[1] $q_{12} = C_{12}\sqrt{p_1 - p_2}$, $q_0 = C_0\sqrt{p_2 - p_0}$

ⓐ
$$\dot{V}_1 = A_1\dot{h}_1 = q_i - C_{12}\sqrt{p_1 - p_2}$$
$$= q_i - C_{12}\sqrt{\rho g}\sqrt{h_1 - h_2}$$

$$\Rightarrow \dot{h}_1 = \underline{\frac{1}{A_1}\left(q_i - C_{12}\sqrt{\rho g}\sqrt{h_1 - h_2}\right)}$$

$$\dot{V}_2 = A_2\dot{h}_2 = C_{12}\sqrt{\rho g}\sqrt{h_1 - h_2} - C_0\sqrt{\rho g}\sqrt{h_2}$$

$$\underline{\dot{h}_2} = \frac{\sqrt{\rho g}}{A_2}\underline{\left(C_{12}\sqrt{h_1 - h_2} - C_0\sqrt{h_2}\right)}$$

ⓑ
$$\dot{h}_1 = 0 \Rightarrow q_i^* = C_{12}\sqrt{\rho g}\sqrt{h_1^* - h_2^*}$$
$$\dot{h}_2 = 0 \Rightarrow q_i^* = C_0\sqrt{h_2^*}\sqrt{\rho g}$$

$$\Rightarrow \underline{\underline{h_2^* = \frac{q_i^{*2}}{\rho g C_0^2}}}$$

$$\underline{\underline{h_1^*}} = \frac{C_0^2 + C_{12}^2}{C_{12}^2} h_2^* = \underline{\underline{q_i^{*2}\frac{1}{\rho g}\cdot\frac{C_0^2 + C_{12}^2}{C_0^2 C_{12}^2}}}$$

$$\left.\frac{\partial f}{\partial h}\right|_* = \begin{bmatrix} \frac{\partial f_1}{\partial h_1} & \frac{\partial f_1}{\partial h_2} \\ \frac{\partial f_2}{\partial h_1} & \frac{\partial f_2}{\partial h_2} \end{bmatrix}_* = \begin{bmatrix} -\frac{C_{12}}{2A_1}\sqrt{\rho g}(h_1 - h_2)^{-1/2} & \frac{C_{12}}{2A_1}\sqrt{\rho g}(h_1 - h_2)^{-1/2} \\ \frac{C_{12}}{2A_2}\sqrt{\rho g}(h_1 - h_2)^{-1/2} & -\frac{C_{12}}{2A_2}\sqrt{\rho g}(h_1 - h_2)^{-1/2} - \frac{\sqrt{\rho g}}{2A_2}C_0 h_2^{-1/2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{C_{12}^2}{2A_1 q_i^*}\rho g & \frac{C_{12}^2}{2A_1 q_i^*}\rho g \\ \frac{C_{12}^2}{2A_2 q_i^*}\rho g & -\frac{C_{12}^2}{2A_2 q_i^*}\rho g - \frac{C_0^2 \rho g}{2A_2 q_i^*} \end{bmatrix}$$

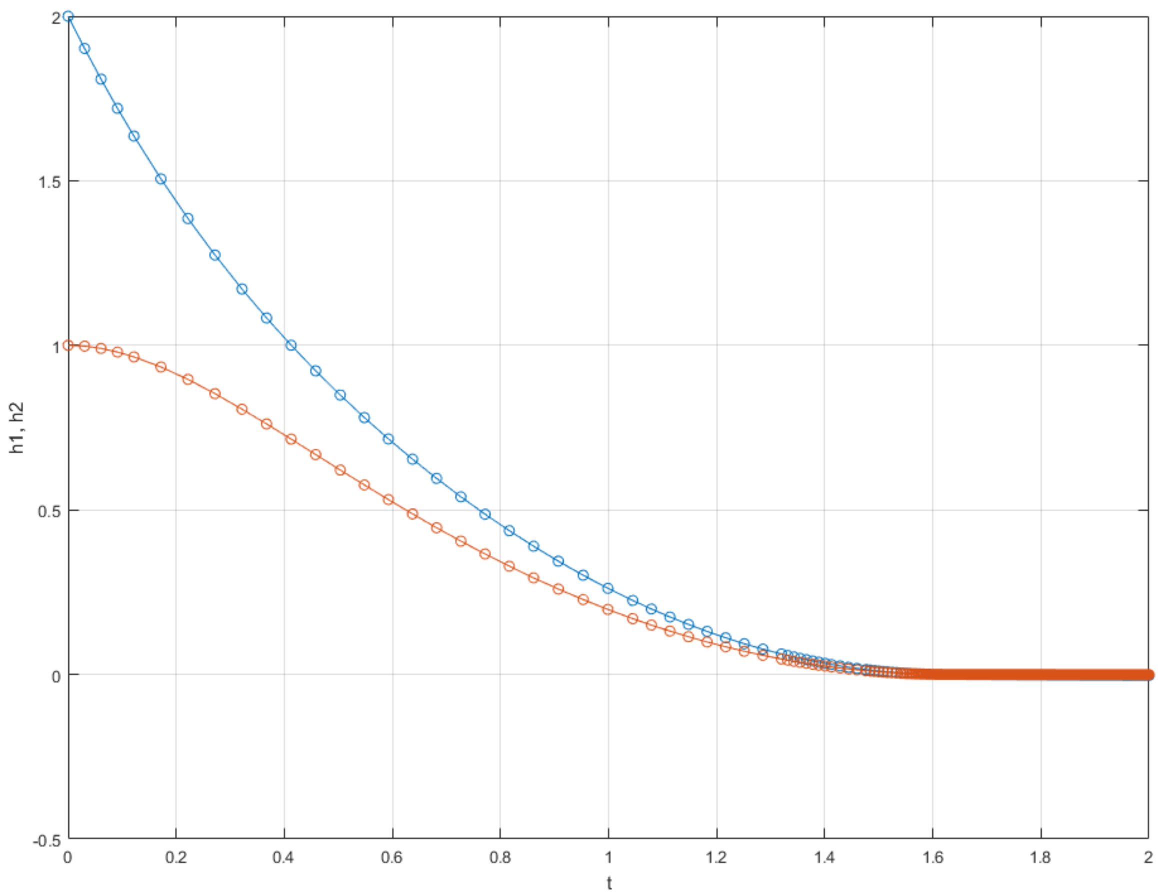
$$A = \frac{\rho g}{2 q_i^{*}} \begin{bmatrix} -\frac{C_{12}^2}{A_1} & \frac{C_{12}^2}{A_1} \\ \frac{C_{12}^2}{A_2} & -\frac{C_{12}^2 + C_0^2}{A_2} \end{bmatrix}$$

$$\underline{B} = \frac{\partial f}{\partial q_i} = \underline{\begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix}},$$

$$\Delta \dot{x} = \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = A \Delta x + B \Delta u, \quad u = \Delta q_i^{*}$$

When $q_i^{*} \to 0$ or $h_1 \to h_2$ we get a singularity, so the linearized model is not very useful.

The valve equations do not take laminar vs. turbulent flow into account.



2
1.5
1
0.5
0
-0.5
h1, h2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t

$\boxed{2}$ 1.

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 5/24 & 1/3 & -1/24 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

ⓐ $R(s) = \dfrac{\det(\mathbb{I} - s(A - \mathbb{1}b^T))}{\det(\mathbb{I} - sA)}$

$$\mathbb{I} - sA = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5s}{24} & 1-\frac{s}{3} & \frac{s}{24} \\ -\frac{s}{6} & -\frac{2s}{3} & 1-\frac{s}{6} \end{bmatrix}, \quad \mathbb{I} - sA + s\mathbb{1}b^T = \begin{bmatrix} 1+\frac{s}{6} & \frac{2s}{3} & \frac{s}{6} \\ -\frac{s}{24} & 1+\frac{s}{3} & \frac{5s}{24} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{R(s)}} = \frac{(1+\frac{s}{6})(1+\frac{s}{3}) + \frac{s^2}{36}}{(1-\frac{s}{3})(1-\frac{s}{6}) + \frac{s^2}{36}} = \underline{\underline{\frac{1+\frac{s}{2}+\frac{s^2}{12}}{1-\frac{s}{2}+\frac{s^2}{12}}}}$$

ⓑ When $Re\{s\} \leq 0$ the denominator is larger than the numinator so $|R(s)| \leq 1$ and $R(s)$ is A-stable. But not L-stable, as $R(jw) \xrightarrow{w\to\infty} 1$. Then it cannot be stiffly accurate.

$$M = \text{diag}(b)A + A^T\text{diag}(b) - bb^T$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 5/36 & 2/9 & -1/36 \\ 1/36 & 1/9 & 1/36 \end{bmatrix} + \begin{bmatrix} 0 & 5/36 & 1/36 \\ 0 & 2/9 & 1/9 \\ 0 & -1/36 & 1/36 \end{bmatrix} - \begin{bmatrix} 1/36 & 2/18 & 1/36 \\ 2/18 & 4/9 & 2/18 \\ 1/36 & 2/18 & 1/36 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{1}{36}$$

This matrix has both positive and negative eigenvalues and is therefore not positive semi definite. So the method is not algebraically stable.

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2.
$$\begin{array}{c|ccc} 0 & 1/6 & -1/6 & 0 \\ 1/2 & 1/6 & 1/3 & 0 \\ 1 & 1/6 & 5/6 & 0 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

ⓐ $\underline{\mathbb{I} - sA} = \begin{bmatrix} 1-\frac{s}{6} & \frac{s}{6} & 0 \\ -s/6 & 1-s/3 & 0 \\ -s/6 & -5s/6 & 1 \end{bmatrix}$, $\mathbb{I} - sA + s\mathbb{1}b^T = \begin{bmatrix} 1 & \frac{5s}{6} & s/6 \\ 0 & 1+\frac{s}{3} & s/6 \\ 0 & -\frac{s}{6} & 1+\frac{s}{6} \end{bmatrix}$

$$\Rightarrow \underline{\underline{R(s)}} = \frac{(1+\frac{s}{3})(1+\frac{s}{6}) + \frac{s^2}{36}}{(1-\frac{s}{6})(1-\frac{s}{3}) + \frac{s^2}{36}} = \underline{\underline{\frac{1+\frac{s}{2}+\frac{s^2}{12}}{1-\frac{s}{2}+\frac{s^2}{12}}}}$$

ⓑ The same arguments for A-stability, L-stability and stiffly accuracy holds as in 1, as the stability function is the same.

$\underline{M} = \mathrm{diag}(b)A + A^T\mathrm{diag}(b) - bb^T$

$$= \begin{bmatrix} 1/36 & -1/36 & 0 \\ 1/9 & 2/9 & 0 \\ 1/36 & 5/36 & 0 \end{bmatrix} + \begin{bmatrix} 1/36 & 1/9 & 1/36 \\ -1/36 & 2/9 & 5/36 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1/36 & 2/18 & 1/36 \\ 2/18 & 4/9 & 2/18 \\ 1/36 & 2/18 & 1/36 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \frac{1}{36}}$$

By the same argument as before in 1, this method is also not algebraically stable.

3. (a)

$$\begin{array}{c|ccc} 0 & 1/6 & -1/3 & 1/6 \\ 1/2 & 1/6 & 5/12 & -1/12 \\ 1 & 1/6 & 2/3 & 1/6 \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

$$(\mathbb{I} - sA) = \begin{bmatrix} 1-\frac{s}{6} & \frac{s}{3} & -\frac{s}{6} \\ -\frac{s}{6} & 1-\frac{5s}{12} & \frac{s}{12} \\ -\frac{s}{6} & -\frac{2s}{3} & 1-\frac{s}{6} \end{bmatrix},$$

$$\mathbb{I} - sA + s\mathbb{1}b^T = \begin{bmatrix} 1 & s & 0 \\ 0 & 1-\frac{s}{4} & \frac{s}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(s) = \frac{1-\frac{s}{4}}{(1-\frac{s}{6})\left((1-\frac{5s}{12})(1-\frac{s}{6}) + \frac{s^2}{18}\right) + \frac{s}{6}\left(\frac{s}{3}(1-\frac{s}{6}) - \frac{s}{6}(\frac{2s}{3})\right) - \frac{s}{6}\left(\frac{s^2}{36} + \frac{s}{6}(1-\frac{5s}{12})\right)}$$

$$R(s) = \frac{1-\frac{s}{4}}{-s^3/24 + s^2/4 - 3s/4 + 1}$$

(b) This method is clearly stiffly accurate, and

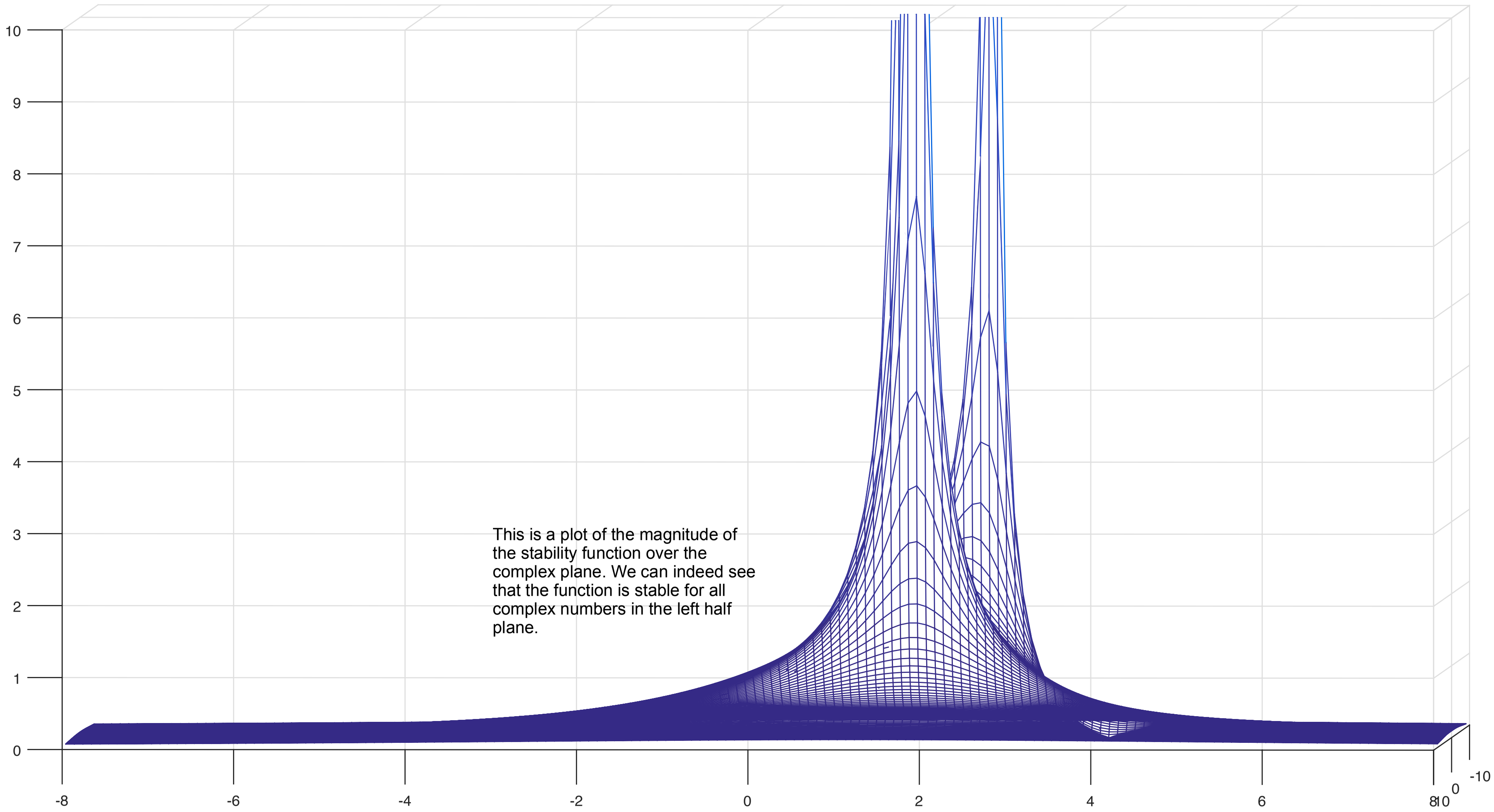
$|R(s)| \to 0$ when $s = jw \to \infty$.

Assuming that the method is A-stable it is therefore L-stable.

This seems like an awful calculation so I verified it graphically instead.

$$M = \begin{bmatrix} 1/36 & -1/18 & 1/36 \\ 1/9 & 5/18 & -1/18 \\ 1/36 & 1/9 & 1/36 \end{bmatrix} + \begin{bmatrix} 1/36 & 1/9 & 1/36 \\ -1/18 & 5/18 & 1/9 \\ 1/36 & -1/18 & 1/36 \end{bmatrix} - \begin{bmatrix} 1/36 & 1/9 & 1/36 \\ 1/9 & 4/9 & 1/9 \\ 1/36 & 1/9 & 1/36 \end{bmatrix}$$

$$= \begin{bmatrix} 1/36 & -1/18 & 1/36 \\ -1/18 & 1/9 & -1/18 \\ 1/36 & -1/18 & 1/36 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \geq 0 \implies$$ Algebraic stability!



This is a plot of the magnitude of the stability function over the complex plane. We can indeed see that the function is stable for all complex numbers in the left half plane.
10
9
8
7
6
5
4
3
2
1
0
-8
-6
-4
-2
0
2
4
6
8
10
0
-10

(c) An A-stable method is stable for all stable systems, i.e. $|R(s)| \le 1 \;\forall\, Re\{\lambda\} \le 0$

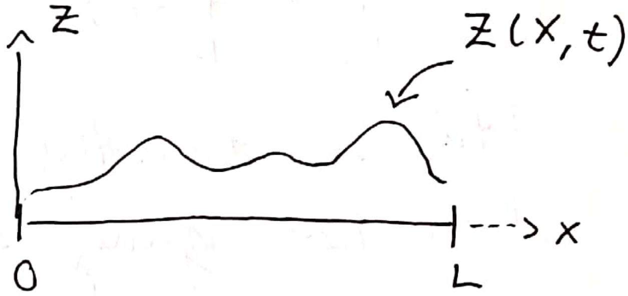
L-stable methods are A-stable and also have the property that fast frequencies are damped out i.e. $R(s) \xrightarrow{\omega \to \infty} 0$.

For oscillatory systems this means that too fast oscillations are removed, which in many cases is desirable. Sometimes we might want to keep the oscillations at a lower frequency instead, so we then want only A-stability.

$\boxed{3}$ $\dfrac{\partial Z}{\partial t} = D\dfrac{\partial^2 Z}{\partial x^2}$, $Z(x,0) = \begin{cases} Z_0, & 0 \le x \le L_0 \\ 0, & L_0 < x \end{cases}$,

$$\frac{\partial Z}{\partial x}(0,t) = \frac{\partial Z}{\partial x}(L,t) = 0$$

ⓐ $Z(x,t) = f(x)g(t)$



$$f(x)\dot{g}(t) = Df''(x)g(t)$$

$$\frac{f''(x)}{f(x)} = \frac{1}{D}\frac{\dot{g}(t)}{g(t)} = k = -\lambda^2$$

$$f''(x) - kf(x) = 0 \Rightarrow f(x) = A\cos\lambda x + B\sin\lambda x$$

$$\frac{\partial Z}{\partial x}(0,t) = f'(0)\,g(t) = 0$$

$$\Rightarrow f'(0) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x\Big|_{x=0} = B\lambda = 0$$

$$\Rightarrow \underline{B=0}$$

$$f'(L) = -A\lambda\sin\lambda L = 0 \Rightarrow \lambda L = \pi n$$

$$\Rightarrow \underline{\lambda = \frac{\pi n}{L}}$$

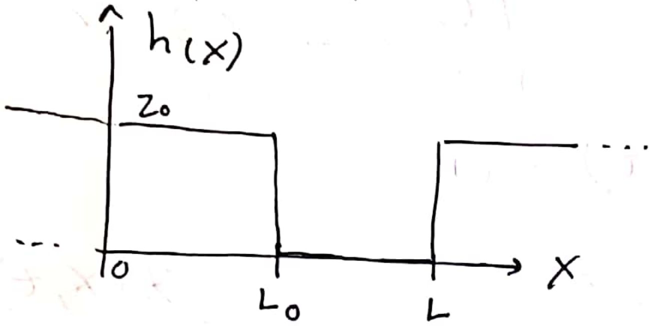
$$\Rightarrow f(x) = A\cos\frac{\pi n}{L}x$$

$$\frac{1}{D}\dot{g}(t) - kg(t) = 0, \quad \dot{g}(t) + \lambda^2 D\,g(t) = 0$$

$$\Rightarrow g_n(t) = B_n e^{-\lambda_n^2 D t}, \quad \alpha_n = AB_n$$

$$\Rightarrow Z(x,t) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{\pi n}{L}x\, e^{-\left(\frac{\pi n}{L}\right)^2 D t}$$

$$Z(x,0) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{\pi n}{L} x = h(x)$$



This is honestly kind of unnecessary... I'm not going to bother finding the Fourier series, so here is the pulse function:

$$h(x) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t \,, \quad a_n = \frac{2A}{n\pi} \sin n\pi \frac{T_p}{T}$$
$$= \frac{2Z_0}{n\pi} \sin n\pi \frac{L_0}{L}$$

$$\Rightarrow \underline{\alpha_n = \frac{2Z_0}{n\pi} \sin n\pi \frac{L_0}{L}} \,, \quad \underline{\alpha_0 = Z_0 \frac{L_0}{L}}$$

$$\underline{Z(x,t) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{\pi n}{L} x \, e^{-\left(\frac{\pi n}{L}\right)^2 Dt}} ,$$

The exponential decays to zero as $t \to \infty$, so the stationary value is 0.

ⓑ $$Z(x, t-\Delta t) = Z(x,t) - \frac{\partial Z}{\partial t}(x,t)\Delta t + \dots$$

$$\Rightarrow \frac{\partial Z}{\partial t}(x,t) \approx \frac{Z(x,t) - Z(x,t-\Delta t)}{\Delta t}$$

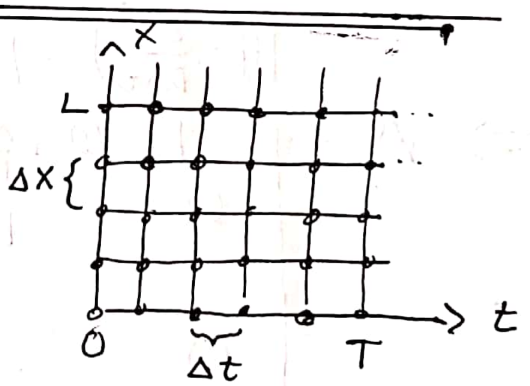
$$Z(x+\Delta x, t) = Z(x,t) + \frac{\partial Z}{\partial x}(x,t)\Delta x + \frac{\partial^2 Z}{\partial x^2}(x,t)\frac{\Delta x^2}{2} + \frac{\partial^3 Z}{\partial x^3}(x,t)\frac{\Delta x^3}{6} + \frac{\partial^4 Z}{\partial x^4}(x,t)\frac{\Delta x^4}{24}$$

$$Z(x-\Delta x, t) = Z(x,t) - \frac{\partial Z}{\partial x}(x,t)\Delta x + \frac{\partial^2 Z}{\partial x^2}(x,t)\frac{\Delta x^2}{2} - \frac{\partial^3 Z}{\partial x^3}(x,t)\frac{\Delta x^3}{6} + \frac{\partial^4 Z}{\partial x^4}(x,t)\frac{\Delta x^4}{24}$$

$$\Rightarrow Z(x+\Delta x, t) + Z(x-\Delta x, t) = 2Z(x,t) + \frac{\partial^2 Z}{\partial x^2}(x,t)\Delta x^2 + O(\Delta x^4)$$

$$\Rightarrow \frac{\partial^2 Z}{\partial x^2}(x,t) \approx \frac{Z(x+\Delta x, t) + Z(x-\Delta x, t) - 2Z(x,t)}{\Delta x^2}$$

ⓒ $Z_{n,m} = Z(x_n, t_m)$



$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t)$$

$$= \frac{Z(x,t) - Z(x,t-\Delta t)}{\Delta t} = D\,\frac{Z(x+\Delta x,t) + Z(x-\Delta x,t) - 2Z(x,t)}{\Delta x^2}$$

$$= \frac{Z_{n,m} - Z_{n,m-1}}{\Delta t} = D\,\frac{Z_{n+1,m} + Z_{n-1,m} - 2Z_{n,m}}{\Delta x^2}$$

$$\Leftrightarrow Z_{n,m}\Delta x^2 + 2D\Delta t\, Z_{n,m} - Z_{n,m-1}\Delta x^2 = D\Delta t\,(Z_{n+1,m} + Z_{n-1}$$

$$Z_{n,m}\left(\frac{\Delta x^2}{D\Delta t}+2\right)-Z_{n-1,m}-Z_{n+1,m}$$

$$=\frac{\Delta x^2}{D\Delta t}Z_{n,m-1}$$

I have no idea where you got those extra 2's from?

Want to write the system as $Ax=b$, with $x_m=[Z_{0,m}, Z_{1,m}, \dots, Z_{N,m}]$.

$$-Z_{0,m}+\left(2+\frac{\cancel{2}\Delta x^2}{D\Delta t}\right)Z_{0,m}-Z_{1,m}=\frac{\Delta x^2}{D\Delta t}Z_{0,m-1}$$

$$-Z_{0,m}+\left(2+\frac{\Delta x^2}{D\Delta t}\right)Z_{1,m}-Z_{2,m}=\frac{\Delta x^2}{D\Delta t}Z_{1,m-1}$$

$$\vdots$$

$$-Z_{N-1,m}+\left(2+\frac{\Delta x^2}{D\Delta t}\right)Z_{N,m}-Z_{N,m}=\frac{\Delta x^2}{D\Delta t}Z_{N,m-1}$$

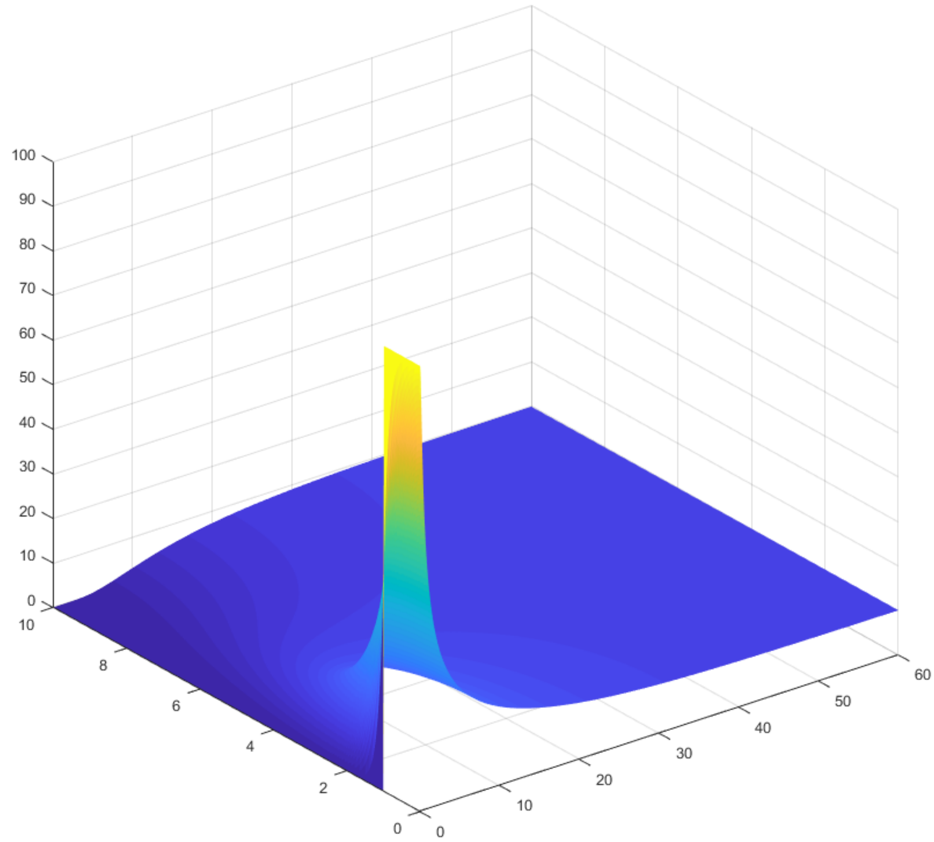
$$\Rightarrow A=\begin{bmatrix} \left(1+\frac{\Delta x^2}{D\Delta t}\right) & -1 & 0 & \cdots & & \cdots & 0 \\ -1 & \left(2+\frac{\Delta x^2}{D\Delta t}\right) & -1 & 0 & \cdots & & \vdots \\ 0 & -1 & \left(2+\frac{\Delta x^2}{D\Delta t}\right) & -1 & \cdots & & \\ \vdots & & -1 & \ddots & \ddots & & \\ 0 & & \cdots & & & -1 & \left(1+\frac{\Delta x^2}{D\Delta t}\right) \end{bmatrix}$$

$$b=\frac{\Delta x^2}{D\Delta t}\begin{bmatrix} Z_{0,m-1} \\ Z_{1,m-1} \\ \vdots \\ Z_{N,m-1} \end{bmatrix}$$

```
D = 1;
Z_0 = 100;
L = 10;
L_0 = 1;
T = 60;
N = 500;
M = 3000;
delta_t = T/M;
delta_x = L/N;
t = 0:delta_t:T;
x = 0:delta_x:L;
Z = [Z_0 * ones(L_0/delta_x,1); zeros((L-L_0)/delta_x + 1, 1)];

for i = delta_t:delta_t:T
    temp = [zeros(1,N+1); -eye(N) zeros(N,1)];
    A_i = eye(N+1) * (2 + delta_x^2/(D*delta_t)) + temp + temp';
    A_i(1,1) = A_i(1,1) - 1;
    A_i(N+1,N+1) = A_i(N+1,N+1) - 1;
    b_i = delta_x^2/(D*delta_t) * Z(:,end);
    Z = [Z A_i\b_i];
end

[T,X] = meshgrid(t,x);
s = surf(T, X, Z);
set(s,'LineStyle','none')
```



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We observe that the initial condition is indeed
diffused over time over the valid range.
The system quickly goes to a steady state where
the entities are evenly spread out.