TTK4135 EXERCISE 2

1 (a) $f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$,

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $f(x) = x_1^3 + 3x_1 x_2^2$

$$\nabla f(x+\alpha p)^T = \begin{bmatrix} 3(x_1+2\alpha)^2 + 3(x_2+\alpha)^2 \\ 6(x_1+2\alpha)(x_2+\alpha) \end{bmatrix} = \begin{bmatrix} 15\alpha^2 \\ 12\alpha^2 \end{bmatrix}$$

$\nabla f(x+\alpha p)^T p = 42\alpha^2$

$f(x+p) = p_1^3 + 3p_1 p_2^2 = 8 + 3 \cdot 2 = 14$

$= f(x) + \nabla f(x+\alpha p)^T p = 42\alpha^2$

$\Rightarrow \underline{\underline{\alpha = 1/\sqrt{3}}}$

(b) $f(x) = x^{\frac{1}{2}}$

~~Ka~~ Lipschitz continuity:

$\|f(x_1) - f(x_0)\| \leq L \|x_1 - x_0\|$ for all $x_0, x_1 \in N$

for some $L > 0$.

That means that

$$\frac{\|f(x_1) - f(x_0)\|}{\|x_1 - x_0\|} = \frac{\|x_1^{\frac{1}{2}} - x_0^{\frac{1}{2}}\|}{\|x_1 - x_0\|} \leq L$$

If we choose $x_0 = 0$ and $x_1 = \delta$ we get:

$$\frac{\|\delta^{\frac{1}{2}}\|}{\|\delta\|} = \frac{1}{\|\delta^{\frac{1}{2}}\|} \leq L$$

We can choose δ arbitrarily small as we approach 0, and so $f(x)$ is not lipschitz continous in 0.

2 $\min_x c^T x \quad s.t. \quad Ax = b, \; x \geq 0$

Start by defining the lagrangian:

$$\mathcal{L}(x, \lambda, s) = f(x) - \sum_{i \in \mathcal{E}} \lambda_i c_i(x) - \sum_{i \in \mathcal{I}} s_i c_i(x)$$

$$= c^T x - \lambda^T (Ax - b) - s^T x$$

$$\nabla_x \mathcal{L}(x, \lambda, s) = c - A^T \lambda^* - s^* = 0 \quad (1),$$

$$Ax^* = b, \; x^* \geq 0 \quad (2),$$

Furthermore the inequality multipliers must be positive:

$$s^* \geq 0 \quad (3),$$

and they must either be active or the multipliers must be zero:

$$s^* x^* = 0 \quad (4)$$

(1) - (4) is thus the KKT conditions for the linear program.

$\boxed{3}$ ⓐ $R = 3A + 2B$,

$S = 2A + 2B$,

$T = A + 3B$,

We want to maximize the cost function

$f(x) = 100x_1 + 75x_2 + 55x_3$,

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R \\ S \\ T \end{bmatrix}$,

which is the same as minimizing $-f(x)$.

We want the production to satisfy

$x_i \geq 0$,

as well as utilizing all the available hours, which can be written as:

A : $3x_1 + 2x_2 + x_3 = 7200$,

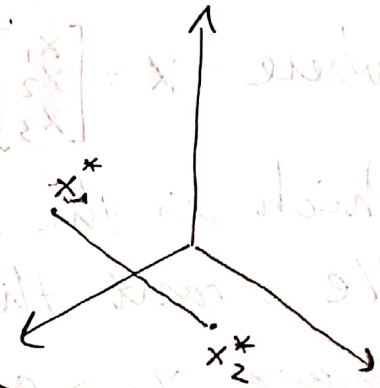
B : $2x_1 + 2x_2 + 3x_3 = 6000$

Which can be written as a LP problem:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

With $c = -\begin{bmatrix} 100 \\ 75 \\ 55 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$

ⓑ Our constraints are basically two planes in $\mathbb{R}^3$ + the limitation that we are only in the first quadrant. that means the feasible region is a line in the first quadrant where the planes meet.



Since $m=2$ we have that B has two elements, which means we have three possible intersections to test:

$$B=\begin{bmatrix}1\\2\end{bmatrix}: \left[\begin{array}{cc|c}3&2&7200\\2&2&6000\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&1200\\0&1&1800\end{array}\right]$$

$$B=\begin{bmatrix}1\\3\end{bmatrix}: \left[\begin{array}{cc|c}3&1&7200\\2&3&6000\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&15600/7\\0&1&3600/7\end{array}\right]$$

$$B=\begin{bmatrix}2\\3\end{bmatrix}: \left[\begin{array}{cc|c}2&1&7200\\2&3&6000\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&3900\\0&1&-600\end{array}\right]$$

Which means that we have two basic feasible points:

$$x_1^* = \begin{bmatrix}1200\\1800\\0\end{bmatrix}, \quad x_2^* = \begin{bmatrix}15600/7\\0\\3600/7\end{bmatrix}$$

(c) From the condition $x_i s_i = 0$ we have ~~that~~ for $x_1^* = (1200, 1800, 0)^T$ ~~that~~ that $s_1 = s_2 = 0$. Which means the equation $A^T\lambda + s = c$ yields:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & -100 \\ 2 & 2 & 0 & -75 \\ 1 & 3 & 1 & -55 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -25 \\ 0 & 1 & 0 & -25/2 \\ 0 & 0 & 1 & 15/2 \end{array}\right]$$

$$\Rightarrow \lambda = \begin{bmatrix} -25 \\ -25/2 \end{bmatrix}, \quad s = \begin{bmatrix} 0 \\ 0 \\ 15/2 \end{bmatrix}$$

For the point $(15600/7, 0, 3600/7)^T$ we have $s_1 = s_3 = 0$, so

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & -100 \\ 2 & 2 & 1 & -75 \\ 1 & 3 & 0 & -55 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -190/7 \\ 0 & 1 & 0 & -65/7 \\ 0 & 0 & 1 & -15/7 \end{array}\right]$$

$$\Rightarrow \lambda = \begin{bmatrix} -190/7 \\ -65/7 \end{bmatrix}; \quad s = \begin{bmatrix} 0 \\ -15/7 \\ 0 \end{bmatrix}$$

As $s_i \geq 0$ is a condition, the only feasible point that satisfies KKT is x_1^*.

So the solution is $x^* = \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix}$.

(d) The dual is

$$\max_{\lambda} b^T \lambda \quad \text{s.t.} \quad A^T \lambda \leq c$$

or alternatively:

$$\min_{\lambda} -b^T \lambda \quad \text{s.t.} \quad c - A^T \lambda \geq 0$$

(e) It is trivial to check that

$$c^T x^* = [-100 \ -75 \ -55] \cdot \begin{bmatrix} 1200 \\ 1800 \\ 0 \end{bmatrix} = -255000$$

$$= b^T \lambda^* = [7200 \quad 6000] \cdot \begin{bmatrix} -25 \\ -25/2 \end{bmatrix} = -255000$$

In general we have that:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c - A^T \lambda^* - s^* = 0$$

$$\Leftrightarrow c^T x^* = (A^T \lambda^* + s^*)^T x^*$$

$$= \lambda^{*T} A x^* + s^{*T} x^*$$

We have that $Ax^* = b$ and $s_i x_i = 0$, such that

$$c^T x^* = \lambda^{*T} b$$

Since $\lambda^{*T} b$ is just the dot product, we can change the order:

$$\underline{c^T x^* = b^T \lambda^*} \quad \square$$

(f) We have that

$$\lambda^* = \begin{bmatrix} -25 \\ -25/2 \end{bmatrix},$$

which means the multiplier is more sensitive to the constraint governing A than B. So make A more available!

By inputting $A = 7201$ into a LP solver we get $x^* = \begin{bmatrix} 1201 \\ 1799 \\ 0 \end{bmatrix}$, $f(x^*) = 255025$

and with $B = 6001$ we get

$$x^* = \begin{bmatrix} 1199 \\ 1801{,}5 \\ 0 \end{bmatrix}, \; f(x^*) = 255012{,}5.$$

As expected $f(x)$ is twice as sensitive to an increase in A compared to B.