Norwegian University of Science and Technology
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Exercise 7
Ricatti Equation and State Estimation

In this exercise we consider the second-order system

$$\ddot{x} + k_1\dot{x} + k_2 x = k_3 u \tag{1}$$

In state-space form, with $x_1 = x$ and $x_2 = \dot{x}$, we get

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -k_2 & -k_1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ k_3\end{bmatrix} u \tag{2}$$

Discretizing the system using the explicit Euler scheme with sampling time T gives

$$\frac{x_{t+1} - x_t}{T} = \begin{bmatrix}0 & 1 \\ -k_2 & -k_1\end{bmatrix} x_t + \begin{bmatrix}0 \\ k_3\end{bmatrix} u_t \tag{3}$$

and hence

$$x_{t+1} = \underbrace{\begin{bmatrix}1 & T \\ -k_2 T & 1 - k_1 T\end{bmatrix}}_{A} x_t + \underbrace{\begin{bmatrix}0 \\ k_3 T\end{bmatrix}}_{B} u_t \tag{4}$$

Let $k_1 = k_2 = k_3 = 1$ and $T = 0.1$. The initial condition is $x_0 = \begin{bmatrix}5 & 1\end{bmatrix}^\top$; the initial state estimate is $\hat{x}_0 = \begin{bmatrix}6 & 0\end{bmatrix}^\top$ when an observer is used.

Problem 1 (20 %) The Riccati Equation

The algebraic or stationary Riccati equation is stated as

$$P = Q + A^\top P(I + BR^{-1}B^\top P)^{-1}A \tag{5}$$

in the MPC note. Another common form of this equation is

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q = 0 \tag{6}$$

(see, e.g., the MATLAB documentation for the `dlqr` function). Use the matrix inversion lemma (also known as the Sherman-Morrison-Woodbury formula)

$$(S + UTV)^{-1} = S^{-1} - S^{-1}U(T^{-1} + VS^{-1}U)^{-1}VS^{-1} \tag{7}$$

to derive (6) from (5).

Problem 2 (30 %) LQR and State Estimation

We will in this problem assume that only x_1 is measured; that is,

$$y_t = Cx_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \tag{8}$$

We use LQR and an observer to control the output.

a We want to minimize the infinite-horizon objective function

$$f^{\infty}(z) = \frac{1}{2} \sum_{t=0}^{\infty} \left\{ \hat{x}_{t+1}^{\top} Q \hat{x}_{t+1} + u_t^{\top} R u_t \right\} \tag{9a}$$

with

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad R = 1 \tag{9b}$$

Note that the objective function is formulated in $\hat{x}_{t+1}$ (the state estimate) as opposed to x_{t+1} (the actual state). Use the MATLAB function `dlqr` to find the optimal feedback gain K, assuming that the full state is available for feedback. What is K and the resulting closed-loop poles (the eigenvalues of $A - BK$)?

b The state observer needs to be faster than the controller, meaning that the poles of $A - K_F C$ are faster than the poles of $A - BK$. Use the MATLAB function `place` to place the observer poles. You can chose the poles yourself or use the poles $p_{1,2} = 0.5 \pm 0.03j$ (in the z-plane); these poles correspond to a time constant of approximately 1/5 of the fastest control time constant. Simulate the system for 50 time steps with feedback from the estimator and plot both x_t and $\hat{x}_t$. Comment on the performance and tune the controller and/or the observer if you wish to improve the performance.

c The control and estimation equations can be written

$$\xi_{t+1} = \begin{bmatrix} x_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - K_F C \end{bmatrix}}_{\Phi} \xi_t \tag{10a}$$

$$\tilde{x}_t = x_t - \hat{x}_t \tag{10b}$$

State the full matrix Φ with your numerical values and verify that the eigenvalues of Φ are the poles of $A - BK$ and $A - K_F C$.

Problem 3 (30 %) MPC and State Estimation

We now add the input constraint

$$-4 \leq u_t \leq 4 \qquad t = 1, \ldots, N-1 \tag{11}$$

and use MPC with Q and R as given in (9b).

a Modify your code and use output-feedback MPC and the observer you designed in Problem 2 to control the system. The output y_t is the same as in the previous problem. Let the MPC minimize the open-loop objective function

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ \hat{x}_{t+1}^\top Q \hat{x}_{t+1} + u_t^\top R u_t \right\} \tag{12}$$

at every time instant with $N = 10$. Tune the controller if necessary. Simulate the closed-loop system for 50 time steps.

b We now assume that both states are available for feedback; that is, $C = I$. Repeat problem 1) with state feedback (do not use the observer). This means the open-loop objective function is

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t \right\} \tag{13}$$

Compare the closed-loop response to what you obtained in 1) and comment.

Problem 4 (20 %) Infinite-Horizon MPC

a Calculate the Riccati matrix P using the MATLAB function `dlqr`.

b Modify your code from Problem 3 2) and minimize the open-loop objective function

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t \right\} + x_N^\top P x_N \tag{14}$$

This can be done by modifying G in the formulation

$$f(z) = \frac{1}{2} z^\top G z \tag{15}$$

Specifically, the last Q on the diagonal of G must be replaced by P. Use $N = 10$ and compare the closed-loop response with what you obtained in Problem 3 2) and comment. Change N and look at the open-loop solutions. Are the input constraints always inactive toward the end of the horizon? When does N become important for performance?