SIMULATION

TTK4130 SUMMARY

- The one-step method $y_{n+1} = y_n + h\phi(y_n, t_n)$ is of order p if $e_{n+1} = O(h^{p+1})$

ERK: $y_{n+1} = y_n + h(b_1k_1 + \dots + b_\sigma k_\sigma)$, $k_1 = f(y_n, t_n)$,
$k_2 = f(y_n + ha_{21}k_1, t_n + c_2h), \dots, k_\sigma = f(y_n + ha_{\sigma 1}k_1 + \dots + ha_{\sigma\sigma-1}k_{\sigma-1}, t_n + c_\sigma h)$

Butcher array: $\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccc} 0 & & & & \\ c_2 & a_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_\sigma & a_{\sigma 1} & \cdots & a_{\sigma\sigma-1} & \\ \hline & b_1 & \cdots & b_{\sigma-1} & b_\sigma \end{array}$ $\quad \sum_{i=1}^{\sigma} b_i = 1$, $0 \le c_i \le 1$, $c_1 \le c_2 \le \dots \le c_\sigma$, $\sum_{j=1}^{i-1} a_{ij} = c_i \le 1$

Euler: $\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$, Modified ~~Improved~~ Euler: $\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$, Improved Euler: $\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$

ERK4: $\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$

ERK stability function: $\boxed{R_E(h\lambda) = \det(\mathbb{I} - \lambda h(A - 1b^T))}$

- Stiff system: large spread of eigenvalues

IRK: $\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1\sigma} \\ c_2 & a_{21} & & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ c_\sigma & a_{\sigma 1} & & \cdots & a_{\sigma\sigma} \\ \hline & b_1 & b_2 & \cdots & b_\sigma \end{array}$ Implicit Euler: $\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$, Trapezoidal: $\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$,
Implicit midpoint: $\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$

Stability function: $R(h\lambda) = 1 + \lambda h b^T(\mathbb{I} - h\lambda A)^{-1}1 = \dfrac{\det(\mathbb{I} - \lambda h(A - 1b^T))}{\det(\mathbb{I} - \lambda hA)}$

* A-stability: $|R(h\lambda)| \le 1 \ \forall \ Re\{\lambda\} \le 0$
* Stiffly accurate: A is nonsingular and $b = A^T e_\sigma$, where $e_\sigma = [0\,0 \cdots 1]^T$.
* L-stability: A-stable + $\lim_{\omega \to \infty} |R(jh\omega)| = 0 \ \forall \ \dot{y} = \lambda y$ where $\lambda = j\omega$

Padé-approximations: want $R(s) \approx e^s \to P^k_m(s)$ is best!

- * $k \le m \le k+2 \Rightarrow |P^k_m(s)| \le 1$, $Re\{s\} \le 0$
  * $\lim_{\omega \to \infty} |P^m_m(j\omega)| = 1$
  * $m > k \Rightarrow \lim_{\omega \to \infty} |P^k_m(s)| = 0$

AN-stability: $Re\{\lambda_i\} \le 0 \Rightarrow |R(h\Lambda)| \le 1$ and that $(1 - h\Lambda A)$ is nonsingular.

An A-stable stiffly accurate RK method is L-stable.

Algebraic stability: $M = diag(b)A + A^T diag(b) - bb^T \ge 0$.
$\Rightarrow$ B-stability $\Rightarrow$ AN-stability $\Rightarrow$ A-stability

* Automatic adjustment of h:
$e_{n+1} \approx \hat{y}_{n+1} - y_{n+1}$, $\varepsilon_{n+1} = \|e_{n+1}\|_p$
if $\varepsilon_{n+1} \gg e_{tol}$ or $\varepsilon_{n+1} \ll e_{tol}$:
$h_{new} = h\left(\frac{e_{tol}}{\varepsilon_{n+1}}\right)^{1/1+p}$
* Event detection:
solve $g(y_n(\alpha), t + \alpha h) = 0$ for α.
- * Multi-step methods:
$y_{n+1} + \alpha y_n + \dots = h\beta_0 f(y_n, t_n) + h\beta_1 f(y_{n+1}, t_{n+1}) + \dots$

DAEs:

$F(\dot{x}, x, u)$ is a DAE if $\frac{\partial F}{\partial \dot{x}}$ is rank deficient.
* Fully-implicit: $F(\dot{x}, x, z, u) = 0$
solvable if $[\frac{\partial F}{\partial \dot{x}} \ \frac{\partial F}{\partial z}]$ has full rank.
* Semi-implicit: $\dot{x} = f(x, z, u)$, $0 = g(x, z, u)$
solvable if $\frac{\partial g}{\partial z}$ has full rank.

Differential index: # $\frac{d}{dt}$ needed to transform DAE to ODE.

RIGID BODY DYNAMICS:

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$$u^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Rigid body kinematics:

Rotation matrix: $v^a = R^a_b v^b$, $R^b_a = (R^a_b)^T = (R^a_b)^{-1}$

$R^a_b \in SO(3) = \{R \mid R \in \mathbb{R}^{3\times 3},\ R^T R = \mathbb{I},\ \det(R) = 1\}$

Homogeneous transformation matrix: $T^a_b = \begin{bmatrix} R^a_b & r^a_{ab} \\ 0 & 1 \end{bmatrix} = (T^b_a)^{-1}$

Euler angles: $R^a_b = R_z(\psi) R_y(\theta) R_x(\varphi)$

Angle axis representation: k, θ

Euler parameters: $q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \begin{bmatrix} \cos\theta/2 \\ k \sin\theta/2 \end{bmatrix}$

$\boxed{\omega^a_{ac} = \omega^a_{ab} + \omega^a_{bc}}$

Angular velocity: $\dot{R}^a_b = (\omega^a_{ab})^\times R^a_b = R^a_b (\omega^b_{ab})^\times$

Vector derivative: $\dot{u}^a = R^a_b (\dot{u}^b + (\omega^b_{ab})^\times u^b)$

$\mathbf{r}_p = \mathbf{r}_o + \mathbf{r}$, $\mathbf{v}_p = \mathbf{v}_o + \frac{{}^b d}{dt}\mathbf{r} + \omega_{ib} \times \mathbf{r}$,

$\mathbf{a}_p = \mathbf{a}_o + \frac{{}^b d^2}{dt^2}\mathbf{r} + 2\omega_{ib} \times \frac{{}^b d}{dt}\mathbf{r} + \alpha_{ib} \times \mathbf{r} + \omega_{ib} \times (\omega_{ib} \times \mathbf{r})$

Newton-Euler equations of motion:

Inertia matrix: $M^b_{b/c} = \int ((r^b)^2 \mathbb{I} - r^b (r^b)^T)\, dm$

Parallel axis theorem: $M^b_{b/o} = M^b_{b/c} + m((r^b_g)^2 \mathbb{I} - r^b_g (r^b_g)^T)$

Newton-Euler: $\mathbf{F}_{bc} = m\mathbf{a}_c$, $\mathbf{T}_{bc} = M_{b/c} \cdot \alpha_{ib} + \omega_{ib} \times (M_{b/c} \cdot \omega_{ib})$

$$\Leftrightarrow \begin{bmatrix} m\mathbb{I} & 0 \\ 0 & M^b_{b/c} \end{bmatrix} \begin{bmatrix} \dot{v}^b_c \\ \alpha^b_{ib} \end{bmatrix} + \begin{bmatrix} m(\omega^b_{ib})^\times v^b_c \\ (\omega^b_{ib})^\times M^b_{b/c} \omega^b_{ib} \end{bmatrix} = \begin{bmatrix} F^b_{bc} \\ T^b_{bc} \end{bmatrix}$$

Lagrangian mechanics:

Generalized coordinates: $\mathbf{r}_k = \mathbf{r}_k(q(t), t)$

Virtual displacement: $\delta \mathbf{r}_k = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i$

d'Alembert's principle: $\sum_{k=1}^{N} \frac{\partial \mathbf{r}_k}{\partial q_i} \cdot \left(m_k \frac{d^2 \mathbf{r}_k}{dt^2} - \mathbf{F}_k\right) = 0$

Lagrange's equation of motion: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$, $i = 1, \dots, n$

where $\mathcal{L}(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q)$

Lagrange's equation of first kind: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \tau_i$

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Passivity:

Energy function: $V(x) > 0$, $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t) < 0 \rightarrow$ stability

Passivity: $\boxed{\int_0^T y(t)u(t)\,dt \geq -E_0 \quad \forall u, t \geq 0}$

H is positive real if:
* $H(s)$ is analytic $\forall\, Re\{s\} > 0$.
* $H(s) \in \mathbb{R}\ \forall\ s \in \mathbb{R} > 0$.
* $Re\{H(s)\} \geq 0\ \forall\, Re\{s\} > 0$

A rational proper $H(s)$ is positive real if:
* $H(s)$ has no poles in $Re\{s\} > 0$.
* $Re\{H(j\omega)\} \geq 0\ \forall\, \omega$ s.t. $j\omega$ is not a pole.
* If $j\omega_0$ is a pole of H it is simple and $\underset{s=j\omega_0}{Res} H(s)$ is real and positive.

$y(s) = H(s)u(s)$ is passive if and only if H is positive real.

Storage function: $\dot{V} = \frac{\partial V}{\partial x} f(x,u,t) = u^T y - g(x)$ is passive for $V(x) \geq 0$, $g(x) \geq 0$.

$\beta = -V\frac{\partial p}{\partial V}$

FRICTION:

Static:

* Coulomb: $F_f = F_c \,\mathrm{sgn}(v)$, $v \neq 0$, $F_c = \mu F_N$
* Karnopp: $F_f = \begin{cases} \mathrm{sat}(F_a, F_c), & v = 0 \\ F_c \,\mathrm{sgn}(v), & v \neq 0 \end{cases}$
* Stribeck: $F_f = \left(F_c + (F_s - F_c)\, e^{-\left(\frac{v}{v_s}\right)^2}\right) \mathrm{sgn}(v)$
* Viscous: $F_f = F_v v$

Dynamic:

* Dahl: $\dot{F} = \sigma\left(v - |v| \frac{F}{F_c}\right)$
* LuGre: $F = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v$, $\dot{z} = v - \sigma_0 \frac{|v|}{g(v)} z$, $g(v) = F_c + (F_s - F_c)\, e^{-\left(\frac{v}{v_s}\right)^2}$

Transmission line: $\frac{\partial p}{\partial t} = -\frac{\beta}{A}\frac{\partial q}{\partial x}$, $\frac{\partial q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial x} - \frac{F(q)}{\rho}$

ELECTRICAL MOTORS:

Gear: $\omega_{out} = n\,\omega_{in}$, $T_{out} = \frac{1}{n} T_{in}$

DC motor: $L_a \frac{d}{dt} i_a = -R_a i_a - K_E \omega_m + u_a$

$J_m \dot{\omega}_m = K_T i_a - T_L$, $\dot{\theta}_m = \omega_m$

Balance laws:

Reynold's transport theorem:

$$\frac{d}{dt}\int_{V_c} \varphi(x,t)\,dV = \int_{V_c} \frac{\partial \varphi(x,t)}{\partial t}\,dV + \int_{\partial V_c} \varphi v_c \cdot n\,dA$$

* Mass balance: $\frac{V_c}{\beta}\dot{p} + \dot{V}_c = q_1 - q_2$
* Momentum balance: $\frac{d}{dt}\int_{V_c} \rho v\,dV = F - \int_{\partial V_c} \rho v (v - v_c) \cdot n\,dA$
* Energy balance: $\frac{d}{dt}\int_{V_c} \rho e\,dV = -\int_{\partial V_c} \rho e v \cdot n\,dA + \dot{Q} - \dot{W}$
  ($e = u + \frac{1}{2}v^2 + gz$)

General integral balance: $\frac{d}{dt}\int_\Omega \Psi\,dV = -\int_\Gamma \Phi_\psi \cdot n\,dA + \int_\Omega \sigma_\psi\,dV$

General differential balance: $\frac{\partial \Psi}{\partial t} + \nabla \cdot \Psi v + \nabla \Phi_\psi = \sigma_\psi$

Material Derivative: $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \nabla f$

* Mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = 0$
* Momentum: $\rho \frac{Dv}{Dt} = B + \nabla \sigma$
* Energy: $\rho \frac{De}{Dt} = -\nabla q + q''' + \nabla(\sigma \cdot v)$

Valves:

$q = C_d A \sqrt{\frac{2}{\rho}\Delta p}$ (turbulent)

$q = C_l \Delta p$ (laminar)

Hydraulic motors:

$\frac{V}{\beta}\dot{p} + \dot{V} = q_{in} - q_{out}$, rotational: $T_m = D_m(p_1 - p_2)$

$E = mu = mc_p T$
(T, U neglected)