SISO Pole Placement (Fossen, 2011)

Table 12.2: PID and acceleration feedback pole-placement algorithm.

1.	Specify the bandwidth $\omega_b > 0$ and the relative damping ratio $\zeta > 0$
2.	Compute the natural frequency: $\omega_n = \frac{1}{\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}}\omega_b$
3.	Specify the gain: $K_m \geq 0$ (optionally acceleration feedback)
4.	Compute the P-gain: $K_p = (m + K_m)\omega_n^2 - k$
5.	Compute the D-gain: $K_d = 2\zeta\omega_n(m + K_m) - d$
6.	Compute the I-gain: $K_i = \frac{\omega_n}{10}K_p$

Definition 12.1 (Control Bandwidth)
The control bandwidth of a system $y = h(s)u$ with negative unity feedback is defined as the frequency ω_b at which the loop transfer function $l(s) = h(s) \cdot 1$ is:

$$|l(j\omega)|_{\omega=\omega_b} = \frac{\sqrt{2}}{2}$$

or equivalently:

$$20 \log |l(j\omega)|_{\omega=\omega_b} = -3 \; dB$$

From this definition it can be shown that the control bandwidth of a second-order system:

$$h(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta = 1.0,$$

$$\omega_b = \omega_n\sqrt{\sqrt{2} - 1} \approx 0.64\,\omega_n$$

Extensions to MIMO Nonlinear Systems (Ch. 12.2.4)

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \\ \mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) &= \boldsymbol{\tau} + \mathbf{w} \end{aligned}$$

$$\boldsymbol{\tau} = \mathbf{g}(\boldsymbol{\eta}) - \mathbf{H}_m(s)\dot{\boldsymbol{\nu}} - \mathbf{J}^\top(\boldsymbol{\eta})\boldsymbol{\tau}_{\mathrm{PID}}$$

$$\boldsymbol{\tau}_{\mathrm{PID}} = \mathbf{K}_p\tilde{\boldsymbol{\eta}} + \mathbf{K}_d\dot{\boldsymbol{\eta}} + \mathbf{K}_i\int_o^t \tilde{\boldsymbol{\eta}}(\tau)d\tau \qquad \tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \boldsymbol{\eta}_d$$

Second-order closed-loop system:

$$\mathbf{H}\dot{\boldsymbol{\nu}} + [\mathbf{C}(\boldsymbol{\nu}) + \mathbf{D}(\boldsymbol{\nu}) + \mathbf{K}_d^*(\boldsymbol{\eta})]\boldsymbol{\nu} + \mathbf{J}^\top(\boldsymbol{\eta})\mathbf{K}_p\tilde{\boldsymbol{\eta}} = \mathbf{w} \qquad \mathbf{K}_d^*(\boldsymbol{\eta}) = \mathbf{J}^\top(\boldsymbol{\eta})\mathbf{K}_d\mathbf{J}(\boldsymbol{\eta}) \qquad \mathbf{H} = \mathbf{M} + \mathbf{K}_m$$

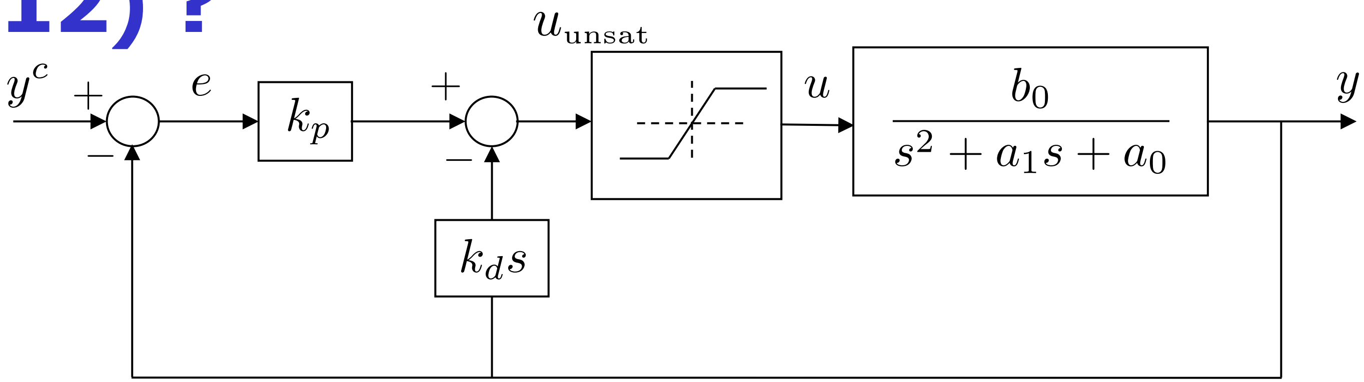
$$\begin{aligned} \mathbf{K}_p &= (\mathbf{M} + \mathbf{K}_m)\,\mathrm{diag}(\omega_{1,\mathrm{n}}^2, \cdots, \omega_{6,\mathrm{n}}^2) \\ \mathbf{K}_d^* &= (\mathbf{M} + \mathbf{K}_m)\,\mathrm{diag}(2\zeta_1\omega_{1,\mathrm{n}}, \cdots, 2\zeta_6\omega_{6,\mathrm{n}}) - \mathbf{D} \\ \mathbf{K}_i &= \mathbf{K}_p\,\mathrm{diag}(\omega_{1,\mathrm{n}}/10, \cdots, \omega_{6,\mathrm{n}}/10) \end{aligned}$$

Usually, $\mathbf{K}_m = \mathbf{0}$
that is no acceleration feedback

Relationship Between Fossen (2011) and Beard & Mclain (2012) ?

$$\boxed{\begin{aligned} k_p &= m\omega_n^2 - k \\ k_d &= 2\zeta\omega_n m - d \end{aligned}}$$

$$a_1 = \frac{d}{m} \qquad a_0 = \frac{k}{m} \qquad b_0 = \frac{1}{m}$$



The control signal u is largest immediately after a step on y_c, at which point the output of the differentiator is essentially zeros. Therefore $u \approx k_p e$. Let $u^{\max}$ be the input saturation limit, and $e^{\max}$, the largest expected step, then set

$$\boxed{k_p = \frac{u^{\max}}{e^{\max}}.}$$

The closed loop transfer function is

$$Y(s) = \frac{b_0 k_p}{s^2 + (a_1 + b_0 k_d)s + (a_0 + b_0 k_p)} Y^c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} Y^c(s)$$

Equating terms gives

$$\omega_n = \sqrt{a_0 + b_0 k_p} \qquad \boxed{k_d = \frac{2\zeta\omega_n - a_1}{b_0}.}$$

Conclusion:

k_d is equal for both methods

k_p is tuned using e^{max}, which relates to ω_n as:

$$e^{\max} = \frac{u^{\max}}{m\omega_n^2 - k}$$