TTK4130 assignment 1

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1 Task 1

1.1

The model keyword defines a new model object. The equation keyword indicates that the following lines of code are equations that govern the dynamics of the model. The parameter keyword says that the variable you are declaring is a parameter of the system, not a constant or a state. This tells Modelica that the variable is fixed in an instance of the simulation, but you may vary it from instance to instance. Real simply states that the variable you are declaring is real. The start keyword is used to set the initial value of a state, and the der keyword is used to write the derivative of a state.

1.2

The quoted text is documentation comments and are added to the documentation of the model. The double slash text is regular comments which are only used to comment the code and is not added to the doc.

1.3

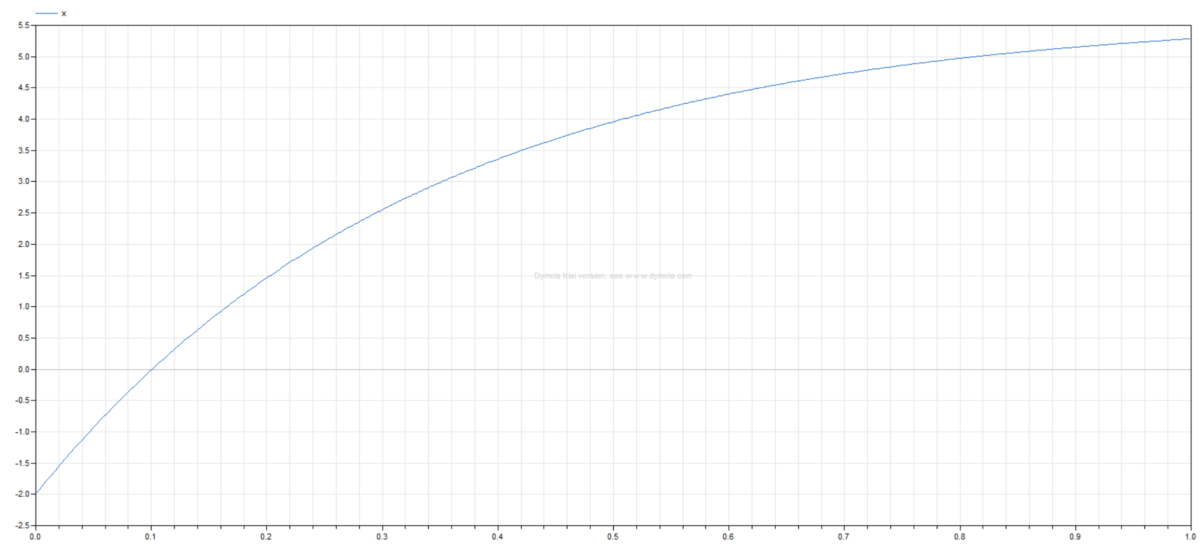

Figure 1: Simulation of the first order system.

1.4

We would obtain the same results, but we would not be able to change the variables a and b for a new instance of the simulation. Since a and b are model parameters that we would like to change, they should not be constants. Constant variables are used for natural constants such as π, which we obviously do not want to alter.

1.5

I set the sim time to 10s, change the solver to a really sloppy Euler, set $a = -1$, $b = 0$, $x_0 = -10$. I set

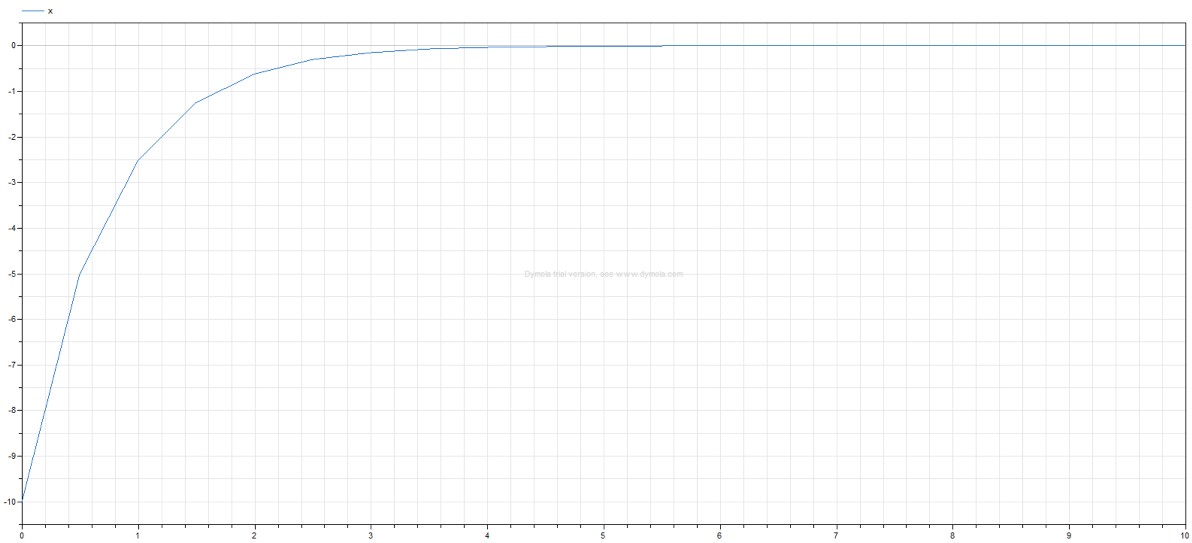

Figure 2: Simulation of the first order system.

the integration time to 0.02s, and set $a = 1$. I simulated the model using euler and dassl.

1.6

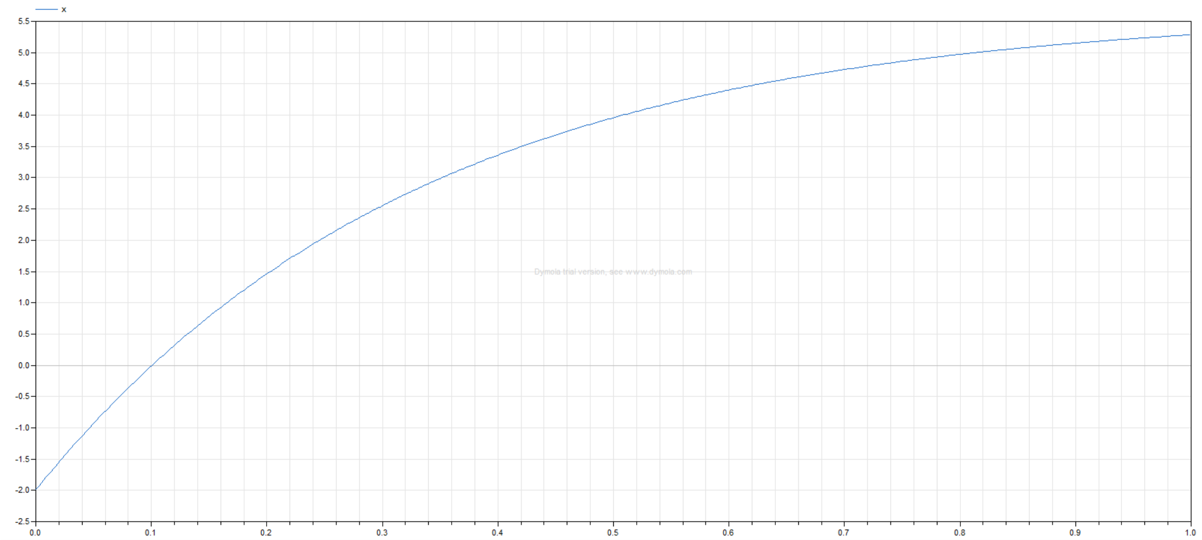

Figure 3: Simulation of the first order system with dassl.

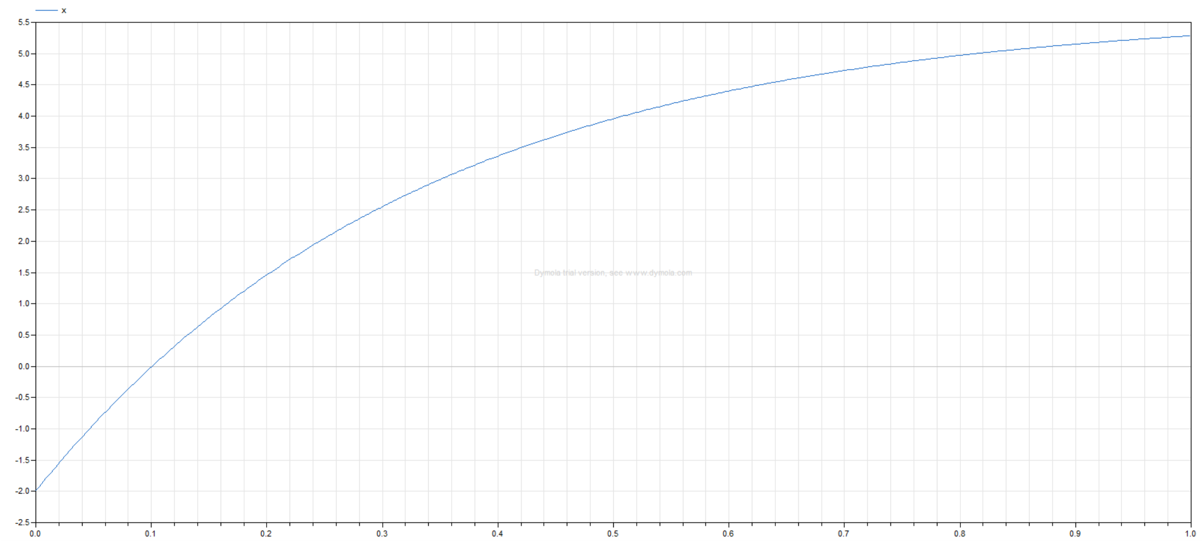

Figure 4: Simulation of the first order system with euler.

TTK4130 ASSIGNMENT 1

$\boxed{2}$ (a) 1. $\dot{x} = -x - \frac{y}{\ln\sqrt{x^2+y^2}} = 0$, $\dot{y} = -y + \frac{x}{\ln\sqrt{x^2+y^2}} = 0$

$x = \frac{-y}{\ln\sqrt{x^2+y^2}}$, $y = \frac{x}{\ln\sqrt{x^2+y^2}}$

This system is only in equilibrium in

$\underline{(x_0, y_0) = (0,0)}$.

2. $\dot{x} = a - x - \frac{4xy}{1+x^2} = 0$, $\dot{y} = bx(1 - \frac{y}{1+x^2}) = 0$

$x=0$ is not valid, as $a>0$, so:

$y = 1+x^2$

$a = x + \frac{4xy}{y} = 5x \iff x_0 = \frac{a}{5}$, $y_0 = 1 + \frac{a^2}{25}$

$\underline{(x_0, y_0) = (\frac{a}{5}, 1 + \frac{a^2}{25})}$

3. $\dot{x} = \left(\frac{y}{1+2y+y^2} - d\right)x = 0$, $\dot{y} = d(4-y) - \frac{2{,}5xy}{1+2y+y^2} = 0$

$x = 0 \Rightarrow y = 4 \quad \underline{(x_{0_1}, y_{0_1}) = (0,4)}$

or $y = d(1+2y+y^2) \iff y^2 + (2-\frac{1}{d})y + 1 = 0$

$\Rightarrow \underline{y} = -\frac{1}{2}(2-\frac{1}{d}) \pm \frac{1}{2}\sqrt{(2-\frac{1}{d})^2 - 4}$

$= -(1-\frac{1}{2d}) \pm \frac{1}{2}\sqrt{\frac{1}{d^2} - \frac{4}{d}} = \frac{1}{2d} - 1 \pm \frac{1}{2d}\sqrt{1-4d}$

$\underline{y = \frac{1}{4d}(\sqrt{1-4d} \pm 1)^2}$

$d(4-y) = 2{,}5xd$

$\Rightarrow x = \frac{8-2y}{5} = \frac{8}{5} - \frac{2}{5}\left(\frac{1}{2d} - 1 \pm \frac{1}{2d}\sqrt{1-4d}\right)$

$= \frac{1}{5d}(10d - 1 \pm \sqrt{1-4d})$

So the two final equilibrium points are: $(x_{02}, y_{02}) = \left(\frac{1}{5d}(10d-1+\sqrt{1-4d}), \frac{1}{4d}(\sqrt{1-4d}+1)^2\right)$

and $(x_{03}, y_{03}) = \left(\frac{1}{5d}(10d-1-\sqrt{1-4d}), \frac{1}{4d}(\sqrt{1-4d}-1)^2\right)$

1. (b)

$$\left.\frac{\partial f_1}{\partial x}\right|_p = -1 + \left.\frac{4yx}{(x^2+y^2)(\ln(x^2+y^2))^2}\right|_p = -1$$

$$\left.\frac{\partial f_1}{\partial y}\right|_p = \frac{4y^2}{(x^2+y^2)(\ln(x^2+y^2))^2} - \left.\frac{2}{\ln(x^2+y^2)}\right|_p = 0$$

$$\left.\frac{\partial f_2}{\partial x}\right|_p = \frac{2}{\ln(x^2+y^2)} - \left.\frac{4x^2}{(x^2+y^2)(\ln(x^2+y^2))^2}\right|_p = 0$$

$$\left.\frac{\partial f_2}{\partial y}\right|_p = -1 - \left.\frac{4xy}{(x^2+y^2)(\ln(x^2+y^2))^2}\right|_p = -1$$

$$\Rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \lambda_1 = \lambda_2 = -1$$

Since we have two eigenvalues with negative real part, the linearized system is asymptotically stable.

2. $$\frac{\partial f_1}{\partial x}\bigg|_p = -1 - \frac{4y(1+x^2) - 8x^2y}{(1+x^2)^2}\bigg|_p$$

$$= -1 - \frac{4y}{1+x^2} - \frac{8x^2y}{(1+x^2)^2}\bigg|_p = -1-4 - \frac{8\frac{a^2}{25}}{(1+\frac{a^2}{25})}$$

$$= \frac{3a^2-125}{25+a^2}$$

$$\frac{\partial f_1}{\partial y}\bigg|_p = -\frac{4x}{1+x^2}\bigg|_p = -\frac{20a}{25+a^2}$$

$$\frac{\partial f_2}{\partial x}\bigg|_p = b - \frac{by(1+x^2) - 2byx^2}{(1+x^2)^2}\bigg|_p$$

$$= b\left(1 - \frac{(1+x^2)^2 - 2x^2(1+x^2)}{(1+x^2)^2}\right)\bigg|_p$$

$$= b\left(\frac{2x^2}{1+x^2}\right)\bigg|_p = \frac{2b\frac{a^2}{25}}{1+\frac{a^2}{25}} = \frac{2a^2b}{25+a^2}$$

$$\frac{\partial f_2}{\partial y}\bigg|_p = \frac{-bx}{1+x^2}\bigg|_p = \frac{-5ab}{25+a^2}$$

$$\Rightarrow A = \frac{1}{25+a^2}\begin{bmatrix} 3a^2-125 & -20a \\ a^2b2 & -5ab \end{bmatrix}$$

$$\Delta(\lambda) = (\lambda - 3a^2+125)(\lambda+5ab) + 40a^3b$$

$$= \lambda^2 + \lambda(5ab-3a^2+125) + 625ab + 25a^3b = 0$$

The system is stable if all coefficients are positive, by Hurwitz' criterion. This means the system ~~is~~ is stable if (a,b) satisfies $5ab-3a^2+125>0$.

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3. $$\frac{\partial f_1}{\partial x} = \frac{y}{1+2y+y^2} - d,$$

$$\frac{\partial f_1}{\partial y} = \frac{x(1+2y+y^2) - xy(2+2y)}{(1+2y+y^2)^2} = \frac{x - xy^2}{(1+2y+y^2)^2}$$

$$\frac{\partial f_2}{\partial x} = -\frac{2{,}5y}{1+2y+y^2},$$

$$\frac{\partial f_2}{\partial y} = -d - \frac{2{,}5x(1+2y+y^2) - 2{,}5xy(2+2y)}{(1+2y+y^2)^2}$$

$$= -d - \frac{2{,}5x - 2{,}5xy^2}{(1+2y+y^2)^2}$$

In $(0,4)$ we get $A = \begin{bmatrix} \frac{4}{25} - d & 0 \\ -\frac{2}{5} & -d \end{bmatrix}$

$$\Delta(\lambda) = (\lambda + d)(\lambda + d - \tfrac{4}{25}) = 0$$

$\Rightarrow \lambda_1 = -d, \ \lambda_2 = \frac{4}{25} - d \leq 0 \Rightarrow d \geq \frac{4}{25}$

So the system is stable for $d \geq \frac{4}{25}$.

In $\left(\frac{1}{5d}(10d - 1 \pm \sqrt{1-4d}), \frac{1}{4d}(\sqrt{1-4d} \pm 1)^2\right)$ we have

$$A = \begin{bmatrix} 0 & g(d) \\ -2{,}5d & -d - 2{,}5g(d) \end{bmatrix}, \text{ where}$$

$$g(d) = \frac{d}{5}(10d - 1 \pm \sqrt{1-4d})\left(\frac{1}{\frac{1}{16d^2}(\sqrt{1-4d} \pm 1)^4} - 1\right)$$

I assume i made a mistake somewhere, because this got really nasty and i will not bother checking the stability.

2c If we only consider the equilibrium points, the linearized system is identical, so if the equilibrium points are stable for the linearized system, it is as well for the non-~~linear~~ system.

If the linearized system is unstable, we do not know anything about the non-linear system.

$\boxed{3}$ a) 1. $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 5 \end{bmatrix}$

$$\Delta(\lambda) = \begin{vmatrix} \lambda-4 & -2 & 0 & 0 \\ 0 & \lambda-3 & 0 & -1 \\ 0 & 0 & \lambda+2 & 0 \\ 0 & 1 & 0 & \lambda-5 \end{vmatrix} = (\lambda-4)\left((\lambda-3)(\lambda+2)(\lambda-5) + \lambda+2\right)$$

$$= (\lambda-4)(\lambda^3 - 6\lambda^2 + 32) = (\lambda-4)^3(\lambda+2)$$

$$\underline{\underline{\lambda_1 = \lambda_2 = \lambda_3 = 4 \,, \quad \lambda_4 = -2}}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 & 0 & 0 \\ 0 & 3-\lambda & 0 & 1 \\ 0 & 0 & -2-\lambda & 0 \\ 0 & -1 & 0 & 5-\lambda \end{bmatrix}$$

$$\lambda = 4 \Rightarrow A - \lambda I = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -6 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\Rightarrow m_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\lambda = -2 \Rightarrow A - \lambda I = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\Rightarrow m_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

2.
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\underline{\Delta(\lambda)} = \begin{vmatrix} \lambda-1 & -1 & -1 & 0 \\ 2 & \lambda+1 & 0 & 1 \\ 0 & 0 & \lambda+1 & 1 \\ 0 & 0 & -2 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda+1)\big((\lambda+1)(\lambda-1)+2\big)$$
$$+2\big((\lambda+1)(\lambda-1)+2\big)$$

$$= (\lambda+1)^2(\lambda-1)^2 + 2(\lambda+1)(\lambda-1) + 2(\lambda^2+1$$
$$= \lambda^4 + 2\lambda^2 + 1 = \underline{(\lambda^2+1)^2}$$

$$\Rightarrow \underline{\underline{\lambda_1 = \lambda_2 = i, \quad \lambda_3 = \lambda_4 = -i}}$$

$$\lambda = i \Rightarrow A - \lambda I = \begin{bmatrix} i-1 & -1 & -1 & 0 \\ 2 & i+1 & 0 & 1 \\ 0 & 0 & i+1 & 1 \\ 0 & 0 & -2 & i-1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2}(i+1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\underline{m_1 = \begin{bmatrix} -\frac{1}{2}(i+1) \\ 1 \\ 0 \\ 0 \end{bmatrix}}}$$

Since $\lambda = -i$ is the complex conjugate of $\lambda = i$ we know that

$$\underline{\underline{m_2 = \begin{bmatrix} -\frac{1}{2}(1-i) \\ 1 \\ 0 \\ 0 \end{bmatrix}}}$$

3. $A = \begin{bmatrix} 0 & -8 & -2 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\lambda I - A = \begin{bmatrix} \lambda & 8 & 2 & 5 \\ 0 & \lambda & -1 & 0 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{bmatrix}$

$\underline{\Delta(\lambda)} = \lambda^4 - 1\cdot 5 + \lambda(-8-2\lambda) = \lambda^4 + 2\lambda^2 + 8\lambda + 5$

$= \underline{(x+1)^2 (x-(1-2i))(x-(1+2i))}$

$\Rightarrow \underline{\underline{\lambda_1 = \lambda_2 = -1, \ \lambda_3 = 1+2i, \ \lambda_4 = 1-2i}}$

$\lambda = -1 \Rightarrow \lambda I - A = \begin{bmatrix} -1 & 8 & 2 & 5 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \underline{\underline{m_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}}$

$\lambda = 1+2i \Rightarrow \lambda I - A = \begin{bmatrix} 1+2i & 8 & 2 & 5 \\ 0 & 1+2i & -1 & 0 \\ -1 & 0 & 1+2i & 0 \\ 0 & -1 & 0 & 1+2i \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -(1+2i) & 0 \\ 0 & 1 & 0 & -(1+2i) \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 1 & 3-4i \end{bmatrix}$, $m_{24} = 1$, $m_{22} = 1+2i$, $m_{23} = -3+4i$, $m_{21} = (1+2i)(-3+4i)$
$= -3+4i-6i-8 = -11-2i$

$\Rightarrow \underline{\underline{m_2 = \begin{bmatrix} -11-2i \\ 1+2i \\ -3+4i \\ 1 \end{bmatrix}}}$

$\lambda = 1-2i \Rightarrow \underline{\underline{m_3 = \begin{bmatrix} -11+2i \\ 1-2i \\ -3-4i \\ 1 \end{bmatrix}}}$

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(c) $\dot{x} = x + 3z + 4y$, $\dot{y} = -4y - 3z - x$, $\dot{z} = -3z - 2y + x + 32u$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 32 \end{bmatrix} u,$$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix}, \quad A(\lambda) = \begin{vmatrix} \lambda - 1 & -4 & -3 \\ 1 & \lambda + 4 & 3 \\ -1 & 2 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda - 1)((\lambda + 4)(\lambda + 3) - 6) - (-4(\lambda + 3) + 6) - (-12 + 3(\lambda + 4))$$
$$= \lambda^2(\lambda + 6) = 0$$

$$\Rightarrow \underline{\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -6}$$

We have a double integrator in the system, so it is <u>unstable</u>!

$\boxed{4}$ $Ni = \mathcal{R}(R_a + R_c + R_b + R_r)$,

$R_a = \frac{z}{A\mu_0}$, $R_c \approx 0$, $R_b \approx 0$

ⓐ $\underline{Ni} = \mathcal{R}\left(\frac{z}{A\mu_0} + \frac{z_0}{A\mu_0}\right) = \underline{\frac{\mathcal{R}}{A\mu_0}(z+z_0)}$

ⓑ $\underline{\frac{N\varphi}{i} = L(z)} = \frac{N}{i} Ni A\mu_0 \frac{1}{z+z_0} = \underline{\frac{N^2 A\mu_0}{z+z_0}}$

$$\underline{F} = \frac{i^2}{2}\frac{\partial L(z)}{\partial z} = \frac{i^2}{2}\cdot\frac{-N^2A\mu_0}{(z+z_0)^2} = \underline{-\frac{i^2N^2A\mu_0}{2(z+z_0)^2}}$$

$$\Sigma F = ma = mg - \frac{i^2N^2A\mu_0}{2(z+z_0)^2}$$

$$\Rightarrow \underline{\ddot{z} = g - \frac{i^2N^2A\mu_0}{2m(z+z_0)^2}}$$

ⓒ State space representation:

$$\mathbf{z} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \dot{\mathbf{z}} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = f(z, i) = \begin{bmatrix} z_2 \\ g - \frac{i^2N^2A\mu_0}{2m(z_1+z_0)^2} \end{bmatrix}$$

$$\underline{A} = \frac{\partial f}{\partial \mathbf{z}}\Big|_{z_1=z_d,\ i=i_d} = \underline{\begin{bmatrix} 0 & 1 \\ \frac{i_d^2N^2A\mu_0}{m(z_d+z_0)^3} & 0 \end{bmatrix}}$$

∂f

$$\underline{B} = \left.\frac{\partial f}{\partial i}\right|_{i_d, z_d} = \begin{bmatrix} 0 \\ -\dfrac{2 i_d N^2 A \mu_0}{2m(z_d+z_0)^2} \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ -\dfrac{i_d N^2 A \mu_0}{m(z_d+z_0)^2} \end{bmatrix}}$$

The linearized system is

$\dot{z} = Az + Bi$, with A, B specified above