Kahoot

- https://play.kahoot.it/#/k/87256f68-7b17-4aa0-9c9c-c30869da5639

Lecture 8: Stability, Padé approximations

- Stability of Runga-Kutta methods
  - **A-stability**
  - Aliasing
  - **L-stability**
  - Padé approximations
  - (Nonlinear stability: AN-, B- and algebraic stability)

Book: 14.6

Explicit Runge-Kutta (ERK) methods

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- ERK:

$$
\begin{aligned}
k_1 &= f(y_n, t_n) \\
k_2 &= f(y_n + h a_{21} k_1, t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{31} k_1 + a_{32} k_2), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1} k_1 + a_{\sigma,2} k_2 + \ldots + a_{\sigma,\sigma-1} k_{\sigma-1}), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\mathsf{T}
\end{array}
\qquad
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}
$$

Recap: Test system, stability function

- One step method (typically: Runge-Kutta):

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Apply it to scalar test system:

$$\dot{y} = \lambda y$$

- We get:

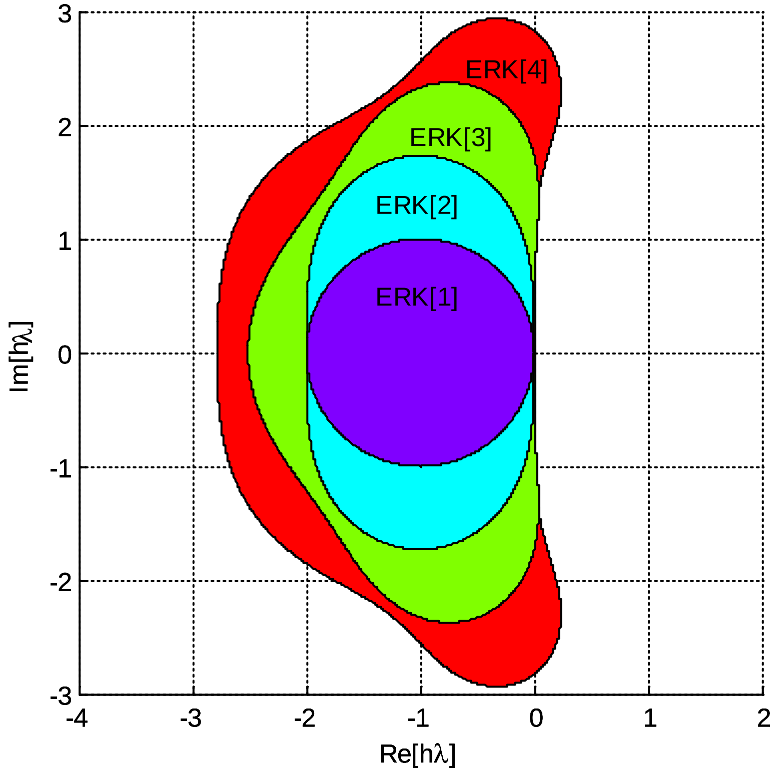
$$y_{n+1} = R(h\lambda) y_n$$

  where $R(h\lambda)$ is stability function

- The method is stable (for test system!) if

$$|R(h\lambda)| \leq 1$$

Stability regions for ERK methods



Implicit Runge-Kutta (IRK) methods

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- IRK:

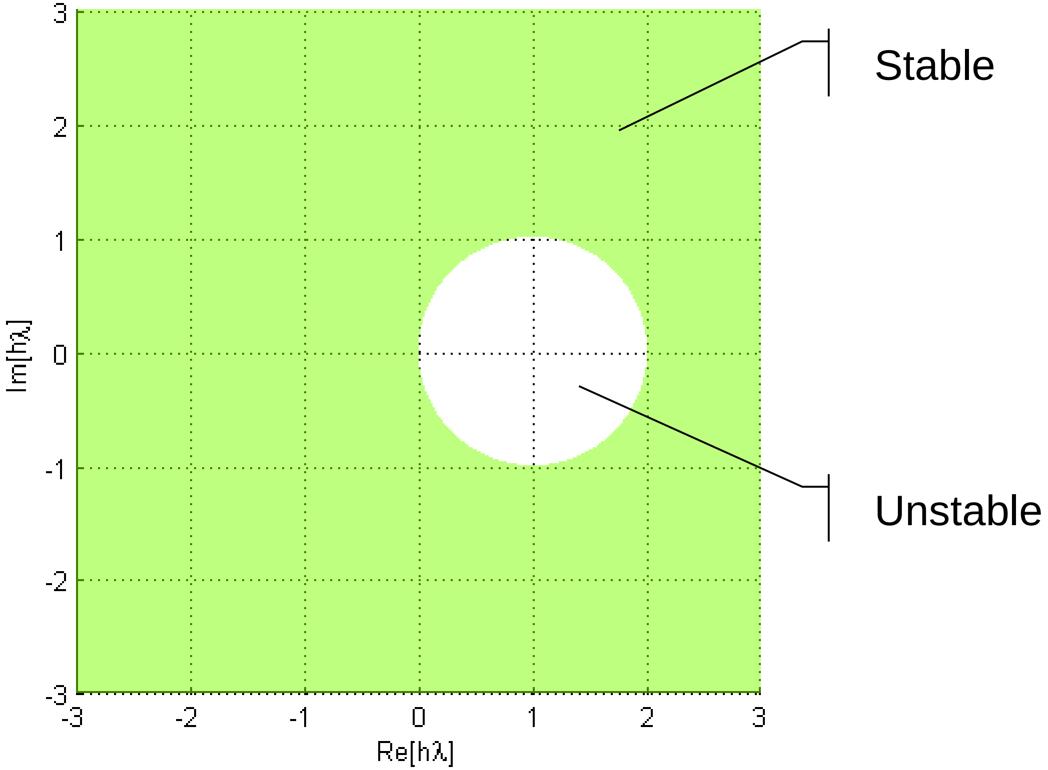
$$
\begin{aligned}
k_1 &= f(y_n + h(a_{1,1}k_1 + a_{1,2}k_2 + \ldots + a_{1,\sigma}k_\sigma), t_n + c_1 h) \\
k_2 &= f(y_n + h(a_{2,1}k_1 + a_{2,2}k_2 + \ldots + a_{2,\sigma}k_\sigma), t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{3,1}k_1 + a_{3,2}k_2 + \ldots + a_{3,\sigma}k_\sigma), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1}k_1 + a_{\sigma,2}k_2 + \ldots + a_{\sigma,\sigma}k_\sigma), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\mathsf{T}
\end{array}
\qquad
\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1,\sigma-1} & a_{1,\sigma} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2,\sigma-1} & a_{2,\sigma} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & a_{\sigma,\sigma} \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}
$$

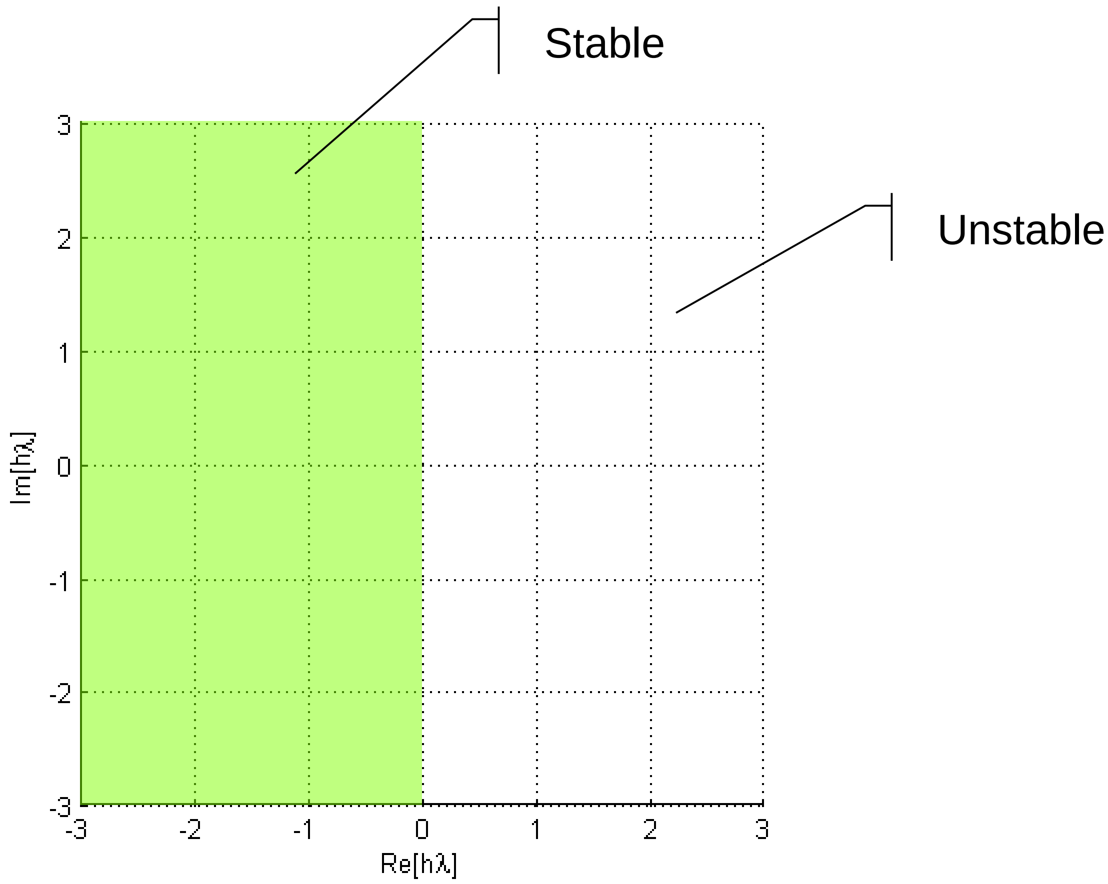
Stability regions for implicit methods

- Implicit Euler



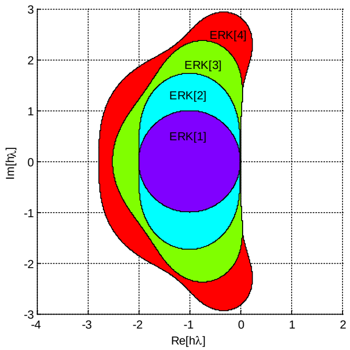
Stability regions for implicit methods

- Trapezoidal rule and implicit midpoint rule:

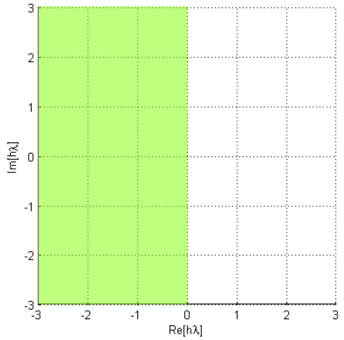


Why use IRK methods?

- IRK methods are much more complex (since we have to solve a set of nonlinear equations for each step) than ERK methods, so why use them?
  - Accuracy? Stability?



- Not because of accuracy
  - Even if an IRK method may have higher accuracy for a given number of stages, it is easy and cheap to achieve same accuracy for ERK by increasing the number of stages



- It's because of the much larger stability region!

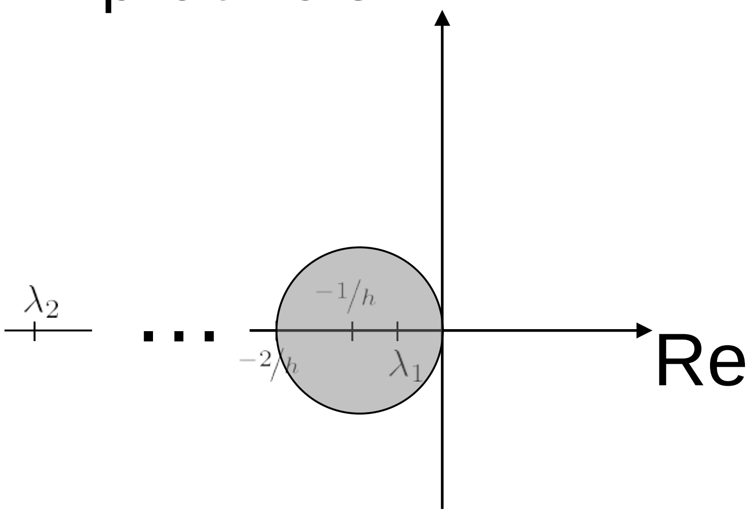
- When is this important?
  - **Stiff systems!**

Stiff system

$$\dot{y}_1 = \lambda_1 y_1$$
$$\dot{y}_2 = \lambda_2 y_2$$

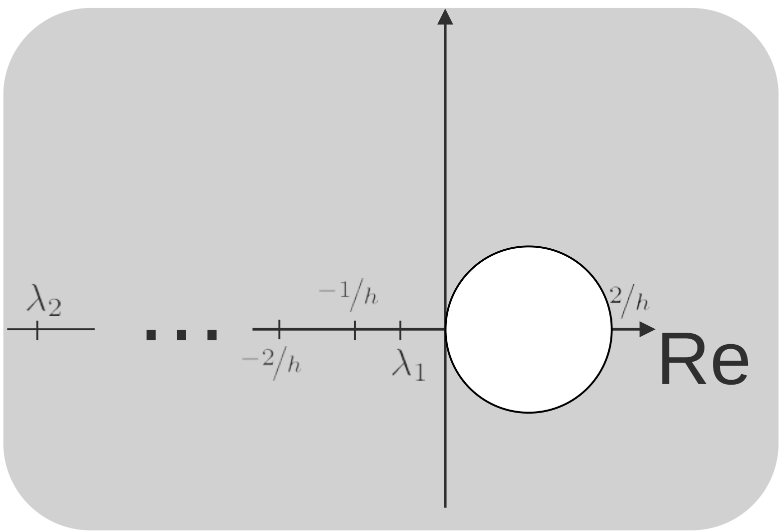
$$\lambda_2 << \lambda_1 < 0$$

Explicit Euler



must choose a small *h* for stability

Implicit Euler



can choose all *h* for stability

D

Def: A method is A stable if

$|R(h\lambda)| \leq 1 \quad \forall \quad \mathrm{Re}[\lambda] \leq 0$

Example 221

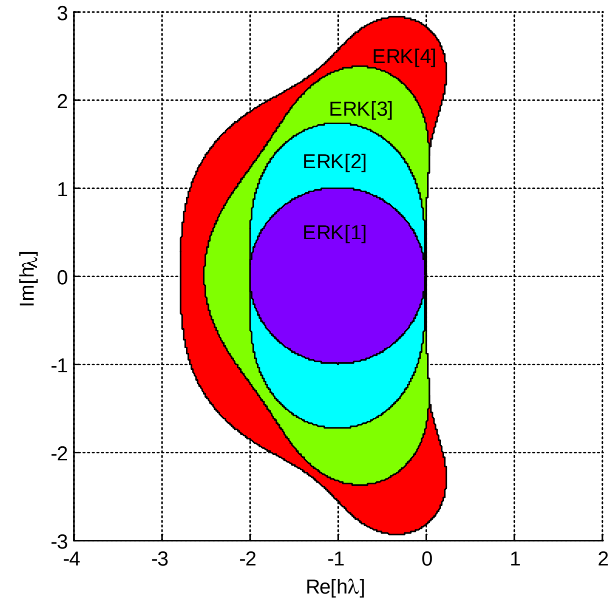
Explicit methods:

$|R(h\lambda)| \to \infty$ if $|\lambda h| \to \infty$

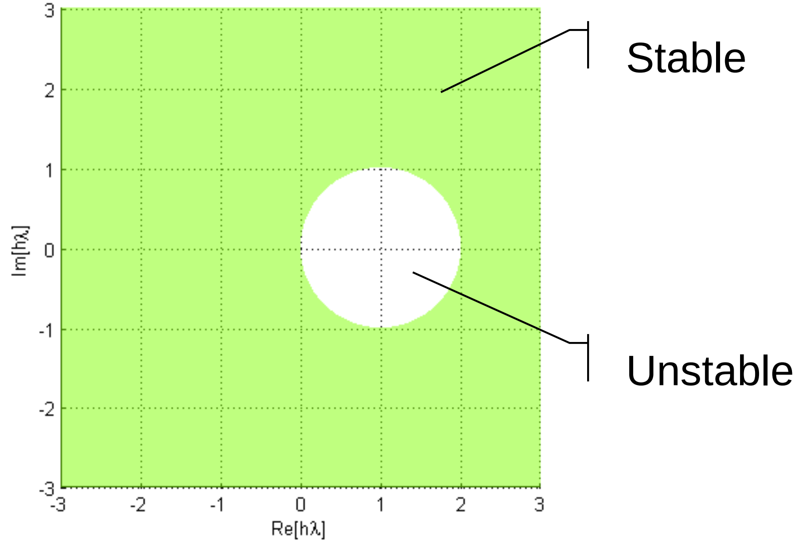
- no explicit methods are A-stable
- Some implicit methods are A-stable

ERK vs IRK stability regions

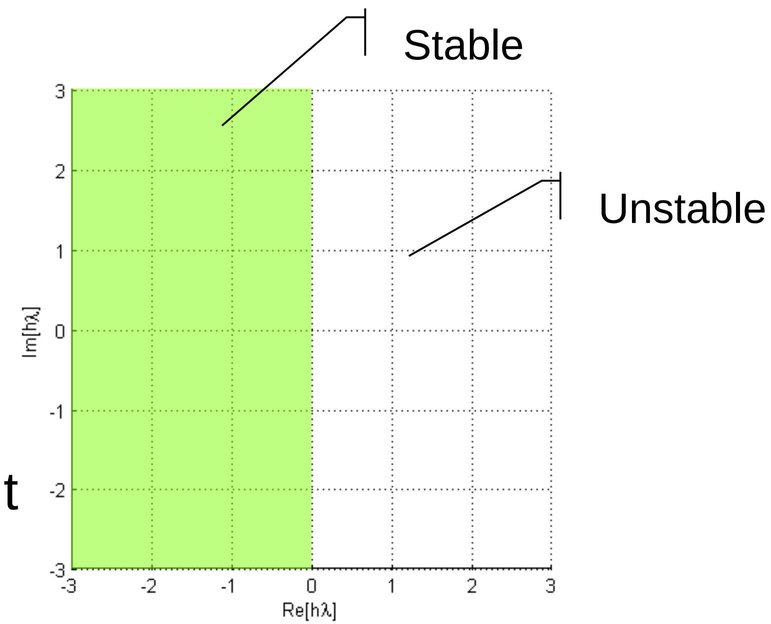
ERK-methods



Implicit Euler



Trapezoidal/ Implicit Midpoint



Aliasing I

Observation : local solution for the test function : $\dot{y} = \lambda y$

$\dot{y}_c(t_n; t) = \lambda y_c(t_n; t)$ $\quad y_c(t_n; t_n) = y_n$

$\rightarrow \quad y_c(t_n, t_{n+1}) = e^{h\lambda} y_n$

Aliasing II

- Assume:

Test system I

$$\dot{y} = \lambda y \; ; \;\; \lambda = \sigma + j\omega$$

$$y_L(t_n; t_{n+1}) = e^{h\lambda} y_n$$

Test system II

$$\dot{y} = \mu y$$

$$y_L(t_n; t_{n+1}) = e^{h\mu} y_n$$

When are these two systems the same?

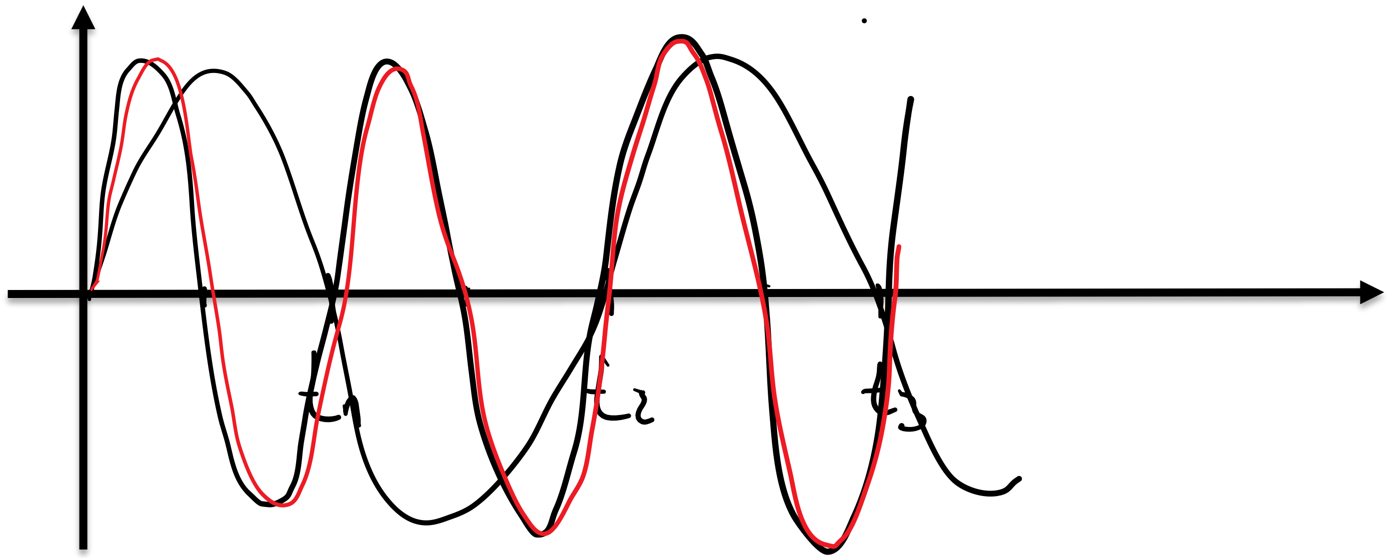
$$e^{h\mu} = e^{h\lambda} = e^{h\sigma}\left(\cos(h\omega) + j\sin(h\omega)\right)$$

$$h\mu = h\sigma + j\left(h\omega + 2k\pi\right)$$

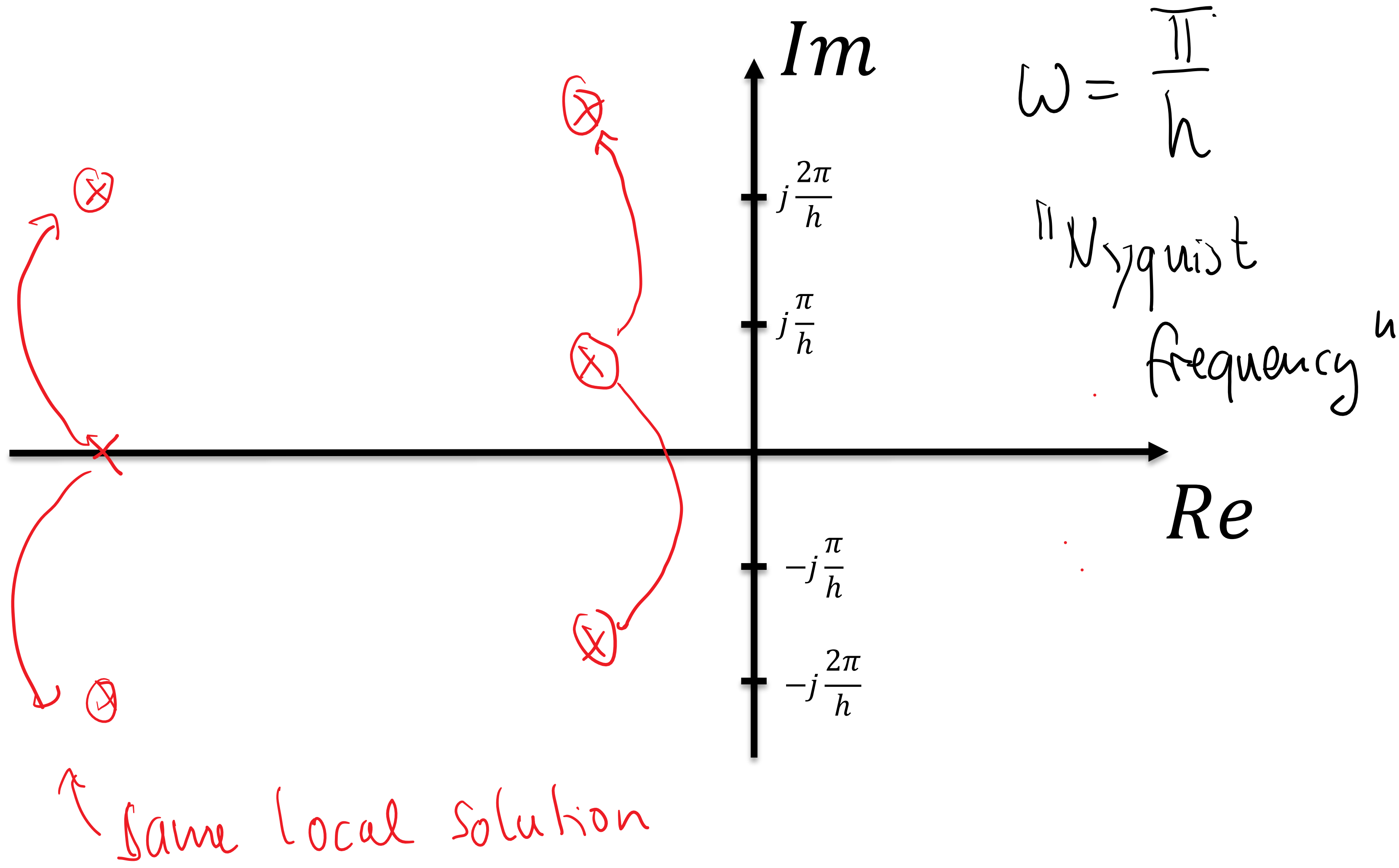
$$\mu = \sigma + j\omega + j\,2k\frac{\pi}{h}$$

$$\mu = \lambda + j\,2k\frac{\pi}{h}$$

Visualisation I

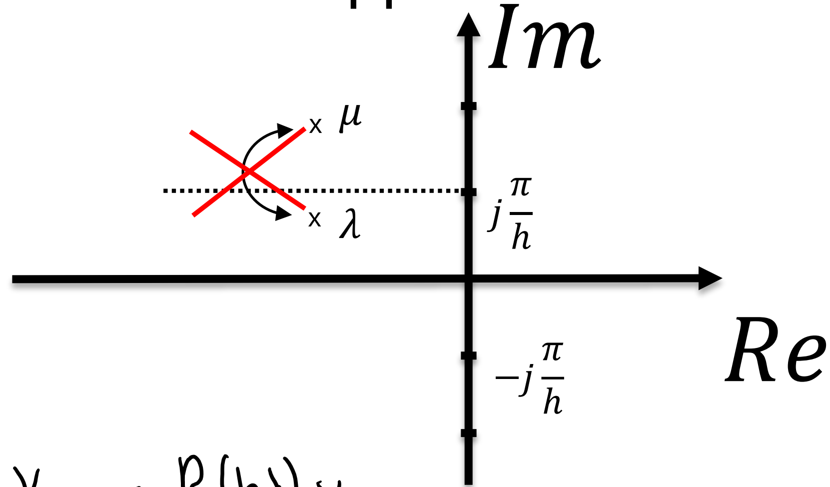


Visualisation II



L-stability I

- Assume μ eigenvalues and we want to suppress:



$$\dot{y} = \mu\lambda$$

exact

$$y_c(t_n; t_{n+1}) = e^{h\mu} y_n$$

$$= e^{h\lambda} y_n$$

$$\mu = \lambda \pm j 2k \frac{\pi}{h} \qquad k = 0, \pm 1 \dots$$

RK

$$y_{n+1} = R(h\lambda) y_n$$

$$R(h\mu) \approx R(h\lambda) \qquad R(h\mu) \approx 0$$

$$y_{n+1} = R(h\lambda) y_n \qquad y_{n+1} \approx 0$$

L-stability II

Def: A method is L-stable if it is A-stable and

$|R(jh\omega)| \to 0$ if $\omega \to \infty$

for all systems $\dot{y} = \lambda y$ where $\lambda = j\omega$

$\to$ L-stable methods damp out fast oscillations

Example – L-stability

ERK : not A-stable, therefore
not L-stable
$(|R(j\omega h)| \to \infty$ if $\omega \to \infty)$

Implicit Euler : $R(s) = \frac{1}{1-s} \to 0$ if $s = j\omega h \to \infty$
$\to$ L-stable

Trapezoidal / implicit midpoint
$$R(s) = \frac{1 + s/2}{1 - s/2} \to 1 \text{ if } s = j\omega h \to \infty$$
$\to$ not L-stable

Padé-approximation I

$$\text{Test system: } \dot{y} = \lambda y$$

$$\xrightarrow{\text{exact}} \quad y_L(t_n; t_{n+1}) = e^{h\lambda} y_n$$

$$\xrightarrow{\text{RK}} \quad y_{n+1} = R(h\lambda) y_n$$

«good» methods fulfill $e^s \approx R(s)$ $\quad s = h\lambda$

ERK: $p = \sigma \leq 4$ $\quad R(s) = 1 + s + \dots + \frac{s^p}{p!}$ Taylor series of e^s

IRK: $\quad R(s) = \frac{1 + \beta_1 s + \dots + \beta_k s^k}{1 + \gamma_1 s + \dots + \gamma_m s^m}$ $\quad k, m \leq \sigma$

Padé-approximation II

Def: Padé approximation $P_m^k(s)$ of e^s is the rational function:

$$P_m^k(s) = \frac{1 + \beta_1 s + \cdots + \beta_k s^k}{1 + \gamma_1 s + \cdots + \gamma_m s^m},$$

which approximates e^s the best given k and m

Padé approximations to e^s

$m \backslash k$	0	1	2	3
0	$\frac{1}{1}$	$\frac{1+s}{1}$	$\frac{1+s+\frac{1}{2}s^2}{1}$	$\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$
1	$\frac{1}{1-s}$	$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$	$\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$	$\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$
2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

Padé approximations to e^s

$m \backslash k$	0	1	2	3
0	$\frac{1}{1}$	$\frac{1+s}{1}$	$\frac{1+s+\frac{1}{2}s^2}{1}$	$\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$
1	$\frac{1}{1-s}$	$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$	$\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$	$\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$
2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

L-stable L-stable A-stable

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

Padé approximations as stability function

1. Assume $k \leq m \leq k+2$

   In that case $\left|P_m^k(s)\right| \leq 1$ if $Re[s] \leq 0$

2. .$|P_m^m(j\omega)| = 1$ if $\omega \rightarrow \infty$

3. Assume $m > k$

   In that case $\left|P_m^k(s)\right| \rightarrow 0$ if $\omega \rightarrow \infty$

One step methods with stability function $R(s) = P_m^k(s)$ are:

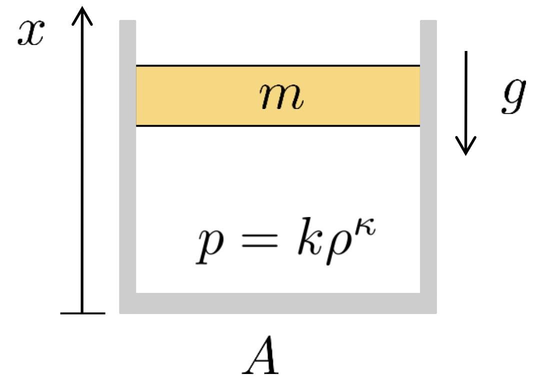
- A-stable if $k = m$
- L-stable if $m = k + 1$ or $m = k + 2$

Pneumatic spring example, again

- Model from Newton's 2nd law:

$$\ddot{x} + g(1 - x^{-\kappa}) = 0$$

  "mass-spring-damper with nonlinear spring"



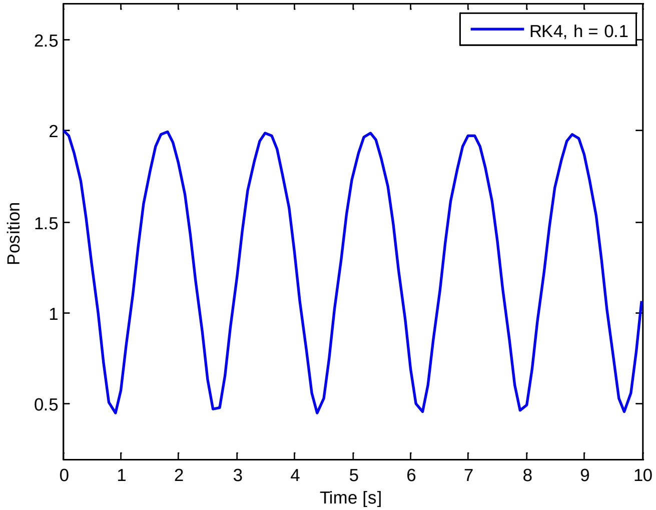
- On state-space form $\dot{y} = f(y, t)$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -g(1 - y_1^{-\kappa}) \end{pmatrix}$$

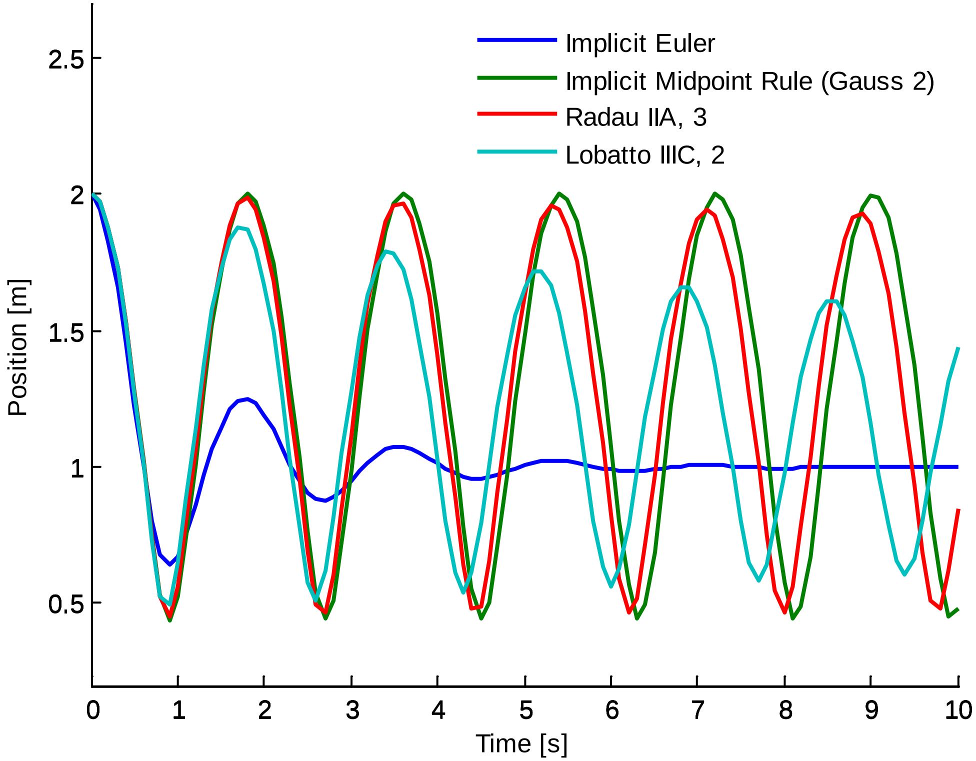
- Linearization about equilibrium:

$$\frac{\partial f}{\partial y} = \begin{pmatrix} 0 & 1 \\ -g\kappa & 0 \end{pmatrix}, \qquad \lambda_{1,2} = \pm j\omega_0, \quad \omega_0 = \sqrt{g\kappa} \approx 3.7$$

Simulation



h = 0.5 (stability limit for RK4)



Padé approximations to e^s

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2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

L-stable L-stable A-stable

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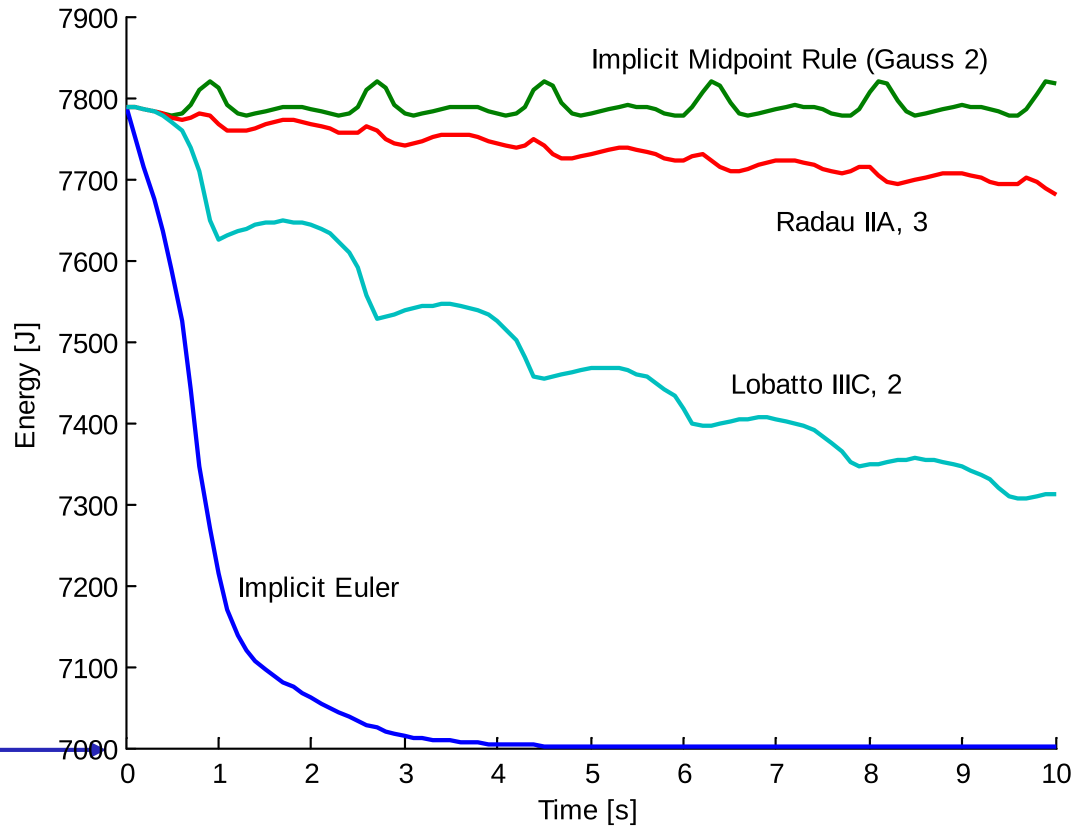
Energy



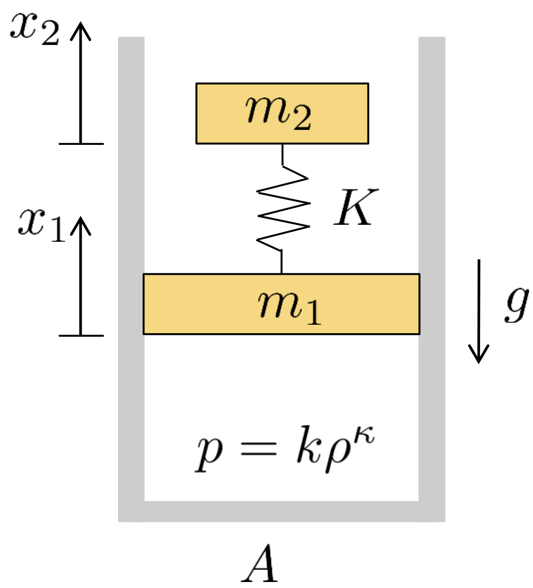
Equilibrium energy

Pneumatic spring with resonant load

- Equations of motion (Newton's law):

$$\ddot{x}_1 + g\left(1 - \frac{m_1+m_2}{m_1}x_1^{-\kappa}\right) + \frac{\omega_2^2}{2}(x_1 - x_2) = 0$$

$$\ddot{x}_2 + g + \frac{\omega_2^2}{2}(x_2 - x_1) = 0$$



- Linearization around equilibrium:

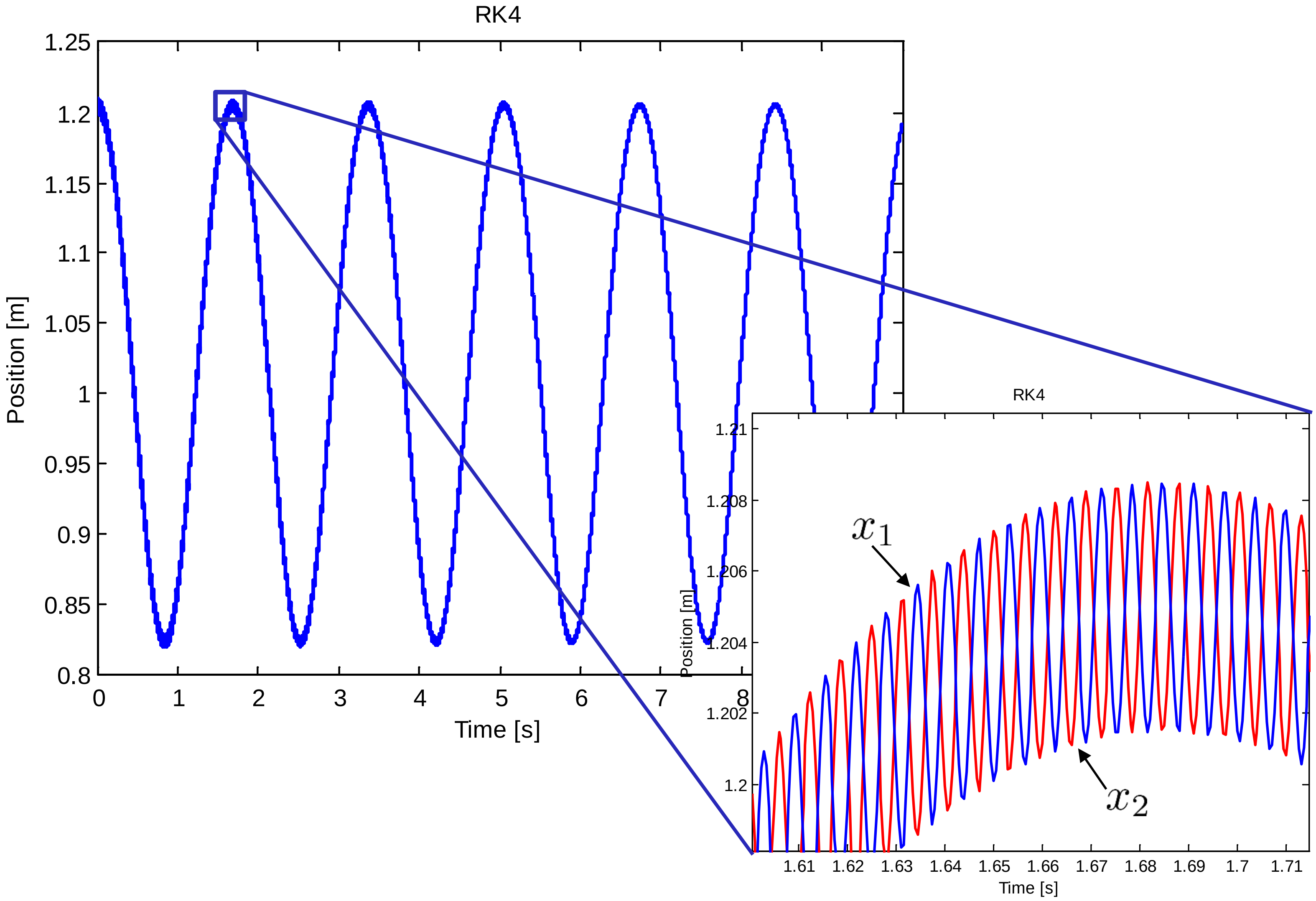
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -g\frac{m_1+m_2}{m_1}\kappa(x_1^*)^{-(\kappa-1)} - \frac{\omega_2^2}{2} & 0 & \frac{\omega_2^2}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\omega_2^2}{2} & 0 & -\frac{\omega_2^2}{2} & 0 \end{pmatrix}$$

- Eigenvalues:

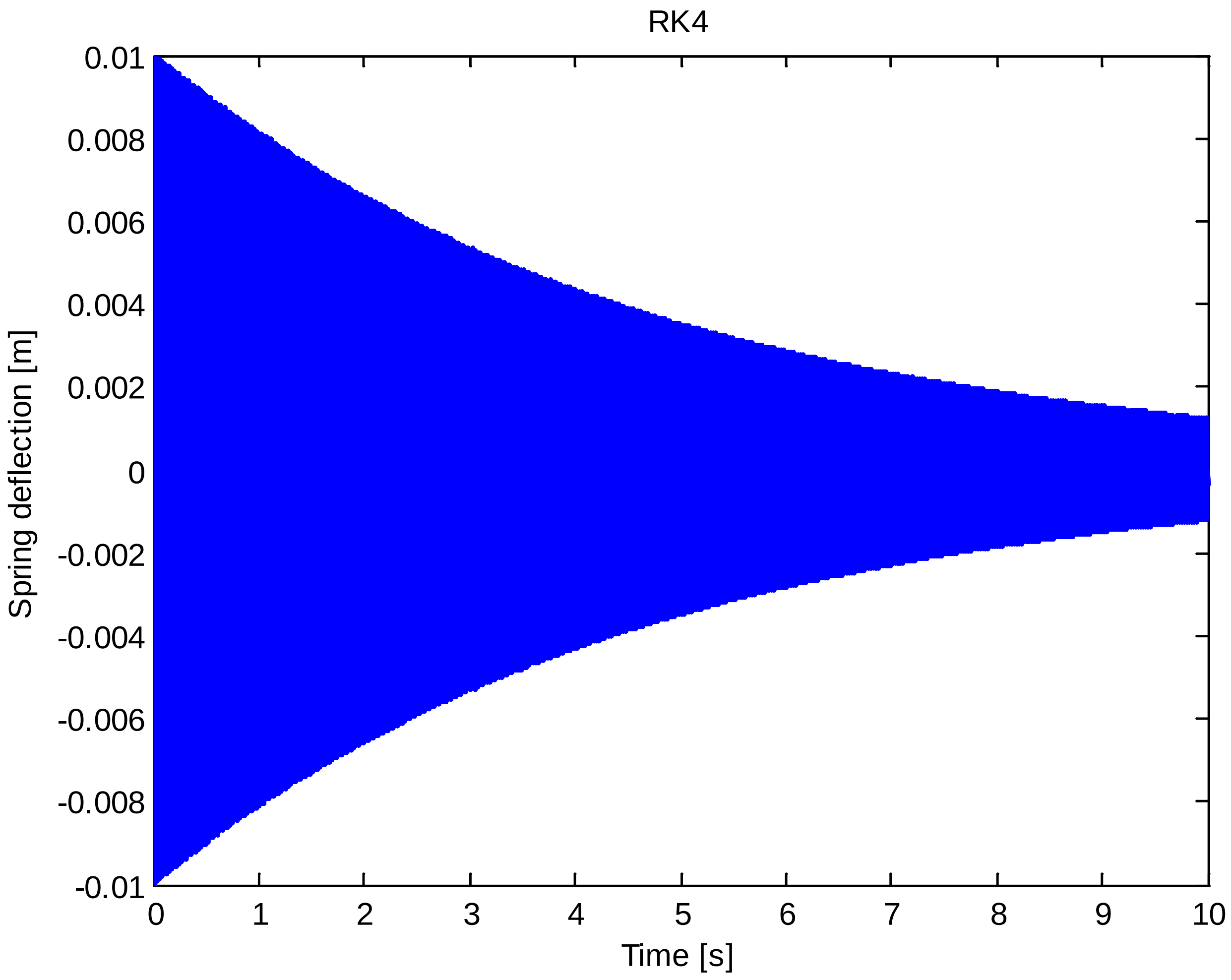
$$\lambda_{1,2} = \pm j\omega_1, \quad \omega_1 = 3.7 \text{ rad/s}$$
$$\lambda_{3,4} = \pm j\omega_2, \quad \omega_2 = 1000 \text{ rad/s}$$

Position of the two masses

RK4 with time step h = 0.0005

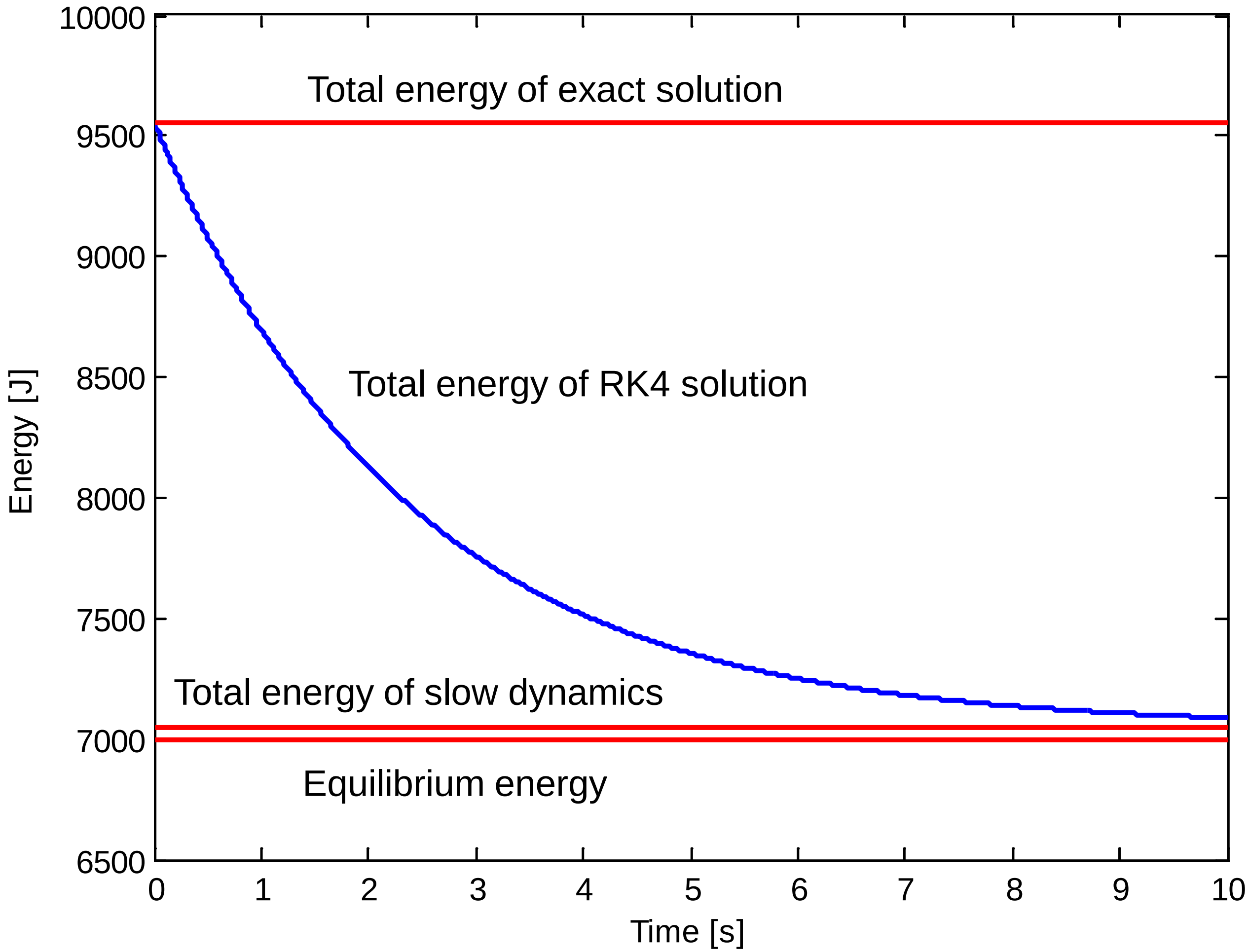


Spring deflection, RK4 with h = 0.0005



- Oscillation is lightly damped by integration method

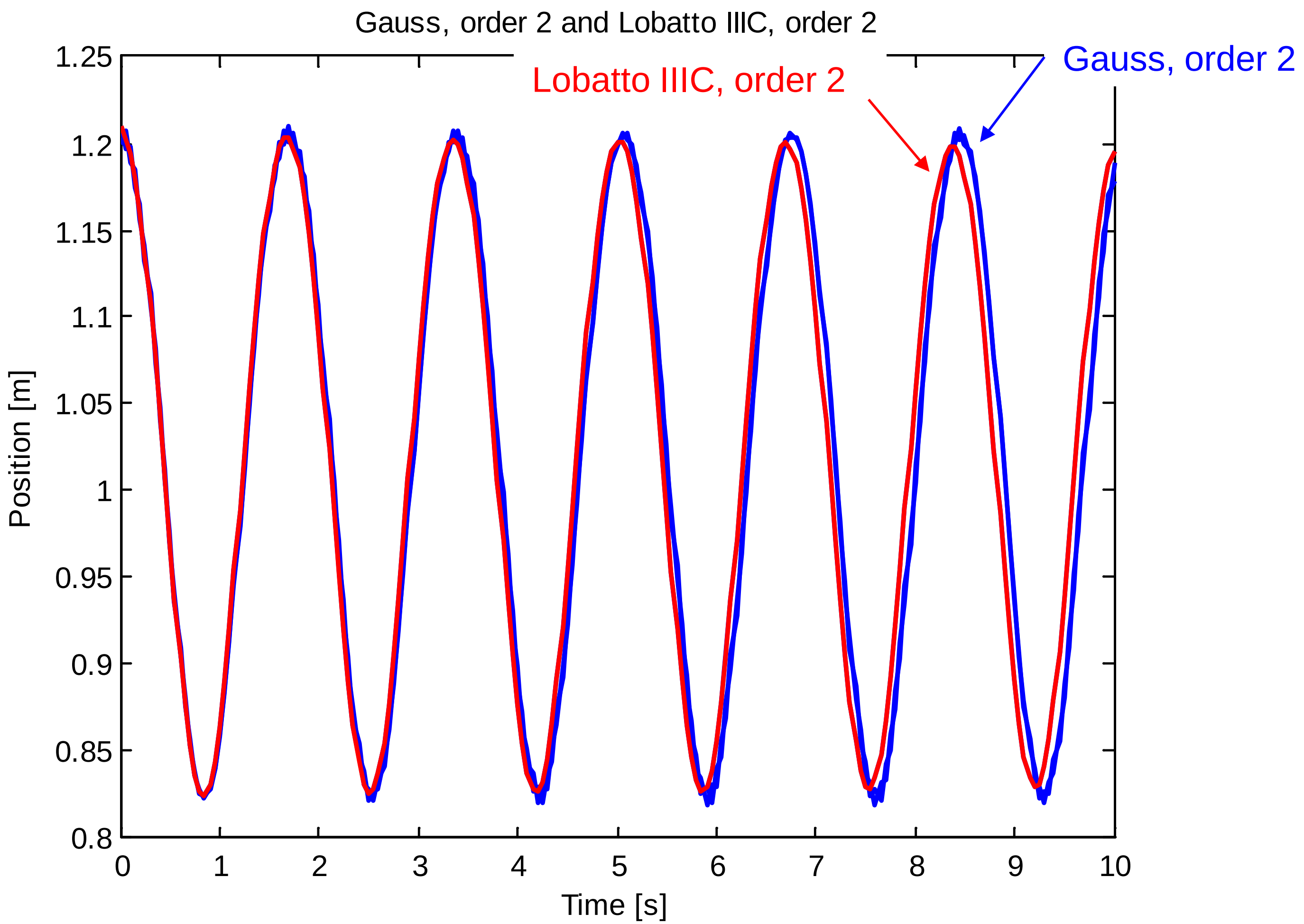
Energy of RK4 solution, h = 0.0005



- Energy related to fast dynamics slowly damped out

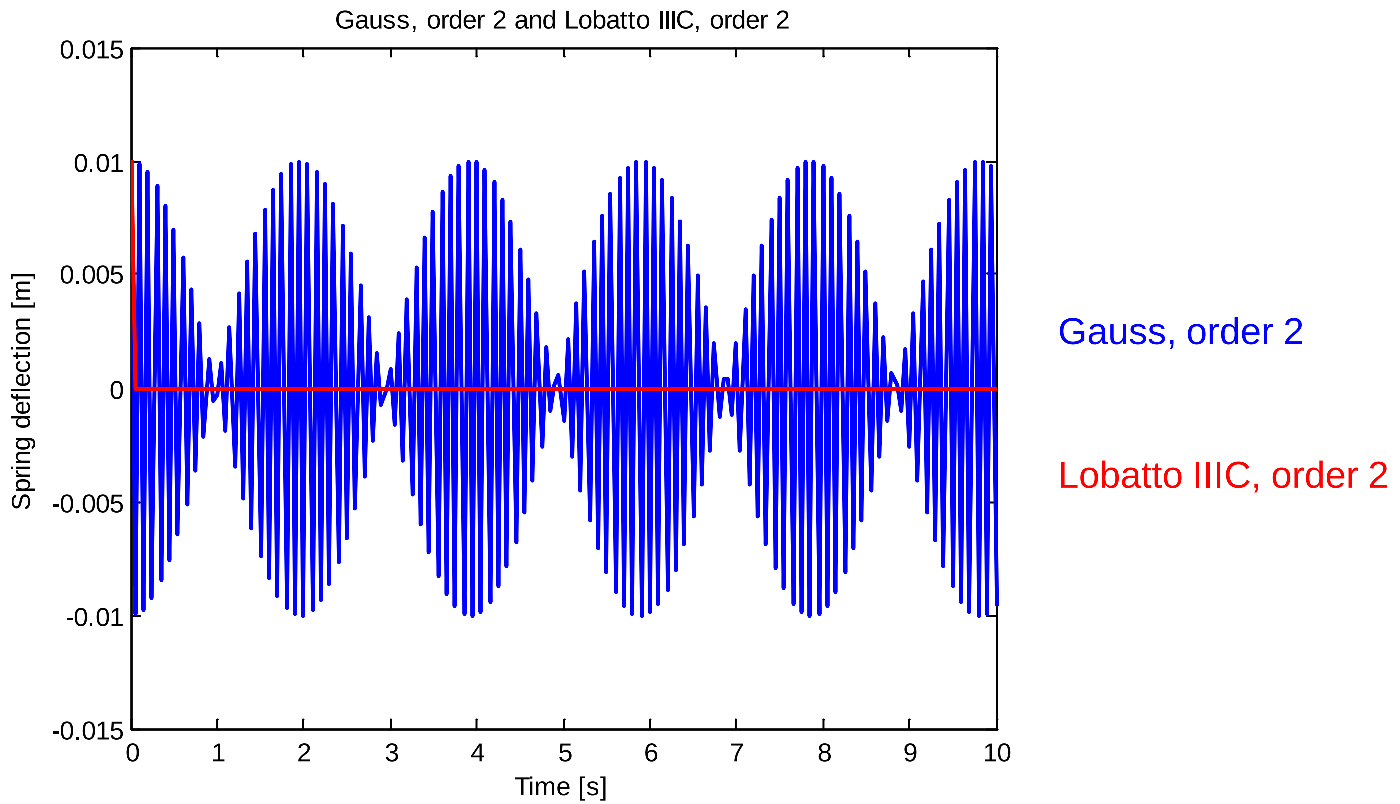
Position of the two masses

Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05



Spring deflection

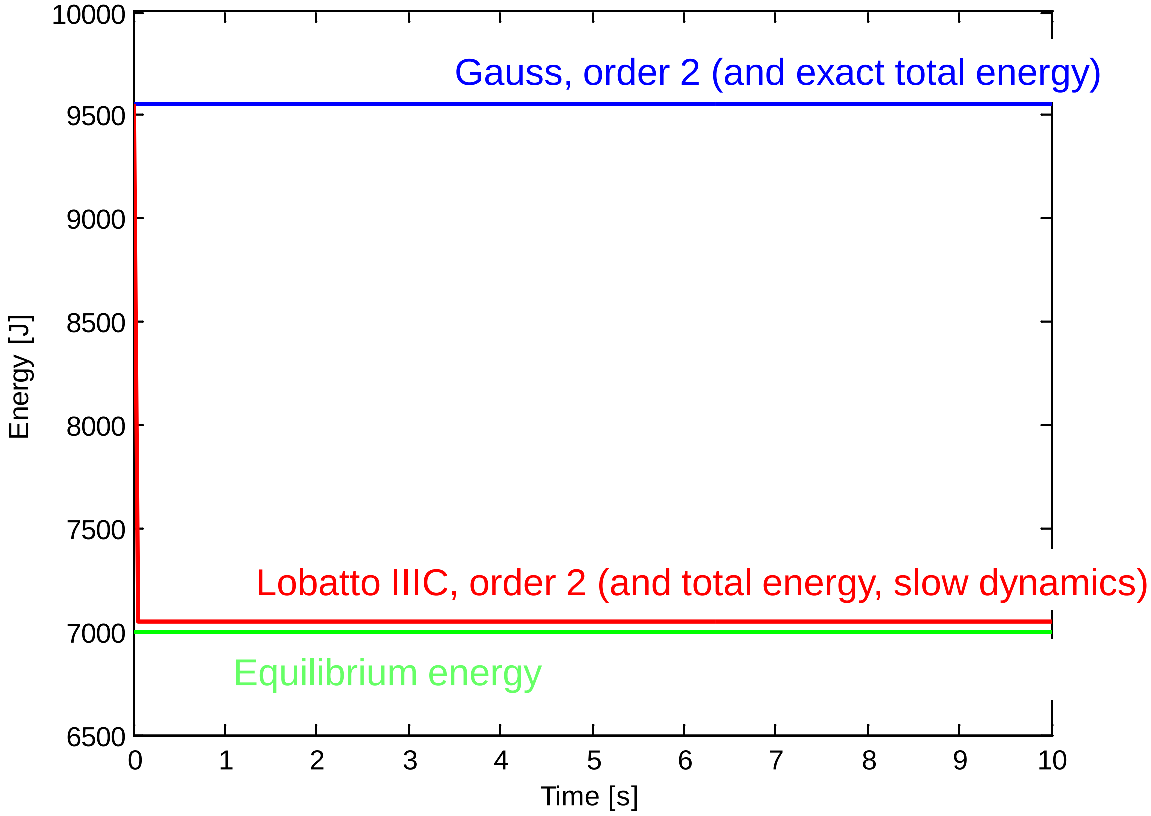
Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05



- Gauss method gives no damping, but shifts fast dynamics and energy to frequencies below Nyquist frequency, $\omega_N = \frac{\pi}{h} = \frac{\pi}{0.05} = 62.8$
- Lobatto IIIC dampens out fast dynamics in one step

Total energies

Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05



- Gauss does not dampen energies at all (same as exact total energy)
- Lobatto IIIC dampens out energy associated with fast dynamics in very few steps, to the energy of slow dynamics

Modelica

- Replacable / redeclare
- Choices

- Check out :
  - http://book.xogeny.com/components/architectures/replaceable/

Homework

- Check out the Modelica Circuit example (uploaded on Blackboard).
  - Look at the structure replaceable / redeclare
  - Read: http://book.xogeny.com/components/architectures/replaceable/

- Read 14.7.

Self-study section

Example: ”Lambert’s problem”

- IVP:

$$\dot{u} = \frac{1}{100} - (\frac{1}{100} + u + v)(1 + (u + 1000)(u + 1)), \quad u(0) = 0$$
$$\dot{v} = \frac{1}{100} - (\frac{1}{100} + u + v)(1 + v^2), \quad v(0) = 0$$

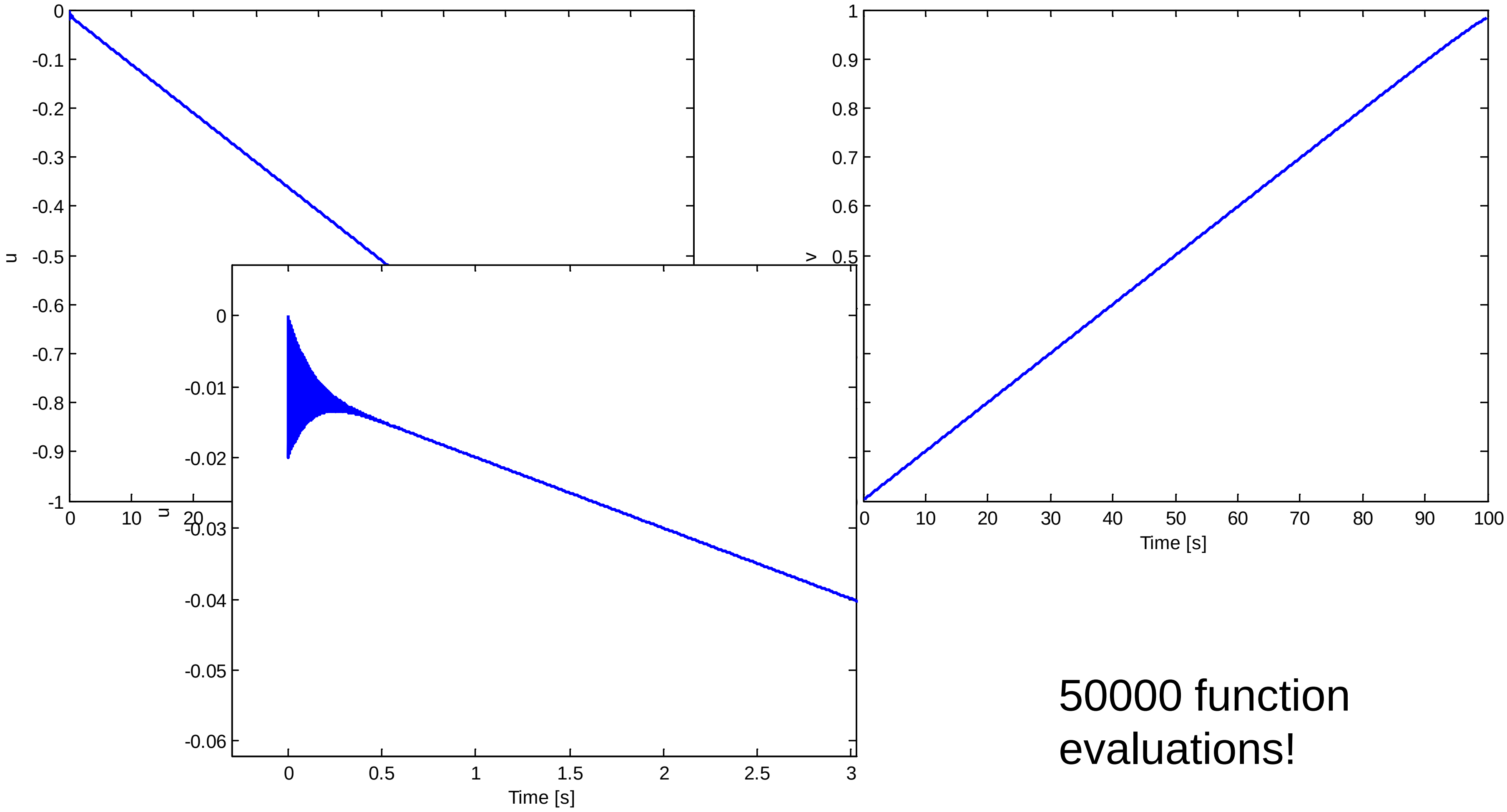
- Task: Simulate from *t* = 0 s til *t* = 100 s

- Eigenvalues:

$$(u, v) = (0, 0) \Rightarrow \lambda_1 \approx -1000, \lambda_2 \approx -0.01$$
$$(u, v) = (-.5, .5) \Rightarrow \lambda_1 \approx -500, \lambda_2 \approx -0.03$$
$$(u, v) = (-1, 1) \Rightarrow \lambda_1 \approx -11, \lambda_2 \approx -1$$

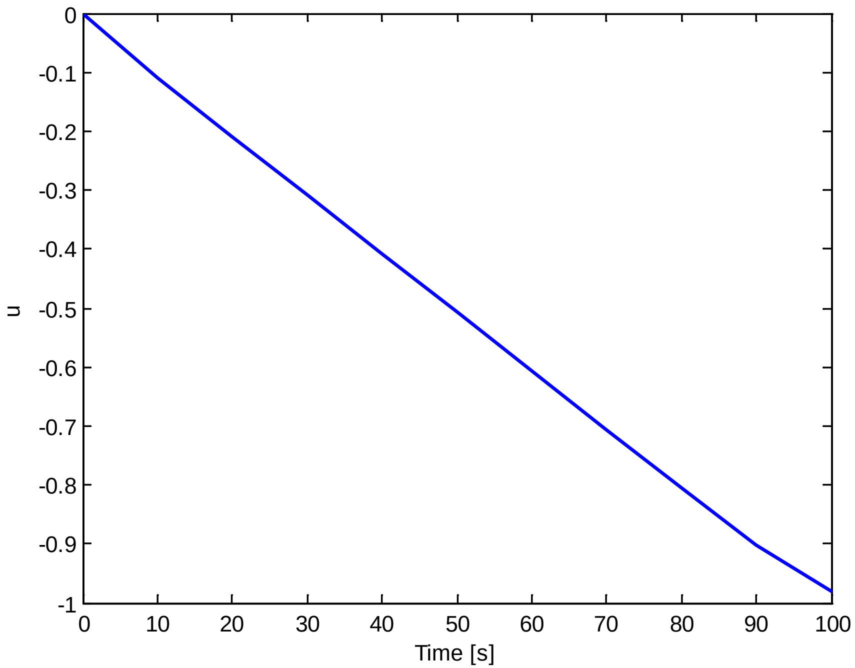
Using Euler (explicit), h = 0.002

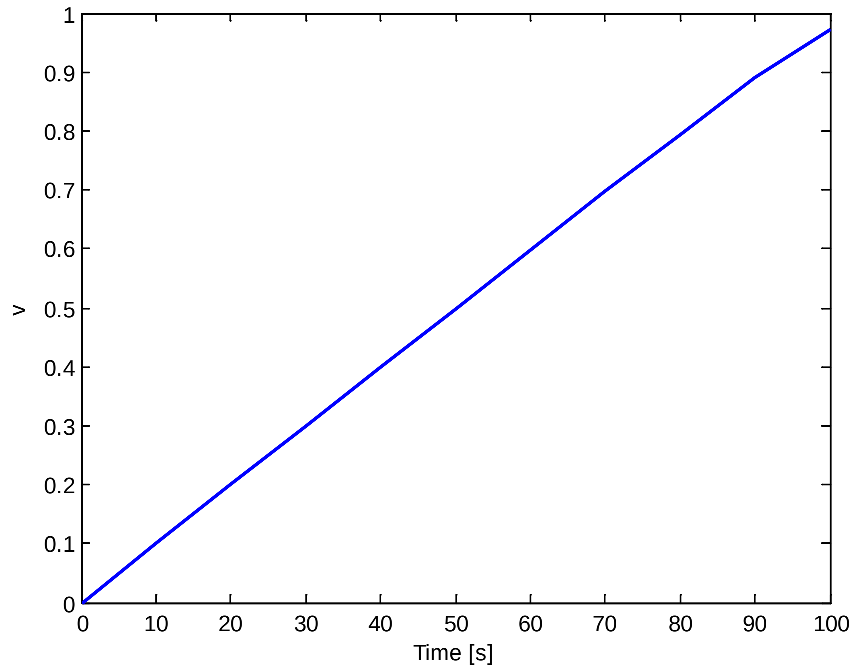
$$h|\lambda| \leq 2$$



50000 function evaluations!

Attempt 3: Euler implicit, h = 10

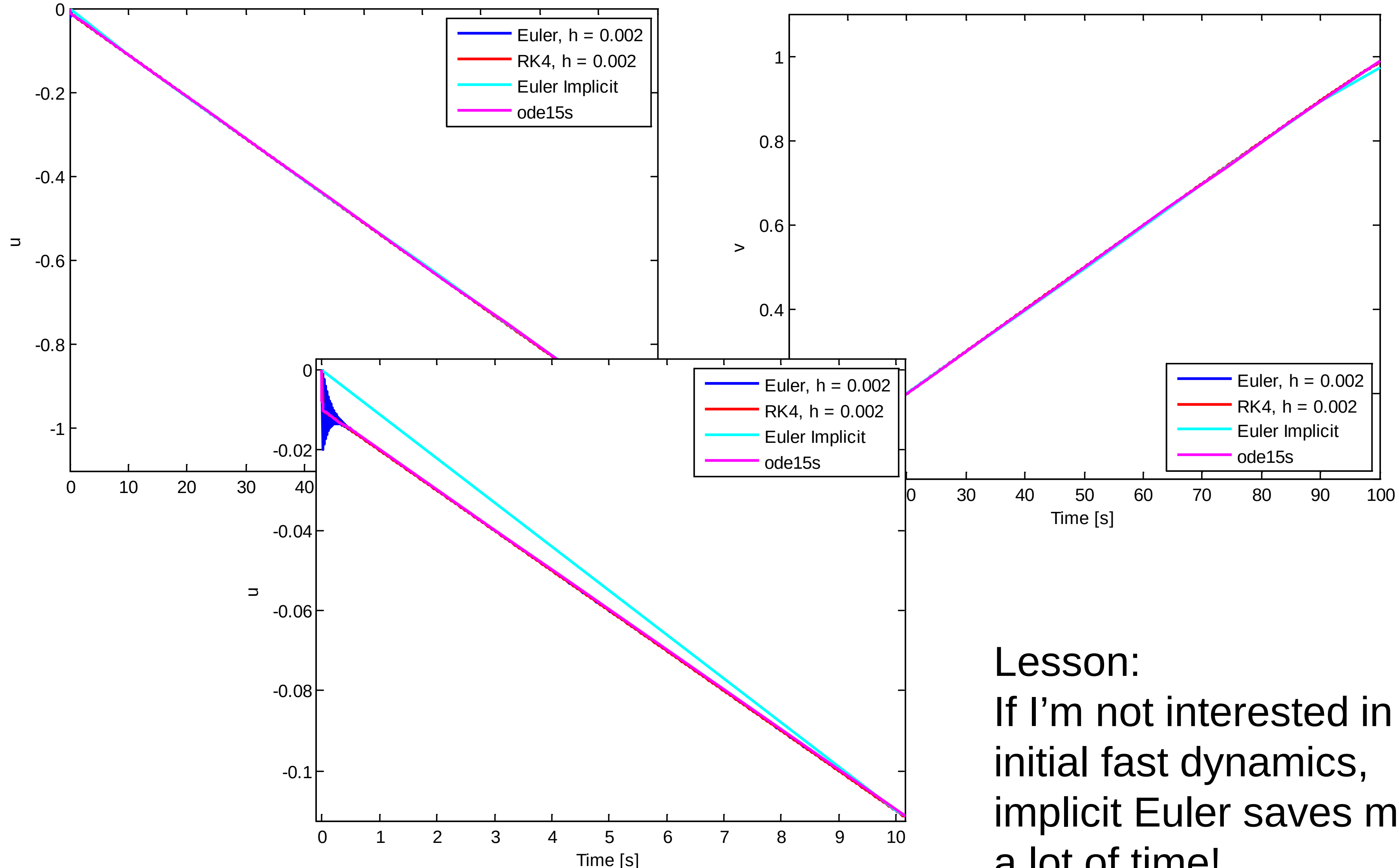




149 function evaluations!

(dependent on solution algorithm)

Comparisons



Lesson:
If I’m not interested in initial fast dynamics, implicit Euler saves me a lot of time!