Lecture 12: Rigid body kinematics – Rotations, angular velocity

Representations of rotation

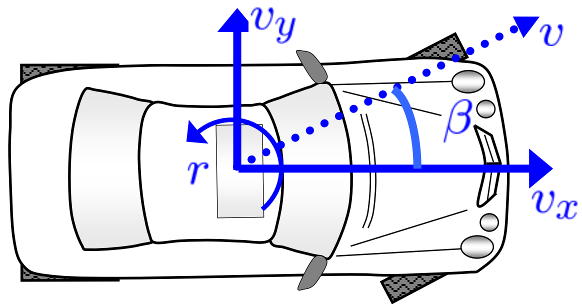
- Rotation matrices
- Euler angles
- 3-parameter specification of rotations
  - Roll-pitch-yaw
- Angle-axis, Euler-parameters
  - 4-parameter specification of rotations

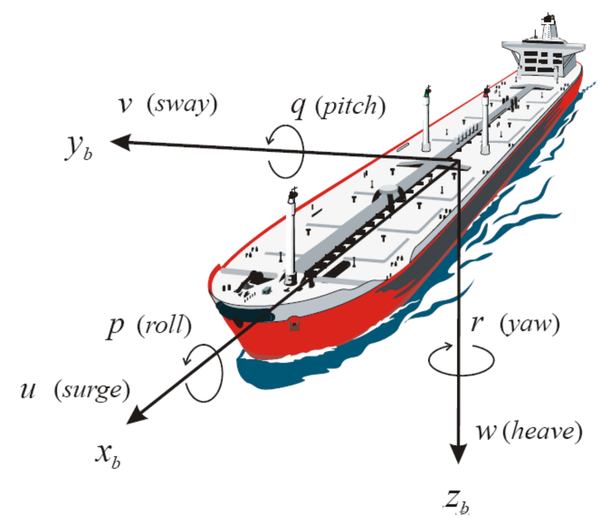
- Angular velocity

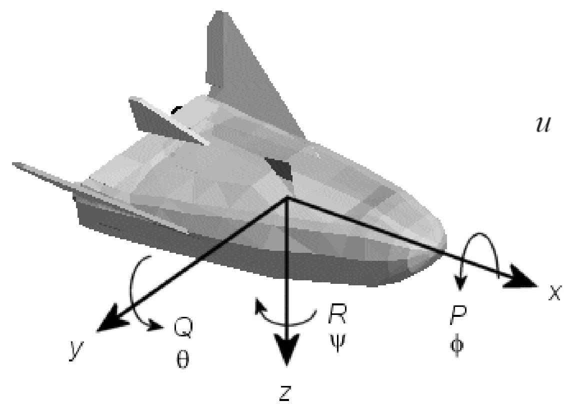
Book: Ch. 6.6, 6.7, 6.8

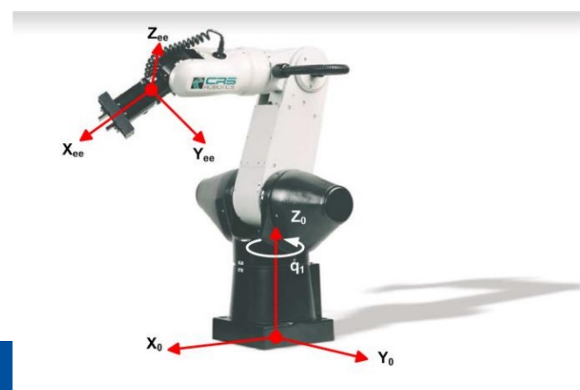
Why rotation matrices?

- Rotation matrices are used to describe rotations and orientations of rigid bodies
  - Road vehicles
  - Marine vessels
  - Airplanes, satellites
  - Robotics







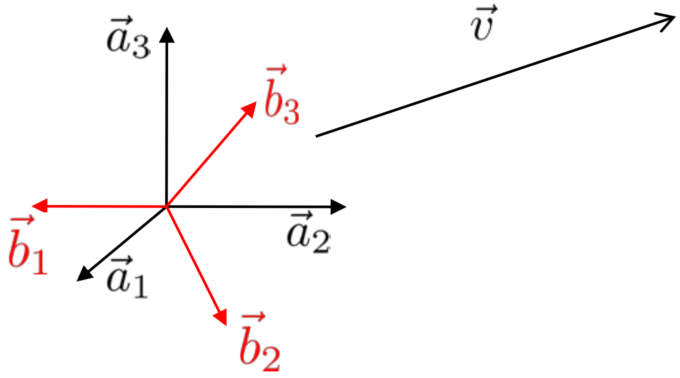


Rotation matrices

The rotation matrix from *a* to *b* $\mathbf{R}_b^a$ is used to

- Transform a coordinate vector from *b* to *a*

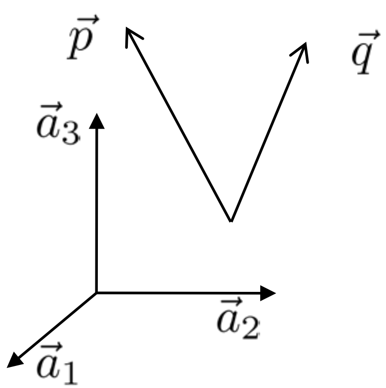
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- Rotate a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

such that $\mathbf{q}^b = \mathbf{p}^a$.



Representations of rotations

- Rotation matrix
  - Simple, but over-parameterized (9 parameters)

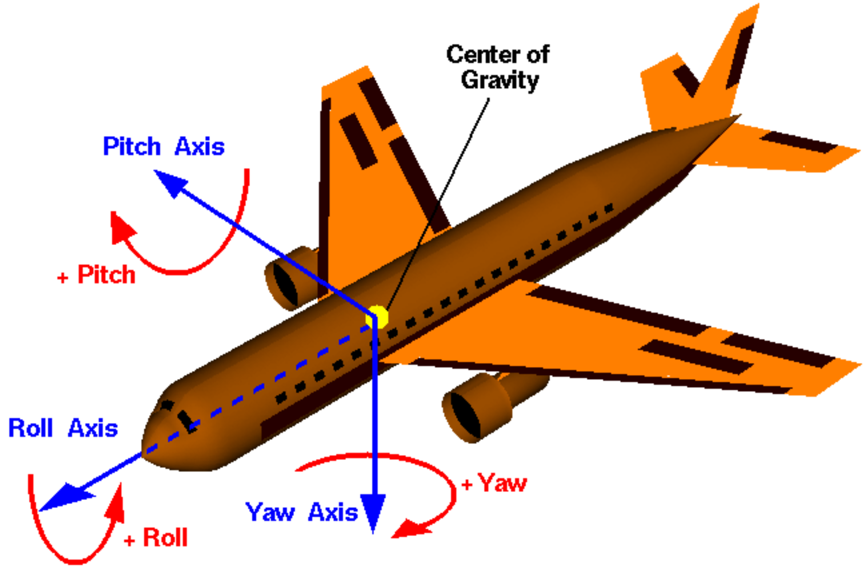
Euler's Theorem:

"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common: Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this later)

- Angle-axis, Euler-parameters
  - 4-parameters are used
  - No singularity problems

Euler-angles: Roll-pitch-yaw



- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

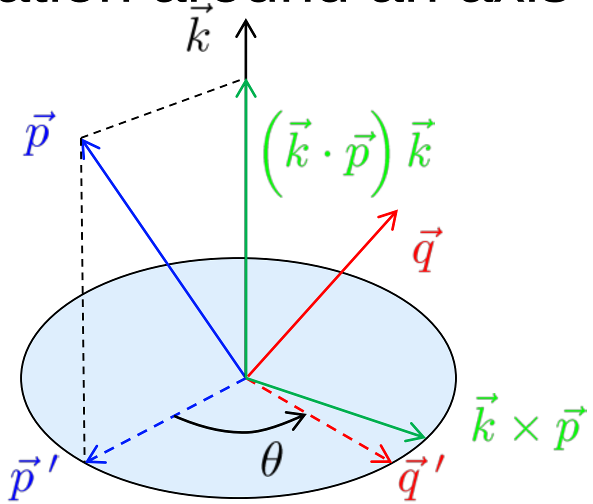
$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Rotation of vectors based on angle-axis representation

- Angle-axis: All rotations can be represented as a simple rotation around an axis

Somewhat different derivation of the rotation dyadic. Compare p. 228 in book.



$$\vec{p}\,' = \vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \vec{q} - \left(\vec{k}\cdot\vec{q}\right)\vec{k} = \vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \cos\theta\,\vec{p}\,' + \sin\theta\,\vec{k}\times\vec{p}$$

$$\vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k} = \cos\theta\,\left(\vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}\right) + \sin\theta\,\vec{k}\times\vec{p}$$

$$\boxed{\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}}$$

Angle-axis rotation dyadic, rotation matrix

- Rotation θ about an axis $\vec{k}$

$$\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\,\vec{k}\left(\vec{k}\cdot\vec{p}\right)$$

- Angle-axis rotation by a dyadic

$$\vec{q} = \Big(\underbrace{\cos\theta\,\vec{I} + \sin\theta\,\vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}}\Big)\cdot\vec{p}$$

$$\vec{q} = \vec{R}_{\vec{k},\theta}\cdot\vec{p}$$

- Angle-axis rotation matrix

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\,\mathbf{I} + \sin\theta\,(\mathbf{k}^a)^{\times} + (1-\cos\theta)\,\mathbf{k}^a(\mathbf{k}^a)^{\mathsf{T}}}$$

- Alternative expression (using $\mathbf{k}^a = \mathbf{k}$ and $\mathbf{k}^{\times}\mathbf{k}^{\times} = \mathbf{k}(\mathbf{k})^{\mathsf{T}} - \mathbf{I}$):

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{I} + \sin\theta\,\mathbf{k}^{\times} + (1-\cos\theta)\,\mathbf{k}^{\times}\mathbf{k}^{\times}}$$

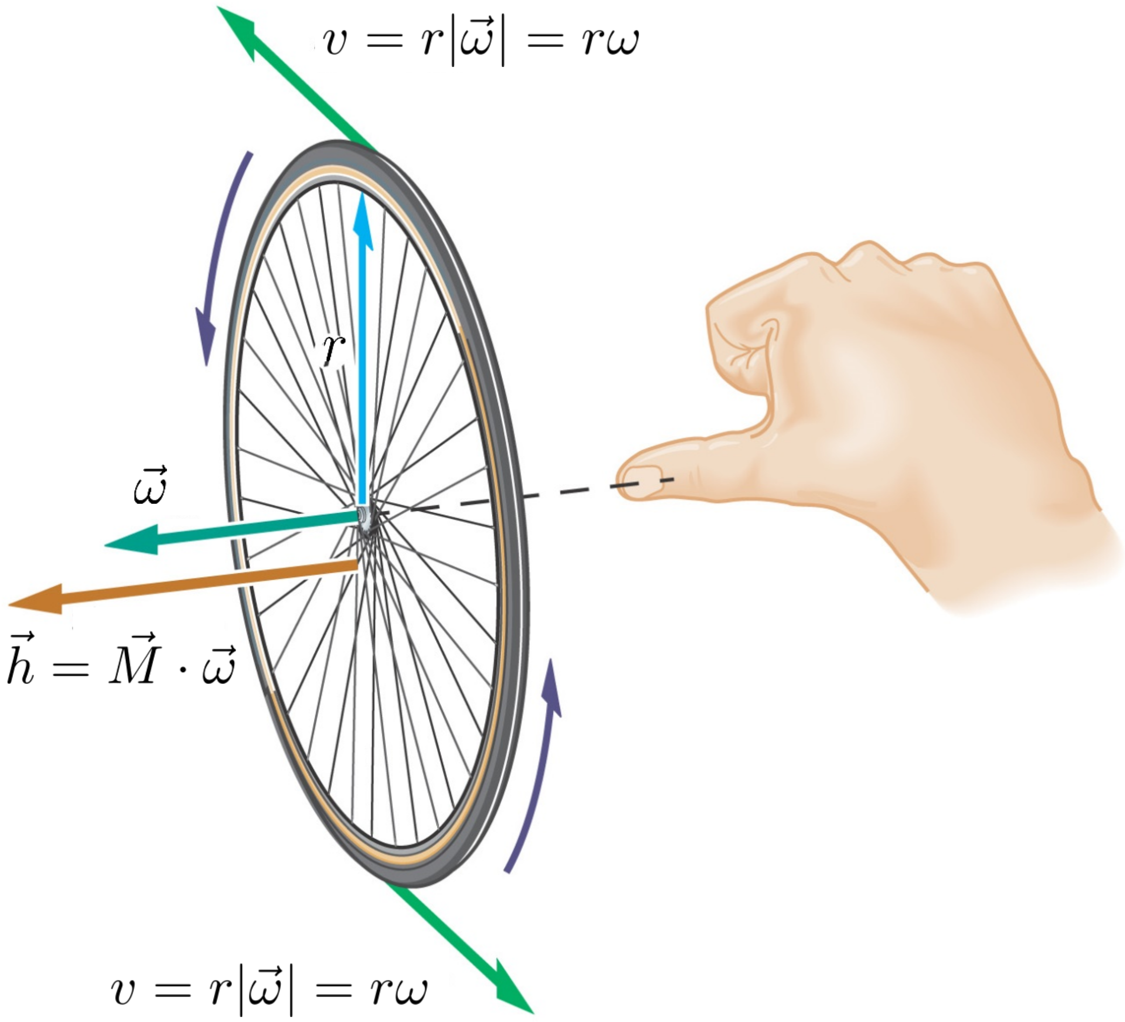
Use of Euler parameters

- ABB robots use Euler parameters (quaternions) internally in the robot control program
  - and Euler angles “externally”

- In Modelica.multibody, one can use either rotation matrices or Euler parameters (quaternions)

- Euler parameters (quaternions) often used in “advanced control” of robots, satellites, etc.

Angular velocity



Kinematic differential equations

- Translation: $\underline{v} \rightarrow \underline{r}$: $\qquad \dot{\underline{r}} = \underline{v}$

- Rotation: $\underline{\omega}_{ab}^{a} \rightarrow \mathbf{R}_b^a$: $\qquad \dot{\mathbf{R}}_b^a = ?$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler angle} \qquad \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler parameter} \qquad \begin{array}{l} \dot{\eta} = ? \\ \dot{\underline{\varepsilon}} = ? \end{array}$$

Definition angular velocity I

R_b^a: orthogonal → $R_b^a (R_b^a)^T = I$

$$\frac{d}{dt}\left[R_b^a (R_b^a)^T \right] = \underbrace{\dot{R}_b^a (R_b^a)^T}_{\text{skew symmetric matrix}} + R_b^a (\dot{R}_b^a)^T = 0$$

$S = -S^T$

that shows: $S = \dot{R}_b^a (R_b^a)^T$

$$S = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$\underline{\omega}_{ab}^a = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$; $(\underline{\omega}_{ab}^a)^\times = \dot{R}_b^a (R_b^a)^T$

$\vec{\omega}_{ab}$ is called angular velocity of b relative to a

Definition angular velocity II

$$(\underline{\omega}_{ab}^{a})^{\times} R_b^a = \dot{R}_b^a (R_b^a)^T R_b^a$$

$$\dot{R}_b^a = (\underline{\omega}_{ab}^{a})^{\times} R_b^a$$

coordinate transformation matrix form of a dyadic
(→ similarity transformation)

$$(\underline{\omega}_{ab}^{a})^{\times} = R_b^a (\underline{\omega}_{ab}^{b})^{\times} R_a^b$$

$$\dot{R}_b^a = R_b^a (\underline{\omega}_{ab}^{b})^{\times} R_a^b R_b^a = R_b^a (\underline{\omega}_{ab}^{b})^{\times}$$

Angular velocity for simple rotation I

$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$[\underline{\omega}_x(\dot{\varphi})]^\times = \dot{\mathbf{R}}_x(\varphi)\mathbf{R}_x(\varphi)^T$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\varphi & -\cos\varphi \\ 0 & \cos\varphi & -\sin\varphi \end{pmatrix} \dot{\varphi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \dot{\varphi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\varphi} \\ 0 & \dot{\varphi} & 0 \end{pmatrix}$$

Angular velocity for simple rotation II

that shows $\qquad \omega_x(\dot{\rho}) = \begin{pmatrix} \dot{\rho} \\ 0 \\ 0 \end{pmatrix}$

$\rightarrow$ on the same $\qquad \omega_y(\dot{\theta}) = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} \qquad \omega_z(\dot{\psi}) = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

Remember: $\begin{pmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

For angle-axis parameterisation

$$\mathbf{R}_b^a = \mathbf{R}_{k,\theta} = \mathbf{I} + \underline{k}^\times \sin\theta + \underline{k}^\times \underline{k}^\times (1 - \cos\theta)$$

- Assume $\underline{k}$ is constant:

$$\begin{aligned}(\underline{\omega}_{ab}^a)^\times &= \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^T \\ &= \dot{\theta}\left(\underline{k}^\times \cos\theta + \underline{k}^\times \underline{k}^\times \sin\theta\right) \\ &\quad \left(\mathbf{I} - \underline{k}^\times \sin\theta + \underline{k}^\times \underline{k}^\times (1 - \cos\theta)\right)\end{aligned}$$

$$[\text{use:} \underline{k}^\times \underline{k}^\times \underline{k}^\times = \underline{k}^\times (\underline{k}\underline{k}^T - \underline{k}^T \underline{k}\mathbf{I}) = -\underline{k}^\times]$$

$$\begin{aligned}&= \ldots \\ &= \dot{\theta}\underline{k}^\times\end{aligned}$$

$$\boxed{\begin{aligned}\underline{\omega}_{ab}^a &= \dot{\theta}\underline{k} \\ \vec{\omega}_{ab} &= \dot{\theta}\vec{k}\end{aligned}}$$

Composite rotations

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$$

$$(\underline{\omega}_{ad}^a)^\times = \dot{R}_d^a (R_d^a)^T$$

$$= [\dot{R}_b^a R_c^b R_d^c + R_b^a \dot{R}_c^b R_d^c + R_b^a R_c^b \dot{R}_d^c] (R_d^c)^T \cdot (R_c^b)^T \cdot (R_b^a)^T$$

$$= \underbrace{\dot{R}_b^a (R_b^a)^T}_{(\underline{\omega}_{ab}^a)^\times} + R_b^a \underbrace{\dot{R}_c^b (R_c^b)^T}_{(\underline{\omega}_{bc}^b)^\times} (R_b^a)^T + R_b^a R_c^b \underbrace{\dot{R}_d^c (R_d^c)^T}_{(\underline{\omega}_{cd}^c)^\times} (R_c^b)^T (R_b^a)^T$$

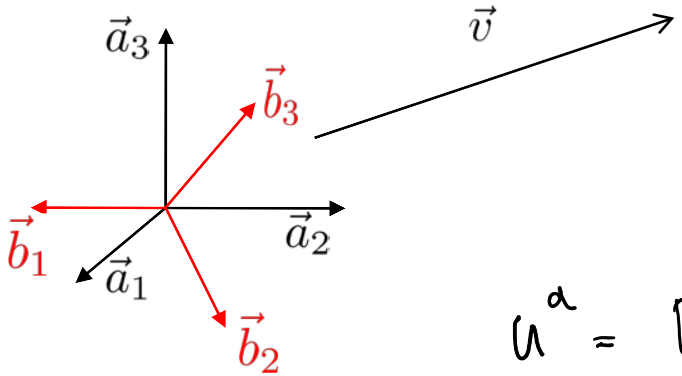
$$= (\underline{\omega}_{ab}^a)^\times + (\underline{\omega}_{bc}^a)^\times + (\underline{\omega}_{cd}^a)^\times$$

that shows $\underline{\omega}_{ad}^a = \underline{\omega}_{ab}^a + \underline{\omega}_{bc}^a + \underline{\omega}_{cd}^a$

or

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd}$$

Differentiation of coordinate vector



$$\underline{u}^a = \begin{pmatrix} u_1^a \\ u_2^a \\ u_3^a \end{pmatrix} \qquad \underline{u}^b = \begin{pmatrix} u_1^b \\ u_2^b \\ u_3^b \end{pmatrix}$$

$$\underline{u}^a = R_b^a \cdot \underline{u}^b \quad \rightarrow \quad \cancel{\dot{\underline{u}}^a = R_b^a \cdot \dot{\underline{u}}^b}$$

$$\begin{aligned} \dot{\underline{u}}^a &= R_b^a \, \dot{\underline{u}}^b + \dot{R}_b^a \, \underline{u}^b \\ &= R_b^a \left[\dot{\underline{u}}^b + \underbrace{R_a^b \, R_b^a}_{I} \, (\underline{\omega}_{ab}^b)^\times \, \underline{u}^b \right] \\ &= R_b^a \left[\dot{\underline{u}}^b + (\underline{\omega}_{ab}^b)^\times \, \underline{u}^b \right] \end{aligned}$$

Differentiation of coordinate-free vector

$$\frac{d}{dt}\vec{u} = ?$$

$\rightarrow$ not defined

$$\vec{u} = u_1^a \vec{a}_1 + u_2^a \vec{a}_2 + u_3^a \vec{a}_3$$

$$\frac{^a d}{dt}\vec{u} = \dot{u}_1^a \vec{a}_1 + \dot{u}_2^a \vec{a}_2 + \dot{u}_3^a \vec{a}_3$$

$$\frac{^b d}{dt}\vec{u} = \dot{u}_1^b \vec{b}_1 + \dot{u}_2^b \vec{b}_2 + \dot{u}_3^b \vec{b}_3$$

$$\frac{^a d}{dt}\vec{u} = \frac{^b d}{dt}\vec{u} + \vec{\omega}_{ab} \times \vec{u}$$

Kinematic differential equations

- Translation: $\underline{v} \to \underline{r}$: $\dot{\underline{r}} = \underline{v}$
- Rotation: $\underline{\omega}_{ab}^{a} \to \mathbf{R}_b^a$: $\dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^{a})^{\times}\mathbf{R}_b^a$

$$\underline{\omega}_{ab}^{a} \to \text{Euler angle} \qquad \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

$$\underline{\omega}_{ab}^{a} \to \text{Euler parameter} \qquad \dot{\eta} = ? \quad \dot{\underline{\varepsilon}} = ?$$

Kinematic differential equation of Euler angles I

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd}$$

$$= \dot{\psi}\,\vec{a}_3 + \dot{\theta}\,a\vec{b}_2 + \dot{\phi}\,\vec{c}_1$$

- If $\theta = 90^o$:
  - $\vec{a}_3$ is parallel to $\vec{c}_1$ (ψ and ϕ have the same axis)
  - Angular velocity components along $\vec{a}_3 \times \vec{b}_2$ (evtl. $\vec{c}_1 \times \vec{b}_2$) cannot be described
  - Singularity of the Euler angles

Kinematic differential equation of Euler angles II

$$\underline{\omega}_{ad}^{a} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + R_z(\psi)\begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + R_z(\psi)\,R_y(\theta)\begin{pmatrix} \dot{\varphi} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\psi\,\dot{\theta} + \cos\psi\,\cos\theta\,\dot{\varphi} \\ \cos\psi\,\dot{\theta} + \sin\psi\,\cos\theta\,\dot{\varphi} \\ \dot{\psi} - \sin\theta\,\dot{\varphi} \end{pmatrix}$$

$$= E_a(\underline{\varphi})\,\dot{\varphi} \qquad \varphi = \begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix}$$

$$E_a(\underline{\varphi}) = \begin{bmatrix} c\psi\, c\theta & -s\psi & 0 \\ s\psi\, c\theta & c\psi & 0 \\ -s\theta & 0 & 1 \end{bmatrix}$$

Kinematic differential equation of Euler angles III

$$\det\left(E_a(\varphi)\right) = \cos\theta\left(\cos^2\psi + \sin^2\psi\right) = \cos\theta$$

$$\rightarrow E_a(\underline{\varphi}) \text{ singular } \theta = 90^\circ \quad \left(\frac{\pi}{2} + k\pi \,;\; k = 0, \pm 1, \dots\right)$$

$$\dot{\underline{\varphi}} = E_a^{-1}(\underline{\varphi})\, \underline{\omega}^a_{ad}$$

Kinematic differential equation of Euler parameter

$$\mathbf{R}_b^a = \mathbf{R}(\eta, \underline{\varepsilon}) \qquad \dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^a)^\times \mathbf{R}_b^a$$

- It can be derived (quaternion algebra p. 248)

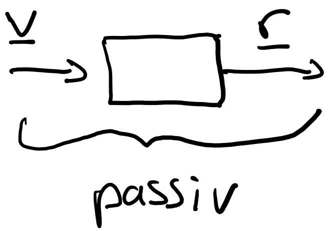
$$\dot{\eta} = -\frac{1}{2}\underline{\varepsilon}^T \underline{\omega}_{ab}^a$$
$$\dot{\underline{\varepsilon}} = \frac{1}{2}(\eta \mathbf{I} - \underline{\varepsilon}^\times)\underline{\omega}_{ab}^a$$

Passivity of kinematic differential equation

Translation: $\dot{r} = v$

$$V = \frac{1}{2}\underline{r}^T\underline{r} > 0$$

$$\dot{V} = \underline{r}^T\underline{v}$$



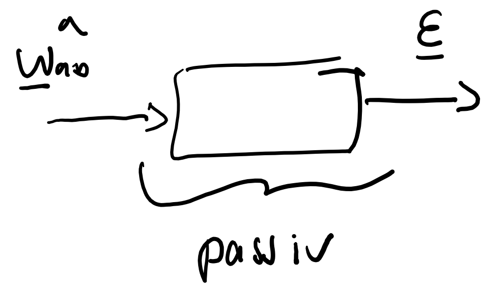
Rotation:

$$V = 2(1-\eta) \geq 0$$

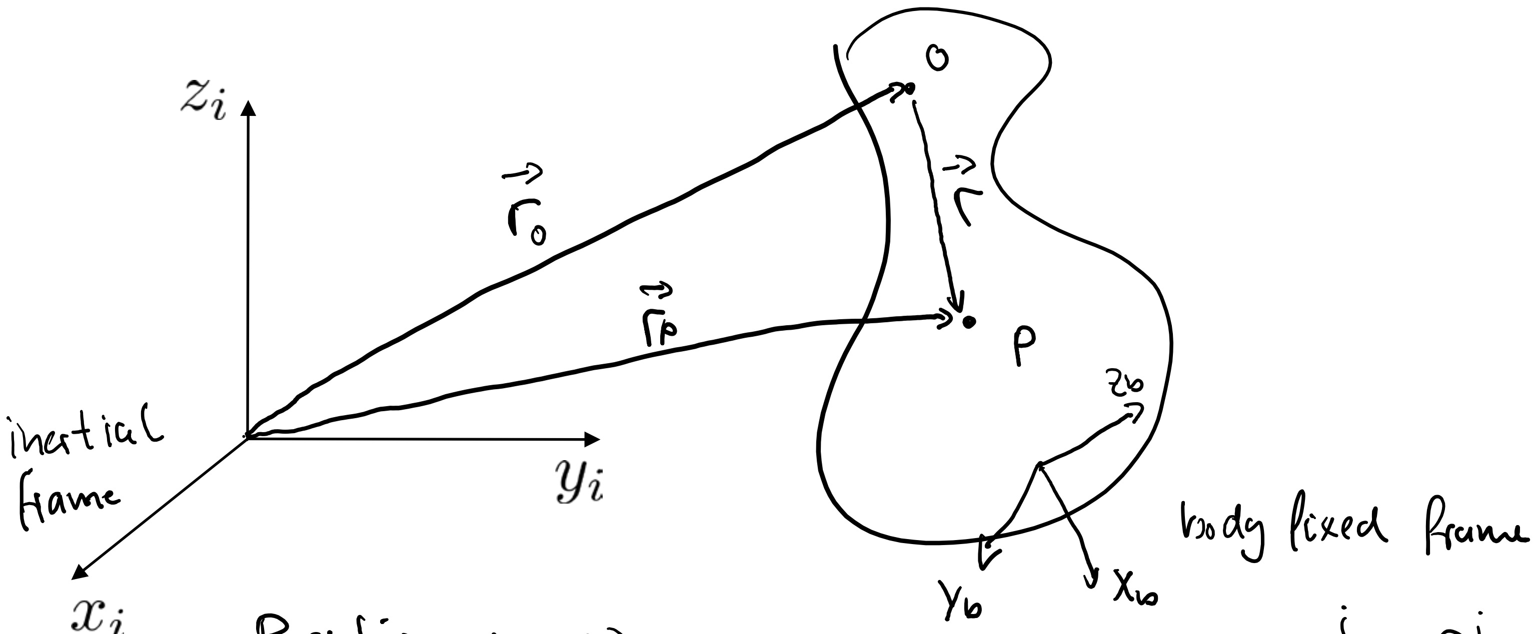
$$\left(|\eta| = \left|\cos\frac{\theta}{2}\right| \leq 1\right)$$

$$\dot{V} = -2\dot{\eta}$$

$$= \underline{\varepsilon}^T \underline{\omega}_{ab}^{a}$$



Kinematics of rigid body I



Position: $\vec r_p = \vec r_o + \vec r$ $\quad\quad \underline{r}^i = R^i_b \, \underline{r}^b$

Velocity: $\vec v_o = \frac{{}^i d}{dt} \vec r_o$ $\quad\quad \vec v_p = \frac{{}^i d}{dt} \vec r_p$

$$\vec v_p = \frac{{}^i d}{dt} \vec r_p = \frac{{}^i d}{dt} \vec r_o + \frac{{}^i d}{dt} \vec r$$

$$= \vec v_o + \frac{{}^b d}{dt} \vec r + \vec \omega_{ib} \times \vec r$$

Kinematics of rigid body II

$$\vec{a}_o = {}^{i}\frac{d^2}{dt^2}\vec{r}_o \qquad \vec{a}_p = {}^{i}\frac{d^2}{dt^2}\vec{r}_p$$

$$\vec{\alpha}_{ib} = {}^{i}\frac{d}{dt}\vec{\omega}_{ib} = {}^{b}\frac{d}{dt}\vec{\omega}_{ib} + \cancel{\vec{\omega}_{ib} \times \vec{\omega}_{ib}}$$

$${}^{i}\frac{d^2}{dt^2}\vec{r}_p = {}^{i}\frac{d^2}{dt^2}\vec{r}_o + {}^{i}\frac{d^2}{dt^2}\vec{r}$$

$$\vec{a}_p = \vec{a}_o + {}^{i}\frac{d}{dt}\left({}^{i}\frac{d}{dt}\vec{r}\right)$$

$$= \vec{a}_o + {}^{i}\frac{d}{dt}\left({}^{b}\frac{d}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r}\right)$$

$$= a_o + {}^{b}\frac{d}{dt}\,{}^{b}\frac{d}{dt}\vec{r} + \vec{\omega}_{ib} \times {}^{b}\frac{d}{dt}\vec{r} + {}^{b}\frac{d}{dt}\vec{\omega}_{ib} \times \vec{r} +$$

$$\vec{\omega}_{ib} \times \left(\vec{\omega}_{ib} \times \vec{r}\right) + \vec{\omega}_{ib} \times {}^{b}\frac{d}{dt}\vec{r}$$

Kinematics of rigid body III

$$\vec{a}_c = \vec{a}_o + \frac{{}^b d^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r}_g + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)$$

Homework

- Derive $[\omega_y(\dot{\theta})]^\times$ and $[\omega_z(\dot{\psi})]^\times$ from $R_y(\theta)$ and $R_z(\psi)$, respectively.
- Derive w_{ad}^b for the Euler angles using the roll-pitch-yaw case (check 6.9.4). *Think good about the order and direction of transformations.*
- Read 6.12
- Read 7.1-7.2