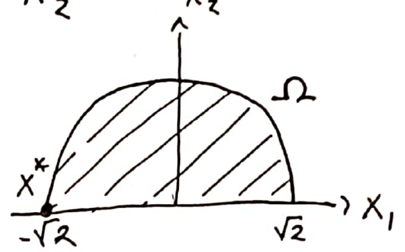
TTK4135 EXERCISE 1

$\boxed{1}$ a

$\min\ x_1 + 2x_2 \quad \text{s.t.}\quad 2 - x_1^2 - x_2^2 \geq 0,\ x_2 \geq 0$

$c_1(x) = 2 - x_1^2 - x_2^2, \quad c_2(x) = x_2$

$\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1 - \lambda_2 c_2$



It is trivial to see that the optimal point is

$x^* = (-\sqrt{2}, 0)^T$

b $\mathcal{L}(x,\lambda) = x_1 + 2x_2 - \lambda_1(2 - x_1^2 - x_2^2) - \lambda_2 x_2$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} 1 + 2\lambda_1 x_1 \\ 2 + 2\lambda_1 x_2 - \lambda_2 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*,\lambda^*) = \begin{bmatrix} 1 - 2\sqrt{2}\lambda_1^* \\ 2 - \lambda_2^* \end{bmatrix} = 0$$

$$\Rightarrow \underline{\lambda^* = \begin{bmatrix} 1/2\sqrt{2} \\ 2 \end{bmatrix}} > 0$$

All the constraints are active in x^*, i.e. $c_1^* = 2 - x_1^{*2} - x_2^{*2} = 2 - \sqrt{2}^2 = 0$

$c_2(x^*) = x_2^* = 0$

Since $c_i(x^*) = 0$ it also follows that

$\lambda_i^* c_i(x^*) = 0$ for $i \in [1,2]$,

so all the KKT conditions are satisfied.

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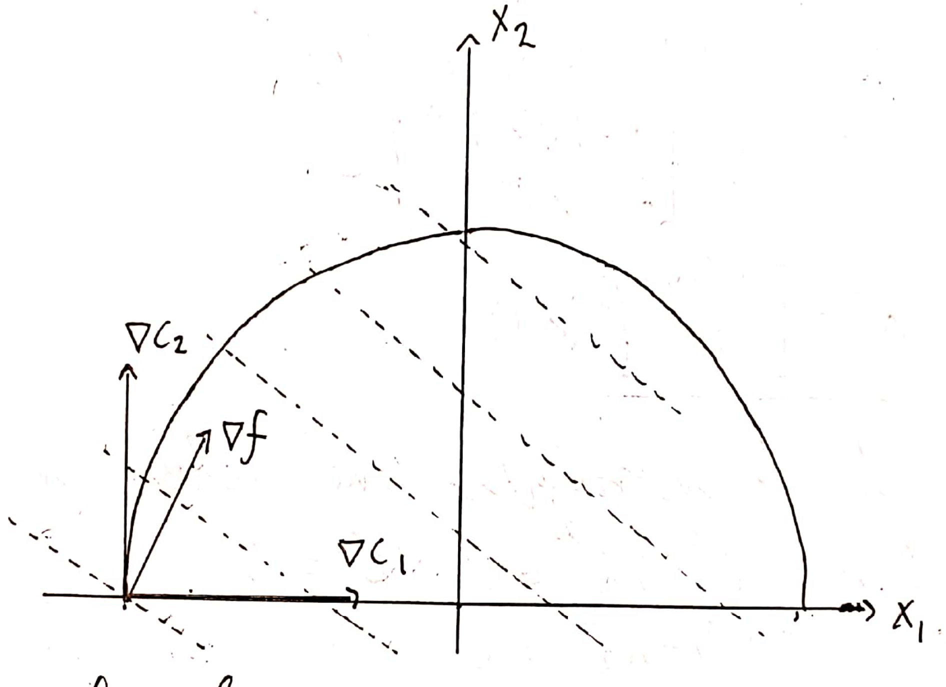
(c)

$$\nabla f(x^*) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \nabla c_1(x^*) = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}, \quad \nabla c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



(d) The Lagrange multipliers has to be positive, as we want the gradient of the objective function and the gradient of the constraints to point in the same direction, not opposite.

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ⓒ The feasible set is clearly convex, and since $f(x)$ is linear it is also convex.
Viz the problem is convex.

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[2] $\min\ 2x_1 + x_2 \quad s.t. \quad x_1^2 + x_2^2 - 2 = 0$

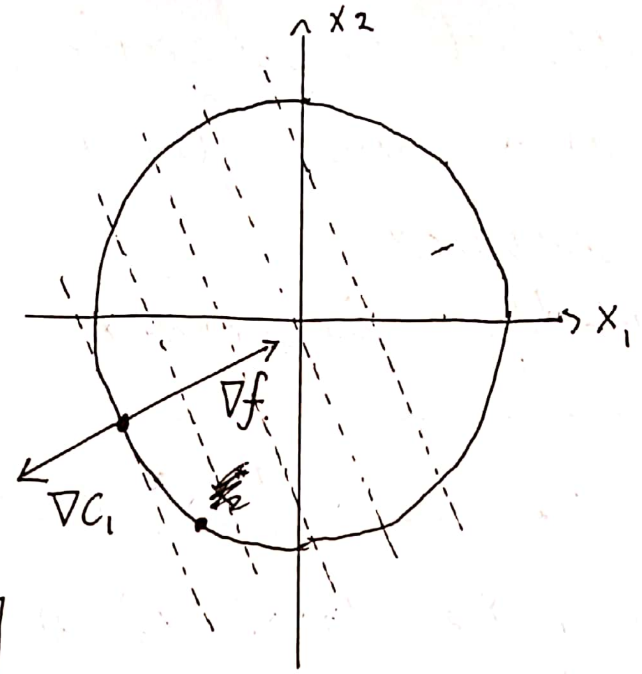
ⓐ $\mathcal{L}(x, \lambda) = 2x_1 + x_2 - \lambda(x_1^2 + x_2^2 - 2)$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 - 2\lambda^* x_1^* \\ 1 - 2\lambda^* x_2^* \end{bmatrix} = 0$$

So an extreme point has the form
$(1/\lambda^*, 1/2\lambda^*)^T$.
We also have that
$x_1^{*2} + x_2^{*2} - 2 = 1/\lambda^{*2} + 1/4\lambda^{*2} - 2 = 0$
$\Rightarrow \lambda^* = \pm\sqrt{\frac{5}{8}}$
Which means the extreme points are
$(\sqrt{\frac{8}{5}}, \sqrt{\frac{2}{5}})$ and $(-\sqrt{\frac{8}{5}}, -\sqrt{\frac{2}{5}})$.
The solution is obviously $(-\sqrt{\frac{8}{5}}, -\sqrt{\frac{2}{5}})$.

ⓑ We have already shown that the stationary condition $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ and the feasability condition $c_1(x^*) = 0$ are met in the two extreme points.
The complementary condition is also met, as $c_1(x^*) = 0 \Rightarrow \lambda_1^* c_1(x^*) = 0$. The two extreme points are the max and min in Ω, so KKT should be true in both cases.

(c)



$$\nabla f(x^*) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\nabla c_1(x^*) = \begin{bmatrix} -2\sqrt{\frac{8}{5}} \\ -2\sqrt{\frac{2}{5}} \end{bmatrix} = \begin{bmatrix} -4\sqrt{2/5} \\ -2\sqrt{2/5} \end{bmatrix}$$

(d) We found previously that

$$\underline{\underline{\lambda^* = -\sqrt{\frac{5}{8}}}}$$

Since we have an equality constraint the sign is not important and therefore not a part of the KKT conditions, so it is consistent with KKT. (Only multipliers associated with an inequality constraint need to be nonnegative).

(e) The second order condition states that:

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w \geq 0 \quad \forall w \in C(x^*, \lambda^*)$$

Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w = [w_1 \; w_2] \cdot \begin{bmatrix} -2\lambda^* & 0 \\ 0 & -2\lambda^* \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= -2\lambda^* (w_1^2 + w_2^2)$$

We observe that the sign of this product is only dependent on λ^*.

So for the extreme point $(2\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}})$ we have that $\lambda^* > 0$ so the 2. order condition is not satisfied. This is to be expected, as this point is the maxima.

The other extreme point $(-2\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}})$ has $\lambda^* < 0$ so the 2. order condition is satisfied, and this point is therefore the optimal point.

(f) While the objective function is convex, the feasible set is not. Any two points on the circle $c_1(x) = 0$ which are not identical will leave the circle, so it is not a convex problem.

$\boxed{3}$ $\min f(x) = -2x_1 + x_2$ s.t. $c_1(x) = (1-x_1)^3 - x_2 \geq 0,$
$c_2(x) = x_2 + 0{,}25\,x_1^2 - 1 \geq 0$

ⓐ $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\nabla c_1(x^*) = \begin{bmatrix} -3(1-x_1^*)^2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$\nabla c_2(x^*) = \begin{bmatrix} 0{,}5\,x_1^* \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are clearly linearly independent, so LICQ holds in x^*.

ⓑ

$$\mathcal{L}(x,\lambda) = -2x_1 + x_2 - \lambda_1((1-x_1)^3 - x_2) - \lambda_2(x_2 + 0{,}25\,x_1^2 - 1)$$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} -2 + 3\lambda_1(1-x_1)^2 - 0{,}5\,\lambda_2 x_1 \\ 1 + \lambda_1 - \lambda_2 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*,\lambda^*) = \begin{bmatrix} -2 + 3\lambda_1^* \\ 1 + \lambda_1^* - \lambda_2^* \end{bmatrix} = 0$$

$$\Rightarrow \lambda^* = \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix} \geq 0 \quad c_1(x^*) = 1-1 = 0, \quad c_2(x^*) = 1-1 = 0$$

Since all constraints are active the complementary conditions are also satisfied. Thus KKT is satisfied.

(d) $$\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*) = \begin{bmatrix} -6\lambda_1^*(1-x_1^*)-0{,}5\lambda_2^* & 0 \\ 0 & 0 \end{bmatrix}$$

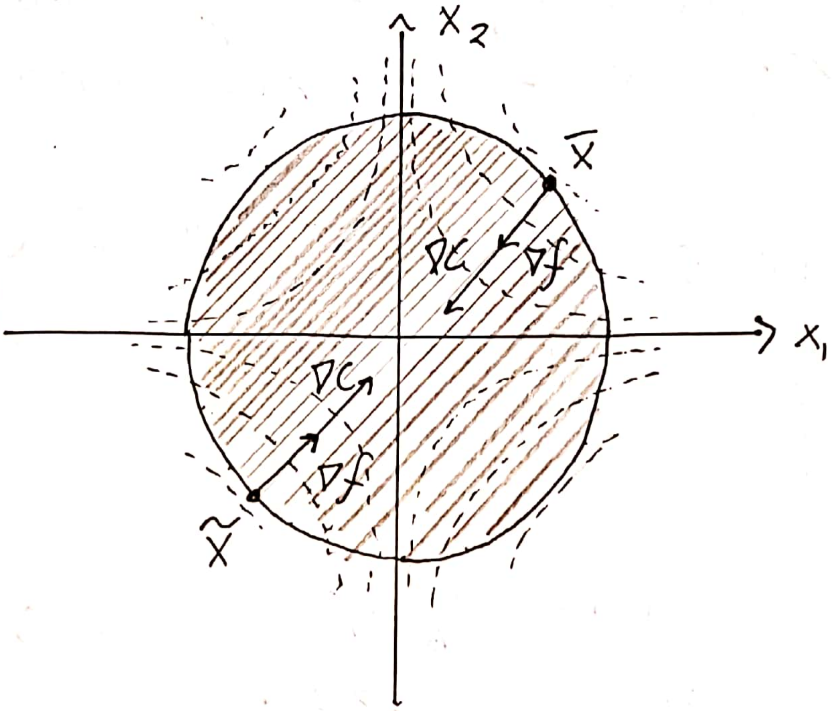
$$= \begin{bmatrix} -\frac{29}{6} & 0 \\ 0 & 0 \end{bmatrix}$$

$$w^T \nabla^2_{xx}\mathcal{L}(x^*,\lambda^*) = -\frac{29}{6} w_1^2 \geq 0$$

The above equation has to hold for $\nabla c_1(x^*)^T w = 0$ and $\nabla c_2(x^*)^T w = 0$, which is only true for $w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so

$w^T \nabla^2_{xx}\mathcal{L}(x^*,\lambda^*) = 0$, and therefore the 2. order neccesary conditions are satisfied, but the sufficient conditions are not.

$\boxed{4}$ min $f(x) = -x_1 x_2$ s.t. $c_1(x) = 1 - x_1^2 - x_2^2 \geq 0$



$$\nabla \bar{f} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$
$$\nabla \bar{c} = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$
$$\nabla \tilde{f} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$\nabla \tilde{c} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\mathcal{L}(x, \lambda) = -x_1 x_2 - \lambda(1 - x_1^2 - x_2^2)$$

$$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} -x_2 + 2\lambda x_1 \\ -x_1 + 2\lambda x_2 \end{bmatrix} = 0$$

$\Rightarrow x_1 = 2\lambda x_2, \quad x_2 = 2\lambda x_1 = (2\lambda)^2 x_2$

$\Rightarrow 4\lambda^2 = 1$

$2\lambda = 1$ (as $\lambda^* \geq 0$)

Assume that c_1 is active.
It is trivial to see graphically that this has to be the case.

$\rightarrow 1 = x_1^2 + x_2^2 = (1 + 4\lambda^2) x_2^2 = 2x_2^2$

$\Rightarrow x_2 = \pm \frac{1}{\sqrt{2}}$,

$x_1 = 2\lambda x_2 = x_2 = \pm \frac{1}{\sqrt{2}}$

So the two solutions are $x^* = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.