Out: January 14, 2019, 8:00
Deadline: January 27, 2019, 20:00

NTNU – Trondheim
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Assignment 1
TTK4130 Modeling and Simulation

About this assignment

The objective of this assignment is to give an introduction to the Modelica language, as well as to recapitulate some results and techniques that have been learned in previous courses, and that are useful -if not necessary- for solving exercises in this course.

Problem 1 handles about solving a simple differential equation with Modelica, and will serve as an introduction to this modeling language. We recommend to use the program Dymola for this exercise. However, any modeling program based on Modelica can be used, as for example openModelica.

Problems 2, 3 and 4 constitute the recapitulation part of this assignment, where the keywords are: equilibrium points, linearization, stability of linear systems, eigenvalues and eigenvectors, Jordan canonical form and linear ODEs.

Problem 1 (The Modelica language, simulation. 15 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

The differential equation

$$\dot{x} = -3x + 17, \quad x(0) = -2,$$

can be represented in Modelica by the following model:

```
model FirstOrder  "A linear 1. order diff. eq."
  // Parameters and variables
  parameter Real a = -3  "Growth rate";
  parameter Real b = 17  "Steady-state value";
  Real x(start= -2)  "State";
equation
  // The differential equation
  der(x) = a*x + b  "1. order diff. eq.";
end FirstOrder;
```

(a) What are the keywords *model*, *equation*, *end*, *parameter*, *Real*, *start*, *der* in Modelica? What are they used for?

Solution: The keyword *model* and *end* are used to indicate the start and end of the model definition, respectively.

The type *Real* is used to represent real numbers. The keyword *parameter* is used to indicate variables whose value is known before the model is simulated. This value has to be provided in the model code, but it can be modified when resimulating.

After the different variables are defined and their initial values are determined, the keyword *equation* is used to indicate the start of the section where the differential and algebraic equations of the model are defined. The keyword *der* is used to indicate the derivative of a variable.

The keyword *start* is a type attribute that is used to declare the initial value of a variable. Moreover, in the case where the initial value of a variable is given as a solution of an equation, which is solved using an iterative method, the *start* attribute gives the starting point for the iterative scheme.

(b) What are the quoted texts and the texts behind two slashes called? What are they used for?

Solution: The texts behind two slashes are usual code comments.

The quoted texts are *descriptive strings*. Unlike comments, descriptive strings can only be inserted in specific places in the code, and they provide additional documentation about the elements of the model they are associated with.

(c) Implement the Modelica code above and simulate it as it is. Add a plot with the obtained values for x to your answer.

NB: The .mo file for this model has been uploaded together with this file.

Solution: The default values of the simulation parameters will simulate the model for 1 second, and the results should be very close the actual solution:

$$x(t) = 17 - 19e^{-3t}.$$

(d) Would you obtain the same results if you replaced the keyword *parameter* with *constant* in the code above? What is the difference between these keywords?

Solution: Yes, we would obtain the same results. The keyword *constant* is used for variables whose value never changes in the scope of the model. For example, it would be natural to use this keyword for variables that represent universal constants. If a variable has the qualifier *constant*, its value can only be changed in the model code (i.e. in the ”modeling” window).

On the other hand, the value of variables with the qualifier *parameter* are only constant during simulation. Parameter variables typically affect the behaviour of the model, and how the quantities they represent affect the model, is usually the object of study. Parameter variables can be changed in the ”plotting” or ”simulation” window. Hence, it is not necessary to modify the model code. This is a nice feature to have when performing experiments.

(e) Change the simulation time, number of intervals and solver used in the simulation, as well as the parameters and initial condition of the model to the values of your choosing. Resimulate without modifying the model code.

Explain which changes you made, and add a plot with the obtained values for x.

Solution: There are infinite many correct answers. The obtained results should correspond to the chosen values.

(f) Change the growth rate parameter to a positive value. Simulate the system with an explicit (e.g. ”euler”) and an implicit solver (e.g. ”dassl”). All other simulation and model parameter values should be the same for both simulations, and they should verify the following conditions:

- There are between 20 and 100 simulation intervals per second.
- The product between the growth rate and the simulation time lies between 5 and 20.

Explain which changes you made, and add a plot with the values for x obtained from each solver.

Solution: A clear deviation between the two curves should noticeable after some time.

(g) **(Optional)** Implement and simulate a Modelica model that solves the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

for the parameter values and initial conditions of your choosing.

Add your Modelica model and a plot with the obtained results for x and $\dot{x}$ to your answer. Explain the simulation setup you used.

Solution: There are infinite many correct solutions. One of them is:

```
model SecondOrder  "A second order diff. eq."
  parameter Real zeta = 0.5  "Damping ratio";
  parameter Real omega_n = 10  "Natural resonance frequency";
  Real x(start= -2)  "State";
  Real x_dot(start=  1)  "Derivative of state";
equation
  der(x) = x_dot;
  der(x_dot) = -2*zeta*omega_n*x_dot-omega_n^2*x;
end SecondOrder;
```

Problem 2 (Equilibrium points, linearization, stability. 30 %)

Consider the systems

1.
$$\dot{x} = \begin{cases} -x - \frac{y}{\ln\sqrt{x^2+y^2}} & ,[x,y] \neq [0,0] \\ 0 & ,[x,y] = [0,0] \end{cases} \qquad \dot{y} = \begin{cases} -y + \frac{x}{\ln\sqrt{x^2+y^2}} & ,[x,y] \neq [0,0] \\ 0 & ,[x,y] = [0,0] \end{cases}$$

NB: The vector field is continuously differentiable.

2.
$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad \dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

3.
$$\dot{x} = \left(\frac{y}{1+2y+y^2} - d\right)x \qquad \dot{y} = d(4-y) - \frac{2.5xy}{1+2y+y^2},$$

where $a, b, d > 0$ are model parameters. For each system, do the following exercises:

(a) Find all equilibrium points of the system, and parameterize them as a function of the model parameters, if any.

Solution: We set $\dot{x} = \dot{y} = 0$ in order to find the equilibrium points of the system.

1. $0 = x\dot{x} + y\dot{y} = -(x^2+y^2)$. Hence, $[0,0]^T$ is the only equilibrium point of the system.
2. $\dot{y} = 0$ implies $x = 0$ or $y = 1 + x^2$.
   If $x = 0$, then $0 = \dot{x} = a > 0$, which is a contradiction.
   If $y = 1 + x^2$, then $0 = \dot{x} = a - 5x$. Hence, $x = \frac{a}{5}$ and $y = 1 + \frac{a^2}{25}$.
   Also, the only equilibrium point of the system is $[\frac{a}{5}, 1 + \frac{a^2}{25}]^T$.
3. $\dot{x} = 0$ implies $x = 0$ or $y = d(y+1)^2$.
   If $x = 0$, then $0 = \dot{y} = d(4-y)$, i.e. $y = 4$.
   If $y = d(y+1)^2$, then $0 = \dot{y} = d(4-y) - 2.5xd$, i.e. $5x + 2y = 8$.
   Furthermore, the solutions of the equation $y = d(1+2y+y^2)$ are
   $$y = \frac{-(2d-1) \pm \sqrt{(2d-1)^2 - 4d^2}}{2d} = \frac{1 - 2d \pm \sqrt{1-4d}}{2d}.$$
   These solutions are real if and only if $d \leq \frac{1}{4}$.

Hence, $[0,4]^T$ is an equilibrium point for all $d > 0$, and for $d \in (0, \frac{1}{4}]$, we have the additional equilibrium points $[x_0^+, y_0^+]$ and $[x_0^-, y_0^-]$, where

$$\begin{aligned} x_0^+ &= \frac{2}{5}(4 - y_0) & x_0^- &= \frac{2}{5}(4 - y_0) \\ y_0^+ &= \frac{1 - 2d + \sqrt{1-4d}}{2d} & y_0^- &= \frac{1 - 2d - \sqrt{1-4d}}{2d}. \end{aligned}$$

Note that $y_0^+ = y_0^- = 1$ for $d = \frac{1}{4}$.

(b) Linearize the system around each of its equilibrium points. Determine whether the linearized systems are stable, asymptotically stable or unstable.

Solution: We write the systems as $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} = [x, y]^T$. Hence, the system matrix of the linearization is obtained by evaluating the Jacobian of f at the particular equilibrium point.

1. The limit of the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{\ln\sqrt{x^2+y^2}}\right) &= \frac{1}{\ln\sqrt{x^2+y^2}} - \frac{1}{\ln\sqrt{x^2+y^2}}\left(\frac{x^2}{x^2+y^2}\right) \\ \frac{\partial}{\partial y}\left(\frac{x}{\ln\sqrt{x^2+y^2}}\right) &= -\frac{1}{\ln\sqrt{x^2+y^2}}\left(\frac{xy}{x^2+y^2}\right) \\ \frac{\partial}{\partial y}\left(\frac{y}{\ln\sqrt{x^2+y^2}}\right) &= \frac{1}{\ln\sqrt{x^2+y^2}} - \frac{1}{\ln\sqrt{x^2+y^2}}\left(\frac{y^2}{x^2+y^2}\right) \\ \frac{\partial}{\partial x}\left(\frac{y}{\ln\sqrt{x^2+y^2}}\right) &= -\frac{1}{\ln\sqrt{x^2+y^2}}\left(\frac{xy}{x^2+y^2}\right) \end{aligned}$$

when $\mathbf{x} \to 0$ is 0. Hence, the system matrix of the linearized system at $[0,0]^T$ is $-I_2$, where I_2 is the 2-by-2 identity matrix. In particular, the linearized system is asymptotically stable.

2. The Jacobian is

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -1 - 4y\frac{1-x^2}{(1+x^2)^2} & -\frac{4x}{1+x^2} \\ b\left(1 - y\frac{1-x^2}{(1+x^2)^2}\right) & -\frac{bx}{1+x^2} \end{bmatrix}$$

Evaluation of the Jacobian at $\mathbf{x}_0 = [\frac{a}{5}, 1 + \frac{a^2}{25}]^T$ gives the system matrix

$$\frac{1}{25+a^2}\begin{bmatrix} -(125 - 3a^2) & -20a \\ 2a^2 b & -5ab \end{bmatrix}$$

The characteristic polynomial of this matrix is

$$\lambda^2 + \frac{125 - 3a^2 + 5ab}{25 + a^2}\lambda + \frac{5ab(125 + a^2)}{(25 + a^2)^2}$$

Since the coefficients of the quadratic and constant terms are positive, the linearized system is asymptotically stable if and only if the coefficient of the linear term is also positive, i.e., if and only if $b > \frac{3(a^2-25)}{5a}$.

Moreover, the linearized system is unstable if and only if $b < \frac{3(a^2-25)}{5a}$.

Finally, if $b = \frac{3(a^2-25)}{5a}$, the linearized system corresponds to a harmonic oscillator. In particular, the linearized system is stable.

3. The Jacobian is

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{y}{(1+y)^2} - d & x\frac{1-y}{(1+y)^3} \\ -\frac{5}{2}\frac{y}{(1+y)^2} & -d - \frac{5}{2}x\frac{1-y}{(1+y)^3} \end{bmatrix}$$

Evaluation of the Jacobian at $\mathbf{x}_0 = [0, 4]^T$ gives the system matrix

$$\begin{bmatrix} \frac{4}{25} - d & 0 \\ -\frac{2}{5} & -d \end{bmatrix}$$

Hence, the linearized system at $[0, 4]^T$ is asymptotically stable if and only if $d > \frac{4}{25}$.

Furthermore, if $d = \frac{4}{25}$ the linearized system is only stable, and if $d < \frac{4}{25}$ it is unstable.

In order to study the stability of the linearized systems for the two curves of equilibrium points: $[x_0^+, y_0^+]^T$ and $[x_0^-, y_0^-]^T$, we use $y = y_0^{\pm}$ to express the system matrix instead of d, since the expressions for $x_0^{\pm}$ and $y_0^{\pm}$ as a function of d are complicated. This gives

$$\frac{1}{(1+y)^2}\begin{bmatrix} 0 & \frac{2}{5}\frac{(4-y)(1-y)}{1+y} \\ -\frac{5}{2}y & -\frac{2}{1+y}((1-y)^2+1) \end{bmatrix}$$

The characteristic polynomial of this matrix is

$$\lambda^2 + \frac{2}{(1+y)^3}((1-y)^2+1)\lambda + \frac{y(4-y)(1-y)}{(1+y)^5}$$

Since the coefficients of the quadratic and linear terms are positive, the linearized system is asymptotically stable if and only if the coefficient of the constant term is also positive, i.e., if and only if $y < 1$ or $y > 4$.

Moreover, the linearized system is unstable if and only if the coefficient of the constant term is negative, i.e., if and only if $1 < y < 4$.

If $y = 1$ or $y = 4$, it is immediate to verify that the eigenvalues of the system matrix are zero and a negative number. Hence, the linearized system is only stable in these cases.

Trivial algebra allows us to translate these results as a function of the parameter d:

If $d \in (0, \frac{1}{4})$, the equilibrium point $[x_0^+, y_0^+]^T$ is asymptotically stable.

If $d = \frac{1}{4}$, then $y_0^+ = y_0^- = 1$, and the equilibrium point is only stable.

If $\frac{4}{25} < d < \frac{1}{4}$, the equilibrium point $[x_0^-, y_0^-]^T$ is unstable.

If $d = \frac{4}{25}$, the equilibrium point $[x_0^-, y_0^-]^T$ is only stable.

Finally, if $d < \frac{4}{25}$, the equilibrium point $[x_0^-, y_0^-]^T$ is asymptotically stable.

(c) Based on the results from the previous part, what can be concluded about the stability of the equilibrium points of the original non-linear system?

Solution: For an equilibrium point, if the linearized system at that point is asymptotically stable, the equilibrium point is asymptotically stable for the original system as well. An analogous result is valid when the linearized system is unstable.

If the linearized system is only stable, the equilibrium point can be stable or unstable. Further study is required.

Problem 3 (Eigenvectors, Jordan canonical form, linear ODEs. 35 %)

Consider the matrices

1. $\begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & -8 & -2 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 1 & 2 & 2 \\ 1 & 1 & -1 & -1 \\ -2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 5 \end{bmatrix}$

(a) For each matrix, find all complex eigenvalues and eigenvectors.

Solution: For each matrix A, we factorize the characteristic polynomial $p(\lambda) = p_A(\lambda) = \det(A - \lambda I)$ in order to obtain the eigenvalues. Finally, for each eigenvalue λ_0, we calculate the nullspace of the matrix $A - \lambda_0 I$, $E^1(\lambda_0)$, which gives the eigenvectors associated to λ_0.

1. $p(\lambda) = (-2-\lambda)(4-\lambda)((3-\lambda)(5-\lambda)+1) = (\lambda-4)^3(\lambda+2)$.

$$E^1(-2) = \ker(A+2I) = \ker \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 7 \end{bmatrix} = \operatorname{span}\{e_3\}.$$

$$E^1(4) = \ker(A-4I) = \ker \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -6 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \operatorname{span}\{e_1\}.$$

Hence, the eigenvectors associated to -2 and 4 can be parameterized by re_3 and se_1 with $r, s \neq 0$, respectively.

2. $p(\lambda) = ((1-\lambda)(-1-\lambda)+2)((-1-\lambda)(1-\lambda)+2) = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2$.

$$\begin{aligned} E^1(i) = \ker(A-iI) &= \ker \begin{bmatrix} 1-i & 1 & 1 & 0 \\ -2 & -1-i & 0 & -1 \\ 0 & 0 & -1-i & -1 \\ 0 & 0 & 2 & 1-i \end{bmatrix} \\ &= \operatorname{span}\{e_1 - (1-i)e_2\}. \\ (A \text{ real}) \Rightarrow E^1(-i) &= \operatorname{span}\{e_1 - (1+i)e_2\}. \end{aligned}$$

Hence, the eigenvectors associated to i and $-i$ can be parameterized by $r(e_1 - (1-i)e_2)$ and $s(e_1 - (1+i)e_2)$ with $r, s \neq 0$, respectively.

3. $p(\lambda) = \lambda^4 + 2\lambda^2 + 8\lambda + 5 = (\lambda+1)^2(\lambda^2 - 2\lambda + 5) = (\lambda+1)^2(\lambda - 1 + 2i)(\lambda - 1 - 2i)$.

$$E^1(-1) = \ker(A+I) = \ker \begin{bmatrix} 1 & -8 & -2 & -5 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \operatorname{span}\{e_1 + e_2 - e_3 - e_4\}.$$

$$E^1(1+2i) = \ker(A - (1+2i)I) = \ker \begin{bmatrix} -1-2i & -8 & -2 & -5 \\ 0 & -1-2i & 1 & 0 \\ 1 & 0 & -1-2i & 0 \\ 0 & 1 & 0 & -1-2i \end{bmatrix}$$

$$= \operatorname{span}\{-(11+2i)e_1 + (1+2i)e_2 - (3-4i)e_3 + e_4\}.$$

$$(A \text{ real}) \Rightarrow E^1(1-2i) = \operatorname{span}\{-(11-2i)e_1 + (1-2i)e_2 - (3+4i)e_3 + e_4\}.$$

Hence, the eigenvectors associated to the eigenvalues -1, $1+2i$ and $1-2i$ can be parameterized by $r(e_1 + e_2 - e_3 - e_4)$, $s(-(11+2i)e_1 + (1+2i)e_2 - (3-4i)e_3 + e_4)$ and $t(-(11-2i)e_1 + (1-2i)e_2 - (3+4i)e_3 + e_4)$ with $r, s, t \neq 0$, respectively.

4. $p(\lambda) = \lambda^4 - 12\lambda^3 + 54\lambda^2 - 108\lambda + 81 = (\lambda - 3)^4$.

$$E^1(3) = \ker(A - 3I) = \ker \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -2 & -1 & -1 \\ -2 & 1 & -1 & -1 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \operatorname{span}\{e_1 + e_2 - e_3,\ e_3 - e_4\}.$$

Hence, the eigenvectors associated to 3 can be parameterized by $r(e_1 + e_2 - e_3) + s(e_3 - e_4)$ with $r \neq 0$ or $s \neq 0$.

(b) **(Optional)** For at least two of the matrices, find the real Jordan canonical form, J, and an associated similarity transformation matrix, i.e., a matrix P such that $P^{-1}AP = J$, where A is the considered matrix.

NB: Notes on the Jordan canonical form have been uploaded together with this file.

Solution: See the notes about the Jordan canonical form. We use $E^j(\lambda_i) = \ker(A - \lambda_i I)^j$.

1. Since $p(\lambda) = (\lambda - 4)^3(\lambda + 2)$ and since $E^1(-2)$ and $E^2(-2)$ have dimension 1, $E^2(4)$ and $E^3(4)$ have to be calculated:

$$E^2(4) = \ker(A - 4I)^2 = \ker \begin{bmatrix} 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{e_1,\ e_2 + e_4\} = E^1(4) \oplus \operatorname{span}\{e_2 + e_4\}.$$

$$E^3(4) = \ker(A - 4I)^3 = \ker \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -216 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{e_1,\ e_2 + e_4,\ e_4\} = E^2(4) \oplus \operatorname{span}\{e_4\}.$$

In particular, there is only one Jordan block associated to 4 and its size is 3. A set of vectors

$\{v_1, v_2, v_3\}$ that give this Jordan block can be constructed in the following way:

$$v_3 = e_4 \in E^3(4) - E^2(4).$$
$$v_2 = (A - 4I)v_3 = e_2 + e_4.$$
$$v_1 = (A - 4I)v_2 = 2e_1.$$

On the other side, the eigenvector $v_4 = e_3$ gives the only Jordan block associated to -2. Hence, a solution is

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

2. Since $p(\lambda) = (\lambda - i)^2(\lambda + i)^2$ and since $E^1(i)$ has dimension 1 and A is real, only $E^2(i)$ needs to be calculated:

$$\begin{aligned} E^2(i) = \ker(A - iI)^2 &= \ker \begin{bmatrix} -2i - 2 & -2i & -2i & -2 \\ 4i & -2 + 2i & -4 & 2i \\ 0 & 0 & -2 + 2i & 2i \\ 0 & 0 & -4i & -2 - 2i \end{bmatrix} \\ &= \operatorname{span}\{e_1 - (1 - i)e_2,\ -ie_2 + e_3 - (1 + i)e_4\} \\ &= E^1(i) \oplus \operatorname{span}\{-ie_2 + e_3 - (1 + i)e_4\}. \end{aligned}$$

Since the eigenvalues have non-zero imaginary parts, the vectors that give the real Jordan canonical form are generated by taking the real and imaginary parts of the vectors that give the complex Jordan canonical form:

$$w_2 = -ie_2 + e_3 - (1 + i)e_4 \in E^2(i) - E^1(i).$$
$$w_1 = (A - iI)w_2 = (1 - i)e_1 + 2ie_2.$$
$$u_1 = \Re(w_1) = e_1.$$
$$v_1 = \Im(w_1) = -e_1 + 2e_2.$$
$$u_2 = \Re(w_2) = e_3 - e_4.$$
$$v_2 = \Im(w_2) = -e_2 - e_4.$$

Hence, a solution is

$$J = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

3. Since $p(\lambda) = (\lambda + 1)^2(\lambda - 1 + 2i)(\lambda - 1 - 2i)$ and $E^1(-1)$ has dimension 1, $E^2(-1)$ has to be calculated:

$$\begin{aligned} E^2(-1) = \ker(A + I)^2 &= \ker \begin{bmatrix} -1 & -21 & -12 & -10 \\ 1 & 1 & 2 & 0 \\ 2 & -8 & -1 & -5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \\ &= \operatorname{span}\{3e_1 + e_2 - 2e_3,\ 2e_1 - e_3 + e_4\}. \end{aligned}$$

Hence, the vectors $\{v_1, v_2\}$ that give the Jordan block of size 2 associated to -1, can be constructed in the following way:

$$v_2 = 2e_1 - e_3 + e_4 \in E^2(-1) - E^1(-1).$$
$$v_1 = (A+I)v_2 = -e_1 - e_2 + e_3 + e_4.$$

Finally, the vectors $\{v_3, v_4\}$ that give the Jordan block associated to the complex eigenvalues $1 \pm 2i$, can be constructed by taking the real and imaginary parts of the vector $w_1 = -(11+2i)e_1 + (1+2i)e_2 - (3-4i)e_3 + e_4$, which generates $E^1(1+2i)$:

$$v_3 = \Re(w_1) = -11e_1 + e_2 - 3e_3 + e_4.$$
$$v_4 = \Im(w_1) = -2e_1 + 2e_2 + 4e_3.$$

Hence, a solution is

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & 2 & -11 & -2 \\ -1 & 0 & 1 & 2 \\ 1 & -1 & -3 & 4 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

4. Since $p(\lambda) = (\lambda - 3)^4$ and since $E^1(3)$ has dimension 2, it follows that $E^3(3) = \mathbb{R}^4$. Hence, only $E^2(3)$ has to be calculated:

$$E^2(3) = \ker(A - 3I)^2 = \ker \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & -6 & -6 & -6 \\ 0 & 3 & 3 & 3 \end{bmatrix} = \operatorname{span}\{e_1,\, e_2 - e_3,\, e_3 - e_4\}.$$

In particular, there are two Jordan block associated to 3: one of size 3 and the other of size 1. A set of vectors $\{v_1, v_2, v_3\}$ that give the Jordan block of size 3 can be constructed in the following way:

$$v_3 = e_4 \in E^3(3) - E^2(3).$$
$$v_2 = (A - 6I)v_3 = 2e_1 - e_2 - e_3 + 2e_4.$$
$$v_1 = (A - 6I)v_2 = 3e_1 + 3e_2 - 6e_3 + 3e_4.$$

The Jordan block of size 1 is given by a vector v_4 such that v_1 and v_4 span $E^1(3)$. We take $v_4 = e_3 - e_4$. Hence, a solution is

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ -6 & -1 & 0 & 1 \\ 3 & 2 & 1 & -1 \end{bmatrix}.$$

Consider now the system

$$
\begin{aligned}
\dot{x} &= x + 3z + 4y \\
\dot{y} &= -4y - 3z - x \\
\dot{z} &= -3z - 2y + x + 32u\,,
\end{aligned}
$$

where $u(t)$ is a known function.

(c) Assume $u \equiv 0$. Determine whether the linear system is stable, asymptotically stable or unstable.

Solution: For $u = 0$, the ODE can be written as $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = [x, y, z]^T$ and

$$
A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix}
$$

Some calculations give that $p_A(\lambda) = -\lambda^3 - 6\lambda = -\lambda^2(\lambda + 6)$. Since 0 is eigenvalue of A, the linear system is not asymptotically stable.
In order to determine whether the system is stable or unstable, we calculate $E^1(0)$:

$$
E^1(0) = \ker(A) = \ker \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix} = \operatorname{span}\{e_1 - e_2 + e_3\}.
$$

Since $p_A(\lambda) = -\lambda^2(\lambda + 6)$ and $E^1(0)$ has dimension less than 2, there is a Jordan block of size larger than 1 associated to 0. Hence, the linear system is unstable.

(d) **(Optional)** Find $[x(t), y(t), z(t)]^T$, $t \geq 0$ for $[x(0), y(0), z(0)]^T = [1, 2, -4]^T$ and $u(t) = te^{-2t}$.

Solution: The system can be written as $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, where $B = [0, 0, 32]^T$. Hence, the solution of this differential equation is given by

$$
\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau
$$

Since $A = PJP^{-1}$ with

$$
J = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix},
$$

it follows that, $e^{At} = Pe^{Jt}P^{-1}$, where

$$
e^{Jt} = \begin{bmatrix} e^{-6t} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.
$$

Hence,

$$
e^{At}\mathbf{x}(0) = \begin{bmatrix} 3t + e^{-6t} \\ -3t + 3 - e^{-6t} \\ 3t - 3 - e^{-6t} \end{bmatrix}
$$

and

$$\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau = P\int_0^t e^{J(t-\tau)} P^{-1} Bu(\tau)\,d\tau$$
$$= P\int_0^t \begin{bmatrix} e^{-6(t-\tau)} & 0 & 0 \\ 0 & 1 & t-\tau \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -16 \\ 16 \\ 0 \end{bmatrix} \tau e^{-2\tau}\,d\tau$$
$$= \begin{bmatrix} 4 - 3(4t+1)e^{-2t} - e^{-6t} \\ -4 + 3(4t+1)e^{-2t} + e^{-6t} \\ 4 - (4t+5)e^{-2t} + e^{-6t} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{x}(t) = \begin{bmatrix} 3t + 4 - 3(4t+1)e^{-2t} \\ -3t - 1 + 3(4t+1)e^{-2t} \\ 3t + 1 - (4t+5)e^{-2t} \end{bmatrix}, \quad t \geq 0.$$

Problem 4 (Modeling, linearization. 20 %)

An iron ball of radius R and mass m is lifted by a magnet with a coil of N turns and a current i around a core of length l_c and cross section $A = \pi R^2$. The vertical position of the ball is z, which is positive in the downwards direction. The flux ϕ flows through the iron core, then over the air gap, through the ball, and finally along the return path through the open air as shown in Figure 1. The magnetomotive force on the ball is Ni, which can be expressed as

$$Ni = \phi\left(\mathcal{R}_a + \mathcal{R}_c + \mathcal{R}_b + \mathcal{R}_r\right) \tag{1}$$

where

$$\mathcal{R}_a = \frac{z}{A\mu_0} \tag{2}$$

is the reluctance of the air gap, and $\mathcal{R}_c$, $\mathcal{R}_b$ and $\mathcal{R}_r$ are the reluctances of the core, ball and return path, respectively. The reluctances $\mathcal{R}_c$ and $\mathcal{R}_b$ are negligible, and $\mathcal{R}_r$ may be assumed to be constant as the total return path will not change significantly as the ball moves.

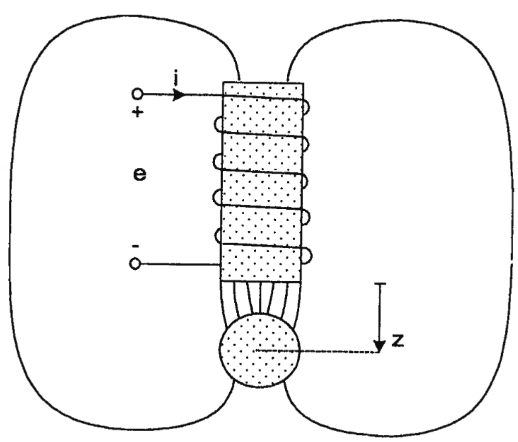

Figure 1: Magnetic levitation experiment

(a) Let the length of the return path be denoted z_0 (assumed constant), and assume a relationship such as (2) for $\mathcal{R}_r$. Write up the total magnetomotive force Ni.

Solution: With

$$R_r = \frac{z_0}{A\mu_0} \tag{3}$$

we get

$$Ni = \phi \frac{z + z_0}{A\mu_0}. \tag{4}$$

Based on the above, we can calculate the inductance

$$L(z) = \frac{N\phi}{i} = \frac{N^2 A\mu_0}{z + z_0}. \tag{5}$$

From this, the magnetic force on the ball can be found from

$$F = \frac{i^2}{2} \frac{\partial L(z)}{\partial z}. \tag{6}$$

(b) Use Newton's second law to find the equation of motion for the ball.

Solution: Doing the differentiation,

$$F = -\frac{N^2 A\mu_0}{2} \frac{i^2}{(z + z_0)^2}, \tag{7}$$

which we insert into Newton's second law to get

$$m\ddot{z} = mg - \frac{N^2 A\mu_0}{2} \frac{i^2}{(z + z_0)^2}. \tag{8}$$

(c) Linearize about a constant position z_d (and a corresponding constant current input, i_d).

Solution: The current input i_d corresponding to constant solution (steady state) z_d is found from

$$0 = m\ddot{z}_d = -\frac{AN^2\mu_0}{2} \frac{i_d^2}{(z_d + z_0)^2} + mg \tag{9}$$

giving

$$i_d = \sqrt{\frac{2mg}{AN^2\mu_0}} (z_d + z_0). \tag{10}$$

Define Δz and Δi by

$$z = z_d + \Delta z,$$
$$i = i_d + \Delta i.$$

The equations of motion linearized around $z = z_d$ becomes

$$\begin{aligned} m\Delta\ddot{z} &= \frac{AN^2\mu_0 i_d^2}{(z_d + z_0)^3} \Delta z - AN^2\mu_0 \frac{i_d}{(z_d + z_0)^2} \Delta i \\ &= \frac{2mg}{z_d + z_0} \Delta z - \frac{\sqrt{2AN^2\mu_0 mg}}{z_d + z_0} \Delta i. \end{aligned}$$

We could also have written the model on state-space form before linearization (or, equivalently, written the linearized model above on state-space form).