial solution  $x(t) = 0$ .

**Lemma 17** The origin of  $\dot{x} = f(x, 0)$  is asymptotically stable if  $\Sigma$  is state-strictly passive, or output-strictly passive and zero state observable. If  $V(x)$  is radially unbounded  $\dot{x} = f(x, 0)$  is GAS.

**Theorem 18** If  $H_1$  and  $H_2$  is passive, then the feedback interconnection of  $H_1$  and  $H_2$  is passive with  $V = V_1 + V_2$ .

**Theorem 19** If  $H_1$  and  $H_2$  satisfies  $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i, i = 1, 2$  and  $\varepsilon_1 + \delta_2 > 0, \varepsilon_2 + \delta_1 > 0$ , then  $\Sigma$  is finite-gain  $\mathcal{L}_2$  stable.

**Theorem 20** If  $H_1$  and  $H_2$  are state-strictly passive,  
or  $H_1$  and  $H_2$  are output-strictly passive and zero state observable,  
or  $H_1$  is state-strictly passive and  $H_2$  is output-strictly passive and zero state observable or opposite,  
then  $\Sigma$  is 0-AS. If  $V_1$  and  $V_2$  are radially unbounded then  $\Sigma$  is 0-GAS.

## 6 | NONLINEAR CONTROL

### 6.1 | Lyapunov control design

- \* Propose a LFC  $V(t, x)$ , typically as desired energy in system.
- \* Choose  $u = g(t, x)$  s.t.  $\dot{V} < 0$  or  $\dot{V} \leq 0$  with La Salle / Barbalat.

## 6.2 | Passivity-based control

**Theorem 21** For the LTI system  $y(s) = h(s)u(s)$  with  $\text{Re}(p_i) < 0, \forall i$  we have:

- \* Passivity  $\Leftrightarrow \text{Re}[h(j\omega)] \geq 0 \forall \omega$  (note that if  $h(s)$  has an integrator as well and  $\text{Re}(z_i) < 0$  this still holds)
- \* Input-strict passivity  $\Leftrightarrow \text{Re}[h(j\omega)] \geq \delta > 0 \forall \omega$ , with  $\varphi(u) = \delta u$
- \* Output-strict passivity  $\Leftrightarrow \exists \varepsilon > 0$  s.t.  $\text{Re}[h(j\omega)] \geq \varepsilon |h(j\omega)|^2 \forall \omega$ , with  $\rho(y) = \varepsilon y$

**Remark** This says that with non-negative real part the system is passive, with a strictly positive real part it is input-strictly passive and if it is lower bounded by the square of the magnitude it is output-strictly passive.

**Remark** Note that the previously discussed passivity theorems can be applied for passivity-based control.

Notably that means that if the controller and plant are passive, then the closed-loop system is passive.

Also if the controller is input- and output strictly passive and the plant is passive, then the system is finite gain  $\mathcal{L}_2$  stable. The final passivity theorem can naturally also be applied.

**Theorem 22** Consider  $H_1$  with  $\dot{x} = f(x, u)$  locally Lipschitz and  $y = h(x)$  continuous, both zero in the origin.

Also consider  $H_2 : \phi(y)$  locally Lipschitz, memoryless and zero in origin.

If  $H_1$  is passive with  $V > 0$ , radially unbounded and zero state observable, and  $H_2$  satisfies  $y^\top \phi(y) > 0, y \neq 0$  (passive, but not lossness) then the origin is GAS.

**Choice of  $y$ :** for  $\dot{x} = f(x) + G(x)u$ , if  $\exists$  LF  $V(x)$  radially unbounded, let  $y = \left[ \frac{\partial V}{\partial x} G(x) \right]^\top$  for the system to be passive.

**Feedback passivation:** choose  $u = \alpha(x) + \beta(x)v, y = h(x)$  s.t.  $\dot{x} = f(x) + G(x)u$  has desired passivity properties  $v \mapsto y$ .

## 6.3 | Feedback linearization

### 6.3.1 | Input-state linearization

Consider  $\dot{x} = f(x) + G(x)u$ . Find a state transformation  $z = T(x)$  and input transformation  $u = \alpha(x) + \beta(x)v$  s.t. the new system in  $z$  coordinates is linear and controllable. This is rarely possible to do, so we consider input-output linearization instead:

### 6.3.2 | Input-output linearization

We now consider the system 
$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

**Definition Lie derivative:**  $L_f h = \frac{\partial h}{\partial x} f, \quad L_f^2 h = \frac{\partial L_f h}{\partial x} f, \quad \dots \quad L_f^i h = L_f \left( L_f^{i-1} h \right)$

**Definition** The system has **relative degree  $\rho$**  in a region  $\mathbb{D}_0 \subset \mathbb{D} \subset \mathbb{R}^n$  if

$$\left. \begin{aligned} L_g L_f^{i-1} h &= 0, \quad 1 \leq i \leq \rho - 1 \\ L_g L_f^{\rho-1} h &\neq 0 \end{aligned} \right\} \forall x \in \mathbb{D}_0 \quad (4)$$

**Remark** (the number of differentiations of  $y$  before  $u$  appears)

**Remark** For linear systems we have  $\rho = n - m$ .

**Remark** If the relative degree is well defined in the region of interest  $\mathbb{D}$ , then the system can be input-output linearized.

**Definition** **Zero dynamics**: internal dynamics when output is kept at zero by the input i.e.  $\psi = 0, \dot{\phi} = f_o(\phi, 0)$ .

**Definition** **Minimum-phase system**: the zero-dynamics are asymptotically stable.

**Input-output linearization:**

1. Find the relative degree  $\rho$

2. Write the system in normal form

Let  $\psi_1 = y, \psi_2 = \dot{y}, \dots$ . The **external dynamics** are then  $\dot{\psi}_1 = \psi_2, \dots, \dot{\psi}_\rho = L_f^\rho h + L_g L_f^{\rho-1} h \cdot u$ .

Let  $\phi_1, \dots, \phi_{n-\rho}$  and  $z = T(x) = [\phi^\top \psi^\top]^\top$ .

Choose  $\phi$  s.t.  $T$  is a diffeomorphism,  $L_g \phi_i = 0$  and  $\phi_i(0) = 0$ .

If the Jacobian  $\frac{\partial T}{\partial x}|_{x_0}$  is nonsingular, then  $T$  is a diffeomorphism. Then the **internal dynamics** are  $\dot{\psi} = \dots$ .

We can then finally write the system in **normal form**:  $\dot{z} = \dots$

3. Choose  $u$  to cancel nonlinearities

$$u = \frac{1}{L_g L_f^{\rho-1} h} \left( -L_f^\rho h + v \right) \implies \psi_\rho = v \quad (5)$$

4. Analyze the zero-dynamics

5. Choose  $v$  to solve the control problem

This is a special case of the system  $\dot{\phi} = f_0(\phi, \psi), \quad \dot{\psi} = A\psi + Bv$ .

**Lemma 23** If the system is minimum phase and  $v = -K\psi$  is chosen s.t.  $(A - BK)$  is Hurwitz, then the origin of the system is asymptotically stable.

**Lemma 24** If  $\dot{\phi} = f_0(\phi, \psi)$  is ISS, then the system is GAS.

For tracking we have  $v = -Ke + y_d^{(\rho)}$ , but the same results apply, except  $\phi$  is now only bounded.

## 6.4 | Adaptive control

## 6.5 | Backstepping

## A | LINEAR METHODS

**Definition** We define the p-norm as:

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \quad (6)$$

$$\|f\|_{\mathcal{L}_p} = \left( \int_0^\infty |f(t)|^p dt \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \quad (7)$$

**Theorem 25** Schwarz' inequality:

$$| \langle x, y \rangle | \leq \|x\| \cdot \|y\| \quad (8)$$

**Definition**  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , then the Jacobian is defined as:

$$\frac{\partial f}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad (9)$$

Which in the scalar case  $m = 1$  is the gradient.