Out: February 18, 2019, 8:00
Deadline: March 3, 2019, 20:00

NTNU – Trondheim
Norwegian University of
Science and Technology

Assignment 6
TTK4130 Modeling and Simulation

Problem 1 (Rotation matrices, dyadics, linear algebra. 50 %)

Let $a = \{O, \vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $b = \{O, \vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be two reference frames, where O is the common origin, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are the orthogonal unit vectors that give the axes of frame a, and $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are the orthogonal unit vectors that give the axes of frame b.

Consider the rotation matrix from a to b, $\mathbf{R}_b^a$.

(a) The columns of $\mathbf{R}_b^a$ are the coordinates of some particular vectors in some particular frame. What vectors and what frame are these?
Hint: Read sections 6.4.1 and 6.4.2 in the book.

> **Solution:** The first second and third column of $\mathbf{R}_b^a$ are the coordinates of $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ in the frame a, respectively.

(b) Determine whether the following identities are true or false:
Hint: Read sections 6.4.1 and 6.4.2 in the book.

$$1.\quad \mathbf{R}_b^a = \begin{bmatrix} (\mathbf{a}_1^a)^T \mathbf{b}_1^a & (\mathbf{a}_1^a)^T \mathbf{b}_2^a & (\mathbf{a}_1^a)^T \mathbf{b}_3^a \\ (\mathbf{a}_2^a)^T \mathbf{b}_1^a & (\mathbf{a}_2^a)^T \mathbf{b}_2^a & (\mathbf{a}_2^a)^T \mathbf{b}_3^a \\ (\mathbf{a}_3^a)^T \mathbf{b}_1^a & (\mathbf{a}_3^a)^T \mathbf{b}_2^a & (\mathbf{a}_3^a)^T \mathbf{b}_3^a \end{bmatrix} \qquad 2.\quad \mathbf{R}_b^a = \begin{bmatrix} \left(\mathbf{b}_1^b\right)^T \mathbf{a}_1^b & \left(\mathbf{b}_1^b\right)^T \mathbf{a}_2^b & \left(\mathbf{b}_1^b\right)^T \mathbf{a}_3^b \\ \left(\mathbf{b}_2^b\right)^T \mathbf{a}_1^b & \left(\mathbf{b}_2^b\right)^T \mathbf{a}_2^b & \left(\mathbf{b}_2^b\right)^T \mathbf{a}_3^b \\ \left(\mathbf{b}_3^b\right)^T \mathbf{a}_1^b & \left(\mathbf{b}_3^b\right)^T \mathbf{a}_2^b & \left(\mathbf{b}_3^b\right)^T \mathbf{a}_3^b \end{bmatrix} \tag{1}$$

> **Solution:** The columns of the matrix in identity 1 are the coordinates of $\vec{b}_1, \vec{b}_2, \vec{b}_3$ in the frame a. Therefore, this matrix is $\mathbf{R}_b^a$.
>
> Analogously, the matrix on the right side of identity 2 is actually $\mathbf{R}_a^b$, which is in general different from $\mathbf{R}_b^a$.
>
> In other words, 1. is correct and 2. is false.

Consider now the vectors

$$\mathbf{u}^a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{w}^b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{2}$$

and the matrix

$$\mathbf{R}_b^a = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}. \tag{3}$$

(c) Show that $\mathbf{R}_b^a$ is a rotation matrix by probing that $\mathbf{R}_b^a \in SO(3)$.
Hint: $SO(3)$ is defined in sections 6.4.2 in the book.

> **Solution:** SO(3) is defined as
>
> $$\mathrm{SO}(3) = \left\{ R | R \in \mathbb{R}^{3\times 3},\ R^T R = I, \text{ and } \det R = 1 \right\}.$$
>
> We have that
>
> $$\left(\mathbf{R}_b^a\right)^T \mathbf{R}_b^a = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

and

$$\det\left(\mathbf{R}_b^a\right) = \det\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = 1,$$

which imply that $\mathbf{R}_b^a \in \mathrm{SO}(3)$. In other words, $\mathbf{R}_b^a$ is a rotation matrix.

(d) What simple rotation does $\mathbf{R}_b^a$ represent?
Hint: "Simple rotations" are defined in section 6.4.4 in the book.

Solution: We recognize $\mathbf{R}_b^a$ to be a rotation about the y-axis:

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

where $\theta = -\pi/3$. Hence, $\mathbf{R}_b^a$ is a rotation by $-\pi/3$ about the y-axis.

(e) What is $\mathbf{R}_a^b$?

Solution: Since $\mathbf{R}_a^b = \left(\mathbf{R}_b^a\right)^T$, then

$$\mathbf{R}_a^b = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

(f) Compute $\mathbf{u}^b$ and $\mathbf{w}^a$.

Solution:

$$\mathbf{u}^b = \mathbf{R}_a^b \mathbf{u}^a = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{3}}{2} \\ 2 \\ \frac{-1+3\sqrt{3}}{2} \end{bmatrix}$$

$$\mathbf{w}^a = \mathbf{R}_b^a \mathbf{w}^b = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-2+\sqrt{3}}{2} \\ -1 \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}$$

(g) Let $\mathbf{R}$ be a rotation matrix and let $\mathbf{a}, \mathbf{b}$ be two vectors. Show that

i) $(\mathbf{R}\mathbf{a})^\times = \mathbf{R}\mathbf{a}^\times\mathbf{R}^T$.
Hint 1: The notation $(\cdot)^\times$ is defined in section 6.2.3 in the book.
Hint 2: The expression is linear in $\mathbf{a}$. *Hence, it is sufficient to prove it for a convenient right-handed orthonormal basis.*

ii) $\mathbf{R}(\mathbf{a} \times \mathbf{b}) = (\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{b})$.
What is the geometrical interpretation of this identity?
Hint 3: Use the previous result.

Solution:

i) Let $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = \mathbf{I}$ and $[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3] = \mathbf{R}^T$. Then $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a right-handed orthonormal basis. For $\mathbf{a} = \mathbf{r}_i$, we have

$$\begin{aligned}
(\mathbf{R}\mathbf{r}_1)^\times &= (\mathbf{e}_1)^\times \\
\mathbf{R}\mathbf{r}_1^\times \mathbf{R}^T &= \mathbf{R}[\mathbf{0}, \mathbf{r}_3, -\mathbf{r}_2] = (\mathbf{e}_1)^\times \\
(\mathbf{R}\mathbf{r}_2)^\times &= (\mathbf{e}_2)^\times \\
\mathbf{R}\mathbf{r}_2^\times \mathbf{R}^T &= \mathbf{R}[-\mathbf{r}_3, \mathbf{0}, \mathbf{r}_1] = (\mathbf{e}_2)^\times \\
(\mathbf{R}\mathbf{r}_3)^\times &= (\mathbf{e}_3)^\times \\
\mathbf{R}\mathbf{r}_3^\times \mathbf{R}^T &= \mathbf{R}[\mathbf{r}_2, -\mathbf{r}_1, \mathbf{0}] = (\mathbf{e}_3)^\times.
\end{aligned}$$

ii)

$$(\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{b}) = (\mathbf{R}\mathbf{a})^\times (\mathbf{R}\mathbf{b}) = \mathbf{R}\mathbf{a}^\times \mathbf{R}^T \mathbf{R}\mathbf{b} = \mathbf{R}\mathbf{a}^\times \mathbf{b} = \mathbf{R}(\mathbf{a} \times \mathbf{b}).$$

By taking $\mathbf{R} = \mathbf{R}_b^a$, $\mathbf{a} = \mathbf{u}^b$ and $\mathbf{b} = \mathbf{v}^b$, we have that

$$\mathbf{u}^a \times \mathbf{v}^a = \mathbf{R}_b^a(\mathbf{u}^b \times \mathbf{v}^b),$$

i.e. the rotation of a cross product is equal to the cross product of the rotations.

(h) Let $\mathbf{R}_b^a$ be given by

$$\mathbf{R}_b^a = \mathbf{R}_y(\theta)\mathbf{R}_z(\psi)\mathbf{R}_x(\phi) \tag{4}$$

where $\mathbf{R}_x(\phi)$, $\mathbf{R}_y(\theta)$ and $\mathbf{R}_z(\psi)$ are simple rotations as defined in Section 6.4.4 in the book. Calculate the elements in $\mathbf{R}_b^a$ as a function of the angles ψ, θ and ϕ.
Show the details of your calculations.

Solution:

$$\begin{aligned}
\mathbf{R}_b^a &= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\psi & \sin\theta\sin\phi - \cos\theta\sin\psi\cos\phi & \sin\theta\cos\phi + \cos\theta\sin\psi\sin\phi \\ \sin\psi & \cos\psi\cos\phi & -\cos\psi\sin\phi \\ -\sin\theta\cos\psi & \cos\theta\sin\phi + \sin\theta\sin\psi\cos\phi & \cos\theta\cos\phi - \sin\theta\sin\psi\sin\phi \end{bmatrix}
\end{aligned}$$

(i) Find the exact values of the elements marked with $*$ in the following rotation matrices:

$$\mathbf{R}_1 = \begin{bmatrix} * & * & * \\ * & * & 1 \\ * & 1 & * \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} -\frac{3}{5} & * & * \\ \frac{4}{5} & * & * \\ 0 & * & 1 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} \frac{1}{2} & * & * \\ * & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{2} & * & * \end{bmatrix}.$$

Show the details of your calculations.
Hint: Use that $\mathbf{R}_i \in SO(3)$.

Solution: Since $\mathbf{R}_i \in SO(3)$, then the columns of $\mathbf{R}_i$ have norm 1 and are orthogonal to each other (the scalar product is zero). This also applies to the rows of $\mathbf{R}_i$. Furthermore, these orthonormal bases are right-handed, i.e. $\det \mathbf{R}_i = \det \mathbf{R}_i^T = 1$.
These properties give

$$\mathbf{R}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}.$$

Problem 2 (Angle-axis representation, Sheeperd's method. 20 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

The rotation (or orientation) specified by a rotation matrix

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

can be represented by a rotation by an angle θ about an axis $\mathbf{k}$. This is known as the **angle-axis representation**. Furthermore, the rotation matrix $\mathbf{R}$ can be written as

$$\mathbf{R} = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\mathbf{I} + \sin\theta\mathbf{k}^{\times} + (1-\cos\theta)\mathbf{k}\mathbf{k}^{T}. \tag{5}$$

(a) Let $\mathbf{R} = \mathbf{R}_b^a$.

Show that the vectors represented by $\mathbf{k}$ in the frames a and b are the same, i.e. $\mathbf{k} = \mathbf{k}^a = \mathbf{k}^b$.

Hint: What is $\mathbf{R}_b^a\mathbf{k}$?

Solution:

$$\begin{aligned} \mathbf{R}_b^a\mathbf{k} = \mathbf{R}_{\mathbf{k},\theta}\mathbf{k} &= \cos\theta\mathbf{k} + \sin\theta\mathbf{k}^{\times}\mathbf{k} + (1-\cos\theta)\mathbf{k}\mathbf{k}^{T}\mathbf{k} \\ &= \cos\theta\mathbf{k} + (1-\cos\theta)\mathbf{k}(\mathbf{k}^{T}\mathbf{k}) = \cos\theta\mathbf{k} + (1-\cos\theta)\mathbf{k} = \mathbf{k}. \end{aligned}$$

Hence, $\mathbf{k} = \mathbf{k}^a = \mathbf{k}^b$.

When implementing control systems involving rotations (for instance for robotic manipulators or satellites), it is often desirable to find $\mathbf{k}$ and θ for a given rotation matrix.

An algorithm that can be used for achieving this, is the Shepperd's method, which is explained in section 6.7.6 in the book.

(b) Implement a Matlab function that calculates the rotation axis $\mathbf{k}$ and the rotation angle θ for an arbitrary rotation matrix $\mathbf{R}$. Add the Matlab script to your answer.

Furthermore, find the rotation axis and rotation angle for each of the rotation matrices found in problem (1.i). Are the obtained results reasonable?

Hint: Combine the results from sections 6.6.5, 6.7.1 and 6.7.6 in the book.

Solution:

```
R1 = [-1 0 0; 0 0 1; 0 1 0];
R2 = [-0.6 -0.8 0; 0.8 -0.6 0; 0 0 1];
R3 = [1/2 -sqrt(6)/4 sqrt(6)/4;
      0 sqrt(2)/2 sqrt(2)/2;
      -sqrt(3)/2 -sqrt(2)/4 sqrt(2)/4];
[theta1, k1] = getAngleAxis(R1)
[theta2, k2] = getAngleAxis(R2)
[theta3, k3] = getAngleAxis(R3)

function  [theta, k] = getAngleAxis(R)
    r11 = R(1,1);
    r22 = R(2,2);
    r33 = R(3,3);
    r00 = r11 + r22 + r33;
    [rii,i] = max([r00, r11, r22, r33]); i = i-1;
```

```
    if i == 0,
        z0 = sqrt(1 + 2*rii - r00);
        z(1) = (R(3,2)-R(2,3))/z0;
        z(2) = (R(1,3)-R(3,1))/z0;
        z(3) = (R(2,1)-R(1,2))/z0;
    elseif i == 1,
        z(1) = sqrt(1 + 2*rii - r00);
        z0 = (R(3,2)-R(2,3))/z(1);
        z(2) = (R(2,1)+R(1,2))/z(1);
        z(3) = (R(1,3)+R(3,1))/z(1);
    elseif i == 2,
        z(2) = sqrt(1 + 2*rii - r00);
        z0 = (R(1,3)-R(3,1))/z(2);
        z(1) = (R(2,1)+R(1,2))/z(2);
        z(3) = (R(3,2)+R(2,3))/z(2);
    elseif i == 3,
        z(3) = sqrt(1 + 2*rii - r00);
        z0 = (R(2,1)-R(1,2))/z(3);
        z(1) = (R(1,3)+R(3,1))/z(3);
        z(2) = (R(3,2)+R(2,3))/z(3);
    end
    % Euler parameters
    eta = z0/2;
    epsilon = [z(1); z(2); z(3)]/2;
    % Angle axis
    theta= 2*acos(eta);
    k = epsilon/(sin(theta/2));
end
```

$$\theta_1 = 3.1416\,, \quad \mathbf{k}_1 = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\,, \quad \theta_2 = 2.2143\,, \quad \mathbf{k}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\,, \quad \theta_3 = 1.2867\,, \quad \mathbf{k}_3 = \begin{bmatrix} -0.5525 \\ 0.7701 \\ 0.3190 \end{bmatrix}.$$

It is straightforward to check that the results for $\mathbf{R}_1$ are correct, i.e. that $\mathbf{k}_1$ is indeed the invariant direction of $\mathbf{R}_1$, and that $\mathbf{R}_1$ represents a rotation by $\theta_1 = \pi \simeq 3.1416$ about $\mathbf{k}_1$.

Moreover, it is immediate to check that $\mathbf{R}_2$ is a rotation by $\theta_2 = \pi - \arccos(0.6) \simeq 2.2143$ about the z-axis. Hence, $\mathbf{k}_2$ and θ_2 are correct.

Finally, one can check that $\mathbf{k}_3$ and θ_3 are correct by replacing these values in (5), and verifying that one obtains $\mathbf{R}_3$.

Problem 3 (Homogeneous transformation matrices, Denavit-Hartenberg convention. 30 %)

The Denavit-Hartenberg (D-H) convention is used to specify the relations between the different coordinate systems used in robotic manipulators. In this convention, each homogeneous transformation matrix $\mathbf{T}^i_{i+1}$ is given as the composition of four basic transformations

$$\mathbf{T}^i_{i+1} = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}, \tag{6}$$

where θ_i, d_i, a_i and α_i are parameters related to the joint i, and

$$
\begin{aligned}
\mathrm{Rot}_{z,\theta_i} &: \text{Rotation } \theta_i \text{ about } z\text{-axis} \\
\mathrm{Trans}_{z,d_i} &: \text{Translation } d_i \text{ along } z\text{-axis} \\
\mathrm{Trans}_{x,a_i} &: \text{Translation } a_i \text{ along } x\text{-axis} \\
\mathrm{Rot}_{x,\alpha_i} &: \text{Rotation } \alpha_i \text{ about } x\text{-axis}
\end{aligned}
$$

This decomposition in simpler transformations is illustrated in Figure 1.

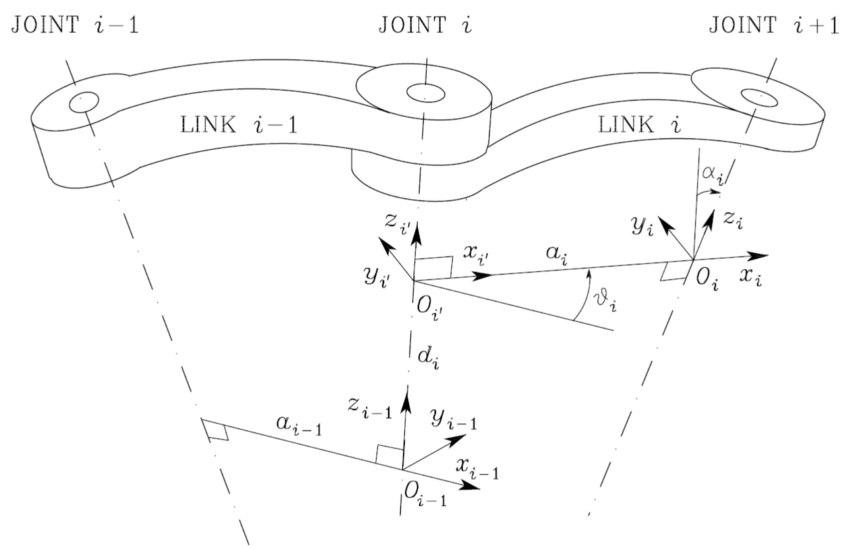

Figure 1: Illustration of transformations involved in the Denavit-Hartenberg convention.

(a) Find a general expression for $\mathbf{T}^i_{i+1}$ as a function of θ_i, d_i, a_i and α_i.

Solution:

$$
\begin{aligned}
\mathbf{T}^i_{i+1} &= \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \\
&= \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

We now want to describe the kinematics of the two manipulators shown in Figure 2.

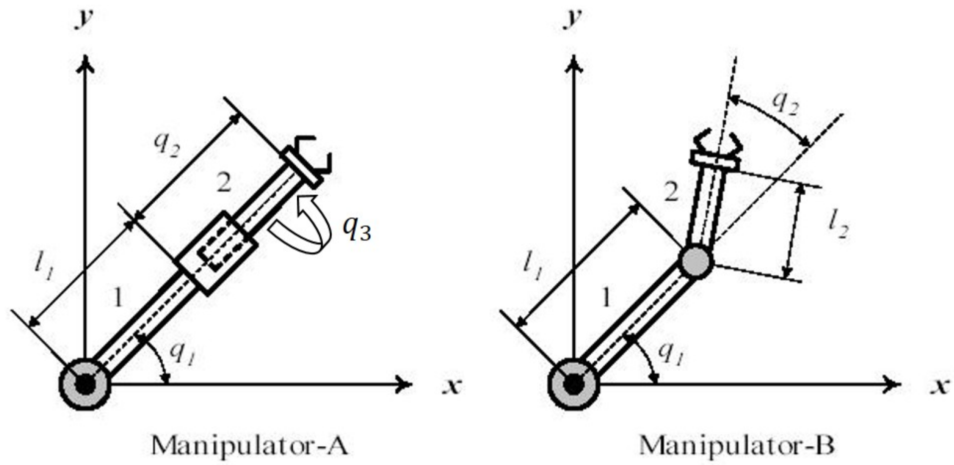

Figure 2: Two robotic manipulators.

The Denavit-Hartenberg parameters for these manipulators can be tabulated as follows:

- Manipulator A has one rotational joint and one translational joint that can also rotate (roll-rotation). The variables q_1, q_2 and q_3 are the joint variables (degrees of freedom), while l_1 is constant:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	0	0	q_2	$\pm q_3$

- Manipulator B has two rotational joints. The variables q_1 and q_2 are the joint variables, while l_1 and l_2 are constants:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	q_2	0	l_2	0

(b) Find the correct sign for the rotation angle $\pm q_3$ based on Figure 2. Justify your answer.

Solution: The correct sign is minus since the direction of the arrow for q_3 corresponds to a clockwise rotation (left-handed) about the q_1-rotated x-axis.

(c) For each of the manipulators, find the homogeneous transformation matrices for each of their joints: $\mathbf{T}_1^0$ and $\mathbf{T}_2^1$. Justify your answer.

Solution: For manipulator A:

$$\mathbf{T}_1^0 = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_2^1 = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & \cos q_3 & \sin q_3 & 0 \\ 0 & -\sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For manipulator B:

$$\mathbf{T}_1^0 = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_2^1 = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d) For each of the manipulators, find the overall transformation matrix $\mathbf{T}_2^0$.
Show the details of your calculations.

Solution: For manipulator A:

$$\begin{aligned} \mathbf{T}_2^0 &= \mathbf{T}_1^0 \mathbf{T}_2^1 \\ &= \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & \cos q_3 & \sin q_3 & 0 \\ 0 & -\sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 & -\sin q_1 \cos q_3 & -\sin q_1 \sin q_3 & (q_2 + l_1) \cos q_1 \\ \sin q_1 & \cos q_1 \cos q_3 & \cos q_1 \sin q_3 & (q_2 + l_1) \sin q_1 \\ 0 & -\sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

For manipulator B:

$$\begin{aligned} \mathbf{T}_2^0 &= \mathbf{T}_1^0 \mathbf{T}_2^1 \\ &= \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & l_1 \cos q_1 \\ \sin q_1 & \cos q_1 & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 & l_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & l_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(q_1 + q_2) & -\sin(q_1 + q_2) & 0 & l_2 \cos(q_1 + q_2) + l_1 \cos q_1 \\ \sin(q_1 + q_2) & \cos(q_1 + q_2) & 0 & l_2 \sin(q_1 + q_2) + l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(e) Let $\vec{g}$ be a vector with the following coordinates in the tool frame:

$$\mathbf{g}^2 = \begin{bmatrix} l_1 \cos q_1 & -l_1 \sin q_1 & 0 & 1 \end{bmatrix}^T. \tag{7}$$

For each of the manipulators, what are the coordinates of this vector in the base frame?
Show the details of your calculations, and make sure to simplify the result.
Hint: The coordinate system of the base frame is shown in Figure 2, while the tool frame is the frame obtained by transforming the base frame with $\mathbf{T}_2^0$.

Solution: The base and tool frames correspond to the frames denoted by 0 and 2, respectively. Since $\mathbf{T}_2^0$ transforms frame 0 to frame 2, it is the coordinate transformation matrix from frame 2 to frame 0. These considerations give:

- For manipulator A:

$$
\begin{aligned}
\mathbf{g}^0 &= \mathbf{T}_2^0 \mathbf{g}^2 \\
&= \begin{bmatrix} \cos q_1 & -\sin q_1 \cos q_3 & -\sin q_1 \sin q_3 & (q_2 + l_1) \cos q_1 \\ \sin q_1 & \cos q_1 \cos q_3 & \cos q_1 \sin q_3 & (q_2 + l_1) \sin q_1 \\ 0 & -\sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \cos q_1 \\ -l_1 \sin q_1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} l_1 \left(1 - \sin^2 q_1 (1 - \cos q_3)\right) + (q_2 + l_1) \cos q_1 \\ l_1 \sin q_1 \cos q_1 (1 - \cos q_3) + (q_2 + l_1) \sin q_1 \\ l_1 \sin q_1 \sin q_3 \\ 1 \end{bmatrix}.
\end{aligned}
$$

- For manipulator- B:

$$
\begin{aligned}
\mathbf{g}^0 &= \mathbf{T}_2^0 \mathbf{g}^2 \\
&= \begin{bmatrix} \cos(q_1 + q_2) & -\sin(q_1 + q_2) & 0 & l_2 \cos(q_1 + q_2) + l_1 \cos q_1 \\ \sin(q_1 + q_2) & \cos(q_1 + q_2) & 0 & l_2 \sin(q_1 + q_2) + l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \cos q_1 \\ -l_1 \sin q_1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} l_1 (\cos q_1 + \cos q_2) + l_2 \cos(q_1 + q_2) \\ l_1 (\sin q_1 + \sin q_2) + l_2 \sin(q_1 + q_2) \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}
$$