NTNU – Trondheim
Norwegian University of
Science and Technology

Assignment 9

TTK4130 Modeling and Simulation

Problem 1 (Cylindrical and spherical coordinates, Lagrange's equations of motion, friction. 60 %)

Consider a point mass in space with mass $m > 0$ and position vector $\vec{r}_m$.

Let $\mathbf{r}^i_m = [x\,,\, y\,,\, z]^T$ be the coordinates of the position vector respect to an inertial frame i.

As we known from Calculus, the position $\mathbf{r}^i_m$ can be represented using cylindrical coordinates:

$$x = r\cos\theta \tag{1a}$$
$$y = r\sin\theta \tag{1b}$$
$$z = z\,, \tag{1c}$$

as well as and spherical coordinates:

$$x = r\sin\phi\cos\theta \tag{2a}$$
$$y = r\sin\phi\sin\theta \tag{2b}$$
$$z = r\cos\phi. \tag{2c}$$

Moreover, a local frame can be defined for each of these coordinate transformations: The cylindrical and the spherical reference frames.

These frames are centered at the position of the point mass, and are denoted by c and s, respectively. For the cylindrical coordinates, the axes are given by the vectors

$$\mathbf{c}^i_r = \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix} \quad \mathbf{c}^i_\theta = \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} \quad \mathbf{c}^i_z = \begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{3}$$

and for the spherical coordinates, the axes are given by the vectors

$$\mathbf{s}^i_r = \begin{bmatrix}\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{bmatrix} \quad \mathbf{s}^i_\phi = \begin{bmatrix}\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{bmatrix} \quad \mathbf{s}^i_\theta = \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix}. \tag{4}$$

(a) Show that the vector sets $\{\vec{c}_r\,,\,\vec{c}_\theta\,,\,\vec{c}_z\}$ and $\{\vec{s}_r\,,\,\vec{s}_\phi\,,\,\vec{s}_\theta\}$ define right-handed reference frames.
Show the details of your calculations.
Hint: Read section 6.4 in the book.

(b) Express the time derivatives of $\mathbf{c}^i_r$, $\mathbf{c}^i_\theta$ and $\mathbf{c}^i_z$ as a function of themselves and the variables r, θ, z and their time derivatives.
Finally, find ω^c_{ic}.
Show the details of your calculations.
Hint: Read section 6.8 in the book. What is $(\omega^c_{ic})^\times$?

(c) Express the time derivatives of $\mathbf{s}^i_r$, $\mathbf{s}^i_\phi$ and $\mathbf{s}^i_z$ as a function of themselves and the variables r, ϕ, θ and their time derivatives.
Finally, find ω^s_{is}.
Show the details of your calculations.
Hint: Read section 6.8 in the book. What is $(\omega^s_{is})^\times$?

Let $\vec{v}_m = \dot{\vec{r}}_m$ and $\vec{a}_m = \ddot{\vec{r}}_m$ be the velocity and acceleration of the point mass, respectively.

(d) Show that the position, velocity and acceleration of the point mass are given in the cylindrical reference frame by

$$\vec{r}_m = r\vec{c}_r + z\vec{c}_z \tag{5a}$$
$$\vec{v}_m = \dot{r}\vec{c}_r + r\dot{\theta}\vec{c}_\theta + \dot{z}\vec{c}_z \tag{5b}$$
$$\vec{a}_m = (\ddot{r} - r\dot{\theta}^2)\vec{c}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{c}_\theta + \ddot{z}\vec{c}_z. \tag{5c}$$

Show the details of your calculations.
Hint 1: Use the results found in part b.
Hint 2: Read section 6.12 in the book.

(e) Show that the position, velocity and acceleration of the point mass are given in the spherical reference frame by

$$\vec{r}_m = r\vec{s}_r \tag{6a}$$
$$\vec{v}_m = \dot{r}\vec{s}_r + r\dot{\phi}\vec{s}_\phi + r\sin\phi\dot{\theta}\vec{s}_\theta \tag{6b}$$
$$\begin{aligned}\vec{a}_m =& (\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\dot{\theta}^2)\vec{s}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\sin\phi\cos\phi\dot{\theta}^2)\vec{s}_\phi \\ &+ (r\sin\phi\ddot{\theta} + 2\dot{r}\sin\phi\dot{\theta} + 2r\dot{\phi}\cos\phi\dot{\theta})\vec{s}_\theta.\end{aligned} \tag{6c}$$

Show the details of your calculations.
Hint 1: Use the results found in part c.
Hint 2: Read section 6.12 in the book.

We will now assume that the point mass in constrained to move on the surface of a sphere. The sphere in question has radius $R > 0$ and is centered at the origin of the frame i. Hence, it is convenient to express the position of the particle using spherical coordinates:

$$x = R\sin\phi\cos\theta \tag{7a}$$
$$y = R\sin\phi\sin\theta \tag{7b}$$
$$z = R\cos\phi. \tag{7c}$$

Let $\vec{N}$ be the force of constraint, i.e. the virtual force that keeps the point mass on this surface.
Moreover, assume that the only force acting on the point mass besides $\vec{N}$ is the gravitational force $\vec{G}$, where $\mathbf{G}^i = -mg\mathbf{e}_3$.

(f) Express the force of constraint $\vec{N}$ as a function of other variables and parameters of this problem. Show the details of your calculations.
Hint: Use Newton's 2. Law and the results from part e.

(g) Find the equations of motion for the point mass, i.e. find the differential equations for ϕ and θ. Show the details of your calculations.
Hint: Read section 8.2 in the book.

We will now constrain the movement of the point mass even more: We will assume that the point mass can only move along a spherical spiral, which is given in spherical coordinates by:

$$x = \frac{R}{\sqrt{1 + a^2\theta^2}}\cos\theta \tag{8a}$$
$$y = \frac{R}{\sqrt{1 + a^2\theta^2}}\sin\theta \tag{8b}$$
$$z = \frac{Ra\theta}{\sqrt{1 + a^2\theta^2}}, \tag{8c}$$

where $a > 0$ is a parameter that describes how steep the spiral is.

(h) Find ϕ as a function of other parameters and variables of this problem.
Show the details of your calculations.
Hint: Use the definition of spherical coordinates.

(i) Find the equations of motion for the point mass, i.e find the differential equation for θ.
Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: The solution is very similar to the solution to part j.

(j) Assume that an additional friction force given by $\vec{F}_f = -k\vec{v}_m$ acts on the mass particle.
Show that the differential equation for the generalized coordinate θ for this new situation is

$$\ddot{\theta} = \frac{a^2\theta(1+2a^2+a^2\theta^2)}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)}\dot{\theta}^2 - \frac{g}{R}\frac{a\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2} - \frac{k}{m}\dot{\theta}. \tag{9}$$

Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: Find the generalized force.

Problem 2 (Friction models, integration methods, event detection. 40 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

In this problem we will implement different static and dynamic friction models. In order to test these models, we will consider a small box of mass $m = 2\,\text{kg}$ sliding on a table due to the application of an external force $F_a = F_a(t) = k_a t$ (ramp function). A friction force F_f will oppose this motion. This simple testbench model is illustrated in Figure 1. Moreover, we will use the parameters

$F_c = 1.5$	Coulomb friction
$F_s = 1.75$	Stiction (static friction)
$F_v = 0.1$	Viscous friction
$v_s = 0.2$	Characteristic Stribeck velocity.

Hint for the whole problem: Read sections 5.2 and 5.3 in book.

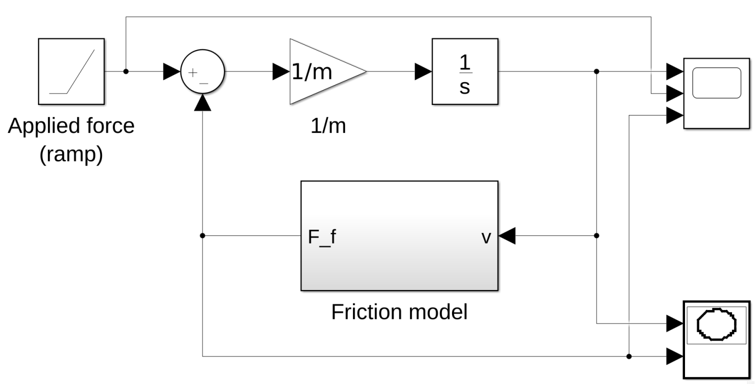

Figure 1: Testbench for friction models.

We start with static friction models, more specifically Coulomb's model, which is given by

$$F_f = F_c \text{sign}(v)\,, \quad v \neq 0\,, \tag{10}$$

where v is the velocity.

(a) Implement Coulomb's model in Simulink using a sign-block. Do not use the built-in Coulomb friction block. Simulate the model over 10 s with a ramp slope k_a of 1 and 3, and with $v(0) = 0$. Use first a variable step solver. Explain what happens.

Thereafter, choose a fixed-step solver with sample time 0.01 instead. How does the model simulate now for both values of the ramp?

Add a figure with the Simulink block diagrams that implement the Coulomb's friction model to your answer.

Hint: Does the sign block include zero-crossing detection?

The Coulomb's friction model has the disadvantage that it is not defined at $v = 0$. The Karnopp's model of Coulomb friction solves this by defining

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_c)\,, & v = 0 \\ F_c \mathrm{sign}(v)\,, & v \neq 0\,, \end{cases} \tag{11}$$

where the saturation function sat is defined in page 198 in the book, and can be implemented using a saturation block in Simulink. For the Karnopp's model to work properly, we must either use variable-step methods with event-detection to determine exactly when $v = 0$, or we have to use some kind of dead-zone around zero-velocity to treat the velocity as zero when it is small. In this problem, we will implement both of these approaches.

(b) Implement Karnopp's friction model using the setup shown in Figure 3, i.e. implement the innards of the two If Action Subsystems. Note that the if-block generates events by default when the value of the if-clause changes.

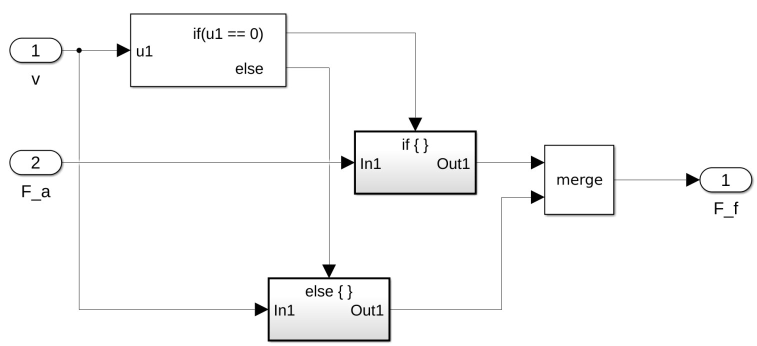

Figure 3: Setup for implementing Karnopp's friction model with event detection.

Simulate using a variable step method. Set $k_a = 1$ and $v(0) = 0$. Comment on the results.

Which role does event-detection play in the simulation?

Add a figure with the Simulink block diagrams implemented inside each If Action Subsystem to your answer.

(c) Implement Karnopp's model without relying on event-detection, by using a dead-zone:

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_c)\,, & |v| \leq \delta \\ F_c \mathrm{sign}(v)\,, & |v| > \delta. \end{cases} \tag{12}$$

Choose $\delta = 1$ and $k_a = 1$. Use a fixed-step solver, and simulate for initial velocity $v(0) = 0$ (as until now) and for $v(0) = -2$. Comment on both results.

Add a figure with the Simulink block diagrams that implement the Karnopp's friction model with dead-zone to your answer.

(d) Extend the friction model from part b. with sticking, Stribeck-effect and linear viscous friction:

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_s)\,, & v = 0 \\ \left(F_c + (F_s - F_c)e^{-(v/v_s)^2}\right)\mathrm{sign}(v) + F_v v\,, & v \neq 0. \end{cases} \tag{13}$$

Simulate with $k_a = 1$ and $v(0) = 0$.

Add a figure with the Simulink block diagrams that implement the friction model (13) to your answer. Moreover, enclose plots of the velocity v and the forces F_a and F_f as a function of time.

Comment on the results, and compare with the results obtained for Coulomb's friction model.

NB: There is a typo in (5.23) in the book.

(e) Finally, we will implement the LuGre dynamic friction model:

$$\dot{z} = v - \sigma_0 \frac{|v|}{g(v)} z \tag{14a}$$

$$g(v) = F_c + (F_s - F_c)e^{-(v/v_s)^2} \tag{14b}$$

$$F_f = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v\,, \tag{14c}$$

where z represents a small displacement in the stick-zone and σ_0 represents the "spring-stiffness" of the asperities of the surface. Set $\sigma_0 = 750$, $\sigma_1 = 0$ and $\sigma_2 = F_v$.

Simulate with $k_a = 1$ and $v(0) = 0$.

Add a figure with the Simulink block diagrams that implement LuGre model to your answer. Moreover, enclose plots of the velocity v and the forces F_a and F_f as a function of time.

Comment on the results, and compare with the results obtained in part d.

Furthermore, play around with the parameters σ_0 and σ_1, and comment on the model behaviour.

Hint: Do you obtain an oscillatory response? If yes, how would you modify σ_0 or σ_1 to get rid of this?

NB: There is a typo in (5.43) in the book.