



Chapter 2

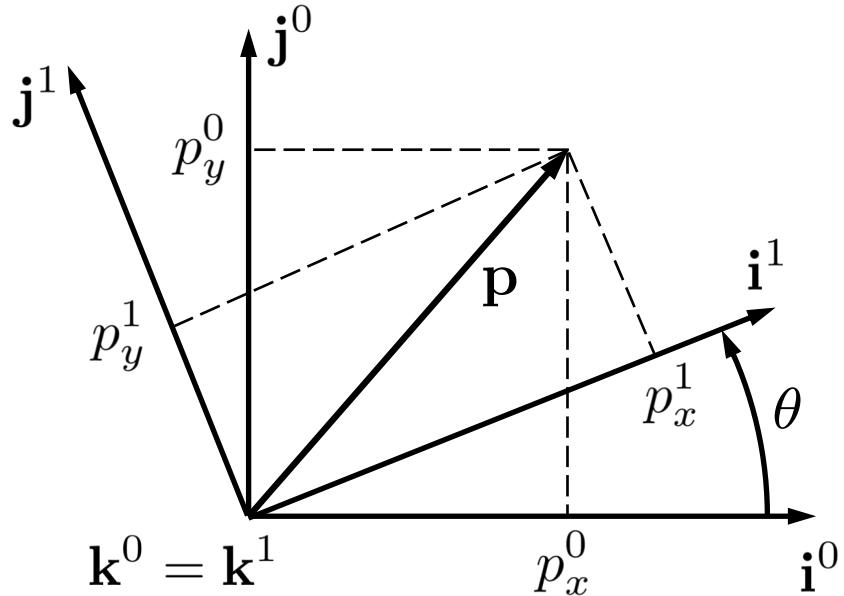
Coordinate Frames

Reference Frames

- In guidance and control of aircraft, reference frames used *a lot*
- Describe relative position and orientation of objects
 - Aircraft relative to direction of wind
 - Camera relative to aircraft
 - Aircraft relative to inertial frame
- Some things most easily calculated or described in certain reference frames
 - Newton's law
 - Aircraft attitude
 - Aerodynamic forces/torques
 - Accelerometers, rate gyros
 - GPS
 - Mission requirements

Must know how to transform between different reference frames

Rotation of Reference Frame



$$\mathbf{p} = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

$$\mathbf{p} = p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1$$

$$p_x^1 \mathbf{i}^1 + p_y^1 \mathbf{j}^1 + p_z^1 \mathbf{k}^1 = p_x^0 \mathbf{i}^0 + p_y^0 \mathbf{j}^0 + p_z^0 \mathbf{k}^0$$

$$\mathbf{p}^1 \triangleq \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \mathbf{i}^1 \cdot \mathbf{i}^0 & \mathbf{i}^1 \cdot \mathbf{j}^0 & \mathbf{i}^1 \cdot \mathbf{k}^0 \\ \mathbf{j}^1 \cdot \mathbf{i}^0 & \mathbf{j}^1 \cdot \mathbf{j}^0 & \mathbf{j}^1 \cdot \mathbf{k}^0 \\ \mathbf{k}^1 \cdot \mathbf{i}^0 & \mathbf{k}^1 \cdot \mathbf{j}^0 & \mathbf{k}^1 \cdot \mathbf{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

$$\mathbf{p}^1 = \mathcal{R}_0^1 \mathbf{p}^0 \quad \text{where} \quad \mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(rotation about \mathbf{k} axis)

Rotation of Reference Frame

Right-handed rotation about \mathbf{j} axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

Right-handed rotation about \mathbf{i} axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

Orthonormal matrix properties:

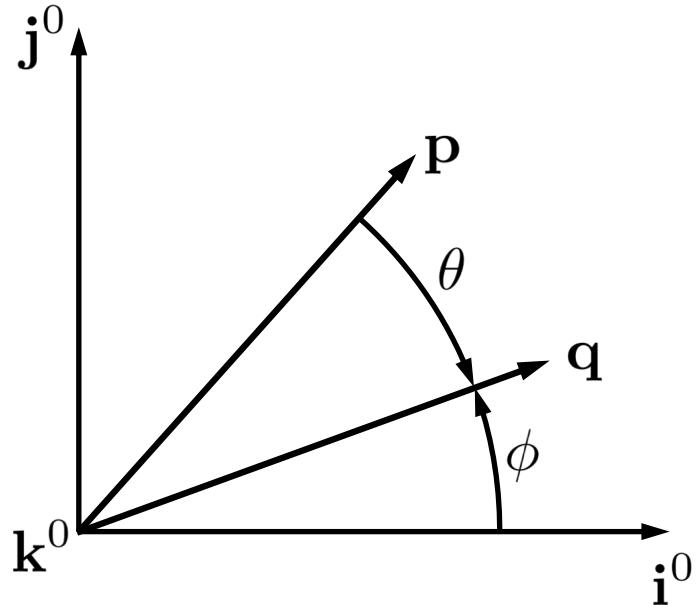
P.1. $(\mathcal{R}_a^b)^{-1} = (\mathcal{R}_a^b)^\top = \mathcal{R}_b^a$

P.2. $\mathcal{R}_b^c \mathcal{R}_a^b = \mathcal{R}_a^c$

P.3. $\det(\mathcal{R}_a^b) = 1$

Rotation of a Vector

Let and $q \triangleq |\mathbf{q}|$, then



R_0^1 can be interpreted as left-handed rotation of vector by angle θ .

$$\begin{aligned}\mathbf{p} &= \begin{pmatrix} p \cos(\theta + \phi) \\ p \sin(\theta + \phi) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p \cos \theta \cos \phi - p \sin \theta \sin \phi \\ p \sin \theta \cos \phi + p \cos \theta \sin \phi \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \cos \phi \\ p \sin \phi \\ 0 \end{pmatrix}\end{aligned}$$

where $p \triangleq |\mathbf{p}|$.
Define

$$\mathbf{q} = \begin{pmatrix} p \cos \phi \\ p \sin \phi \\ 0 \end{pmatrix}$$

then

$$\mathbf{p} = (\mathcal{R}_0^1)^\top \mathbf{q} \quad \implies \quad \mathbf{q} = \mathcal{R}_0^1 \mathbf{p}$$

Passive vs. Active Rotation

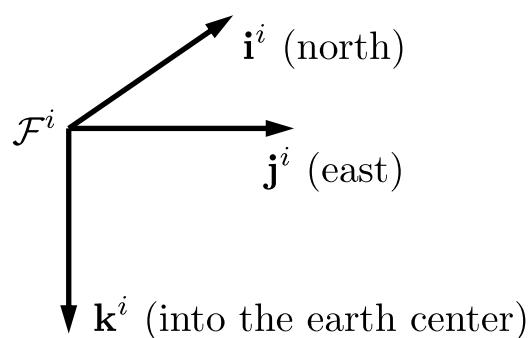
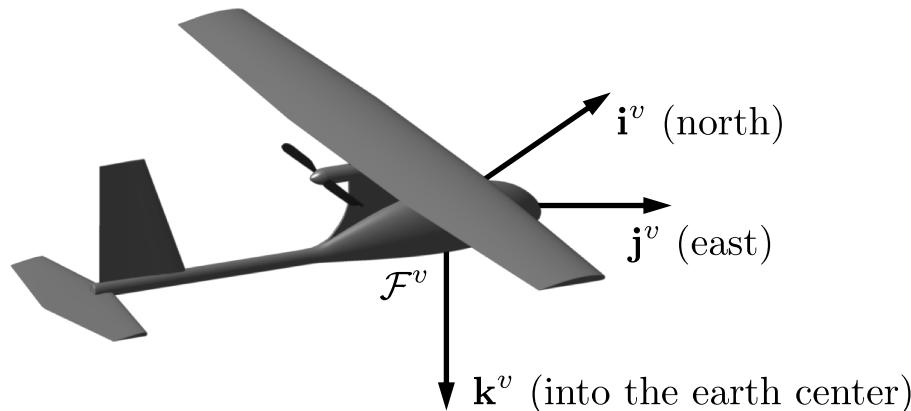
Passive Rotation/Transformation: The point \mathbf{p} is stationary and the coordinate frame changes.

Active Rotation/Transformation: The coordinate frame is the same, but the point it rotated.

R_0^1 as defined earlier, represents a right-handed passive transformation, or a left-handed active transformation.

If we stick with right handed rotations, then R_0^1 represents a right handed passive rotation, and $R_0^{1\top}$ represents a right handed active rotation.

Inertial Frame and Vehicle Frame



- Vehicle frame has same orientation as inertial frame
- Vehicle frame is fixed at cm of aircraft
- Inertial and vehicle frames are referred to as NED frames
- $N \rightarrow x, E \rightarrow y, D \rightarrow z$

Euler Angles

- Need way to describe attitude of aircraft
- Common approach: Euler angles

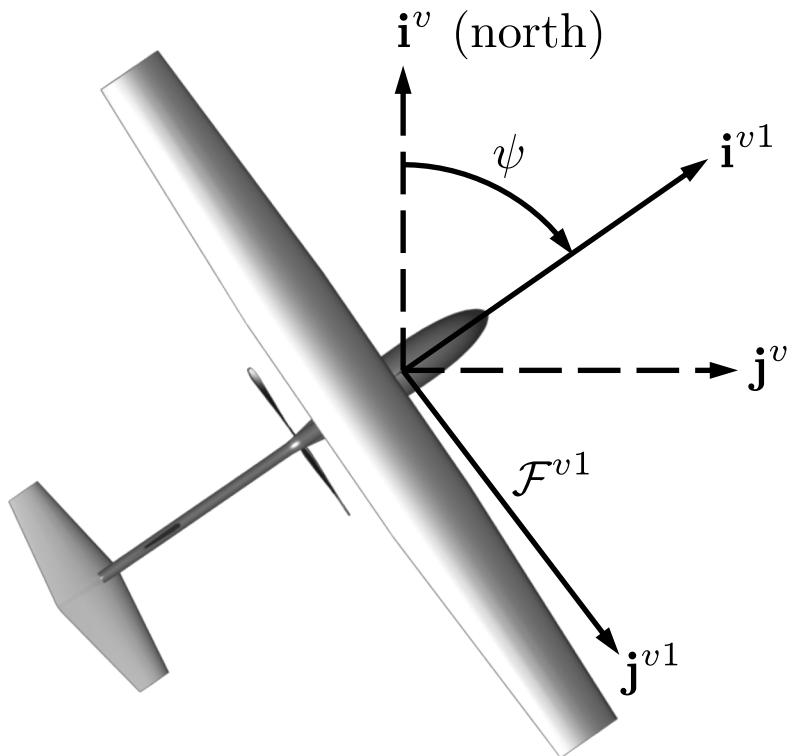
ψ : heading (yaw)

θ : elevation (pitch)

ϕ : bank (roll)

- Pro: Intuitive
- Con: Mathematical singularity
 - Quaternions are alternative for overcoming singularity

Vehicle-1 Frame



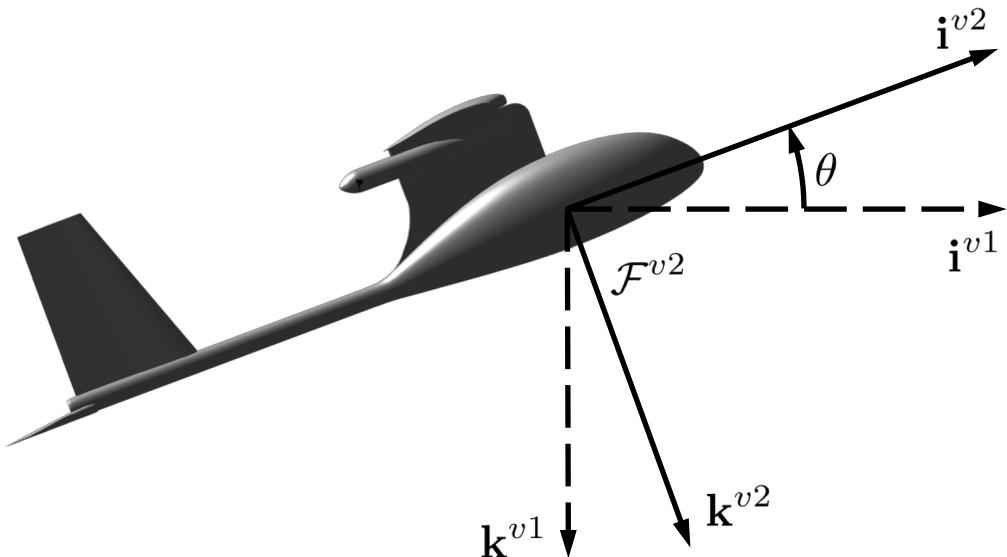
$$\mathcal{R}_v^{v1}(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives

$$\mathbf{p}^{v1} = \mathcal{R}_v^{v1}(\psi)\mathbf{p}^v$$

where ψ is the heading.

Vehicle-2 Frame



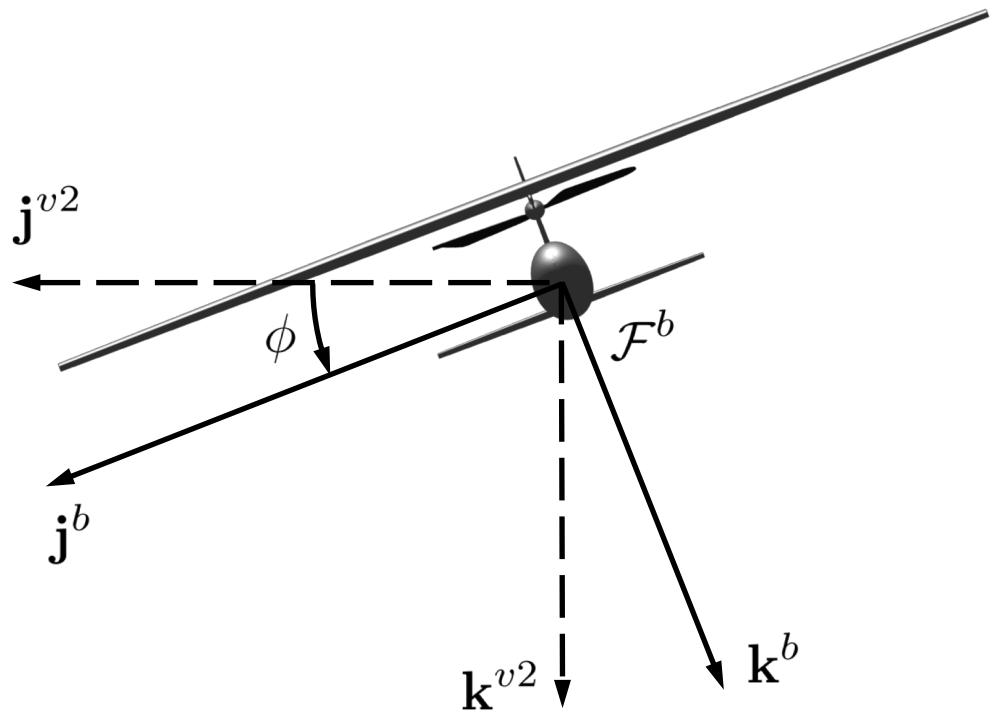
$$\mathcal{R}_{v1}^{v2}(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

gives

$$\mathbf{p}^{v2} = \mathcal{R}_{v1}^{v2}(\theta)\mathbf{p}^{v1}$$

where θ is the pitch angle.

Body Frame



$$\mathcal{R}_{v2}^b(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}$$

gives

$$\mathbf{p}^b = \mathcal{R}_{v2}^b(\phi) \mathbf{p}^{v2}$$

where ϕ is the roll (bank) angle.

Inertial Frame to Body Frame Transformation

Define

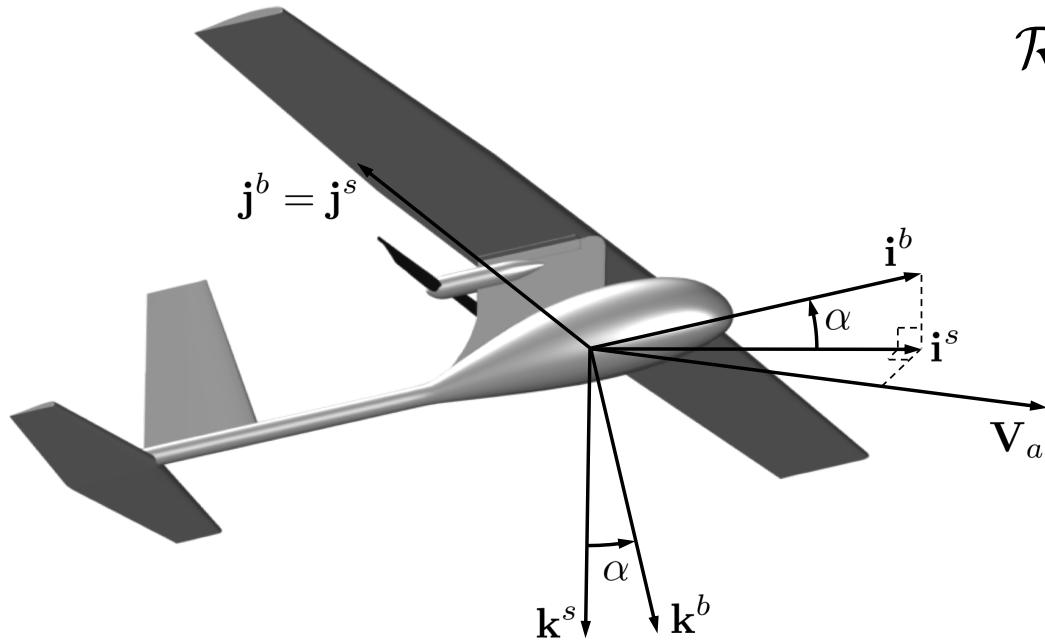
$$\begin{aligned}\mathcal{R}_v^b(\phi, \theta, \psi) &= \mathcal{R}_{v2}^b(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_v^{v1}(\psi) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}\end{aligned}$$

to give

$$\mathbf{p}^b = \mathcal{R}_v^b(\Theta)\mathbf{p}^v,$$

where, $\Theta = (\phi, \theta, \psi)^\top$.

Stability Frame

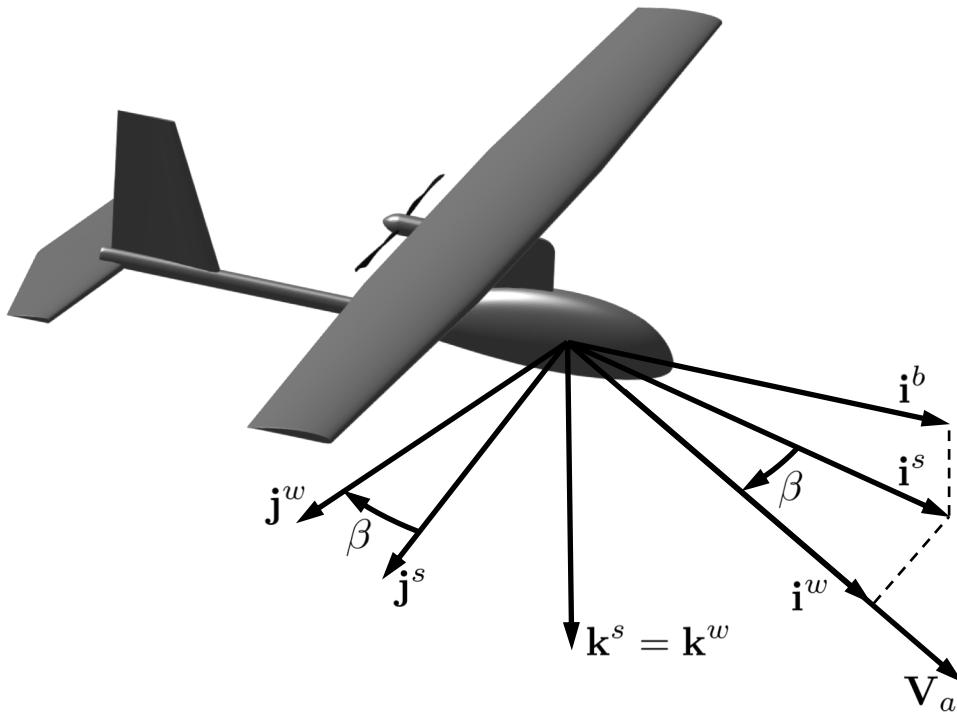


$$\mathcal{R}_s^b(\alpha) = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

$$\mathbf{p}^b = \mathcal{R}_s^b(\alpha)\mathbf{p}^s$$

Stability frame helps us rigorously define angle of attack. Forces and moments are most naturally expressed in the stability frame.

Wind Frame



$$\mathcal{R}_s^w(\beta) = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

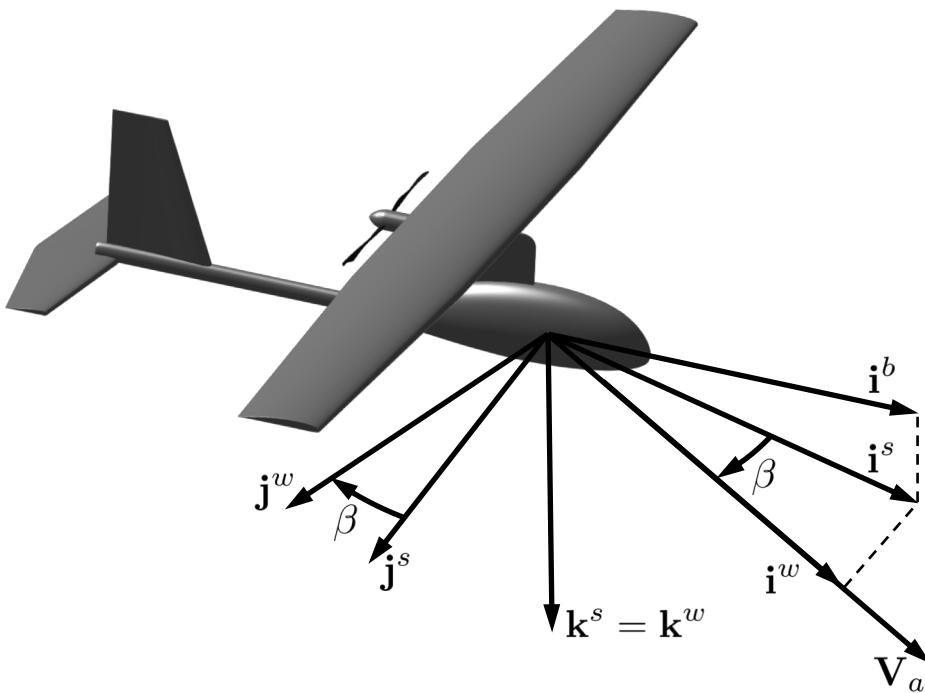
$$\mathbf{p}^w = \mathcal{R}_s^w(\beta)\mathbf{p}^s$$

$$\mathcal{R}_b^w(\alpha, \beta) = \mathcal{R}_s^w(\beta)\mathcal{R}_b^s(\alpha)$$

$$= \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos \beta \cos \alpha & \sin \beta & \cos \beta \sin \alpha \\ -\sin \beta \cos \alpha & \cos \beta & -\sin \beta \sin \alpha \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$

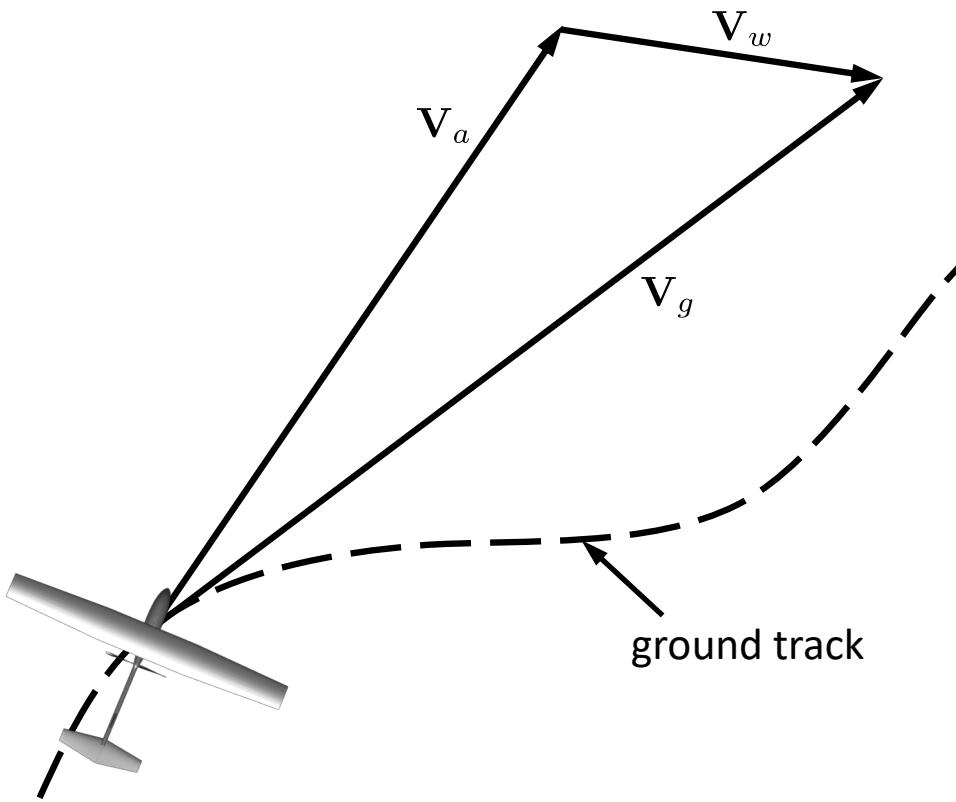
Wind Frame (continued)



- Wind frame helps us rigorously define side-slip angle
- Forces and moments are most naturally defined in wind frame
- Side-slip angle is nominally zero for tailed aircraft

$$\mathcal{R}_w^b(\alpha, \beta) = (\mathcal{R}_b^w)^\top(\alpha, \beta) = \begin{pmatrix} \cos \beta \cos \alpha & -\sin \beta \cos \alpha & -\sin \alpha \\ \sin \beta & \cos \beta & 0 \\ \cos \beta \sin \alpha & -\sin \beta \sin \alpha & \cos \alpha \end{pmatrix}$$

Airspeed, Wind Speed, Ground Speed



a/c wrt to surrounding air
expressed in body frame

$$\mathbf{V}_a = \mathbf{V}_g - \mathbf{V}_w$$

$$\mathbf{V}_g^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

a/c wrt to inertial frame
expressed in body frame

$$\mathbf{V}_w^b = \begin{pmatrix} u_w \\ v_w \\ w_w \end{pmatrix} = \mathcal{R}_v^b(\phi, \theta, \psi) \begin{pmatrix} w_n \\ w_e \\ w_d \end{pmatrix}$$

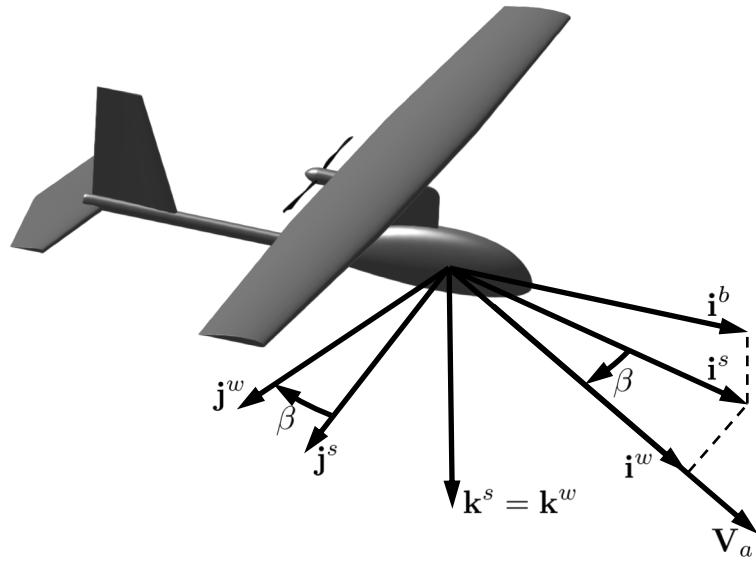
$$\mathbf{V}_a^w = \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{V}_a^b = \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \begin{pmatrix} u - u_w \\ v - v_w \\ w - w_w \end{pmatrix}$$

Airspeed, Angle of Attack, Sideslip Angle

$$\begin{aligned}\mathbf{V}_a^b &= \begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = \mathcal{R}_w^b \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta \cos \alpha & -\sin \beta \cos \alpha & -\sin \alpha \\ \sin \beta & \cos \beta & -\sin \beta \sin \alpha \\ \cos \beta \sin \alpha & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} u_r \\ v_r \\ w_r \end{pmatrix} = V_a \begin{pmatrix} \cos \alpha \cos \beta \\ \sin \beta \\ \sin \alpha \cos \beta \end{pmatrix}$$



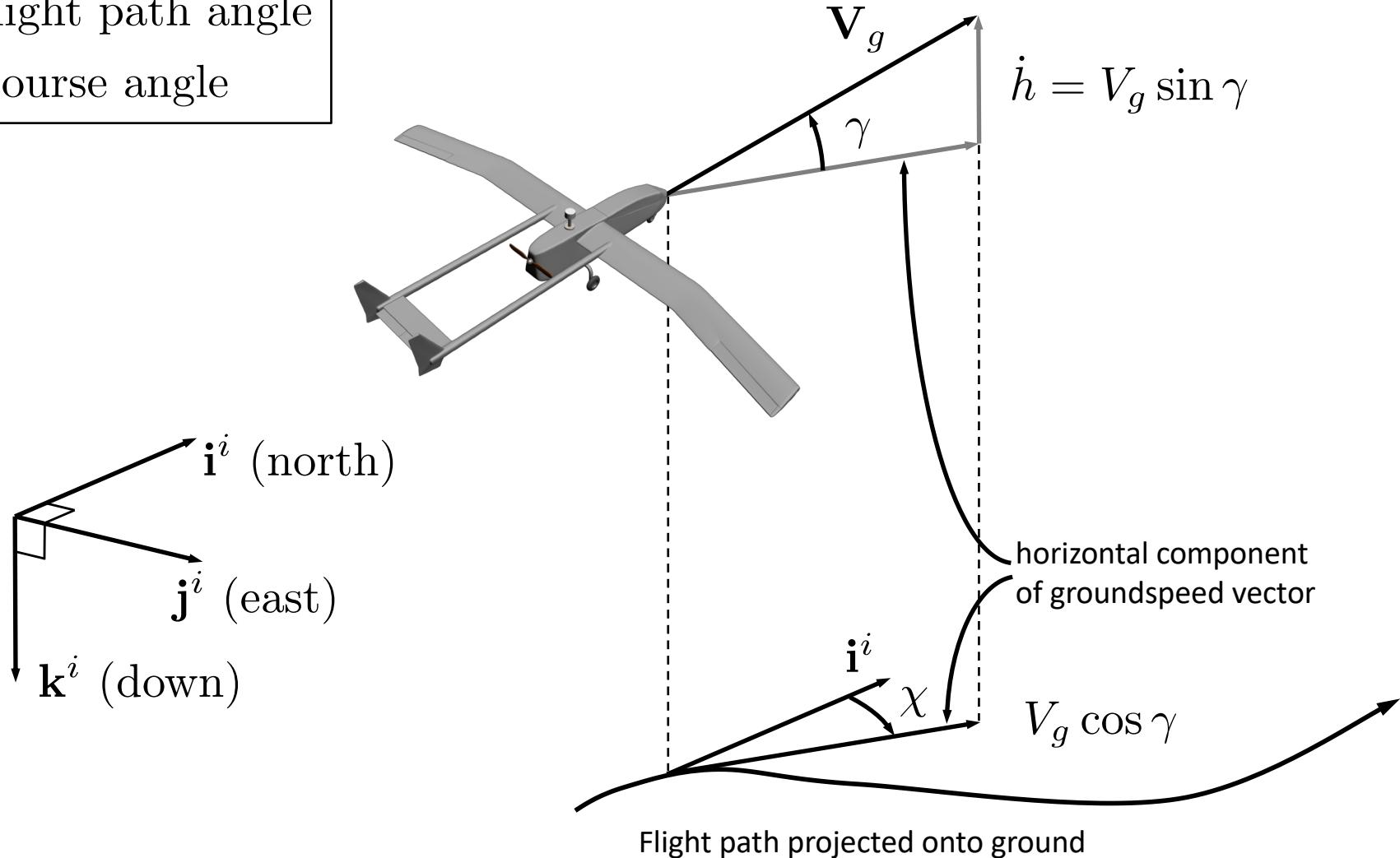
$$V_a = \sqrt{u_r^2 + v_r^2 + w_r^2}$$

$$\alpha = \tan^{-1} \left(\frac{w_r}{u_r} \right)$$

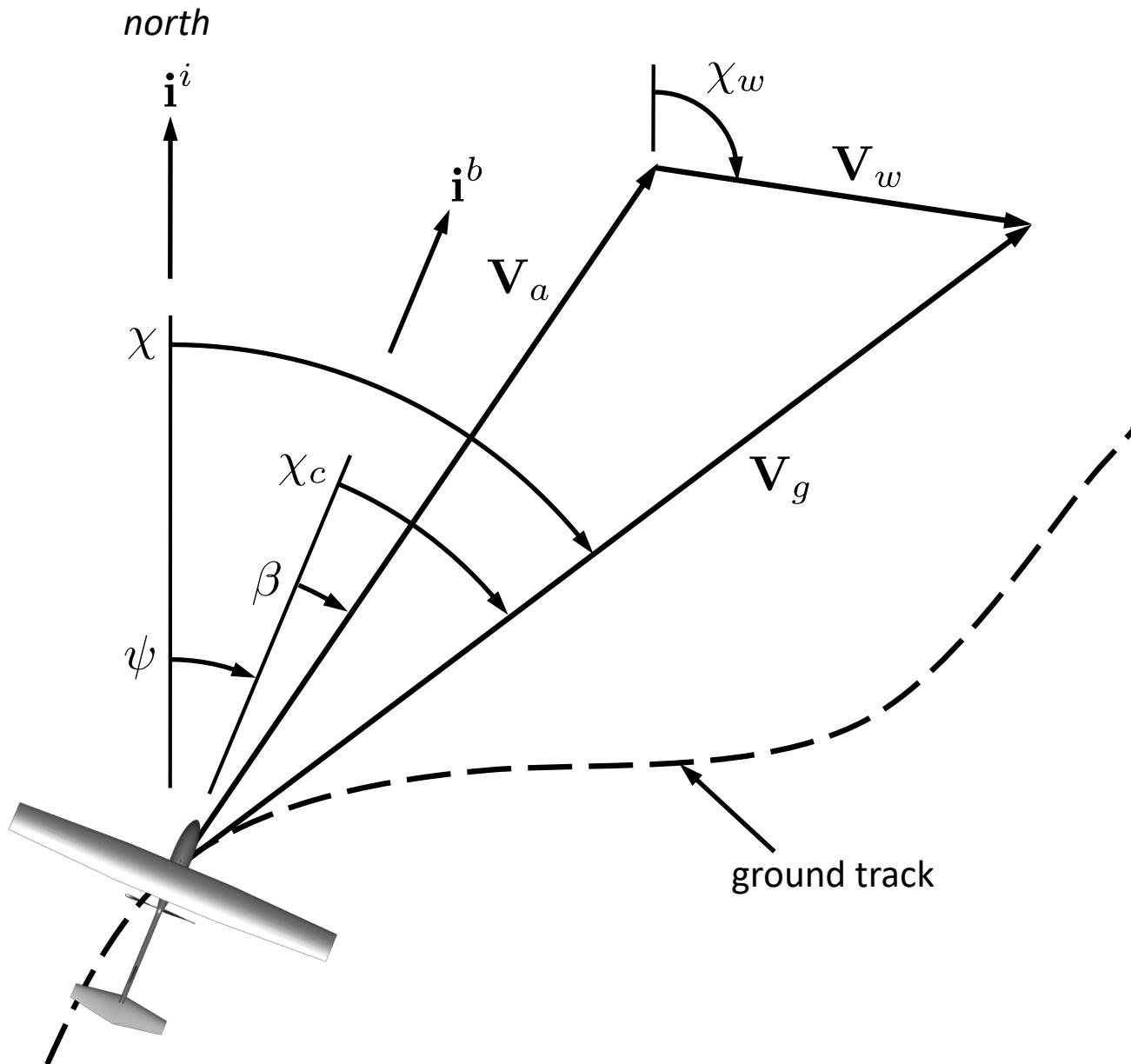
$$\beta = \sin^{-1} \left(\frac{v_r}{\sqrt{u_r^2 + v_r^2 + w_r^2}} \right)$$

Course and Flight Path Angles

γ : flight path angle
 χ : course angle



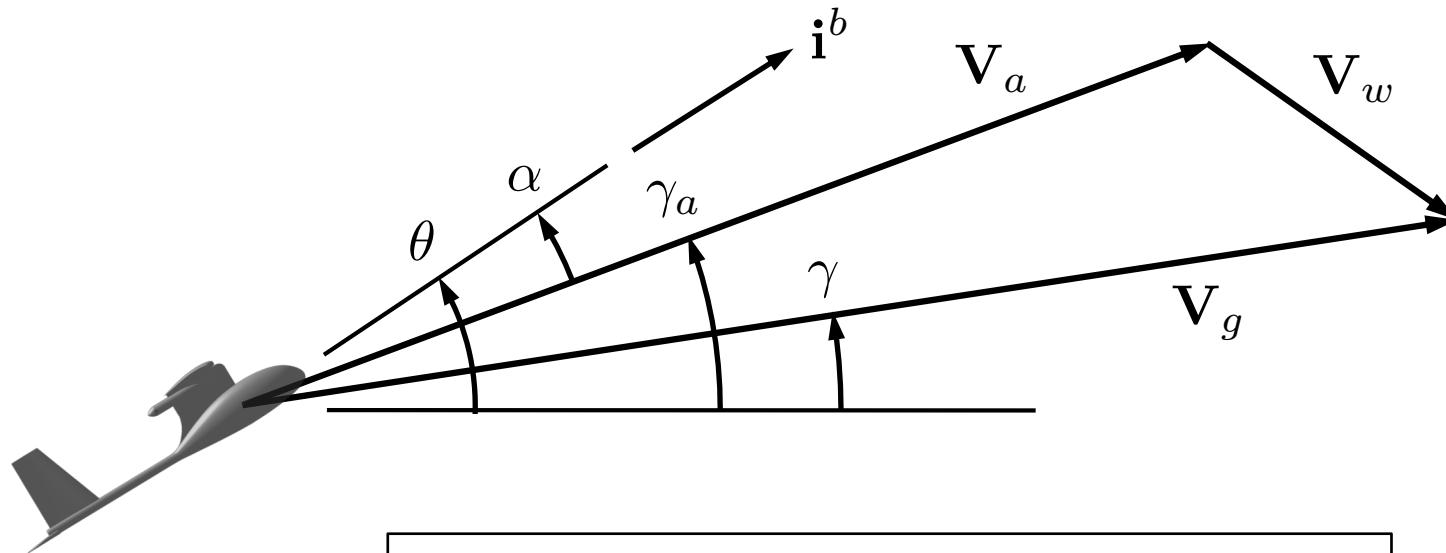
Wind Triangle



- χ : course angle
- ψ : heading angle
- β : side slip angle
- χ_c : crab angle
- χ_w : wind direction

$$\chi_c \stackrel{\Delta}{=} \chi - \psi$$

Wind Triangle



θ : pitch angle

α : angle of attack

γ : flight path angle

γ_a : air-mass-relative flight path angle

$$\gamma_a = \theta - \alpha$$

When wind speed and sideslip are zero...

If both the windspeed and the sideslip angles are zero, i.e.,

$$V_w = 0$$

$$\beta = 0$$

then we have the following simplifications

$V_a = V_g$ Airspeed equals groundspeed

$u = u_r$ Velocity equals velocity relative to the air mass

$v = v_r$

$w = w_r$

$\psi = \chi$ Heading equals course

$\gamma = \gamma_a$ Flight path angle equals air-mass-referenced flight path angle

Differentiation of a Vector

$$\mathbf{p} = p_x \mathbf{i}^b + p_y \mathbf{j}^b + p_z \mathbf{k}^b$$

$$\frac{d}{dt_i} \mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b + p_x \frac{d}{dt_i} \mathbf{i}^b + p_y \frac{d}{dt_i} \mathbf{j}^b + p_z \frac{d}{dt_i} \mathbf{k}^b$$

$$\frac{d}{dt_b} \mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b$$

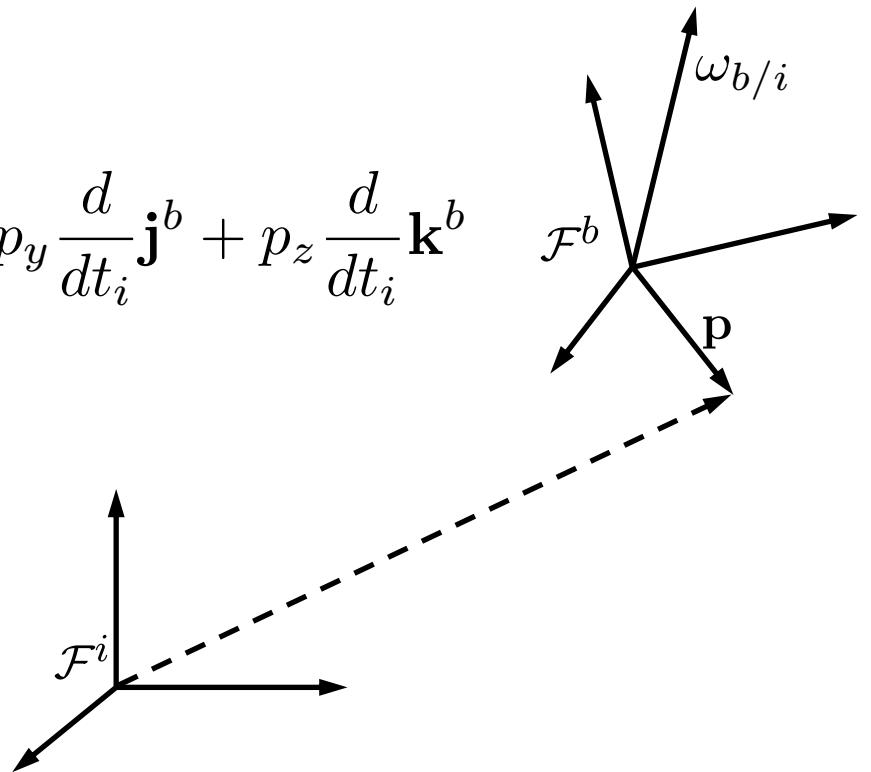
$$\dot{\mathbf{i}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{i}^b$$

$$\dot{\mathbf{j}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{j}^b$$

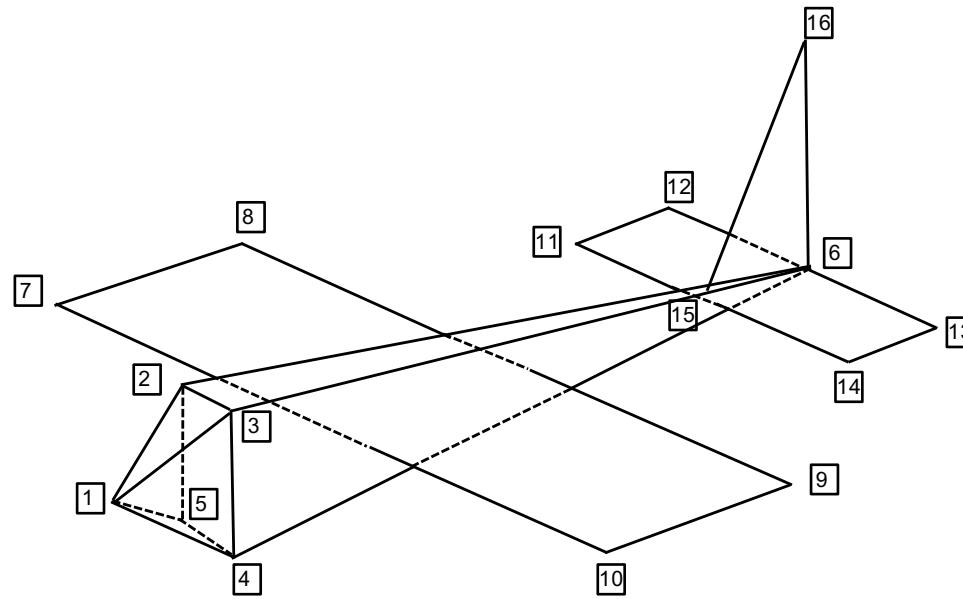
$$\dot{\mathbf{k}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{k}^b$$

$$\begin{aligned} p_x \dot{\mathbf{i}}^b + p_y \dot{\mathbf{j}}^b + p_z \dot{\mathbf{k}}^b &= p_x (\boldsymbol{\omega}_{b/i} \times \mathbf{i}^b) + p_y (\boldsymbol{\omega}_{b/i} \times \mathbf{j}^b) + p_z (\boldsymbol{\omega}_{b/i} \times \mathbf{k}^b) \\ &= \boldsymbol{\omega}_{b/i} \times \mathbf{p} \end{aligned}$$

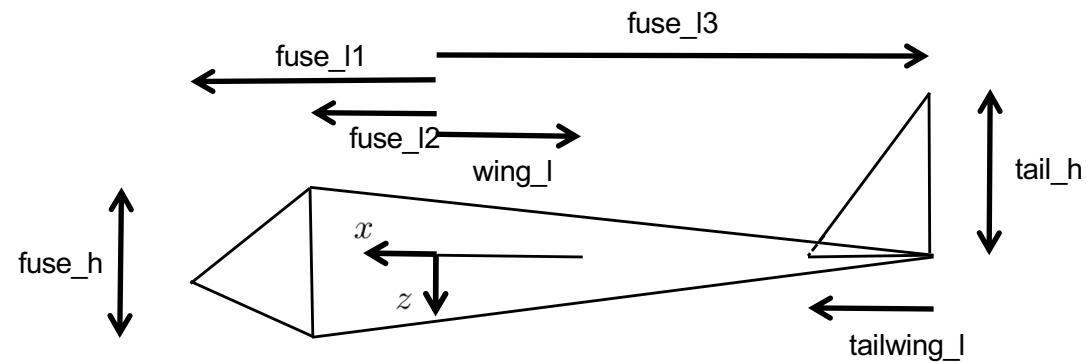
$$\frac{d}{dt_i} \mathbf{p} = \frac{d}{dt_b} \mathbf{p} + \boldsymbol{\omega}_{b/i} \times \mathbf{p}$$



Project Aircraft



Project Aircraft



Project Aircraft

