1

(a) $R_b^a = [b_1^a \; b_2^a \; b_3^a]$

So the coloumn vectors of R_b^a are the axises of the b frame in the a frame.

(b) 1. This is correct per definition.
2. This is R_a^b, not R_b^a!

(c) $r^a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $w^b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $R_b^a = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$

Observe that R_b^a is a simple rotation about the y-axis.

$SO(3) = \{ R \mid R \in \mathbb{R}^3, R^T R = \mathbb{I}, \det R = 1 \}$

Obviously R_b^a is in $\mathbb{R}^3$.

$\underline{R_b^{aT} R_b^a} = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}+\frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4}+\frac{1}{4} \end{bmatrix} = \underline{\mathbb{I}}$

$\underline{\det R_b^a} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \underline{1}$

As $R_b^a \in SO(3)$ it is a rotation matrix.

ⓓ $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

R^a_b represent a $\frac{\pi}{6}$ rotation about the y-axis.

ⓔ $R^b_a = (R^a_b)^T = \begin{bmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \sqrt{3}/2 \end{bmatrix}$

ⓕ

$$\underline{\underline{u^b}} = R^b_a u^a = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{\sqrt{3}+3}{2} \\ 2 \\ \frac{3\sqrt{3}-1}{2} \end{bmatrix}}}$$

$$\underline{\underline{w^a}} = R^a_b w_b = \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \sqrt{3}/2 - 1 \\ -1 \\ \sqrt{3} + 1/2 \end{bmatrix}}}$$

⑨ i) Since the expression is linear in a we only need to prove the identity for rotation around one axis e.g. $R = R_x(\varphi)$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \lfloor R\mathbf{a} \times \rfloor = \begin{bmatrix} 0 & (-a_2 s\varphi - a_3 c\varphi) & (a_2 c\varphi - a_3 s\varphi) \\ (a_2 s\varphi + a_3 c\varphi) & 0 & -a_1 \\ (-a_2 c\varphi + a_3 s\varphi) & a_1 & 0 \end{bmatrix}$$

$$R\lfloor \mathbf{a} \times \rfloor = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 c\varphi + a_2 s\varphi & -a_1 s\varphi & -a_1 c\varphi \\ a_3 s\varphi - a_2 c\varphi & a_1 c\varphi & -a_1 s\varphi \end{bmatrix}$$

$$R\lfloor \mathbf{a} \times \rfloor R^T = \begin{bmatrix} 0 & -a_3 c\varphi - a_2 s\varphi & -a_3 s\varphi + a_2 c\varphi \\ a_3 c\varphi + a_2 s\varphi & 0 & -a_1 (c\varphi^2 + s\varphi^2) \\ a_3 s\varphi - a_2 c\varphi & a_1 (c\varphi^2 + s\varphi^2) & 0 \end{bmatrix}$$

$$R\lfloor \mathbf{a} \times \rfloor R^T = \begin{bmatrix} 0 & -a_3 c\varphi - a_2 s\varphi & -a_3 s\varphi + a_2 c\varphi \\ a_3 c\varphi + a_2 s\varphi & 0 & -a_1 \\ a_3 s\varphi - a_2 s\varphi & a_1 & 0 \end{bmatrix}$$

$$\underline{R\lfloor \mathbf{a} \times \rfloor R^T = \lfloor R\mathbf{a} \times \rfloor} \quad \square$$

ii) $\underline{R(a \times b)} = R\lfloor a \times \rfloor b = R \lfloor a \times \rfloor R^T R b$

$= \lfloor Ra \times \rfloor Rb = \underline{Ra \times Rb}$ $\square$

Geometrical interpretation: rotating the normal to a and b is the same as finding the normal to the rotated vectors i.e. rotation

preserve crossproducts.

(b) $R^a_b = R_y(\theta) R_z(\psi) R_x(\varphi)$

$$= \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \cdot \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}$$

$$= \begin{bmatrix} c\theta c\psi & -c\theta s\psi & s\theta \\ s\psi & c\psi & 0 \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}$$

$$\underline{\underline{= \begin{bmatrix} c\theta c\psi & -c\theta s\psi c\varphi + s\theta s\varphi & c\theta s\psi s\varphi + s\theta c\varphi \\ s\psi & c\psi c\varphi & -c\psi s\varphi \\ -s\theta c\psi & s\theta s\psi c\varphi + c\theta s\varphi & -s\theta s\psi s\varphi + c\theta c\varphi \end{bmatrix}}}$$

(i) $R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 1 \\ a_{31} & 1 & a_{33} \end{bmatrix}$

$$R_1 R^T = \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21} + a_{12}a_{22} + a_{13} & a_{11}a_{31} + a_{12} + a_{13}a_{33} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13} & a_{21}^2 + a_{22}^2 + 1 & a_{21}a_{31} + a_{22} + a_{33} \\ a_{11}a_{31} + a_{12} + a_{13}a_{33} & a_{21}a_{31} + a_{22} + a_{33} & a_{31}^2 + 1 + a_{33}^2 \end{bmatrix} = \mathbb{I}$$

$\Rightarrow \quad a_{21} = a_{22} = a_{31} = a_{33} = 0$

$\Rightarrow \quad a_{13} = a_{12} = 0 \quad , \quad a_{11}^2 = 1 \quad \Leftrightarrow \quad a_{11} = \pm 1$

$\det R_1 = a_{11} \cdot -1 = 1 \Rightarrow a_{11} = -1$

$$\Rightarrow R_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$R_2 = \begin{bmatrix} -3/5 & a_{12} & a_{13} \\ 4/5 & a_{22} & a_{23} \\ 0 & a_{32} & 1 \end{bmatrix}$$

$$R_2 R_2^+ = \begin{bmatrix} 9/25 + a_{12}^2 + a_{13}^2 & -12/25 + a_{12}a_{22} + a_{13}a_{23} & \cdots \\ -12/25 + a_{12}a_{22} + a_{13}a_{23} & 16/25 + a_{22}^2 + a_{23}^2 & \cdots \\ a_{12}a_{32} + a_{13} & a_{22}a_{32} + a_{23} & a_{32}^2 + 1 \end{bmatrix}$$

$= \mathbb{I} \Rightarrow a_{32} = a_{13} = a_{23} = 0,$

$a_{12}a_{22} = 12/25,$

$\frac{16}{25} + a_{22}^2 = 1, \quad \frac{9}{25} + a_{12}^2 = 1$

$\Rightarrow a_{22} = \pm\frac{3}{5}, \quad a_{12} = \pm\frac{4}{5}$

$\det R_2 = -\frac{3}{5}a_{22} - \frac{4}{5}a_{12} = 1 \Rightarrow a_{12} = -\frac{4}{5}, a_{22} = -\frac{3}{5}$

$$\Rightarrow R_2 = \begin{bmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that this is a simple rotation around z-axis.

$$R_3 = \begin{bmatrix} 1/2 & a_{12} & a_{13} \\ a_{21} & \sqrt{2}/2 & \sqrt{2}/2 \\ -\frac{\sqrt{3}}{2} & a_{32} & a_{33} \end{bmatrix}$$

$$R_3 R_3^T = \begin{bmatrix} \frac{1}{4} + a_{12}^2 + a_{13}^2 & \cdots & \cdots \\ \frac{1}{2} a_{21} + \frac{\sqrt{2}}{2} a_{12} + \frac{\sqrt{2}}{2} a_{13} & a_{21}^2 + 1 & \cdots \\ -\frac{\sqrt{3}}{4} + a_{12} a_{32} + a_{13} a_{33} & -\frac{\sqrt{3}}{2} a_{21} + \frac{\sqrt{2}}{2} a_{32} + \frac{\sqrt{2}}{2} a_{33} & \frac{3}{4} + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

$$= \mathbb{I} \Rightarrow \underline{a_{21} = 0}, \ \frac{\sqrt{2}}{2}(a_{32} + a_{33}) = 0$$

$$\Rightarrow \underline{a_{32} = -a_{33}}, \ \frac{\sqrt{2}}{2}(a_{12} + a_{13}) = 0$$

$$\Rightarrow \underline{a_{13} = -a_{12}}$$

$$\frac{3}{4} + a_{32}^2 + a_{33}^2 = 1 \iff \underline{a_{33} = \pm \frac{1}{2\sqrt{2}}}$$

$$\frac{1}{4} + a_{12}^2 + a_{13}^2 = 1 \iff \underline{a_{12} = \pm \frac{\sqrt{3}}{2\sqrt{2}}}$$

$$\det R_3 = \frac{1}{2} \cdot \frac{\sqrt{2}}{2}(a_{33} - a_{32}) - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}(a_{12} - a_{13}) = 1$$

$$= \frac{\sqrt{2}}{2} a_{33} - \frac{\sqrt{6}}{2} a_{12} = 1 \iff \pm \frac{1}{2} - \pm \frac{3}{2} = 1$$

Which is only satisfied if

$$a_{12} = \frac{\sqrt{3}}{2\sqrt{2}}, \quad a_{33} = -\frac{1}{2\sqrt{2}}$$

$$\Rightarrow R_3 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{bmatrix}$$

Gee, I sure learned a lot from doing all this algebra :)

$\boxed{2}$ $R_{k,\theta} = \cos\theta \, \mathbb{I} + \sin\theta \, S(k) + (1-\cos\theta) k k^T$

ⓐ $R = R^a_b$

$$k^a = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$k^b = R_{k,\theta} k^a = \cos\theta k^a + \sin\theta \, S(k^a) k^a + (1-\cos\theta) k^a k^{aT} k^a$$

$$= \cos\theta k^a + \sin\theta \cdot 0 + (1-\cos\theta) k^a$$

$$= k^a + \cos\theta k^a - \cos\theta k^a$$

$$\underline{\underline{k^b = k^a}} \quad \square$$

ⓑ We can find η, ε from Shepperd.

Then we have:

$$\theta = 2 \arccos \eta, \quad k = \frac{1}{\sin\frac{\theta}{2}} \varepsilon$$

Using the function we find:

$$k_1, \theta_1 = \begin{bmatrix} 0 \\ 0{,}71 \\ 0{,}71 \end{bmatrix}, \pi$$

$$k_2, \theta_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 2{,}2143$$ ← simple z rotation as previously mensioned

$$k_3, \theta_3 = \begin{bmatrix} -0{,}1773 \\ 0{,}1272 \\ -0{,}3070 \end{bmatrix}, \quad 1{,}6441$$

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```
function [ k, theta ] = rotmat2angleaxis( R )
    T = trace(R);
    [ri, i] = max([T R(1,1) R(2,2) R(3,3)]);
    zi = sqrt(1 + 2*ri - T);
    z = [];
    switch i
        case 1
            z = [zi (R(3,2) - R(2,3))/zi
                (R(1,3) - R(3,1))/zi (R(2,1) - R(1,2))/zi];
        case 2
            z = [(R(3,2) - R(2,3))/zi zi
                (R(2,1) + R(1,2))/zi (R(1,3) + R(3,1))/zi];
        case 3
            z = [(R(1,3) - R(3,1))/zi (R(2,1) + R(1,2))/zi
                zi (R(3,2) + R(2,3))/zi];
        case 4
            z = [(R(2,1) - R(1,2))/zi (R(1,3) + R(3,1))/zi
                (R(3,2) + R(2,3))/zi zi];
    end
    n = z(1)/2;
    e = z(2:4)/2;
    theta = 2*acos(n);
    k = e/sin(theta/2);
end
```

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3 (a)

$$T^i_{i+1} = \begin{bmatrix} R_{z,\theta_i} & 0 \\ 0^T & 1 \end{bmatrix} \cdot \begin{bmatrix} I & r_{z,d_i} \\ 0^T & 1 \end{bmatrix} \cdot \begin{bmatrix} I & r_{x,a_i} \\ 0^T & 1 \end{bmatrix} \cdot \begin{bmatrix} R_{x,\alpha_i} & 0 \\ 0^T & 1 \end{bmatrix}$$

$$T^i_{i+1} = \begin{bmatrix} R_{z,\theta_i} R_{x,\alpha_i} & R_{z,\theta_i}(r_{x,a_i} + r_{z,d_i}) \\ 0^T & 1 \end{bmatrix}$$

$$R_{z,\theta_i} R_{x,\alpha_i} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}$$

$$R_{z,\theta_i}(r_{x,a_i} + r_{z,d_i}) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ 0 \\ d_i \end{bmatrix} = \begin{bmatrix} c\theta_i a_i \\ s\theta_i a_i \\ d_i \end{bmatrix}$$

$$\Rightarrow T^i_{i+1} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	0	0	q_2	$\pm q_3$

B	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	q_2	0	l_2	0

(b) Right hand rule says that q_3 has a negative sign.

(c) $$T_{A\,1}^{\,0} = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{A\,2}^{\,1} = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & cq_3 & sq_3 & 0 \\ 0 & -sq_3 & cq_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{B\,1}^{\,0} = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_{A\,1}^{\,0}$$

$$T_{B\,2}^{\,1} = \begin{bmatrix} cq_2 & -sq_2 & 0 & l_2 cq_2 \\ sq_2 & cq_2 & 0 & l_2 sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There was really no point in showing details, I simply put the values into the expression for T^{i}_{i+1} and used $c0 = 1$, $s0 = 0$.

(d) $T_{A\,2}^{\,0} = T_{A\,1}^{\,0}\, T_{A\,2}^{\,1}$

$$= \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & cq_3 & sq_3 & 0 \\ 0 & -sq_3 & cq_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{A\,2}^{\,0} = \begin{bmatrix} cq_1 & -sq_1 cq_3 & -sq_1 sq_3 & q_2 cq_1 + l_1 cq_1 \\ sq_1 & cq_1 cq_3 & cq_1 sq_3 & q_2 sq_1 + l_1 sq_1 \\ 0 & -sq_3 & cq_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_{B\,2}^{\,0} = T_{B\,1}^{\,0}\, T_{B\,2}^{\,1}$

$$= \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} cq_2 & -sq_2 & 0 & l_2 cq_2 \\ sq_2 & cq_2 & 0 & l_2 sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{B\,2}^{\,0} = \begin{bmatrix} cq_1 cq_2 - sq_1 sq_2 & -cq_1 sq_2 - sq_1 cq_2 & 0 & l_2 cq_1 cq_2 - l_2 sq_1 sq_2 + l_1 cq_1 \\ sq_1 cq_2 + cq_1 sq_2 & -sq_1 sq_2 + cq_1 cq_2 & 0 & l_2 sq_1 cq_2 + l_2 cq_1 sq_2 + l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yikes, this is bound to be incorrect...

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ⓔ

- $g^2 = \begin{bmatrix} l_1 \cos q_1 \\ -l_1 \sin q_1 \\ 0 \\ 1 \end{bmatrix}$

$$g^0 = T^2_0 g^2 = (T^0_2)^{-1} g^2$$

$$(T^0_2)^{-1} = \begin{bmatrix} R^{0\,T}_2 & -R^{0\,T}_2 r^0_{02} \\ 0^T & 1 \end{bmatrix}$$

- This calculation is monstrous.
What about

$$T^0_g = T^0_2 T^2_g = T^0_2 \cdot \begin{bmatrix} \mathbb{I} & g_2 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R^0_2 & r^0_{02} + R^0_2 r^2_g \\ 0^T & 1 \end{bmatrix}$$

$$r_A{}^0_{02} + R_A{}^0_2 r^2_g = \begin{bmatrix} c_1 (q_2 + l_1) \\ s_1 (q_2 + l_1) \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 c_1^2 + l_1 s_1^2 c_3 \\ l_1 s_1 c_1 - l_1 s_1 c_1 c_3 \\ l_1 s_1 s_3 \end{bmatrix}$$

- $g_A{}^0 = \begin{bmatrix} c_1 (q_2 + l_1) + l_1 c_1^2 + l_1 s_1^2 c_3 \\ s_1 (q_2 + l_1) + l_1 s_1 c_1 - l_1 s_1 c_1 c_3 \\ l_1 s_1 s_3 \end{bmatrix}$

ps: I would have really appreciated some sort of verification of the result at least once in this task, e.g. show that, because all these matrix multiplications are bound to be erroneous.

$$\underline{\underline{g_B{}^0}} = r_B{}^0_{02} + R_B{}^0_2 r^2_g = \begin{bmatrix} l_2 c_1 c_2 - l_2 s_1 s_2 + l_1 c_1 \\ l_2 s_1 c_2 + l_2 c_1 s_2 + l_1 s_1 \\ 0 \end{bmatrix}$$

- $+ \begin{bmatrix} l_1 c_1^2 c_2 - l_1 c_1 s_1 s_2 + l_1 s_1 c_1 s_2 + l_1 s_1^2 c_2 \\ l_1 c_1 s_1 c_2 + l_1 c_1^2 s_2 + l_1 s_1^2 s_2 - l_1 s_1 c_1 c_2 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} l_1 c_2 \\ l_1 s_2 \\ 0 \end{bmatrix}}}$