TTK4130 assignment 2

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January 25, 2019

1 Task 1

1.1

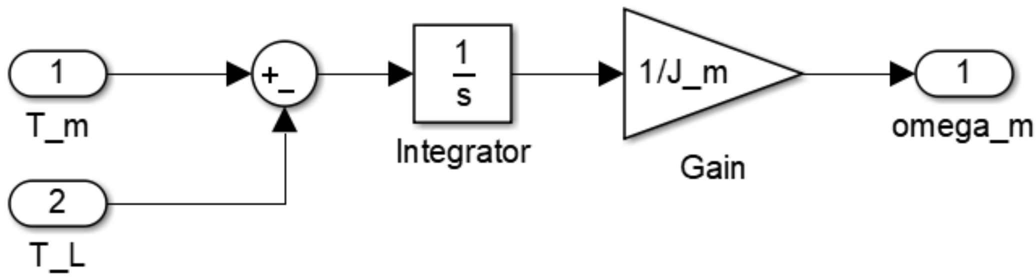

Figure 1: Motor system implemented in Simulink.

1.2

It is evident that ω_{i-1} and T_i are only inputs and not states in the system. θe is only an internal state. Finally ω_i and T_{i-1} are then the outputs of the system, when it is regarded in a signal-flow manner.

1.3

1.4

All the angular velocities have some small oscillations, but the loads are able to follow the reference pretty well.

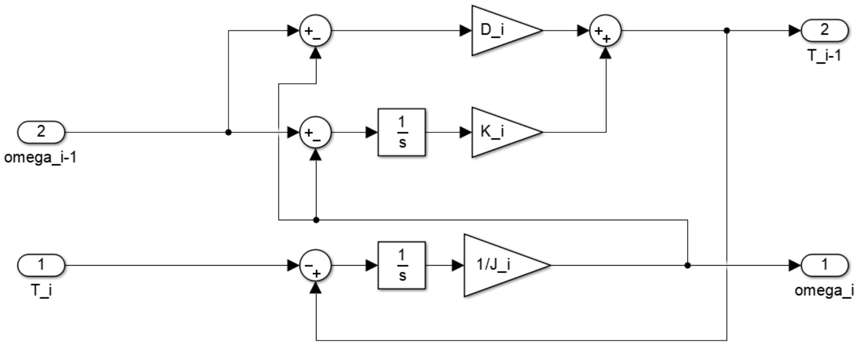

Figure 2: Load implemented in Simulink.

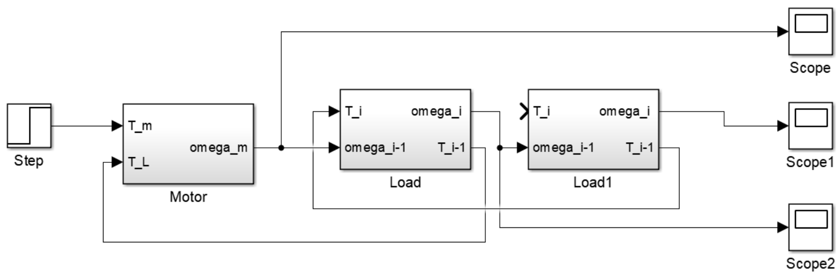

Figure 3: Complete system.

1.5

From the Bode plot we notice that we have two large resonance peaks, as well as a huge phase shift for certain frequencies. Furthermore, the system does not appear to be stable, as the phase is below -180 at cutoff.

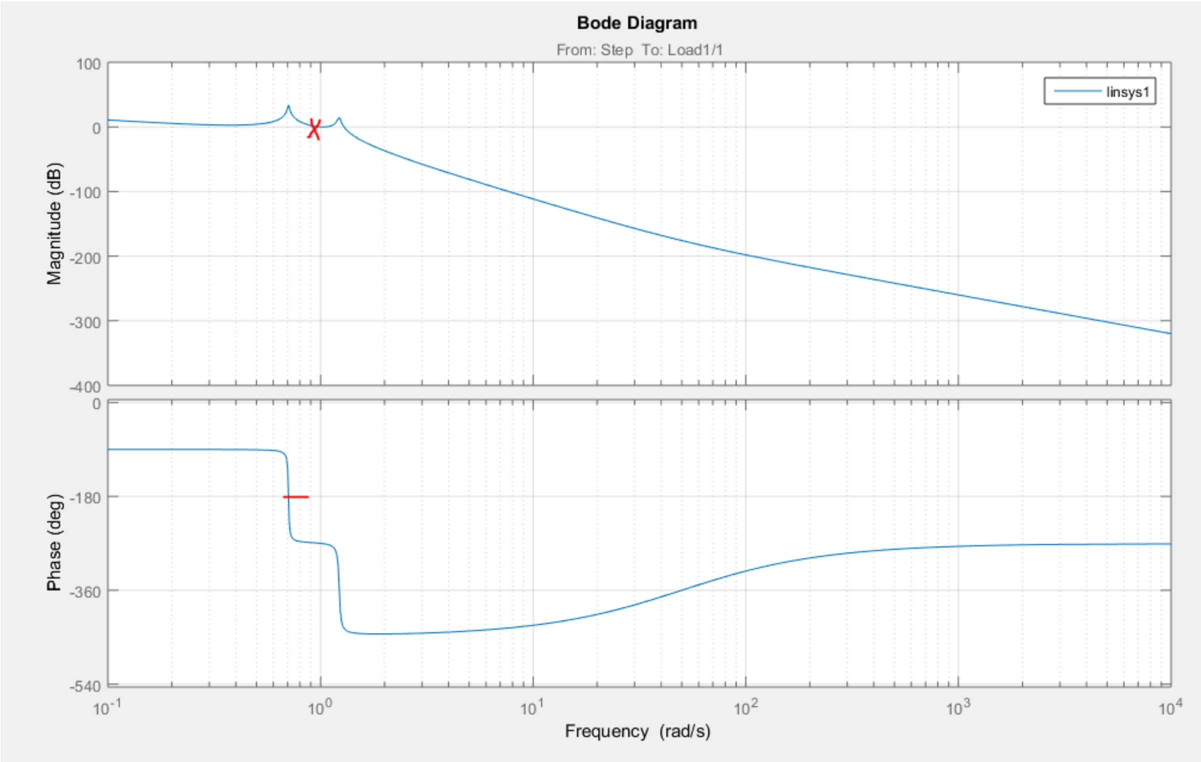

Figure 4: Bode plot of system.

1.6

The result is the same when we model the system using the standard blocks in Dymola. The model in Dymola is obviously much more readable and much faster and less errorprone to model, but higher level. The Simulink model gives us insight into the dynamics of the system and how it behaves as a state space system, which is lost in the Dymola representation.

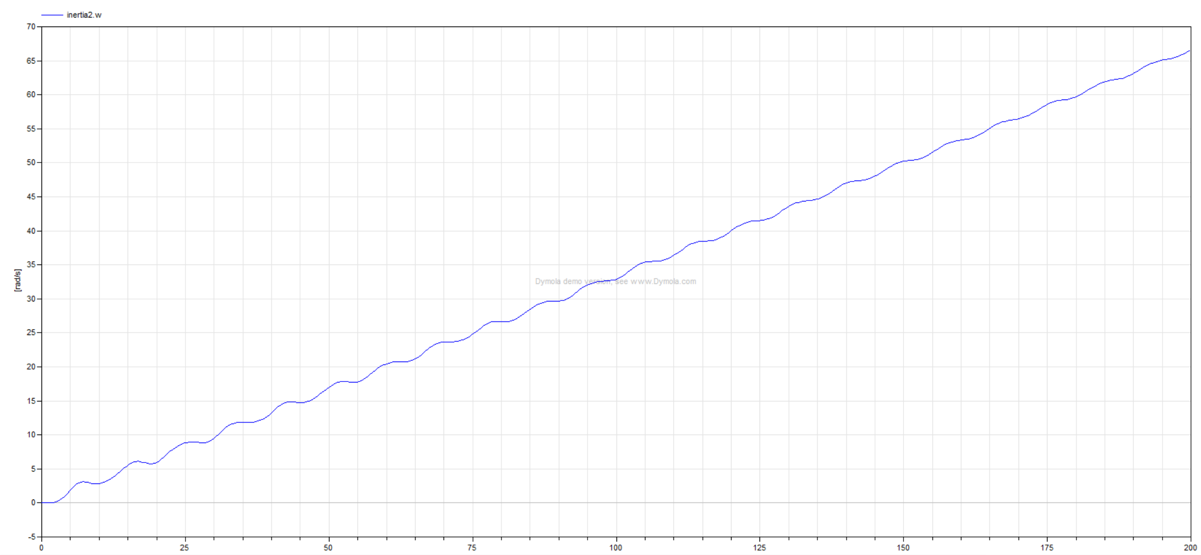

Figure 5: Dymola simulation result.

1.7

The information stored in the flange i.e. the I/O ports for the rotational components are angle and torque, as opposed to angular velocity and torque which we used in the Simulink system.

1.8

The Bode plots are indeed the same, oh yeah!

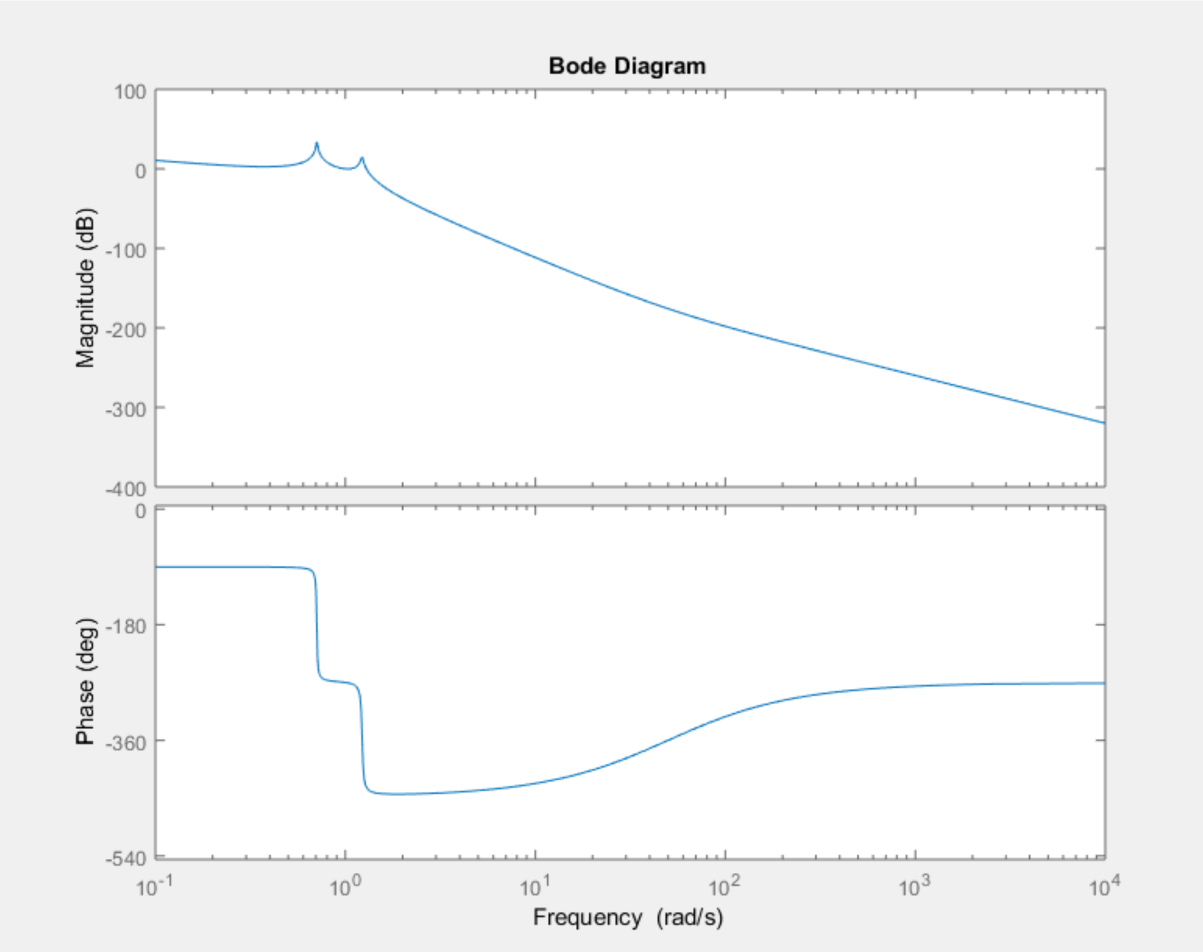

Figure 6: Dymola simulation bode plot.

2 A rational $H(s)$ is positive real if:

i) All poles have real parts less than or equal to zero.

ii) $Re\{H(j\omega)\} \geq 0 \ \forall \omega$ such that $j\omega$ is not a pole of $H(s)$.

iii) If $j\omega_0$ is a pole of $H(s)$, it is simple and $\underset{s=j\omega_0}{Res} H(s) = \lim_{s \to j\omega_0} (s-j\omega_0)H(s)$ is real and positive.

1. $H(s) = \dfrac{as}{1+bs}$

To satisfy i) we have that $b \geq 0$.

To satisfy ii) we get:

$$Re\{H(j\omega)\} = Re\left\{\frac{aj\omega}{1+bj\omega}\right\} = \frac{ab\omega^2}{1+b^2\omega^2} \geq 0$$

$\Rightarrow a \geq 0$

iii) does not happen, and so $H(s)$ is P.R. for $a, b \geq 0$.

2. $H(s) = \dfrac{s+a}{s^2+b^2} = \dfrac{s+a}{(s+jb)(s-jb)}$

The poles are strictly imaginary, so i) is satisfied.

$$H(j\omega) = \frac{j\omega + a}{j(\omega+b)(\omega-b)} = \frac{\omega - aj}{\omega^2 - b^2}$$

$Re\{H(j\omega)\} = \dfrac{\omega}{\omega^2 - b^2} \geq 0$ is not possible to satisfy for all ω, and so $H(s)$ is not P.R. ⍨

3. $$H_3(s) = \frac{s+a}{s+b}$$

i) $\underline{b \geq 0}$

ii) $$H_3(j\omega) = \frac{j\omega + a}{j\omega + b} = \frac{ab + \omega^2 + j\omega(b-a)}{b^2 + \omega^2}$$

$$Re\{H_3(j\omega)\} = \frac{ab+\omega^2}{b^2+\omega^2} \geq 0 \Rightarrow \underline{a \geq 0}$$

iii) Not the case.

$H_3(s)$ is P.R. if $a, b \geq 0$.

4. $$H_4(s) = \frac{s(s+a)}{(s+b)(s+c)}$$

i) If $a=b$ or $a=c$ we only require that the remaining pole is negative for P.R. Otherwise we get $b, c \geq 0$.

ii) $$H_4(j\omega) = \frac{j\omega(j\omega+a)}{(j\omega+b)(j\omega+c)}$$

$$= \frac{\omega^4 + ab\omega^2 - j\omega^3(a-b) + jc\omega^3 + j\omega abc + \omega^2(a-b)c}{(\omega^2+b^2)(\omega^2+c^2)}$$

$$Re\{H_4(j\omega)\} = \frac{\omega^4 + ab\omega^2 + \omega^2(a-b)c}{(\omega^2+b^2)(\omega^2+c^2)} \geq 0$$

$$\omega^2(ab + ac - bc) \geq 0$$

$$a \geq \frac{bc}{b+c}$$

~~Res~~

5. $$H_5(s) = \frac{1}{(s+a)(s+b)}$$

i) $a, b \geq 0$

$$H_5(j\omega) = \frac{(ab - \omega^2) - j\omega(a+b)}{(a^2+\omega^2)(b^2+\omega^2)}$$

$$Re\{H_5(j\omega)\} = \frac{ab - \omega^2}{(a^2+\omega^2)(b^2+\omega^2)}$$

We observe that ii) is not possible to fullfill, and so the transfer function is not PR.

6. $$H_6(s) = \frac{s^2+a^2}{s^2+b^2} = \frac{(s+ja)(s-ja)}{(s+jb)(s-jb)}$$

i) The poles are imaginary and thus has a non-positive real part.

ii) $$H_6(j\omega) = \frac{(\omega+a)(\omega-a)}{(\omega+b)(\omega-b)} = \frac{\omega^2-a^2}{\omega^2-b^2} \geq 0$$

If $a = \pm b$ this criterion is fulfilled, otherwise we cannot garantee this for all ω.

So H_6 is only PR for $a = \pm b$.

$\boxed{3}$ (a) $m\ddot{x} + d_1\dot{x} + d_3\dot{x}^3 + kx = F$

Define $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{F}{m} - \frac{k}{m}x_1 - \frac{d_1}{m}x_2 - \frac{d_3}{m}x_2^3$$

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} = kx_1\dot{x}_1 + mx_2\dot{x}_2$$

$$= kx_1x_2 + mx_2\left(\frac{F}{m} - \frac{k}{m}x_1 - \frac{d_1}{m}x_2 - \frac{d_3}{m}x_2^3\right)$$

$$= Fx_2 - d_1x_2^2 - d_3x_2^4$$

Let the system input be $u = F$, and the system output be $y = x_2$.

$$\dot{V} = uy - g(x),$$

$$g(x) = x_2^2(d_1 + d_3x_2^2)$$

As $d_1, d_3, x_2^2 \geq 0$, so is $g(x)$.

We have thereby showed that $\dot{V} = uy - g(x)$, $g(x) \geq 0$, and the system is therefore passive.

(b) $\alpha T_d \dot{x}_1 + x_1 = (\alpha - 1) e,$

$\beta T_i \dot{x}_2 + x_2 = \frac{\beta - 1}{\alpha} (e + x_1),$

$u = K_p \left(\frac{e + x_1}{\alpha} + x_2 \right)$

Let the input be e and the output be u (a controller).

(b) $$\int_0^T e u \, dt = \int_0^T e K_p \left(\frac{e + x_1}{\alpha} + x_2 \right) dt$$

$$= \int_0^T \underbrace{\frac{K_p}{\alpha} e^2}_{\geq 0} + \frac{K_p}{\alpha} e x_1 + K_p e x_2 \, dt$$

$$\geq \int_0^T \frac{K_p}{\alpha} x_1 \left(\frac{1}{\alpha - 1} (\alpha T_d \dot{x}_1 + x_1) \right) + K_p \alpha \left(\frac{\beta T_i \dot{x}_2 + x_2}{\beta - 1} - x_1 \right) x_2 \, dt$$

$$= \int_0^T \frac{K_p T_d}{\alpha - 1} x_1 \dot{x}_1 + \frac{K_p x_1^2}{\alpha(\alpha-1)} + \frac{K_p \alpha \beta T_i}{\beta - 1} x_2 \dot{x}_2 + \underbrace{\frac{K_p \alpha}{\beta - 1} x_2^2}_{\geq 0} - K_p \alpha x_1 x_2 \, dt$$

$$\geq \int_{x_1(0)}^{x_1(T)} \frac{K_p T_d}{\alpha - 1} x_1 \, dx_1 + \int_{x_2(0)}^{x_2(T)} \frac{K_p \alpha \beta T_i}{\beta - 1} x_2 \, dx_2 + \int_0^T \frac{K_p x_1^2}{\alpha(\alpha - 1)} - K_p \alpha x_1 x_2 \, dt$$

$$= \frac{K_p T_d}{\alpha - 1} \left(\tfrac{1}{2} x_1^2(T) - \tfrac{1}{2} \overset{\gtrless 0}{x_1^2(0)} \right) + \frac{K_p \alpha \beta T_i}{\beta - 1} \left(\underbrace{\tfrac{1}{2} x_2^2(T)}_{\geq 0} - \tfrac{1}{2} x_2^2(0) \right)$$
$$+ \int_0^T \frac{K_p x_1^2}{\alpha(\alpha - 1)} - K_p \alpha x_1 x_2 \, dt$$

$$\geq \frac{K_p T_d}{\alpha - 1} \cdot \tfrac{1}{2} x_1^2(T) - \frac{K_p \alpha \beta T_i}{\beta - 1} \cdot \tfrac{1}{2} x_2^2(0) + \int_0^T \frac{K_p x_1^2}{\alpha(\alpha-1)} - K_p \alpha x_1 x_2$$

All of there remaining terms are negative and the $x_1 x_2$ term I do not know how to deal with... ::(

(b) Alternatively we try to show that $H(s)$ is PR:

$$u(s) = \frac{K_p}{\alpha} e(s) + \frac{K_p}{\alpha} X_1(s) + K_p X_2(s),$$

$$X_1(s) = \frac{\alpha - 1}{1 + s\alpha T_d} e(s),$$

$$X_2(s) = \frac{(\beta - 1)(1 + T_d s)}{(1+\beta T_i s)(1+\alpha T_d s)} e(s)$$

$$\Rightarrow H(s) = \frac{u(s)}{e(s)} = K_p\left(\frac{1}{\alpha} + \frac{\alpha - 1}{\alpha(1+s\alpha T_d)} + \frac{(\beta-1)(1+T_d s)}{(1+\beta T_i s)(1+T_d \alpha s)}\right)$$

$$H(s) = \ldots = K_p \beta \frac{1 + (T_d + T_i)s + T_d T_i s^2}{(1+\beta T_i s)(1+\alpha T_d s)}$$

Now evidently the poles are in the left half plane, so we just need to show that $Re\{H(j\omega)\} \geq 0 \ \forall \omega$:

$$H(j\omega) = K_p\beta \frac{(1 - T_d T_i \omega^2 + (T_d+T_i)j\omega)(1-\beta T_i j\omega)(1 - \alpha T_d j\omega)}{(1+\beta^2 T_i^2\omega^2)(1+\alpha^2 T_d^2 \omega^2)}$$

$$Re\{H(j\omega)\} = K_p\beta \frac{(1 - T_d T_i\omega^2)(1-\alpha\beta T_i T_d \omega^2) + (T_d+T_i)(\alpha T_d + \beta T_i)\omega^2}{(1+\beta^2 T_i^2 \omega^2)(1+\alpha^2 T_d^2 \omega^2)}$$

$$= K_p\beta \frac{1 + \omega^2(\alpha T_d^2 + \beta T_i T_d + \alpha T_i T_d + \beta T_i^2 - \alpha\beta T_i T_d - T_d T_i) + \alpha\beta T_i^2 T_d^2 \omega^4}{(1+\beta^2 T_i^2\omega^2)(1+\alpha^2 T_d^2\omega^2)}$$

$$\geq 0 \Rightarrow \alpha T_d^2 + \beta T_i T_d + \alpha T_i T_d + \beta T_i^2 - \alpha\beta T_i T_d - T_d T_i \geq 0$$

$$= \alpha T_d^2 + \beta T_i^2 + T_d T_i(\alpha + \beta - 1 - \alpha\beta) \geq 0$$

$$\alpha + \beta - 1 - \alpha\beta \geq 0 \iff \underline{(\beta - 1)(1-\alpha) \geq 0} \quad \square$$

Passive system!

since $\beta > 1$ and $\alpha \in [0,1]$ this is satisfied $\rightarrow$ PR

(c) $y = f(u)$

$$\int_{t_0}^{t} y u \, dt = \int_{t_0}^{t} f(u) u \, dt \geq -E_0(t_0)$$

If f and u has the same sign the integrand is always positive, and such the inequality is always satisfied for any E_0, which means the system is passive.