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$$r_m^i = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} = \begin{bmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{bmatrix}$$

$$c_r^i = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad c_\theta^i = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad c_z^i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$s_r^i = \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}, \quad s_\varphi^i = \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{bmatrix}, \quad s_\theta^i = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

a) $$R_c^i = [c_r^i \; c_\theta^i \; c_z^i] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is simply a rotation about the z-axis, which is right-handed.

$$R_s^i = [s_r^i \; s_\varphi^i \; s_\theta^i] = \begin{bmatrix} \sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \\ \cos\varphi & -\sin\varphi & 0 \end{bmatrix}$$

$$s_r^i \times s_\varphi^i = \begin{bmatrix} -\sin^2\varphi\sin\theta - \cos^2\varphi\sin\theta \\ \cos^2\varphi\cos\theta + \sin^2\varphi\cos\theta \\ \sin\varphi\cos\varphi\sin\theta\cos\theta - \sin\varphi\cos\varphi\sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = s_\theta^i$$

Since $s_r^i \times s_\varphi^i = s_\theta^i$ the reference frame is right-handed.

b) $$\dot{c}_r^i = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}\dot\theta, \quad \dot{c}_\theta^i = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{bmatrix}\dot\theta, \quad \dot{c}_z^i = 0$$

$$\underline{\underline{\dot{c}_r^i = c_\theta^i\dot\theta, \quad \dot{c}_\theta^i = -c_r^i\dot\theta, \quad \dot{c}_z^i = 0}}$$

$$\dot{R}_c^i = R_c^i(\omega_{ic}^c)^\times$$

$$(\omega_{ic}^c)^\times = R_c^{i\,-1}\dot{R}_c^i = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \dot\theta$$

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$$(\omega^c_{ic})^\times = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\dot\theta \iff \omega^c_{ic} = \begin{bmatrix}0\\0\\\dot\theta\end{bmatrix}$$

(c)

$$\dot s^i_r = \begin{bmatrix}\dot\varphi\cos\varphi\cos\theta - \dot\theta\sin\varphi\sin\theta\\ \dot\varphi\cos\varphi\sin\theta + \dot\theta\sin\varphi\cos\theta\\ -\dot\varphi\sin\varphi\end{bmatrix} = s^i_\varphi\dot\varphi + \begin{bmatrix}-\dot\theta\sin\varphi\sin\theta\\ \dot\theta\sin\varphi\cos\theta\\ 0\end{bmatrix}$$

$$\dot s^i_\varphi = \begin{bmatrix}-\dot\varphi\sin\varphi\cos\theta - \dot\theta\cos\varphi\sin\theta\\ -\dot\varphi\sin\varphi\sin\theta + \dot\theta\cos\varphi\cos\theta\\ -\dot\varphi\cos\varphi\end{bmatrix} = -s^i_r\dot\varphi + \begin{bmatrix}-\dot\theta\cos\varphi\sin\theta\\ \dot\theta\cos\varphi\cos\theta\\ 0\end{bmatrix}$$

$$\dot s^i_\theta = \begin{bmatrix}-\dot\theta\cos\theta\\ -\dot\theta\sin\theta\\ 0\end{bmatrix} = \dot\theta\begin{bmatrix}-\cos\theta\\ -\sin\theta\\ 0\end{bmatrix}$$

$$\dot s^i_r = s^i_\varphi\dot\varphi + \dot\theta\sin\varphi\, s^i_\theta, \quad \dot s^i_\varphi = -s^i_r\dot\varphi + \dot\theta\cos\varphi\, s^i_\theta$$

Let $\dot R^i_s = \overbrace{\begin{bmatrix}s^i_\varphi & -s^i_r & 0\end{bmatrix}}^{A}\dot\varphi + \overbrace{\begin{bmatrix}s\varphi\, s^i_\theta & c\varphi\, s^i_\theta & \begin{smallmatrix}-c\theta\\-s\theta\\0\end{smallmatrix}\end{bmatrix}}^{B}\dot\theta$

$$(\omega^s_{is})^\times = R^{iT}_s\dot R^i_s = R^{iT}_s A\dot\varphi + R^{iT}_s B\dot\theta$$

$$R^{iT}_s A = \begin{bmatrix}s\varphi c\theta & s\varphi s\theta & c\varphi\\ c\varphi c\theta & c\varphi s\theta & -s\varphi\\ -s\theta & c\theta & 0\end{bmatrix}\cdot\begin{bmatrix}c\varphi c\theta & -s\varphi c\theta & 0\\ c\varphi s\theta & -s\varphi s\theta & 0\\ -s\varphi & -c\varphi & 0\end{bmatrix}$$

$$= \begin{bmatrix}s\varphi c\varphi - s\varphi c\varphi & -s\varphi^2 - c\varphi^2 & 0\\ c\varphi^2 + s\varphi s\varphi & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

$$R_s^{i\,T} B = \begin{bmatrix} s\varphi c\theta & s\varphi s\theta & c\varphi \\ c\varphi c\theta & c\varphi s\theta & -s\varphi \\ -s\theta & c\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} -s\varphi s\theta & -c\varphi s\theta & -c\theta \\ s\varphi c\theta & c\varphi c\theta & -s\theta \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -s\varphi \\ 0 & 0 & -c\varphi \\ s\varphi & c\varphi & 0 \end{bmatrix}$$

$$(\omega_{is}^{s})^{\times} = \begin{bmatrix} 0 & -\dot{\varphi} & -\dot{\theta}s\varphi \\ \dot{\varphi} & 0 & -\dot{\theta}c\varphi \\ \dot{\theta}s\varphi & \dot{\theta}c\varphi & 0 \end{bmatrix} \iff \omega_{is}^{s} = \begin{bmatrix} \dot{\varphi} \\ -\dot{\theta}s\varphi \\ \dot{\theta}c\varphi \end{bmatrix}$$

d) $$\underline{r_m} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} = r\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{r c_r + z c_z} \quad \square$$

$$V_m = \dot{r}c_r + r\dot{c}_r + \dot{z}c_z + z\dot{c}_z$$

$$\underline{V_m = \dot{r}c_r + c_\theta^i \dot{\theta} r + \dot{z}c_z} \quad \square$$

$$a_m = \ddot{r}c_r^i + \dot{r}\dot{c}_r^i + \dot{c}_\theta^i\dot{\theta}r + c_\theta^i(\ddot{\theta}r + \dot{\theta}\dot{r}) + \ddot{z}c_z^i + \dot{z}\dot{c}_z^i$$

$$= \ddot{r}c_r^i + \dot{r}\dot{\theta}c_\theta^i - c_r^i\dot{\theta}^2 r + r\ddot{\theta}c_\theta^i + \dot{\theta}\dot{r}c_\theta^i + \ddot{z}c_z^i$$

$$\underline{a_m = (\ddot{r} - r\dot{\theta}^2)c_r^i + (2\dot{r}\dot{\theta} + r\ddot{\theta})c_\theta^i + \ddot{z}c_z^i} \quad \square$$

e)

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$$\underline{r_m} = \begin{bmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{bmatrix} = \underline{r\, s^i_r} \quad \square$$

$$V_m = \dot{r}\, s^i_r + r\,\dot{s}^i_r = \dot{r}\, s^i_r + r\left(s^i_\varphi \dot{\varphi} + \dot{\theta}\sin\varphi\, s^i_\theta \right)$$

$$\underline{V_m = \dot{r}\, s^i_r + r\dot{\varphi}\, s^i_\varphi + r\dot{\theta}\sin\varphi\, s^i_\theta} \quad \square$$

$$\begin{aligned} a_m = {} & \ddot{r}\, s^i_r + \dot{r}\,\dot{s}^i_r + \dot{r}\dot{\varphi}\, s^i_\varphi + r\left(\ddot{\varphi}\, s^i_\varphi + \dot{\varphi}\,\dot{s}^i_\varphi\right) \\ & + \dot{r}\dot{\theta}\sin\varphi\, s^i_\theta + r\left(\ddot{\theta}\sin\varphi\, s^i_\theta + \dot{\theta}\left(\dot{\varphi}\cos\varphi\, s^i_\theta + \sin\varphi\,\dot{s}^i_\theta\right)\right) \\ = {} & \ddot{r}\, s^i_r + \dot{r}\,\dot{s}^i_r + \left(\dot{r}\dot{\varphi} + r\ddot{\varphi} \cancel{+\ddot{\varphi}}\right) s^i_\varphi + r\dot{\varphi}\,\dot{s}^i_\varphi \\ & + \left(\dot{r}\dot{\theta}\sin\varphi + r\ddot{\theta}\sin\varphi + r\dot{\theta}\dot{\varphi}\cos\varphi\right) s^i_\theta + r\dot{\theta}\sin\varphi\,\dot{s}^i_\theta \\ = {} & \ddot{r}\, s^i_r + \dot{r}\left(s^i_\varphi\dot{\varphi} + \dot{\theta}\sin\varphi\, s^i_\theta\right) + \left(\dot{r}\dot{\varphi} + r\ddot{\varphi}\right) s^i_\varphi \\ & + r\dot{\varphi}\left(\dot{\theta}\cos\varphi\, s^i_\theta - \dot{\varphi}\, s^i_r\right) + \left(\dot{r}\dot{\theta}\sin\varphi + r\ddot{\theta}\sin\varphi + r\dot{\theta}\dot{\varphi}\cos\varphi\right) s^i_\theta \\ & + r\dot{\theta}\sin\varphi\,\dot{s}^i_\theta \\ = {} & \left(\ddot{r} - r\dot{\varphi}^2\right) s^i_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right) s^i_\varphi \\ & + \left(r\ddot{\theta}\sin\varphi + 2\dot{r}\dot{\theta}\sin\varphi + 2r\dot{\varphi}\dot{\theta}\cos\varphi\right) s^i_\theta + r\dot{\theta}\sin\varphi\,\dot{s}^i_\theta \end{aligned}$$

Now observe that:

$$\dot{s}^i_\theta = -\dot{\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = -\dot{\theta}\left(\sin\varphi \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} + \cos\varphi \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{bmatrix} \right)$$

$$= -\dot{\theta}\begin{bmatrix} \cos\theta\left(\sin^2\varphi + \cos^2\varphi\right) \\ \sin\theta\left(\sin^2\varphi + \cos^2\varphi\right) \\ \sin\varphi\cos\varphi - \sin\varphi\cos\varphi \end{bmatrix} = -\dot{\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

Such that:

$$\dot{s}^i_\theta = -\dot{\theta}\left(\sin\varphi\, s^i_r + \cos\varphi\, s^i_\varphi\right)$$

$$\Rightarrow \quad r\dot{\theta}\sin\varphi\,\dot{s}^i_\theta = -r\dot{\theta}^2\sin^2\varphi\, s^i_r - r\dot{\theta}^2\sin\varphi\cos\varphi\, s^i_\varphi$$

$$\Rightarrow \quad \underline{\begin{aligned} a_m = {} & \left(\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\sin^2\varphi\right) s^i_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} - r\dot{\theta}^2\sin\varphi\cos\varphi\right) s^i_\varphi \\ & + \left(r\ddot{\theta}\sin\varphi + 2\dot{r}\dot{\theta}\sin\varphi + 2r\dot{\varphi}\dot{\theta}\cos\varphi\right) s^i_\theta \end{aligned}} \quad \square$$

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(f) $m a_m = N - mg e_3$

Let r be fixed at $r = R$.

$\Rightarrow \dot{r} = 0$

$\Rightarrow m a_m = \big((-R\dot{\varphi}^2 - R\sin^2\varphi\,\dot{\theta}^2) s_r^i$

$+ (R\ddot{\varphi} - R\sin\varphi\cos\varphi\,\dot{\theta}^2) s_\varphi^i + (R\sin\varphi\,\ddot{\theta} + 2R\dot{\varphi}\cos\varphi\,\dot{\theta}) s_\theta^i\big) \cdot m$

$$e_3^s = R_i^s e_3^i = \begin{bmatrix} c\varphi \\ -s\varphi \\ 0 \end{bmatrix}$$

$$\underline{N} = m\Big(\underline{(-R\dot{\varphi}^2 - R\sin^2\varphi\,\dot{\theta}^2) s_r^i + (R\ddot{\varphi} - R\sin\varphi\cos\varphi\,\dot{\theta}^2) s_\varphi^i}$$

$$\underline{+ (R\sin\varphi\,\ddot{\theta} + 2R\dot{\varphi}\cos\varphi\,\dot{\theta}) s_\theta^i + \begin{bmatrix} g\cos\varphi \\ -g\sin\varphi \\ 0 \end{bmatrix}\Big)}$$

(g) $V_m = R(\dot{\varphi} s_\varphi^i + \sin\varphi\,\dot{\theta} s_\theta^i)$

$T = \frac{1}{2} m V_m^T V_m = \frac{1}{2} m (\dot{\varphi}^2 + \sin^2\varphi\,\dot{\theta}^2) R^2$

$U = mgh \cancel{} = mgR\sin\varphi$

$\mathcal{L} = \frac{1}{2} m R^2 (\dot{\varphi}^2 + \sin^2\varphi\,\dot{\theta}^2) - mgR\sin\varphi$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = mR^2 \sin\varphi\cos\varphi\,\dot{\theta}^2 - mgR\cos\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mR^2\dot{\varphi}, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mR^2\ddot{\varphi}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\sin^2\varphi\,\dot{\theta}, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2(2\sin\varphi\cos\varphi\,\dot{\varphi}\dot{\theta} + \sin^2\varphi\,\ddot{\theta})$$

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$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} - \frac{\partial \mathcal{L}}{\partial \theta} = mR^2(2\sin\varphi\cos\varphi\,\dot\varphi\dot\theta + \sin^2\varphi\,\ddot\theta) = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = mR^2\ddot\varphi + mgR\cos\varphi - mR^2\sin\varphi\cos\varphi\,\dot\theta^2 = 0$$

$$\ddot\theta = \frac{-2\cos\varphi\,\dot\varphi\dot\theta}{\sin\varphi} = -\frac{2\dot\varphi\dot\theta}{\tan\varphi}$$

$$\ddot\varphi = \frac{1}{R}\left(R\sin\varphi\cos\varphi\,\dot\theta^2 - g\cos\varphi\right) = \sin\varphi\cos\varphi\,\dot\theta^2 - \frac{g}{R}\cos\varphi$$

(h) $$r_m = \begin{bmatrix} R\cos\theta/\sqrt{1+a^2\theta^2} \\ R\sin\theta/\sqrt{1+a^2\theta^2} \\ Ra\theta/\sqrt{1+a^2\theta^2} \end{bmatrix} = \begin{bmatrix} R\sin\varphi\cos\theta \\ R\sin\varphi\sin\theta \\ R\cos\varphi \end{bmatrix}$$

$$\Rightarrow \varphi = \arcsin\left(\frac{1}{\sqrt{1+a^2\theta^2}}\right)$$

(i) $$\sin\varphi = \frac{1}{\sqrt{1+a^2\theta^2}}, \quad \cos\varphi = \frac{a\theta}{\sqrt{1+a^2\theta^2}}, \quad \tan\varphi = \frac{1}{a\theta}$$

$$\frac{d}{dt}\sin\varphi = \dot\varphi\cos\varphi = \frac{a\theta\dot\varphi}{\sqrt{1+a^2\theta^2}} = -\frac{1}{2}(1+a^2\theta^2)^{-\frac{3}{2}}\cdot 2a^2\theta\dot\theta$$

$$\Rightarrow \dot\varphi = \frac{-a\dot\theta}{1+a^2\theta^2}, \quad \ddot\varphi = \frac{-a\ddot\theta(1+a^2\theta^2) + a\dot\theta\cdot 2a^2\theta\dot\theta}{(1+a^2\theta^2)^2}$$

Assuming the equations above are correct we get:

$$\ddot\theta = -2\dot\varphi\dot\theta\,a\theta = \frac{2a^2\theta\dot\theta^2}{1+a^2\theta^2}$$

Which can't possibly be correct. Don't know what's wrong tho...

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(i) $T = \frac{1}{2}m(\dot{\varphi}^2 + \sin^2\varphi\ \dot{\theta}^2)R^2$,

$U = mgR\sin\varphi$

Constraint:

$$f(\theta,\varphi) = \sin\varphi - \frac{1}{\sqrt{1+a^2\theta^2}} = 0$$

$$\frac{\partial f}{\partial\theta} = a^2\theta\cancel{\varphi}(1+a^2\theta^2)^{-\frac{3}{2}}, \quad \frac{\partial f}{\partial\varphi} = \cos\varphi$$

$$\mathcal{L} = \frac{1}{2}mR^2(\dot{\varphi}^2 + \dot{\theta}^2\sin^2\varphi) - mgR\sin\varphi$$

All the calculations are the same, but we use Lagrangian of first kind:

$$(*)\quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} - \lambda\frac{\partial f}{\partial\theta} = mR^2\sin^2\varphi\,\ddot{\theta} + 2mR^2\sin\varphi\cos\varphi\,\dot{\varphi}\dot{\theta}$$

$$-\lambda a^2\theta(1+a^2\theta^2)^{-\frac{3}{2}} = 0$$

$$(**)\quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} - \lambda\frac{\partial f}{\partial\varphi} = mR^2\ddot{\varphi} + mgR\cos\varphi - mR^2\sin\varphi\cos\varphi\,\dot{\theta}^2$$

$$-\lambda\cos\varphi = 0$$

$$\rightarrow (*)\quad \frac{mR^2\ddot{\theta}}{1+a^2\theta^2} + \frac{2mR^2a\theta\dot{\varphi}\dot{\theta}}{1+a^2\theta^2} - \frac{\lambda a^2\theta}{\sqrt{1+a^2\theta^2}(1+a^2\theta^2)} = 0$$

$$mR^2\sqrt{1+a^2\theta^2}(\ddot{\theta} + 2a\theta\dot{\varphi}\dot{\theta}) - \lambda a^2\theta = 0$$

$$\rightarrow (**)\quad \cancel{m\varphi}\ mR^2\ddot{\varphi} + \frac{mgRa\theta}{\sqrt{1+a^2\theta^2}} - \frac{mR^2\dot{\theta}^2a\theta}{1+a^2\theta^2} - \frac{\lambda a\theta}{\sqrt{1+a^2\theta^2}} = 0$$

$$\lambda = \frac{1}{a\theta}\left(mR^2\ddot{\varphi}\sqrt{1+a^2\theta^2} + mgRa\theta - \frac{mR^2\dot{\theta}^2a\theta}{\sqrt{1+a^2\theta^2}}\right)$$

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$$mR^2\sqrt{1+a^2\theta^2}\,(\ddot\theta + 2a\theta\dot\varphi\dot\theta) - \lambda a^2\theta = 0$$

$$\ddot\theta + 2a\theta\dot\varphi\dot\theta - \frac{\lambda a^2\theta}{mR^2\sqrt{1+a^2\theta^2}} = 0$$

$$\ddot\theta + 2a\theta\dot\theta\cdot\frac{-a\dot\theta}{1+a^2\theta^2} - \frac{a^{\cancel{2}}\cancel{\theta}}{\cancel{m}R^{\cancel{2}}\sqrt{1+a^2\theta^2}}\cdot\frac{1}{a\cancel{\theta}}\left(\cancel{m}R^{\cancel{2}}\ddot\varphi\sqrt{1+a^2\theta^2} + \cancel{m}gR a\theta - \frac{\cancel{m}R^{\cancel{2}}\dot\theta^2 a\theta}{\sqrt{1+a^2\theta^2}}\right) = 0$$

$$\ddot\theta - \frac{2a^2\theta\dot\theta^2}{1+a^2\theta^2} - \frac{a}{\sqrt{1+a^2\theta^2}}\left(\frac{-a\ddot\theta(1+a^2\theta^2)+a^3\dot\theta^2 2\theta}{(1+a^2\theta^2)^2}\sqrt{1+a^2\theta^2} + \frac{g}{R}a\theta - \frac{a\theta\dot\theta^2}{\sqrt{1+a^2\theta^2}}\right) = 0$$

$$\ddot\theta\left(1+\frac{a^2}{1+a^2\theta^2}\right) = \frac{2a^2\theta\dot\theta^2}{1+a^2\theta^2} + \frac{2a^4\theta\dot\theta^2}{(1+a^2\theta^2)^2} - \frac{g}{R}\frac{a^2\theta}{\sqrt{1+a^2\theta^2}} + \frac{a^2\theta\dot\theta^2}{1+a^2\theta^2}$$

$$\ddot\theta\,(1+a^2+a^2\theta^2) = 2a^2\theta\dot\theta^2 - 2a^4\theta\dot\theta^2/(1+a^2\theta^2) - \frac{g}{R}a^2\theta\sqrt{1+a^2\theta^2} + a^2\theta\dot\theta^2$$

$$\ddot\theta = \frac{3a^2\theta\dot\theta^2(1+a^2\theta^2) - 2a^4\theta\dot\theta^2}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)} - \frac{g}{R}\frac{a^2\theta\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2}$$

$$= \frac{3a^2\theta\dot\theta^2 + 3a^4\theta^3\dot\theta^2 - 2a^4\theta\dot\theta^2}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)} - \frac{g}{R}\frac{a^2\theta\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2}$$

$$\ddot\theta = \frac{a^2\theta(3+3a^2\theta^2-2a^2)}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)}\dot\theta^2 - \frac{g}{R}\frac{a^2\theta\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2}$$

Close enough... It is honestly impossible to find algebra errors when you only show the result in the last task...

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j) $r_m = \begin{bmatrix} \cos\theta \\ \sin\theta \\ a\theta \end{bmatrix} \cdot \frac{R}{\sqrt{1+a^2\theta^2}}$

$$V_m = \frac{R}{(1+a^2\theta^2)^{3/2}} \begin{bmatrix} -(1+a^2\theta^2)\sin\theta\,\dot\theta - \cos\theta\, a^2\theta\dot\theta \\ (1+a^2\theta^2)\cos\theta\,\dot\theta - \sin\theta\, a^2\theta\dot\theta \\ a\dot\theta(1+a^2\theta^2) - a^3\theta^2\dot\theta \end{bmatrix}$$

$$= \frac{R\dot\theta}{(1+a^2\theta^2)^{3/2}} \begin{bmatrix} -\sin\theta - a^2\sin\theta\,\theta^2 - \cos\theta\, a^2\theta \\ \cos\theta + a^2\theta^2\cos\theta - a^2\theta\sin\theta \\ a + a^3\theta^2 - a^3\theta^2 \end{bmatrix}$$

$$F_f = \frac{Rk\dot\theta}{(1+a^2\theta^2)^{3/2}} \begin{bmatrix} a^2\theta^2\sin\theta + a^2\theta\cos\theta + \sin\theta \\ -a^2\theta^2\cos\theta + a^2\theta\sin\theta - \cos\theta \\ a \end{bmatrix}$$

Lagrangian is the same with generalized force applied:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial\dot\theta} - \frac{\partial \mathcal{L}}{\partial\theta} - \lambda\frac{\partial f}{\partial\theta} = F_{f\theta}$$

Don't know if this is correct, but I will try to find the θ-component in the sphere frame:

$$F_f^S = R_i^S F_f = \begin{bmatrix} s\varphi c\theta & s\varphi s\theta & c\varphi \\ c\varphi c\theta & c\varphi s\theta & -s\varphi \\ -s\theta & c\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} a^2\theta^2 s\theta + a^2\theta c\theta + s\theta \\ -a^2\theta^2 c\theta + a^2\theta s\theta - c\theta \\ a \end{bmatrix} \frac{Rk\dot\theta}{(1+a^2\theta^2)^{3/2}}$$

$$= \frac{Rk\dot\theta}{(1+a^2\theta^2)^{3/2}} \begin{bmatrix} * \\ * \\ -s\theta(a^2\theta^2 s\theta + a^2\theta c\theta + s\theta) + c\theta(a^2\theta s\theta - a^2\theta^2 c\theta - c\theta) \end{bmatrix}$$

$$= \frac{Rk\dot\theta}{(1+a^2\theta^2)^{3/2}} \begin{bmatrix} * \\ * \\ -a^2\theta^2 - 1 \end{bmatrix}$$

In the prior calculations we multiplied by $\frac{1}{mR^2} \cdot \frac{1}{1+a^2+a^2\theta^2}$.

I think we need to do this first tho:

$$mR^2\ddot{\varphi} + \frac{mgRa\theta}{\sqrt{1+a^2\theta^2}} - \frac{mR^2\dot{\theta}^2 a\theta}{1+a^2\theta^2} - \frac{\lambda a\theta}{\sqrt{1+a^2\theta^2}} = \cancel{\theta} F_{f\varphi}$$

$$F_{f\varphi} = \frac{Rk\dot{\theta}}{(1+a^2\theta^2)^{3/2}} \big(c\varphi c\theta (a^2\theta^2 s\theta + a^2\theta c\theta + s\theta) + c\varphi s\theta(-a^2\theta^2 c\theta + a^2\theta s\theta - c\theta) - s\varphi a \big)$$

$$= \frac{Rk\dot{\theta}}{(1+a^2\theta^2)^{3/2}} (a^2\theta c\varphi - s\varphi a) = \frac{Rk\dot{\theta}(a^3\theta^2 - a)}{(1+a^2\theta^2)^2}$$

$$\Rightarrow \lambda = \frac{1}{a\theta}\left(mR^2\ddot{\varphi}\sqrt{1+a^2\theta^2} + mgRa\theta - \frac{mR^2\dot{\theta}^2 a\theta}{\sqrt{1+a^2\theta^2}} - \frac{Rk\dot{\theta}(a^3\theta^2 - a)}{(1+a^2\theta)^{3/2}} \right)$$

$$\ddot{\theta} + 2a\theta\dot{\varphi}\dot{\theta} - \frac{\lambda a^2\theta}{mR^2\sqrt{1+a^2\theta^2}} = \frac{-Rk\dot{\theta}}{(1+a^2\theta^2)^{3/2}}(a^2\theta^2+1)\cdot\frac{(a^2\theta^2+1)}{mR^2}$$

$$= -\frac{k\dot{\theta}}{Rm}\sqrt{1+a^2\theta^2}$$

$$\ddot{\theta} + 2a\theta\dot{\theta}\cdot\frac{-a\dot{\theta}}{1+a^2\theta^2} - \frac{a^2\theta}{mR^2\sqrt{1+a^2\theta^2}}\cdot\frac{1}{a\theta}\left(mR^2\ddot{\varphi}\sqrt{1+a^2\theta^2} + mgRa\theta - \frac{mR^2\dot{\theta}^2 a\theta}{\sqrt{1+a^2\theta^2}} - \frac{Rk\dot{\theta}(a^2\theta^2-1)a}{(1+a^2\theta^2)^2} \right)$$

$$= \ddot{\theta} \cancel{+} - \frac{2a^2\theta\dot{\theta}^2}{1+a^2\theta^2} - \frac{a}{mR^2\sqrt{1+a^2\theta^2}}\left(mR^2\,\frac{-a\ddot{\theta}(1+a^2\theta^2) + a^3\dot{\theta}^2 2\theta}{(1+a^2\theta^2)^{3/2}} + mgRa\theta - \frac{mR^2 a\theta\dot{\theta}^2}{\sqrt{1+a^2\theta^2}} + \frac{Rka\dot{\theta}(1-a^2\theta^2)}{(1+a^2\theta^2)^2} \right) = -\frac{k\dot{\theta}}{Rm}\sqrt{1+a^2\theta^2}$$

$$\ddot{\theta}(1+a^2+a^2\theta^2) = (1+a^2\theta^2)\left(\frac{2a^2\theta\dot{\theta}^2}{1+a^2\theta^2} + \frac{2a^4\theta\dot{\theta}^2}{(1+a^2\theta^2)^2} - \frac{g}{R}\frac{a^2\theta}{\sqrt{1+a^2\theta^2}} + \frac{a^2\theta\dot{\theta}^2}{1+a^2\theta^2} \cancel{\ } - \frac{\cancel{R}ka^2\dot{\theta}(1-a^2\theta^2)}{mR(1+a^2\theta^2)^{5/2}} - \frac{k\dot{\theta}}{Rm}\sqrt{1+a^2\theta^2} \right)$$

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$$\ddot{\theta} = \frac{a^2\theta(3+3a^2\theta^2-2a^2)}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)}\dot{\theta}^2 - \frac{g}{R}\frac{a^2\theta\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2}$$

$$-\frac{ka^2\dot{\theta}(1-a^2\theta^2)}{mR(1+a^2+a^2\theta^2)(1+a^2\theta^2)^{5/2}} - \frac{k\dot{\theta}\sqrt{1+a^2\theta^2}}{mR(1+a^2+a^2\theta^2)}$$

Yeah this is obviously incorrect so I will throw in the towel.

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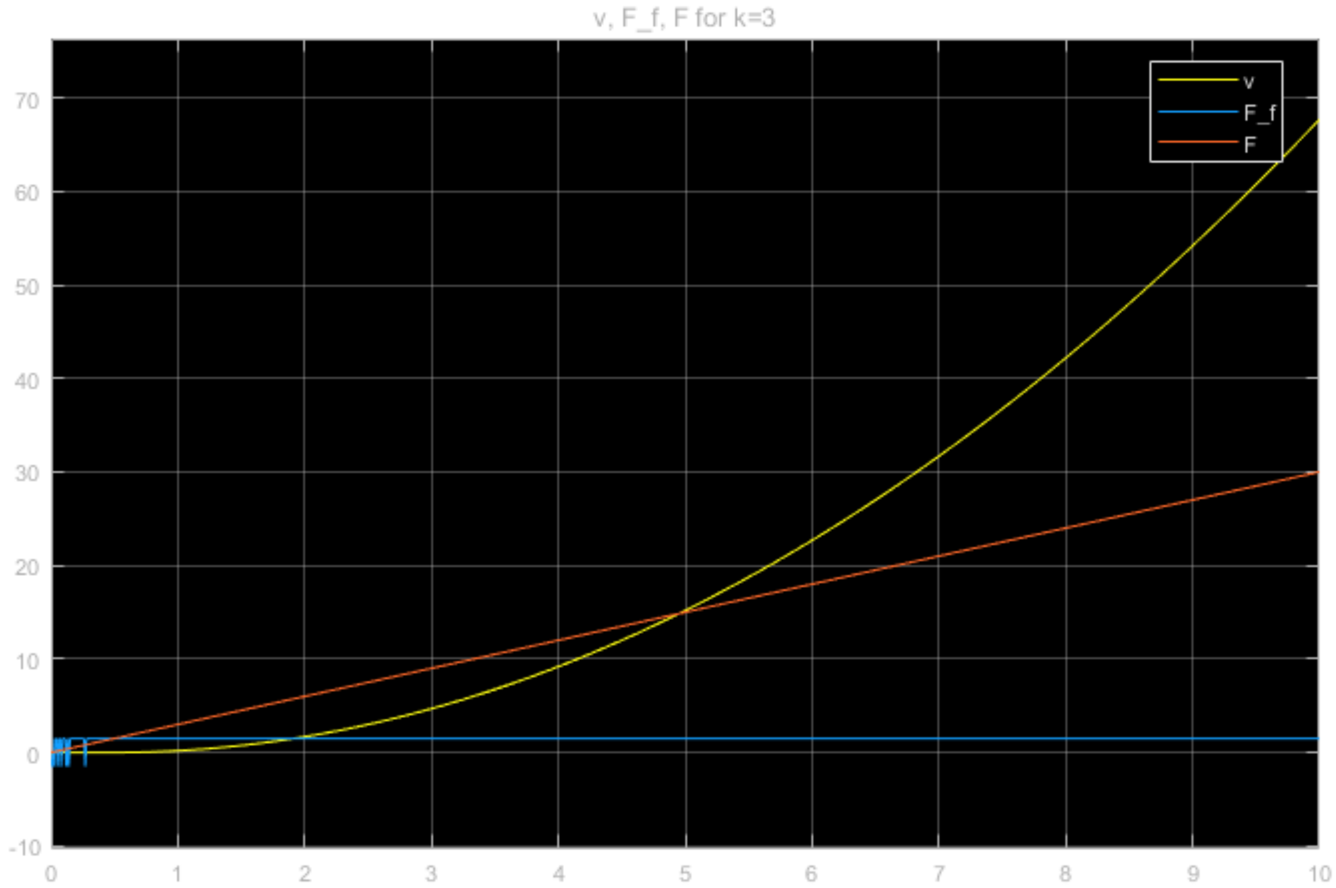
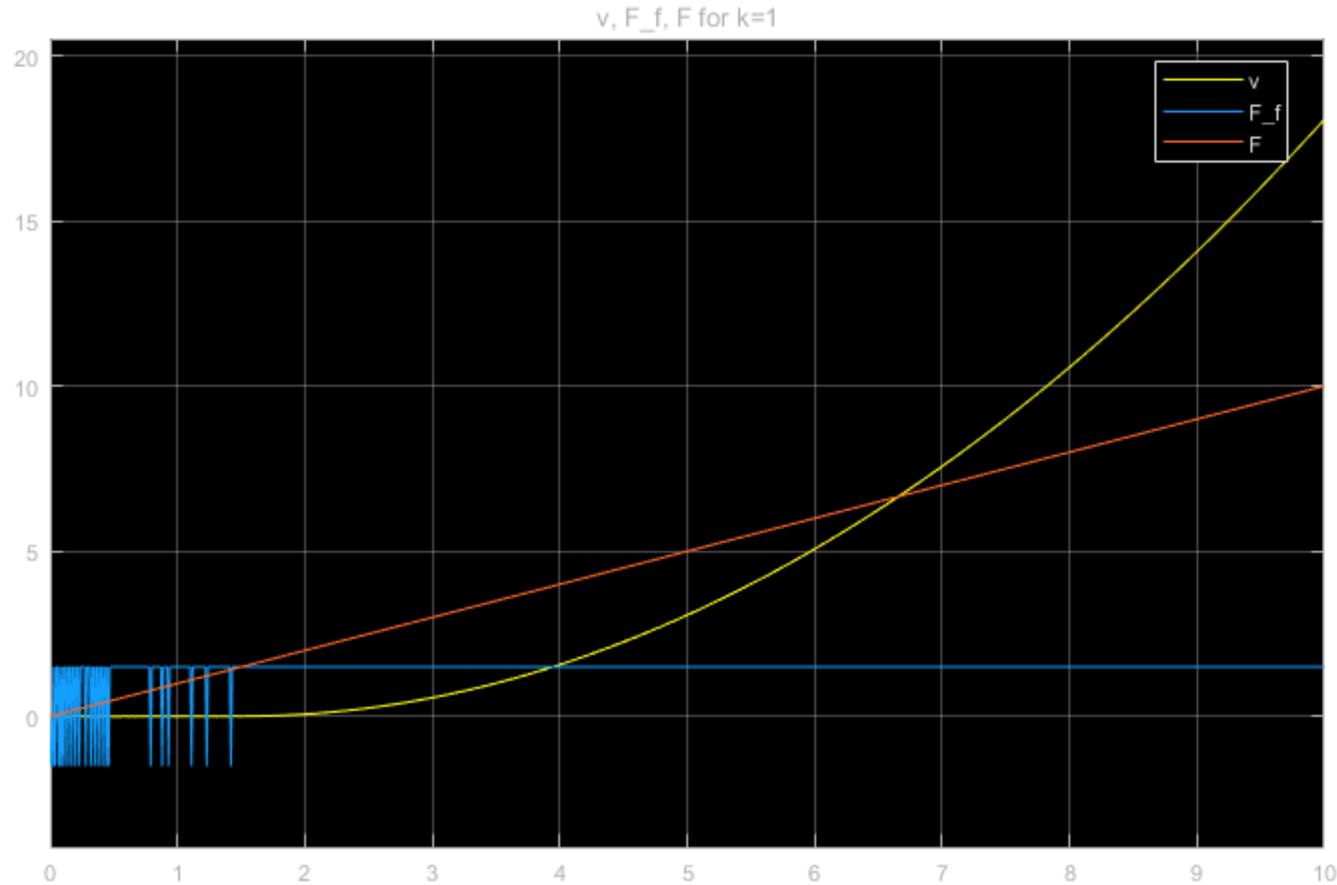
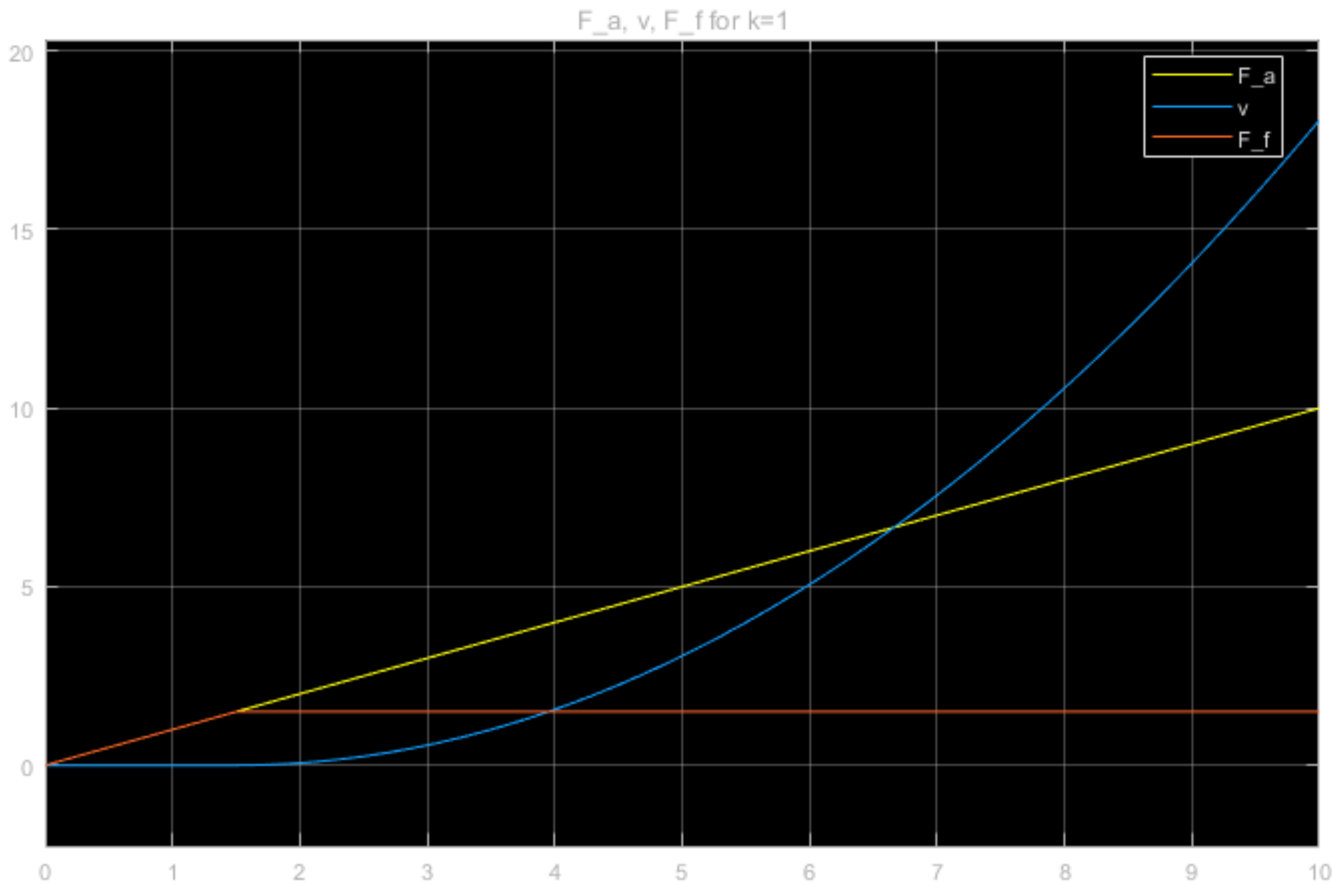
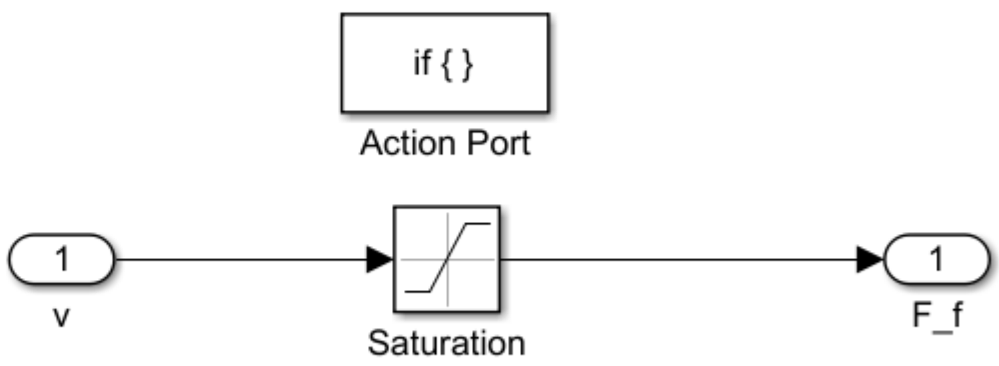
2a
Since we have discontinuities at v = 0 the variable step solver uses very small time steps in the beginning, as the Coulomb model is not defined at v=0.
When using a fixed step solver we still have the problem with discontinuities at v=0.

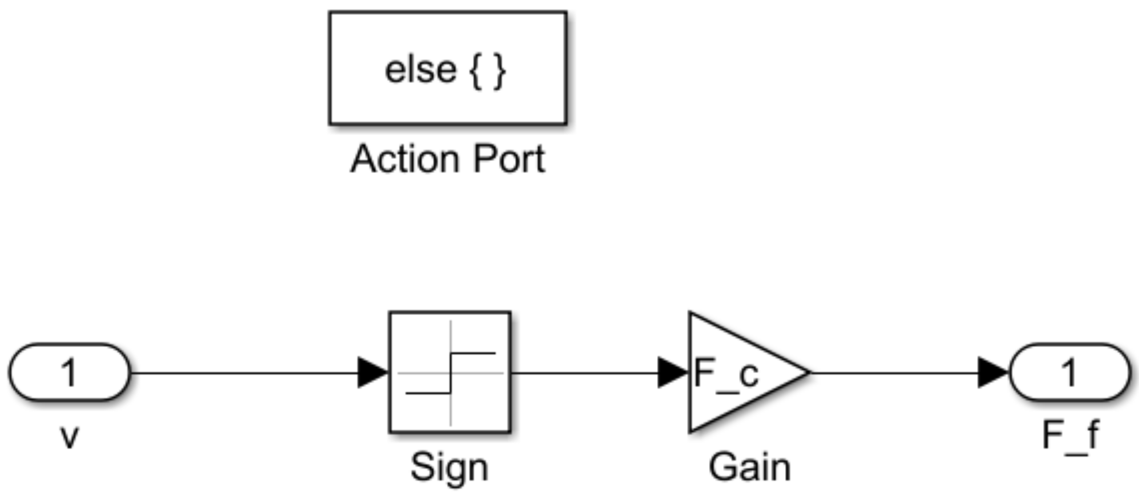



v, F_f, F for k=1
v
F_f
F
20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10

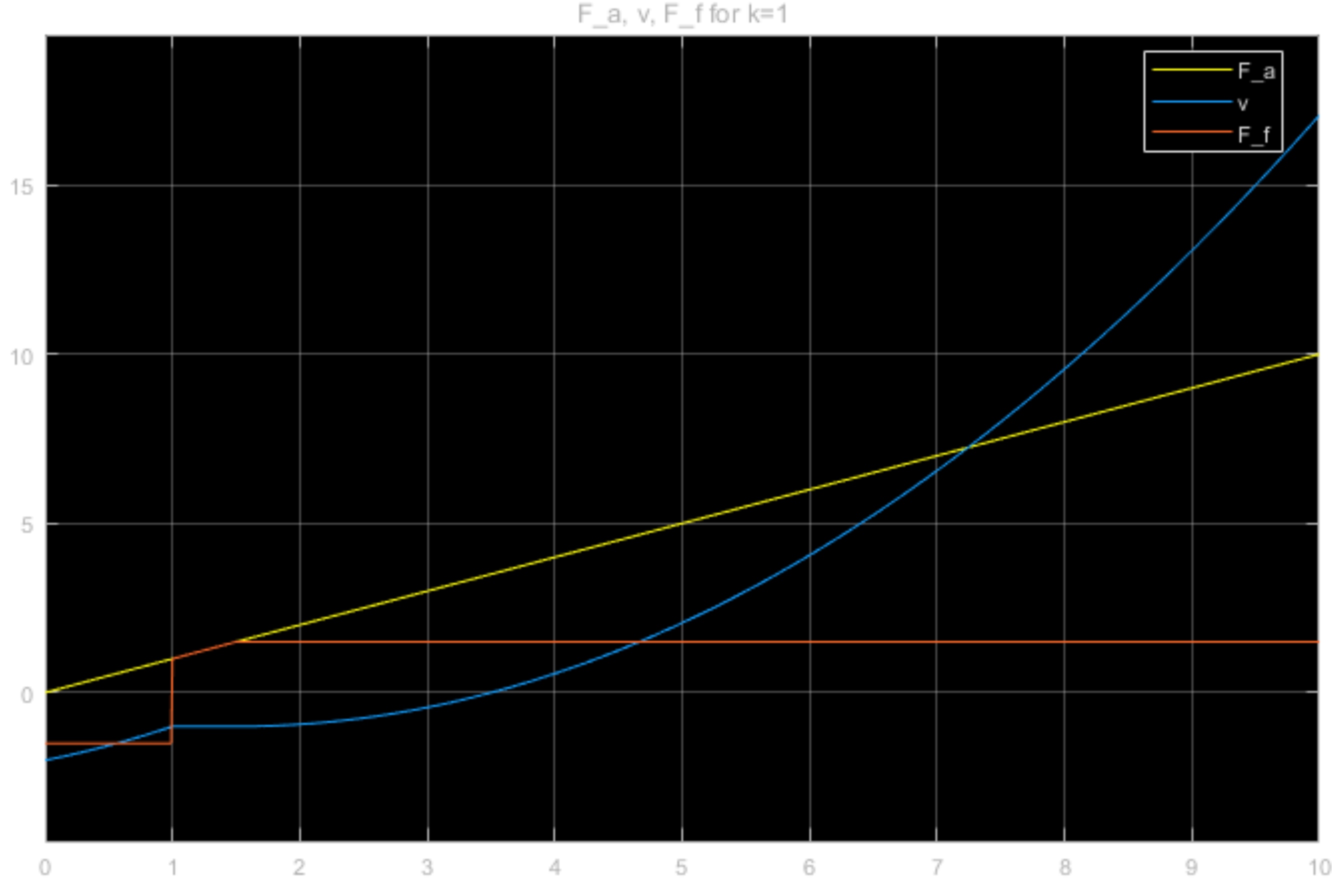
2b
Observe how we longer have nasty discontinuities at v=0.
Furthermore the velocity stays at zero until we pass the threshold
where the ramp force is larger than the friction force,
and then we start to accelerate.



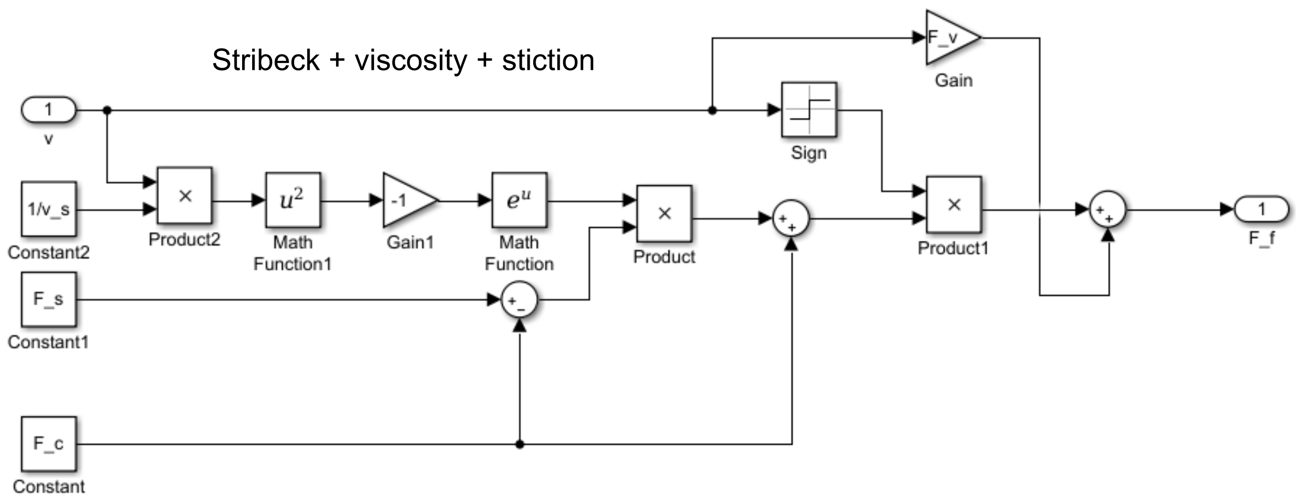

if { }
Action Port
1
v
Saturation
1
F_f


else { }
Action Port
1
v
Sign
F_c
Gain
1
F_f

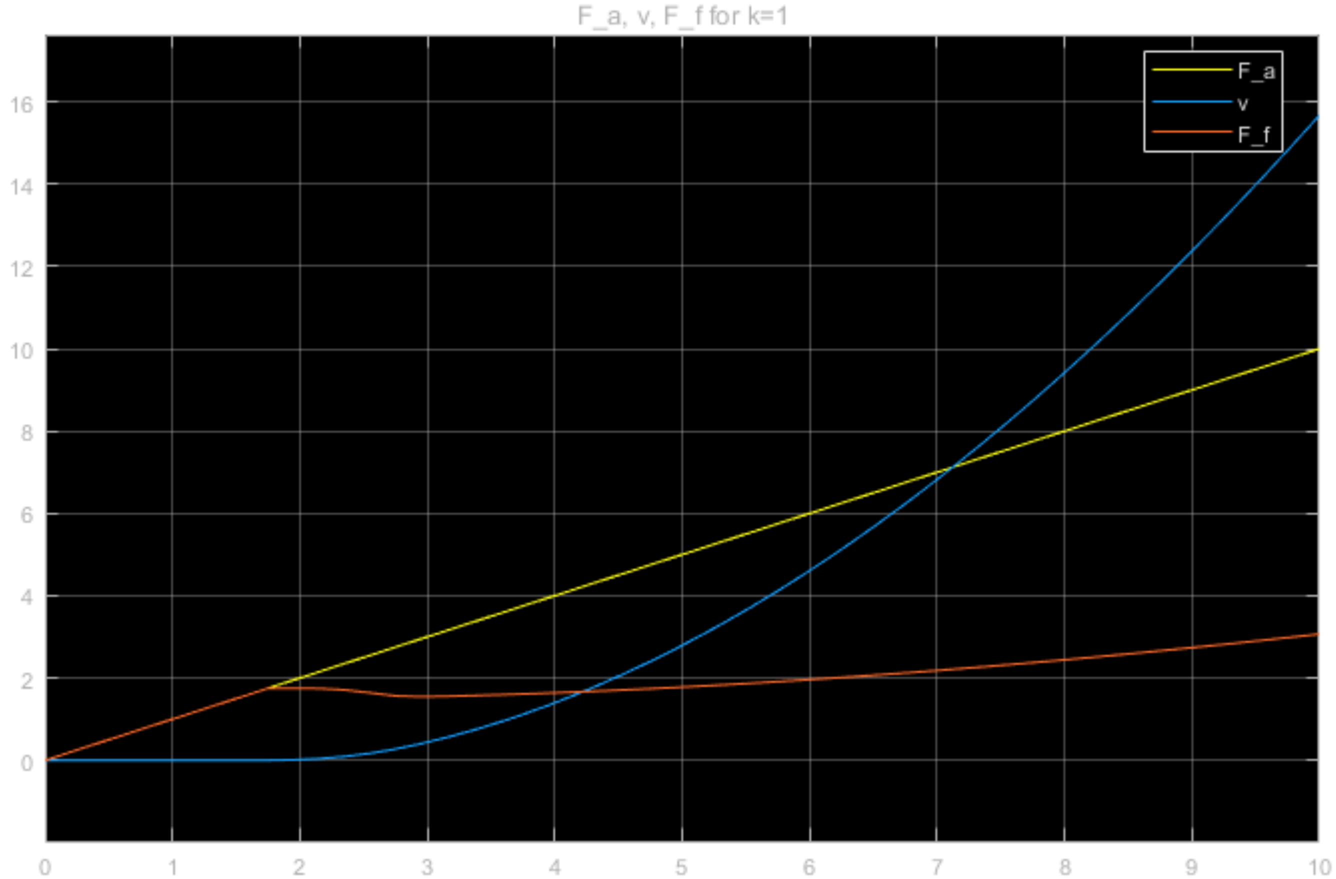
2c
We get the same results. When v(0) = -2 we hit v = -1 for 0.5 seconds
until the ramp force is larger than the friction force and we continue to accelerate again.



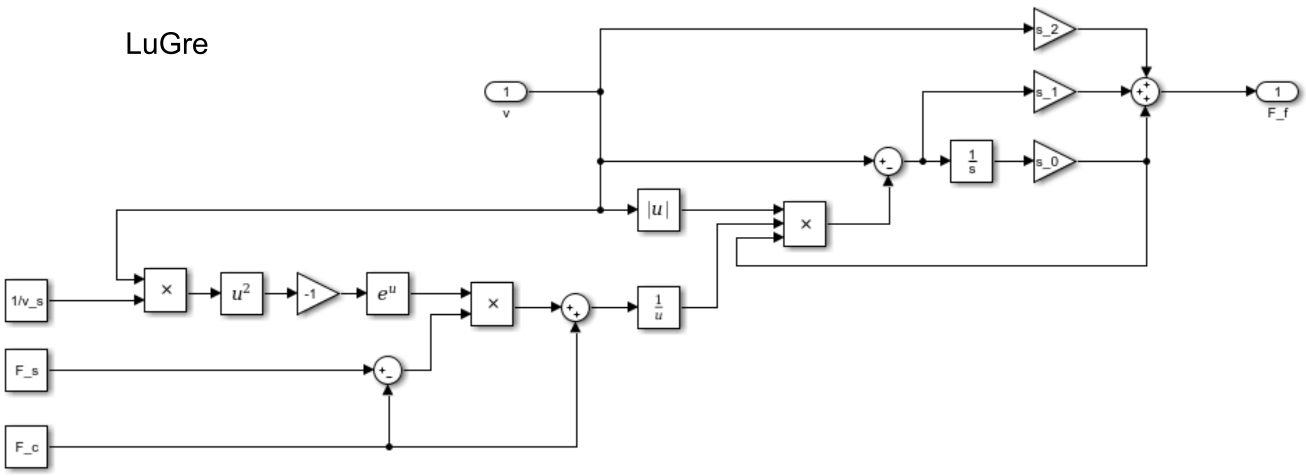
Stribeck + viscosity + stiction



2d
Now we observe how we get viscosity as the friction increases with speed,
as well as a dip in the viscous friction from the Stribeck effect.

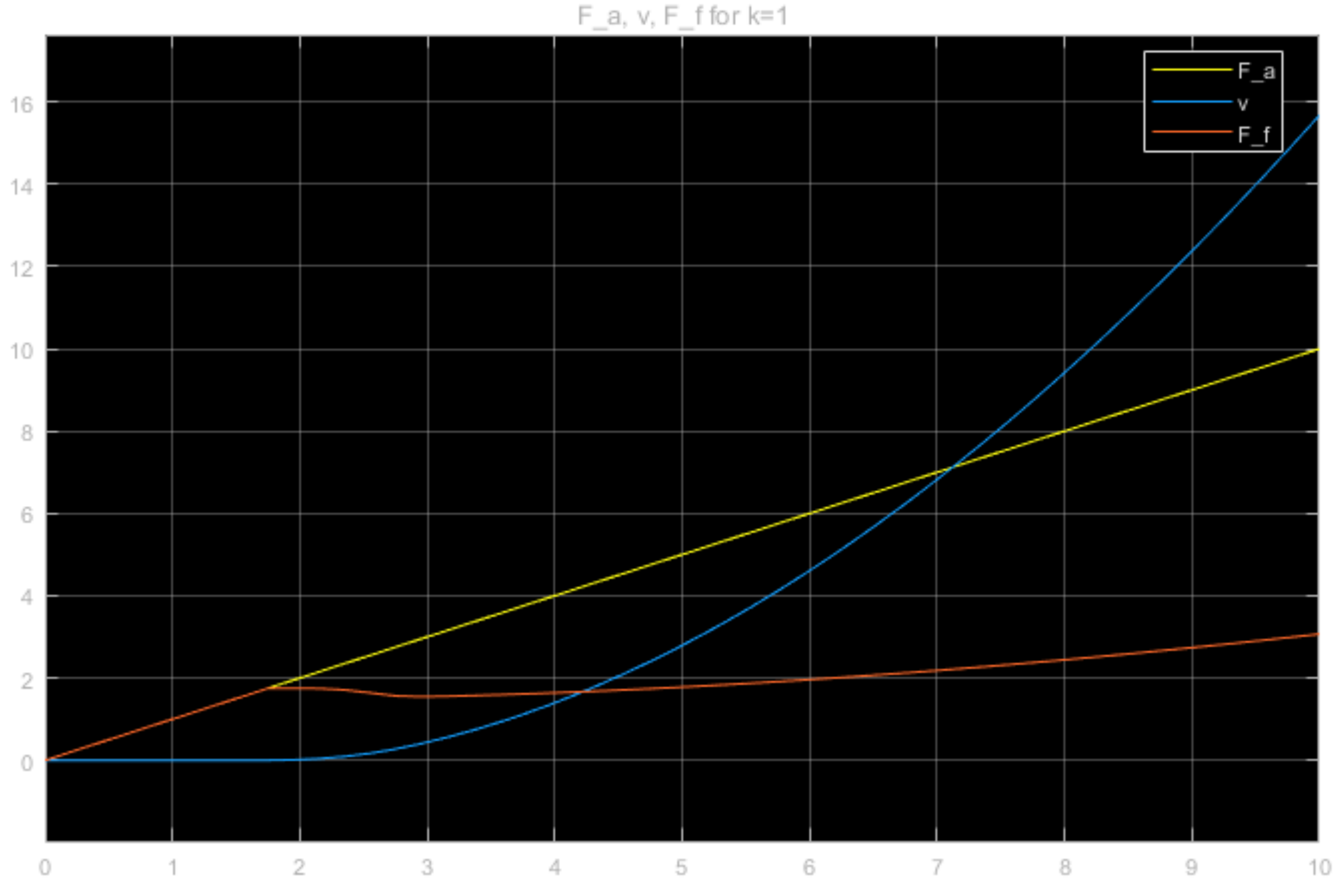


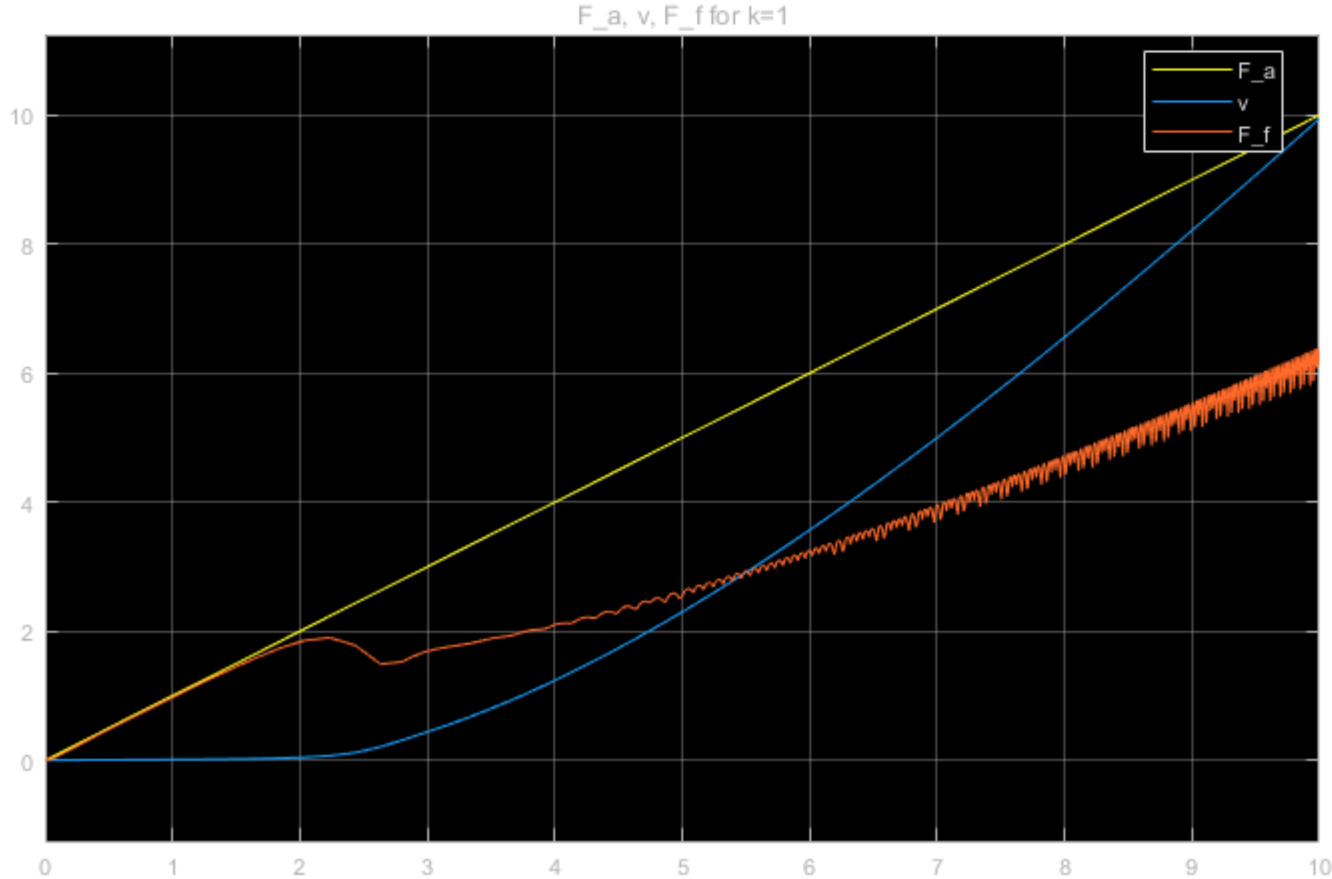
LuGre



2e

Interestingly the results look exactly the same as with linear viscosity and Stribeck.
By adding a nonzero sigma_1 we get oscillations, with a frequency determined also by sigma_0.
sigma_2 determine how hard the linear viscosity is.




F_a, v, F_f for k=1
F_a
v
F_f
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
9
10