Chapter 3 – Rigid-Body Kinetics

3.1 Newton–Euler Equations of Motion about CG
3.2 Newton–Euler Equations of Motion about CO
3.3 Rigid-Body Equations of Motion

In order to derive the marine craft equations of motion, it is necessary to study of the motion of rigid bodies, hydrodynamics and hydrostatics.

The overall goal of Chapter 3 is to show that the rigid-body kinetics can be expressed in a vectorial setting according to:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\mathbf{M}_{RB}$ Rigid-body mass matrix
$\mathbf{C}_{RB}$ Rigid-body Coriolis and centripetal matrix due to the rotation of {b} about {n}
$\boldsymbol{\nu} = [u,v,w,p,q,r]^T$ generalized velocity expressed in {b}
$\boldsymbol{\tau}_{RB} = [X,Y,Z,K,M,N]^T$ generalized force expressed in {b}

Chapter 3 – Rigid-Body Kinetics

The equations of motion will be represented in two body-fixed reference points:

1) Center of gravity (CG), subscript *g*
2) Origin CO of {b}, subscript *b*

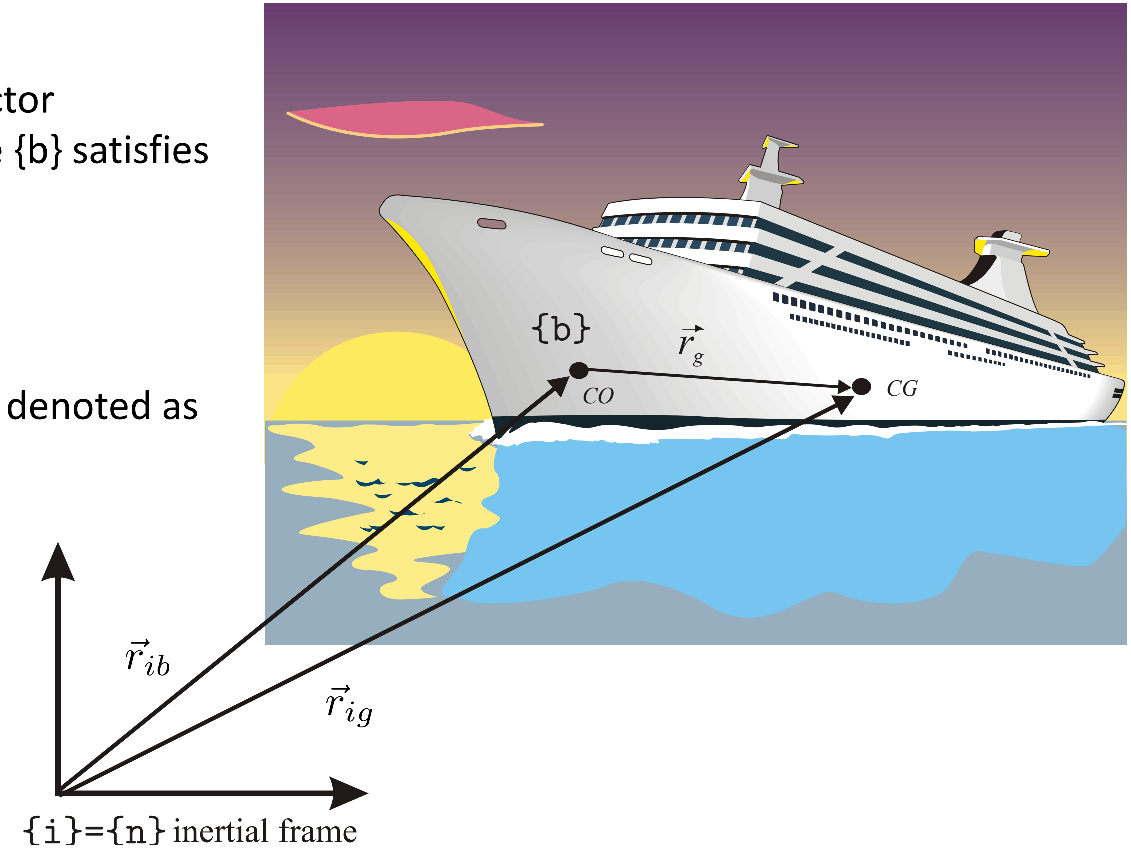
These points coincides if the vector $\vec{r}_g = \vec{0}$

Time differentiation of a vector
in a moving reference frame {b} satisfies

$$\frac{{}^{i}\mathrm{d}}{\mathrm{d}t}\vec{a} = \frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{a} + \vec{\omega}_{ib} \times \vec{a}$$

Time differentiation in {b} is denoted as

$$\dot{\vec{a}} := \frac{{}^{b}d}{dt}\vec{a}$$



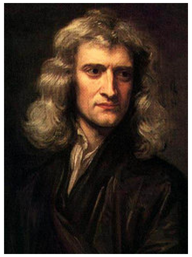
3.1 Newton-Euler Equations of Motion about CG

Coordinate-free vector: A vector $\vec{v}_{nb}$, velocity of {b} with respect to {n}, is defined by its magnitude and direction but without reference to a coordinate frame.
Coordinate vector: A vector $\vec{v}_{nb}$ decomposed in the inertial reference frame is denoted by

Newton-Euler Formulation

Newton's Second Law relates mass *m*, acceleration $\dot{\vec{v}}_{ig}$ and force $\vec{f}_g$ according to:

$$\boxed{m\dot{\vec{v}}_{ig} = \vec{f}_g}$$



Isaac Newton (1642-1726)

where the subscript *g* denotes the center of gravity (CG).

Euler's First and Second Axioms

Euler suggested to express Newton's Second Law in terms of conservation of both linear momentum $\vec{p}_g$ and angular momentum $\vec{h}_g$ according to:

Leonhard Euler (1707-1783)

$$\boxed{\begin{aligned} \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p}_g &= \vec{f}_g \qquad & \vec{p}_g &= m\vec{v}_{ig} \\ \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{h}_g &= \vec{m}_g \qquad & \vec{h}_g &= I_g\vec{\omega}_{ig} \end{aligned}}$$

$\vec{f}_g$ and $\vec{m}_g$ are forces/moments about CG
$\vec{\omega}_{ib}$ is the angular velocity of frame *b* relative frame *i*
I_g is the inertia dyadic about the body's CG

3.1 Translational Motion about CG

When deriving the equations of motion it will be assumed that:

(1) The vessel is rigid
(2) The NED frame is inertial—that is, $\{n\} \approx \{i\}$

The first assumption eliminates the consideration of forces acting between individual elements of mass while the second eliminates forces due to the Earth's motion relative to a star-fixed inertial reference system such that:

$$\vec{v}_{ig} \approx \vec{v}_{ng}$$
$$\vec{\omega}_{ig} = \vec{\omega}_{ib} \approx \vec{\omega}_{nb}$$

For guidance and navigation applications in space it is usual to use a star-fixed reference frame or a reference frame rotating with the Earth. Marine crafts are, on the other hand, usually related to the NED reference frame. This is a good assumption since forces on a marine craft due to the Earth's rotation:

$$\omega_{ie} = 7.2921 \times 10^{-5} \text{ rad/s}$$

are quite small compared to the hydrodynamic forces.

3.1 Translational Motion about CG

{n} is inertial

$$\vec{r}_{ng} = \vec{r}_{nb} + \vec{r}_g \quad \longleftarrow \quad \vec{r}_{ig} = \vec{r}_{ib} + \vec{r}_g$$

Time differentiation of $\vec{r}_{ng}$ in a moving reference frame {b} gives

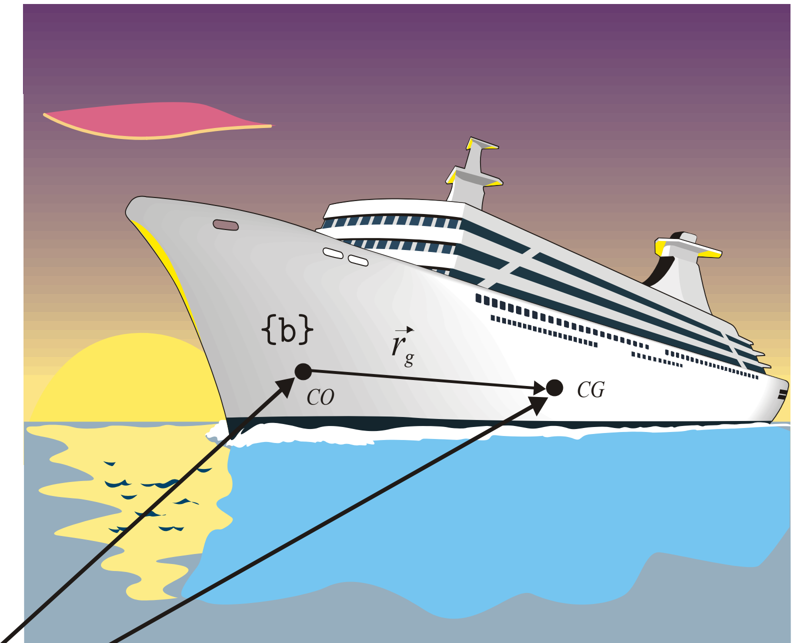
$$\vec{v}_{ng} = \vec{v}_{nb} + \left(\frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r}_g + \vec{\omega}_{nb} \times \vec{r}_g \right)$$

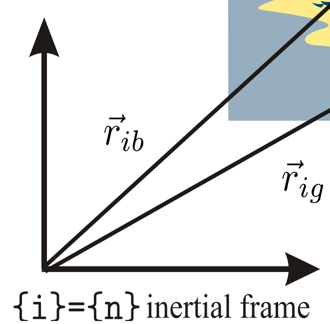
For a rigid body, CG satisfies

$$\frac{{}^b d}{dt}\vec{r}_g = \vec{0}$$

$$\vec{v}_{ng} = \vec{v}_{nb} + \vec{\omega}_{nb} \times \vec{r}_g$$

$$\begin{aligned} \vec{f}_g &= \frac{{}^i\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{ig}) \\ &= \frac{{}^i\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{ng}) \\ &= \frac{{}^b\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{ng}) + m\vec{\omega}_{nb} \times \vec{v}_{ng} \\ &= m(\dot{\vec{v}}_{ng} + \vec{\omega}_{nb} \times \vec{v}_{ng}) \end{aligned}$$





Body-fixed reference frame *{b}* is fixed in the point *CO* and rotating with respect to the inertial frame {i}

Translational Motion about CG Expressed in {b}

$$m[\dot{\boldsymbol{v}}^b_{ng} + \boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{v}^b_{ng}] = \boldsymbol{f}^b_g$$

3.1 Rotational Motion about CG

The derivation starts with the Euler's 2nd axiom:

$$
\begin{aligned}
\vec{m}_g &= \frac{^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{ib}) \\
&= \frac{^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{nb}) \\
&= \frac{^b\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{nb}) + \vec{\omega}_{nb} \times (I_g\vec{\omega}_{nb}) \\
&= I_g\dot{\vec{\omega}}_{nb} - (I_g\vec{\omega}_{nb}) \times \vec{\omega}_{nb}
\end{aligned}
$$

Rotational Motion about CG Expressed in {b}

$$\boldsymbol{I}_g^b\dot{\boldsymbol{\omega}}_{nb}^b - \boldsymbol{S}(\boldsymbol{I}_g^b\boldsymbol{\omega}_{nb}^b)\,\boldsymbol{\omega}_{nb}^b = \boldsymbol{m}_g^b$$

where $\boldsymbol{I}_g^b$ is the *inertia dyadic*

where I_x, I_y, and I_z are the *moments of inertia* about {b} and $I_{xy}=I_{yx}$, $I_{xz}=I_{zx}$ and $I_{yz}=I_{zy}$ are the *products of inertia* defined as:

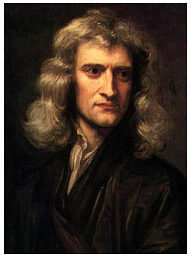
$$\boldsymbol{I}_g^b := \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}, \qquad \boldsymbol{I}_g^b = (\boldsymbol{I}_g^b)^\top > 0$$

$$
\begin{aligned}
I_x &= \int_V (y^2+z^2)\ \rho_m dV; & I_{xy} &= \int_V xy\ \rho_m dV = \int_V yx\ \rho_m dV = I_{yx} \\
I_y &= \int_V (x^2+z^2)\ \rho_m dV; & I_{xz} &= \int_V xz\ \rho_m dV = \int_V zx\ \rho_m dV = I_{zx} \\
I_z &= \int_V (x^2+y^2)\ \rho_m dV; & I_{yz} &= \int_V yz\ \rho_m dV = \int_V zy\ \rho_m dV = I_{zy}
\end{aligned}
$$

3.1 Equations of Motion about CG

The Newton-Euler equations can be represented in matrix form according to:

$$\boldsymbol{M}_{RB}^{CG} \begin{bmatrix} \dot{\boldsymbol{v}}_{ng}^{b} \\ \dot{\boldsymbol{\omega}}_{nb}^{b} \end{bmatrix} + \boldsymbol{C}_{RB}^{CG} \begin{bmatrix} \boldsymbol{v}_{ng}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{g}^{b} \\ \boldsymbol{m}_{g}^{b} \end{bmatrix}$$



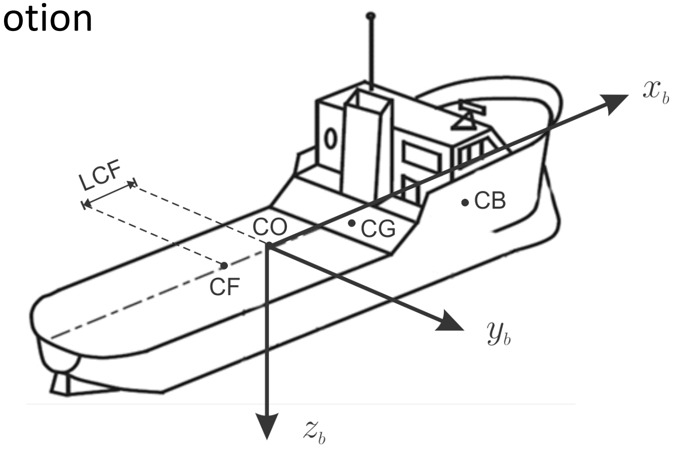
Isaac Newton (1642-1726)
Leonhard Euler (1707-1783)

Expanding the matrices give:

$$\underbrace{\begin{bmatrix} m\boldsymbol{I}_3 & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_g^b \end{bmatrix}}_{\boldsymbol{M}_{RB}^{CG}} \begin{bmatrix} \dot{\boldsymbol{v}}_{ng}^{b} \\ \dot{\boldsymbol{\omega}}_{nb}^{b} \end{bmatrix} + \underbrace{\begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & -\boldsymbol{S}(\boldsymbol{I}_g^b\boldsymbol{\omega}_{nb}^b) \end{bmatrix}}_{\boldsymbol{C}_{RB}^{CG}} \begin{bmatrix} \boldsymbol{v}_{ng}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{g}^{b} \\ \boldsymbol{m}_{g}^{b} \end{bmatrix}$$

3.2 Newton-Euler Equations of Motion about CO

For marine craft it is desirable to derive the equations of motion for an arbitrary origin CO to take advantage of the craft's geometric properties. Since the hydrodynamic forces and moments often are computed in CO, Newton's laws will be formulated in CO as well.



Transform the equations of motion from CG to CO using a coordinate transformation based on:

$$\begin{aligned} \boldsymbol{v}_{ng}^b &= \boldsymbol{v}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{r}_g^b \\ &= \boldsymbol{v}_{nb}^b - \boldsymbol{r}_g^b \times \boldsymbol{\omega}_{nb}^b \\ &= \boldsymbol{v}_{nb}^b + \boldsymbol{S}^\top(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{nb}^b \end{aligned}$$

$$\boxed{\begin{bmatrix} \boldsymbol{v}_{ng}^b \\ \boldsymbol{\omega}_{nb}^b \end{bmatrix} = \boldsymbol{H}(\boldsymbol{r}_g^b) \begin{bmatrix} \boldsymbol{v}_{nb}^b \\ \boldsymbol{\omega}_{nb}^b \end{bmatrix}}$$

Transformation matrix:

$$\boldsymbol{H}(\boldsymbol{r}_g^b) := \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{S}^\top(\boldsymbol{r}_g^b) \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_3 \end{bmatrix}, \qquad \boldsymbol{H}^\top(\boldsymbol{r}_g^b) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times3} \\ \boldsymbol{S}(\boldsymbol{r}_g^b) & \boldsymbol{I}_3 \end{bmatrix}$$

3.2 Newton-Euler Equations of Motion about CO

Newton-Euler equations in matrix-vector form (about CG)

$$\boldsymbol{M}_{RB}^{CG}\begin{bmatrix}\dot{\boldsymbol{v}}_{ng}^{b}\\ \dot{\boldsymbol{\omega}}_{nb}^{b}\end{bmatrix}+\boldsymbol{C}_{RB}^{CG}\begin{bmatrix}\boldsymbol{v}_{ng}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\begin{bmatrix}\boldsymbol{f}_{g}^{b}\\ \boldsymbol{m}_{g}^{b}\end{bmatrix}$$

$$\boxed{\begin{bmatrix}\boldsymbol{v}_{ng}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\boldsymbol{H}(\boldsymbol{r}_{g}^{b})\begin{bmatrix}\boldsymbol{v}_{nb}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}}$$

See App. C for more details

Newton-Euler equations in matrix-vector form (about CO)

$$\boldsymbol{H}^{\top}(\boldsymbol{r}_{g}^{b})\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{g}^{b})\begin{bmatrix}\dot{\boldsymbol{v}}_{nb}^{b}\\ \dot{\boldsymbol{\omega}}_{nb}^{b}\end{bmatrix}+\boldsymbol{H}^{\top}(\boldsymbol{r}_{g}^{b})\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{g}^{b})\begin{bmatrix}\boldsymbol{v}_{nb}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\boldsymbol{H}^{\top}(\boldsymbol{r}_{g}^{b})\begin{bmatrix}\boldsymbol{f}_{g}^{b}\\ \boldsymbol{m}_{g}^{b}\end{bmatrix}$$

$$\boldsymbol{M}_{RB}:=\boldsymbol{H}^{\top}(\boldsymbol{r}_{g}^{b})\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{g}^{b})$$
$$\boldsymbol{C}_{RB}:=\boldsymbol{H}^{\top}(\boldsymbol{r}_{g}^{b})\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{g}^{b})$$

Expanding the matrices

$$\boldsymbol{M}_{RB}=\begin{bmatrix}m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_{g}^{b})\\ m\boldsymbol{S}(\boldsymbol{r}_{g}^{b}) & \boldsymbol{I}_{g}^{b}-m\boldsymbol{S}^{2}(\boldsymbol{r}_{g}^{b})\end{bmatrix}$$
$$\boldsymbol{C}_{RB}=\begin{bmatrix}m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b}) & -m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b})\boldsymbol{S}(\boldsymbol{r}_{g}^{b})\\ m\boldsymbol{S}(\boldsymbol{r}_{g}^{b})\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b}) & -\boldsymbol{S}\left(\left(\boldsymbol{I}_{g}^{b}-m\boldsymbol{S}^{2}(\boldsymbol{r}_{g}^{b})\right)\boldsymbol{\omega}_{nb}^{b}\right)\end{bmatrix}$$

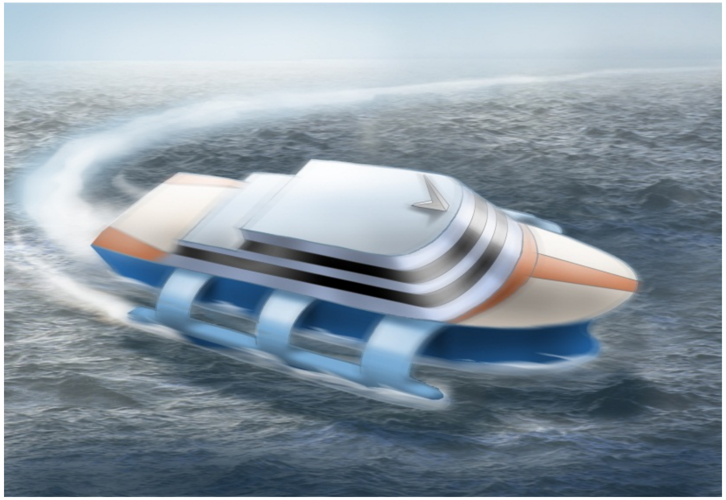
3.2 Translational Motion about CO

Translational Motion about CO Expressed in {b}

$$m[\dot{\boldsymbol{v}}^b_{nb} + \boldsymbol{S}(\dot{\boldsymbol{\omega}}^b_{nb})\boldsymbol{r}^b_g + \boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{v}^b_{nb} + \boldsymbol{S}^2(\boldsymbol{\omega}^b_{nb})\boldsymbol{r}^b_g] = \boldsymbol{f}^b_b$$

An alternative representation using vector cross products is:

$$m[\dot{\boldsymbol{v}}^b_{nb} + \dot{\boldsymbol{\omega}}^b_{nb} \times \boldsymbol{r}^b_g + \boldsymbol{\omega}^b_{nb} \times \boldsymbol{v}^b_{nb} + \boldsymbol{\omega}^b_{nb} \times (\boldsymbol{\omega}^b_{nb} \times \boldsymbol{r}^b_g)] = \boldsymbol{f}^b_b$$



3.2 Rotational Motion about CO

Rotational Motion about CO Expressed in {b}

$$\boldsymbol{I}_b^b \dot{\boldsymbol{\omega}}_{nb}^b + \boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{I}_b^b \boldsymbol{\omega}_{nb}^b + m\boldsymbol{S}(\boldsymbol{r}_g^b)\dot{\boldsymbol{v}}_{nb}^b + m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_{nb}^b = \boldsymbol{m}_b^b$$

An alternative representation using vector cross products is:

$$\boldsymbol{I}_b^b \dot{\boldsymbol{\omega}}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{I}_b^b \boldsymbol{\omega}_{nb}^b + m\boldsymbol{r}_g^b \times (\dot{\boldsymbol{v}}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{v}_{nb}^b) = \boldsymbol{m}_b^b$$

Theorem 3.1 (Parallel Axis or Huygens-Steiner Theorem)

The inertia dyadic $\boldsymbol{I}_b^b$ about an arbitrary origin o_b is given by:

$$\boldsymbol{I}_b^b = \boldsymbol{I}_g^b - m\boldsymbol{S}^2(\boldsymbol{r}_g^b) = \boldsymbol{I}_g^b + m\left((\boldsymbol{r}_g^b)^\top \boldsymbol{r}_g^b \boldsymbol{I}_3 - \boldsymbol{r}_g^b(\boldsymbol{r}_g^b)^\top\right)$$

where $\boldsymbol{I}_g^b$ is the inertia dyadic about the body's center of gravity.

Christian Huygens (1629-1695)
Jakob Steiner (1796-1863)

3.3 Rigid-Body Equations of Motion

$$
\begin{array}{lll}
\boldsymbol{f}_b^b & = [X, Y, Z]^\top & \text{force through } o_b \text{ expressed in } \{b\} \\
\boldsymbol{m}_b^b & = [K, M, N]^\top & \text{moment about } o_b \text{ expressed in } \{b\} \\
\boldsymbol{v}_{nb}^b & = [u, v, w]^\top & \text{linear velocity of } o_b \text{ relative } o_n \text{ expressed in } \{b\} \\
\boldsymbol{\omega}_{nb}^b & = [p, q, r]^\top & \text{angular velocity of } \{b\} \text{ relative to } \{n\} \text{ expressed in } \{b\} \\
\boldsymbol{r}_g^b & = [x_g, y_g, z_g]^\top & \text{vector from } o_b \text{ to CG expressed in } \{b\}
\end{array}
$$

Component form (SNAME 1950)

$$
\begin{aligned}
m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] &= X \\
m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] &= Y \\
m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] &= Z \\
I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} & \\
+m\left[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)\right] &= K \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{xy} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} & \\
+m\left[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)\right] &= M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{yz} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} & \\
+m\left[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)\right] &= N
\end{aligned}
$$

3.3 Rigid-Body Equations of Motion

Matrix-Vector Form (Fossen 1991)

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ generalized velocity

$\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^\top$ generalized force

Property 3.1 (Rigid-Body System Inertia Matrix)

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b) & \boldsymbol{I}_b^b \end{bmatrix}$$
$$= \begin{bmatrix} m & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & my_g & -mx_g & 0 \\ 0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\ mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\ -my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

$$\mathbf{M}_{RB} = \mathbf{M}_{RB}^\top > 0, \qquad \dot{\mathbf{M}}_{RB} = \mathbf{0}_{6\times 6}$$

3.3 Rigid-Body Equations of Motion

Matlab:
The rigid-body system inertia matrix M_{RB} can be computed in Matlab as

```
r_g = [10 0 1]';          % location of CG with respect to CO
R44 = 10;                 % radius of gyration in roll
R55 = 20;                 % radius of gyration in pitch
R66 = 5;                  % radius of gyration in yaw
m = 1000;                 % mass
I_g = m * diag([R44^2 R55^2 R66^2]); % inertia dyadic in CG

% rigid-body system inertia matrix
S = Smtrx(r_g);
MRB = [ m * eye(3)   -m * S
        m * S        I_g - m * S^2 ]

MRB =
      1000         0         0         0      1000         0
         0      1000         0     -1000         0     10000
         0         0      1000         0    -10000         0
         0     -1000         0    101000         0    -10000
      1000         0    -10000         0    501000         0
         0     10000         0    -10000         0    125000
```

The rigid-body system inertia matrix can also be computed using the command

```
MRB = rbody(m,R44,R55,R66,zeros(3,1),r_g)
```

3.3 Rigid-Body Equations of Motion

Theorem 3.2 (Coriolis-Centripetal Matrix from System Inertia Matrix)

Let **M** be a 6×6 *system inertia matrix* defined as:

$$\mathbf{M} = \mathbf{M}^\top = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} > 0$$

where $\mathbf{M}_{21} = \mathbf{M}_{12}^\top$. Then the *Coriolis-centripetal matrix* can always be parameterized such that

$$\mathbf{C}(\boldsymbol{\nu}) = -\mathbf{C}^\top(\boldsymbol{\nu})$$

$$\mathbf{C}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -\mathbf{S}(\mathbf{M}_{11}\boldsymbol{\nu}_1 + \mathbf{M}_{12}\boldsymbol{\nu}_2) \\ -\mathbf{S}(\mathbf{M}_{11}\boldsymbol{\nu}_1 + \mathbf{M}_{12}\boldsymbol{\nu}_2) & -\mathbf{S}(\mathbf{M}_{21}\boldsymbol{\nu}_1 + \mathbf{M}_{22}\boldsymbol{\nu}_2) \end{bmatrix}$$

where $\boldsymbol{\nu}_1 = [u, v, w]^\top$, $\boldsymbol{\nu}_2 = [p, q, r]^\top$

Proof: Sagatun and Fossen (1991).

3.3 Rigid-Body Equations of Motion

Property 3.2 (Rigid-Body Coriolis and Centripetal Matrix)

The *rigid-body Coriolis and centripetal matrix* $\mathbf{C}_{RB}(\boldsymbol{\nu})$ can always be represented such that $\mathbf{C}_{RB}(\boldsymbol{\nu})$ is skew-symmetric, that is

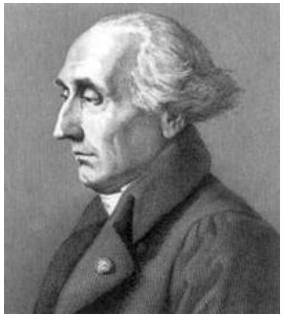
$$\mathbf{C}_{RB}(\boldsymbol{\nu}) = -\mathbf{C}_{RB}^{\top}(\boldsymbol{\nu}), \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^{6}$$

The skew-symmetric property is very useful when designing nonlinear motion control system since the quadratic form $\boldsymbol{\nu}^{\top}\mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv 0$.

This is exploited in energy-based designs where Lyapunov functions play a key role. The same property is also used in nonlinear observer design.

There exists several parameterizations that satisfies Property 3.2. Two of them are presented on the forthcoming pages:

3.3 Rigid-Body Equations of Motion



Joseph-Louis Lagrange (1736-1813)

Lagrangian Parameterization

Application of the Theorem 3.2 with $\boldsymbol{M} = \boldsymbol{M}_{RB}$ yields the following expression

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_g^b) - \boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

which can be rewritten according to

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) + m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

To ensure that $\mathbf{C}_{RB}(\mathbf{v}) = -\mathbf{C}_{RB}(\mathbf{v})^T$, it is necessary to use $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_1 = 0$ and add $\mathbf{S}(\mathbf{v}_1)$ in $C_{RB}^{\{21\}}$

NTNU

3.3 Rigid-Body Equations of Motion

Lagrangian Parameterization

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_g^b) - \boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

Component form

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -m(y_g q + z_g r) & m(y_g p + w) & m(z_g p - v) \\ m(x_g q - w) & -m(z_g r + x_g p) & m(z_g q + u) \\ m(x_g r + v) & m(y_g r - u) & -m(x_g p + y_g q) \end{matrix}\right.$$

$$\left.\begin{matrix} m(y_g q + z_g r) & -m(x_g q - w) & -m(x_g r + v) \\ -m(y_g p + w) & m(z_g r + x_g p) & -m(y_g r - u) \\ -m(z_g p - v) & -m(z_g q + u) & m(x_g p + y_g q) \\ 0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\ I_{yz} q + I_{xz} p - I_z r & 0 & -I_{xz} r - I_{xy} q + I_x p \\ -I_{yz} r - I_{xy} p + I_y q & I_{xz} r + I_{xy} q - I_x p & 0 \end{matrix}\right]$$

3.3 Rigid-Body Equations of Motion

Linear Velocity-Independent Parameterization

By using the cross-product property $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2 = -\mathbf{S}(\mathbf{v}_2)\mathbf{v}_1$, it is possible to move $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2$ from $C_{RB}^{\{12\}}$ to $C_{RB}^{\{11\}}$. This gives an expression for $\mathbf{C}_{RB}(\mathbf{v})$ that is independent of linear velocity $\mathbf{v}_1$ (Fossen and Fjellstad 1995):

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

Remark 1: This expression is useful when ocean currents enter the equations of motion. The main reason for this is that $\mathbf{C}_{RB}(\mathbf{v})$ does not depend on linear velocity $\mathbf{v}_1$. This can be further exploited when considering a marine craft exposed to irrotational ocean currents. According to Property 8.1 in Section 8.3:

$$\mathbf{M}_{RB}\dot{\mathbf{v}} + \mathbf{C}_{RB}(\mathbf{v})\mathbf{v} \equiv \mathbf{M}_{RB}\dot{\mathbf{v}}_r + \mathbf{C}_{RB}(\mathbf{v}_r)\mathbf{v}_r$$

if the relative velocity vector $\mathbf{v}_r = \mathbf{v} - \mathbf{v}_c$ is defined such that only linear ocean current velocities are used:

$$\mathbf{v} := [u_c, v_c, w_c, 0, 0, 0]^\top$$

3.3 Rigid-Body Equations of Motion

Linear Velocity-Independent Parameterization

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

This formula can also be expressed in terms of the $\boldsymbol{C}_{\mathrm{RB}}$ matrix in CG, and the transformation matrix from CG to CO

$$\boxed{\begin{bmatrix} \boldsymbol{v}_{ng}^b \\ \boldsymbol{\omega}_{nb}^b \end{bmatrix} = \boldsymbol{H}(\boldsymbol{r}_g^b) \begin{bmatrix} \boldsymbol{v}_{nb}^b \\ \boldsymbol{\omega}_{nb}^b \end{bmatrix}}$$

See App. C for more details

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) = \boldsymbol{H}^{\top}(\boldsymbol{r}_g) \begin{bmatrix} 0 & -mr & mq & 0 & 0 & 0 \\ mr & 0 & -mp & 0 & 0 & 0 \\ -mq & mp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_z r & -I_y q \\ 0 & 0 & 0 & -I_z r & 0 & I_x p \\ 0 & 0 & 0 & I_y q & -I_x p & 0 \end{bmatrix} \boldsymbol{H}(\boldsymbol{r}_g)$$

Matlab:
The Lagrangian parametrization (Theorem 3.2) is implemented in the Matlab MSS toolbox in the function `m2c.m`, while the linear-velocity independent parametrization (3.60) is implemented in the more generic functions `rbody.m`. The following example demonstrates how $C_{RB}(\boldsymbol{\nu})$ can be computed numerically

```
r_g = [10 0 1]';          % location of CG with respect to CO
R44 = 10;                 % radius of gyration in roll
R55 = 20;                 % radius of gyration in pitch
R66 = 5;                  % radius of gyration in yaw
m = 1000;                 % mass
nu = [8 0.5 0.1 0.2 -0.3 0.2]'; % velocity vector

% Method 1: Linear velocity-independent parametrization
nu2 = nu(4:6);
[MRB, CRB] = rbody(m,R44,R55,R66,nu2,r_g)

MRB =
      1000        0        0        0     1000        0
         0     1000        0    -1000        0    10000
         0        0     1000        0   -10000        0
         0    -1000        0   101000        0   -10000
      1000        0   -10000        0   501000        0
         0    10000        0   -10000        0   125000
CRB =
         0     -200     -300      200     3000    -2000
       200        0     -200        0     2200        0
       300      200        0     -200      300     2000
      -200        0      200        0     2800   120000
     -3000    -2200     -300    -2800        0    -2000
      2000        0    -2000  -120000     2000        0

% Method 2: Lagrangian parametrization
CRB = m2c(MRB,nu)

CRB =
        0        0        0        0     3100    -2300
        0        0        0    -3100        0     7700
        0        0        0     2300    -7700        0
        0     3100    -2300        0    28000   143300
    -3100        0     7700   -28000        0    17700
     2300    -7700        0  -143300   -17700        0
```

Even though the numerical values for the two $C_{RB}(\boldsymbol{\nu})$ matrices are different, they both produce the same product $C_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}$.

3.3 Linearized 6-DOF Rigid-Body Equations of Motion

The nonlinear rigid-body equations of motion

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

can be linearized about $\boldsymbol{\nu}_o = [U,0,0,0,0,0]^\top$ for a marine craft moving at forward speed U.

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}^{*}\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$$\mathbf{C}_{RB}^{*} = \mathbf{M}_{RB}\mathbf{L}U$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linearized Coriolis and centripetal forces are recognized as:

$$\mathbf{f}_c = \mathbf{C}_{RB}^{*}\boldsymbol{\nu} = \begin{bmatrix} 0 \\ mUr \\ -mUq \\ -my_g Uq - mz_g Ur \\ mx_g Uq \\ mx_g Ur \end{bmatrix}$$

3.3 Linearized 6 DOF Rigid-Body Equations of Motion

Matlab:
The linearized model (3.64) is computed using the following Matlab commands

```
U = 1;
MRB = [
      1000       0       0       0    1000       0
         0    1000       0   -1000       0   10000
         0       0    1000       0  -10000       0
         0   -1000       0  101000       0  -10000
      1000       0  -10000       0  501000       0
         0   10000       0  -10000       0  125000];

L = zeros(6,6); L(2,6) = 1; L(3,5) = -1;
CRB = MRB * L * U;

CRB =
         0       0       0       0       0       0
         0       0       0       0       0    1000
         0       0       0       0   -1000       0
         0       0       0       0       0   -1000
         0       0       0       0   10000       0
         0       0       0       0       0   10000
```

Notice that the skew-symmetric property is destroyed by linearization. Moreover, $\boldsymbol{C}^*_{\mathrm{RB}} \neq -(\boldsymbol{C}^*_{\mathrm{RB}})^\top$