Project 1: Interpolation and Derivatives

This project has two parts:

1. Interpolate a known function using specific sampled points
2. Compute the derivative of the function using the basis functions

1 Interpolation

Let us use the function:

$$f(x) = \sin\left(\frac{\pi}{2}x\right), \quad x \in [-1, +1] \tag{1}$$

Using nodal basis functions $L_i(x)$ construct an interpolant of $f(x)$ as follows:

$$f_N(x_k) = \sum_{i=0}^{N} L_i(x_k) f_i \tag{2}$$

where x_k are $k = 1, \ldots, 50$ evenly spaced points and the Nth order interpolation points are:

1. **Equally-spaced points**
2. **The roots of the Lobatto polynomials** (these are called the Legendre-Gauss-Lobatto points)
3. **The roots of the Legendre polynomials** (these are called the Legendre-Gauss points)

Once you construct the Lagrange polynomials $L_i(x)$, check to see how accurate your interpolation becomes as you increase N. Run values of $N = 1, \ldots, 64$.

To determine how well your interpolant is, use the normalized error norms defined at the end of this document. Plot the error as a function of order N.

2 Derivative

For the same analytic function:

$$f(x) = \sin\left(\frac{\pi}{2}x\right), \quad x \in [-1, +1] \tag{3}$$

using the Lagrange polynomials evaluate the derivative of $f(x)$ and compare with the exact solution using error norms. Use all three types of points as above.

3 Write-Up

Turn in a few pages (enough to cover all the points) discussing what you see for each of the three interpolation points. Which points allow you to construct a good solution and which do not. Corroborate your arguments with error norm plots and show me a figure or two showing what happens when one of the interpolation point sets breaks down. Please also submit your code.

Be sure to build a routine (I call it `Lagrange_basis` or something like that) that gives you a matrix L_{ij} of Lagrange polynomials for the interpolation points (i) evaluated at the sampling points (j). This will make it easier to use for Project 2 where you will need all this machinery again.

4 Helpful Relations

The normalized error norms that you should use are:

L^1 **norm:**

$$\|error\|_{L^1} = \frac{\sum_{k=1}^{N_s} |f_N(x_k) - f(x_k)|}{\sum_{k=1}^{N_s} |f(x_k)|} \tag{4}$$

L^2 **norm:**

$$\|error\|_{L^2} = \sqrt{\frac{\sum_{k=1}^{N_s} (f_N(x_k) - f(x_k))^2}{\sum_{k=1}^{N_s} f(x_k)^2}} \tag{5}$$

L^∞ **norm:**

$$\|error\|_{L^\infty} = \frac{\max_{1\le k\le N_s} |f_N(x_k) - f(x_k)|}{\max_{1\le k\le N_s} |f(x_k)|} \tag{6}$$

where $k = 1, \ldots, N_s$ are the number of points for which you are computing the error norms. Recall that the equation for the Lagrange polynomials is:

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{N} \frac{(x - x_j)}{(x_i - x_j)} \tag{7}$$

Differentiating this expression yields:

$$\frac{dL_i}{dx}(x) = \sum_{\substack{k=0 \\ k\neq i}}^{N} \left(\frac{1}{x_i - x_k} \right) \prod_{\substack{j=0 \\ j\neq i \\ j\neq k}}^{N} \frac{(x - x_j)}{(x_i - x_j)} \tag{8}$$