TTK4130 assignment 3

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1 Task 1

1.1

We can model the dynamics of the populations as the following state space system:

$$\dot{x} = \begin{bmatrix} \dot{H} \\ \dot{I} \\ \dot{Z} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} bH - b_d H^2 - dH - iHZ \\ iHZ - AI - dI \\ aI + rD - nHZ \\ dH + dI + nHz - rD \end{bmatrix} \tag{1}$$

Which can be implemented in Modelica like so:

```
model IncompleteZombieApocalypse "Zombie apocalypse model"
  parameter Real a = 1.4e-6;
  parameter Real b = 3.1e-8;
  parameter Real b_d = 5.6e-16;
  parameter Real d = 2.8e-8;
  parameter Real i = 2.6e-6;
  parameter Real n = 1.4e-6;
  parameter Real r = 2.8e-7;
  Real H(min=0);
  Real I(min=0);
  Real Z(min=0);
  Real D(min=0);

initial equation
  H = 10^7;
  I = 10^6;
  Z = 0.5*10^6;
  D = 10^5;

equation
  der(H) = b*H - b_d*H^2 - d*H - i*H*Z;
  der(I) = i*H*Z - a*I - d*I;
  der(Z) = a*I + r*D - n*H*Z;
  der(D) = d*H + d*I + n*H*Z - r*D;
  annotation(experiment(StartTime=0, StopTime=8640000, Tolerance=1e-8));
end IncompleteZombieApocalypse;
```

Figure 1: Modelica code of the population system.

As the figure below illustrates, the model will end with all the populations being turned into zombies, for these specific parameters. We observe that the Z population dies out immediately, but when all of H is dead, it starts growing to a final equilibrium where I=0, H=0 and D=0. If H=0 we have no healthy humans, which is a lost cause.

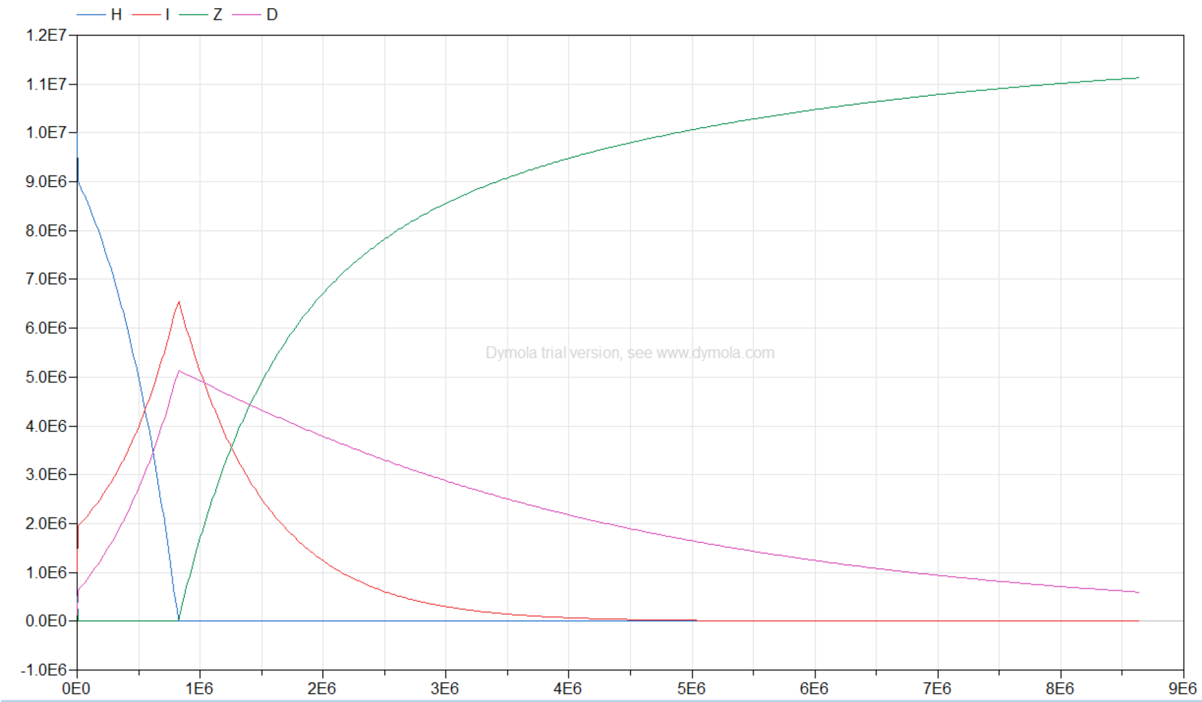


Figure 2: Simulation of the zombie apocalypse.

1.2

If we assume that all the populations are mutually exclusive, we can model the augmented system like so:

$$\dot{x} = \begin{bmatrix} \dot{H} \\ \dot{I} \\ \dot{Z} \\ \dot{D} \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} bH - b_d H^2 - dH - iHZ \\ iHZ - AI - dI - q_i I \\ aI + rD - nHZ - q_z Z \\ dH + dI + nHz + d_q Q - rD \\ q_i I + q_z Q - d_q Q \end{bmatrix} \tag{2}$$

Here we also assume that the amount of zombies/infected we can quarantine is not dependent on the amount of humans, which is not realistic. The augmented system looks like this in Modelica code: We observe that the majority of the entities in the simulation end up dead instead of as zombies, but the humans die out anyway.

```
model IncompleteZombieApocalypse "Zombie apocalypse model"
  parameter Real a = 1.4e-6;
  parameter Real b = 3.1e-8;
  parameter Real b_d = 5.6e-16;
  parameter Real d = 2.8e-8;
  parameter Real i = 2.6e-6;
  parameter Real n = 1.4e-6;
  parameter Real r = 2.8e-7;
  parameter Real q_i = 2.7e-6;
  parameter Real q_z = 2.7e-6;
  parameter Real d_q = 2.8e-5;
  Real H(min=0);
  Real I(min=0);
  Real Z(min=0);
  Real D(min=0);
  Real Q(min=0);

initial equation
  H = 10^7;
  I = 10^6;
  Z = 0.5*10^6;
  D = 10^5;
  Q = 0;

equation
  der(H) = b*H - b_d*H^2 - d*H - i*H*Z;
  der(I) = i*H*Z - a*I - d*I - q_i*I;
  der(Z) = a*I + r*D - n*H*Z - q_z*Z;
  der(D) = d*H + d*I + n*H*Z + d_q*Q - r*D;
  der(Q) = q_i*I + q_z*Z - d_q*Q;
  annotation(experiment(StartTime=0, StopTime=8640000, Tolerance=1e-8));
end IncompleteZombieApocalypse;
```

Figure 3: Modelica code of augmented system.

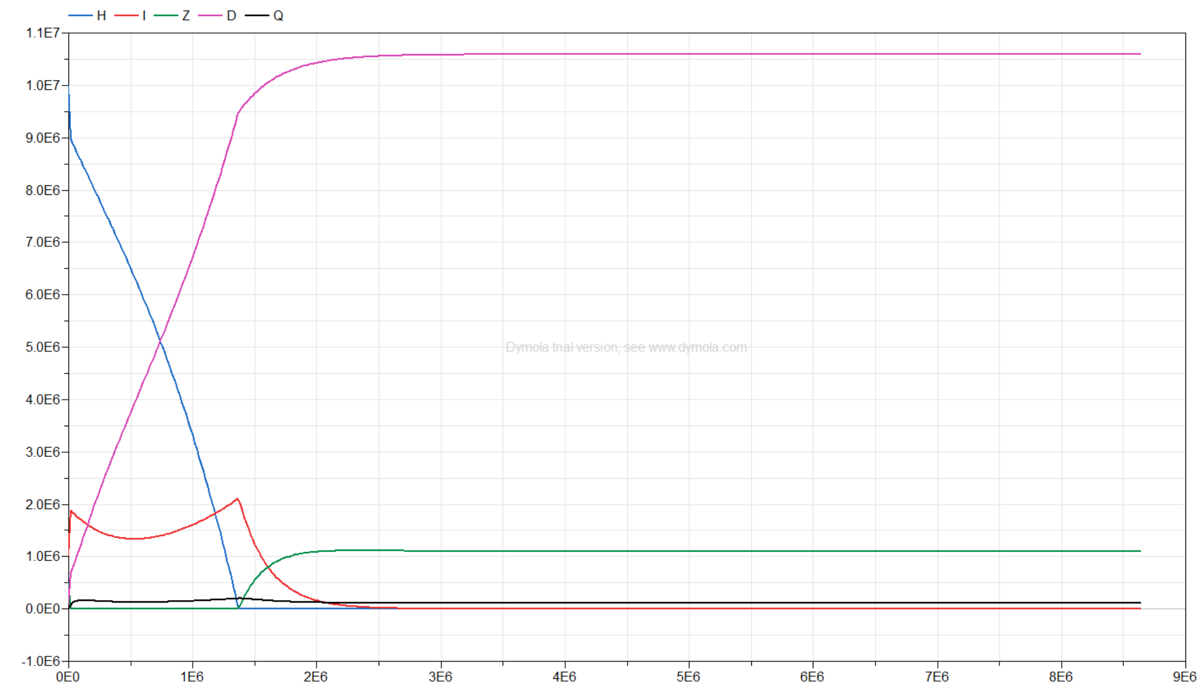


Figure 4: Simulation of augmented system.

1.3

The model looks like this if we add the cure dynamics:

$$\dot{x} = \begin{bmatrix} \dot{H} \\ \dot{I} \\ \dot{Z} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} bH + cI + cZ - b_d H^2 - dH - iHZ \\ iHZ - AI - dI - cI \\ aI + rD - nHZ - cZ \\ dH + dI + nHz - rD \end{bmatrix} \tag{3}$$

We observe from the simulation that the stationary value is $H = 2 * 10^6$.

```
model IncompleteZombieApocalypse "Zombie apocalypse model"
  parameter Real a = 1.4e-6;
  parameter Real b = 3.1e-8;
  parameter Real b_d = 5.6e-16;
  parameter Real d = 2.8e-8;
  parameter Real i = 2.6e-6;
  parameter Real n = 1.4e-6;
  parameter Real r = 2.8e-7;
  parameter Real c = 2.7e-3;
  Real H(min=0);
  Real I(min=0);
  Real Z(min=0);
  Real D(min=0);

initial equation
  H = 10^7;
  I = 10^6;
  Z = 0.5*10^6;
  D = 10^5;

equation
  der(H) = b*H + c*I + c*Z - b_d*H^2 - d*H - i*H*Z;
  der(I) = i*H*Z - a*I - d*I - c*I;
  der(Z) = a*I + r*D - n*H*Z - c*Z;
  der(D) = d*H + d*I + n*H*Z - r*D;
  annotation(experiment(StartTime=0, StopTime=864000000, Tolerance=1e-8));
end IncompleteZombieApocalypse;
```

Figure 5: Modelica code of augmented system.

1.4

Since most of the code is just declaration of the parameters and initializing the population states, this could be its own model that we simply extend for the different augmentations of the system.

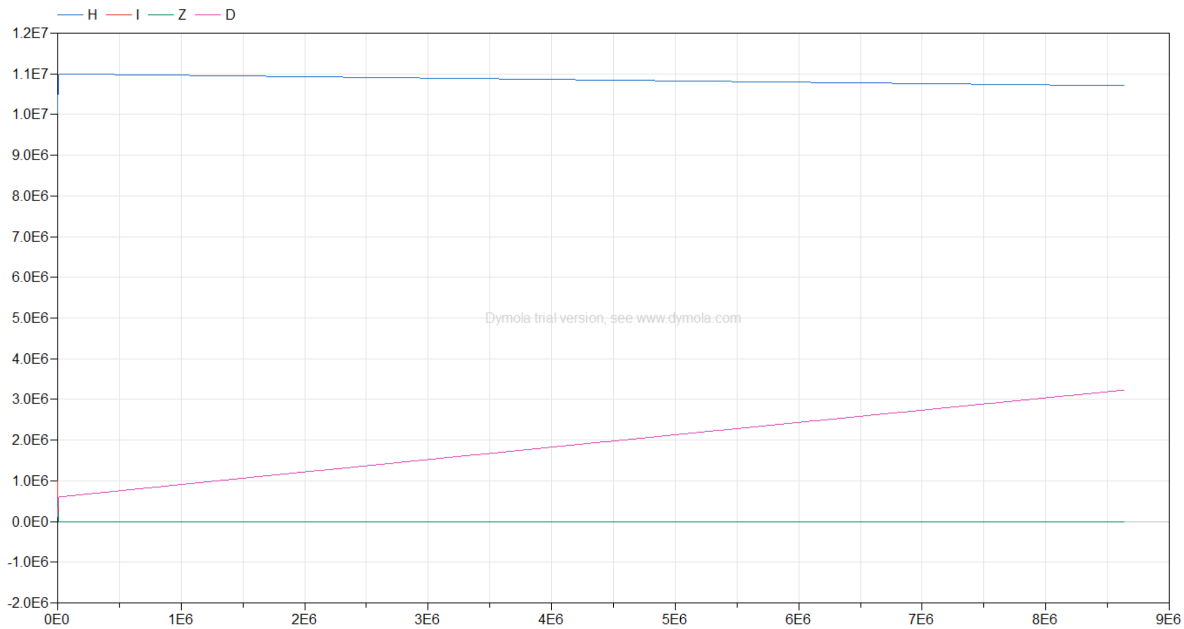


Figure 6: Simulation of augmented system.

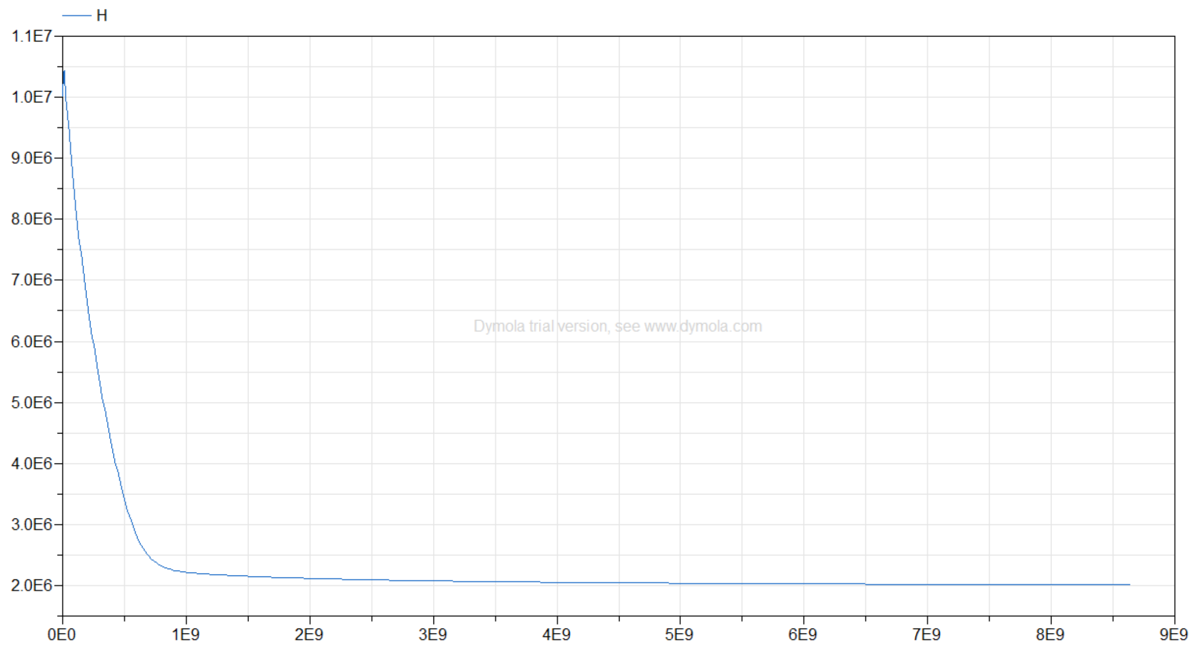


Figure 7: Simulation of H over a long time horizon.

1.5

The structure of this model is basically just a bunch of states in a continous state machine, where entities flow from one state to the other given certain very simple rules. Any system where you have entities that transform from one state to another can be modelled using this structure. The obvious choice is for biological systems, like perhaps modelling different discrete behaviours in some species, or the spread of a virus in a species. It could maybe be used to model microbiological systems, like a system with bacteria and parasites.

$\boxed{2}$ $\ddot{x} + c\dot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0$

ⓐ $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$,

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} x_2 \\ -g\left(1 - \left(\frac{x_d}{x_1}\right)^{\kappa}\right) - c x_2 \end{bmatrix}$$

ⓑ $y_{n+1} = y_n + h f(y_n, t_n)$

$$y_{n+1} = y_n + h \begin{bmatrix} y_{2n} \\ -g\left(1 - \left(\frac{x_d}{x_{1n}}\right)^{\kappa}\right) - c\, y_{2n} \end{bmatrix}$$

ⓒ $k_1 = f(y_n, t_n) = \begin{bmatrix} y_{n2} \\ -g\left(1 - \left(\frac{x_d}{y_{n1}}\right)^{\kappa}\right) - c y_{n2} \end{bmatrix}$,

$$k_2 = f\left(y_n + \frac{h}{2} k_1,\ t_{n+\frac{h}{2}}\right)$$

$$= \begin{bmatrix} y_{n2} + \frac{h}{2} k_{1_2} \\ -g\left(1 - \left(\frac{x_d}{y_{n1} + h/2\, k_{1_1}}\right)^{\kappa}\right) - c\left(y_{n2} + \frac{h}{2} k_{1_2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} y_{n2} - \frac{h}{2}\left(g\left(1 - \left(\frac{x_d}{y_{n1}}\right)^{\kappa}\right) + c y_{n2}\right) \\ -g\left(1 - \left(\frac{x_d}{y_{n1} + \frac{h}{2} y_{n2}}\right)^{\kappa}\right) - c\left(y_{n2} - \frac{h}{2}\left(g\left(1 - \left(\frac{x_d}{y_{n1}}\right)^{\kappa}\right) - c y_{n2}\right)\right) \end{bmatrix}$$

$$y_{n+1} = y_n + h k_2$$

(d) We see that $x = x_d$ is the equilibrium point of the system.

$$\bar{x} = \begin{bmatrix} x_d \\ 0 \end{bmatrix},$$

$$\underline{\underline{A}} = \frac{\partial f}{\partial x}\Big|_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ g x_d^{\kappa} \cdot \frac{-\kappa}{x_1^{\kappa+1}} & -c \end{bmatrix}\Bigg|_{\bar{x}} = \underline{\underline{\begin{bmatrix} 0 & 1 \\ \frac{-g\kappa}{x_d} & -c \end{bmatrix}}}$$

$$\underline{\underline{\dot{x} = A x}}$$

(e) $$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ \frac{g\kappa}{x_d} & \lambda + c \end{vmatrix} = \lambda(\lambda + c) + \frac{g\kappa}{x_d} = 0$$

$$\Rightarrow \lambda = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4g\kappa/x_d}$$

i) $c = 0 \Rightarrow \underline{\lambda} = \pm \frac{1}{2}\sqrt{4g\kappa/x_d}\, j = \underline{\pm\sqrt{\frac{g\kappa}{x_d}}\, j}$

The Euler method is not stable for only imaginary eigenvalues!

ii) $R(h\lambda) = 1 + h\lambda$, $\lambda = -3{,}019$, $-5{,}9079$

$$|1 + h\lambda| \leq 1 \Rightarrow -2 < h\lambda < 0$$

$$\Rightarrow -\frac{2}{\lambda} > h > 0$$

In which the limiting factor is the eigenvalue $\lambda = -5{,}9079$, which gives us the criterion: $\underline{\underline{h < 0{,}3385}}$

$\boxed{3}$ $L\frac{di}{dt} + Ri = u, \quad i(0) = i_0$

ⓐ $u = 0$

$$\frac{di}{dt} = -\frac{R}{L} i$$

$$V(x) = \frac{1}{2} L i^2$$

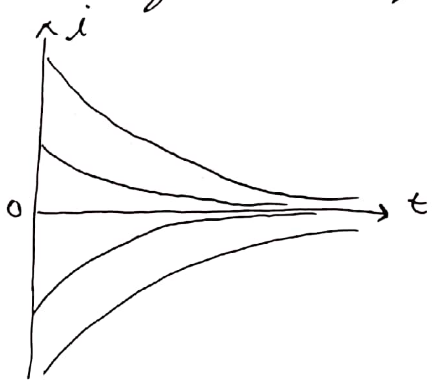
$$\dot{V} = \frac{\partial V}{\partial i} \cdot \frac{di}{dt} = Li \cdot -\frac{R}{L} i = -Ri^2$$

$$= uy - g(i)$$

where $u = 0$ and

$g(i) = Ri^2 \geq 0 \Rightarrow$ System is passive and stable when autonomous.

This is a simple first order system that decays to zero:



ⓑ $e = i - i_r$

$$V(e) = \frac{1}{2} L e^2$$

$$\dot{V} = \frac{\partial V}{\partial e} \frac{de}{dt} = Le\left(-\frac{R}{L} i + \frac{u}{L}\right)$$

$$= ue - Rei = ue - Re(e + i_R)$$

$$= ue - Re^2 - Rei_R < 0$$

$\underline{u = -ke + Ri_R} \Rightarrow \dot{V} = -(R+k)e^2 < 0$

(Which for $t \to \infty$ has $\dot{V} = 0 \Rightarrow \underline{e = 0}$)

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1.6

Note that I have plotted the simulation as well as the ode45 solution as verification.

```
k = 1;
i_0 = 0;
i_r = 0.001;
L = 0.003;
R = 2500;
t_f = 10 * 10^(-6);
h = 10 * 10^(-9);
N = t_f / h;
f = @(i) (k+R)*(i_r - i)/L;
i = (i_0);

for j = 1:N
    k_1 = f(i(j));
    k_2 = f(i(j) + k_1 * h/2);
    k_3 = f(i(j) + k_2 * h/2);
    k_4 = f(i(j) + k_3 * h);
    i(j+1) = i(j) + h*(k_1 + 2*k_2 + 2*k_3 + k_4)/6;
end

plot(i);
hold on;


[t,y] = ode45(@(t,y) f(y), [0 t_f], 0);

plot(t*100000000, y);
```

Figure 8: Matlab code for simulating the circuit using ERK4.

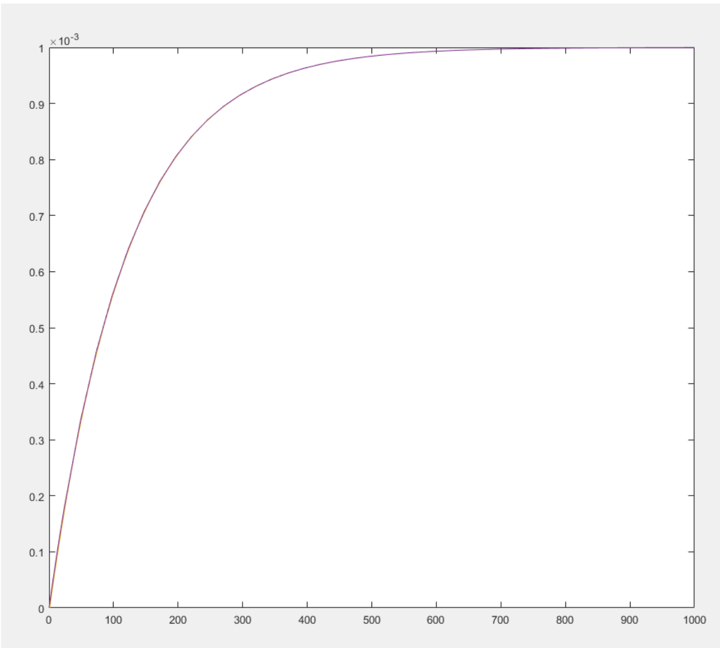

Figure 9: Simulation.

I will not bother doing the same simulation in Dymola, as it would simply be implementing a first order system and choosing the ERK4 solver...

$\boxed{4}$ $\dot{x} = ax + b$

ⓐ A linear system has the following solution:

$$\underline{x(t)} = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b \, d\tau$$

$$= e^{at}x_0 + e^{at} \cdot b \cdot -\frac{1}{a}e^{-a\tau}\Big|_0^t$$

$$= e^{at}x_0 + e^{at} \cdot -\frac{b}{a}(e^{-at} - 1)$$

$$\underline{\underline{= e^{at}x_0 - \frac{b}{a}(1 - e^{at})}}$$

ⓑ $\underline{x_{n+1}} = x_n + h(ax_n + b) = \underline{(1+ha)x_n + hb}$

$$x_{n+1} = (1+ha)x_n + hb = (1+ha)((1+ha)x_{n-1} + hb) + hb$$

$$= (1+ha)^2 x_{n-1} + (1+ha)hb + hb$$

$$= (1+ha)^3 x_{n-2} + (1+ha)^2 hb + (1+ha)hb + hb$$

$$\Rightarrow x_{n+1} = (1+ha)^{n+1}x_0 + hb\sum_{j=0}^{n}(1+ha)^j$$

$$\underline{x_n = (1+ha)^n x_0 + hb\sum_{j=0}^{n-1}(1+ha)^j} \quad \square$$

$$\sum_{j=0}^{n-1}(1+ha)^j = \frac{1-(1+ha)^n}{1-(1+ha)} = \frac{1-(1+ha)^n}{-ha}$$

$$\underline{x_n} = (1+ha)^n x_0 + hb \cdot \frac{1-(1+ha)^n}{-ha}$$

$$\underline{= (1+ha)^n x_0 + \frac{b}{a}((1+ha)^n - 1), \quad a \neq 0}$$

In the case where $a = 0$ we get:

$$\underline{x_n} = (1+ha)^n x_0 + hb\sum_{j=0}^{n-1} 1 \; \underline{= (1+ha)^n x_0 + hbn, \; a = 0}$$

ⓒ For $a = 0$ the solution obviously does not converge.

For $a \neq 0$ we get:

$$(1+ah) < 1 \quad , \quad \text{but also} \quad ah > -2$$

$$\Rightarrow \underline{\underline{a < 0}} \quad , \quad \underline{\underline{a > -\frac{2}{h}}}$$

This is to be expected, as otherwise we have exponential growth, or oscillations.

ⓓ $x_n = (1+ha)^n x_0 - \frac{b}{a}(1-(1+ha)^n)$

$$x_n = \left(1+\frac{t_n a}{n}\right)^n x_0 - \frac{b}{a}\left(1-\left(1+\frac{t_n a}{n}\right)^n\right)$$

$$\underline{\lim_{n\to\infty} x_n} = e^{at_n} x_0 - \frac{b}{a}(1-e^{at_n}),$$

$$= \underline{X(t_n)} \; ☐$$

by using the fact that

$$\lim_{n\to\infty} \left(1+\frac{t_n a}{n}\right)^n = e^{at_n}$$

ⓔ $E_n = x_n - X(t_n)$

$$= \left((1+ah)^n - e^{at_n}\right) x_0$$

$$+ \frac{b}{a}\left((1+ah)^n - 1 + 1 - e^{at_n}\right)$$

$$E_n = \left((1+ah)^n - e^{at_n}\right) x_0 + \frac{b}{a}\left((1+ah)^n - e^{at_n}\right)$$

$$\underline{E_n = \left((1+ah)^n - e^{at_n}\right)\left(x_0 + \frac{b}{a}\right)}$$

$$\lim_{h\to 0} \frac{E_n(h)}{h} = \lim_{h\to 0} \left(X_0 + \frac{b}{a}\right)\left(\frac{(1+ah)^n}{h} - \frac{e^{anh}}{h}\right) = 0$$

Where the last equality holds as $(1+ah)^n$ and e^{anh} decays much faster to 1 than h decays to zero, such that the difference $(1+ah)^n - e^{anh}$ decays much faster to 0 than h.

(f) Stability only tells us if the integration "blows up" or oscillates or not, it doesn't tell us anything about accuracy, i.e. how large the global error gets.
While the two terms both depend on h and λ of the system, they describe two completely different things.