TTK4135 EXERCISE 3

$\boxed{1}$ $\min_{x \in \mathbb{R}^n} c^T x \quad$ s.t. $\quad Ax = b, \; x \geq 0$

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$$

KKT:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c - A^T \lambda^* - s^* = 0,$$
$$Ax^* = b,$$
$$x^* \geq 0, \; s^* \geq 0,$$
$$s_i x_i = 0 \; \forall i \in I$$

ⓐ Newton direction: $p_k^N = -(\nabla^2 f_k)^{-1} \nabla f_k$

Since $\nabla_x^2 f(x) = 0$ it is trivial to see that p_k^N is not defined, as the inverse of the laplacian of f is not defined.

ⓑ $f(\alpha x + (1-\alpha)y) = c^T \alpha x + c^T (1-\alpha) y$
$= \alpha f(x) + (1-\alpha) f(y) = \alpha c^T x + (1-\alpha) c^T y$
as α is just a scalar.

So $f(x) = c^T x$ is convex.

But is the feasible set Ω convex?

$\Omega : \{ x \in \mathbb{R}^n \mid Ax = b, \; x \geq 0 \}$

Let $x, y \in \Omega$, then the point $z = \alpha x + (1-\alpha) y$ satisfies

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$Az = \alpha Ax + (1-\alpha)Ay = \alpha b + (1-\alpha)b = b$

And furthermore it is easy to see that for $x \geq 0$, $y \geq 0$ we get

$z = \alpha x + (1-\alpha)y \geq 0$, $\alpha \in [0,1]$

So Ω is convex.

Since $f(x)$ and Ω are convex, (1) is a convex problem.

ⓒ $\max_{\lambda} b^T\lambda$ s.t. $A^T\lambda \leq c$

$\iff \min_{\lambda} -b^T\lambda$ s.t. $A^T\lambda \leq c$

$\mathcal{L}(\lambda, x) = -b^T\lambda - x^T(c - A^T\lambda)$

$\nabla_x \mathcal{L}(\lambda^*, x^*) = -b + Ax^* = 0$

$\Rightarrow Ax^* = b$

We also have that

$x^* \geq 0$, and

$x^{*T}(c - A^T\lambda^*) = 0$, which when defining $s = c - A^T\lambda^*$ gives $s^Tx^* = 0$. We also have

$c - A^T\lambda^* \geq 0 \iff s^* \geq 0$

So all the KKT conditions are indeed the same.

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(d) $c^T x^* = b^T \lambda^*$

(e) x is a BFP if it is feasible, and there exists an index set $\beta(x)$ of m indices such that

$$i \notin \beta \Rightarrow x_i = 0$$

and

$$B = [A_i]_{i \in \beta}$$

is nonsingular.

(f) LICQ is satisfied if the set of all active constraints gradiants are linearly independent.

Since the gradiant of the constraint is A^T, LICQ is satisfied in the LP case if A^T has full rank, a.k.a. A has full rank.

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2 (a) Let $A = x_1$, $B = x_2$

$f(x) = \left(\frac{3}{2} x_1 + x_2\right) p$

For simplicity set $p = 2$

$\Rightarrow f(x) = 3x_1 + 2x_2$

The constraints are that the total hours of R_I and R_{II} are limited i.e.

$2x_1 + x_2 \leq 8$, $x_1 + 3x_2 \leq 15$

Transform to standard form by slacking variables:

$2x_1 + x_2 + z_1 \leq 8$, $x_1 + 3x_2 + z_2 = 15$,

$x_1, x_2, z_1, z_2 \geq 0$

$\Rightarrow \min_{x \in \mathbb{R}^4} c^T x$ s.t. $Ax = b$, $x \geq 0$,

$c^T = [-3\ -2\ 0\ 0]$, $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$

(c) $x^* = \begin{bmatrix} 1{,}8 \\ 4{,}4 \end{bmatrix}$, $f(x^*) = 14{,}2$

This is in a corner of Ω as expected. All the constraints are not active, but the 2 "main" ones are.

(e) From the plot we see that the simplex method moves along the polytope, exchanging the "bad" elements of the basis for new ones, as theory says.

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$\boxed{3}$ $\min\limits_{x} q(x) = \frac{1}{2} x^T G x + x^T c$

s.t. $a_i^T x = b_i,\ i \in \mathcal{E}$, $a_i^T x \geq b_i,\ i \in \mathcal{I}$

ⓐ The active set $\mathcal{A}(x^*)$ is the set of all the indecies of the active constraints, i.e.

$$\mathcal{A}(x^*) = \{ i \in \mathcal{E} \cup \mathcal{I} \mid a_i^T x^* = b_i \}$$

$$= \{ i \in \mathcal{E},\ i \in \mathcal{I} \mid a_i^T x^* = b_i \}$$

ⓑ $\mathcal{L}(x, \lambda) = \frac{1}{2} x^T G x + x^T c - \sum\limits_{i \in \mathcal{A}(x^*)} \lambda_i (a_i^T x - b_i)$

$$\begin{cases} \nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \sum\limits_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0, \\ a_i^T x^* - b_i = 0,\ i \in \mathcal{A}(x^*), \\ a_i^T x^* - b_i \geq 0,\ i \in \mathcal{I} / \mathcal{A}(x^*), \\ \lambda_i^* \geq 0,\ i \in \mathcal{I} \cap \mathcal{A}(x^*) \end{cases}$$