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NTNU – Trondheim
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Assignment 10

TTK4130 Modeling and Simulation

Problem 1 (Tank with liquid, mass balance. 20 %)

A tank with area A is filled with an incompressible liquid with constant density ρ and level h. The liquid volume is then $V = Ah$, and the mass of the liquid in the tank is $m = V\rho$.

Liquid enters the tank through a pipe with mass flow $w_i = \rho A_i v_i$, where A_i is the pipe cross section, and v_i is the velocity over this cross section, which is assumed constant. Moreover, liquid leaves the tank through a second pipe with mass flow $w_u = \rho A_u v_u$, where A_u is the cross section of this pipe, and v_u is the velocity over its cross section, which is also assumed constant.

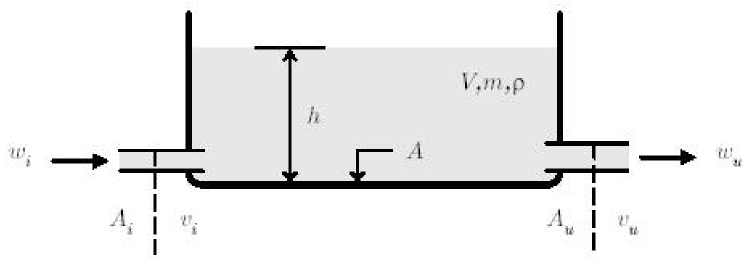

Figure 1: Tank with liquid

Use a mass balance for the tank to set up a differential equation for the level h.

Hint: Read sections 10.2, 10.4 and 11.1 in the book.

Solution: The principle of mass conservation is

$$\frac{D}{Dt}\iiint_V \rho\, dV = 0,$$

i.e. the mass is constant in a material volume. Using eq. (10.90) in the book, the liquid mass balance for a fixed volume V_f (the total volume of the tank) becomes (eq. (11.8) in the book):

$$\underbrace{\frac{D}{Dt}\iiint_{V_f} \rho\, dV}_{\substack{\text{rate of change}\\ \text{of liquid mass}\\ \text{in } V_f}} = - \underbrace{\iint_{\partial V_f} \rho \mathbf{v}^T \mathbf{n}\, dA}_{\substack{\text{net increase of mass}\\ \text{by flow in and out}\\ \text{of } V_f}} \quad .$$

(Alternatively, one could assume the liquid volume as "control volume" and use eq. (11.10) in the book.)

We have that

$$\frac{d}{dt}\iiint_{V_f} \rho\, dV = \frac{d}{dt}\rho V = \rho A \dot{h},$$

i.e. the total mass of liquid in the tank, and

$$-\iint_{\partial V_f} \rho \mathbf{v}^T \mathbf{n}\, dA = \rho v_i A_i - \rho v_u A_u,$$

i.e. the flow in and out of the tank.

Inserting this identities into the equation above gives

$$A\rho\dot{h} = \rho v_i A_i - \rho v_u A_u$$

$$\boxed{\dot{h} = \frac{A_i}{A} v_i - \frac{A_u}{A} v_u}$$

Problem 2 (Compressor, momentum balance, Bernoulli's equation. 15 %)

A compressor takes in air with pressure p_0 and velocity $v_0 = 0$ from the surroundings. The air flows through a duct into the compressor. For control purposes, it would be beneficial to have a measurement of the mass flow into the compressor. However, this measurement is not available.

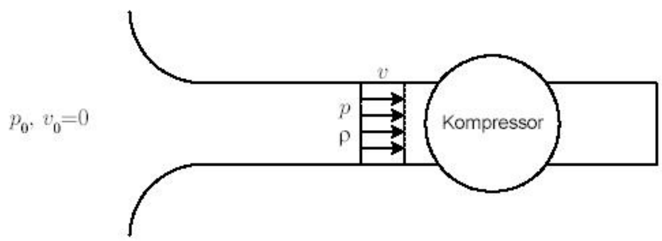

Figure 2: Compressor

Instead, there is a pressure measurement in the duct, giving a measurement p.

How can the mass flow w and velocity v be found from this measurement?

Assume that the density ρ in the duct is constant and known, that there is no friction, and that the velocity is uniform over the cross-section where the pressure transmitter is located.

Hint: Read section 11.2 in the book.

Solution: Bernoulli's equation for frictionless, incompressible flow along a streamline (eq. (11.86) in the book) relates pressure, velocity and elevation at two points on a streamline:

$$\frac{p_1 - p_0}{\rho} + \frac{1}{2}\left(v_1^2 - v_0^2\right) + (z_1 - z_0)g = 0.$$

Choosing point 1 to be the location of the pressure transmitter ($p_1 = p$, $v_1 = v$ and $z_1 = 0$) and point 0 to be the duct inlet ($p_0 = p_0$, $v_0 = 0$ and $z_0 = 0$), we get

$$p = p_0 - \frac{1}{2} v^2 \rho\,,$$

i.e.

$$\boxed{v = \sqrt{\frac{2\left(p_0 - p\right)}{\rho}}}$$

(Note that p must be smaller than p_0.)

This gives mass flow

$$\boxed{w = A\rho v = A\sqrt{2\rho\left(p_0 - p\right)}}$$

Problem 3 (Stirred tank, mass and energy balance. 30 %)

Figure 3 shows a stirred tank that cools an inlet stream. The tank is cooled by a "jacket" that contains a fluid of presumably lower temperature than the tank. The inlet stream to the tank has density ρ, temperature T_1 and mass flow rate w_1. Moreover, the outlet stream from the tank is given by

$$w_2 = Cu\sqrt{h}\,, \tag{1}$$

where C is a constant, u is the valve opening and h is the level of the liquid in the tank. Furthermore, assume that the outflow is controlled such that the level h does not exceed the height of the jacket.

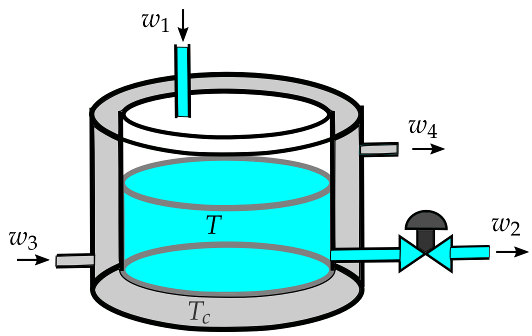

Figure 3: Tank with cooling jacket.

The inlet and outlet mass flow rates for the jacket are matched such that the jacket is always filled with fluid. In symbols, $w_3 = w_4$. Moreover, the cooling fluid has density ρ_c, and the inlet stream to the jacket has temperature T_3. Since the tank is stirred, we assume homogeneous conditions, i.e that the temperature T is the same everywhere in the tank. Similarly, we assume that the temperature T_c is the same everywhere in the jacket.

The cross-sectional area of the tank is A and the volume of the jacket is V_c.

The heat transfer from the tank to the jacket is given by

$$Q = Gh(T - T_c)\,, \tag{2}$$

where G is the heat transfer coefficient (a constant). We assume that the jacket and tank are well insulated from the surroundings, i.e. there are no other heat losses. Furthermore, we assume that both fluids are incompressible, i.e. that their specific internal energy and enthalpy can be assumed equal and proportional to the temperature, with constant of proportionality being c_p and c_{pc} for the two fluids, respectively.

Set up differential equations for the temperatures T in the tank and T_c in the jacket, and the level h in the tank.

Hint: Read sections 11.1 and 11.4 in the book.

Solution: We must first set up mass balances. For the tank, the mass balance is

$$\frac{d}{dt}(\rho A h) = w_1 - w_2\,,$$

which gives

$$\boxed{\dot{h} = \frac{1}{\rho A}\left(w_1 - Cu\sqrt{h}\right)}$$

For later use, the mass balance for the jacket is

$$\frac{d}{dt}\left(\rho_c V_c\right) = w_3 - w_4 = 0.$$

The energy balance for the tank gives

$$\begin{aligned}
\frac{d}{dt}\left(\rho c_p T A h\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{d}{dt} T + \rho c_p A T \frac{d}{dt} h &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{d}{dt} T + \rho c_p A T \frac{1}{\rho A}\left(w_1 - w_2\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{d}{dt} T &= w_1 c_p (T_1 - T) - Gh(T - T_c)
\end{aligned}$$

$$\boxed{\frac{d}{dt} T = \frac{w_1}{\rho A h}(T_1 - T) - \frac{G}{\rho c_p A}(T - T_c)}$$

Finally, the energy balance for the jacket gives

$$\begin{aligned}
\frac{d}{dt}\left(\rho_c c_{p,c} T_c V_c\right) &= w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c) \\
\rho_c c_{p,c} V_c \frac{d}{dt} T_c &= w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c).
\end{aligned}$$

Since $w_3 = w_4$, it follows that

$$\boxed{\frac{d}{dt} T_c = \frac{w_3}{\rho_c V_c}(T_3 - T_c) + \frac{Gh}{\rho_c c_{p,c} V_c}(T - T_c)}$$

Problem 4 (Mixing, reactions, mass balance. 35 %)

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then contains a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

(a) Assume that the average density ρ is constant, and set up a differential equation for the level of the tank.

Hint: Use the ordinary overall mass balance. Read section 11.1 in the book.

Solution: The mass balance equation is

$$\frac{d}{dt}\iiint_{V_f} \rho \, dV = -\iint_{\partial V_f} \rho \vec{v} \cdot \vec{n} \, dA$$

$$\frac{d}{dt}\left(\rho A h\right) = w_C - w$$

$$\boxed{\frac{d}{dt} h = \frac{w_C - w}{\rho A}}$$

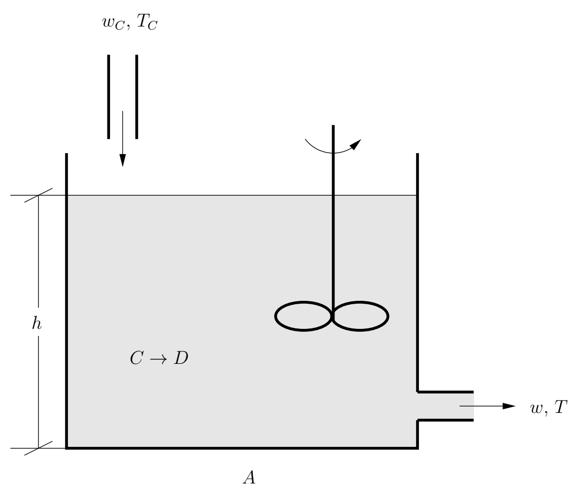

Figure 4: Tank reactor

(b) In a material volume V_m, the following holds:

$$\frac{D}{Dt}\iiint_{V_m} \rho_C \, dV = -\iiint_{V_m} J \, dV.$$

Use this together with an appropriate form of the transport theorem to show that the mass balance for substance C in integral form in a fixed control volume V_f is

$$\frac{d}{dt}\iiint_{V_f} \rho_C \, dV = -\iiint_{V_f} J \, dV - \iint_{\partial V_f} \rho_C \vec{v} \cdot \vec{n} \, dA.$$

Hint: Read section 10.4 in the book.

Solution: Set $\phi = \rho_C$ in eq. (10.90) in the book, and insert the first equation to obtain the result.

In this problem, the natural control volume is the volume of the liquid in the tank. Although this volume is not fixed, this can be ignored since the expansion of the volume does not accumulate more substance C. In symbols, $\rho_C \vec{v}_c \cdot \vec{n} = 0$.

Assume from now onwards that J is proportional to the density of substance C: $J = k\frac{m_C}{V}$. Moreover, assume that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow. In symbols, $w_{C,out} = \frac{m_C}{m_C + m_D} w$.

(c) Set up differential equations for the mass of substance C in the tank ($\frac{d}{dt} m_C = \ldots$).
Hint: Use the mass balance from part b.

Solution:

$$\frac{d}{dt}(\rho_C A h) = w_C - JV - w_{C,out}$$

$$\boxed{\frac{d}{dt} m_C = w_C - k m_C - \frac{m_C}{m_C + m_D} w}$$

(d) What is the mass balance equation in integral form for substance D in a fixed volume?
Use this to write up the mass balance for substance D.

Hint: This task is similar to the ones in part b. and part c.

Solution: For substance D, the integral mass balance is

$$\frac{D}{Dt}\iiint\limits_{V_m}\rho_D\,dV=\iiint\limits_{V_m}J\,dV.$$

Insertion into (10.90) gives

$$\frac{d}{dt}\iiint\limits_{V_f}\rho_D\,dV=\iiint\limits_{V_f}J\,dV-\iint\limits_{\partial V_f}\rho_D\vec{v}\cdot\vec{n}\,dA.$$

Solving the integrals, give

$$\boxed{\frac{d}{dt}m_D=JV-w_D=km_C-\frac{m_D}{m_C+m_D}w}$$

(e) Verify that the differential equations from part c. and part d. agree with the answer in part a.

Solution:

$$\begin{aligned}\frac{d}{dt}m&=\frac{d}{dt}m_C+\frac{d}{dt}m_D\\&=w_C-km_C-\frac{m_C}{m_C+m_D}w+km_C-\frac{m_D}{m_C+m_D}w\\&=w_C-w.\end{aligned}$$

This agrees with the solution to part a.

(f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Furthermore, assume no "heat flux", i.e. that the tank is well insulated, and that the internal energy is $u=c_pT$.
Hint: The book does not treat energy balances with "internally generated" energy. Therefore, you must derive the energy balance in integral form for this case, as you did for the mass balance.

Solution: Under the assumptions made, eq. (11.164) in the book takes the form

$$\frac{D}{Dt}\iiint\limits_{V_m}\rho u\,dV=\iiint\limits_{V_m}cJ\,dV$$

($e=u$, pressure work and heat flux ignored, but heat from reaction added.)
Insertion of this identity into (11.169) for a fixed volume gives

$$\frac{d}{dt}\iiint\limits_{V_f}\rho u\,dV=\iiint\limits_{V_f}cJ\,dV-\iint\limits_{\partial V_f}\rho u\vec{v}\cdot\vec{n}\,dA.$$

By setting $u=c_pT$ and resolving the integrals, we obtain that

$$\begin{aligned}\frac{d}{dt}\left(\rho c_pTV\right)&=cJV+w_Cc_pT_C-wc_pT\\ \rho c_pV\frac{d}{dt}T+\rho c_pTA\frac{d}{dt}h&=cJV+w_Cc_pT_C-wc_pT\end{aligned}$$

Finally, it follows from the result from part a. and $JV = km_C$ that

$$\rho c_p A h \frac{d}{dt} T + c_p T (w_C - w) = ckm_C + w_C c_p T_C - w c_p T$$

$$\boxed{\frac{d}{dt} T = \frac{ckm_C + c_p w_C (T_C - T)}{\rho c_p A h}}$$