Solution Final Exam 2018

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This documents is meant to serve as a guide for the external examiner. For some problems there might be more than one solution even thought the guide only shows on solution. In addition, the solution might be much more detailed than what is expected to obtain a maximum score.

Problem 1: Inertial Navigation System Aided by GNSS (35%)

Consider the kinematic equations:

$$\dot{\boldsymbol{p}}^n = \boldsymbol{R}(\psi)\boldsymbol{v}^b \tag{1}$$
$$\dot{\psi} = r \tag{2}$$

where $\boldsymbol{p}^n = [x, y, z]^\top$ and $\boldsymbol{v}^b = [u, v, w]^\top$. Assume that you measure:

$$\boldsymbol{p}^n_{gnss} = \boldsymbol{p}^n + \boldsymbol{v}_1 \tag{3}$$
$$\psi_{compass} = \psi + v_2 \tag{4}$$
$$\boldsymbol{a}^b_{\text{imu}} = \boldsymbol{R}^\top(\psi)\left(\dot{\boldsymbol{v}}^n - \boldsymbol{g}^n\right) + \boldsymbol{b}^b_{\text{acc}} \tag{5}$$

where $\boldsymbol{v}_1 \in \mathbb{R}^3$ and $v_2 \in \mathbb{R}$ are white measurement noise. The gravity vector in NED is $\boldsymbol{g}^n = [0, 0, g]^\top$. The accelerometer bias is modeled as:

$$\dot{\boldsymbol{b}}^b_{\text{acc}} = -\frac{1}{T_b}\boldsymbol{b}^b_{\text{acc}} + \boldsymbol{w} \tag{6}$$

where $\boldsymbol{w} \in \mathbb{R}^3$ is white noise.

1a (2%) Explain what we mean with aided inertial navigation and dead reckoning.

- Aided navigation means that the INS drift (integration of accelerometers and angular rate sensors) is corrected by a position (aiding) measurement, typically GNSS. The main tool for this is the Kalman filter.
- Dead reckoning corresponds to the case when you loose the GNSS positions measurements and the position and velocity measurements/estimates are based on integration of accelerometers and angular rate sensors. These estimates will drift with time and is only accurate in a short period after the aiding measurement is lost.

1b (3%) Which assumptions are (1)–(6) based on?

- (1)-(2) is based on neglecting roll and pitch (assumed to be zero).
- (3)-(4) are based on having position and heading measurements only affected by Gaussian measurement noise
- (5) is based on having zero roll and pitch and that the accelerometer measurements are affected by a time-varying bias.
- The acceleration bias (6) is assumed to be a Gauss-Markov process (first-order model driven by white noise).

1c (6%) Find a linear time-varying (LTV) state-space model:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u} + \boldsymbol{E}\boldsymbol{w} \tag{7}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{v} \tag{8}$$

where $\boldsymbol{u} = \boldsymbol{a}^b_{\text{imu}} + \boldsymbol{R}^\top(\psi)\boldsymbol{g}^n$ for estimation of

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}^n \\ \boldsymbol{v}^n \\ \boldsymbol{b}^b_{\text{acc}} \end{bmatrix} \tag{9}$$

You are supposed to write down the expressions for all the matrices.

The rotation matrix in yaw is known. Hence, $\boldsymbol{R}(t) = \boldsymbol{R}(\psi(t))$. Moreover, we can rewrite the accelerometer equation so that

$$\dot{\mathbf{v}^n} = \mathbf{R}(t)\mathbf{a}^b_{imu} + \mathbf{g}^n - \mathbf{R}(t)\mathbf{b}^b_{acc} = \mathbf{R}(t)\mathbf{u} - \mathbf{R}(t)\mathbf{b}^b_{acc} \tag{10}$$

The state-space model becomes:

$$\boldsymbol{A}(t) = \begin{bmatrix} 0_3 & \boldsymbol{I}_3 & 0_3 \\ 0_3 & 0_3 & -\boldsymbol{R}(t) \\ 0_3 & 0_3 & -\frac{1}{T_b}\boldsymbol{I}_3 \end{bmatrix}, \quad \boldsymbol{B}(t) = \begin{bmatrix} 0_3 \\ \boldsymbol{R}(t) \\ 0_3 \end{bmatrix} \tag{11}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_3, 0_3, 0_3 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0_3 \\ 0_3 \\ \boldsymbol{I}_3 \end{bmatrix} \tag{12}$$

1d (8%) Write down the pseudocode for a discrete-time Kalman filter (for-loop) using the LTV model under 1c.

```
% initialization of Kalman filter
x_prd = zeros(1,9);
P_prd = eye(9);
Q = eye(3);
R = eye(3);

% Constants
```

```
C = [ eye(3) zeros(3,6) ];
E = [zeros(6,3)
     eye(3)     ];
g = 9.81;

% sampling and measurements
h_pos = 1;  % GNSS position measurement (1 Hz)
h = 0.01;   % sampling time (equal to IMU/yaw measurements at 100 Hz)

for i = 1:N
   t = (i-1) * h;                            % time (s)

   % GNSS measurements are slower than the sampling time
   if mod( t, h_pos ) == 0
       y = "measurement"
       C      = [ eye(3) zeros(3,6) ];
   else
       C      = zeros(3,9);                % no measurement
   end

   % KF gain
   K = P_prd * C' * inv( C * P_prd * C' + R );

   % Corrector
   IKC = eye(9) - K * C;
   P_hat = IKC * P_prd * IKC' + K * R * K';
   eps = z - C * x_prd;
   x_hat = x_prd + K * eps;

   % Predictor (k+1)
   a_imu = "IMU acceleration measurement"
   R = "compute the rotation matrix when measuring the yaw angle"
   u = a_imu + R' * [0 0 g]';             % compute input u

   A = [ zeros(3,3) eye(3)       zeros(3,3)  % time-varying matrices
         zeros(3,3) zeros(3,3) -R
         zeros(3,3) zeros(3,3) -1/Tb * eye(3) ];
   B = [ zeros(3,3)
         R
         zeros(3,3) ];
   Ad = eye(9) + h * A;                      % discrete-time matrices
   Bd = h * B;

   x_prd = Ad * x_hat + Bd * u;
   P_prd = Ad * P_hat * Ad' + E * Q * E';

end
```

1e (8%) Assume that you have a single axis yaw gyro measuring:

$$r_{gyro} = r + b_{gyro} \tag{13}$$

Furthermore, assume that b_{gyro} is constant and show by using a Lyapunov function that the fixed-gain estimator:

$$\dot{\hat{\psi}} = r_{gyro} - \hat{b}_{gyro} + K_1(\psi - \hat{\psi}) \tag{14}$$
$$\dot{\hat{b}}_{gyro} = -K_2(\psi - \hat{\psi}) \tag{15}$$

renders the equilibrium point $(\psi - \hat{\psi}, b_{gyro} - \hat{b}_{gyro}) = \mathbf{0}$ exponentially stable for proper choices of the observer gains K_1 and K_2. State the conditions for choosing the gains.

We want to prove that the error dynamics ($\tilde{\psi} = \psi - \hat{\psi}$ and $\tilde{b}_{gyro} = b_{gyro} - \hat{b}_{gyro}$) are asymptotically stable. The derivative of the error dynamics are

$$\begin{aligned}
\dot{\tilde{\psi}} &= \dot{\psi} - \dot{\hat{\psi}} \\
&= r - (r + b_{gyro} - \hat{b}_{gyro} + K_1(\psi - \hat{\psi})) \\
&= -\tilde{b}_{gyro} - K_1\tilde{\psi} \\
\dot{\tilde{b}}_{gyro} &= \dot{b}_{gyro} - \dot{\hat{b}}_{gyro} \\
&= 0 - (-K_2(\psi - \hat{\psi})) \\
&= K_2\tilde{\psi}
\end{aligned} \tag{16}$$

Lets choose the following Lyapunov function candidate:

$$V = \frac{1}{2}\tilde{\psi}^2 + \frac{1}{2K_2}\tilde{b}_{gyro}^2 \tag{17}$$

The derivative can be calculated as

$$\dot{V} = \tilde{\psi}\dot{\tilde{\psi}} + \frac{1}{K_2}\tilde{b}_{gyro}\dot{\tilde{b}}_{gyro} \tag{18}$$
$$= -\tilde{\psi}\left(\tilde{b}_{gyro} + K_1\tilde{\psi}\right) + \tilde{b}_{gyro}\tilde{\psi} \tag{19}$$
$$= -K_1\tilde{\psi}^2 \tag{20}$$

We need to choose $K_1, K_2 > 0$ so that V is positive definite. Moreover, by choosing $K_1 > 0$, $\dot{V}$ is negative semi definite. It is not negative definite since $\tilde{b}_{gyro}$ is not a part of $\dot{V}$. Thus, Lyapunovs direct method can only show stability, but not asymptotic stability. We need to use LaSalle. The set Ω where $\dot{V}$ is zero is defined as

$$\begin{aligned}
\Omega &= \left\{\tilde{\psi} \in \mathbb{R}, \tilde{b}_{gyro} \in \mathbb{R} | \dot{V} = 0\right\} \\
&= \left\{\tilde{\psi} \in \mathbb{R}, \tilde{b}_{gyro} \in \mathbb{R} | \tilde{\psi} = 0, \tilde{b}_{gyro} \in \mathbb{R}\right\}
\end{aligned} \tag{21}$$

The largest and only invariant set in Ω is $M = \left\{\tilde{\psi} = 0, \tilde{b}_{gyro} = 0\right\}$ since every other value for $\tilde{b}_{gyro}$ create a nonzero $\dot{\tilde{\psi}}$ and pushes $\tilde{\psi}$ out of the set Ω. Thus, the only way to get stuck in Ω is at the equilibrium point. Therefore, the equilibrium point $(\tilde{\psi}, \tilde{b}_{gyro}) = \mathbf{0}$ is asymptotically stable (AS) by application of LaSalle-Krasowski's theorem. Moreover, since the error dynamics are linear, the system is also exponentially stable. This holds globally as long as $\psi \in \mathbb{R}$ and not restricted to $[-\pi, \pi)$.

1f (8%) Show that:

$$\dot{\boldsymbol{v}}^n = \boldsymbol{R}(\psi)\left(\dot{\boldsymbol{v}}^b + \boldsymbol{S}(r)\boldsymbol{v}^b\right) \tag{22}$$

where

$$\boldsymbol{S}(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

and show how you can find expressions for $\hat{\boldsymbol{v}}^b$ and $\dot{\hat{\boldsymbol{v}}}^b$.

$$\boldsymbol{v}^n = \boldsymbol{R}\boldsymbol{v}^b \tag{24}$$

$$\dot{\boldsymbol{v}}^n = \boldsymbol{R}\dot{\boldsymbol{v}}^b + \dot{\boldsymbol{R}}\boldsymbol{v}^b \tag{25}$$

$$= \boldsymbol{R}\dot{\boldsymbol{v}}^b + \boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega}^b)\boldsymbol{v}^b \tag{26}$$

$$= \boldsymbol{R}\left(\dot{\boldsymbol{v}}^b + \boldsymbol{S}(\boldsymbol{\omega}^b)\boldsymbol{v}^b\right) \tag{27}$$

Choosing $\boldsymbol{\omega}^b = [0, 0, r]^\top$ and $\boldsymbol{\Theta} = [0, 0, \psi]^\top$ give (22)–(23), q.e.d.

$\hat{\boldsymbol{v}}^b$ can be calculated as

$$\hat{\boldsymbol{v}}^b = \mathbf{R}^\top(\psi)\hat{\boldsymbol{v}}^n \tag{28}$$

From the state-space model we have

$$\dot{\boldsymbol{v}}^n = \boldsymbol{R}(\psi)\boldsymbol{u} - \boldsymbol{R}(\psi)\boldsymbol{b}^b_{\text{acc}} \tag{29}$$

Hence, by rearranging the terms in (27)

$$\dot{\boldsymbol{v}}^b = \boldsymbol{R}^\top(\psi)\dot{\boldsymbol{v}}^n - \boldsymbol{S}(r)\boldsymbol{v}^b \tag{30}$$

$$= \boldsymbol{u} - \boldsymbol{b}^b_{\text{acc}} - \boldsymbol{S}(r)\boldsymbol{v}^b \tag{31}$$

The body-fixed acceleration estimate can then be computed as:

$$\dot{\hat{\boldsymbol{v}}}^b = \boldsymbol{u} - \hat{\boldsymbol{b}}^b_{\text{acc}} - \boldsymbol{S}(r)\hat{\boldsymbol{v}}^b \tag{32}$$

Problem 2: Semi-Submersible Drilling Rig (40%)

Consider the semi-submersible drilling rig shown in Figure 1. The 6-DOF model is chosen as:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{\nu} \tag{33}$$

$$(\boldsymbol{M}_{RB} + \boldsymbol{M}_A)\dot{\boldsymbol{\nu}} + \boldsymbol{D}\boldsymbol{\nu} + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} \tag{34}$$

where the numerical values are given in Appendix A and $\boldsymbol{J}(\boldsymbol{\eta})$ is the 6-DOF kinematic transformation matrix.

Figure 1: Semi-submersible drilling rig. The position and heading of the semi-submersible are controlled by using 4 azimuth thrusters, which have the following coordinates (-40,-20, 10), (-40, 20, 10), (40, -20, 10) and (40, 20, 10). The body-fixed coordinate system CO is located on the centerline midships.

2a (6%) Compute the decoupled time constants and natural periods in 6 DOF.

When considering the decoupled states we only look at the terms on the diagonal. Since the terms in $\mathbf{G}$ are zero for surge, sway and yaw, they get one pole each in zero (pure integrator). The second pole is given by the following expression:

$$\lambda_i = -\frac{\mathbf{D}(i,i)}{\mathbf{M}_{RB}(i,i) + \mathbf{M}_A(i,i)} \tag{35}$$

where $\mathbf{M}_A(a,b)$ corresponds to the element on row a and column b in the matrix $\mathbf{M}_A$ and i is 1,2 and 6 (surge sway and yaw). The poles and time constants are given as:

State	Pole	Time constants
Surge	-0.00909	110 s
Sway	-1/230	230 s
Yaw	-0.0125	80 s

For heave, roll and pitch, the elements in $\mathbf{G}$ are nonzero and we will end up with second-order systems on the form

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{36}$$

The natural frequencies for these states are, therefore,

$$\omega_{n_i} = \sqrt{\frac{\mathbf{G}(i,i)}{\mathbf{M}_{RB}(i,i) + \mathbf{M}_A(i,i)}} \tag{37}$$

where i is 3,4 and 5. Thus, the natural frequencies are:

State	Natural frequency
Heave	0.354 rad/s
Roll	0.545 rad/s
Pitch	0.659 rad/s

2b (5%) Assume that the drilling rig is exposed to an irrotational ocean current, which is constant in NED. Show how you will modify (34) to include the effect of ocean currents.

We introduce relative velocities to let current affect the drilling rig. The current is constant in NED, but we need to express it in body since the velocities $\boldsymbol{\nu}$ are given in body. The relative velocity is defined as

$$\boldsymbol{\nu}_r = \begin{bmatrix} \mathbf{v}^b - \mathbf{v}_c^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} \tag{38}$$

where $\mathbf{v}_c^b$ is the current velocity decomposed in body. The current only affects the linear velocities since it is irrotational. To include the effects of ocean current we use the relative velocity in (34):

$$(\boldsymbol{M}_{RB} + \boldsymbol{M}_A)\dot{\boldsymbol{\nu}}_r + \boldsymbol{D}\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} \tag{39}$$

where $\mathbf{v}_c^b = \mathbf{R}_n^b \mathbf{v}_c^n$. Moreover, $\dot{\mathbf{v}}_c^b$ can be calculated as

$$\begin{aligned} \dot{\mathbf{v}}_c^n &= \dot{\mathbf{R}}_b^n \mathbf{v}_c^b + \mathbf{R}_b^n \dot{\mathbf{v}}_c^b := 0 \\ \dot{\mathbf{v}}_c^b &= -\mathbf{R}_n^b \dot{\mathbf{R}}_b^n \mathbf{v}_c^b \\ &= -\mathbf{R}_n^b \mathbf{R}_b^n \mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\mathbf{v}_c^b \\ &= -\mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\mathbf{v}_c^b \end{aligned} \tag{40}$$

2c (5%) Reduce the 6-DOF model to a 3-DOF horizontal-plane model for surge, sway and yaw motions. State the necessary assumptions to write the model as:

$$\dot{\boldsymbol{\eta}}_3 = \boldsymbol{R}(\psi)\boldsymbol{\nu}_3 \tag{41}$$

$$\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + \boldsymbol{D}_3\boldsymbol{\nu}_3 = \boldsymbol{\tau}_3 \tag{42}$$

where $\boldsymbol{\eta}_3 = [x, y, \psi]^\top$ and $\boldsymbol{\nu}_3 = [u, v, r]^\top$. Write down the expressions for $\boldsymbol{M}_3$ and $\boldsymbol{D}_3$. Numerical values are required.

Note that surge, sway and yaw are not affected by $\mathbf{G}$ so we can neglect it from the analysis. The assumptions for reducing the model to 3-DOF are

- Neglect roll, pitch and heave.
- Roll and pitch are zero so that $J(\boldsymbol{\eta}) = \boldsymbol{R}(\psi)$
- $w = p = q = 0$

The matrices can be written as

$$\begin{aligned} \boldsymbol{M}_3 &= 10^{10} * \begin{bmatrix} 0.0027 + 0.0017 & 0 & 0 \\ 0 & 0.0027 + 0.0042 & -0.0014 \\ 0 & -0.0014 & 3.7192 + 3.2049 \end{bmatrix} \\ \boldsymbol{D}_3 &= 10^9 * \begin{bmatrix} 0.0004 & 0 & 0 \\ 0 & 0.0003 & -0.0002 \\ 0 & -0.0002 & 0.8656 \end{bmatrix} \end{aligned} \tag{43}$$

2d (6%) Design a nonlinear PD controller for regulation of generalized position and velocity to zero using pole placement. Write down the formula for the control law and find the expressions for the controller gains $\boldsymbol{K}_p$ and $\boldsymbol{K}_d$ as a function of the model parameters. Specify, which numerical value you would choose for the natural frequency in surge, sway and yaw.

We want to design a nonlinear PD controller that regulates the generalized position and velocity to zero. Lets choose the following nonlinear PD controller without acceleration feedback:

$$\boldsymbol{\tau} = -\boldsymbol{R}^\top(\psi)(\boldsymbol{K}_d\dot{\tilde{\boldsymbol{\eta}}}_3 + \boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3) \tag{44}$$

where $\tilde{\boldsymbol{\eta}}_3 = \boldsymbol{\eta}_3 - \boldsymbol{\eta}_{3d}$. Moreover, $\dot{\tilde{\boldsymbol{\eta}}}_3 = \dot{\boldsymbol{\eta}}_3 - \dot{\boldsymbol{\eta}}_{3d} = \dot{\boldsymbol{\eta}}_3$. If we insert this into the 3-DOF model we get the following closed loop dynamics:

$$\begin{aligned}
\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + \boldsymbol{D}_3\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)(\boldsymbol{K}_d\dot{\tilde{\boldsymbol{\eta}}}_3 + \boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3) &= \boldsymbol{0}_3 \\
\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + \boldsymbol{D}_3\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_d\dot{\tilde{\boldsymbol{\eta}}}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3 &= \boldsymbol{0}_3 \\
\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + \boldsymbol{D}_3\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_d\boldsymbol{R}(\psi)\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3 &= \boldsymbol{0}_3 \\
\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + (\boldsymbol{D}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_d\boldsymbol{R}(\psi))\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3 &= \boldsymbol{0}_3 \\
\boldsymbol{M}_3\dot{\boldsymbol{\nu}}_3 + (\boldsymbol{D}_3 + \boldsymbol{K}_d^*)\boldsymbol{\nu}_3 + \boldsymbol{R}^\top(\psi)\boldsymbol{K}_p\tilde{\boldsymbol{\eta}}_3 &= \boldsymbol{0}_3
\end{aligned} \tag{45}$$

where $\boldsymbol{K}_d^* = \boldsymbol{R}^\top(\psi)\boldsymbol{K}_d\boldsymbol{R}(\psi)$. This can be compared with the general mass-springer-damper structure so that the controller gains can be chosen as

$$\begin{aligned}
\boldsymbol{K}_p &= \boldsymbol{M}_3\,\mathrm{diag}(\omega_{1,n}^2, \omega_{2,n}^2, \omega_{3,n}^2) \\
\boldsymbol{K}_d^* &= \boldsymbol{M}_3\,\mathrm{diag}(2\zeta_1\omega_{1,n}, 2\zeta_2\omega_{2,n}, 2\zeta_3\omega_{3,n}) - \boldsymbol{D}_3
\end{aligned} \tag{46}$$

and $\boldsymbol{K}_d$ is calculated from $\boldsymbol{K}_d^*$.

By designing the system so that $\zeta_i = 1$ for surge, sway and yaw, the controller gains can be chosen directly based on the desired natural frequency. The book states that when applying a feedback control system to stabilize motions in surge, sway and yaw, the natural periods will be in the range of 100-200s. Therefore, the natural frequencies should be chosen close to the interval given by 0.03-0.10 rad/s.

2e (6%) The semi-submersible is controlled by 4 azimuth thrusters as shown in Figure 1. The azimuth thrust is modeled as $(i = 1, 2, 3, 4)$:

$$F_i = 200\,|n_i|n_i \quad [\mathrm{kN}] \tag{47}$$

and all thrusters can be rotated an angle α_i. Find a formula for the roll moment produced by the 4 thrusters.

Azimuth thrusters can generate forces in both the body x and y axes:

$$\begin{aligned}
F_{x_i} &= F_i\cos(\alpha_i) \\
F_{y_i} &= F_i\sin(\alpha_i)
\end{aligned} \tag{48}$$

The roll moment is given as

$$K = F_z l_y - F_y l_z \tag{49}$$

where l_y and l_z are the moment arms (displacement from CO). Since F_z is zero for the azimuth thrusters, the roll moment is

$$\begin{aligned}K &= \sum_{i=1}^{4} -F_i \sin(\alpha_i) l_{z_i} \\ &= \sum_{i=1}^{4} -200|n_i|n_i \sin(\alpha_i) l_{z_i} \\ &= \sum_{i=1}^{4} -2000|n_i|n_i \sin(\alpha_i)\end{aligned} \tag{50}$$

since l_{z_i} is $10m$ for all thrusters.

2f (6%) The control forces and moment acting on the semi-submersible can be written as:

$$\boldsymbol{\tau}_3 = k\boldsymbol{T}_e\boldsymbol{u}_e \tag{51}$$

with

$$\boldsymbol{T}_e = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -20 & 40 & -20 & -40 & 20 & -40 & 20 & 40 \end{bmatrix} \tag{52}$$

where $\boldsymbol{\tau}_3 = [\tau_x, \tau_y, \tau_\psi]^\top \in \mathbb{R}^3$, $k \in \mathbb{R}$ and $\boldsymbol{u}_e \in \mathbb{R}^8$. Find the numerical value for k. Assume that the PD controller generates the signal $\boldsymbol{\tau}_3 \in \mathbb{R}^3$ and show how you can compute the control inputs n_i for $(i = 1, 2, 3, 4)$ and azimuth angles α_i for $(i = 1, 2, 3, 4)$ using an unconstrained control allocation method. It is not necessary to compute the numerical values. Only the formulas are required.

The input can be chosen as

$$\boldsymbol{u}_e = \begin{bmatrix} F_{e1,x} \\ F_{e1,y} \\ F_{e2,x} \\ \vdots \\ F_{e4,y} \end{bmatrix} \tag{53}$$

where $F_{ei,x} = |n_i|n_i \cos(\alpha_i)$ and $F_{ei,y} = |n_i|n_i \sin(\alpha_i)$. Since the total input F_i is equal to $200|n_i|n_i$, we can see from the relationship $F = \boldsymbol{K}_e\boldsymbol{u}_e$ that $\boldsymbol{K}_e$ is a diagonal matrix with 200 on the diagonal. Therefore, it can be simplified to be a scalar since

$$\boldsymbol{\tau} = \boldsymbol{T}_e * 200\boldsymbol{I}_8 * \boldsymbol{u}_e = 200\boldsymbol{T}_e\boldsymbol{u}_e = k\boldsymbol{T}_e\boldsymbol{u}_e \tag{54}$$

Thus, it follows that the scalar gain k is equal to 200.

The optimization problem can be formulated as

$$\begin{aligned} J = \underbrace{min}_{\boldsymbol{u}_e} \left\{ \boldsymbol{u}_e^\top \boldsymbol{W} \boldsymbol{u}_e \right\} \\ \text{subject to: } \boldsymbol{\tau} - k\boldsymbol{T}_e\boldsymbol{u}_e = 0 \end{aligned} \tag{55}$$

where $\boldsymbol{W}$ is the weighting matrix. Since every thruster is equal and no limitations are stated (we want to use the thrusters equally much), the weighting matrix should simply be the identity matrix. The solution to this optimization problem is (see page 405 in the book):

$$\boldsymbol{u}_e = \frac{1}{k}\boldsymbol{T}_w^{\dagger}\boldsymbol{\tau} = \frac{1}{k}\boldsymbol{T}^{\top}(\boldsymbol{T}\boldsymbol{T}^{\top})^{-1}\boldsymbol{\tau} \tag{56}$$

The optimization problem solves the system so that the forces in (53) are known. However, we are looking for solutions for n_i and α_i. They can be extracted by calculating

$$\begin{aligned} U_i &= \sqrt{F_{ei,x}^2 + F_{ei,y}^2} = |n_i|n_i \\ n_i &= \text{sign}(U_i)\sqrt{|U_i|} \\ \alpha_i &= \arctan\left(\frac{F_{ei,y}}{F_{ei,x}}\right) \end{aligned} \tag{57}$$

2g (3%) What are the main limitations/problems with the solution in 2f?

The solution in 2f) is based on an unconstrained optimization problem. Therefore, saturation limits on both the azimuth angle and the revolution n_i are neglected so the solution may not be physically feasible. Moreover, the dynamics of the thrusters are neglected (time constants for α_i and n_i) and they cannot be changed instantaneously. We also assume that the system model is correct (both for $\boldsymbol{\nu}_3$ and $\boldsymbol{\tau}_3$). Another issue can be that controllability may be lost if all thrusters are aligned in the same direction. This can for instance happen to counteract a constant disturbance. Normally, it is necessary to restrict the problem so that a maximum percentage of the thrust can be applied in the same direction.

2h (3%) Will the PD control law $\boldsymbol{\tau}_3$ generate roll and pitch motions during stationkeeping when applied to (34)? Justify your answer.

If we apply the control law for the 3-DOF system to the 6-DOF dynamics, it will generate roll and pitch motions during stationkeeping. This can be seen from the equation for the roll and pitch moments. Azimuth thrusters generate a force in both the body x and y axes. We showed in problem 2e) that a roll moment is created by $\boldsymbol{\tau}_3$ and the expression for the pitch moment is

$$M = F_x l_z - F_z l_x \tag{58}$$

and since both F_x and l_z are nonzero, a pitch moment will also be generated.

Problem 3: ROV and AUV Tracking Systems (20%)

Consider the simplified ship model:

$$\dot{\psi} = r \tag{59}$$

$$T\dot{r} + r = K\delta \tag{60}$$

$$m\dot{u} + d|u|u = \tau \tag{61}$$

where the rudder angle δ and thrust τ are the control inputs. It is assumed that $u \gg 0$ and $v = 0$ such that speed $U = u$. The model parameters m, d, K and T are known.

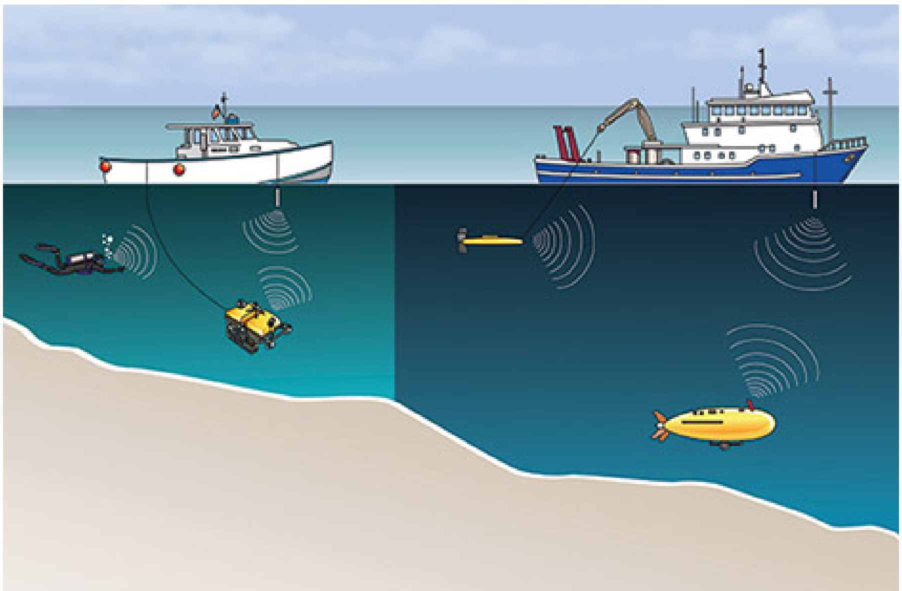

Figure 2: ROV and AUV tracking systems.

3a (4%) Design a feedback linearizing controller for τ such that u converges to u_d exponentially.

We want to find expression

$$\begin{aligned}\tau &= ma^b + \text{nonlinear terms} \\ &= ma^b + d|u|u\end{aligned} \tag{62}$$

Lets choose the commanded acceleration as a PI controller with reference feed forward:

$$a^b = m(\dot{u}_d - K_p(u - u_d) - K_i \int (u - u_d)) \tag{63}$$

The closed-loop dynamics are then:

$$\begin{aligned}&m\dot{u} + d|u|u = m(\dot{u}_d - K_p(u - u_d) - K_i \int_0^t (u(\tau) - u_d(\tau)) + d|u|u \\ &m\dot{\tilde{u}} + mK_p\tilde{u} + mK_i \int_0^t \tilde{u}(\tau) = 0 \\ &\dot{\tilde{u}} + K_p\tilde{u} + K_i \int_0^t \tilde{u}(\tau) = 0\end{aligned} \tag{64}$$

This is a linear system, which is GES if the gains are chosen as $K_p = 2\lambda$ and $K_i = \lambda^2$ with $\lambda > 0$.

3b (4%) Design a PD controller for δ such that ψ converges to ψ_d exponentially.

A PD controller has the form

$$\delta = -K_p(\psi - \psi_d) - K_d(\dot{\psi} - \dot{\psi}_d) = -K_p\tilde{\psi} - K_d\dot{\tilde{\psi}} \tag{65}$$

We assume that ψ_d is constant and insert it into the Nomoto model:

$$
\begin{aligned}
T\dot{r} + r &= K(-K_p\tilde{\psi} - K_d\dot{\tilde{\psi}}) \\
T\ddot{\tilde{\psi}} + \dot{\tilde{\psi}} &= K(-K_p\tilde{\psi} - K_d\dot{\tilde{\psi}}) \\
T\ddot{\tilde{\psi}} + (1 + KK_d)\dot{\tilde{\psi}} + KK_p\tilde{\psi} &= 0
\end{aligned} \tag{66}
$$

This can be compared with a mass-spring-damper system so that

$$
\begin{aligned}
1 + KK_d = 2\zeta\omega_0 &\rightarrow K_d = \frac{2\zeta\omega_0 - 1}{K} \\
KK_p = \omega_0^2 &\rightarrow K_p = \frac{\omega_0^2}{K}
\end{aligned} \tag{67}
$$

3c (7%) Assume that you measure the NED position $\boldsymbol{p}_t^n$ and velocity $\boldsymbol{v}_t^n$ of a moving underwater vehicle (target), which you want to follow with your ship. It is also assumed that you measure your own position $\boldsymbol{p}^n$ and heading angle ψ. Show how you can use the constant bearing guidance law:

$$
\boldsymbol{v}_d^n = \boldsymbol{v}_t^n - k\frac{\tilde{\boldsymbol{p}^n}}{||\tilde{\boldsymbol{p}^n}||} \tag{68}
$$

where $\tilde{\boldsymbol{p}^n} = \boldsymbol{p}^n - \boldsymbol{p}_t^n$ together with the control laws from 3a and 3b to track a moving underwater vehicle. The control objective is $u = u_t$ and $v = 0$.

We have a controller for surge and heading. We cannot control the sway velocity so our goal is to align our ship with the direction of the desired NED velocity so that the body x axis points in the direction of the velocity. The guidance law gives us a desired NED velocity $\mathbf{v}_d^n$ which we want to achieve. Lets assume that we only consider 3-DOF so that roll, pitch and heave can be neglected. The direction (course over ground) of the desired velocity is

$$
\chi_d = \arctan\left(\frac{\mathbf{v}_d^n(2)}{\mathbf{v}_d^n(1)}\right) \tag{69}
$$

If we assume that sideslip and current can be neglected (crab is zero on straight-line segments without current), the solution is to send χ_d into the heading controller. When $\psi = \chi_d$ the ship is aligned with the direction of $\mathbf{v}_d^n$. The remaining goal then is to get the ship to a speed given by $||\mathbf{v}_d^n||$ and this is achieved by sending the desired total speed over ground to the surge controller.

It is also be possible to transform the system in to vessel parallel coordinates and use the linear path-following scheme from assignment 1 where the y coordinate gives the cross-track error.

3d (5%) Explain what will happen if you replace $\tilde{\boldsymbol{p}^n}$ in (68) with:

$$
\tilde{\boldsymbol{p}^n} = \boldsymbol{p}^n - \left(\boldsymbol{p}_t^n + \boldsymbol{R}(\psi)\begin{bmatrix} 100 \\ 0 \end{bmatrix}\right) \tag{70}
$$

In general, the goal of the guidance law is to push $\tilde{\boldsymbol{p}}^n$ to zero. When we define $\tilde{\boldsymbol{p}}^n = \boldsymbol{p}^n - \boldsymbol{p}_t^n$, which is the normal definition of the position error, $\boldsymbol{p}^n$ will converge to the target position $\boldsymbol{p}_t^n$ since the error is pushed to zero. When the error term is changed, the guidance law will push the new error to zero and this leads to:

$$\begin{aligned}\tilde{\boldsymbol{p}}^n &= \boldsymbol{p}^n - \left(\boldsymbol{p}_t^n + \boldsymbol{R}(\psi)\begin{bmatrix}100\\0\end{bmatrix}\right) = 0\\ \boldsymbol{p}^n &= \boldsymbol{p}_t^n + \boldsymbol{R}(\psi)\begin{bmatrix}100\\0\end{bmatrix}\end{aligned} \tag{71}$$

Since $\boldsymbol{R}(\psi)$ is the rotation matrix $\boldsymbol{R}_b^n$, the vector $[100, 0]^\top$ is expressed in body. Therefore, the ship position will converge to the target position and additionally 100 meters along the body x axis. This means that our own ship will go past the target and stay 100 meters in front of the target on the same course.

3e (5%) Assume that you measure range and bearing from the ship to the underwater vehicle instead of $\boldsymbol{p}^n$ and $\boldsymbol{p}_t^n$. Show how the constant bearing guidance law (68) can be modified to use range-bearing measurements instead of positions.

We assume that we still can measure $\mathbf{v}_t^n$, but that we cannot measure absolute positions of our own ship and the target. Range and bearing measurements give us the relative position between the ships decomposed in the body frame. The measurement is displayed in Figure 3. The range gives the distance between the target and our own ship. Therefore, the range $r = ||\tilde{\mathrm{p}}^n||$. Moreover, the error in the desired position is defined as

$$\tilde{\boldsymbol{p}}^n = \boldsymbol{R}_b^n \begin{bmatrix} r\cos(\alpha)\\ r\sin(\alpha)\end{bmatrix} \tag{72}$$

By inserting this term into the constant-bearing guidance law we get:

$$\boldsymbol{v}_d^n = \boldsymbol{v}_t^n - k\frac{\tilde{\boldsymbol{p}}^n}{||\tilde{\boldsymbol{p}}^n||} = \boldsymbol{v}_t^n - \frac{k}{r}\boldsymbol{R}_b^n\begin{bmatrix} r\sin(\alpha)\\ r\cos(\alpha)\end{bmatrix} = \boldsymbol{v}_t^n - k\boldsymbol{R}_b^n\begin{bmatrix}\sin(\alpha)\\ \cos(\alpha)\end{bmatrix} \tag{73}$$

Moreover, k can be defined:

$$k = U_{A,max}\frac{||\tilde{\boldsymbol{p}}^n||}{\sqrt{(\tilde{\boldsymbol{p}}^n)^\top \tilde{\boldsymbol{p}}^n + \Delta_{\tilde{p}}^2}} = U_{A,max}\frac{r}{\sqrt{r + \Delta_{\tilde{p}}^2}} \tag{74}$$

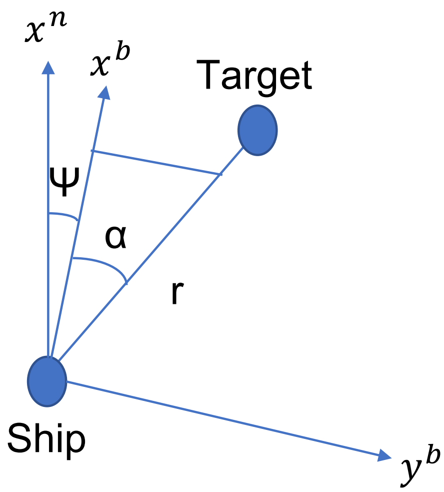

Figure 3: Range (r) and bearing (α) measurements

A Numerical Values for Semi-Submersible Drilling Rig

```
Mass of the vessel = 27 162 500 kg

MRB =
   1.0e+10 *
    0.0027         0         0         0   -0.0530         0
         0    0.0027         0    0.0530         0   -0.0014
         0         0    0.0027         0    0.0014         0
         0    0.0530         0    3.4775         0   -0.0265
   -0.0530         0    0.0014         0    3.8150         0
         0   -0.0014         0   -0.0265         0    3.7192

MA =
   1.0e+10 *
    0.0017         0         0         0   -0.0255         0
         0    0.0042         0    0.0365         0         0
         0         0    0.0021         0         0         0
         0    0.0365         0    1.3416         0         0
   -0.0255         0         0         0    2.2267         0
         0         0         0         0         0    3.2049

D =
   1.0e+09 *
    0.0004         0         0         0   -0.0085         0
         0    0.0003         0    0.0067         0   -0.0002
         0         0    0.0034         0    0.0017         0
         0    0.0067         0    4.8841         0   -0.0034
   -0.0085         0    0.0017         0    7.1383         0
         0   -0.0002         0   -0.0034         0    0.8656

G =
   1.0e+10 *
         0         0         0         0         0         0
         0         0         0         0         0         0
```

```
         0         0    0.0006         0         0         0
         0         0         0    1.4296         0         0
         0         0         0         0    2.6212         0
         0         0         0         0         0         0
```