TTK4190 Guidance and Control

Exam Suggested Solution

Spring 2011

Problem 1

A) The weight and buoyancy of the vehicle can be found as follows:

$$W = mg = 125 \times 9.81 = 1226.3 \text{ N} \tag{1}$$

$$B = 1026 \times \frac{4}{3}\pi \left(\frac{0.62}{2}\right)^3 \times 9.81 = 1256 \text{ N} \tag{2}$$

The result confirms the statement about the vehicle having slightly positively buoyant behavior.

B)The moments of inertia, as well as the products of inertia, about CO (assuming that CG coincides with CO) are computed as:

$$I_x = I_y = I_z = \frac{2}{5}mr^2 = \frac{2}{5}125\left(\frac{0.62}{2}\right)^2 = 4.805 \text{ kgm}^2 \tag{3}$$

$$I_{xy} = I_{xz} = I_{yz} = 0 \tag{4}$$

The rigid-body mass matrix is:

$$\mathbf{M}_{RB} = \text{diag}\{m, m, m, I_x, I_y, I_z\} \tag{5}$$

The added mass is then found to be:

$$X_{\dot{u}} = Y_{\dot{v}} = Z_{\dot{w}} = -\rho\frac{2}{3}\pi r^3 = -1026\frac{2}{3}\pi\left(\frac{0.62}{2}\right)^3 = -64.0164 \text{ kg} \tag{6}$$

which gives the following added mass matrix:

$$\mathbf{M}_A = \text{diag}\{-X_{\dot{u}}, -Y_{\dot{v}}, -Z_{\dot{w}}, 0, 0, 0\} \tag{7}$$

The added mass is the mass that a marine craft has to displace while moving and, similarly, the added inertia is due to the water mass that has to be displaced when the marine craft rotates. Since in this case the vehicle is a sphere, its rotational motion does not induce any mass displacement, this is why the three last elements of the diagonal matrix are equal to zero.

C) From section 4.1 we recall that the linear restoring forces and moments for an underwater vessel can be written:

$$\mathbf{G} = \text{diag}\{0, 0, 0, (z_g - z_b)W, (z_g - z_b)W, 0\} \tag{8}$$

$$= \text{diag}\{0, 0, 0, 61.315, 61.315, 0\} \tag{9}$$

D) From equation (6.88) in the book, we have:

$$X_{|u|u} = Y_{|v|v} = Z_{|w|w} = -\frac{1}{2}\rho C_d A \tag{10}$$

$$= -0.5 \times 1026 \times 0.8 \times \pi\left(\frac{0.62}{2}\right)^2 = -123.9027 \text{ kg/m} \tag{11}$$

E) The natural frequencies in roll and pitch can be computed from equations (7.266) – (7.267) and (7.274) – (7.275) in the book:

$$\omega_{\text{roll}} = \sqrt{\frac{(z_g - z_b)W}{I_x}} = \sqrt{\frac{0.05 * 1226.3}{4.805}} = 3.5722 \;\; \text{rad/s} \tag{12}$$

$$\omega_{\text{pitch}} = \sqrt{\frac{(z_g - z_b)W}{I_y}} = \sqrt{\frac{0.05 * 1226.3}{4.805}} = 3.5722 \;\; \text{rad/s} \tag{13}$$

$$T_{\text{roll}} = T_{\text{pitch}} = \frac{2\pi}{3.5722} = 1.7589 \text{ s} \tag{14}$$

F)Similarly to example 12.5 in the book, we compute the linear damping coefficient in roll and pitch:

$$\begin{aligned} K_p = M_q &= -2\sqrt{1 - r^2}\sqrt{(z_g - z_b)W I_x} && (15)\\ &= -2\sqrt{1 - 0.995^2}\sqrt{0.05 \times 1226.3 \times 4.805} && (16)\\ &= -3.4286 \text{ kg m/s} && (17) \end{aligned}$$

G)The linear damping coefficient N_r can be found by following example 12.6, and more specifically equation (12.81) from the book:

$$\frac{I_z}{-N_r} = T_{\text{yaw}} \tag{18}$$

$$N_r = -\frac{I_z}{T_{\text{yaw}}} = -\frac{4.805}{20} = -0.24 \text{ kg m/s} \tag{19}$$

H) Since $\dot{\psi} = r$, we can re-write equation (12.79) as

$$I_z \ddot{\psi} - N_r \dot{\psi} = N_\delta \delta \tag{20}$$

If we choose the scaling factor $N_\delta = 1$ and use the PD control law $\delta = -K_d\dot{\psi} - K_p\psi$, equation (20) turns into the 2nd order system

$$I_z \ddot{\psi} + (K_d - N_r)\dot{\psi} + K_p \psi = 0 \tag{21}$$

Consequently, equations (12.75)-(12.76) give:

$$2\zeta\omega_n = \frac{-N_r + K_d}{I_z}, \quad \omega_n^2 = \frac{K_p}{I_z} \tag{22}$$

which results in the proportional gain

$$\begin{aligned} K_p &= \omega_n^2 I_z && (23)\\ &= \left(\frac{2\pi}{10}\right)^2 4.805 && (24)\\ &= 1.8969 && (25) \end{aligned}$$

I)According to equation (12.75) from the book:

$$\begin{aligned} K_d &= 2\zeta\omega_n I_z + N_r && (26)\\ &= 2 \times 0.8 \left(\frac{2\pi}{10}\right)^2 \times 4.8050 - 0.24 && (27)\\ &= 2.8 && (28) \end{aligned}$$

K) Section 7.5.4
L) Local asymptotically stable since there is a kinematic singularity when using Euler angles
M) There is no restoring force in heave for a submerged sphere with $W = B$. Hence, only stability can be concluded (not asymptotically stable).

Problem 2

A)
Starting with:

$$(m - X_{\dot{u}})\dot{u} - Z_{u|u|}u|u| + mg\sin\theta = T \tag{29}$$

We try the Lyapunov function candidate:

$$V_1(\tilde{u}) = \frac{1}{2}(m - X_{\dot{u}})\tilde{u}^2 \tag{30}$$

and by differentiating with respect to time:

$$\begin{aligned}
\dot{V}_1(\tilde{u}) &= (m - X_{\dot{u}})(\dot{u} - \dot{u}_d)\tilde{u} && (31)\\
&= \tilde{u}\left[(m - X_{\dot{u}})\dot{u} - (m - X_{\dot{u}})\dot{u}_d\right] && (32)\\
&= \tilde{u}\left[Z_{u|u|}u|u| - mg\sin\theta + T - (m - X_{\dot{u}})\dot{u}_d\right] && (33)
\end{aligned}$$

we choose the control input T as:

$$T = -Z_{u|u|}u|u| + mg\sin\theta + (m - X_{\dot{u}})\dot{u}_d - k\tilde{u} \tag{34}$$

and the Lyapunov function derivative takes the form:

$$\dot{V}_1(\tilde{u}) = -k\tilde{u}^2 \leq -cV \tag{35}$$

This inequality implies that V_1 converges to 0 exponentially, and hence, as $V_1(\tilde{u})$ is upper and lower bounded:

$$\frac{1}{4}(m - X_{\dot{u}})|\tilde{u}|^2 \leq V_1(\tilde{u}) \leq (m - X_{\dot{u}})|\tilde{u}|^2 \tag{36}$$

This implies that the equilibrium $\tilde{u} = 0$ is UGES.

B) According to linear theory, for very small θ we can approximate the trigonometric functions as

$$\cos\theta \simeq 1, \quad \sin\theta \simeq 0. \tag{37}$$

Following the procedure described in section 7.5.3 (we only have nonlinear damping here) in the book we get the system of equations:

$$\begin{aligned}
\dot{\tilde{z}} &= -U_0\theta + w && (38)\\
\dot{\theta} &= q && (39)\\
\dot{q} &= \frac{M_q q}{(I_y - M_{\dot{q}})} + \frac{M_{\delta_s}\delta_s}{(I_y - M_{\dot{q}})}. && (40)
\end{aligned}$$

Notice that the linearized equation for z has the opposite sign compared to h. This means that z is expressed in the North - East - Down frame while the altitude h is directed upwards.
This linear system can be written in matrix form:

$$\begin{bmatrix} \dot{\tilde{h}} \\ \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & U_0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \dfrac{M_q q}{(I_y - M_{\dot{q}})} \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \theta \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{M_{\delta_s}}{(I_y - M_{\dot{q}})} \end{bmatrix} \delta_s + \begin{bmatrix} -w \\ 0 \\ 0 \end{bmatrix}. \tag{41}$$

The system is controllable and we can use a controller of the form $\mathbf{u} = -\mathbf{K}\mathbf{x}$

C) Since $u = U_0$ it follows that

$$\dot{h} = U_0\theta - w$$

$$z_1 = h - h_d \tag{42}$$

$$\dot{z}_1 = U_0\theta - w - \dot{h}_d \tag{43}$$

Virtual control:

$$\theta = \alpha_1 + z_2 \tag{44}$$

$$\dot{z}_1 = U_0(\alpha_1 + z_2) - w - \dot{h}_d \tag{45}$$

$$\alpha_1 = \frac{1}{U_0}\left(-k_1 z_1 + w + \dot{h}_d\right) \tag{46}$$

$$\dot{z}_1 = -k_1 z_1 + U_0 z_2 \tag{47}$$

$$V_1 = \frac{1}{2} z_1^2 \tag{48}$$

$$\dot{V}_1 = -k_1 z_1^2 + U_0 z_1 z_2 \tag{49}$$

$$\dot{z}_2 = \dot{\theta} - \dot{\alpha}_1 \tag{50}$$

$$= q - \dot{\alpha}_1 \tag{51}$$

$$V_2 = V_1 + \frac{1}{2} z_2^2 \tag{52}$$

$$\dot{V}_2 = -k_1 z_1^2 + U_0 z_1 z_2 + z_2(q - \dot{\alpha}_1) \tag{53}$$

$$q = \alpha_2 + z_3 \tag{54}$$

$$\dot{V}_2 = -k_1 z_1^2 + U_0 z_1 z_2 + z_2(\alpha_2 + z_3 - \dot{\alpha}_1) \tag{55}$$

$$\alpha_2 = -k_2 z_2 + \dot{\alpha}_1 - U_0 z_1 \tag{56}$$

$$\dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 + z_2 z_3 \tag{57}$$

$$\dot{z}_3 = \dot{q} - \dot{\alpha}_2 \tag{58}$$

$$(I_y - M_{\dot{q}})\dot{z}_3 = (I_y - M_{\dot{q}})\dot{q} - (I_y - M_{\dot{q}})\dot{\alpha}_2 \tag{59}$$

$$= -M_q q + M_{\delta_s}\delta_s - (I_y - M_{\dot{q}})\dot{\alpha}_2 \tag{60}$$

$$V_3 = V_2 + \frac{1}{2}(I_y - M_{\dot{q}}) z_3^2 \tag{61}$$

$$\dot{V}_3 = -k_1 z_1^2 - k_2 z_2^2 + z_2 z_3 + z_3(-M_q q + M_{\delta_s}\delta_s - (I_y - M_{\dot{q}})\dot{\alpha}_2) \tag{62}$$

$$\delta_s = \frac{1}{M_{\delta_s}}\left(-k_3 z_3 + (I_y - M_{\dot{q}})\dot{\alpha}_2 + M_q q - z_2\right) \tag{63}$$

$$\dot{V}_3 = -k_1 z_1^2 - k_2 z_2^2 - k_3 z_3^2 \tag{64}$$

Hence, GES follows.
D) In steady-state we have

$$\dot{h} = u\theta - w = 0 \tag{65}$$

$$\tan(\alpha) = \frac{w}{u} \tag{66}$$

$$= \theta \tag{67}$$

Hence, for small angle of attack $\alpha \approx \theta$.

Problem 3

A. Assuming an aerial vehicle, for low-speed simulation of aircraft flying over a small region of the Earth and with no requirement for precise simulation of position, it is usual to neglect the centripetal and Coriolis terms. Neglecting the centripetal terms is equivalent to assuming that the Earth is flat and neglecting the Coriolis term is equivalent to assuming that the Earth is an inertial frame. In the flat-earth equations of motion, the equations have significant errors for velocities over the Earth with magnitude greater than about 600 m/s. Global navigation, on the other hand takes all these into account (Earth is an oblate spheroid which cannot be considered as an inertial frame) and offers an accurate and reliable way of navigating around the Earth no matter of how large the distances, or the vehicle velocity, are.

B. One of the most common problems is that magnetic compasses are affected by magnetic disturbances that might occur close to the instrument. This is very common in the case of UAVs where the available space is limited and there might be many wires and other instruments close to the compass. Unless properly insulated from such disturbances, the compass measurements can be practically useless. The most important problem, however is that of *compass variation* which is the difference between true north and magnetic north. The amount of variation changes by as much as 20 degrees as one circumnavigates the Earth. If this compass error is not compensated for, a navigator can find himself far off course.

C. First of all, the Extended Kalman Filter is not a pure nonlinear approach since it linearizes the system at the current state value and then implements the linear Kalman filter equations. Moreover, it is a stochastic approach, which means that it takes into account the uncertainty of the signals (noise, or model uncertainty) and gives a result that actually consists of an estimate of the mean value and the covariance of the state(s). A deterministic observer relies on the tools of nonlinear theory, the Lyapunov theorems for instance, in order to determine whether the observer estimation is stable, asymptotically stable, etc. This method neglects the measurements' noise. Consequently, in practice, even in the case where the observer is *GES*, the result will be ultimately bounded and this bound will depend on the uncertainty which was not included in the analysis.

D. According to paragraph 8.1 in the book, he wind velocity V_w and its direction β_w can be measured by an anemometer and a weather vane respectively. The wind forces can then be computed from the equations (8.23). Apparently, in order to do this we need to know the dimensions of the vehicle etc.

The wave forces and moments are described in paragraph 8.2 and the ocean current forces and moments in paragraph 8.3.

F. The environmental forces problem is not observable because it is impossible to distinguish several unknown quantities that might have been introduced in the estimation model as a constant bias, for example the current and the uncertainties due to modeling errors.