1a) $$x[n] = \sum_{i=1}^{p} A_i r_i^n + w[n], \quad n = 0, 1, \dots, N-1$$

Estimator:

$$\hat{A} = (H^T H)^{-1} H^T x$$

where

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ & \vdots & & \\ r_1^n & r_2^n & \cdots & r_p^n \end{bmatrix}$$

and

$$H^T H = \left\{ \sum_{n=0}^{N-1} (r_i r_j)^n \right\}_{i,j}$$

If $p=2$, $r_1 = 1$, $r_2 = -1$, N even, we get

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix}, \quad H^T H = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$

Finally

$$\hat{A} = (H^T H)^{-1} H^T x$$

$$= \begin{bmatrix} \frac{1}{N} & \\ & \frac{1}{N} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & -1 \end{bmatrix} \cdot x$$

$$= \begin{bmatrix} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ \frac{1}{N} \sum_{n=0}^{N-1} (-1)^n x[n] \end{bmatrix}$$

$$Cov(\hat{A}) = \sigma^2 (H^T H)^{-1} = \frac{\sigma^2}{N} \cdot I$$

1b) The observation matrix becomes

$$H = \begin{bmatrix} 1 & t_0 & \sin 2\pi t_0 \\ 1 & t_1 & \sin 2\pi t_1 \\ & & \vdots \\ 1 & t_{N-1} & \sin 2\pi t_{N-1} \end{bmatrix}$$

We build H using t.txt, and solve the estimation problem using

$$[\hat{A}, \hat{B}, \hat{C}]^T = (H^T H)^{-1} H^T x \quad (x \text{ from x.txt})$$

$$= [1.0023, 2.0133, 2.9099]^T$$

The CRLB is given by the diagonal of $\sigma^2 (H^T H)^{-1}$:

$$\mathrm{var}(\hat{A}) \geq 0.0445$$

$$\mathrm{var}(\hat{B}) \geq 0.0345$$

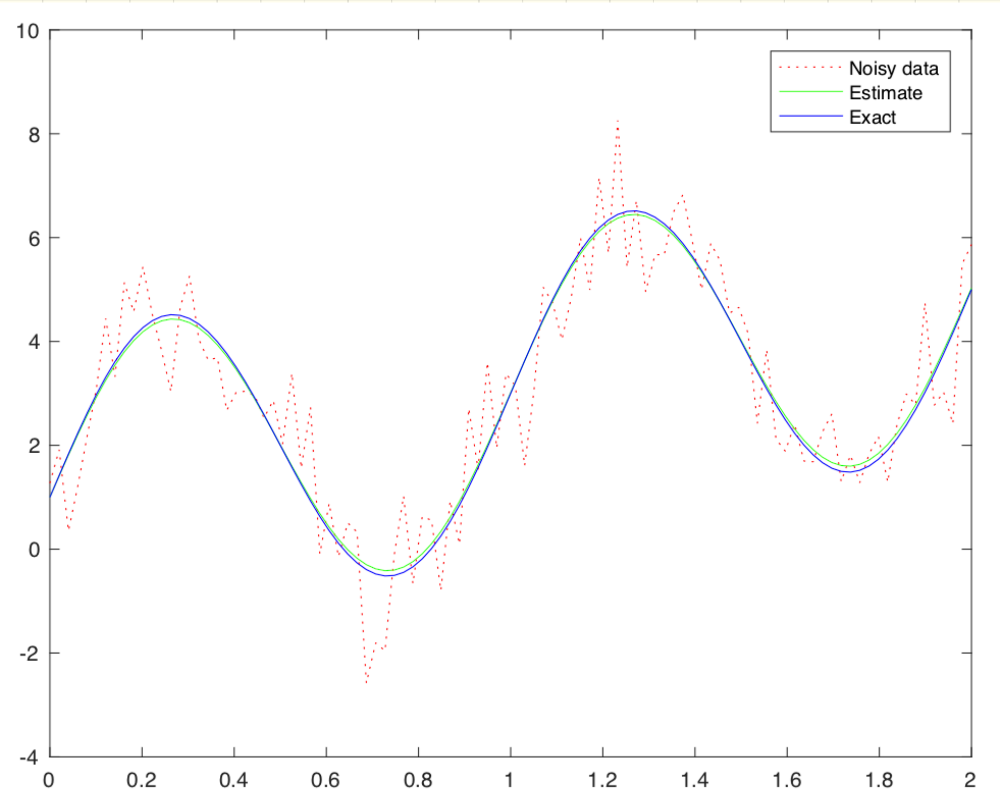
$$\mathrm{var}(\hat{C}) \geq 0.0237$$

Below is a plot of the curve

$$x = \hat{A} + \hat{B}t + \hat{C}\sin(2\pi t)$$

as well as the curve corresponding to the true values, $A=1, B=2, C=3$

The noisy signal x is also plotted



1c) For both cases we have

$$x = \mu \cdot \mathbf{1} + w \quad , \quad \mathbf{1} = [1, 1, \dots, 1]^T$$

where

$$E\{w\} = 0 \quad , \quad E\{ww^T\} = \operatorname{var}\{w[n]\} \cdot I$$

Then the BLUE is

$$\hat{\mu} = \frac{1^T C^{-1} x}{1^T C^{-1} 1} = \frac{1^T x}{1^T 1} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

For the Gaussian case the BLUE is also the MVU. This is not the case for the Laplacian case.

2a) 1) Model

$$x = A + w \quad , \quad w \sim N(0, A)$$

Then

$$p(x;A) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{1}{2A}(x-A)^2}$$

2) $\frac{\partial}{\partial A} \log P(X ; A)$

$$= \frac{\partial}{\partial A} \log \prod_{n=0}^{N-1} P(x[n];A)$$

$$= \frac{\partial}{\partial A} \sum_{n=0}^{N-1} -\frac{1}{2}\log 2\pi - \frac{1}{2}\log A - \frac{1}{2A}(x[n]-A)^2$$

$$= \sum_{n=0}^{N-1} -\frac{1}{2A} + \frac{1}{2A^2}(x[n]-A)^2 + \frac{1}{A}(x[n]-A) = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} -\frac{1}{2} + \frac{x^2[n] - 2Ax[n] + A^2}{2A} + x[n] - A = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} \frac{x^2[n] - 2Ax[n] + A^2 + 2A(x[n]-A)}{2A} = \frac{N}{2}$$

$$\Rightarrow \sum_{n=0}^{N-1} x^2 - A^2 = AN$$

$$\Rightarrow \quad NA^2 + NA - \sum_{n=0}^{N-1} x^2[n] = 0$$

$$\Rightarrow \quad A^2 + A - \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = 0$$

This is a quadratic equation in A, so the solutions are:

$$\Rightarrow \quad A = \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}}{2}$$

As the square root is always larger than one, and A needs to be positive, we get

$$\hat{A}_{ML} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}$$

3) First compute the second derivative of the log-pdf.

$$\frac{\partial^2}{\partial A} \log p(x;A)$$

$$= \frac{\partial}{\partial A} \sum_{n=0}^{N-1} -\frac{1}{2A} + \frac{1}{2A^2}(x[n]-A)^2 + \frac{1}{A}(x[n]-A)$$

$$= \sum_{n=0}^{N-1} \frac{1}{2A^2} - \frac{1}{A^3}(x[n]-A)^2 - \frac{x[n]-A}{A^2} - \frac{x[n]}{A^2}$$

Taking the expectation:

$$E\left\{\frac{\partial^2}{\partial A^2} \log p(x;A)\right\}$$

$$= \frac{N}{2A^2} - \frac{\sum_{n=0}^{N-1} E\{(x[n]-A)^2\}}{A^3} - \frac{\sum_{n=0}^{N-1} E\{x[n]-A\}}{A^2} - \frac{\sum_{n=0}^{N-1} E\{x[n]\}}{A^2}$$

$$= \frac{N}{2A^2} - \frac{NA}{A^3} - \frac{0}{A^2} - \frac{NA}{A^2}$$

$$= -\frac{N}{2A^2} - \frac{N}{A} = -\frac{N+2AN}{2A^2}$$

The CRLB is

$$\mathrm{var}(\hat{A}) \geq \frac{2A^2}{N+2AN}$$

4) The sample mean estimator for a DC level in Gaussian noise has variance $\frac{\sigma^2}{N}$, which in this case becomes

$$\mathrm{var}(\hat{A}_{sm}) \geq \frac{A}{N}$$

5) We rewrite the CRLB,

$$\frac{2A^2}{(N+2AN)} = \frac{A}{N}\left(\frac{2A}{1+2A}\right) < \frac{A}{N}$$

we see that the variance of the sample mean estimator is strictly larger for all N.

Since the MLE variance converges to the CRLB for large N, it will eventually be better.

2b) The pdf. for one observations is

$$p(x[n];f) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x[n]-\cos(2\pi f n))^2}$$

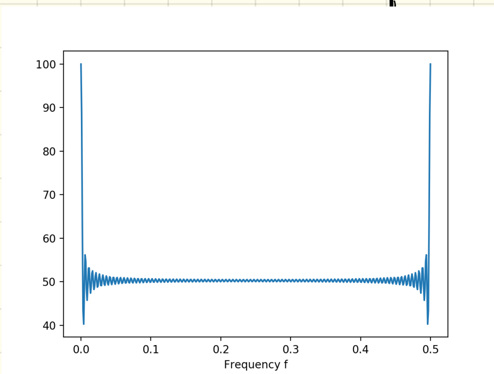
For the entire dataset we get

$$\hat{f}_{ML} = \underset{f}{\operatorname{argmax}} \log \prod_{n=0}^{N-1} p(x[n];f)$$

$$= \underset{f}{\operatorname{argmax}} \sum_{n=0}^{N-1} -\frac{1}{2}\left(x[n]-\cos(2\pi f n)\right)^2$$

$$= \underset{f}{\operatorname{argmax}} \sum_{n=0} x[n]\cos(2\pi f n) - \frac{1}{2}\cos^2(2\pi f n)$$

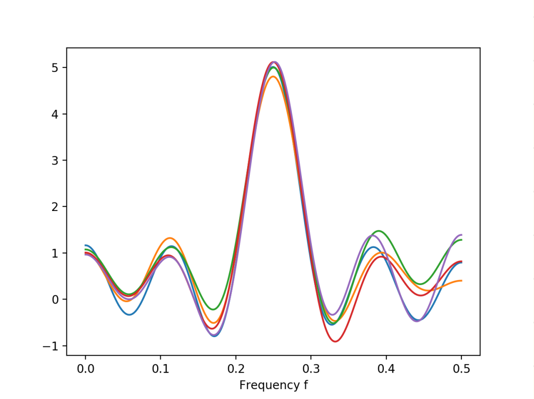
Plotting $\cos^2(2\pi f n)$ for $f \in (0, \frac{1}{2})$ we see it is almost constant for $f \in (\varepsilon, \frac{1}{2} - \varepsilon)$



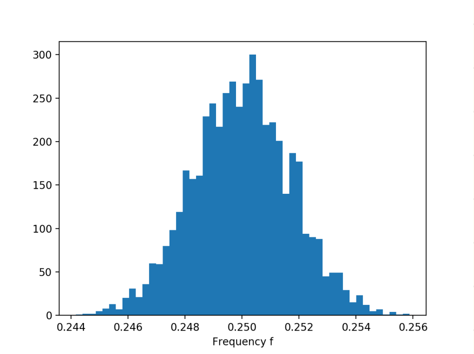
We can then approximate our estimate

$$\hat{f}_{ML} \approx \underset{f}{\operatorname{argmax}} \sum_{n=0}^{N-1} x[n]\cos(2\pi f n)$$

Plotting the objective function for a few realizations of x :



Using a grid search with $\Delta f = 1/2 \cdot \frac{1}{1000}$ and $M = 5000$ sets of observations yields the following histogram



We see that the estimates are distributed around $f = 0.25$.

2c) For the MLE we have

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}} \; p(x;\theta)$$

$$= \underset{\theta}{\operatorname{argmax}} \; \log p(x;\theta)$$

which is true if

$$\frac{\partial}{\partial\theta} \log p(x;\theta) \Big|_{\theta=\hat{\theta}_{ML}} = 0$$

But then, if an efficient est. exists

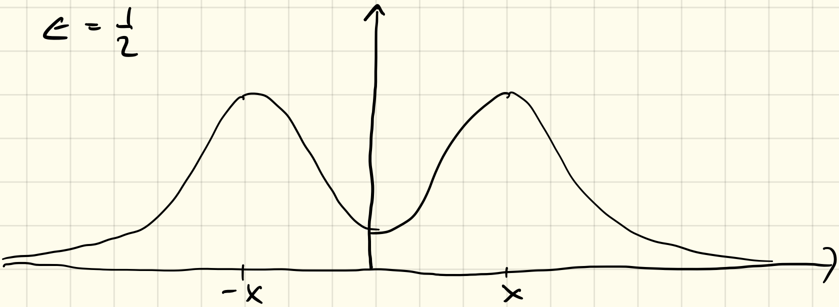
$$I(\theta)(g(x)-\theta) \Big|_{\theta=\hat{\theta}_{ML}} = 0$$

So

$$I(\hat{\theta}_{ML})(g(x)-\hat{\theta}_{ML}) = 0$$

$$\Rightarrow \hat{\theta}_{ML} = g(x)$$

3a) $\epsilon = \frac{1}{2}$



MMSE estimate is $E\{\theta | x\} = 0$ due to symmetry

MAP estimate is $\pm x$. Not unique

$\epsilon = 3/4$

MMSE estimate:

$$\hat{\theta}_{MMSE} = \int_{-\infty}^{\infty} \epsilon \theta \frac{1}{2\pi} e^{-\frac{1}{2}(\theta - x)^2} dx + \int_{-\infty}^{\infty} (1-\epsilon)\theta \frac{1}{2\pi} e^{-\frac{1}{2}(\theta + x)^2} dx$$

$$= \epsilon x + (1-\epsilon)(-x) = 3/4\, x - 1/4\, x = \underline{\frac{x}{2}}$$

MAP estimate (from figure): $\hat{\theta}_{map} = x$

3b) MMSE:

$$\hat{\theta}_{MMSE} = E\{\theta | x\}$$

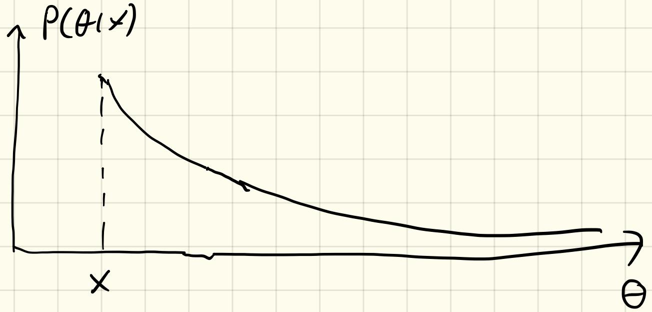
$$= \int_x^\infty \theta e^{-(\theta - x)} dx$$

$$= e^x \left[-\theta e^{-\theta} - e^{-\theta} \right]_x^\infty$$

$$= e^x \left(x e^{-x} + e^{-x} \right)$$

$$= \underline{x + 1}$$

MAP: Pdf



$$\hat{\theta}_{MAP} = \underset{\theta}{\arg\max}\ P(\theta | x) = \underline{x}$$