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Solution for Exam
TTK 4190 Guidance and Control of Vehicles

Problem 1: Spacecraft Control System (25%)

1a (4%) The model is valid under the following assumptions:

- No external forces in space
- Center of gravity $\mathbf{r} = [0, 0, 0]^\top$
- Symmetrical about the xy- and xz-planes such that $I_{xy} = I_{xz} = 0$
- Constant and positive mass $m > 0$
- The body is rigid

1b (4%) The vectors and matrices are:

$$\mathbf{M}_1 = m\,\mathbf{I}_3 \tag{1}$$

$$\mathbf{M}_2 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & -I_{yz} \\ 0 & -I_{yz} & I_z \end{bmatrix} \tag{2}$$

$$\mathbf{C}_1(\boldsymbol{\nu}_2) = m \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{3}$$

$$\mathbf{C}_2(\boldsymbol{\nu}_2) = \begin{bmatrix} 0 & -I_{yz}q + I_z r & -I_y q + I_{yz} r \\ I_{yz}q - I_z r & 0 & I_x p \\ I_y q - I_{yz} r & -I_x p & 0 \end{bmatrix} \tag{4}$$

$$\boldsymbol{\nu}_1 = [u, v, w]^\top \tag{5}$$

$$\boldsymbol{\nu}_2 = [p, q, r]^\top \tag{6}$$

$$\boldsymbol{\tau}_1 = [\tau_1, \tau_2, \tau_3]^\top \tag{7}$$

$$\boldsymbol{\tau}_2 = [\tau_4, \tau_5, \tau_6]^\top \tag{8}$$

1c (4%) Since $m > 0$ it follows that the quadratic forms $\mathbf{x}^\top \mathbf{M}_i \mathbf{x} > 0$ $(i = 1, 2)$ for all $\mathbf{x} \neq \mathbf{0}$. The matrices $\mathbf{C}_1(\boldsymbol{\nu}_2)$ and $\mathbf{C}_2(\boldsymbol{\nu}_2)$ are skew-symmetric by inspection. Moreover, $\mathbf{C}_1(\boldsymbol{\nu}_2) = -\mathbf{C}_1(\boldsymbol{\nu}_2)^\top$ and $\mathbf{C}_2(\boldsymbol{\nu}_2) = -\mathbf{C}_2(\boldsymbol{\nu}_2)^\top$.

1d (4%)

$$\boldsymbol{\tau}_1 = \mathbf{M}_1 \mathbf{a}_1 + \mathbf{C}_1(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1 \tag{9}$$

$$\mathbf{a}_1 = \dot{\tilde{\boldsymbol{\nu}}}_{1d} - k_p \tilde{\boldsymbol{\nu}}_1 - k_i \int_0^t \tilde{\boldsymbol{\nu}}_1(\tau)\,\mathrm{d}\tau \tag{10}$$

where $k_p = 2\lambda$ and $k_i = \lambda^2$.

1e (4%) Lyapunov function candidate:

$$\begin{aligned} V &= \frac{1}{2}\boldsymbol{\nu}_1^\top \mathbf{M}_1 \boldsymbol{\nu}_1 \\ \dot{V} &= \boldsymbol{\nu}_1^\top \mathbf{M}_1 \dot{\boldsymbol{\nu}}_1 \\ &= \boldsymbol{\nu}_1^\top \left[\boldsymbol{\tau}_1 - \mathbf{C}_1(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1\right] \\ &= -k_p \boldsymbol{\nu}_1^\top \boldsymbol{\nu}_1 < 0, \quad \forall \boldsymbol{\nu}_1 \neq \mathbf{0} \end{aligned} \tag{11}$$

Since $V > 0$ and $\dot{V} < 0$ it follows from Lyapunov's direct method that the equilibrium point $\boldsymbol{\nu}_1 = \mathbf{0}$ is globally exponentially stable.

1f (5%) The Euler angles are chosen to represent the attitude of the spacecraft. The kinematics and kinetics can be written as:

$$\dot{\boldsymbol{\Theta}} = \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})\boldsymbol{\nu}_2 \tag{12}$$

$$\mathbf{M}_2 \dot{\boldsymbol{\nu}}_2 = \boldsymbol{\tau}_2 - \mathbf{C}_2(\boldsymbol{\nu}_2)\boldsymbol{\nu}_2 \tag{13}$$

The solution is inspired by the control law in Assignment 1. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\boldsymbol{\nu}_2^\top \mathbf{M}_2 \boldsymbol{\nu}_2 + \frac{1}{2}\tilde{\boldsymbol{\Theta}}^\top \mathbf{K}_p \tilde{\boldsymbol{\Theta}} \tag{14}$$

The derivative of the Lyapunov function candidate is

$$\begin{aligned} \dot{V} &= \boldsymbol{\nu}_2^\top \mathbf{M}_2 \dot{\boldsymbol{\nu}}_2 + \tilde{\boldsymbol{\Theta}}^\top \mathbf{K}_p \dot{\tilde{\boldsymbol{\Theta}}} \\ &= \boldsymbol{\nu}_2^\top (\boldsymbol{\tau}_2 - \mathbf{C}_2(\boldsymbol{\nu}_2)\boldsymbol{\nu}_2) + \tilde{\boldsymbol{\Theta}}^\top \mathbf{K}_p (\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})\boldsymbol{\nu}_2) \\ &= \boldsymbol{\nu}_2^\top (\boldsymbol{\tau}_2 - \mathbf{C}_2(\boldsymbol{\nu}_2)\boldsymbol{\nu}_2) + \boldsymbol{\nu}_2^\top \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})^\top \mathbf{K}_p \tilde{\boldsymbol{\Theta}} \\ &= \boldsymbol{\nu}_2^\top (\boldsymbol{\tau}_2 - \mathbf{C}_2(\boldsymbol{\nu}_2)\boldsymbol{\nu}_2 + \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})^\top \mathbf{K}_p \tilde{\boldsymbol{\Theta}}) \\ &= \boldsymbol{\nu}_2^\top (\boldsymbol{\tau}_2 + \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})^\top \mathbf{K}_p \tilde{\boldsymbol{\Theta}}) \end{aligned} \tag{15}$$

where we have used that skew-symmetry implies that $\boldsymbol{\nu}_2^\top \mathbf{C}_2(\boldsymbol{\nu}_2)\boldsymbol{\nu}_2 = 0$. Moreover, the gain matrix $\mathbf{K}_p$ is designed to be symmetric and positive definite. Choosing the nonlinear PD control law

$$\boldsymbol{\tau} = -\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta})^\top \mathbf{K}_p \tilde{\boldsymbol{\Theta}} - \mathbf{K}_d \boldsymbol{\nu}_2 \tag{16}$$

yields

$$\dot{V} = -\boldsymbol{\nu}_2^\top \mathbf{K}_d \boldsymbol{\nu}_2 \tag{17}$$

where $\mathbf{K}_d$ is symmetric and positive definite. Local stability follows directly from Lyapunov's direct method. Moreover, local asymptotic stability can be shown with the Krasovskii-LaSalle's theorem. Global stability properties cannot be shown because of the singularity of the Euler angles, but since the singularity is a single point we have "near global stability". Global stability can be achieved by using quaternions or rotation matrices as the parametrization.

Problem 2: Ship Control by Successive Loop Closure (35%)

This follows the design procedures of Beard & Mclain (2012).

2a (4%) Closed-loop system

$$\dot{\psi} = r \tag{18}$$

$$T\dot{r} + (1 + KK_d)r + KK_p\,\psi = KK_p\,\psi^c + w \tag{19}$$

Hence,

$$H_{\psi/\psi^c}(s) = \frac{KK_p}{Ts^2 + (1 + KK_d)s + KK_p} = \frac{\frac{KK_p}{T}}{s^2 + \frac{1+KK_d}{T}s + \frac{KK_p}{T}} \tag{20}$$

$$H_{\psi/w}(s) = \frac{\frac{1}{T}}{s^2 + \frac{1+KK_d}{T}s + \frac{KK_p}{T}} \tag{21}$$

2b (8%) Open-loop transfer function

$$\frac{\psi}{\delta}(s) = \frac{K/T}{s^2 + (1/T)s} = \frac{0.001}{s^2 + 0.01s} \tag{22}$$

Hence, $a_0 = 0, a_1 = 0.01$ and $b_0 = 0.001$. The P gain follows from the saturating limits:

$$K_p = \frac{\delta_{\max}}{e_{\max}} = \frac{10^o}{1^o} = 10 \tag{23}$$

The natural frequency is:

$$\omega_{n_\psi} = \sqrt{a_0 + b_0 k_p} = \sqrt{0 + 0.001 \cdot 10} = 0.1 \text{ rad/s} \tag{24}$$

and

$$K_d = \frac{2\zeta_\psi \omega_{n_\psi} - a_1}{b_0} = \frac{2 \cdot 1 \cdot 0.1 - 0.01}{0.001} = 190 \tag{25}$$

2c (2%) The heading loop DC gain is:

$$K_{\psi_{DC}} = H_{\psi/\psi^c}(0) = \frac{\frac{KK_p}{T}}{0^2 + \frac{1+KK_d}{T}0 + \frac{KK_p}{T}} = 1.0 \tag{26}$$

2d (3%) The sway kinematic equation is:

$$\dot{y} = u\sin(\psi) + v\cos(\psi) \approx U\,\psi \tag{27}$$

for small ψ and v. Moreover, $U = \sqrt{u^2 + v^2} \approx u$.

2e (8%) The sway kinematics (neglecting roll and pitch) is:

$$\omega_{n_y} = \frac{1}{10}\,\omega_{n_\psi} = 0.01 \text{ rad/s} \tag{28}$$

The transfer function between y^c and y is:

$$y(s) = \frac{UK_{p_y}s + UK_{i_y}}{s^2 + UK_{p_y}s + UK_{i_y}}\, y^c(s) \tag{29}$$

The gains are:

$$K_{p_y} = \frac{2\zeta_y\omega_{n_y}}{U} = \frac{2\cdot 1\cdot 0.01}{10} = 0.002 \tag{30}$$

$$K_{i_y} = \frac{\omega_{n_y}^2}{U} = \frac{0.01^2}{10} = 0.00001 \tag{31}$$

2f (3%) Minimum sensor configuration:

- Gyro or magnetic compass
- Yaw rate sensor or gyro (IMU)
- GNSS position
- GNSS speed, DVL or pitot tube

2g (4%) The term w consists of:

- Drift forces (2nd-order waves, wind and current moment)
- Unmodeled yaw dynamics due to 6-DOF nonlinear coupling terms
- Unmodeled nonlinear rudder dynamics
- Parametric uncertainty (K and T)

The key assumption is that w is slowly varying such that $\dot{w} \approx 0$.

2h (3%) If the ship moves in another direction than North, the coordinate system must be aligned with the desired direction. This can be done by using vessel-parallel coordinates:

$$\boldsymbol{\eta}_p = \mathbf{R}_\psi^\top(\psi)\boldsymbol{\eta} \tag{32}$$

where $\boldsymbol{\eta}_p = [x_p, y_p, 0]^\top$ and $\boldsymbol{\eta} = [x, y, 0]^\top$. Hence, the cross-track error is computed as:

$$y_p = -\sin(\psi)x + \cos(\psi)y \tag{33}$$

This is based on the assumption that $\dot{\mathbf{R}}_\psi \approx \mathbf{0}$ and that roll and pitch angles are zero.

Problem 3: Nonlinear Control of Autonomous Rotorcraft (15%)

3a (6%) We start with

$$\begin{aligned} \mathbf{s} &= \dot{\tilde{\boldsymbol{\Theta}}} + \lambda\, \tilde{\boldsymbol{\Theta}} \\ &= \mathbf{T}_\Theta(\boldsymbol{\Theta})\boldsymbol{\omega} - (\dot{\boldsymbol{\Theta}}_d - \lambda\, \tilde{\boldsymbol{\Theta}}) \end{aligned} \tag{34}$$

Differentiating this expression with respect to time gives:

$$\dot{\mathbf{s}} = \mathbf{T}_\Theta(\boldsymbol{\Theta})\dot{\boldsymbol{\omega}} + \dot{\mathbf{T}}_\Theta(\boldsymbol{\Theta})\boldsymbol{\omega} - (\ddot{\boldsymbol{\Theta}}_d - \lambda\, \dot{\tilde{\boldsymbol{\Theta}}}) \tag{35}$$

Substituting the attitude dynamics for $\dot{\boldsymbol{\omega}}$ into this expression gives

$$\dot{\mathbf{s}} = \mathbf{T}_\Theta(\boldsymbol{\Theta})\mathbf{I}^{-1}(\boldsymbol{\tau} + \boldsymbol{\Delta}(\boldsymbol{\Theta},\boldsymbol{\tau})) + \dot{\mathbf{T}}_\Theta(\boldsymbol{\Theta})\boldsymbol{\omega} - (\ddot{\boldsymbol{\Theta}}_d - \lambda\, \dot{\tilde{\boldsymbol{\Theta}}}) \tag{36}$$

Choosing the nonlinear control law as

$$\boldsymbol{\tau} = \mathbf{I}\, \mathbf{T}_\Theta(\boldsymbol{\Theta})^{-1} \left(\ddot{\boldsymbol{\Theta}}_d - \lambda\, \dot{\tilde{\boldsymbol{\Theta}}} - \dot{\mathbf{T}}_\Theta(\boldsymbol{\Theta})\boldsymbol{\omega} - \mathbf{K}_d\, \mathbf{s} \right) \tag{37}$$

gives

$$\dot{\mathbf{s}} + \mathbf{K}_d\, \mathbf{s} = \mathbf{T}_\Theta(\boldsymbol{\Theta})\, \mathbf{I}^{-1}\, \boldsymbol{\Delta}(\boldsymbol{\Theta},\boldsymbol{\tau}) \tag{38}$$

3b (7%) The modified control is:

$$\boldsymbol{\tau} = \mathbf{I}\, \mathbf{T}_\Theta(\boldsymbol{\Theta})^{-1} \left(\ddot{\boldsymbol{\Theta}}_d - \lambda\, \dot{\tilde{\boldsymbol{\Theta}}} - \dot{\mathbf{T}}_\Theta(\boldsymbol{\Theta})\boldsymbol{\omega} - \mathbf{K}_d\, \mathbf{s} - k_s \mathrm{sgn}(\mathbf{s}) \right) \tag{39}$$

where $k_s > 0$ is a scalar to be decided. This gives the error dynamics:

$$\dot{\mathbf{s}} + \mathbf{K}_d\, \mathbf{s} + k_s \mathrm{sgn}(\mathbf{s}) = \mathbf{T}_\Theta(\boldsymbol{\Theta})\, \mathbf{I}^{-1}\, \boldsymbol{\Delta}(\boldsymbol{\Theta},\boldsymbol{\tau}) \tag{40}$$

Stability analysis:

$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^\top \mathbf{s} \tag{41}$$

$$\begin{aligned} \dot{V}(\mathbf{s}) &= \mathbf{s}^\top \dot{\mathbf{s}} \\ &= -\mathbf{s}^\top \mathbf{K}_d\, \mathbf{s} - k_s\, \|\mathbf{s}\| + \mathbf{s}^\top \mathbf{T}_\Theta(\boldsymbol{\Theta})\, \mathbf{I}^{-1}\, \boldsymbol{\Delta}(\boldsymbol{\Theta},\boldsymbol{\tau}) \end{aligned} \tag{42}$$

Choosing:

$$k_s > \|\mathbf{T}_\Theta(\boldsymbol{\Theta})\| \cdot \|\mathbf{I}^{-1}\| \cdot \delta \tag{43}$$

ensures that $\dot{V}(\mathbf{s}) < 0$.

3c (2%) The function $\mathrm{sgn}(\mathbf{s})$ can be replaced by a sigmoid function to avoid chattering. For instance $\mathrm{sat}(\mathbf{s})$ as defined in Fossen (2011) or $\tanh(\mathbf{s})$.

Problem 4: Ship Maneuvering (25%)

4a (2%) Body-fixed current velocity:

$$\mathbf{v}_c^b = \mathbf{R}_\psi(\psi)^\top \mathbf{v}_c^n \tag{44}$$

This gives

$$\mathbf{v}_c^b = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_c \cos(\beta_c) \\ V_c \sin(\beta_c) \\ 0 \end{bmatrix} \tag{45}$$

or alternatively

$$u_c^b = V_c \cos(\beta_c - \psi) \tag{46}$$

$$v_c^b = V_c \sin(\beta_c - \psi) \tag{47}$$

4b (5%) Since $\mathbf{v}_c^n = \textit{constant}$ it follows that

$$\dot{\mathbf{v}}_c^n = \dot{\mathbf{R}}_\psi(\psi)\mathbf{v}_c^b + \mathbf{R}_\psi(\psi)\dot{\mathbf{v}}_c^b = \mathbf{0} \tag{48}$$

Exploiting that $\dot{\mathbf{R}}_b^n = \mathbf{R}_b^n \mathbf{S}(\boldsymbol{\omega}_{b/n}^b)$, we get

$$\mathbf{R}_b^n \left(\mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\mathbf{v}_c^b + \dot{\mathbf{v}}_c^b \right) = \mathbf{0} \tag{49}$$

and

$$\dot{\mathbf{v}}_c^b = -\mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\, \mathbf{v}_c^b \tag{50}$$

4c (8%) Modified linear maneuvering model, which includes current velocities, generalized wind forces and generalized 2nd-order wave-induced forces:

$$\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} - \dot{v}_c^b \\ \dot{r} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v - v_c^b \\ r \\ \psi \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} \delta + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{waves}} \tag{51}$$

4d (5%) The maneuvering model including environmental disturbances is valid under the following assumptions:

- Linear model (kinematics and ship kinetics)
- Linear wave theory
- Linear superposition of environmental forces
- Neglecting surge, heave, roll and pitch
- Constant hydrodynamic coefficients (no fluid memory effects)
- Constant mass and inertia
- No parametric uncertainty

4e (5%) The current speed and direction can be simulated using the stochastic models:

$$\dot{V}_c = -\alpha_1 V_c + \text{white noise} \tag{52}$$

$$\dot{\beta}_c = -\alpha_2\, \beta_c + \text{white noise} \tag{53}$$

where $\alpha_i \geq 0$ $(i = 1, 2)$ are two constants. In addition, it is necessary to limit the speed and direction to the intervals $[0, V_c^{\max}]$ and $[-180^o, +180^o]$, respectively. Integration of the differential equations together with

$$\mathbf{v}_c^n = \begin{bmatrix} V_c \cos(\beta_c) \\ V_c \sin(\beta_c) \\ 0 \end{bmatrix} \tag{54}$$

gives a time-varying current simulator for $\mathbf{v}_c^n$.