**1a**

- $$x[n] = \sum_{i=1}^{p} A_i r_i^n + w[n] \quad , \quad n = 0, 1, \ldots, N-1 \quad ,$$

$$w \sim N(0, \sigma^2) = N(0,1)$$

For $x = H\theta + w$ the LSE-estimator is:

$$\hat{\theta} = (H^T H)^{-1} H^T x$$

In our case we have:

- $$x = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r_1 & r_2 & r_3 & \cdots & r_p \\ r_1^2 & r_2^2 & \cdots & & r_p^2 \\ \vdots & & \ddots & & \vdots \\ r_1^{N-1} & & \cdots & & r_p^{N-1} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} + w = HA + w .$$

So the estimator is:

$$\hat{A} = (H^T H)^{-1} H^T x \quad , \quad H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ \vdots & & \ddots & \vdots \\ r_1^{N-1} & & & r_p^{N-1} \end{bmatrix}$$

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$p = 2, \; r_1 = 1, \; r_2 = -1, \; N$ even

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} ,$$

$$H^T H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} ,$$

$$(H^T H)^{-1} = \begin{bmatrix} 1/N & 0 \\ 0 & 1/N \end{bmatrix} = \frac{1}{N} \mathbb{I}$$

- $$\hat{A} = (H^T H)^{-1} H^T x = \frac{1}{N} H^T x = \frac{1}{N} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$\hat{A} = \frac{1}{N} \begin{bmatrix} x_0 + x_1 + \ldots + x_{N-1} \\ x_0 + x_2 + \ldots + x_{N-2} - (x_1 + x_3 + \ldots x_{N-1}) \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} \bar{X} \\ \frac{1}{2}(\bar{X}_{even} - \bar{X}_{odd}) \end{bmatrix}$$

(where $\bar{X}_{even} = \frac{1}{\frac{1}{2}N} \sum_{even} x_i$ , $\bar{X}_{odd} = \frac{1}{\frac{1}{2}N} \sum_{odd} x_i$ )

$\boxed{1c}$ 1. $p(x[n];\mu) = \frac{1}{2} e^{-|x[n]-\mu|}$

$$E\{x[n]\} = \mu, \quad Var\{x[n]\} = 2$$

$$\hat{\theta} = \frac{s^T C_x^{-1} x}{s^T C_x^{-1} s}$$

$$s = 1, \quad C_x = 2\mathbb{I}, \quad C_x^{-1} = \frac{1}{2}\mathbb{I}$$

$$\Rightarrow \hat{\theta} = \frac{\sum_{n=0}^{N-1} \frac{1}{2} x[n]}{\sum_{n=0}^{N-1} \frac{1}{2}} = \frac{\sum_{n=0}^{N-1} x[n]}{N} = \bar{X}$$

$$\hat{\theta}_1 = \bar{X}$$

2. $p(x[n];\mu) = N(\mu, 1)$

$$s = 1, \quad C_x = \mathbb{I} = C_x^{-1}$$

$$\hat{\theta} = \frac{1^T x}{1^T 1} = \frac{\sum_{n=0}^{N-1} x[n]}{\sum_{n=0}^{N-1} 1} = \bar{X}$$

$$\hat{\theta}_2 = \bar{X}$$

Both BLUE estimators are the sample mean, which is the MVU for $\mu$.

1b

```
data = importdata('x.txt')';
time = importdata('t.txt')';
N = 100;
H = [ ones(N,1) time (arrayfun(@(t) sin(2*pi*t),time)) ];
theta = inv(H' * H) * H' * data
x_est = H * theta;
CRLB = inv(H' * H)

plot(time, data);
hold on;
grid on;
plot(time, x_est);
```

Command Window

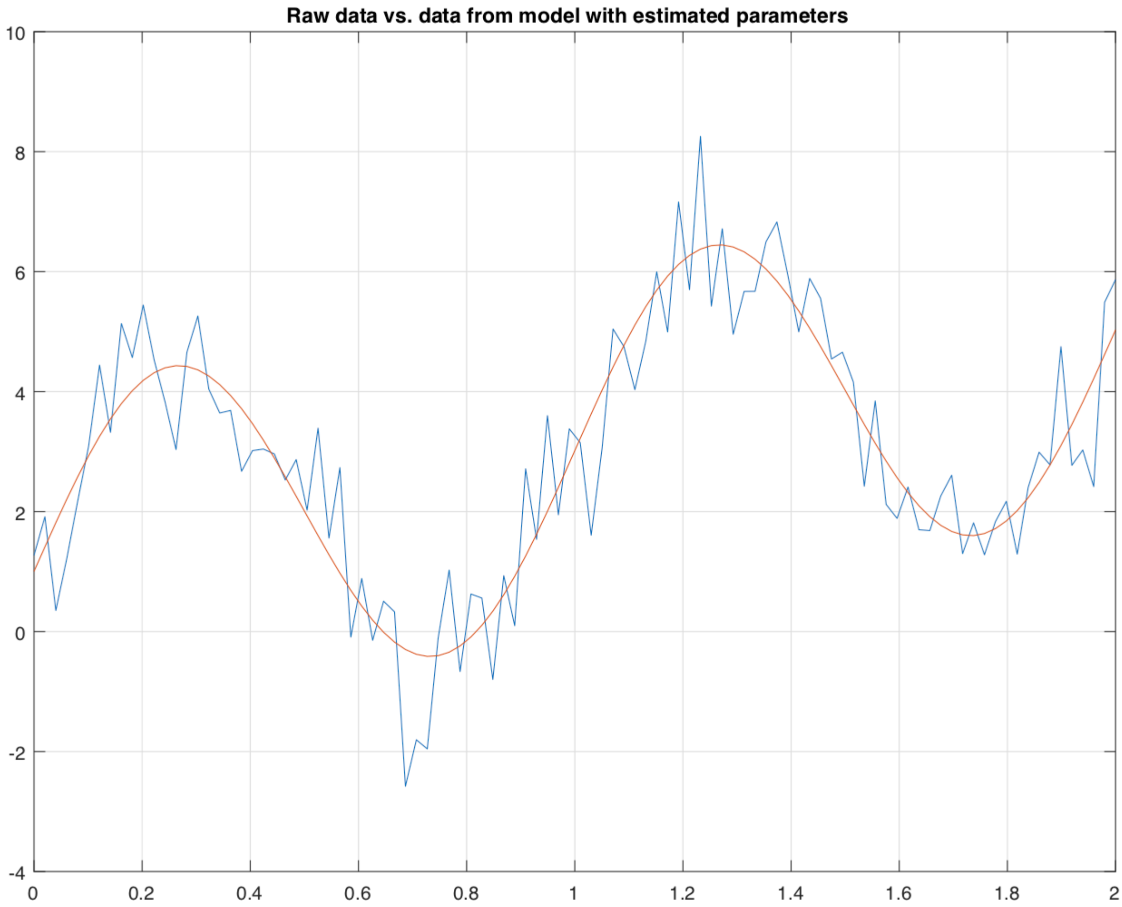
```
theta =

    1.0023
    2.0133
    2.9099


CRLB =

    0.0445   -0.0345   -0.0110
   -0.0345    0.0345    0.0110
   -0.0110    0.0110    0.0237

>>
```


Raw data vs. data from model with estimated parameters
10
8
6
4
2
0
-2
-4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

$\boxed{2a}$ $x = A + w, \quad w \sim N(0, A)$

1. $x \sim N(A, A)$

$$\underline{\underline{p(x;A) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{(x-A)^2}{2A}}}}$$

2. $L(A|x) = p(x;A)$

$$\log L(A|x) = \sum_{n=0}^{N-1} \log p(x(n);A)$$

$$= \sum_{n=0}^{N-1} \left( -\frac{1}{2} \log(2\pi A) - \frac{(x(n)-A)^2}{2A} \right)$$

$$\frac{\partial \log L(A|x)}{\partial A} = \sum_{n=0}^{N-1} \left( -\frac{1}{2A} + \frac{x(n)^2}{2} A^{-2} - \frac{1}{2} \right)$$

$$= -\frac{N}{2A} - \frac{N}{2} + \frac{1}{2A^2} \sum_{n=0}^{N-1} x(n)^2 = 0$$

$$\Rightarrow \underline{A^2 + A - \frac{1}{N} \sum x(n)^2 = 0}$$

$$\Rightarrow \underline{\underline{\hat{A} = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4}{N} \sum_{n=0}^{N-1} x(n)^2}}} \quad ?$$

3. $$\log p(x;A) = \sum_{n=0}^{N-1} \left( -\frac{1}{2} \log 2\pi A - \frac{(x(n)-A)^2}{2A} \right)$$

$$\frac{\partial \log p(x;A)}{\partial A} = \sum_{n=0}^{N-1} \left( -\frac{1}{2A} + \frac{x(n)^2}{2} \frac{1}{A^2} - \frac{1}{2} \right)$$

$$\frac{\partial^2 \log p(x;A)}{\partial A^2} = \sum_{n=0}^{N-1} \left( \frac{1}{2A^2} - x(n)^2 \frac{1}{A^3} \right) = \frac{1}{A^2} \left( \frac{N}{2} - \sum_{n=0}^{N-1} \frac{x(n)^2}{A} \right)$$

$$E\left\{ \frac{\partial^2 \log p(x;A)}{\partial A^2} \right\} = \frac{N}{2A^2} - \frac{1}{A^3} \sum_{n=0}^{N-1} E\{x(n)^2\}$$

$$\mathrm{Var}\{x(n)\} = E\{x(n)^2\} - 2A\,E\{x(n)\} + A^2 = A$$

$$\Rightarrow \underline{E\{x(n)^2\}} = A - A^2 + 2A^2 = \underline{A^2 + A}$$

$$\Rightarrow \underline{E\left\{ \frac{\partial^2 \log p(x;A)}{\partial A^2} \right\}} = \frac{N}{2A^2} - \frac{1}{A^3} N(A^2+A) = \underline{-\frac{N}{A}\left(1 + \frac{1}{2A}\right)}$$

$$\Rightarrow \underline{\underline{\mathrm{CRLB}}} = \frac{-1}{E\left\{ \frac{\partial^2 \log p(x;A)}{\partial A^2} \right\}} = \frac{1}{\frac{N}{A}\left(1 + \frac{1}{2A}\right)} = \underline{\underline{\frac{A^2}{N(A + \frac{1}{2})}}}$$

This is only valid if $E\left\{ \frac{\partial \log p(x;A)}{\partial A} \right\} = 0$.

$$\underline{E\left\{ \frac{\partial \log p(x;A)}{\partial A} \right\}} = -\frac{N}{2A} - \frac{N}{2} + \frac{1}{2A^2} \sum_{n=0}^{N-1} E\{x(n)^2\}$$

$$= -\frac{N}{2A} - \frac{N}{2} + \frac{1}{2A^2} N(A^2+A) = \frac{N}{2} + \frac{N}{2A} - \frac{N}{2} - \frac{N}{2A} = \underline{0}$$

4. $$\hat{A} = \bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x(n)$$

$$\underline{\underline{\mathrm{Var}\{\hat{A}\}}} = \frac{1}{N^2} \sum_{n=0}^{N-1} \mathrm{Var}\{x(n)\} = \underline{\underline{\frac{A}{N}}}$$

5. $$\frac{\mathrm{Var}\{\hat{A}\}}{\mathrm{CRLB}} = \frac{A}{N} \cdot \frac{N(A + \frac{1}{2})}{A^2} = 1 + \frac{1}{2A}$$

Only when $N \to \infty$ do the sample mean approach the CRLB.

$\boxed{3a}$ $p(\theta|x) = \frac{\varepsilon}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta-x)^2} + \frac{(1-\varepsilon)}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta+x)^2}$

- MAP: $\hat{\theta}_{MAP} = \max_{\theta} p(\theta|x)$

$$\log p(\theta|x) = \log \varepsilon - \frac{1}{2}\log 2\pi - \frac{1}{2}(\theta-x)^2 + \log(1-\varepsilon) - \frac{1}{2}\log 2\pi - \frac{1}{2}(\theta+x)^2$$

$$\frac{\partial \log p(\theta|x)}{\partial\theta} = -(\theta-x) - (\theta+x) = -2\theta = 0$$

$$\Rightarrow \underline{\underline{\hat{\theta}_{MAP} = 0}}$$

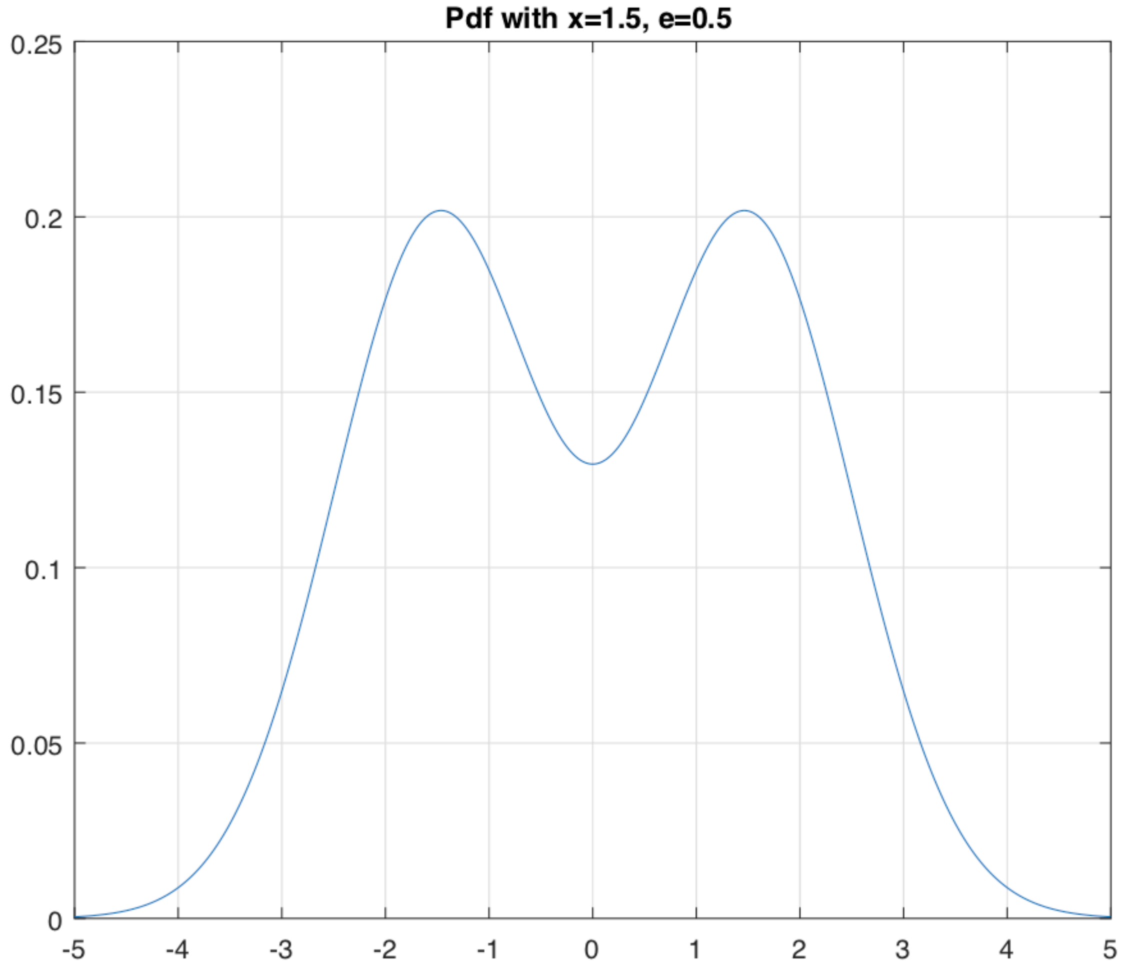
- BMSE: $\hat{\theta}_{BMSE} = \int \theta p(\theta|x)\,dx$

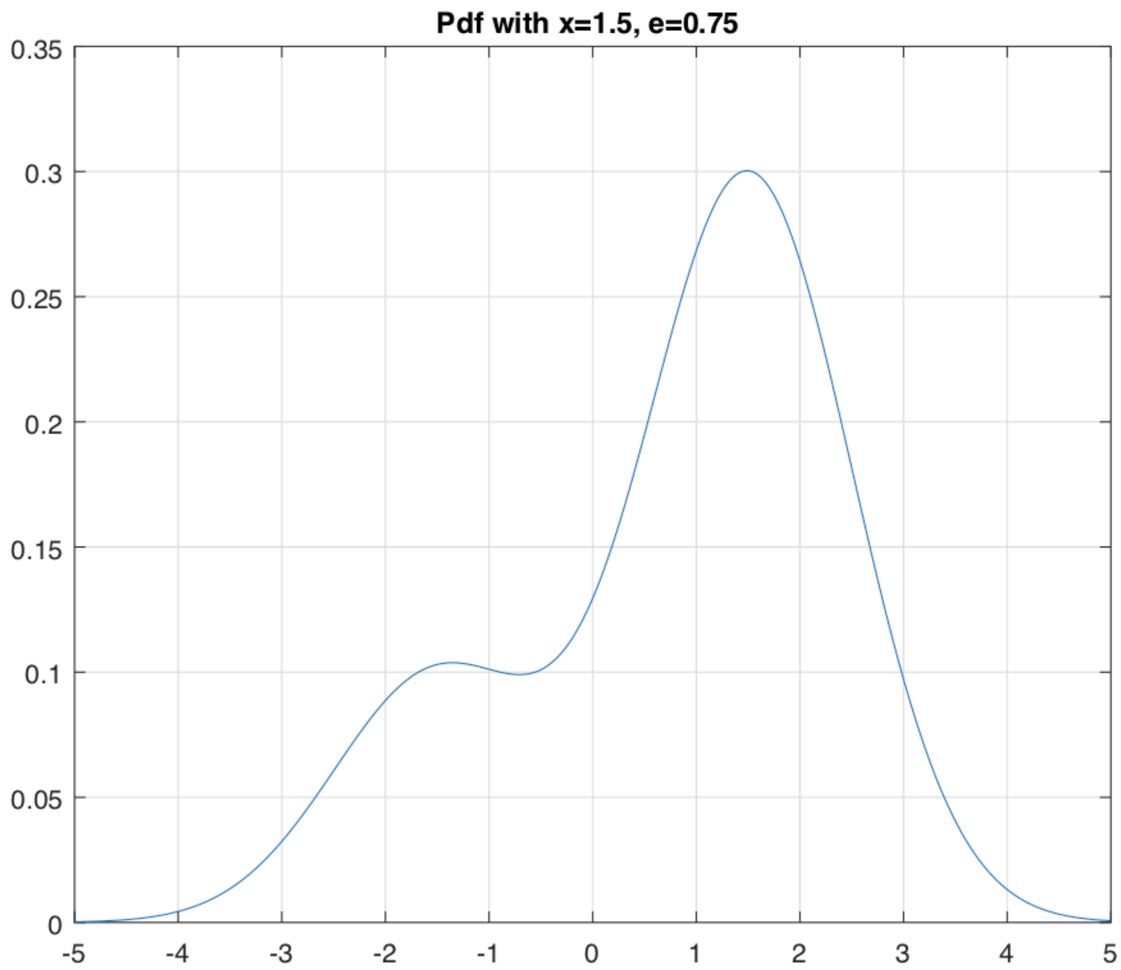
$$= \varepsilon \int \theta \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta-x)^2} d\theta + (1-\varepsilon)\int \theta \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta+x)^2} d\theta$$

$$= \varepsilon x + (1-\varepsilon)\cdot -x$$

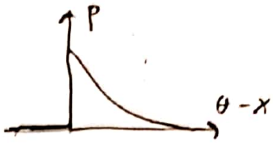
$$\underline{\underline{\hat{\theta}_{BMSE} = x(2\varepsilon-1)}}$$

- Observe that when $\varepsilon = \frac{1}{2}$ the pdf is symmetric and thus it follows that $\hat{\theta}_{MAP}(\frac{1}{2}) = \hat{\theta}_{BMSE}(\frac{1}{2})$.


Pdf with x=1.5, e=0.5
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


Pdf with x=1.5, e=0.75
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

$\boxed{3b}$ $p(\theta|x) = \begin{cases} e^{-(\theta - x)} & , \theta \geq x \\ 0 & , \theta < x \end{cases}$




$$\underline{\underline{\hat{\theta}_{MAP}}} = \arg\max_{\theta} p(\theta|x) = \underline{\underline{x}}$$

$$\underline{\underline{\hat{\theta}_{BMSE}}} = \int \theta p(\theta|x) d\theta = e^{x} \int_{0}^{\infty} \theta e^{-\theta} d\theta$$

$$= e^{x}(-\theta e^{-\theta} - e^{-\theta})\Big|_{0}^{\infty} = \underline{\underline{e^{x}}}$$