Lecture 18: Rigid body dynamics, summing up

- Brief recap: Newton-Euler equations of motion
- Brief recap: Lagrange’s equation of motion

- Pendulum example using both Newton-Euler and Lagrange

- Old exam(s) (using Lagrange)

Lagrange vs Newton-Euler

Newton-Euler

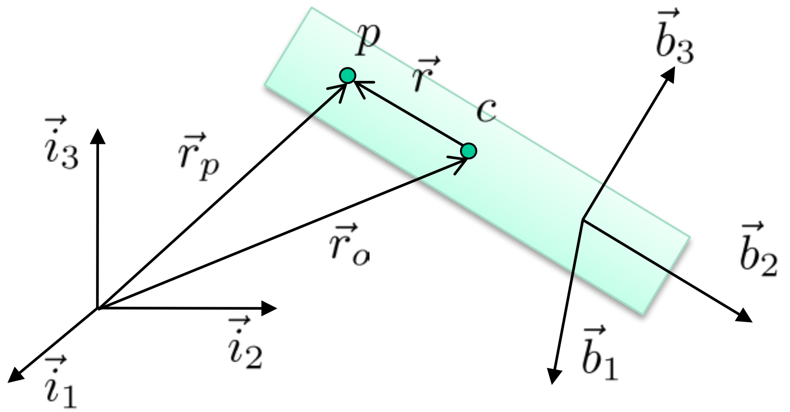
- Vectors
- Forces and moments
- Does not eliminate forces of constraints:
  - Obtains solutions for all forces and kinematic variables
  - ”Inefficient” (large DAE models)
- More general
  - Large systems can be handled (but for some configurations tricks are needed)
  - Used in advanced modeling software

Lagrange

- Algebraic
- Energy
- Eliminates forces of constraints
  - Solutions only for generalized coordinates (and forces)
  - ”Efficient” (smaller ODE models)
- Less general
  - Need independent generalized coordinates
  - Difficult to automate for large/complex problems

Newton-Euler EoM for rigid bodies

- Velocities and accelerations (Ch. 6.12)



$$\vec{v}_c := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_c, \quad \vec{v}_p := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_p$$

$$\vec{a}_c := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_c, \quad \vec{a}_p := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_p$$

$$\begin{aligned}\vec{v}_p &= \vec{v}_c + \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} \\ &= \vec{v}_c + \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ &= \vec{v}_c + \vec{\omega}_{ib} \times \vec{r}, \quad \vec{r}\text{ fixed.}\end{aligned}$$

$$\boxed{\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ib} \times \vec{u}}$$

$$\boxed{\vec{a}_p = \vec{a}_c + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}), \quad \vec{r}\text{ fixed.}}$$

- Newton-Euler equations of motion (Ch. 7.3)

$$\boxed{\begin{aligned}\vec{F}_{bc} &= m\vec{a}_c \\ \vec{T}_{bc} &= \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)\end{aligned}}$$

Lagrange equations of motion I

Generalized coordinates

- Find *n* generalized coordinates that parametrize "degrees of freedom" (allowed motion).
  - That is, all positions are function of generalized coordinates

$$\vec{r}_k = \vec{r}_k(\mathbf{q}) \qquad \mathbf{q} = \begin{pmatrix} q_1 & q_2 & \dots & q_n \end{pmatrix}^\mathsf{T}$$

- Differentiate to find velocity

$$\vec{v}_k(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}_k(\mathbf{q}) = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\dot{q}_i$$

  - For rigid bodies: velocity of center(s) of mass, and also angular velocity $\vec{\omega}_{ib}(\mathbf{q}, \dot{\mathbf{q}})$

- Find the generalized (actuator) forces τ_i associated with q_i
  - If q_i angle, then τ_i torque
  - If q_i displacement, then τ_i force

$$\tau_i = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k$$

- On coordinate form:

$$k = 1, \dots, N \text{ particles: } \mathbf{r}_k^i(\mathbf{q}), \quad \mathbf{v}_k^i(\mathbf{q}, \dot{\mathbf{q}})$$

$$k = 1, \dots, N \text{ rigid bodies: } \mathbf{r}_{ck}^i(\mathbf{q}), \quad \mathbf{v}_{ck}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \boldsymbol{\omega}_{ik}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{M}_{k/c}^b$$

Lagrange equations of motion II

Kinetic and potential energy

- Find kinetic energy:
  - N particles: $T = \sum_{k=1}^{N} \frac{1}{2} m_k \vec{v}_k \cdot \vec{v}_k$
  - Each rigid body (p. 273):

$$T = \int_b \frac{1}{2}\vec{v}_p \cdot \vec{v}_p \mathrm{d}m = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{\omega}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{\omega}_{ib}$$

- On coordinate form:

$$N \text{ particles: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_k^i)^\mathsf{T} \mathbf{v}_k^i = \frac{1}{2} m_k (\mathbf{v}_k^b)^\mathsf{T} \mathbf{v}_k^b$$

$$N \text{ rigid bodies: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_{ck}^b)^\mathsf{T} \mathbf{v}_{ck}^b + \frac{1}{2}(\boldsymbol{\omega}_{ik}^b)^\mathsf{T} \mathbf{M}_{k/c}^b \boldsymbol{\omega}_{ik}^b$$

- Find (total) potential energy $U = U(\mathbf{q}) = \sum U_k(\mathbf{q})$
  - Gravity: $U_k(\mathbf{q}) = m_k g h(\mathbf{q})$
  - Spring: $U_k(\mathbf{q}) = \frac{1}{2} k x^2(\mathbf{q})$
  - ...

Lagrange equations of motion III

- Construct Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q})$$

- Find 2*n* partial derivatives (scalars)

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \frac{\partial \mathcal{L}}{\partial q_i}$$

- Write up *n* equations of motion
  - That is, *n* 2nd order differential equations

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i}$$

Robotic manipulator 8.2.8

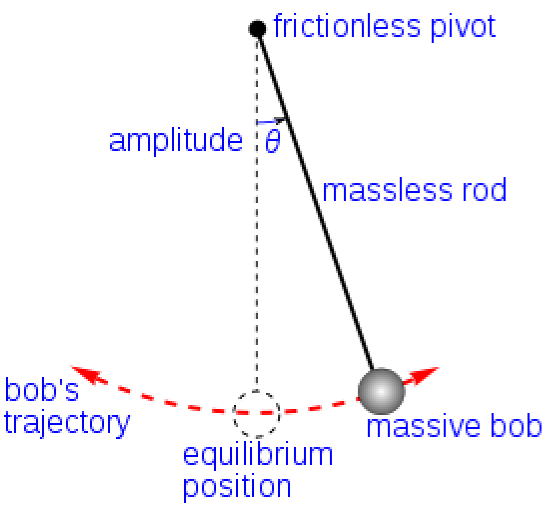
$$T_k = \frac{1}{2} m_k (\underline{v}_{ck}^{i})^T \underline{v}_{ck}^{i} + \frac{1}{2} (\omega_{ik}^{i})^T M_{c/k}^{i} \omega_{ik}^{i}$$

function of $\underline{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_2 \end{pmatrix}$

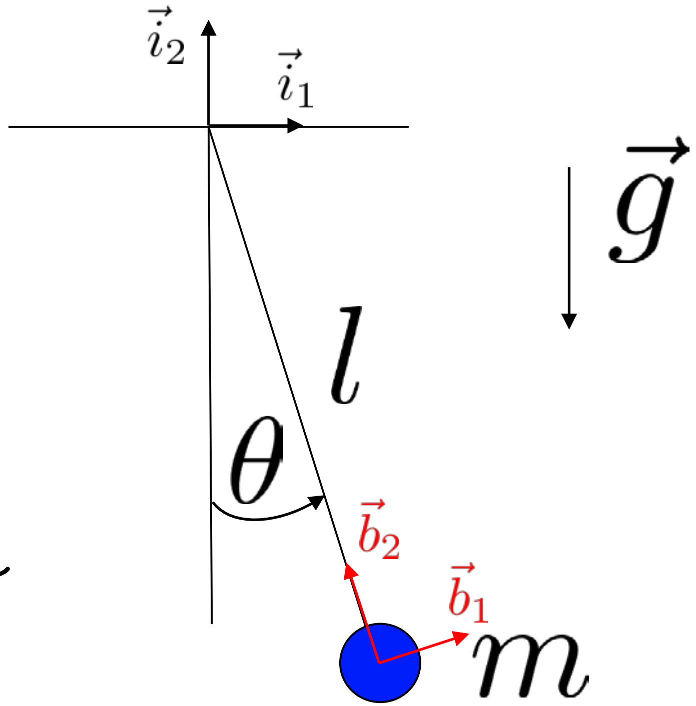
$$\Rightarrow \quad M(\underline{q})\ddot{q} + C(q,\dot{q})\dot{q} + \underline{g}(\underline{q}) = \tau$$

Example: Pendulum

- Pendulum (bob) as particle:
  - Using Newton-Euler EoM, in inertial and body system
  - Using Lagrange EoM
- Pendulum as rigid body
  - Using Lagrange EoM



Example: Pendulum



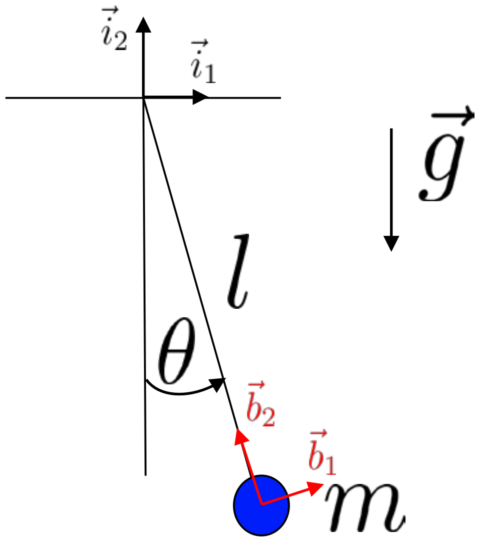
$$\vec{r} = x \cdot \vec{i}_1 - y\,\vec{i}_2$$

$$\vec{b}_1 = \cos\theta\,\vec{i}_1 + \sin\theta\,\vec{i}_2$$

$$\vec{b}_2 = -\sin\theta\,\vec{i}_1 + \cos\theta\,\vec{i}_2$$

Newton-Euler: $m \dfrac{d^2\vec{r}}{dt^2} = \delta\vec{b}_2 - mg\,\vec{i}_2$

Example: Pendulum - inertial



$$m(\ddot{x}\,\vec{i}_1 - \ddot{y}\,\vec{i}_2) = \delta(-\sin\theta\,\vec{i}_1 + \cos\theta\,\vec{i}_2) - mg\,\vec{i}_2$$

$$\vec{i}_1: \quad m\ddot{x} = -\delta\sin\theta \qquad (1)$$

$$\vec{i}_2: \quad -m\ddot{y} = \delta\cos\theta - mg \qquad (2)$$

$$\left[\to \sin\theta = \frac{x}{l} \; ; \; \cos\theta = y/l\right]$$

$x = l\sin\theta$

$\dot{x} = l\cos\theta\,\dot{\theta}$

$\ddot{x} = -l\cos\theta\,\ddot{\theta} - l\sin\theta\,\dot{\theta}^2$

In addition: $x^2 + y^2 = l^2$ (3)

$\to$ DAE - system

Differential index I

$$m\ddot{x} = -\delta \sin\theta$$
$$-m\ddot{y} = \delta \cos\theta - mg$$
$$x^2 + y^2 = l^2 \Leftarrow$$

- How many diff. variables?

$x, \dot{x}, y, \dot{y}$

- How many alg. variables?

δ

$$\frac{\partial g(\delta)}{\partial \delta} = 0$$

→ not diff. index 1

→ Find diff. index by taking time-derivative of algebraic equation(s)

Differential index II

$$m\ddot{x} = -\delta \sin\theta$$

$$-m\ddot{y} = \delta \cos\theta - mg$$

$$x^2 + y^2 = l^2$$

1st diff: $0 = 2x\dot{x} + 2y\dot{y}$

$= x\dot{x} + y\dot{y}$

2nd diff: $0 = \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}$

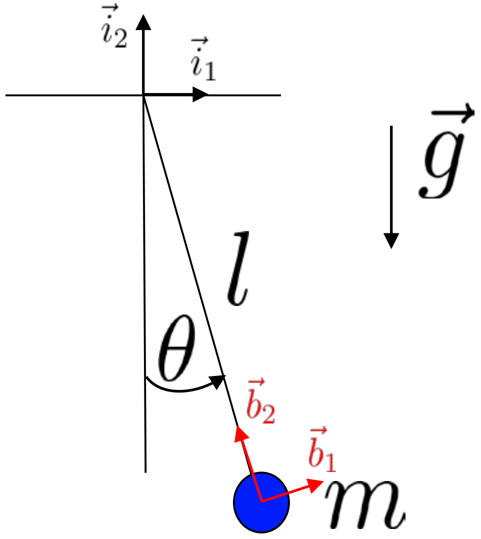
$$0 = \dot{x}^2 + x\left(-\frac{\delta}{m}\frac{x}{l}\right) + \dot{y}^2 + y\left(-\frac{\delta}{m}\frac{y}{l} + g\right)$$

$$\delta = ml\,\frac{\dot{x}^2 + \dot{y}^2 + yg}{x^2 + y^2}$$

3rd diff: $\dot{\delta} = \ldots$

$\rightarrow$ diff Index: 3

Example: Pendulum – body I



$$\vec{r} = -l\vec{b}_2$$

$$\vec{\omega}_{ib} = \dot{\theta}\vec{i}_3 = \dot{\theta}\vec{b}_3$$

$$\frac{{}^i d}{dt}\vec{r} = \frac{{}^b d}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r}$$

$$= 0 + \dot{\theta}\vec{b}_3 \times (-l\vec{b}_2)$$

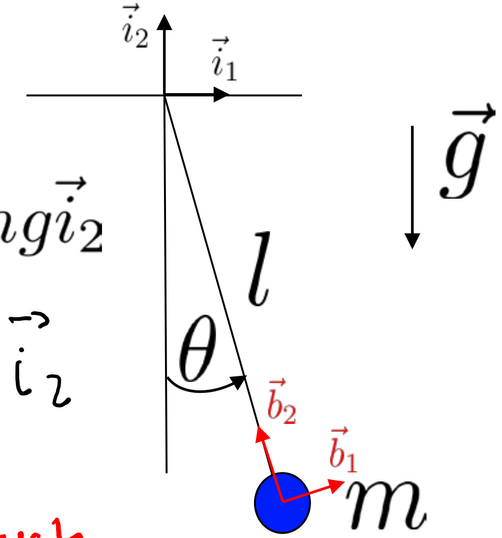
$$= l\dot{\theta}\vec{b}_1$$

$$\frac{{}^i d^2}{dt^2}\vec{r} = \frac{{}^i d}{dt}(l\dot{\theta}\vec{b}_1)$$

$$= \frac{{}^b d}{dt}(l\dot{\theta}b_1) + (\dot{\theta}\vec{b}_3) \times (l\dot{\theta}\vec{b}_1)$$

$$= l\ddot{\theta}\vec{b}_1 + l\dot{\theta}^2\vec{b}_2$$

Example: Pendulum – body II



$$m\frac{{}^i d^2}{dt^2}\vec{r} = \delta\vec{b}_2 - mg\vec{i}_2$$

$$m\left(l\ddot{\theta}\vec{b}_1 + l\dot{\theta}^2\vec{b}_2\right) = \delta\vec{b}_2 - mg\vec{i}_2$$

- $\vec{b}_1$: $\boxed{ml\ddot{\theta} = -mg\sin\theta}$ ← enough for θ
- $\vec{b}_2$: $ml\dot{\theta}^2 = \delta - mg\cos\theta$

← can be used to calculate δ

Example: Pendulum – Lagrange

generalized coordinate : θ

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m (l\dot{\theta})^2$$

$$U = mgh = -mgl\cos\theta$$

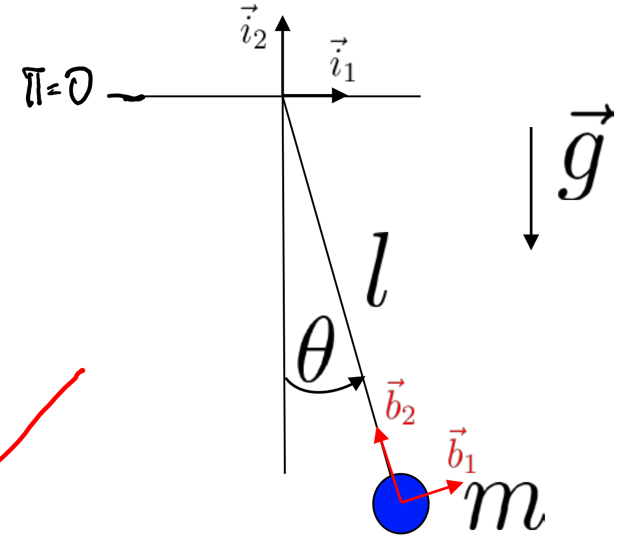
$$\mathcal{L} = T - U \; ; \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} =$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$l\ddot{\theta} + g\sin\theta = 0$$



$$\vec{v} = l\dot{\theta}\,\vec{b}_1$$

$$\vec{r} = l\sin\theta\,\vec{i}_1 - l\cos\theta\,\vec{i}_2$$

$$\vec{v} = l\cos\theta\,\dot{\theta}\,\vec{i}_1 + l\sin\theta\,\dot{\theta}\,\vec{i}_2$$

$$\vec{v} \cdot \vec{v} = l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2$$

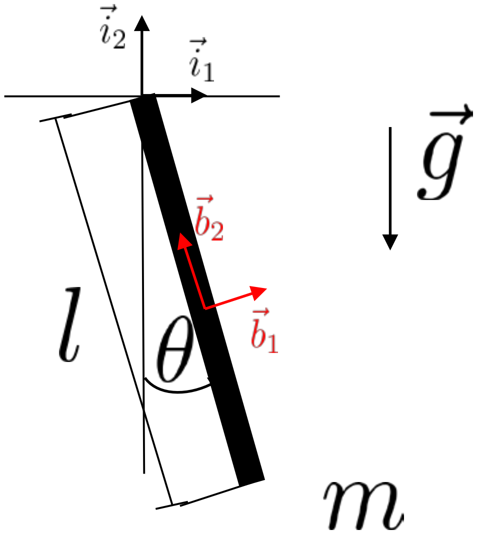
$$= l^2\dot{\theta}^2$$

Rigid-body pendulum with Lagrange I

generalised coordinate : θ

$$\vec{v}_c = \frac{l}{2}\dot{\theta}\,\vec{b}_1 \;;\quad \underline{v}_c^{b} = \begin{bmatrix} l/2\,\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

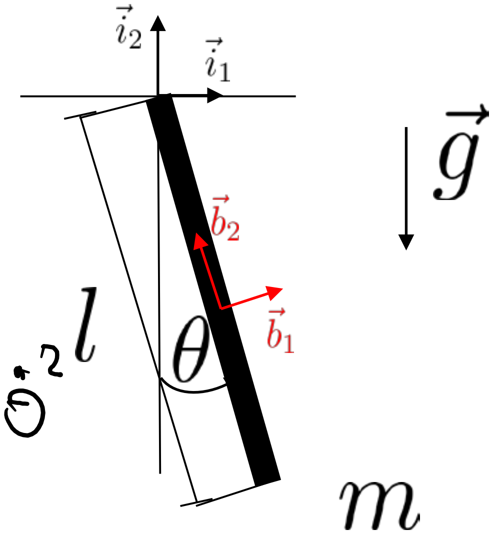
$$\vec{\omega}_{ib} = \dot{\theta}\,\vec{i}_3 = \dot{\theta}\,\vec{b}_3 \;;\quad \omega_{ib}^{b} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$



$$M_{b/c}^{b} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & I_z \end{bmatrix}$$

$$I_z = \frac{1}{12} m l^2$$

Rigid-body pendulum with Lagrange II



$$T = \frac{1}{2} m \left(\underline{v}_c^b\right)^T \underline{v}_c^b + \frac{1}{2} \left(\underline{\omega}_{ib}^b\right)^T M_{b/c}^b \, \underline{\omega}_{ib}^b$$

$$= \frac{1}{2} m \frac{l^2}{4} \dot{\theta}^2 + \frac{1}{2} I_z \dot{\theta}^2$$

$$= \frac{1}{2} \left(m/4 + I_z/l^2 \right) l^2 \dot{\theta}^2 = \frac{1}{6} m l^2 \dot{\theta}^2$$

$$U = m g h_c = -m g \, l/2 \cos\theta$$

$$\mathcal{L} = T - U = 1/6 \, m l^2 \dot{\theta}^2 + m g \, l/2 \cos\theta$$

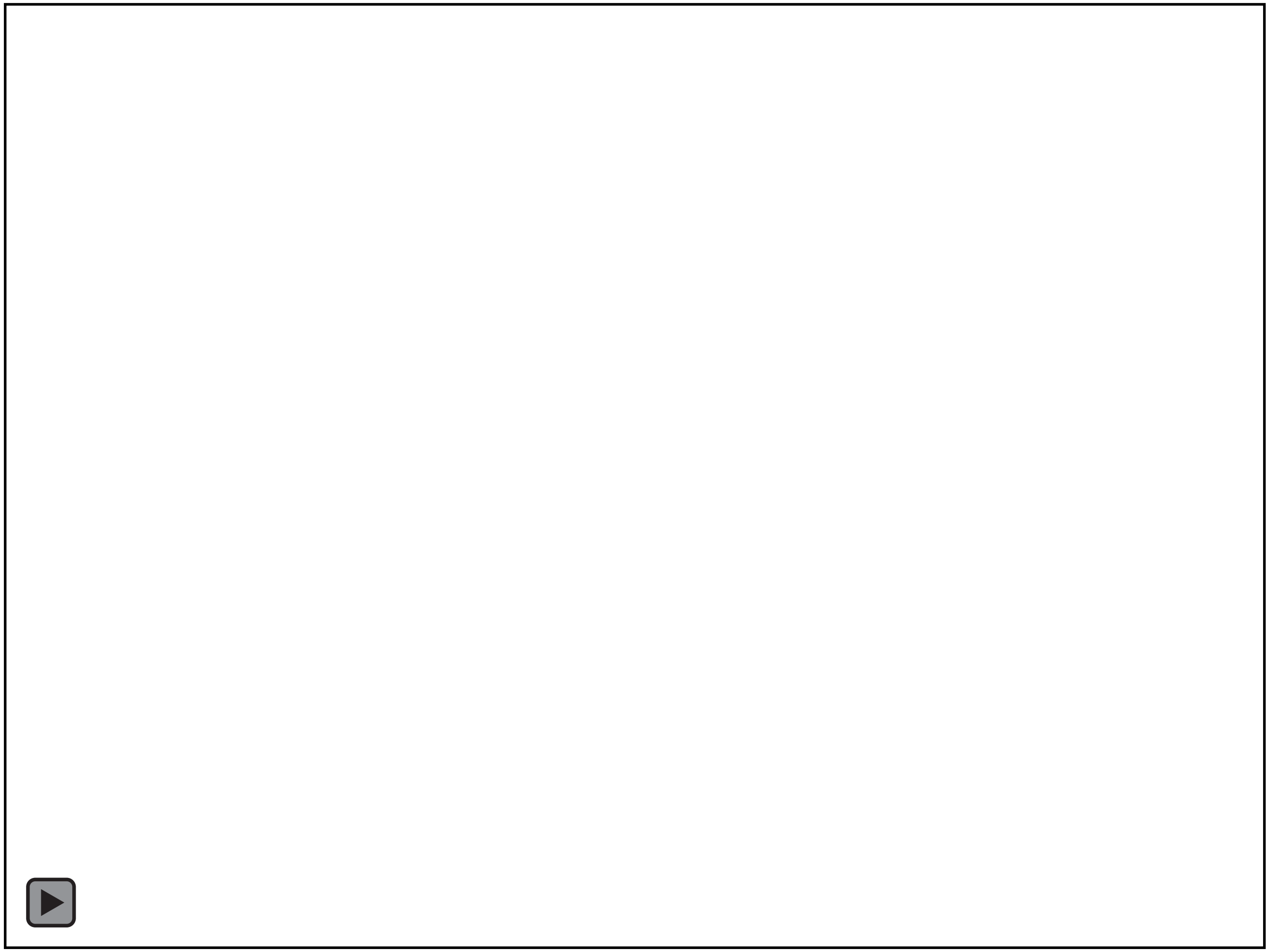
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 1/3 \, m l^2 \dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -m g \, l/2 \sin\theta$$

$$1/3 \, m l^2 \ddot{\theta} + m g \, l/2 \sin\theta = 0$$

$$\ddot{\theta} + 3/2 \, g/l \sin\theta = 0$$

Gyroscopic pendulum

(Inertia wheel pendulum)



Problem 1 (26 %)

The gyroscopic pendulum consists of a physical pendulum with a rotating symmetric disc at the end, spinning about an axis parallel to the axis of rotation of the pendulum. See Figure 1. The stiff rod has mass m_1, length ℓ_1 and moment of inertia I_1. The position of the rod's center of gravity is given by ℓ_{c1} (cf. figure). The disc has mass m_2 and moment of inertia I_2. The pendulum is attached to a fixed coordinate system (axis x and y).

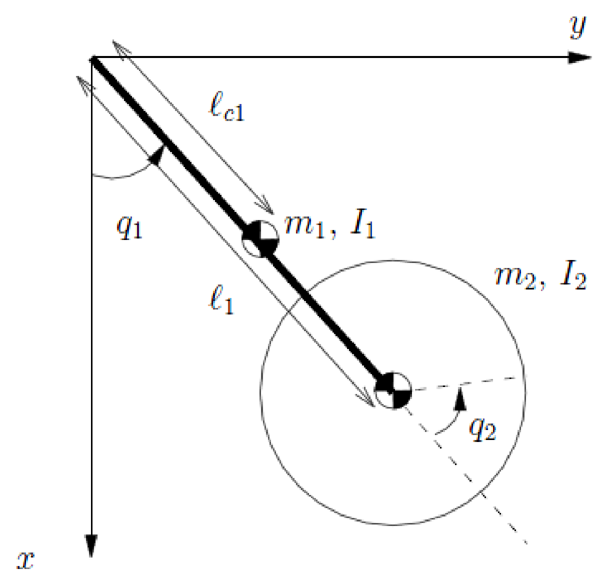


Figure 1: Gyroscopic pendulum

The rotating disc is actuated by a torque τ (which could be generated e.g. by a DC-motor). The gyroscopic pendulum is sometimes used as an experiment to illustrate nonlinear control theory.

We will develop the equations of motion for the gyroscopic pendulum.

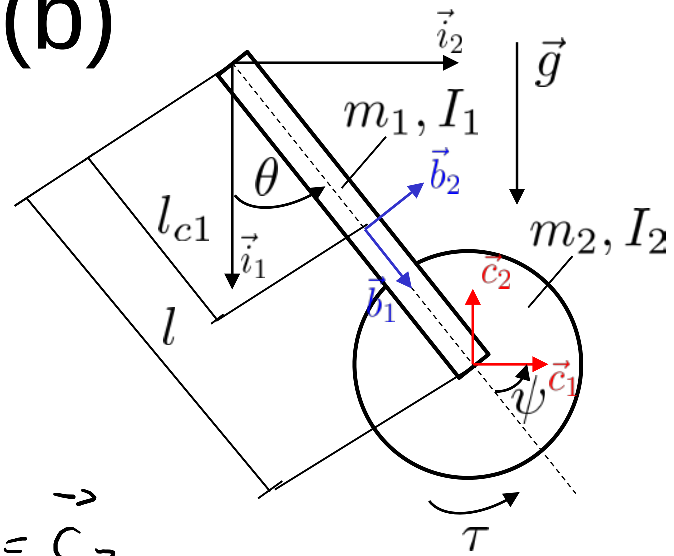
(4 %) (a) Choose appropriate generalized coordinates for this system. The figure should give you some hints. What are the corresponding generalized forces?

(6 %) (b) What is the angular velocity of the disc (that is, of a coordinate system fixed in the disc) in the earth-fixed coordinate system?

(10 %) (c) Find the kinetic and potential energy for the system as functions of the generalized coordinates.

(6 %) (d) Derive the equations of motion for the system.

Gyroscopic pendulum (a),(b)



(a) generalised coord. : $[\Theta, \ \Psi]$
$q_1 \ ; \ q_2$

generalised forces : $0 \ ; \ \tau$

(b) $\vec{\omega}_{ib} = \dot{\Theta}\,\vec{i}_3 = \dot{q}_1\,\vec{i}_3$

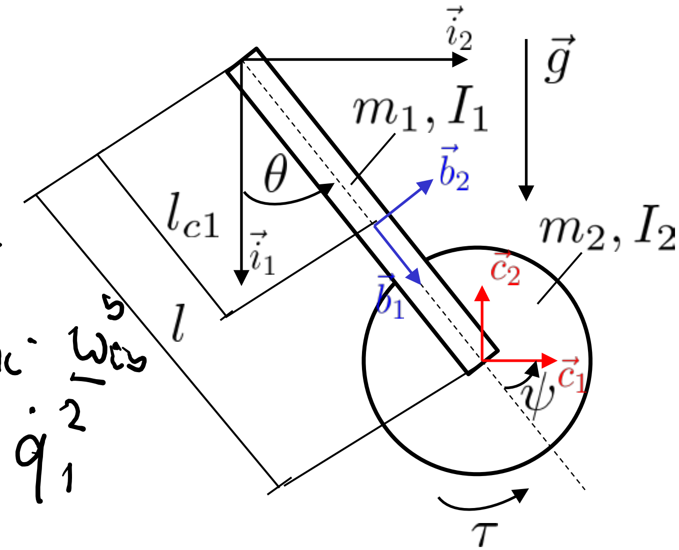
$\vec{\omega}_{bc} = \dot{q}_2\,\vec{b}_3$ $\qquad \vec{b}_3 = \vec{i}_3 = \vec{c}_3$

$\vec{\omega}_{ic} = \vec{\omega}_{ib} + \vec{\omega}_{bc} = (\dot{q}_1 + \dot{q}_2)\,\vec{i}_3$

(c) $\underline{r}_{c1}^{i} = \begin{pmatrix} l_{c1}\cos q_1 \\ l_{c1}\sin q_1 \end{pmatrix}$ $\qquad \underline{v}_{c1}^{i} = \begin{pmatrix} -l_{c1}\sin q_1\,\dot{q}_1 \\ l_{c1}\cos q_1\,\dot{q}_1 \end{pmatrix}$

$\underline{r}_{c2}^{i} = \begin{pmatrix} l\cos q_1 \\ l\sin q_1 \end{pmatrix}$ $\qquad \underline{v}_{c2}^{i} = \begin{pmatrix} -l\sin q_1\,\dot{q}_1 \\ l\cos q_1\,\dot{q}_1 \end{pmatrix}$

Gyroscopic pendulum (c)



Kinetic energy of "rod"

$$T_1 = 1/2\, m_1 (v_{c_1}^i)^T v_{c_1} + \frac{1}{2} (\omega_{ib}^b)^T \cdot M_{bc} \cdot \omega_{ib}^b$$

$$= \frac{1}{2} m_1 (l_{c_1} \dot{q}_1)^2 + 1/2\, I_1 \dot{q}_1^2$$

Kinetic energy of "disc"

$$T_2 = 1/2\, m_2 (l \dot{q}_1)^2 + \frac{1}{2} I_2 (\dot{q}_1 + \dot{q}_2)^2$$

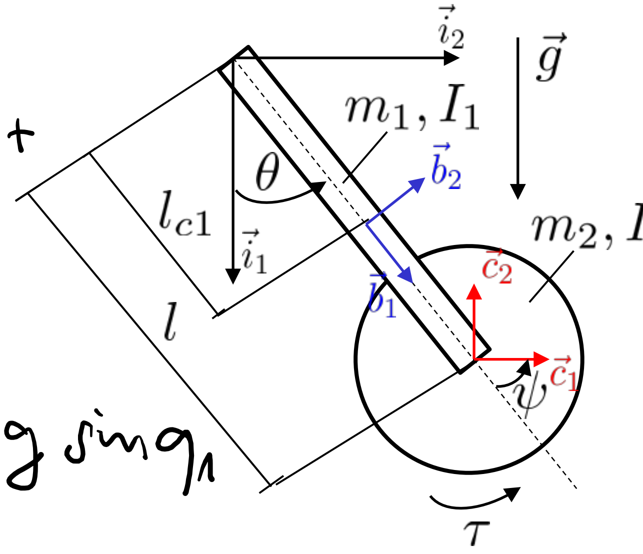
$$U_1 = -m_1 l_{c_1} g \cos q_1 \qquad U_2 = -m_2 l\, g \cos q_1$$

$$T = T_1 + T_2 \qquad U = U_1 + U_2$$

Gyroscopic pendulum (d) I

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$$

$$\mathcal{L} = T - U$$



$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m_1 l_{c1} \dot{q}_1 + I_1 \dot{q}_1 + m_2 l_1^2 \dot{q}_1 + I_2(\dot{q}_1 + \dot{q}_2)$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = - m_1 l_{c1} g \sin q_1 - m_2 l_1 g \sin q_1$$

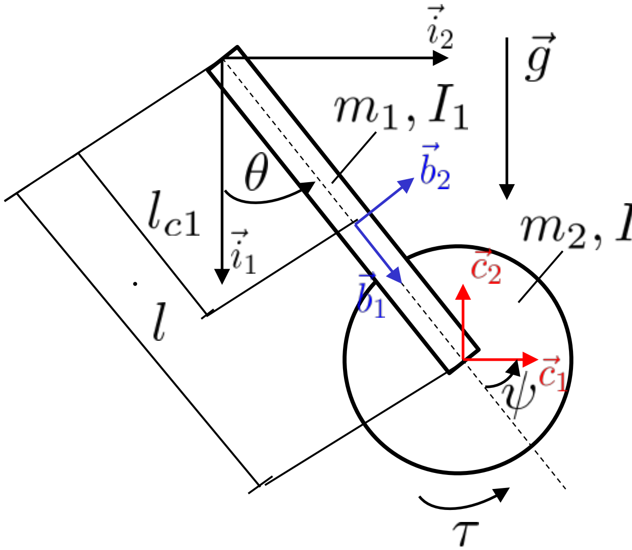
$$0 = m_1 l_{c1} \ddot{q}_1 + I_1 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + I_2(\ddot{q}_1 + \ddot{q}_2) + m_1 l_{c1} g \sin q_1 + m_2 l_1 g \sin q_1$$

Gyroscopic pendulum (d) II

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = I_2(\dot{q}_1 + \dot{q}_2) \qquad \frac{\partial \mathcal{L}}{\partial q_2} = 0$$

$$I_2(\ddot{q}_1 + \ddot{q}_2) = \tau$$



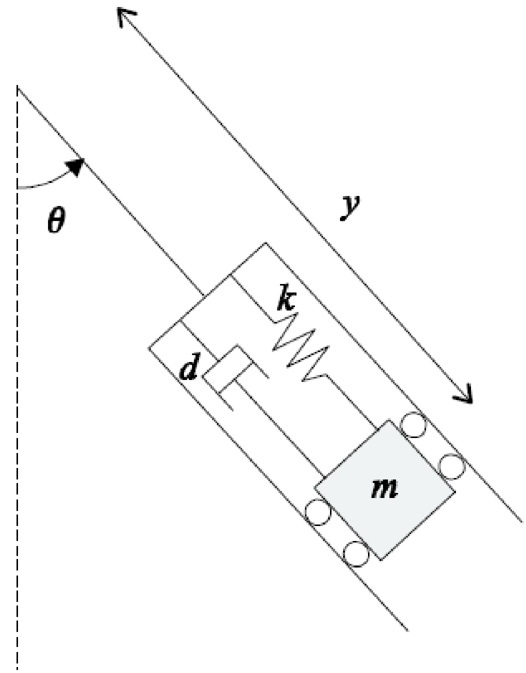


Figure 1: Kloss i rør

Oppgave 3) (15 %)

Figur (1) viser en kloss inne i et rør som svinger om et opphengspunkt. Anta at all masse bortsett fra klossen er neglisjerbar, og at klossens masse er m med massesenter gitt av y som er avtanden mellom massesenteret og opphengspunktet. Videre er fjærkonstanten k og dempekonstanten d. Fjæra er kraftløs når $y = y_0$. Det er ingen friksjon i systemet.

Velg passende generaliserte koordinater $\mathbf{q}$ og bruk Lagranges formulering for å sette opp en matematisk modell.

Block in a pipe I

$$\delta W = F \cdot \delta q$$

Damper: $\delta W = -d\,\dot{q}\,\delta q$

generalised force:

$$Q = \frac{\delta W}{\delta q} = -d\,\dot{q}$$

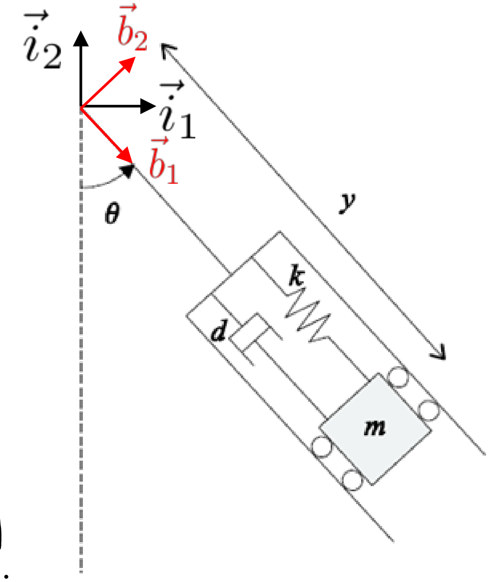


Figure 1: Kloss i rør

generalised coordinates: θ and y

generalised force: 0 and $-d\,\dot{y}$

damper!

Homework

- Try to solve the «Bock in a pipe» task with the Lagrange approach (**Recommendation**: Try to find the Lagrangian yourself before you check your result with the result on the next slide)

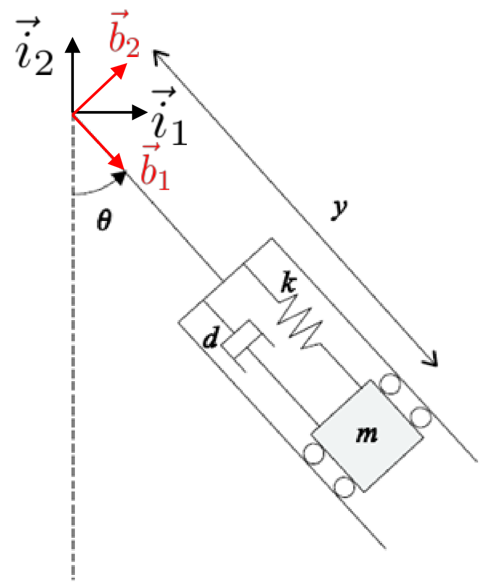


Figure 1: Kloss i rør

Block in a pipe II

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$$

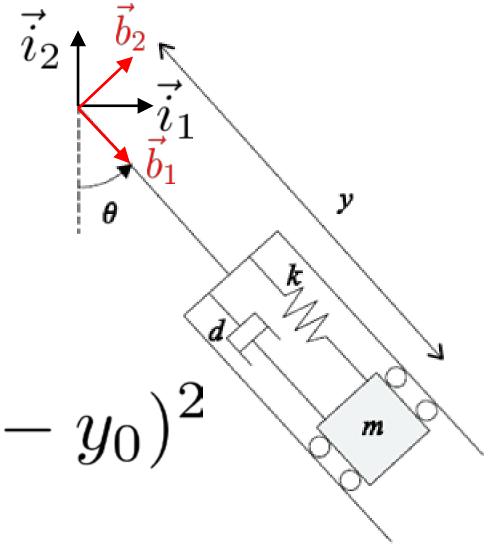


Figure 1: Kloss i rør

$$\begin{aligned}\mathcal{L} &= \mathbf{T} - \mathbf{U} \\ &= \frac{1}{2}m(\dot{y}^2 + y^2\dot{\theta}^2) + mgy\cos\theta - \frac{1}{2}k(y - y_0)^2\end{aligned}$$