Project 2: Numerical Integration / Quadrature

Using the interpolation functions from Homework 1, sample the basis functions at equispaced, LGL and LG integration points in order to perform the Gauss quadrature:

$$\int_{-1}^{+1} f_N(x)dx = \int_{-1}^{+1} \sum_{i=0}^{N} L_i(x) f_i(x) dx = \sum_{k=0}^{n} w_k \left(\sum_{i=0}^{N} L_i(x_k) f_i \right)$$

where x_k are the Nth order Gauss quadrature points and compare against the exact integral of $f(x)$ for $x \in [1, +1]$. Analyse this until $N = 128$.

Turn in a few pages (enough to cover all the points) discussing what you see for each of the three quadrature points. Corroborate your arguments with error norm plots and show a figure or two showing what happens when one of the quadrature point sets breaks down. Please append your code at the end of your report.

Helpful Relations

The normalized error norms that you should use are:

L^1 norm:

$$\|error\|_{L^1} = \frac{\sum_{k=1}^{N_s} |f_N(x_k) - f(x_k)|}{\sum_{k=1}^{N_s} |f(x_k)|} \tag{1}$$

L^2 norm:

$$\|error\|_{L^2} = \sqrt{\frac{\sum_{k=1}^{N_s} (f_N(x_k) - f(x_k))^2}{\sum_{k=1}^{N_s} f(x_k)^2}} \tag{2}$$

L^∞ norm:

$$\|error\|_{L^\infty} = \frac{\max_{1 \le k \le N_s} |f_N(x_k) - f(x_k)|}{\max_{1 \le k \le N_s} |f(x_k)|} \tag{3}$$

where $k = 1, \ldots, N_s$ are the number of points for which you are computing the error norms.

Recall that the equation for the Lagrange polynomials is:

$$L_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{N} \frac{(x - x_j)}{(x_i - x_j)} \tag{4}$$

Differentiating this expression yields:

$$\frac{dL_i}{dx}(x) = \sum_{\substack{k=0 \\ k \ne i}}^{N} \left(\frac{1}{x_i - x_k} \right) \prod_{\substack{j=0 \\ j \ne i \\ j \ne k}}^{N} \frac{(x - x_j)}{(x_i - x_j)} \tag{5}$$