TTK4190 Guidance and Control of Vehicles

Assignment 1

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Problem 1 Attitude Control of Satellite

Problem 1.1

Problem 1.1.1 Finding equilibrium

We are given the following equations of motion (EoM) for the satellite

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega}, \\ \mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} &= \boldsymbol{\tau}.\end{aligned} \tag{1}$$

To linearize this, we must first find an equilibrium point. We set the derivatives $\dot{\mathbf{q}}$ and $\dot{\boldsymbol{\omega}}$ equal to zero and in addition require $\mathbf{q} = [\eta, \epsilon_1, \epsilon_2, \epsilon_3]^\top = [\eta, 0, 0, 0]^\top$ and $\boldsymbol{\tau} = \mathbf{0}$ thus obtaining

$$\mathbf{T}_\mathbf{q}(\mathbf{q})\boldsymbol{\omega}_0 = \mathbf{0}, \tag{2}$$

$$\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega}_0)\boldsymbol{\omega}_0 = \mathbf{0}. \tag{3}$$

Using (2.69) from [1] we can rewrite (2) and obtain

$$\begin{aligned}\mathbf{T}_\mathbf{q}(\mathbf{q})\boldsymbol{\omega}_0 &= \frac{1}{2}\begin{bmatrix} -\boldsymbol{\epsilon}_0^\top \\ \eta\mathbf{I}_3 + \mathbf{S}(\boldsymbol{\epsilon}_0)\end{bmatrix}\boldsymbol{\omega}_0 \\ &= \frac{1}{2}\begin{bmatrix}\mathbf{0}^\top \\ \mathbf{I}_3\end{bmatrix}\boldsymbol{\omega}_0 \\ &= \mathbf{0}.\end{aligned} \tag{4}$$

Hence $\boldsymbol{\omega}_0 = \mathbf{0}^\top$ and we have the equilibrium point $\mathbf{x}_0 = [\boldsymbol{\epsilon}_0^\top, \boldsymbol{\omega}_0^\top]^\top = \mathbf{0}^\top$. This clearly also satisfies (3).

Problem 1.1.2 Linearizing around equilibrium

We wish to linearize this non-linear system around $\mathbf{x}_0$. For that, we need expressions for $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\omega}}$. We find

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\eta\mathbf{I}_3 + \mathbf{S}(\boldsymbol{\epsilon}))\boldsymbol{\omega} = \frac{1}{2}\eta\boldsymbol{\omega} + \frac{1}{2}\mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} = \frac{1}{2}\eta\boldsymbol{\omega} = \frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}}\boldsymbol{\omega}, \quad \text{and} \tag{5}$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}_{CG}^{-1}(\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}) = \frac{1}{mr^2}(\mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} + \boldsymbol{\tau}) = S(\boldsymbol{\omega})\boldsymbol{\omega} + \frac{1}{mr^2}\boldsymbol{\tau} = \frac{1}{mr^2}\boldsymbol{\tau}. \tag{6}$$

So,

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\omega}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}}\boldsymbol{\omega} \\ \frac{1}{mr^2}\boldsymbol{\tau}\end{bmatrix} = \mathbf{f}(\mathbf{x}, \boldsymbol{\tau}). \tag{7}$$

The linearized system has the form

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\boldsymbol{\tau}} \tag{8}$$

where $\mathbf{A} =$ and $\mathbf{B}$ are the Jacobians $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$, and $\frac{\partial \mathbf{f}}{\partial \boldsymbol{\tau}}$ respectively, evaluated at the equilibrium $\mathbf{x} = \mathbf{x_0}$. That is,

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}_0^\top\boldsymbol{\epsilon}_0}\boldsymbol{\epsilon}_0\boldsymbol{\omega}_0^\top & \frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}_0^\top\boldsymbol{\epsilon}_0}\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix} = \begin{bmatrix}\mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix}, \quad \text{and} \tag{9}$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{0}_3 \\ \frac{1}{mr^2}\mathbf{I}_3\end{bmatrix} \tag{10}$$

where $\mathbf{0}_3$ denotes a 3-by-3 zero-valued matrix. (9) follows from the following results

$$\frac{\partial}{\partial \epsilon_i}\frac{1}{2}\sqrt{1-\epsilon_1^2-\epsilon_2^2-\epsilon_3^2}\omega_k = -\frac{1}{2}\sqrt{1-\epsilon_1^2-\epsilon_2^2-\epsilon_3^2}^{-1}\epsilon_i\omega_k, \tag{11}$$

$$\frac{\partial}{\partial \omega_i}\frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}}\omega_i = \frac{1}{2}\sqrt{1-\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}}. \tag{12}$$

Problem 1.2

We now introduce the control law

$$\tau = -\mathbf{K}_d\boldsymbol{\omega} - k_p\boldsymbol{\epsilon}, \tag{13}$$

where $\mathbf{K}_d = k_d\mathbf{I}_3 = 40\mathbf{I}_3$ and $k_p = 2$. Since this input is only dependent on our state vector x, we can find an expression for the closed loop system with a single system matrix. This system becomes

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\boldsymbol{\tau}} \\
&= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(-\mathbf{K}_d\boldsymbol{\omega} - k_p\boldsymbol{\epsilon}) \\
&= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(-\begin{bmatrix}\mathbf{0}_3 & \mathbf{K}_d\end{bmatrix}\hat{\mathbf{x}} - \begin{bmatrix}k_p\mathbf{I}_3 & \mathbf{0}_3\end{bmatrix}\hat{\mathbf{x}}) \\
&= (\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\begin{bmatrix}k_p\mathbf{I}_3 & \mathbf{K}_d\end{bmatrix})\hat{\mathbf{x}} \\
&= \left(\begin{bmatrix}\mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix} - \begin{bmatrix}\mathbf{0}_3 \\ \frac{1}{mr^2}\mathbf{I}_3\end{bmatrix}\begin{bmatrix}k_p\mathbf{I}_3 & \mathbf{K}_d\end{bmatrix}\right)\hat{\mathbf{x}} \\
&= \begin{bmatrix}\mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ -\frac{k_p}{mr^2}\mathbf{I}_3 & -\frac{1}{mr^2}\mathbf{K}_d\end{bmatrix}\hat{\mathbf{x}}.
\end{aligned} \tag{14}$$

The matlab command below is used to find the eigenvalues of this matrix, which will coincide with the poles of the closed loop transfer function.

```
eigs([zeros(3,3), 1/2*eye(3); -k_p/(m*r^2)*eye(3), -k_d/(m*r^2)*eye(3)])
```

These eigenvalues are found to be $\lambda_{1,2} = -0.0278 \pm 0.0248j$, i.e. complex conjugated in the left half plane. This means that the system is stable. We prefer having complex conjugated poles in this case because it results more "aggressive" actuation, albeit with the added side-effect of some oscillations in the response. The reason we want aggressive actuation in this scenario is due to the non-linear nature of the system. Our linearization is only valid in a single point, and gets progressively worse as the state moves away from it. We hence want the controller to control the state back into where the linearization is acceptable as quickly as possible (within reason). In this case, some oscillation is acceptable if it yields faster convergence to the linearization point.

Problem 1.3

With the control law from (13) the system behaves as shown in figure 1. This control law drives the euler angles to zero. To drive them to an arbitrarily chosen reference $\boldsymbol{\epsilon}$, we would instead have to define the control law using the reference error $\tilde{\boldsymbol{\epsilon}}$, as is done in equation (3).

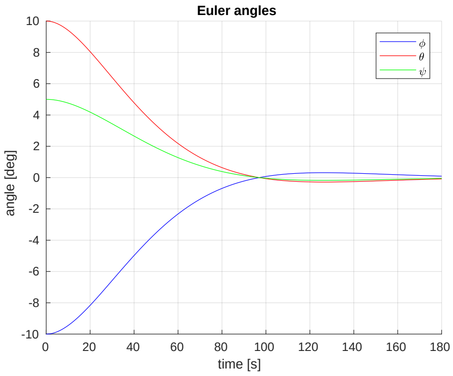
Problem 1.4

To make a control law based on deviation from a reference, we define the quaternion error

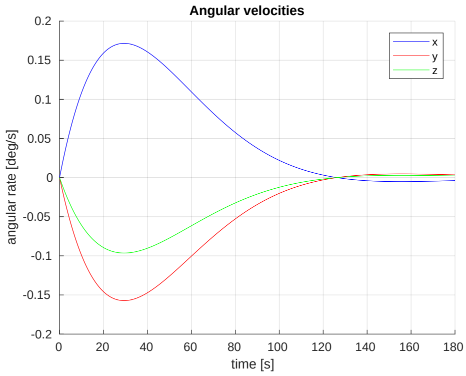
$$\tilde{\mathbf{q}} := \begin{bmatrix}\tilde{\eta} \\ \tilde{\epsilon}\end{bmatrix} = \bar{\mathbf{q}}_d \otimes \mathbf{q}, \tag{15}$$

where $\mathbf{q}_d$ is the reference (and the bar denotes the quaternion conjugate). Using the definition of the quaternion product from the assignment text, we obtain

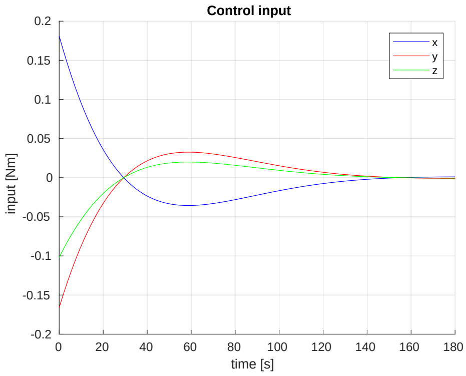
$$\tilde{\mathbf{q}} = \begin{bmatrix}\eta_d\eta + \boldsymbol{\epsilon}_d^\top\boldsymbol{\epsilon} \\ \eta_d\boldsymbol{\epsilon} - \eta\boldsymbol{\epsilon}_d - \mathbf{S}(\boldsymbol{\epsilon}_d)\boldsymbol{\epsilon}\end{bmatrix} = \begin{bmatrix}\eta_d\eta + \epsilon_{d1}\epsilon_1 + \epsilon_{d2}\epsilon_2 + \epsilon_{d3}\epsilon_3 \\ \eta_d\epsilon_1 - \eta\epsilon_{d1} - \epsilon_{d2}\epsilon_3 + \epsilon_{d3}\epsilon_2 \\ \eta_d\epsilon_2 - \eta\epsilon_{d2} - \epsilon_{d3}\epsilon_1 + \epsilon_{d1}\epsilon_3 \\ \eta_d\epsilon_3 - \eta\epsilon_{d3} - \epsilon_{d1}\epsilon_2 + \epsilon_{d2}\epsilon_1\end{bmatrix} \tag{16}$$



(a) Euler angles.



(b) Angular velocities.



(c) Control input.

Figure 1: Simulation results with control law (13)

where we have used

$$\mathbf{S}(\epsilon_d)\epsilon = \begin{bmatrix} 0 & -\epsilon_{d3} & \epsilon_{d2} \\ \epsilon_{d3} & 0 & -\epsilon_{d1} \\ -\epsilon_{d2} & \epsilon_{d1} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} \epsilon_{d2}\epsilon_3 - \epsilon_{d3}\epsilon_2 \\ \epsilon_{d3}\epsilon_1 - \epsilon_{d1}\epsilon_3 \\ \epsilon_{d1}\epsilon_2 - \epsilon_{d2}\epsilon_1 \end{bmatrix}. \tag{17}$$

After convergence, i.e. when $\mathbf{q} = \mathbf{q}_d$, this error becomes

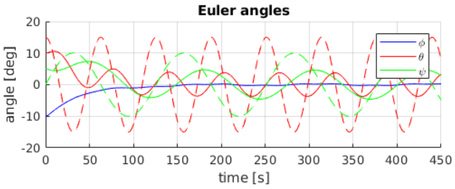
$$\tilde{\mathbf{q}} = \begin{bmatrix} \eta_d^2 + \boldsymbol{\epsilon}_d^\top \boldsymbol{\epsilon}_d \\ \eta_d \boldsymbol{\epsilon}_d - \eta_d \boldsymbol{\epsilon}_d - \mathbf{S}(\boldsymbol{\epsilon}_d)\boldsymbol{\epsilon}_d \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \tag{18}$$

since $\eta^2 = 1 - \boldsymbol{\epsilon}^\top \boldsymbol{\epsilon}$ and $\mathbf{S}(\boldsymbol{\epsilon})\boldsymbol{\epsilon} = \mathbf{0}$.

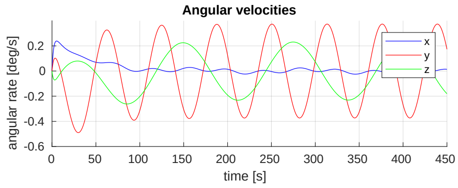
Problem 1.5

We now try to simulate with the control law based on a reference $\mathbf{q}_d$, i.e. with the control law

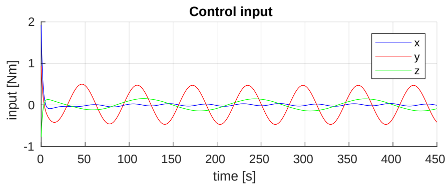
$$\boldsymbol{\tau} = -\mathbf{K}_d - k_p \tilde{\boldsymbol{\epsilon}}. \tag{19}$$



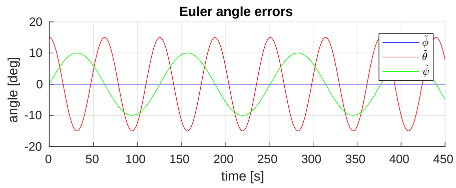
(a) Euler angles. Reference: dashes, state: line.



(b) Angular velocities.



(c) Control input.



(d) Error in euler angles.

Figure 2: Simulation results with control law (19)

Figure 2 shows the system simulated with reference angles $\phi(t) = 0$, $\theta(t) = 15\cos(0.1t)$, $\psi(t) = 10\sin(0.05t)$. As we can see, the satellite is not able to respond to the fast frequencies, so we get a large phase shift and a big decrease in amplitude. We recall that when a sinusoidal signal is differentiated, or passed through a first order system for that matter, the output is scaled proportional to the period, which in this case is quite small, leading to a decreased amplitude.

Problem 1.6

We now introduce an angular velocity reference $\boldsymbol{\omega}_d$ to our control law. We define the error of this as $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_d$ and the new control law as

$$\boldsymbol{\tau} = -\mathbf{K}_d\tilde{\boldsymbol{\omega}} - k_p\tilde{\boldsymbol{\epsilon}}. \tag{20}$$

From equation (2.31) in [1] we have the following expression for the reference,

$$\boldsymbol{\omega}_d = \mathbf{T}_{\Theta_d}^{-1}(\Theta_d)\dot{\Theta}_d, \tag{21}$$

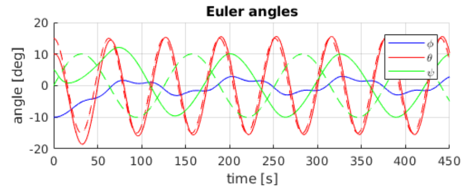
where $\mathbf{T}_{\boldsymbol{\Theta}_d}^{-1}$, given by equation (2.33), is

$$\mathbf{T}_{\boldsymbol{\Theta}_d}^{-1} = \begin{bmatrix} 1 & 0 & -\sin(\theta_d) \\ 0 & \cos(\phi_d) & \cos(\theta_d)\sin(\phi_d) \\ 0 & \sin(\phi_d) & \cos(\theta_d)\cos(\phi_d) \end{bmatrix}. \tag{22}$$

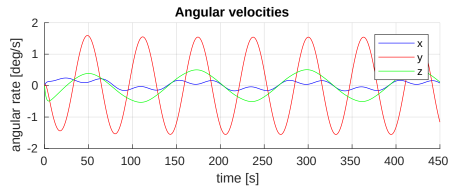
We analytically differentiate our reference euler angle signals to obtain

$$\dot{\boldsymbol{\Theta}}_d = \begin{bmatrix} \dot{\phi}_d \\ \dot{\theta}_d \\ \dot{\psi}_d \end{bmatrix} = \begin{bmatrix} 0 \\ -1.5\sin(0.1t) \\ 0.5\cos(0.05t) \end{bmatrix}. \tag{23}$$

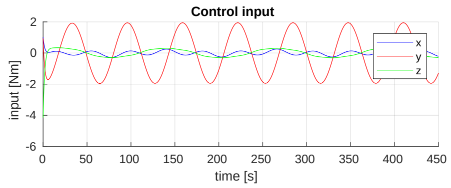
Simulating the system this reference and control law as well as $\mathbf{K}_d = k_d\mathbf{I}_3 = 400\mathbf{I}_3$ and $k_p = 20$ as before, yields the results shown in figure 3.



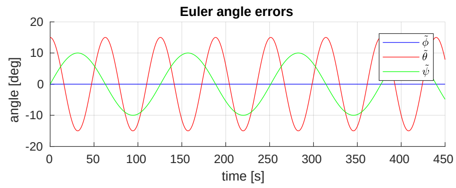
(a) Euler angles. Reference: dashes, state: line.



(b) Angular velocities.



(c) Control input.



(d) Error in euler angles.

Figure 3: Simulation results with control law (20)

This time, the satellite is able to follow the reference very closely in θ. In ψ however, it manages to correctly follow the amplitude, but still has a significant phase shift. Notice also how ϕ is more oscillatory than before. This is likely due to the system being non-linear. Nevertheless, we still have an overall improved performance, and this is very much expected. The previous control law would constantly try to drive the angular velocity to zero. This is clearly unattainable if the satellite is supposed to follow a sineusoidal signal, as its angular velocity would also have to vary with a sine. Since we are able to find the analytical expression for this expected velocity, we can expect great improvements over the earlier control law, and as we have seen, we were able to obtain correct amplitude control. The phase shift in ψ is however hard to combat with this control scheme and it is partly due to the fact that controller is unable to predict where the velocity is going to go, as it is constantly changing. Adding another, higher order term, i.e. the accelleration (which we can also find an analytical expression from) to the control law could help remove this phase shift.

Problem 1.7

We now assume setpoint regulation, i.e. $\boldsymbol{\omega}_d = \mathbf{0}$ and $\boldsymbol{\epsilon}_d$ and η_d are constant. The Lyapunov function for this situation can be written as

$$V = \frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}} + 2k_p(1-\tilde{\eta}) \tag{24}$$

and the derivative as

$$\dot{V} = -k_d\boldsymbol{\omega}^\top\boldsymbol{\omega}. \tag{25}$$

We know this to be positive and radially unbounded. Clearly $\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{\omega}}$ is positive, since $\mathbf{I}_{CG} = mr^2\mathbf{I}$ is positive definite. Furthermore, we have for any quaternion that

$$||\mathbf{q}|| = 1 = \sqrt{\eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2} = \sqrt{\eta^2 + \boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}} \tag{26}$$

and so

$$||\mathbf{q}||^2 = \eta^2\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon} = 1 \implies \boldsymbol{\epsilon}^\top\boldsymbol{\epsilon} = 1 - \eta^2 \leq 1 \tag{27}$$

which further implies

$$\eta^2 = 1 - \boldsymbol{\epsilon}^\top\boldsymbol{\epsilon} \implies \eta = \sqrt{1-\boldsymbol{\epsilon}^\top\boldsymbol{\epsilon}} \leq 1. \tag{28}$$

Now since $\tilde{\mathbf{q}}$ is a unit quaternion (it is the product of two unit quaternions, and we know the unit quaternions map (doubly) to SO(3) and this group is closed under multiplication [2]), we have $\tilde{\eta} \leq 1$ and hence $2k_p(1-\tilde{\eta})$ is also positive and so $V \geq 0$ always. We know V to be radially unbounded because, since the angular velocity terms $\tilde{\omega}_i$ are squared, $|\tilde{\omega}_i| \to \infty \implies V \to \infty$. Furthermore, since $\tilde{\boldsymbol{\omega}}$ is the only part of the input to the Lyapunov function (call it $\mathbf{x} = [\tilde{\boldsymbol{\omega}}, \tilde{\eta}]^\top$)

that has an unbounded domain, we know that $||\mathbf{x}|| \to \infty \iff |\tilde{\omega}_i| \to \infty$ for some $\tilde{\omega}_i$, and so $||\mathbf{x}|| \to \infty \implies V \to \infty$, meaning that V is radially unbounded.

We now show that the derivative of V is 25. Differentiating the first term yields

$$\frac{d}{dt}(\frac{1}{2}\tilde{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\tilde{\boldsymbol{w}}_d) = \frac{d}{dt}(\frac{1}{2}\boldsymbol{\omega}^\top \mathbf{I}_{CG}\boldsymbol{w}_d) = \dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega}. \tag{29}$$

Likewise, the derivative of the second term becomes

$$\frac{d}{dt}(2k_p(1-\tilde{\eta})) = \frac{d}{dt}(-2k_p\tilde{\eta}) = -2k_p\dot{\eta} = k_p\tilde{\boldsymbol{\epsilon}}^\top \tilde{\omega} \tag{30}$$

where the last expression is obtained from inserting equation (7) from the assignment. Now, inserting the control law from 20 and using that $\boldsymbol{\omega}_d = \mathbf{0}$, we obtain

$$\frac{d}{dt}(2k_p(1-\tilde{\eta})) = (-\boldsymbol{\tau} - \mathbf{K}_d\tilde{\boldsymbol{\omega}})^\top \tilde{\boldsymbol{\omega}} = (-\boldsymbol{\tau} - \mathbf{K}_d\boldsymbol{\omega})^\top \boldsymbol{\omega} = -\boldsymbol{\tau}^\top \boldsymbol{\omega} - k_d\boldsymbol{\omega}^\top \boldsymbol{\omega}. \tag{31}$$

Now, inserting the dynamics, i.e. equation (1) in the assignment, we get

$$\begin{aligned}\frac{d}{dt}(2k_p(1-\tilde{\eta})) &= -(\mathbf{I}_{CG} - \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega})^\top \boldsymbol{\omega} - k_d\boldsymbol{\omega}^\top \boldsymbol{\omega} \\ &= \dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega} + \boldsymbol{\omega}^\top \mathbf{S}(\mathbf{I}_{CG}\boldsymbol{\omega})\boldsymbol{\omega} - k_d\boldsymbol{\omega}^\top \boldsymbol{\omega} \\ &= -\dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega} + \boldsymbol{\omega}^\top \mathbf{I}_{CG}\mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} - k_d\boldsymbol{\omega}\boldsymbol{\omega} \\ &= -\dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega} - k_d\boldsymbol{\omega}^\top \boldsymbol{\omega},\end{aligned} \tag{32}$$

where we have used $\mathbf{S}(\boldsymbol{\omega})\boldsymbol{\omega} = \mathbf{0}$. Combining 29 and 32 we get

$$\dot{V} = \dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega} - \dot{\boldsymbol{\omega}}^\top \mathbf{I}_{CG}\boldsymbol{\omega} - k_d\boldsymbol{\omega}^\top \boldsymbol{\omega} = -k_d\boldsymbol{\omega}^\top \boldsymbol{\omega}. \tag{33}$$

We see that since $\dot{V} \geq 0$, it is stable, but we do not know if it is asymptotically stable. We see that when $\dot{V} = 0$, $\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}} = \mathbf{0}$ which clearly implies $\dot{\boldsymbol{\omega}} = 0$ as well, and so our dynamics become $\mathbf{I}_{CG}\dot{\boldsymbol{\omega}} - \mathbf{S}(\mathbf{I}_{CG}\omega)\boldsymbol{\omega} = \boldsymbol{\tau} = \mathbf{0}$. Thus, from the control law, we get $\tilde{\boldsymbol{\epsilon}} = \mathbf{0}$, and hence have two equilibrium points, found from $\tilde{\eta} = \pm\sqrt{1 - \tilde{\boldsymbol{\epsilon}}^\top \tilde{\boldsymbol{\epsilon}}} = \pm 1$. $\tilde{\eta} = 1$ yields $V = 0$, and hence is an asymptotically stable, $\tilde{\eta} = -1$ on the other hand yields $V = 4k_p$. Since V is monotonically decreasing, any small perturbation away from this point will send V towards $\tilde{\eta} = 1$, and $\tilde{\eta} = -1$ is hence an unstable equilibrium point. So in conclusion, the system is *not* globally asymptotically stable, because it has multiple equilibrium points, some of which are unstable. It is however locally asymptotically stable in $\boldsymbol{\omega} = 0, \tilde{\eta} = 1$.

Problem 2 Straight-line path following in the horizontal plane

Problem 2.1

From (2.25) we have:

$$\dot{\mathbf{p}}^b_{nb} = \mathbf{R}(\mathbf{\Theta_{nb}})\mathbf{v}^b_{nb} \tag{34}$$

Then:

$$\begin{aligned}
\dot{x} &= u\cos(\psi)\cos(\theta) + v[\cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\theta)] \\
&+ w\ [\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi)\sin(\theta)] \\
&= u\cos(\psi) * 1 + v[\cos(\psi) * 0 * 0 - \sin(\psi) * 1] + w\ [\sin(\psi) * 0 + \cos(\psi) * 1 * 0] \\
&= u\cos(\psi) - v\sin(\psi) \\
\dot{y} &= u\sin(\psi)\cos(\theta) + v[\cos(\psi)\cos(\phi) + \sin(\phi)\sin(\theta)\sin(\psi)] \\
&+ w[\sin(\theta)\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi)] \\
&= u\sin(\psi) * 1 + v[\cos(\psi) * 1 + \sin(\phi) * 0 * 0] + w[0 * \sin(\psi) * 1 + \cos(\psi) * 0] \\
&= u\sin(\psi) + v\cos(\psi)
\end{aligned}$$

Using:

$$\sin(\arctan(x)) = \frac{x}{1+x^2} \tag{35}$$
$$\cos(\arctan(x)) = \frac{1}{1+x^2} \tag{36}$$
$$U = \sqrt{u^2+v^2} \tag{37}$$
$$\beta = \arctan(\frac{v}{u}) \tag{38}$$
$$\chi = \psi + \beta \tag{39}$$

We have:

$$\begin{aligned}
\dot{x} &= u\cos(\psi) - v\sin(\psi) = \sqrt{u^2+v^2}[\cos(\psi)\frac{u}{\sqrt{u^2+v^2}} - \sin(\psi)\frac{v}{\sqrt{u^2+v^2}}] \\
&= U[\cos(\psi)\frac{1}{\sqrt{1+\frac{v^2}{u^2}}} - \sin(\psi)\frac{\frac{v}{u}}{\sqrt{1+\frac{v^2}{u^2}}}] \\
&= U[\cos(\psi)\cos(\arctan(\frac{v}{u})) - \sin(\psi)\sin(\arctan(\frac{v}{u}))] = U[\cos(\psi)\cos(\beta)) - \sin(\psi)\sin(\beta))] \\
&= U\cos(\psi+\beta) = U\cos(\chi) \\
\dot{y} &= u\sin(\psi) + v\cos(\psi) = \sqrt{u^2+v^2}[\sin(\psi)\frac{u}{\sqrt{u^2+v^2}} + \cos(\psi)\frac{v}{\sqrt{u^2+v^2}}] \\
&= U[\sin(\psi)\frac{1}{\sqrt{1+\frac{v^2}{u^2}}} + \cos(\psi)\frac{\frac{v}{u}}{\sqrt{1+\frac{v^2}{u^2}}}] = U[\sin(\psi)\cos(\arctan(\frac{v}{u})) + \cos(\psi)\sin(\arctan(\frac{v}{u}))] \\
&= U\sin(\psi + \arctan(\frac{v}{u})) = U\sin(\psi+\beta) = U\sin(\chi)
\end{aligned}$$

Problem 2.2

$$\dot{x} = U\cos(\psi+\beta) \tag{40}$$
$$\dot{y} = U\sin(\psi+\beta) \tag{41}$$

With β small (and a disturbance) and ψ also small. Then (12) can be seen from Taylor expansion of the sine and cosine functions for small values, with $\dot{x} = U$ and $\dot{y} = U\psi$.

As the vessel is following a straight line in the horizontal plane, and the assumtions above hold, it is clear that the only part of the cross-track error, $e(t) = -[x(t) - x_k]\sin(\alpha_k) + [y(t) - y_k]\cos(\alpha_k)$, which will affect it is y, such that we can give $e(t) = y$.

Problem 2.3

The Nomoto model can be written as

$$\begin{aligned} T\dot{r} + r &= K\delta + b \\ \dot{\psi} &= r \end{aligned} \tag{42}$$

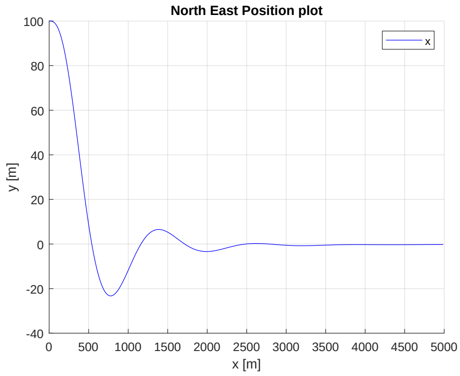
Then we can find the transfer functions of (42) as:

$$\begin{aligned} Ts^2\psi(s) + s\psi(s) &= K\delta(s) + b(s) \\ s(Ts+1)\frac{s}{U}y(s) &= K\delta(s) + b(s) \\ y(s) &= \frac{KU}{s^2(Ts+1)}\delta(s) + \frac{U}{s^2(Ts+1)}b(s) \\ h_1(s) &= \frac{KU}{s^2(Ts+1)} \\ h_2(s) &= \frac{U}{s^2(Ts+1)} \end{aligned}$$

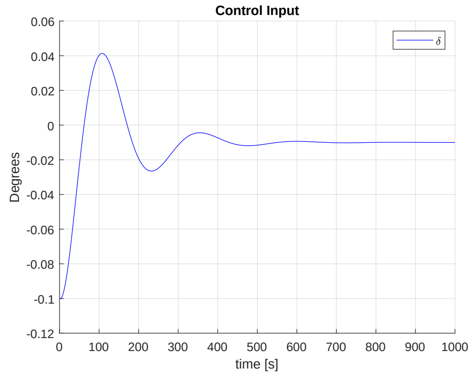
The reason we'd like to use the integral term is to remove the bias. The integral can then essentially sum up the bias over time, so that it would be removed. This, however, introduces another integrator into the system, and as we can see above, there are already more integrators there. Therefore, a derivative term is needed to stabilize the system.

Problem 2.4

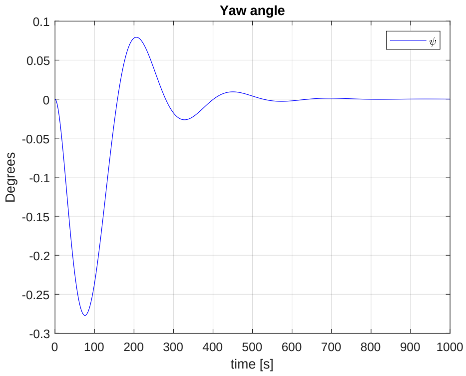
Simulating the system with MATLAB, and tuning the PID gains until a satisfying result is reached (for $k_p = 1\text{e}{-3}$¸ $k_i = 2\text{e}{-7}$ and $k_d = 5.5\text{e}{-2}$). Then, the following figures are generated:



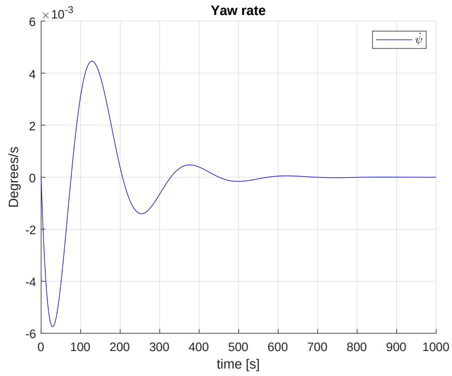
(a) North east position.



(b) Control input.



(c) Yaw angle.



(d) Yaw rate.

Figure 4: Nomoto simulation results.

As we can see from the figure, when tuned like this the system will follow a straight line. It is worth noting that when not tuned like this (very slight differences), the system is no longer this stable, and it is therefore important to properly tune a system like this. Further more, it is also impossible to properly tune the system with no integral or no derivative terms.

References

[1] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control.* John Wiley & Sons, 2011.

[2] J. Solà, "Quaternion kinematics for the error-state kalman filter," *CoRR*, vol. abs/1711.02508, 2017. [Online]. Available: http://arxiv.org/abs/1711.02508