TTK4250 - Sensor Fusion - Assignment 1

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Contents

1 Task 1: The CDF of any random variable is Uniform(0, 1)

Given the equations:

$$P_X(x) = P(X \leq x)$$
$$Y = P_X(X)$$

Assuming that there exists a well-defined $P_X^{-1}(x)$, such that $P_X(P_X^{-1}(x)) = x$.

$$P_Y(y) = P(Y \leq y) = P(P_X(X) \leq y) = P(X \leq P_X^{-1}(y)) = P_X(P_X^{-1}(y)) = y$$

As all probability values must be in the range $[0, 1]$, then $P_Y(y)$ is also only defined for this range. Combined with knowing that the PDF of a CDF is its derivative, we can conclude with eq. (1).

$$Y \sim Uniform(0, 1) \tag{1}$$

2 Task 2: Some results regarding the Poisson distribution

Some useful equations:

$$Poisson(x;\lambda) = p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x \in \{0,1,2,\dots\} \tag{2}$$

$$Binomial(x;r,n) = p(x) = \binom{n}{x} r^x (1-r)^{n-x}, x \in \{0,1,\dots,n\} \tag{3}$$

$$DiscreteGenFunc(t) = G(t) = E_X[t^X] = \sum_{n=-\infty}^{\infty} p(x_n) t^{x_n} \tag{4}$$

$$ExponentialFunction: e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} (1+\frac{x}{n})^n \tag{5}$$

$$BinomialTheorem: \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{(n-i)} = (a+b)^n \tag{6}$$

$$SumOfPMF: P(A+B=x) = \sum_{i=-\infty}^{\infty} P(A=i)\cdot P(B=x-i) \tag{7}$$

2.1 a)

Utilizing eq. (4), eq. (2) and eq. (5) we get:

$$G(t) = \sum_{n=-\infty}^{\infty} p(x_n) t^{x_n} = \sum_{n=0}^{\infty} \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} t^{x_n} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(t\lambda)^{x_n}}{x_n!}$$

Such that the generating function of a Poisson distributed random variable N is:

$$G(t) = e^{-\lambda} e^{t\lambda} = e^{\lambda(t-1)} \tag{8}$$

2.2 b)

Utilizing eq. (4), eq. (3) and eq. (6) we get:

$$G(t) = \sum_{i=-\infty}^{\infty} p(m_i) t^{m_i} = \sum_{i=0}^{n} \binom{n}{m_i} r^{m_i} (1-r)^{n-m_i} t^{m_i} \tag{9}$$

$$= \sum_{i=0}^{n} \frac{n!}{m_i!(n-m_i)} (rt)^{m_i} (1-r)^{n-m_i} = (rt+1-r)^n \tag{10}$$

Such that the generating function of a Binomial distributed random variable M is:

$$G(t) = (1-r+rt)^n \tag{11}$$

2.3 c)

Utilizing the resulting function eq. (11), $r = \frac{\lambda}{n}$ and eq. (5), we get:

$$\lim_{n\to\infty} G(t) = \lim_{n\to\infty}(1 - r + rt)^n = \lim_{n\to\infty}(1 + (t-1)r)^n$$
$$= \lim_{n\to\infty}(1 + \frac{(t-1)\lambda}{n})^n = e^{(t-1)\lambda}$$

As we can see, the result generating function of a Binomial distributed random variable, where n goes to infinity, is the same as the generating function for the Poisson distributed random variable.

These two distributions are obviously closely linked, and as noted the textbook, the binomial distribution can be approximated by the poisson distribution when n is large / tends towards infinity[1, page 19].

2.4 d)

First, defining:

$$N_1 \sim Poisson(\lambda_1) \tag{12}$$
$$N_2 \sim Poisson(\lambda_2) \tag{13}$$
$$N = N_1 + N_2 \tag{14}$$

As N_1, N_2 are independant, we can use eq. (7) and eq. (2). Then we have:

$$P(N = n) = P(N_1 + N_2 = n) = \sum_{i=-\infty}^{\infty} P(N_1 = i) \cdot P(N_2 = n - i)$$
$$= \sum_{i=0}^{\infty} P(N_1 = i) \cdot P(N_2 = n - i) = \sum_{i=0}^{\infty} \frac{\lambda_1^i e^{-\lambda_1}}{i!} \cdot \frac{\lambda_2^{(n-i)} e^{-\lambda_2}}{(n-i)!}$$
$$= \sum_{i=0}^{\infty} \frac{\lambda_1^i \lambda_2^{(n-i)} e^{-(\lambda_1+\lambda_2)}}{i!(n-i)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{i=0}^{\infty} \frac{n! \lambda_1^i \lambda_2^{(n-i)}}{i!(n-i)!}$$

Finally, utilizing eq. (6) and comparing the result with eq. (2), we can conclude that:

$$P(N = n) = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1 + \lambda_2)^n = Poisson(n; \lambda = \lambda_1 + \lambda_2) \tag{15}$$

As we can note from the textbook [1, page 25], the second property of the generating function states that the generating function of a sum of independant random variables is the product of the generating functions. Using eq. (8), we can show:

$$G_N(t) = G_{N_1}(t) \cdot G_{N_2}(t) = e^{\lambda_1(t-1)} \cdot e^{\lambda_2(t-1)} = e^{(\lambda_1+\lambda_2)(t-1)}$$

Comparing this result with eq. (8) and using the first property of the generating function, we can conclude that

$$P(N = n) = Poisson(n; \lambda = \lambda_1 + \lambda_2) \tag{16}$$

3 Task 3: Estimationg the number of boats in a region

Some more useful equations:

$$ExponentialPDF(t_i - t_{i-1}; \lambda) = p(t_i - t_{i-1}) \tag{17}$$

$$= \begin{cases} \lambda e^{-\lambda(t_i - t_{i-1})} & t_i - t_{i-1} \geq 0 \\ 0 & t_i - t_{i-1} < 0 \end{cases} \tag{18}$$

$$MomentGenerating(s) = M_X(s) = E_X[e^{sx}] = \int_{-\infty}^{\infty} p(x) e^{sx} dx \tag{19}$$

$$\Gamma \quad Distribution \quad PDF = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)} \tag{20}$$

$$\Gamma \quad Distribution \quad MGF = (\frac{1}{1 - \theta s})^k \tag{21}$$

The MGF of the Gamma distribution eq. (21) found in the textbook[1, page 26].

3.1 a)

$$\begin{aligned} p(t_1 | t_1 \geq t_0) &= \frac{p(t_1 \geq t_0 | t_1) p(t_1)}{p(t_1 \geq t_0)} \\ &= \begin{cases} \frac{\lambda e^{-\lambda t_1}}{\int_{t_0}^{\infty} \lambda e^{-\lambda t_1} dt_1} & if \quad t_1 \geq t_0 \\ 0 & else \end{cases} \\ &= \begin{cases} \frac{\lambda e^{-\lambda t_1}}{\lambda \ [-\frac{1}{\lambda} e^{-\lambda t_n}]_{t_0}^{\infty}} & if \quad t_1 \geq t_0 \\ 0 & else \end{cases} \\ &= \begin{cases} \lambda e^{-\lambda(t_1 - t_0)} & if \quad t_1 \geq t_0 \\ 0 & else \end{cases} \end{aligned}$$

As the exponential distribution refers to the amount of time between two events, memorylessness states that this distribution is not dependant on how much time has passed. One useful feature of this property would be that we can view the time between events as completely independant, and don't have to record previous results, as it is completely useless to future predictions. This also means that there is no real difference in when you start measuring/observing the process as well.

3.2 b)

Noteing that we know the probability density functions of $t_i - t_{i-1}$ as eq. (17), and knowing that a sum of PDFs can be represented as a product of each of the PDFs' generating function, we can see:

$$t_n - t_0 = \sum_{i=1}^{n} t_i - t_{i-1}, t_1 \geq t_0 \tag{22}$$

$$M_X(s) = \int_{-\infty}^{\infty} p(x)e^{sx}dx = \int_0^{\infty} \lambda e^{-\lambda x}e^{sx}dx = \frac{\lambda}{\lambda - s} \tag{23}$$

$$M_{\sum X} = (\frac{\lambda}{\lambda - s})^n \tag{24}$$

Comparing with eq. (21), we can conclude that $t_n - t_0$ is distributed according to the gamma distribution, with $\theta = \frac{1}{\lambda}$ and $k = n$.

3.3 c)

$$\begin{aligned}
Pr(t_{n+1} > T|t_n) &= Pr(t_{n+1} - t_n > T - t_n|t_n) \\
&= 1 - P_{t_{n+1}-t_n}(T - t_n) = 1 - \int_0^{T-t_n} \lambda e^{-\lambda(t_i - t_{i-1})}d(t_i - t_{i-1}) \\
&= 1 - [-e^{-\lambda(t_{n+1}-t_n)}]_0^{T-t_n} \\
&= e^{-\lambda(T-t_n)}
\end{aligned}$$

3.4 d)

Using eq. (20) and the results from the previous tasks, we have:

$$\begin{aligned}
Pr(n) &= Pr(t_n \leq T, t_{n+1} > T|t_1 \leq t_0) \\
p(t_n, t_{n+1} > T|t_1 \leq t_0) &= GammaPDF(t_n - t_0; n, \frac{1}{\lambda})e^{-\lambda(T-t_n)} \\
\to Pr(n) &= \int_{t_0}^{T} \frac{\lambda^n (t_n - t_0)^{n-1} e^{-\lambda(t_n - t_0)}}{\Gamma(n)} e^{-\lambda(T-t_n)}dt_n \\
&= [\frac{\lambda^n (t_n - t_0)^n e^{-\lambda(T-t_0)}}{n!}]_{t_0}^{T} \\
&= \frac{\lambda^n (T - t_0)^n e^{-\lambda(T-t_0)}}{n!} \\
&= Poisson(n; \lambda(T - t_0))
\end{aligned}$$

3.5 e)

Combining the results from above, as well as the fact that $1 + P_D - P_D = 1$, we get:

$$\begin{aligned}
p(n_D, n) = p(n_D|n)p(n) &= \binom{n}{n_D} P_D^{n_D}(1-P_D)^{n-n_D} \frac{(\lambda(T-t_0))^n e^{-\lambda(T-t_0)}}{n!} \\
&= \frac{n!}{n_D!(n-n_D)!n!} P_D^{n_D}(1-P_D)^{n-n_D}(\lambda(T-t_0))^n e^{-\lambda(T-t_0)(P_D+1-P_D)} \\
&= \frac{(\lambda(T-t_0)P_D)^{n_D}}{n_D!} e^{-\lambda(T-t_0)P_D} \frac{(\lambda(T-t_0)(1-P_D))^{n_U}}{n_U!} e^{-\lambda(T-t_0)(1-P_D)} \\
&= Poisson(n_D; \lambda(T-t_0)P_D) Poisson(n_U; \lambda(T-t_0)(1-P_D))
\end{aligned}$$

3.6 f)

Using the generating function of the Poisson distribution, eq. (8), and the fact that a sum of variables becomes a product as a generating function, we can find:

$$\begin{aligned}
M_m(t) = M_{n_D} + M_{fa} &= \exp([\lambda(T-t_0)P_D](t-1)) \exp(\Lambda(t-1)) \\
&= \exp([\Lambda + \lambda(T-t_0)P_D](t-1))
\end{aligned}$$

Therefore, $m \sim Poisson(m; (\Lambda + \lambda(T-t_0)P_D))$.

$$\begin{aligned}
p(n_D|m) = \frac{p(m|n_D)p(n_D)}{p(m)} &= \frac{p(m = n_D + m_{fa}|n_D)p(n_D)}{p(m)} \\
&= \frac{\Lambda e^{-\Lambda}}{m_{fa}!} \frac{(\lambda P_D(T-t_0))^{n_D} e^{-\lambda P_D(T-t_0)}}{n_D!} \frac{m!}{[\Lambda + \lambda P_D(T-t_0)]^m e^{-[\Lambda + \lambda P_D(T-t_0)}} \\
&= \frac{m!}{n_D!(m-n_D)!} [\frac{\Lambda}{\Lambda + \lambda P_D(T-t_0)}]^{m-n_D} [1 - \frac{\Lambda}{\Lambda + \lambda P_D(T-t_0)}]^{n_D} \\
&= Binomial(x = n_D; r = \frac{\Lambda}{\Lambda + \lambda P_D(T-t_0)}, n = m)
\end{aligned}$$

3.7 g)

With

$$
\begin{aligned}
&Binomial(x = n_D; r = \frac{\Lambda}{\Lambda + \lambda P_D(T - t_0)}, n = m) \\
&n_D Binomial(n_D; r, m) = mr Binomial(n_D - 1; r, m - 1) \\
&\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x + y)^n
\end{aligned}
$$

we get:

$$
\begin{aligned}
\sum_{0}^{\infty} n_D Binomial(n_D; r, m) &= \sum_{0}^{\infty} mr \binom{m-1}{n_D - 1} r^{m-1-n_D+1} (1 - r)^{n_D - 1} \\
&= mr(r + 1 - r)^{m-1} = mr
\end{aligned}
$$

4 Task 4: Transformation of Gaussian random variables

Some more useful equations:

$$Let \quad x \in \mathbb{R} \quad \mathcal{N}(\mu, \Sigma) \tag{25}$$

$$\rightarrow g(x) = \mathcal{N}(\mu, \Sigma) \tag{26}$$

$$\mathcal{N}(x; \mu, \Sigma) = \frac{\exp(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu))}{\sqrt{(2\pi)^n |\Sigma|}} \tag{27}$$

$$det(A^{\frac{1}{2}}) = det(A)^{\frac{1}{2}} \tag{28}$$

$$(A^{-1})^\top = (A^\top)^{-1} \tag{29}$$

$$det(A^\top) = det(A), \quad when \quad rank(A) \quad full \tag{30}$$

$$\chi^2 \quad Distribution = p(x; n) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp(-\frac{x}{2}) \tag{31}$$

Theorem 2.4.1

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \tag{32}$$

$$pdf: \quad \mathbf{x}, \mathbf{y} \rightarrow g(\mathbf{x}), h(\mathbf{y}) \tag{33}$$

$$h(\mathbf{y}) = \sum_i g(\mathbf{f}_i^{-1}(\mathbf{y}))|\det(\mathbf{F}_i^{-1}(\mathbf{y}))| \tag{34}$$

4.1 a)

As $z = f(x)$ is invertible, the problem reduces to:

$$\begin{aligned}
z &= f(x) = \Sigma^{-\frac{1}{2}}(x-\mu) \\
\Sigma &= \Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\top \\
\mathbf{f^{-1}} &= \Sigma^{\frac{1}{2}} z + \mu \\
\mathbf{F^{-1}} &= \Sigma^{\frac{1}{2}} \\
h(\mathbf{z}) &= \frac{\exp(-\frac{1}{2}(\Sigma^{\frac{1}{2}}z + \mu - \mu)^\top \Sigma^{-1}(\Sigma^{\frac{1}{2}}z + \mu - \mu))}{(\sqrt{2\pi})^n |\Sigma|} |\det(\Sigma^{\frac{1}{2}})| \\
&= \frac{\exp(-\frac{1}{2}(\Sigma^{\frac{1}{2}}z)^\top \Sigma^{-1}(\Sigma^{\frac{1}{2}}z))}{(\sqrt{(2\pi)})^n |\Sigma|} |\det(\Sigma)|^{\frac{1}{2}} \\
&= \frac{\exp(-\frac{1}{2}z^\top (\Sigma^{\frac{1}{2}})^\top (\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^\top)^{-1}\Sigma^{\frac{1}{2}}z)}{(\sqrt{2\pi})^n} \\
&= \frac{\exp(-\frac{1}{2}z^\top z)}{(\sqrt{2\pi})^n} = \mathcal{N}(z; 0, \mathbf{I})
\end{aligned}$$

We may conclude that z is given by the standard normal distribution.

4.2 b)

Noteing:

$$z \sim \mathcal{N}(0, \mathbf{I}) \to z_i \sim \mathcal{N}(0, 1)$$

Further noteing that $\mathbf{f}$ is not invertible, we have:

$$\begin{aligned} \mathbf{f_1}^{-1} &= \sqrt{y_i} && \to \mathbf{F_1}^{-1} && = \frac{1}{2\sqrt{y}} \\ \mathbf{f_2}^{-1} &= -\sqrt{y_i} && \to \mathbf{F_2}^{-1} && = -\frac{1}{2\sqrt{y}} \end{aligned}$$

Combining this, we get:

$$h(y_i) = \frac{1}{\sqrt{2\pi y}}(\frac{1}{2}\exp(-\frac{1}{2}(\sqrt{y})^2) + \frac{1}{2}\exp(-\frac{1}{2}(-\sqrt{y})^2)) = \frac{1}{\sqrt{2\pi y}}\exp(-\frac{y}{2}) \tag{35}$$

This distribution is the χ^2 distribution, with 1 degree of freedom.

4.3 c)

y is now defined as:

$$y = (x - \mu)^\top \Sigma^{-1}(x - \mu) = z^\top z = \sum_i z_i^2 = \sum_i y_i$$

Using the result from 4b), section 4.2, we know the distribution of y_i. Knowing that the χ^2 distribution is a special case of the Γ distribution, and knowing the MGF of the Γ distribution, we can conclude that the MGF of the χ^2 distribution as eq. (36), with the degrees of freedom $n = 1$.

$$M_{y_i}(s) = (\frac{1}{1 - 2s})^{\frac{n}{2}} \tag{36}$$

Using one of the properties of the generating functions, we can combine a sum of random variables to a product of MGF's:

$$M_{y_i}(s) = ((\frac{1}{1 - 2s})^{\frac{1}{2}})^n = (\frac{1}{1 - 2s})^{\frac{n}{2}} \tag{37}$$

eq. (37) can be compared to eq. (36), and we can conclude that y is distributed as χ^2 with n degrees of freedom.

5 Task 5: Sensor fusion

Summarizing the equations given in this task:

$$
\begin{aligned}
y &= x + 2 \\
z^c &= H^c x + v^c \\
z^r &= H^r x + v^r \\
x^+ &= Fx + w
\end{aligned}
$$

Summarizing the distributions of the variables given in this task:

$$
\begin{aligned}
x &\sim \mathcal{N}(\bar{x}, P) \\
v^c &\sim \mathcal{N}(0, R^c) \\
v^r &\sim \mathcal{N}(0, R^r) \\
w &\sim \mathcal{N}(0, Q)
\end{aligned}
$$

Summarizing the values given in this task:

$$
\begin{aligned}
\bar{x} &= \begin{bmatrix} 0 & 0 \end{bmatrix}^\top, \qquad P = 25 I_2, \qquad H^c = H^r = I_2, \\
R^c &= \begin{bmatrix} 79 & 36 \\ 36 & 36 \end{bmatrix}, \quad R^r = \begin{bmatrix} 28 & 4 \\ 4 & 22 \end{bmatrix}, \quad z_c = \begin{bmatrix} 2 & 14 \end{bmatrix}^\top, \quad z_r = \begin{bmatrix} -4 & 6 \end{bmatrix}^\top
\end{aligned}
$$

5.1 a)

As x is given, adding a constant to normally distributed random variable will shift the expected value, and leave the variance. Therefore:

$$
p(z^c|x) \sim \mathcal{N}(H^c x, R^c)
$$

5.2 b)

From Bayes' rule and Theorem 3.3.1 from the textbook, [1, page 43], we have:

$$
\begin{aligned}
p(x, z^c) = p(z^c|x)p(x) &= \mathcal{N}(z^c; H^c x, R^c)\mathcal{N}(x; \bar{x}, P) \\
&= \mathcal{N}(z; \bar{z}, S)\mathcal{N}(x; \hat{x}, \hat{P})
\end{aligned}
$$

As we can find the values used to relate the distributions used in the identity, this means that the two distributions can be expressed independantly of each other. , we can conclude that the result is also Gaussian.

5.3 c)

As the sum of Gaussians are Gaussian, and we have the linearity theorem, we can note that:

$$p(z^c) = \mathcal{N}(H^c\bar{x}, H^c P (H^c)^\top + R^c)$$

From Bayes rule, we can also find:

$$p(x|z^c) = \frac{p(z^c|x)p(x)}{p(z^c)}$$

5.4 d)

5.5 e)

5.6 f)

5.7 g)

5.8 h)

6 Task 6: Information matrix update

$$z = Hx + w$$

$$\begin{aligned} x &\sim \mathcal{N}(\bar{x}, P) \\ w &\sim \mathcal{N}(0, R) \end{aligned}$$

Then with the Kalman Filter form, taken from the textbook, [1, page 49]:

$$\begin{aligned} x_k = Fx_{k-1} + v_k, \quad v_k & \sim \mathcal{N}(0, Q) \\ z_k = Hx_k + w_k, \quad w_k & \sim \mathcal{N}(0, R) \\ x_0 \sim \mathcal{N}(\hat{x_0}, P_0) & \end{aligned}$$

And substituting:

$$\begin{aligned} F = I, Q &= 0, \\ \hat{x_0} = \bar{x}, P_0 = P & \end{aligned}$$

Then we may simplify $x_k = x$ and $k = 1$, and we have this problem. Then, using the posteriori covariance matrix from the Kalman Filter algorithm, we have:

$$\begin{aligned} \hat{P} &= (I - W_k H) P_{k|k-1} \\ &= (I - P_{k|k-1} H^\top S_k^{-1} H) P_{k|k-1} \\ &= (I - P_{k|k-1} H^\top (H P_{k|k-1} H^\top + R)^{-1} H) P_{k|k-1} \\ &= (I - [F P_{k-1} F^\top + Q] H^\top (H [F P_{k-1} F^\top + Q] H^\top + R)^{-1} H)[F P_{k-1} F^\top + Q] \\ &= (I - P H^\top (H P H^\top + R)^{-1} H) P \\ &= P - P H^\top (R + H P H^\top)^{-1} H P \\ &= (P^{-1})^{-1} + (P^{-1})^{-1} (-H)^\top (R - H (P^{-1})^{-1} (-H)^\top)^{-1} H P^{-1})^{-1} \end{aligned}$$

Utilizing eq. (38), we can calculate the inverse covariance matrix:

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \tag{38}$$

$$\begin{aligned} \hat{P}^{-1} &= [(P^{-1})^{-1} + (P^{-1})^{-1} (-H)^\top (R - H (P^{-1})^{-1} (-H)^\top)^{-1} H P^{-1})^{-1}]^{-1} \\ &= [(P^{-1} + H^\top R^{-1} H)^{-1}]^{-1} \\ &= P^{-1} + H^\top R^{-1} H \end{aligned}$$

References

[1] Edmund Brekke. *Fundamentals of Sensor Fusion*. 2019.