Problems

1. Generate sequence $\{x[n]\}$ of $N = 10^4$ *i.i.d.* Gaussian random variables with zero mean and unit variance ($\sigma^2 = 1$). Plot 100 samples together with a constant threshold $\lambda = 0.3$. How many samples exceed the threshold? Now consider all sample values: What is the fraction that exceeds the threshold? Compare this value to the theoretical value. Plot histogram. Plot true PDF, etc. (Useful Matlab functions: hist, normpdf, normcdf, etc.)

2. Consider the following binary hypothesis testing problem

$$\begin{aligned} H_0 &: x[0] = w[0] \\ H_1 &: x[0] = w[0] + 1 \end{aligned}$$

   where $w(0)$ is $N(0,1)$. Write the LRT for the NP detector. Compute the optimal threshold to be used in the LRT so that the probability of false alarm is $P_{FA} = 10^{-3}$. What is the power of the test, i.e., what is the probability of detection P_D? How can you increase the detection performance?

3. Verify that $T(\mathbf{x}) = \sum_{n=0}^{N-1} x(n)$ is $N(0, N\sigma^2)$ under H_0 and $N(NA, N\sigma^2)$ under H_1.

4. Consider the following binary hypothesis testing problem

$$\begin{aligned} H_0 &: x[n] \sim N(0, \sigma_0^2) \\ H_1 &: x[n] \sim N(0, \sigma_1^2) \end{aligned}$$

   with $\sigma_1^2 > \sigma_0^2$. For the general case of N samples, derive the LRT and threshold associated with the NP test. For the special case of $N = 1$, illustrate the decision regions Ω_0 and Ω_1.

5. Consider the following binary hypothesis testing problem

$$\begin{aligned} H_0 &: x[n] = w[n] \\ & \qquad\qquad\qquad\qquad n = 0, 1, \ldots, N-1 \\ H_1 &: x[n] = w[n] + s[n] \end{aligned}$$

   where $\{w[n]\}$ is a sequence of *i.i.d.* Gaussian random variables with zero mean and variance σ^2, while $\{s[n]\}$ is a sequence of deterministic constants. Derive a Neyman-Pearson (N-P) optimum test with a prescribed level α. Simplify your test statistic as much as you can.

6. You are given the task to design a signal that renders the best detection performance in a WGN channel. The following two signals are suggested

$$\begin{aligned} s_0(n) &= 4 \\ & \qquad\qquad\qquad n = 0, 1, \ldots, N-1 \\ s_1(n) &= 4 \cdot (-1)^n \end{aligned}$$

   Which signal gives the best detection performance?

7. Consider the following binary hypothesis testing problem

$$\begin{aligned} H_0 &: x[n] = w[n] \\ & \qquad\qquad\qquad\qquad n = 0, 1, \cdots, N-1 \\ H_1 &: x[n] = w[n] + s \end{aligned}$$

where $\{w[n]\}$ is a sequence of *i.i.d.* random variables with a first-order pdf given by

$$p(w[n]) = \frac{a}{2} e^{-a|w[n]|} \quad a > 0$$

while s is a constant deterministic signal. Derive an N-P optimum test with a prescribed level α. Simplify your test statistic as much as you can. What if s is replaced by s_i in the alternative hypothesis? Specifically, what would be the corresponding N-P optimum test statistic?

8. In coherent binary phase shift keying, information is transmitted using two equally probable sinusoids with one out of two phases: $s_0[n] = A\cos 2\pi f n$ or $s_1(n) = A\cos(2\pi f n + \pi) = -s_0(n)$. The signals propagates over an additive white Gaussian noise channel and at the receiver we are are faced with the following detection problem:

$$\begin{aligned} H_0 : \quad & x(n) = s_0(n) + w(n) \;\; n = 0, 1, \ldots, N-1 \\ H_1 : \quad & x(n) = s_1(n) + w(n) \;\; n = 0, 1, \ldots, N-1 \end{aligned} \tag{2.64}$$

where $w(n)$ is WGN with variance σ^2.

a) Show that $\rho_s = -1$.
b) Show that $\bar{E}_s = E_{s_0} = E_{s_1} \approx A^2 N/2$.
c) Plot P_e versus the energy-to-noise-ratio (ENR), $\bar{E}/\sigma^2$, and provide the required ENR to yield an error rate of 10^{-3}.

9. In coherent frequency shift keying, information is transmitted using two equally probable sinusoids with one out of two frequencies: $s_0[n] = A\cos 2\pi f_0 n$ or $s_1(n) = A\cos 2\pi f_1 n$. The detection problem is the same as in Problem 8 above.

a) Show that $\rho_s \approx 0$ for $|f_0 - f_1| >> \frac{1}{2N}$.
b) Show that $\bar{E}_s = E_{s_0} = E_{s_1} \approx A^2 N/2$.
c) Plot P_e versus the energy-to-noise-ratio (ENR), $\bar{E}/\sigma^2$, and provide the required ENR to yield an error rate of 10^{-3}.