```
% Matlab code to Problem 1

close all
clear all

N      = 10^4;
lambda = 0.3;
x      = randn(N,1);
n      = linspace(0,100-1);


disp(['Number of samples above threshold among first 100 samples: ' num2str(nnz(x(1:100)>lambda))])
```

Number of samples above threshold among first 100 samples: 44

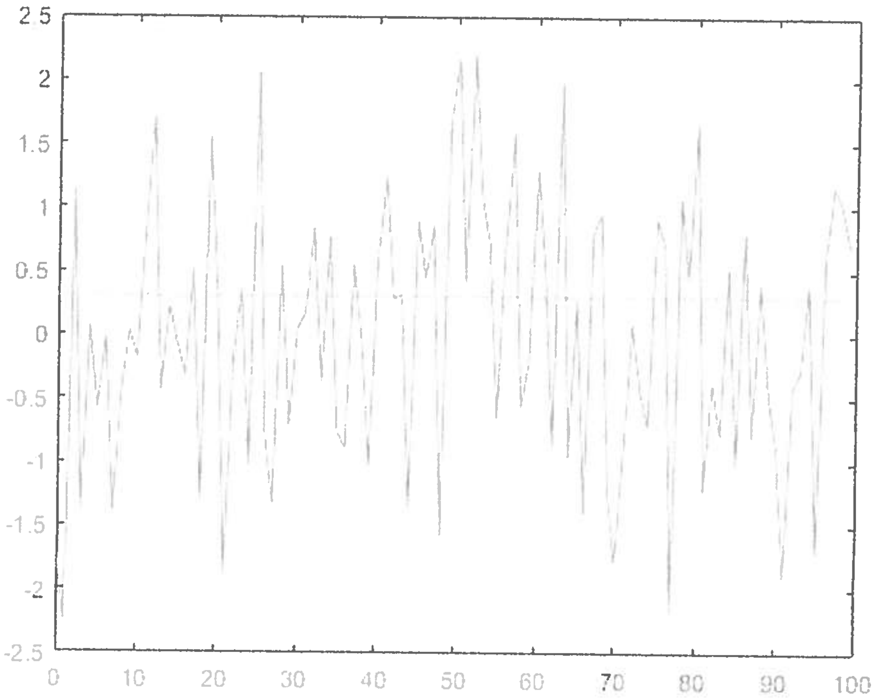
```
disp(['Fraction of samples exceeding threshold: ' num2str(nnz(x>lambda)/N)])
```

Fraction of samples exceeding threshold: 0.3861

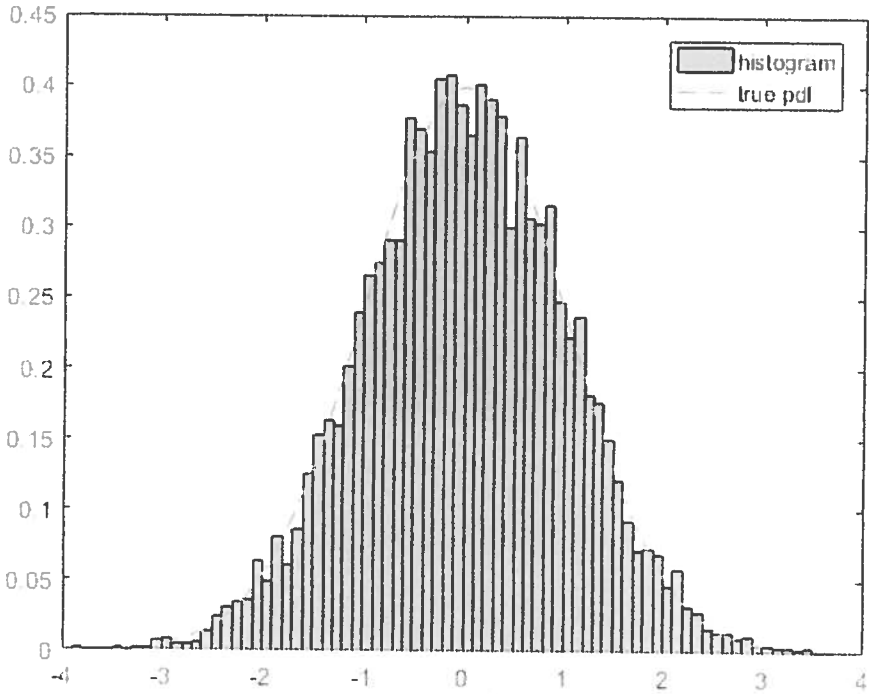
```
disp(['Theoretical value: ' num2str(normcdf(lambda,0,1,'upper'))])
```

Theoretical value: 0.38209

```
figure
plot(n,x(1:100)); hold on
plot(n,lambda*ones(1,100),'--')
```




```
figure
histogram(x,'BinWidth',0.1,'Normalization','pdf'), hold on
mu = 0; s = 1; x  = (-3:0.1:3)';
px = makedist('Normal',mu,sqrt(s));
plot(x,px.pdf(x),'--','Linewidth',1)
legend('histogram','true pdf');
```



0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-4
-3
-2
-1
0
1
2
3
4
histogram
true pdf

2. $$L(\underset{\sim}{x}) = \frac{P_1(\underset{\sim}{x})}{P_0(\underset{\sim}{x})} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x(0)-1)^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2(0)}}$$

$$= e^{-\frac{1}{2}\{\cancel{x^2(0)} - 2x(0) + 1 - \cancel{x^2(0)}\}}$$

Decide H_1 whenever $\ln L(\underset{\sim}{x}) > \ln \lambda$, or

$$\ln L(\underset{\sim}{x}) = x(0) - \frac{1}{2} > \ln \lambda$$

$$\Leftrightarrow \quad x(0) > \ln \lambda + \frac{1}{2} = \lambda'$$

$x(0) \sim N(0,1)$ under H_0

$x(0) \sim N(1,1)$ under H_1

$$\Rightarrow \quad P_{FA} = \text{Prob}\{x(0) > \lambda';\, H_0\} = \int_{\lambda'}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = Q(\lambda')$$

$$P_D = \text{Prob}\{x(0) > \lambda';\, H_0\} = \int_{\lambda'}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}\, dx = Q(\lambda'-1)$$

$$P_{FA} = 10^{-3} = Q(\lambda') \Rightarrow \lambda' = Q^{-1}(10^{-3}) \approx$$

$$\Rightarrow P_D = Q(Q^{-1}(10^{-3}) - 1) =$$

- Ways of increasing P_D
  - increase P_{FA}
  - Collect more samples.

3. Detection problem:

$$H_0: \quad x[n] = w[n],$$
$$H_1: \quad x[n] = A + w[n], \quad n = 0, 1, \ldots, N-1$$

with i.i.d. $w[n] \sim \mathcal{N}(0, \sigma^2)$

$$T(\underset{\sim}{x}) = \sum_{n=0}^{N-1} x(n)$$

- The test statistic $T(\underset{\sim}{x})$ is Gaussian under each hypothesis (sum of Gaussian variables is Gaussian)
- The means and variances are

$$E\{T(\underset{\sim}{x}); H_0\} = E\left\{\sum_{n=0}^{N-1} w[n]\right\} = \sum_{n=0}^{N-1} E\{w[n]\} = 0$$

$$E\{T(\underset{\sim}{x}); H_1\} = E\left\{\sum_{n=0}^{N-1} (A + w[n])\right\} = NA$$

$$\mathrm{var}\{T(\underset{\sim}{x}); H_0\} = \mathrm{var}\left\{\sum_{n=0}^{N-1} w[n]\right\} = \sum_{n=0}^{N-1} \mathrm{var}\{w(n)\} = N\sigma^2$$

$$\mathrm{var}\{T(\underset{\sim}{x}); H_1\} = \mathrm{var}\left\{\sum_{n=0}^{N-1} (A + w(n))\right\} = N\sigma^2$$

$$\Longrightarrow \quad T(\underset{\sim}{x}) \sim \begin{cases} \mathcal{N}(0, N\sigma^2) & \text{under } H_0 \\ \mathcal{N}(NA, N\sigma^2) & \text{under } H_1 \end{cases}$$

4. NP test decides H_1 if

$$\frac{p_1(\underset{\sim}{x})}{p_0(\underset{\sim}{x})} > \lambda \quad \text{or} \quad \frac{\frac{1}{(2\pi\sigma_1^2)^{N/2}} e^{-\frac{1}{2\sigma_1^2}\sum_{n=0}^{N-1} x^2(n)}}{\frac{1}{(2\pi\sigma_0^2)^{N/2}} e^{-\frac{1}{2\sigma_0^2}\sum_{n=0}^{N-1} x^2(n)}} > \lambda$$

Taking the logarithm on both sides

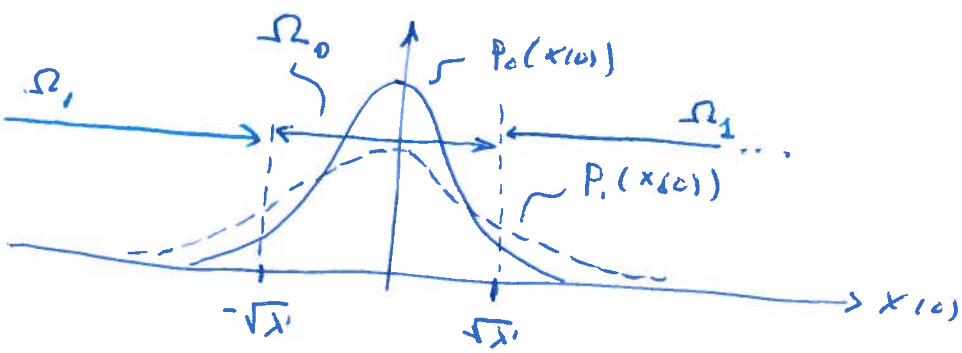
$$-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum_{n=0}^{N-1} x^2(n) > \ln\lambda + \frac{N}{2}\ln\frac{\sigma_1^2}{\sigma_0^2}$$

Since $\sigma_1 > \sigma_0$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2(n) > \lambda'$$

where $\lambda' = \dfrac{\frac{2}{N}\ln\lambda + \ln\frac{\sigma_1^2}{\sigma_0^2}}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}$

- Test statistic $T(\underset{\sim}{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n)$ is the estimate of of the variance, and we decide H_1 if the power is large enough.
- If $N=1$, the detector decides H_1 if $x^2(0) > \gamma$ or $|x(0)| > \gamma$



5.

$$L(\underset{\sim}{x}) = \frac{e^{-\frac{1}{2\sigma^2}\sum_n (x(n) - s(n))^2}}{e^{-\frac{1}{2\sigma^2}\sum x^2(n)}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$\Rightarrow \ln(\underset{\sim}{x}) = -\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x(n) - s(n))^2 - \sum_{n=0} x^2(n)\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \lambda$$

$\Rightarrow$ We decide H_1 if

$$\frac{1}{\sigma^2}\sum_{n=0}^{N-1} x(n)\, s(n) - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1} s^2(n) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \lambda$$

or

$$T(\underset{\sim}{x}) = \underbrace{\sum_{n=0}^{N-1} x(n)\, s(n)}_{\text{correlator}} \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln \lambda + \frac{1}{2}\sum_{n=0}^{N-1} s^2(n) = \lambda'$$

$T(\underset{\sim}{x})$ is Gaussian under both hypothesis.

$$E\{T(\underset{\sim}{x}); H_0\} = E\left\{\sum_{n=0}^{N-1} w[n]\, s[n]\right\} = \sum_{n=0}^{N-1} E\{w[n]\} E\{s[n]\} = 0$$

$$E\{T(\underset{\sim}{x}); H_1\} = E\left\{\sum_{n=0}^{N-1} (s[n] + w(n))\, s[n]\right\} = \sum_{n=0}^{N-1} s^2[n] = \mathcal{E}_s$$

$$\mathrm{Var}\{T(\underset{\sim}{x}); H_0\} = \mathrm{var}\left\{\sum_{n=0}^{N-1} w(n)\, s(n)\right\} = \sum_{n=0}^{N-1} \mathrm{var}\{w(n)\}\, s^2[n]$$

$$= \sigma^2 \mathcal{E}_s = \mathrm{var}\{T(\underset{\sim}{x}); H_1\}$$

$$\Rightarrow T(\underset{\sim}{x}) \sim \begin{cases} N(0, \sigma^2 \mathcal{E}_s) & \text{under } H_0 \\ N(\mathcal{E}_s, \sigma^2 \mathcal{E}_s) & \text{under } H_1 \end{cases}$$

$$P_{FA} = \text{Prob}\{T(\underset{\sim}{x}) > \lambda' ; H_0\} = Q\left(\frac{\lambda'}{\sqrt{\sigma^2 E_s}}\right) \quad (*)$$

$$P_D = \text{Prob}\{T(\underset{\sim}{x}) > \lambda' ; H_1\} = Q\left(\frac{\lambda' - E_s}{\sqrt{\sigma^2 E_s}}\right) \quad (**)$$

$$(*) \Rightarrow \quad \lambda' = \sqrt{\sigma^2 E_s}\, Q^{-1}(\alpha)$$

6. From Problem 5 we see that the performance depends on the energy-to-noise ratio $\frac{\mathcal{E}_s}{\sigma^2}$, i.e.,

$$P_D = Q\left(\frac{\gamma' - \mathcal{E}_s}{\sqrt{\sigma^2 \mathcal{E}_s}}\right) = Q\left(\frac{\sqrt{\sigma^2 \mathcal{E}_s}\, Q^{-1}(\alpha)}{\sqrt{\sigma^2 \mathcal{E}_s}} - \frac{\mathcal{E}_s}{\sqrt{\sigma^2 \mathcal{E}_s}}\right)$$

$$= Q\left(Q^{-1}(\alpha) - \sqrt{\frac{\mathcal{E}_s}{\sigma^2}}\right)$$

$$\mathcal{E}_{s_0} = \sum_{n=0}^{N-1} s_0^2(n) = \sum_{n=0}^{N-1} 4^2 = 4^2 N$$

$$\mathcal{E}_{s_1} = \sum_{n=0}^{N-1} s_1^2(n) = \sum_{n=0}^{N-1} 4^2(-1)^{2n} = 4^2 N$$

$\Rightarrow$ $\mathcal{E}_{s_1} = \mathcal{E}_{s_0}$. Both sequences yield same P_D.

8. a)

$$\rho_s = \frac{s_1^T s_0}{\frac{1}{2}(s_1^T s_1 + s_0^T s_0)}$$

$$s_1^T s_0 = \sum_{n=0}^{N-1} s_1(n)\, s_0(n) = -\sum_{n=0}^{N-1} s_0^2(n) = -s_0^T s_0$$

$$s_1^T s_1 = s_0^T s_0$$

$$\Rightarrow \quad \rho_s = \frac{-s_0^T s_0}{\frac{1}{2}(s_0^T s_0 + s_0^T s_0)} = -1$$

$$\mathcal{E}_{s_0} = \mathcal{E}_{s_1} = s_0^T s_0 = \sum_{n=0}^{N-1} s_0^2(n) = \sum_{n=0}^{N-1} A^2 \cos^2 2\pi f n$$

$$= \frac{A^2}{2} \sum_{n=0}^{N-1} 1 + \cos(2\cdot 2\pi f n) = \frac{NA^2}{2} + \frac{A^2}{2} \underbrace{\sum_{n=0}^{N-1} \cos(2\cdot 2\pi f n)}_{\approx 0}$$

$$\approx \frac{NA^2}{2}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} \cos(2\cdot 2\pi f n) = \frac{1}{N}\mathrm{Re}\left\{\sum_{n=0}^{N-1} e^{j2\cdot 2\pi f n}\right\} = \begin{cases} f \neq 0 \\ f = \frac{1}{2}\cdot k \end{cases} =$$

$$= \frac{1}{N}\mathrm{Re}\left\{\frac{1 - e^{j4\pi f N}}{1 - e^{j4\pi f}}\right\} =$$

$$= \frac{1}{N}\mathrm{Re}\left\{\frac{e^{j2\pi f N}}{e^{j2\pi f}} \frac{\sin 2\pi f N}{\sin 2\pi f}\right\}$$

$$= \frac{1}{N} \cos 2\pi f (N-1) \cdot \frac{\sin 2\pi f N}{\sin 2\pi f}$$

is small in magnitude if $f > 0$ and $f << \frac{1}{2}$

$$P_e = Q\left(\frac{1}{2}\sqrt{\frac{\|s_1 - s_0\|^2}{\sigma^2}}\right) = Q\left(\sqrt{\frac{E_s(1-\rho_s)}{2\sigma^2}}\right)$$

$$= Q\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$$

Plot in Matlab.

$$\Rightarrow \quad s_0^T s_1 \approx \frac{A^2 N}{2}\left(\frac{1}{2N} \frac{\sin 2\pi (f_0+f_1)N}{\sin \pi (f_0+f_1)} + \frac{1}{2N} \frac{\sin 2\pi (f_1-f_0)N}{\sin \pi (f_1-f_0)} \right)$$

$$\approx 0$$

$$\Rightarrow \quad \rho_s = 0$$

b) See solution to 8b

c) $P_e = Q\left(\sqrt{\frac{E_s}{2\sigma^2}}\right).$

Comparing to BPSK (in 8c) we see that FSK must have twice average energy than BPSK to have the same error probability.

```
% Problem 8 and 9
% Plot Pe for BPSK and FSK

ENR = 10.^((0:16)/10); % Energy-to-noise ratio 0 to 16 dB

semilogy((0:16),normcdf(sqrt(ENR),0,1,'upper')), hold on
semilogy((0:16),normcdf(sqrt(ENR/2),0,1,'upper'))

xlabel('Energy-to-noise ratio (dB)')
ylabel('Probability of error, {\itP_e}')
legend('BPSK','FSK')
```