# Problem 1: Ship Path-Following Control System (35%)

Figure 1: NTNU's research vessel, R/V Gunnerus,

Consider the kinematic equations:

$$
\begin{aligned}
\dot{N} &= u\cos(\psi)\cos(\theta) + v[\cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\phi)] \\
&\quad + w\,[\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi)\sin(\theta)] \\
\dot{E} &= u\sin(\psi)\cos(\theta) + v[\cos(\psi)\cos(\phi) + \sin(\phi)\sin(\theta)\sin(\psi)] \\
&\quad + w[\sin(\theta)\sin(\psi)\cos(\phi) - \cos(\psi)\sin(\phi)] \\
\dot{D} &= -u\sin(\theta) + v\cos(\theta)\sin(\phi) + w\cos(\theta)\cos(\phi) \\
\dot{\phi} &= p + q\sin(\phi)\tan(\theta) + r\cos(\phi)\tan(\theta) \\
\dot{\theta} &= q\cos(\phi) - r\sin(\phi) \\
\dot{\psi} &= q\frac{\sin(\phi)}{\cos(\theta)} + r\frac{\cos(\phi)}{\cos(\theta)}, \qquad \theta \neq \pm 90^o
\end{aligned}
$$

and Nomoto model:

$$T\dot{r} + r = K\delta \tag{1}$$

with $T = 22.0$ s and $K = 0.1$ s$^{-1}$.

**1a (2%)** The ship is moving at $U = 10$ m/s. What is the steady-state turning radius of the ship for $\delta = 10$ deg.?

Steady-state turning rate: $r = K\delta$
Turing radius: $R = \frac{U}{r} = \frac{U}{K\delta} = \frac{10}{0.1 \cdot 10\frac{\pi}{180}} \approx 573$ m

**1b (2%)** The control objective is to track a straight line at constant forward speed $U =$ constant. A 2-D Cartesian system is oriented such that the $x-$axis points Northwards, while the $y-$axis points Eastwards. Explain under which conditions:

$$\dot{\psi} = r \tag{2}$$
$$\dot{y}_e = U\psi \tag{3}$$

is a good approximation for the ship cross-track error $y_e$.

It is necessary to assume that $\phi$, $\theta$ and $\psi$ are small angles. In addition, $v = w = q = 0$ and $U = u$. This can be obtained by investigating the equations for $\dot{\psi}$ and $\dot{E}$.

**1c (5%)** Find the expressions for $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ and the state vector $\mathbf{x}$ in

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\,\delta \tag{4}$$

$$y_e = \mathbf{c}^\top \mathbf{x} \tag{5}$$

where the control objective is $y_e = 0$. Show how the linear optimal regulator $\delta$ can be computed and explain how you will choose the weighting matrices.

$$\mathbf{x} = [y_e, \psi, r]^\top$$

$$\mathbf{A} = \begin{bmatrix} 0 & U & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{T} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \frac{K}{T} \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The weighting matrix $\mathbf{R}$ is a scalar because we have a single input and we define it as $R = r_1$. The steady-state solution of this problem is

$$\delta = -\frac{1}{r_1}\mathbf{b}^\top \mathbf{P}_\infty \mathbf{x}$$

$$\mathbf{P}_\infty \mathbf{A} + \mathbf{A}^\top \mathbf{P}_\infty - \frac{1}{r_1}\mathbf{P}_\infty \mathbf{b}\mathbf{b}^\top \mathbf{P}_\infty + \mathbf{c}\mathbf{Q}\mathbf{c}^\top = \mathbf{0}$$

$$r_1 > 0, \qquad \mathbf{Q} = \text{diag}(q_1, q_2, q_3) > 0$$

The weighting matrices $\mathbf{R}$ and $\mathbf{Q}$ penalize the input and the deviation from the control objective, respectively. If you want to limit the use of the rudder input, you can choose a large value $r_1$ for example.

**1d (5%)** Modify the state-space model (expressions $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$) such that the resulting optimal control law includes integral action.

In order to include integral action, we augment the state $\mathbf{x}$ with an integral state:

$$\dot{z} = y_e$$

and define $\mathbf{x_a} = [z, \mathbf{x}^\top]^\top = [z, y_e, \psi, r]^\top$. The augmented system can be written as

$$\dot{\mathbf{x}_a} = \mathbf{A}_a \mathbf{x}_a + \mathbf{b}_a \delta$$

$$y_e = \mathbf{c}_a^\top \mathbf{x_a}$$

where

$$\mathbf{A}_a = \begin{bmatrix} 0 & \mathbf{c}^\top \\ \mathbf{0} & \mathbf{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-1}{T} \end{bmatrix}$$

$$\mathbf{b}_a = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K}{T} \end{bmatrix}, \qquad \mathbf{c_a} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

**1e (8%)** Assume that

$$\dot{y}_e = U \sin(\psi) \tag{6}$$

and choose the line-of-sight (LOS) guidance law according to

$$\psi_d = \tan^{-1}(-K_p y_e) \tag{7}$$

where $K_p > 0$. Assume that $\psi = \psi_d$ and find an expression for the function $f(y_e, U)$ such that:

$$\dot{y}_e = f(y_e, U) \tag{8}$$

$$\dot{y}_e = -U \frac{K_p y_e}{\sqrt{1 + K_p^2 y_e^2}}$$

What is the equilibrium point of (8)?

$\dot{y}_e = 0$ gives the equilibrium point $y_e = 0$

Linearize the cross-track error dynamics (8) about the equilibrium point.

$$\dot{y}_e = -U K_p y_e$$

Under what conditions is the equilibrium point of the linearized system exponentially stable?

$U > 0$ and $K_p > 0$

This can be proven by linear theory or by using a Lyapunov function and theorem A.3 from the book.

Hint: $\sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{1+x^2}}$.

**1f (5%)** Use pole-placement to design a PD-controller for the system (1) such that

$$\frac{\psi}{\psi_d}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{9}$$

Compute the numerical values for the PD gains $K_p$ and $K_d$ for $\zeta = 1.0$ such that the bandwidth of the closed-loop system is $0.3$ rad/s.

$$T\dot{r} + r = -KK_p(\psi - \psi_d) - KK_d(r - r_d)$$

Assuming $r_d = 0$ (set-point control), the closed loop system becomes

$$\frac{\psi}{\psi_d} = \frac{KK_p}{s^2 + \frac{1+KK_d}{T}s + \frac{KK_p}{T}}$$

Moreover, we specify $\zeta = 0.1$ and $\omega_b = 0.3$

$$2\zeta\omega_n = \frac{1 + KK_d}{T}, \qquad \omega_n^2 = \frac{KK_p}{T}$$

$$\omega_n = \frac{1}{0.64}\omega_b \approx 0.47$$

$$K_d = 196.8, \qquad K_p = 48.6$$

**1g (3%)** Draw a block diagram showing how the LOS guidance algorithm (Problem 1e) can be combined with the pole-placement algorithm (Problem 1f) to solve the path-following control problem of a ship moving on a straight line.

The block diagram can look something like Figure 2. The main concern here is to illustrate how the flow of the signals is so the blocks do not have to be in the same order.

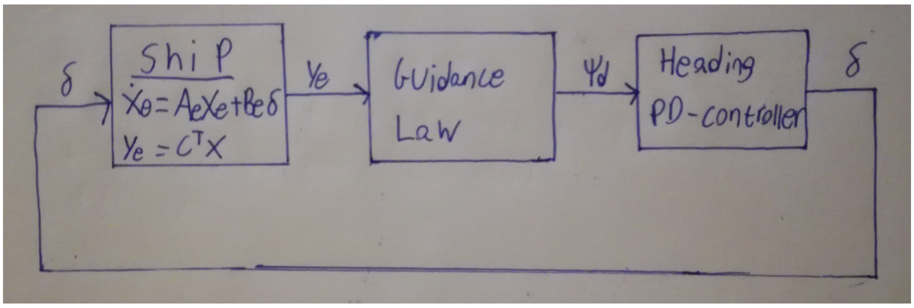


Figure 2: Problem 1g).

**1h (5%)** Explain how the path-following guidance system can be modified to track several straight-line segments. Include equations describing your approach and make a drawing (block diagram) showing how the equations can be used to implement the guidance and control systems.

To solve this type of problem you need to model the path as straight-line segments where waypoints are used to model the lines. In order to switch between different segments you can use the circle of acceptance. The next waypoint (k+1) is chosen if the ship is within a circle with radius R. This can be described mathematically as

$$[x_{k+1} - x(t)]^2 + [y_{k+1} - y(t)]^2 \leq R_{k+1}^2$$

The radius R can in general be chosen with a value of two times the ship length, but it depends on the maneuvering characteristics of the vehicle. The block diagram should extend Figure 2 to include the switching mechanism (a path manager utilizing the circle of acceptance) that decides which waypoint should be used in the guidance system. Crab angle compensation can also be included if current is present and you need a system with very accurate path following. The desired heading from the guidance system can be calculated as

$$\chi_d(y_e) = \chi_p + \chi_r(y_e)$$

where the lookahead guidance law is used. Note that the cross-track error $y_e$ is valid as long as the x-axis of the coordinate system (in which $y_e$ is defined) points in the direction of the path. In such a case, you can remove $\chi_p$ from $\chi_d$ as long as you define the yaw angle with respect to that x-axis (and not the North axis).

# Problem 2: UAV Altitude Control System (25%)

Figure 3: NTNU's Penguin UAV system.

Consider the following UAV model:

$$
\begin{aligned}
\dot{p}_n &= (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w \\
\dot{p}_e &= (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w \\
\dot{h} &= u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\
\dot{u} &= rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right] \\
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{w} &= qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right] \\
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= q\sin\phi\sec\theta + r\cos\phi\sec\theta \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{q} &= \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]
\end{aligned}
$$

**2a (4%)** Explain under which conditions

$$\dot{h} = u\theta - w \tag{10}$$

$$\dot{\theta} = q \tag{11}$$

is a good approximation for aircraft altitude.

It is necessary to assume that $\phi$ and $\theta$ are small angles so that $\sin(\phi) = \phi$ and $cos(\phi) = 1$ (the same for $\theta$ obviously). In addition, $v = 0$ and $r = 0$.

**2b (8%)** Assume that $\delta_t$ is chosen such that $V_a =$ constant. Furthermore, assume that $\dot{u} = 0$, and that the lateral motions and wind can be neglected. Design a backstepping controller for altitude control using elevator $\delta_e$ as control input. Use the kinematic equation (10) and explain why you choose $u$, $\theta$ or $w$ as virtual controller. The desired altitude $h_d =$ constant. It is not necessary to include integral action when you design the control law.

$-w = \alpha_1 + z_2$ is chosen as virtual controller since $\dot{w}$ contains the control input $\delta_e$ and $\dot{u} = 0$.

$$z_1 = h \tag{12}$$

$$\begin{aligned}\dot{z}_1 &= u\theta - w \\ &= u\theta + (\alpha_1 + z_2)\end{aligned}$$

Choosing

$$\alpha_1 = -k_1 z_1 - u\theta \tag{13}$$

gives

$$\dot{z}_1 = -k_1 z_1 + z_2 \tag{14}$$

Next,

$$\begin{aligned}\dot{z}_2 &= -\dot{\alpha}_1 - \dot{w} \\ &= -\dot{\alpha}_1 - f(\cdot) - b(\cdot)\delta_e\end{aligned} \tag{15}$$

where

$$f(\cdot) = qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2m}[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a}] \tag{16}$$

$$b(\cdot) = \frac{\rho V_a^2 S}{2m} C_{Z_{\delta_e}}(\alpha) \tag{17}$$

Choosing the Lyapunov function V and finding the derivative:

$$\begin{aligned}V &= 1/2(z_1^2 + z_2^2) \\ \dot{V} &= z_1(-k_1 z_1 + z_2) + z_2(-\dot{\alpha}_1 - f(\cdot) - b(\cdot)\delta_e) \\ &= -k_1 z_1^2 + z_2(z_1 - \dot{\alpha}_1 - f(\cdot) - b(\cdot)\delta_e)\end{aligned} \tag{18}$$

Finally, we choose

$$\delta_e = \frac{-1}{b(\cdot)}\left(-k_2 z_2 - z_1 + \dot{\alpha}_1 + f(\cdot)\right) \tag{19}$$

to obtain

$$\dot{V} = -k_1 z_1^2 - k_2 z_2^2 \tag{20}$$

and, therefore, the desired stability properties.

**2c (2%)** Find an expression for the time derivative of stabilizing function, $\dot{\alpha}_1$, which is only function of the states and not the time derivative of the states.

$$\begin{aligned}\dot{\alpha}_1 &= -k_1 \dot{z}_1 - \dot{u}\theta - u\dot{\theta} \\ &= -k_1 \dot{z}_1 - uq \\ &= -k_1(-k_1 z_1 + z_2) - uq\end{aligned} \tag{21}$$

since $\dot{u} = 0$. Alternatively, it can be written as

$$\begin{aligned} \dot{\alpha}_1 &= -k_1\dot{h} - uq \\ &= -k_1(u\theta - w) - uq \end{aligned} \tag{22}$$

**2d (6%)** What is the equilibrium point of the closed-loop system? Write the error dynamics in matrix-vector form and discuss if the equilibrium point is locally/globally asymptotically/exponentially stable by using Lyapunov stability theory.

The equilibrium point is $(z_1, z_2) = (0, 0)$ with the change of coordinates from $[x_1, x_2]^\top$ to $[z_1, z_2]^\top$. The equlibrium point is GES since:

$$\dot{\mathbf{z}} = -\mathbf{K}\mathbf{z} + \mathbf{S}\mathbf{z} \tag{23}$$

where

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{24}$$

Moreover,

$$V = \frac{1}{2}\mathbf{z}^\top\mathbf{z} \tag{25}$$

$$\dot{V} = \mathbf{z}^\top\dot{\mathbf{z}} = -\mathbf{z}^\top\mathbf{K}\mathbf{z} \tag{26}$$

and the derivative is obtained by utilizing that $\mathbf{z}^\top\mathbf{S}\mathbf{z} = 0$. The matrix $\mathbf{K}$ is positive definite by design and this means that $\dot{V}$ is negative definite. Therefore, the system can be proven GAS by the direct method and GES by theorem A.3 in the book (see also page 462-463 in the book).

**2e (3%)** What kind of navigation and sensor system do you need to implement the backstepping controller.

We need to measure the attitude, altitude, body-fixed linear velocities and the angular velocities. This can be achieved with INS with GNSS, IMU, altimeter (optionally). In addition, we need an airdata sensor for $V_a, \alpha, \beta$. In practice it may be desirable to estimate the states because the sensors are affected by bias and noise.

**2f (2%)** Is the backstepping controller robust? Explain why/why not.

No, in order to use integrator backstepping we need to know the system model accurately. Because this is difficult in practice, care must be taken when using this type of control law. This is also the case with feedback linearization.

# Problem 3: Estimation and Navigation (30%)

Consider the vehicle model:

$$\dot{\mathbf{p}}^n = \mathbf{R}_b^n(\Theta)\mathbf{v}^b \tag{27}$$

$$\mathbf{M}\dot{\mathbf{v}}^b + \mathbf{D}\mathbf{v}^b = \tau^b \tag{28}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{D}$ is the damping matrix and $\tau^b$ is the control input. Furthermore, $\mathbf{p}^n = [x, y, z]^\top$ and $\mathbf{v}^b = [u, v, w]^\top$. Assume that you measure the attitude vector $\boldsymbol{\Theta} = [\phi, \theta, \psi]^\top$ perfectly such that:

$$\mathbf{R}_b^n(\boldsymbol{\Theta}(t)) := \mathbf{R}(t) \tag{29}$$

**3a (2%)** The measurement equations for linear acceleration and position are:

$$\mathbf{z}_1 = \mathbf{p}^n + \mathbf{w}_1 \tag{30}$$
$$\mathbf{z}_2 = \mathbf{R}_n^b(\boldsymbol{\Theta})(\dot{\mathbf{v}}^n - \mathbf{g}^n) + \mathbf{w}_2 \tag{31}$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are Gaussian white noise and $\mathbf{g}^n = [0, 0, 9.81]^\top$. What kind of sensors/navigation systems can provide measurements $\mathbf{z}_1$ and $\mathbf{z}_2$ for:

- underwater vehicles
  Acoustic system, IMU.
- surface ships
  GNSS, radio navigation systems, IMU.

**3b (4%)** Show that $\mathbf{z}_2$ can be rewritten as

$$\mathbf{z}_2 = \dot{\mathbf{v}}^b + \mathbf{S}(t)\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2 \tag{32}$$

if $\omega_{b/n}^b$ is known and

$$\omega_{b/n}^b \times \mathbf{v}^b := \mathbf{S}(t)\mathbf{v}^b \tag{33}$$

$$\begin{aligned}
\mathbf{z}_2 &= \mathbf{R}_n^b(\boldsymbol{\Theta})\left(\mathbf{R}_n^b(\boldsymbol{\Theta})(\dot{\mathbf{v}}^b + \omega_{b/n}^b \times \mathbf{v}^b) - \mathbf{g}^n\right) + \mathbf{w}_2 \\
&= \dot{\mathbf{v}}^b + \omega_{b/n}^b \times \mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2 \\
&= \dot{\mathbf{v}}^b + \mathbf{S}(t)\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2
\end{aligned} \tag{34}$$

Alternatively:

$$\dot{\mathbf{v}}^n = \frac{d}{dt}(\mathbf{R}_b^n \mathbf{v}^b) = \dot{\mathbf{R}}_b^n \mathbf{v}^{\mathbf{b}} + \mathbf{R}_b^n \dot{\mathbf{v}}^b \tag{35}$$
$$= \mathbf{R}_b^n S(t)\mathbf{v}^b + \mathbf{R}_b^n \dot{\mathbf{v}}^b \tag{36}$$
$$\mathbf{z}_2 = \mathbf{R}_n^b(\Theta)(\mathbf{R}_b^n S(t)\mathbf{v}^b + \mathbf{R}_b^n \dot{\mathbf{v}}^b - \mathbf{g}^n) + \mathbf{w}_2 \tag{37}$$
$$= \dot{\mathbf{v}}^b + \mathbf{S}(t)\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2 \tag{38}$$

**3c (8%)** Find the expressions for $\mathbf{A}(t)$, $\mathbf{B}$, $\mathbf{C}(t)$ and $\mathbf{D}(t)$, the state vector $\mathbf{x}$ and $\mathbf{u}$ in

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{v} \tag{39}$$
$$\mathbf{z} = \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u} + \mathbf{w} \tag{40}$$

where the objective is to estimate $\mathbf{x}$ from the measurements $\mathbf{z}$ and $\mathbf{u}$. Explain how you will model the terms $\mathbf{E}$ and $\mathbf{v}$ if the goal is to estimate $\mathbf{x}$ using a linear time-varying (LTV) Kalman filter.

We add process noise $\mathbf{v}$ only for the kinetic equation (no noise for the kinematic equation). Hence,

$$\mathbf{E} = \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{I}_3 \end{bmatrix}$$

This is because the noise in the velocity will affect the position because the position directly depends on the velocity. The acceleration measurement equation is rewritten as

$$\begin{aligned}
\mathbf{z}_2 &= \dot{\mathbf{v}}^b + \mathbf{S}(t)\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2 \\
&= \mathbf{M}^{-1}(\boldsymbol{\tau}^b - \mathbf{D}\mathbf{v}^b) + \mathbf{S}(t)\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2 \\
&= \mathbf{M}^{-1}\boldsymbol{\tau}^b + (\mathbf{S}(t) - \mathbf{M}^{-1}\mathbf{D})\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{w}_2
\end{aligned}$$

We can see that the gravity vector needs to be a part of the input $\mathbf{u}$ to satisfy the desired state-space form. This gives the the state-space model

$$\mathbf{x} = [(\mathbf{p}^n)^\top, (\mathbf{v}^b)^\top]^\top, \qquad \mathbf{u} = [(\boldsymbol{\tau}^b)^\top, (\mathbf{g}^n)^\top]^\top$$

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{R}(t) \\ \mathbf{0}_3 & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}^{-1} & \mathbf{0}_3 \end{bmatrix}$$

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{S}(t) - \mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \qquad \mathbf{D}(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}^{-1} & -\mathbf{R}^\top(t) \end{bmatrix}$$

**3d (6%)** Explain how you will modify the measurement equation $\mathbf{z}_2$ and the state-space model under 3b) to estimate acceleration bias.

The measurement model with acceleration bias is

$$\begin{aligned}
\mathbf{z}_2 &= \mathbf{M}^{-1}\boldsymbol{\tau}^b + (\mathbf{S}(t) - \mathbf{M}^{-1}\mathbf{D})\mathbf{v}^b - \mathbf{R}(t)^\top \mathbf{g}^n + \mathbf{b} + \mathbf{w}_2 \\
\dot{\mathbf{b}} &= \mathbf{v}_2
\end{aligned}$$

where $\mathbf{b}$ is the bias. We augment the state space model with $\mathbf{b}$ so that the augmented state is defined as $\mathbf{x}_a = [(\mathbf{p}^n)^\top, (\mathbf{v}^b)^\top, \mathbf{b}^\top]^\top$. The state-space model becomes:

$$\mathbf{A}_a(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{R}(t) & \mathbf{0}_3 \\ \mathbf{0}_3 & -\mathbf{M}^{-1}\mathbf{D} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}, \qquad \mathbf{B}_a(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}^{-1} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}$$

$$\mathbf{C}_a(t) = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{S}(t) - \mathbf{M}^{-1}\mathbf{D} & \mathbf{I}_3 \end{bmatrix}, \qquad \mathbf{D}_a(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}^{-1} & -\mathbf{R}^\top(t) \end{bmatrix}$$

$$\mathbf{E}_a(t) = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \qquad \mathbf{v}_a = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$$

**3e (6%)** Assume that the model (27)–(28) is unknown. Show that:

$$\dot{\mathbf{p}}^n = \mathbf{v}^n \tag{41}$$
$$\dot{\mathbf{v}}^n = \mathbf{u}_a^n \tag{42}$$
$$\mathbf{z}_1 = \mathbf{p}^n + \mathbf{w}_1 \tag{43}$$

where

$$\mathbf{u}_a^n := \mathbf{R}(t)\mathbf{z}_2 + \mathbf{g}^n - \mathbf{R}(t)\mathbf{w}_2 \tag{44}$$

Propose a state-estimator for $\mathbf{p}^n$ and $\mathbf{v}^n$, which is globally exponentially stable.

We have that

$$\mathbf{z}_2 = \mathbf{R}_n^b(\boldsymbol{\Theta})(\dot{\mathbf{v}}^n - \mathbf{g}^n) + \mathbf{w}_2$$

This can be rewritten as

$$\mathbf{R}(t)^\top(\dot{\mathbf{v}}^n - \mathbf{g}^n) = \mathbf{z}_2 - \mathbf{w}_2$$

or

$$\dot{\mathbf{v}}^n = \mathbf{R}(t)\mathbf{z}_2 + \mathbf{g}^n - \mathbf{R}(t)\mathbf{w}_2$$

State estimator

$$\dot{\hat{\mathbf{p}}}^n = \hat{\mathbf{v}}^n + \mathbf{K}_1(\mathbf{z}_1 - \hat{\mathbf{p}}^n) \tag{45}$$
$$\dot{\hat{\mathbf{v}}}^n = \mathbf{R}(t)\mathbf{z}_2 + \mathbf{g}^n + \mathbf{K}_2(\mathbf{z}_1 - \hat{\mathbf{p}}^n) \tag{46}$$

**3f (4%)** What are the conceptual differences of the estimators in 3c and 3e. Also set up a list of advantages and disadvantages of these two approaches.

The first one (3c) requires the vehicle model (with uncertainty) and uses a Kalman filter which is a stochastic state estimator. The second one (3e) uses accelerometers directly in the motion model instead through an observer (deterministic approach). This is a classical INS, which is vehicle independent.

**Advantages and disadvantages with the Kalman filter approach**:

- The vehicle model can be used as an predictor if position measurements fail (dead reckoning system).
- The Kalman filter is well known and tested and used in several industrial applications.
- Works well for nonlinear systems in practice even though it is not optimal.
- Provides a measure of the accuracy of the estimates through the covariance matrix.
- Need to known the input from the actuators.
- Not optimal unless the system is linear.
- More computationally demanding (especially some of the matrix inversions which gets worse with the dimension of the state space).

**Advantages and disadvantages with the observer approach**:

- Vehicle independent.
- Computationally effective.
- Can prove stability of the estimates through Lyapunov theory. Global stability properties can often be proved.
- Not used very much in industrial applications in comparison with the Kalman filter.
- No measure of the uncertainty of the estimates so the estimates might diverge if you are not careful.
- Perhaps more dependent on the accuracy of the initial values for the estimates.

## Problem 4: Multiple-Choice Problems (10%)

The YES and NO questions below give you 2 points for correct answer, -1 point for wrong answer and 0 points for no answer. Please answer only YES or NO, alternately no answer at all.

**4a (2%)** The roll and pitch periods of a ship depends on the sea state and load condition.

YES

**4b (2%)** It is possible to use rudders in dynamic positioning systems even though lift and drag are zero for zero water speed.

YES

**4c (2%)** The Nomoto model

$$h(s) = \frac{K}{s(Ts+1)} + d(s) \tag{47}$$

of a ship where $d(s)$ is a constant wind disturbance has an open-loop integrator so it is not necessary to include integral action in the control law in order to avoid steady-state errors.

NO

**4d (2%)** The added mass matrix is independent of the location of the coordinate system.

NO

**4e (2%)** Coriolis forces can destabilize ships and underwater vehicles.

YES