$\boxed{1a}$ $Z = \frac{X-\mu}{\sigma}$, $Z = \pm 2{,}58$ for a 99% certainty.

Now $s = c\frac{t}{2}$, which means

$s \sim N\left(\frac{cT}{2}, \frac{c\sigma^2}{2}\right)$, $\frac{c\sigma}{2} \cdot 2{,}58 = 1$

$\Rightarrow \underline{\sigma} = 2{,}6 \cdot 10^{-9}\,s = \underline{2{,}6\,ns}$

If we have N observations we get

$\frac{c\sigma}{2\sqrt{N}} \cdot 2{,}58 = 1 \iff \underline{N = 15}$

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$\boxed{1b}$ $\hat{\sigma}^2 = \frac{1}{N}\sum_{n=0}^{N-1} x(n)^2$, $x(n) \sim \mathcal{N}(0,\sigma^2)$

$$\underline{E\{\hat{\sigma}^2\}} = \frac{1}{N}\sum_{n=0}^{N-1} E\{x(n)^2\} = \frac{1}{N}\cdot N\sigma^2 = \underline{\sigma^2}$$

So when the mean is known the estimator is unbiased.

$$\underline{Var\{\hat{\sigma}^2\}} = \frac{1}{N^2}\sum_{n=0}^{N-1} Var\{x(n)^2\}$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\left(E\{x^4\} - E\{x^2\}^2\right)$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\left(3\sigma^4 - \sigma^4\right) = \underline{\frac{2\sigma^4}{N}}$$

When $N \to \infty$ the variance of the estimator naturally goes to zero.

$\boxed{1c}$ $x = A + w$, $w \sim \mathcal{N}(0,\sigma^2)$

$$\hat{\theta} = \left(\frac{1}{N}\sum_{n=0}^{N-1} x(n)\right)^2 = \hat{A}^2$$

$$\underline{\underline{E\{\hat{\theta}\}}} = E\left\{\frac{1}{N^2}\left(\sum_{n=0}^{N-1} x(n)\right)^2\right\} = E\{\hat{A}^2\} = E\{\hat{A}\}^2 + Var\{\hat{A}\}$$

$$= A^2 + \frac{1}{N^2}\sum_{n=0}^{N-1} Var\{x(n)\} = A^2 + \frac{1}{N^2}\cdot N\sigma^2 = \underline{\underline{A^2 + \frac{\sigma^2}{N}}}$$

The estimator is biased, but approaches the real value as $N \to \infty$.

$\boxed{2}$ ⓐ $p(\delta_i;\beta) = \frac{1}{\beta} e^{-\delta_i/\beta}$, $E\{\delta_i\} = \beta$, $Var\{\delta_i\} = \beta^2$

i) $p(\delta;\beta) = \prod_{n=0}^{N-1} p(\delta(n);\beta)$

$\log p(\delta;\beta) = \sum_{n=0}^{N-1} \log p(\delta(n);\beta)$

$= \sum_{n=0}^{N-1}\left(-\log\beta - \frac{\delta(n)}{\beta}\right) = -N\log\beta - \frac{1}{\beta}\sum_{n=0}^{N-1}\delta(n)$

$\frac{\partial \log p(\delta;\beta)}{\partial\beta} = -\frac{N}{\beta} + \frac{1}{\beta^2}\sum_{n=0}^{N-1}\delta(n)$

In order to use the CRLB we need to show that $E\{\frac{\partial \log p}{\partial\beta}\} = 0$:

$E\left\{\frac{\partial \log p(\delta;\beta)}{\partial\beta}\right\} = -\frac{N}{\beta} + \frac{1}{\beta^2}\sum_{n=0}^{N-1}E\{\delta(n)\}$

$= -\frac{N}{\beta} + \frac{1}{\beta^2}\sum_{n=0}^{N-1}\beta = -\frac{N}{\beta} + \frac{N}{\beta} = \underline{0}$ □

$\underline{\underline{CRLB}} = -\frac{1}{E\left\{\frac{\partial^2 \log p(\delta;\beta)}{\partial\beta^2}\right\}} = -\frac{1}{E\left\{\frac{N}{\beta^2} - \frac{2}{\beta^3}\sum_{n=0}^{N-1}\delta(n)\right\}}$

$= -\frac{1}{\left(\frac{N}{\beta^2} - \frac{2N}{\beta^2}\right)} = \underline{\underline{\frac{\beta^2}{N}}}$

ii) $\hat{\beta} = \bar{\delta} = \frac{1}{N}\sum_{n=0}^{N-1}\delta(n)$, $E\{\hat{\beta}\} = \frac{1}{N}\sum_{n=0}^{N-1}E\{\delta(n)\}$

$= \frac{1}{N}\cdot N\beta = \beta$

$\underline{Var(\hat{\beta})} = \frac{1}{N^2}\sum_{n=0}^{N-1}Var(\delta(n))$

$= \frac{1}{N^2}\sum_{n=0}^{N-1}\beta^2 = \underline{\frac{\beta^2}{N}}$ This estimator is efficient!

$\boxed{2b}$ $H = h_R + j h_Q$, $h_Q \sim h_R \sim N(0, \sigma^2)$

$$R = \sqrt{h_R^2 + h_Q^2} \, , \quad p(r;\sigma^2) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

$$p(\mathbf{r};\sigma^2) = \prod_{n=0}^{N-1} p(r(n);\sigma^2)$$

$$\log p(\mathbf{r};\sigma^2) = \sum_{n=0}^{N-1} \log p(r(n);\sigma^2)$$

$$= \sum_{n=0}^{N-1}\left(\log r(n) - \log \sigma^2 - \frac{r^2(n)}{2\sigma^2}\right) = \sum_{n=0}^{N-1} \log r(n) - N \log \sigma^2$$

$$- \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} r(n)^2$$

$$\frac{\partial \log p(\mathbf{r};\sigma^2)}{\partial \sigma^2} = -\frac{N}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{n=0}^{N-1} r(n)^2$$

$$E\left\{\frac{\partial \log p(\mathbf{r};\sigma^2)}{\partial \sigma^2}\right\} = -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{n=0}^{N-1} E\{r(n)^2\}$$

$$= -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{n=0}^{N-1} E\{h_R^2 + h_Q^2\} = -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{n=0}^{N-1} 2\sigma^2$$

$$= -\frac{N}{\sigma^2} + \frac{1}{2\sigma^4} \cdot N \cdot 2\sigma^2 = -\frac{N}{\sigma^2} + \frac{N}{\sigma^2} = \underline{0}$$

$$\underline{\underline{CRLB}} = -\frac{1}{E\left\{\frac{\partial^2 \log p(\mathbf{r};\sigma^2)}{\partial \sigma^2}\right\}} = -\frac{1}{E\left\{\frac{N}{\sigma^4} - \frac{1}{\sigma^6} \sum_{n=0}^{N-1} r(n)^2\right\}}$$

$$= -\frac{1}{\frac{N}{\sigma^4} - \frac{1}{\sigma^6} \cdot N \cdot 2\sigma^2} = -\frac{1}{-\frac{N}{\sigma^4}} = \underline{\underline{\frac{\sigma^4}{N}}}$$

I will try the following estimator:

$$\hat{\alpha} = \frac{1}{2N} \sum_{n=0}^{N-1} r(n)^2 , \quad \underline{E\{\hat{\alpha}\}} = \frac{1}{2N} \sum_{n=0}^{N-1} E\{r(n)^2\}$$

$$= \frac{1}{2N} \sum_{n=0}^{N-1} 2\sigma^2 = \underline{\sigma^2}$$ So it is unbiased.

$$\mathrm{Var}\{\hat{\alpha}\} = \frac{1}{4N^2}\sum_{n=0}^{N-1} \mathrm{Var}\{r(n)^2\}$$

$$= \frac{1}{4N^2}\sum_{n=0}^{N-1} \mathrm{Var}\{h_R^2 + h_Q^2\} = \frac{1}{4N^2}\sum_{n=0}^{N-1} 2\,\mathrm{Var}\{h_R^2\}$$

We have that $h_R^2 \sim \sigma^2 \chi_1^2$, where χ_1^2 is the Chi-squared distribution with 1 degree of freedom, so $\mathrm{Var}\{h_R^2\} = 2\sigma^4$.

$$\Rightarrow \underline{\mathrm{Var}\{\hat{\alpha}\}} = \frac{1}{4N^2}\sum_{n=0}^{N-1} 4\sigma^4 = \underline{\frac{\sigma^4}{N}}$$

So this estimator is indeed efficient.