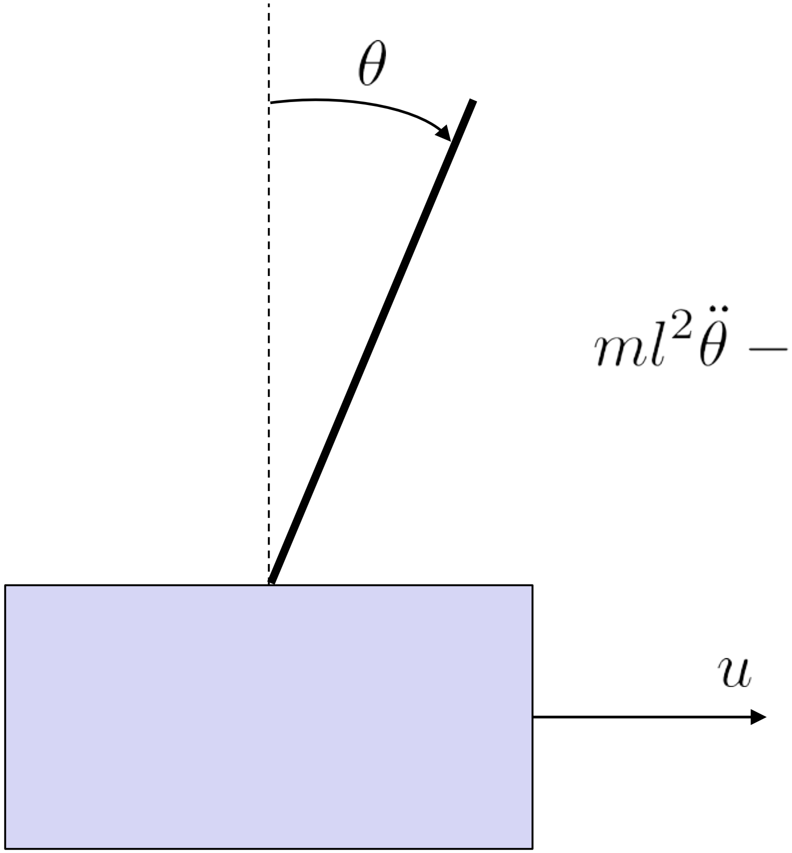
Lecture 2:

- Model types (E1.1-1.3,E2.1-2.2)
  - State space models, transfer functions
  - Linear models, nonlinear models

Example: “Stick balancing”



$$ml^2\ddot{\theta} - mg\sin\theta = -mlu\cos\theta$$

Example: "Stick balancing" [Example 4]

$$ml^2\ddot{\theta} - mg\sin\theta = -mlu\cos\theta$$

Linearise around $\theta = \dot{\theta} = 0 \qquad u = 0$

$$\ddot{\theta} = -\frac{u}{l}\cos\theta + \frac{g}{l^2}\sin\theta$$

$$\Delta\ddot{\theta} \approx \frac{\partial}{\partial\theta}\left[-\frac{u}{l}\cos\theta + \frac{g}{l^2}\sin\theta\right]_{\substack{\theta=\dot{\theta}=0\\u=0}}\Delta\theta + \frac{\partial}{\partial\dot{\theta}}\left[-\frac{u}{l}\cos\theta + \frac{g}{l^2}\sin\theta\right]_{\substack{\theta=\dot{\theta}=0\\u=0}}\Delta\dot{\theta} + \frac{\partial}{\partial u}\left[-\frac{u}{l}\cos\theta + \frac{g}{l^2}\sin\theta\right]_{\substack{\theta=\dot{\theta}=0\\u=0}}\Delta u$$

$$= \left[\frac{u}{l}\sin\theta + \frac{g}{l^2}\cos\theta\right]\Big|_{\substack{\theta=\dot{\theta}=0\\u=0}}\Delta\theta + 0\cdot\Delta\dot{\theta} + \left[-1/l\cos\theta\right]\Big|_{\substack{\theta=\dot{\theta}=0\\u=0}}\Delta u$$

$$= g/l^2\,\Delta\theta - 1/l\,\Delta u$$

1.3 Transfer functions

Linear time invariant model (LTI)

$$\underline{\dot{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}$$

System matrix

Control matrix

$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

Output matrix

Feed-Forward matrix

Laplace notation

+ important analysis
and design methods

$$\underline{x}(s) = \mathcal{L}\{\underline{x}(t)\}$$
$$\underline{u}(s) = \mathcal{L}\{\underline{u}(t)\}$$
$$\underline{y}(s) = \mathcal{L}\{\underline{y}(t)\}$$

$$\mathcal{L}\{\underline{\dot{x}}(t)\} = s\mathcal{L}\{\underline{x}(t)\} - \underbrace{x(t=0)}_{\text{Assume } = 0}$$

Transform LTI system

$$\underline{\dot{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \qquad \underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

$$s\,\underline{x}(s) = A\,\underline{x}(s) + B\,\underline{u}(s) \qquad (1)$$

$$\underline{y}(s) = C\,\underline{x}(s) + D\,\underline{u}(s) \qquad (2)$$

$$(sI - A)\,\underline{x}(s) = B\,\underline{u}(s)$$

$$\underline{x}(s) = [sI - A]^{-1} B\underline{u}(s)$$

$$\underline{y}(s) = \underbrace{\left(C\,[sI - A]^{-1}B + D\right)}_{H(s)\text{: transfer function}}\underline{u}(s)$$

Rational transfer function

$$\frac{y(s)}{u(s)} = H(s)$$

- Rational transfer function if it can be expressed as:

$$H(s) = K\frac{P(s)}{Q(s)}$$
$$= K\frac{(s+z_1)\dots(s+z_m)}{(s+p_1)\dots(s+p_n)}$$

- m: zeros ; n: poles

Partial differential equations

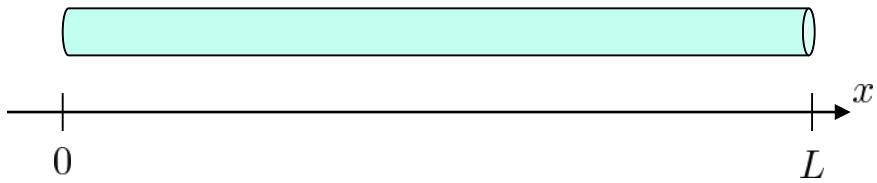
- Partial differential equations (pde) lead to irrational transfer functions

- They can be approximated by rational transfer functions with infinitely order

  $\rightarrow$ infinit dimension system

- Example: Transport equation/advection /wave equation

PDE – Example I



PDE:

$$\frac{\partial v(x,t)}{\partial t} = -C\,\frac{\partial v(x,t)}{\partial x} \qquad v(0,t) = v_1(t)$$

$$\mathcal{L}\left\{\frac{\partial v(x,t)}{\partial t}\right\} = s\,\mathcal{L}\left\{v(x,t)\right\} = s\,v(x,s)$$

$$s\,v(x,s) = -C\,\frac{\partial v(x,s)}{\partial x}$$

$\rightarrow$ ODE of x

$$-\frac{s}{C}\,\partial x = \frac{\partial v(x,s)}{v(x,s)}$$

$$\int_0^x -\frac{s}{C}\,\partial x = \int_{v(0,s)}^{v(x,s)} \frac{\partial v(x,s)}{v(x,s)}$$

PDE – Example II

$$-\frac{s}{c}x = \ln\left(\frac{v(x,s)}{v(0,s)}\right)$$

$$\exp\left(-\frac{s}{c}x\right) = \frac{v(x,s)}{v(0,s)}$$

$$v(x,s) = v(0,s)\exp\left(-\frac{s}{c}x\right)$$

boundary conditions:

$$v(L,s) = v_2(s)$$

$$v_2(s) = v_1(s)\exp\left(-\frac{s}{c}L\right)$$

$$\frac{v_2(s)}{v_1(s)} = \underbrace{\exp(-Ts)}_{\text{irrational transfer function}} \qquad \underbrace{T = \frac{L}{c}}_{\text{propagation time}}$$

Lecture 3: Energy functions and passivity

Using ”energy” as a concept for characterizing system behavior

- Energy functions (aka Lyapunov functions)
  - If the ”internal energy” of a system decreases, the system is stable
  - ”Introvert” (not concerned with surroundings)
- Passivity
  - Does a system produce ”energy” to its surroundings?
  - ”Extrovert” (mainly concerned with surroundings, via inputs and outputs)
- The above concepts are connected via storage functions (next time)

Book: E2.3, E2.4

Energy function

- The system: $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$

- Assume we have a function $V(x,t) \geq 0$, which describes the «energy» of the system

- The derivative of the energy function $V(x,t)$ is

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t)$$

- If we have $\dot{V} \leq 0$

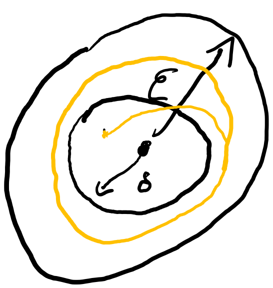
  → Energy of the system decreases monotonically

  → stability

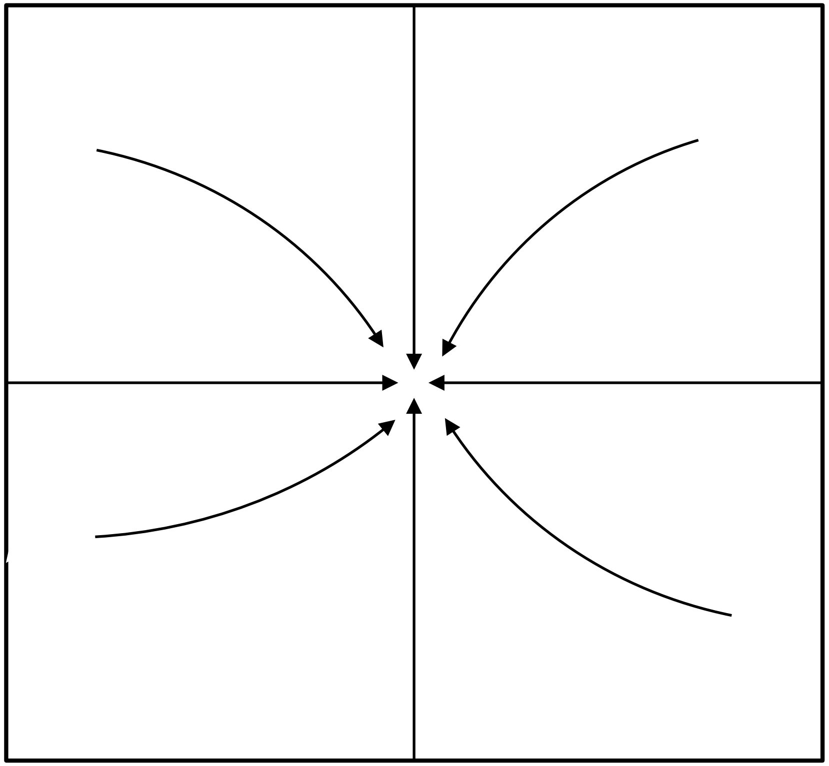
Stable system

- Equilibrium point is stable if for any possible $\varepsilon > 0$ radius around the steady state point a region with the radius δ exist, such that for all initial values $|x_0 - x_e| < \delta$ the solution $x(t)$ fullfils for all $t > t_0$ the following condition:

$$|x(t) - x_e| < \varepsilon$$

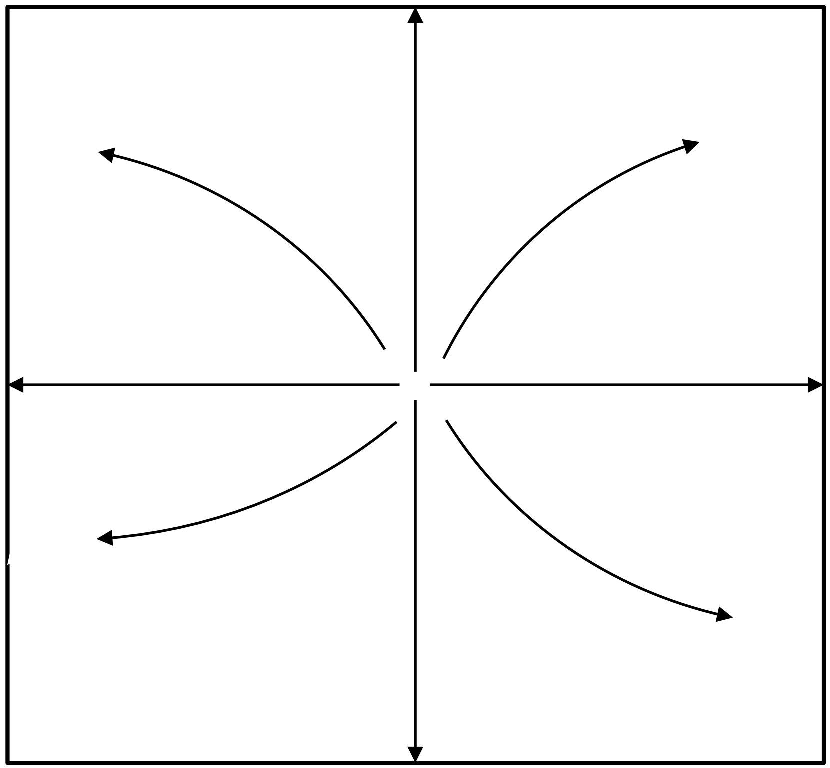


Phase diagram for system with real Eigenvalues



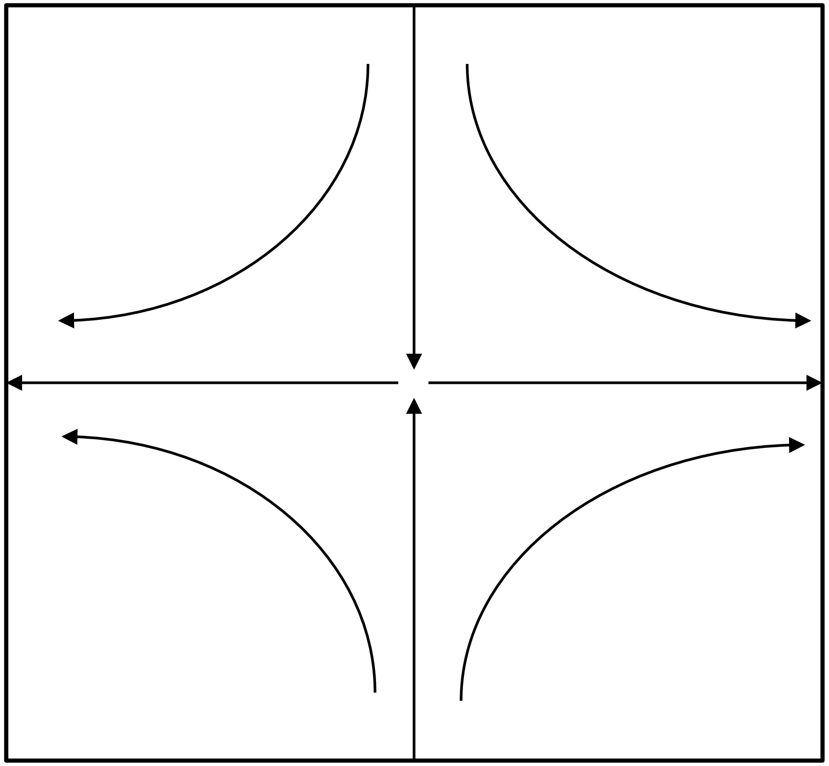
Stable

Phase diagram for system with real Eigenvalues



Unstable

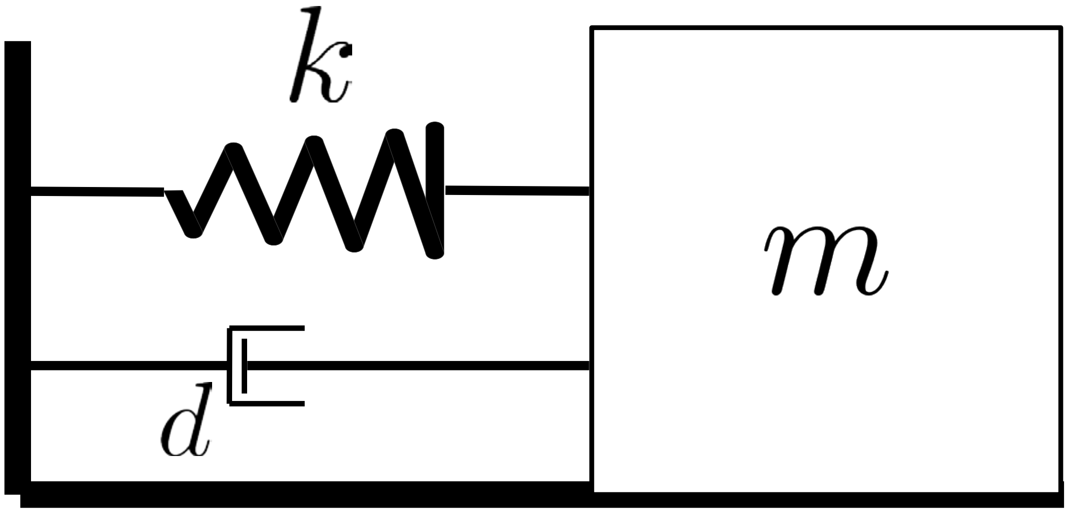
Phase diagram for system with real Eigenvalues



Saddle → unstable

Mass-spring-damper I (2.3.41)

$$m\ddot{x} + d\dot{x} + kx = 0$$



Mass-spring-damper II $m\ddot{x} + d\dot{x} + kx = 0$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m} x_1 - \frac{d}{m} x_2 \end{aligned}$$

$$V = \underbrace{\frac{1}{2} m x_2^2}_{\text{kin. energy}} + \underbrace{\frac{1}{2} k x_1^2}_{\text{pot. energy}} \geq 0$$

Mass-spring-damper III $m\ddot{x} + d\dot{x} + kx = 0$

$$\dot{V} = m x_2 \dot{x}_2 + k x_1 \dot{x}_1$$

$$= m x_2 \left(-\frac{k}{m} x_1 - \frac{d}{m} x_2 \right) + k x_1 x_2$$

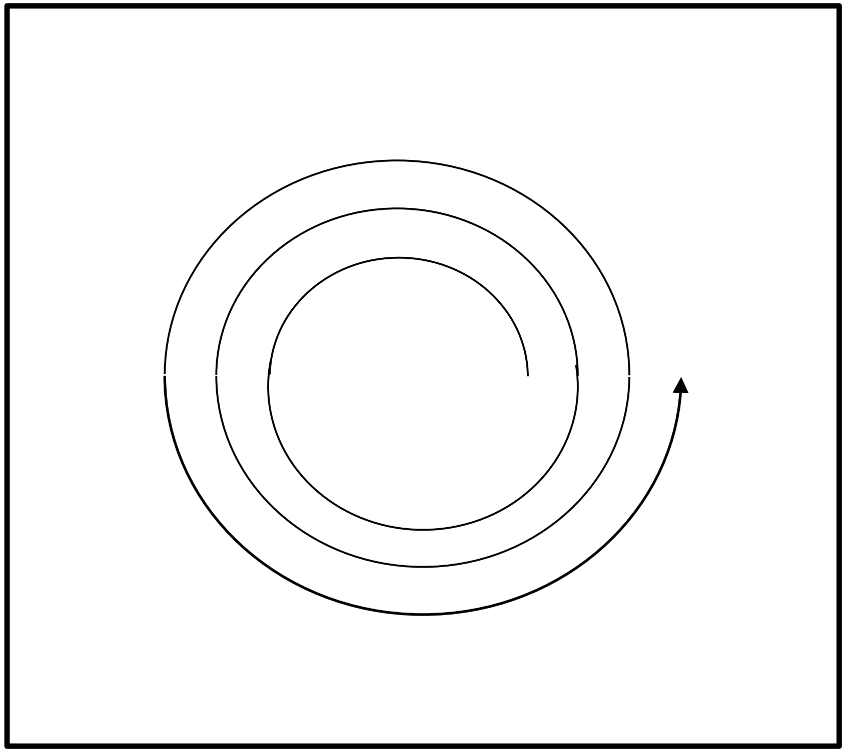
$$= -d x_2^2 \leq 0$$

$$\rightarrow \quad V(t) \leq V(t_0) = V_0$$

$\rightarrow$ stable

Mass-spring-damper

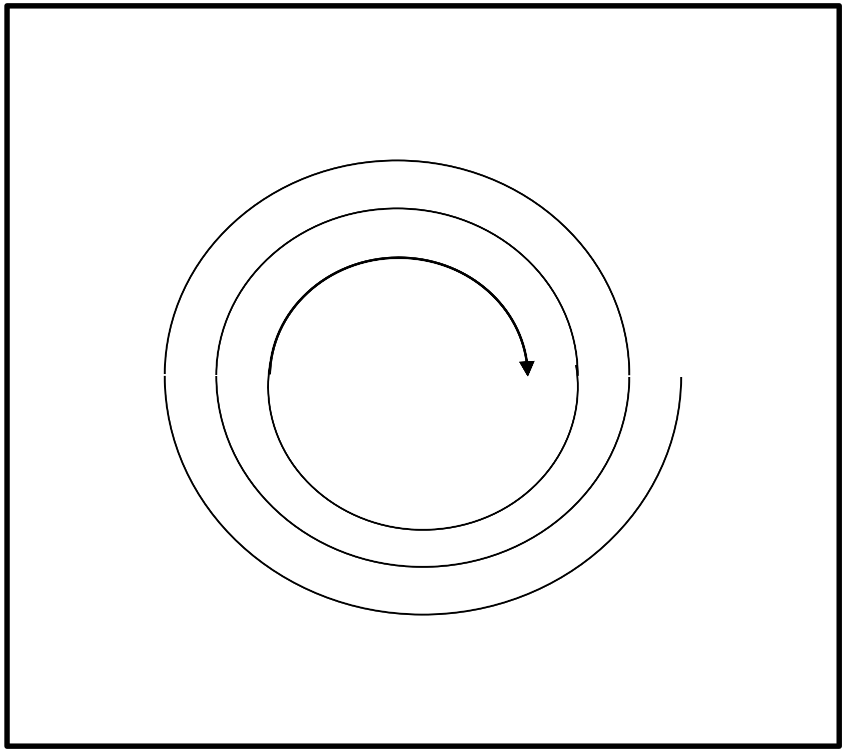
$$\lambda_{1,2} = u \pm iv$$



unstable $u > 0$

Mass-spring-damper

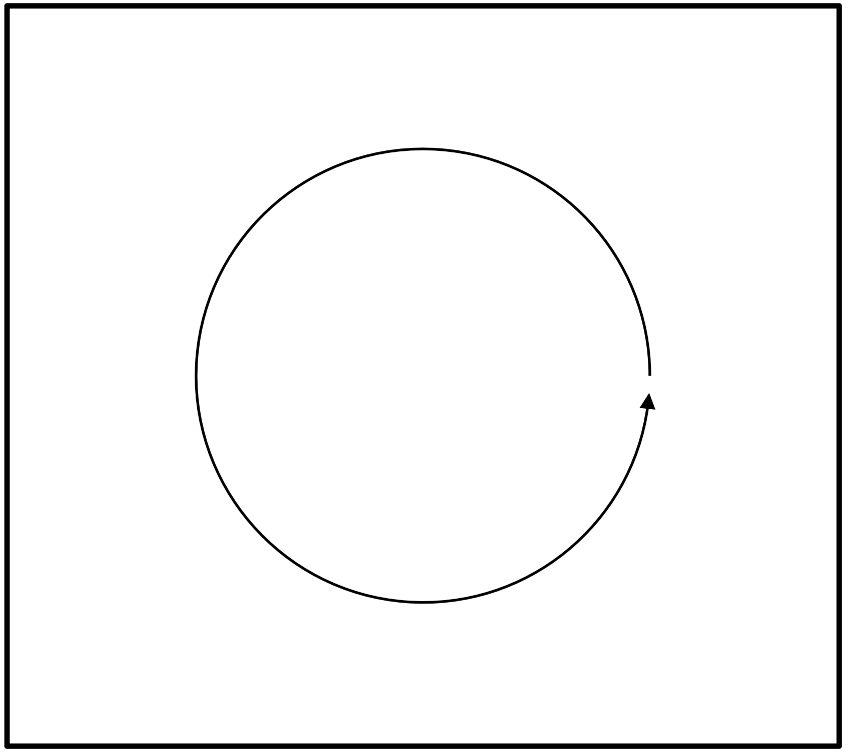
$$\lambda_{1,2} = u \pm iv$$



stable $\quad u < 0$

Mass-spring-damper

$$\lambda_{1,2} = u \pm iv$$

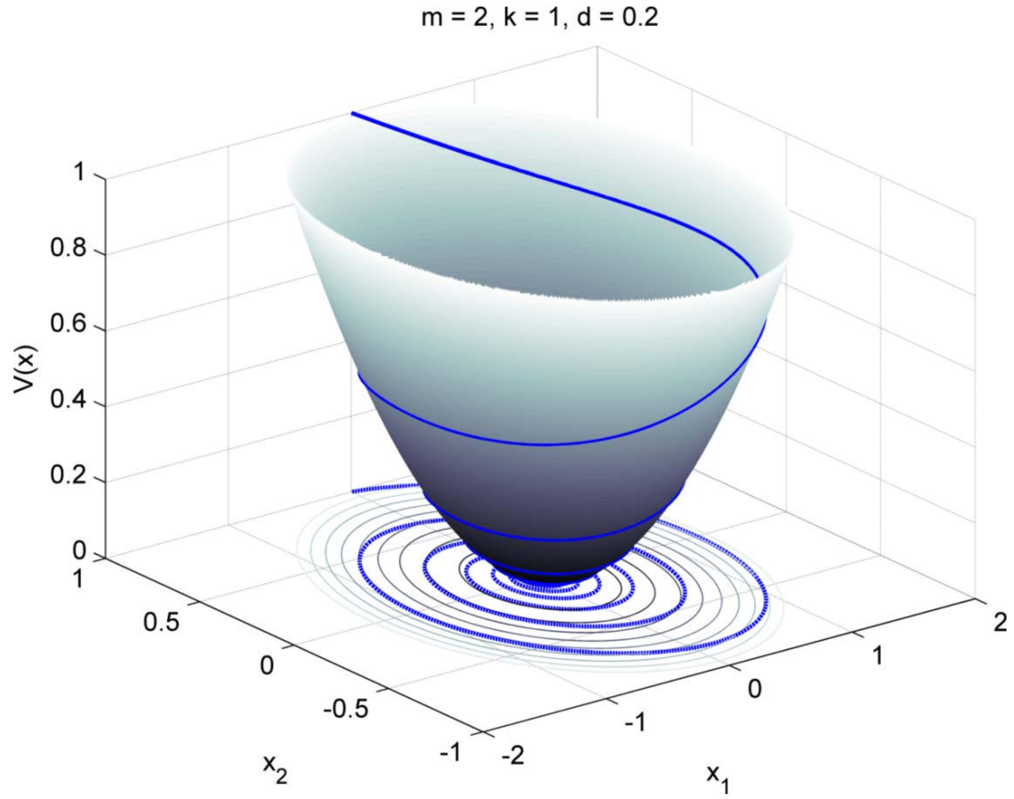


centre $\quad u = 0$

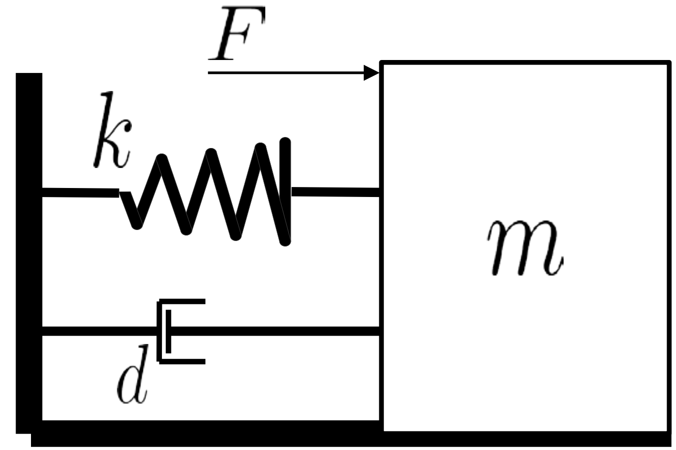
Mass-spring-damper

$$m\ddot{x} + d\dot{x} + kx = 0$$

$$V(x) = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$$



Mass-spring-damper with force



$$m\ddot{x} + d\dot{x} + kx = F$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} m x_2^2$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right] \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{F}{m} - \frac{k}{m}x_1 - \frac{d}{m}x_2 \end{aligned}$$

$$\dot{V} = -d x_2^2 + F x_2$$

If F supplied by a controller: $F = -k_d \cdot x_2$

$\rightarrow \quad \dot{V} = -(k_d + d) x_2^2 \leq 0 \quad \rightarrow$ stable

General: Energy-based controller design

$$\left.\begin{aligned} \dot{x} &= f(x,u,t) \\ \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t) \end{aligned}\right\}$$

Choose a controller that

$$\dot{V} \leq 0$$

Why learn about passivity? Preview...

- Say you have several systems (or models), and you want to interconnect them
  - For instance, a process and a controller, or a motor and a load, or two buffer tanks in series, ...
  - Will the interconnection be stable?

- Bad news: The interconnection of stable systems is not necessarily stable

- **Good news: The interconnection of passive systems is passive (and therefore stable)!**

Homework (recommended)

- Derive the derivative of the energy function of the mass-spring-damper system with force
  - $\dot{V} = -dx_2^2 + Fx_2$

- Read section 2.4.1, 2.4.2, 2.4.3 in the book
  - Try to proof passivity of the transfer-function:

$$H(s) = \frac{1}{1 + Ts}$$

by first transfering the function to the time domain