NTNU – Trondheim
Norwegian University of
Science and Technology

Assignment 9

TTK4130 Modeling and Simulation

Problem 1 (Cylindrical and spherical coordinates, Lagrange's equations of motion, friction. 60 %)

Consider a point mass in space with mass $m > 0$ and position vector $\vec{r}_m$.

Let $\mathbf{r}_m^i = [x\,,\, y\,,\, z]^T$ be the coordinates of the position vector respect to an inertial frame i.

As we known from Calculus, the position $\mathbf{r}_m^i$ can be represented using cylindrical coordinates:

$$x = r\cos\theta \tag{1a}$$
$$y = r\sin\theta \tag{1b}$$
$$z = z\,, \tag{1c}$$

as well as and spherical coordinates:

$$x = r\sin\phi\cos\theta \tag{2a}$$
$$y = r\sin\phi\sin\theta \tag{2b}$$
$$z = r\cos\phi. \tag{2c}$$

Moreover, a local frame can be defined for each of these coordinate transformations: The cylindrical and the spherical reference frames.

These frames are centered at the position of the point mass, and are denoted by c and s, respectively. For the cylindrical coordinates, the axes are given by the vectors

$$\mathbf{c}_r^i = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \quad \mathbf{c}_\theta^i = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad \mathbf{c}_z^i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$

and for the spherical coordinates, the axes are given by the vectors

$$\mathbf{s}_r^i = \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} \quad \mathbf{s}_\phi^i = \begin{bmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{bmatrix} \quad \mathbf{s}_\theta^i = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}. \tag{4}$$

(a) Show that the vector sets $\{\vec{c}_r\,,\, \vec{c}_\theta\,,\, \vec{c}_z\}$ and $\{\vec{s}_r\,,\, \vec{s}_\phi\,,\, \vec{s}_\theta\}$ define right-handed reference frames. Show the details of your calculations.
Hint: Read section 6.4 in the book.

Solution: For the cylindrical coordinates:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

For the spherical coordinates:

$$\begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} = 1$$

(b) Express the time derivatives of $\mathbf{c}_r^i$, $\mathbf{c}_\theta^i$ and $\mathbf{c}_z^i$ as a function of themselves and the variables r, θ, z and their time derivatives.
Finally, find ω_{ic}^c.
Show the details of your calculations.
Hint: Read section 6.8 in the book. What is $(\omega_{ic}^c)^\times$*?*

Solution: Direct derivation gives

$$\begin{aligned}\dot{\mathbf{c}}_r^i &= \mathbf{c}_\theta^i \dot{\theta} \\ \dot{\mathbf{c}}_\theta^i &= -\mathbf{c}_r^i \dot{\theta} \\ \dot{\mathbf{c}}_z^i &= \mathbf{0}.\end{aligned}$$

Hence,

$$(\omega_{ic}^c)^\times = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\omega_{ic}^c = [0\,,0\,,\dot{\theta}]^T$, i.e. $\vec{\omega}_{ic} = \dot{\theta}\vec{e}_z$.

(c) Express the time derivatives of $\mathbf{s}_r^i$, $\mathbf{s}_\phi^i$ and $\mathbf{s}_z^i$ as a function of themselves and the variables r, ϕ, θ and their time derivatives.
Finally, find ω_{is}^s.
Show the details of your calculations.
Hint: Read section 6.8 in the book. What is $(\omega_{is}^s)^\times$*?*

Solution: Direct derivation gives

$$\begin{aligned}\dot{\mathbf{s}}_r^i &= \mathbf{s}_\phi^i \dot{\phi} + \mathbf{s}_\theta^i \sin\phi\dot{\theta} \\ \dot{\mathbf{s}}_\phi^i &= -\mathbf{s}_r^i \dot{\phi} + \mathbf{s}_\theta^i \cos\phi\dot{\theta} \\ \dot{\mathbf{s}}_\theta^i &= -\dot{\mathbf{s}}_r^i \sin\phi\dot{\theta} - \dot{\mathbf{s}}_\phi^i \cos\phi\dot{\theta}.\end{aligned}$$

Hence,

$$(\omega_{is}^s)^\times = \begin{bmatrix} 0 & -\dot{\phi} & -\sin\phi\dot{\theta} \\ \dot{\phi} & 0 & -\cos\phi\dot{\theta} \\ \sin\phi\dot{\theta} & \cos\phi\dot{\theta} & 0 \end{bmatrix}$$

and $\omega_{is}^s = [\cos\phi\dot{\theta}\,, -\sin\phi\dot{\theta}\,, \dot{\phi}]^T$, i.e. $\vec{\omega}_{is} = \cos\phi\dot{\theta}\vec{s}_r - \sin\phi\dot{\theta}\vec{s}_\phi + \dot{\phi}\vec{s}_\theta$.

Let $\vec{v}_m = \dot{\vec{r}}_m$ and $\vec{a}_m = \ddot{\vec{r}}_m$ be the velocity and acceleration of the point mass, respectively.

(d) Show that the position, velocity and acceleration of the point mass are given in the cylindrical reference frame by

$$\vec{r}_m = r\vec{c}_r + z\vec{c}_z \tag{5a}$$
$$\vec{v}_m = \dot{r}\vec{c}_r + r\dot{\theta}\vec{c}_\theta + \dot{z}\vec{c}_z \tag{5b}$$
$$\vec{a}_m = (\ddot{r} - r\dot{\theta}^2)\vec{c}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{c}_\theta + \ddot{z}\vec{c}_z. \tag{5c}$$

Show the details of your calculations.
Hint 1: Use the results found in part b.

Hint 2: Read section 6.12 in the book.

Solution: Equation $\vec{r}_m = r\vec{c}_r + z\vec{c}_z$ follows directly from the definition of cylindrical coordinates and the cylindrical reference frame.
Derivation and the results found in part b. give the rest of equations.

(e) Show that the position, velocity and acceleration of the point mass are given in the spherical reference frame by

$$\begin{aligned} \vec{r}_m &= r\vec{s}_r && \text{(6a)} \\ \vec{v}_m &= \dot{r}\vec{s}_r + r\dot{\phi}\vec{s}_\phi + r\sin\phi\dot{\theta}\vec{s}_\theta && \text{(6b)} \\ \vec{a}_m &= (\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\dot{\theta}^2)\vec{s}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\sin\phi\cos\phi\dot{\theta}^2)\vec{s}_\phi \\ &\quad + (r\sin\phi\ddot{\theta} + 2\dot{r}\sin\phi\dot{\theta} + 2r\dot{\phi}\cos\phi\dot{\theta})\vec{s}_\theta. && \text{(6c)} \end{aligned}$$

Show the details of your calculations.
Hint 1: Use the results found in part c.
Hint 2: Read section 6.12 in the book.

Solution: Equation $\vec{r}_m = r\vec{s}_r$ follows directly from the definition of spherical coordinates and the spherical reference frame.
Derivation and the results found in part c. give the rest of equations.

We will now assume that the point mass in constrained to move on the surface of a sphere. The sphere in question has radius $R > 0$ and is centered at the origin of the frame i. Hence, it is convenient to express the position of the particle using spherical coordinates:

$$\begin{aligned} x &= R\sin\phi\cos\theta && \text{(7a)} \\ y &= R\sin\phi\sin\theta && \text{(7b)} \\ z &= R\cos\phi. && \text{(7c)} \end{aligned}$$

Let $\vec{N}$ be the force of constraint, i.e. the virtual force that keeps the point mass on this surface.
Moreover, assume that the only force acting on the point mass besides $\vec{N}$ is the gravitational force $\vec{G}$, where $\mathbf{G}^i = -mg\mathbf{e}_3$.

(f) Express the force of constraint $\vec{N}$ as a function of other variables and parameters of this problem. Show the details of your calculations.
Hint: Use Newton's 2. Law and the results from part e.

Solution: Since

$$\begin{aligned} m\vec{a}_m &= \vec{N} + \vec{G} \\ \vec{G} &= -mg\cos\phi\vec{e}_r + mg\sin\phi\vec{e}_\phi \\ \vec{a}_m &= (-R\dot{\phi}^2 - R\sin^2\phi\dot{\theta}^2)\vec{s}_r + (R\ddot{\phi} - R\sin\phi\cos\phi\dot{\theta}^2)\vec{s}_\phi \\ &\quad + (R\sin\phi\ddot{\theta} + 2R\dot{\phi}\cos\phi\dot{\theta})\vec{s}_\theta\,, \end{aligned}$$

it follows that

$$\begin{aligned} \vec{N} &= m(g\cos\phi - R\dot{\phi}^2 - R\sin^2\phi\dot{\theta}^2)\vec{s}_r \\ &\quad + m(-g\sin\phi + R\ddot{\phi} - R\sin\phi\cos\phi\dot{\theta}^2)\vec{s}_\phi \\ &\quad + m(R\sin\phi\ddot{\theta} + 2R\dot{\phi}\cos\phi\dot{\theta})\vec{s}_\theta. \end{aligned}$$

(g) Find the equations of motion for the point mass, i.e. find the differential equations for ϕ and θ. Show the details of your calculations.
Hint: Read section 8.2 in the book.

Solution: Since $r = R$ is constant, the velocity vector is $\vec{v}_m = R\dot{\phi}\vec{s}_\phi + R\sin\phi\dot{\theta}\vec{s}_\theta$.
Hence, the kinetic energy is

$$T = \frac{1}{2}mR^2\left(\dot{\phi}^2 + \sin^2\phi\dot{\theta}^2\right).$$

On the other hand, the potential energy is

$$U = mgz = mgR\cos\phi.$$

The Lagrangian is $\mathcal{L} = T - U$. Moreover,

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= mR^2\dot{\phi} \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= mR^2\ddot{\phi} \\
\frac{\partial\mathcal{L}}{\partial\phi} &= mR^2\sin\phi\cos\phi\dot{\theta}^2 + mgR\sin\phi \\
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= mR^2\sin^2\phi\dot{\theta} \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= 2mR^2\sin\phi\cos\phi\dot{\phi}\dot{\theta} + mR^2\sin^2\phi\ddot{\theta} \\
\frac{\partial\mathcal{L}}{\partial\theta} &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\ddot{\phi} &= \sin\phi\cos\phi\dot{\theta}^2 + \frac{g}{R}\sin\phi \\
\ddot{\theta} &= -2\cot\phi\dot{\phi}\dot{\theta}.
\end{aligned}$$

We will now constrain the movement of the point mass even more: We will assume that the point mass can only move along a spherical spiral, which is given in spherical coordinates by:

$$x = \frac{R}{\sqrt{1 + a^2\theta^2}}\cos\theta \tag{8a}$$

$$y = \frac{R}{\sqrt{1 + a^2\theta^2}}\sin\theta \tag{8b}$$

$$z = \frac{Ra\theta}{\sqrt{1 + a^2\theta^2}}, \tag{8c}$$

where $a > 0$ is a parameter that describes how steep the spiral is.

(h) Find ϕ as a function of other parameters and variables of this problem.
Show the details of your calculations.
Hint: Use the definition of spherical coordinates.

Solution: One of many trigonometric identities is

$$\phi = \cot^{-1}(a\theta).$$

(i) Find the equations of motion for the point mass, i.e find the differential equation for θ. Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: The solution is very similar to the solution to part j.

Solution: Since $r = R$ is constant and $\phi = \cot^{-1}(a\theta)$, the velocity vector is

$$\vec{v}_m = R\dot{\phi}\vec{s}_\phi + R\sin\phi\dot{\theta}\vec{s}_\theta = -\frac{Ra\dot{\theta}}{1+a^2\theta^2}\vec{s}_\phi + \frac{R\dot{\theta}}{\sqrt{1+a^2\theta^2}}\vec{s}_\theta$$

Hence, the kinetic energy is

$$T = \frac{1}{2}mR^2\dot{\theta}^2\frac{1+a^2+a^2\theta^2}{(1+a^2\theta^2)^2}.$$

On the other hand, the potential energy is

$$U = mgz = mgR\frac{a\theta}{\sqrt{1+a^2\theta^2}}.$$

The Lagrangian is $\mathcal{L} = T - U$. Moreover,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mR^2\frac{1+a+a^2\theta^2}{(1+a^2\theta^2)^2}\dot{\theta} \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= mR^2\frac{1+a+a^2\theta^2}{(1+a^2\theta^2)^2}\ddot{\theta} - 2mR^2a^2\theta\frac{1+2a+a^2\theta^2}{(1+a^2\theta^2)^3}\dot{\theta}^2 \\
\frac{\partial \mathcal{L}}{\partial \theta} &= -mR^2a^2\theta\frac{1+2a+a^2\theta^2}{(1+a^2\theta^2)^3}\dot{\theta}^2 - mgRa\frac{1}{(1+a^2\theta^2)^{\frac{3}{2}}}.
\end{aligned}$$

Hence,

$$\ddot{\theta} = \frac{a^2\theta(1+2a^2+a^2\theta^2)}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)}\dot{\theta}^2 - \frac{g}{R}\frac{a\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2}.$$

(j) Assume that an additional friction force given by $\vec{F}_f = -k\vec{v}_m$ acts on the mass particle. Show that the differential equation for the generalized coordinate θ for this new situation is

$$\ddot{\theta} = \frac{a^2\theta(1+2a^2+a^2\theta^2)}{(1+a^2\theta^2)(1+a^2+a^2\theta^2)}\dot{\theta}^2 - \frac{g}{R}\frac{a\sqrt{1+a^2\theta^2}}{1+a^2+a^2\theta^2} - \frac{k}{m}\dot{\theta}. \tag{9}$$

Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: Find the generalized force.

Solution: The generalized force for the only generalized coordinate θ is given by

$$\tau = \frac{\partial \vec{r}_m}{\partial \theta} \cdot \vec{F}_f.$$

Since

$$\begin{aligned}
\frac{\partial \vec{r}_m}{\partial \theta} &= R\left(-\frac{a}{1+a^2\theta^2}\vec{s}_\phi + \frac{1}{\sqrt{1+a^2\theta^2}}\vec{s}_\theta\right) \\
\vec{F}_f &= -k\vec{v}_m = -kR\left(-\frac{a\dot{\theta}}{1+a^2\theta^2}\vec{s}_\phi + \frac{\dot{\theta}}{\sqrt{1+a^2\theta^2}}\vec{s}_\theta\right),
\end{aligned}$$

it follows that

$$\tau = -kR^2 \frac{1 + a^2 + a^2\theta^2}{(1 + a^2\theta^2)^2}\dot{\theta}.$$

By applying Lagrange's equations of motion to $\mathcal{L} = T - U$, we obtain the following differential equation

$$\ddot{\theta} = \frac{a^2\theta(1 + 2a^2 + a^2\theta^2)}{(1 + a^2\theta^2)(1 + a^2 + a^2\theta^2)}\dot{\theta}^2 - \frac{g}{R}\frac{a\sqrt{1 + a^2\theta^2}}{1 + a^2 + a^2\theta^2} - \frac{k}{m}\dot{\theta}.$$

Problem 2 (Friction models, integration methods, event detection. 40 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

In this problem we will implement different static and dynamic friction models. In order to test these models, we will consider a small box of mass $m = 2\,\text{kg}$ sliding on a table due to the application of an external force $F_a = F_a(t) = k_a t$ (ramp function). A friction force F_f will oppose this motion. This simple testbench model is illustrated in Figure 1. Moreover, we will use the parameters

$F_c = 1.5$	Coulomb friction
$F_s = 1.75$	Stiction (static friction)
$F_v = 0.1$	Viscous friction
$v_s = 0.2$	Characteristic Stribeck velocity.

Hint for the whole problem: Read sections 5.2 and 5.3 in book.

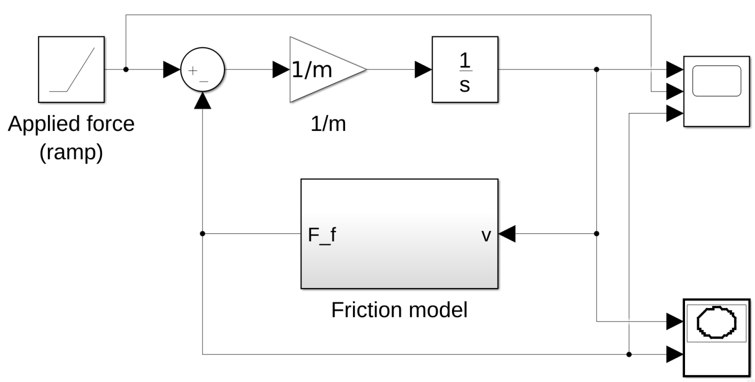


Figure 1: Testbench for friction models.

We start with static friction models, more specifically Coulomb's model, which is given by

$$F_f = F_c \text{sign}(v)\,, \quad v \neq 0\,, \tag{10}$$

where v is the velocity.

(a) Implement Coulomb's model in Simulink using a sign-block. Do not use the built-in Coulomb friction block. Simulate the model over 10 s with a ramp slope k_a of 1 and 3, and with $v(0) = 0$. Use first a variable step solver. Explain what happens.

Thereafter, choose a fixed-step solver with sample time 0.01 instead. How does the model simulate now for both values of the ramp?

Add a figure with the Simulink block diagrams that implement the Coulomb's friction model to your answer.

Hint: Does the sign block include zero-crossing detection?

Solution: The implementation is shown in Figure 2.

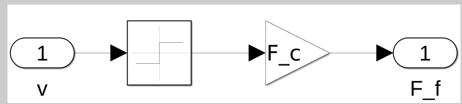


Figure 2: Implementation of Coulomb's friction model in Simulink.

The variable-step solver detects an event (a zero-crossing in the sign-function, v goes from $v = 0$ to $v > 0$) already on the first step, and tries to locate the time of the zero-crossing. However, no matter how small step is chosen, the zero-crossing happens during the first step. Due to settings in the solver, it finally chooses a very small step-size, and tries to continue. Since initially F_a is small, $F_s > F_a$ and $\dot{v} = F_a - F_s < 0$ for $v > 0$, and $\dot{v} = F_a + F_s > 0$ for $v < 0$. Hence, v will unphysically oscillate between a negative and positive value, generating a lot of zero-crossings (one per step). Due to the small steps taken, the maximum number of zero-crossings is quickly reached.

For a fixed-step solver one might get similar oscillations, depending on the algorithm and the time-step chosen. For the settings in this problem, we observe that we have oscillations as long as $F_a < F_s$. Hence, the oscillations of the response last longer for smaller ramp slopes.

The Coulomb's friction model has the disadvantage that it is not defined at $v = 0$. The Karnopp's model of Coulomb friction solves this by defining

$$F_f = \begin{cases} \text{sat}(F_a, F_c)\,, & v = 0 \\ F_c \text{sign}(v)\,, & v \neq 0\,, \end{cases} \tag{11}$$

where the saturation function sat is defined in page 198 in the book, and can be implemented using a saturation block in Simulink. For the Karnopp's model to work properly, we must either use variable-step methods with event-detection to determine exactly when $v = 0$, or we have to use some kind of dead-zone around zero-velocity to treat the velocity as zero when it is small. In this problem, we will implement both of these approaches.

(b) Implement Karnopp's friction model using the setup shown in Figure 3, i.e. implement the innards of the two If Action Subsystems. Note that the if-block generates events by default when the value of the if-clause changes.

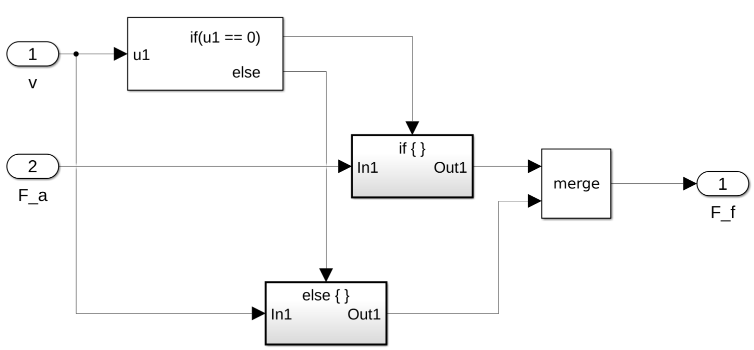
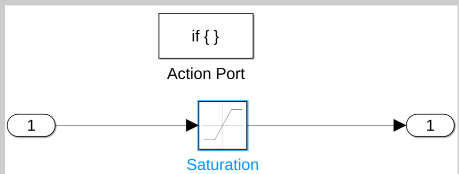


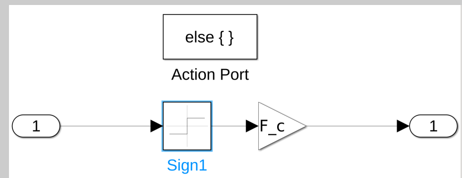
Figure 3: Setup for implementing Karnopp's friction model with event detection.

Simulate using a variable step method. Set $k_a = 1$ and $v(0) = 0$. Comment on the results.
Which role does event-detection play in the simulation?
Add a figure with the Simulink block diagrams implemented inside each If Action Subsystem to your answer.

Solution: The contents of the If Action Subsystems is shown in Figure 4.



(a) Subsystem 1



(b) Subsystem 2

Figure 4: The If Action Subsystems.

In the simulation, we observe that $v = 0$ until F_a becomes larger than F_c, as expected. The event-detection avoids executing the signum-function unless $v \neq 0$.

(c) Implement Karnopp's model without relying on event-detection, by using a dead-zone:

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_c), & |v| \leq \delta \\ F_c \mathrm{sign}(v), & |v| > \delta. \end{cases} \tag{12}$$

Choose $\delta = 1$ and $k_a = 1$. Use a fixed-step solver, and simulate for initial velocity $v(0) = 0$ (as until now) and for $v(0) = -2$. Comment on both results.
Add a figure with the Simulink block diagrams that implement the Karnopp's friction model with dead-zone to your answer.

Solution: Karnopp's friction model with dead-zone can be implemented in several ways. For example, one can reuse the framework from the previous problem as shown in Figure 5.

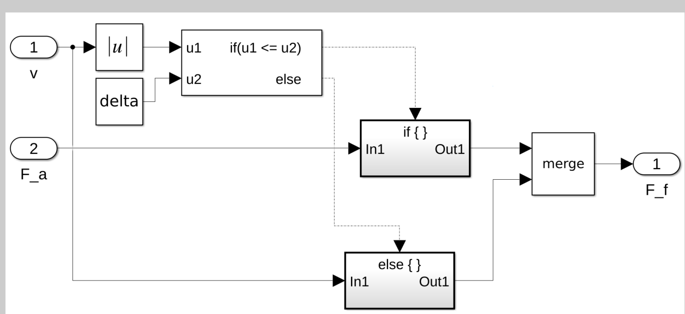


Figure 5: Setup for implementing Karnopp's friction model with dead-zone.

If we use initial velocity $v(0) = 0$ we get identical results as in part b. On the other hand, if the initial velocity is $v(0) = -2$, we observe that the velocity remains $v = -\delta$ when inside the dead-zone, where we ideally want $v = 0$. If this is not an acceptable behavior, we should choose δ smaller. However, if we choose δ too small, we will obtain the oscillations and non-physical behavior from part a.

(d) Extend the friction model from part b. with sticking, Stribeck-effect and linear viscous friction:

$$F_f = \begin{cases} \text{sat}(F_a, F_s)\,, & v = 0 \\ \left(F_c + (F_s - F_c)e^{-(v/v_s)^2}\right)\text{sign}(v) + F_v v\,, & v \neq 0. \end{cases} \tag{13}$$

Simulate with $k_a = 1$ and $v(0) = 0$.

Add a figure with the Simulink block diagrams that implement the friction model (13) to your answer. Moreover, enclose plots of the velocity v and the forces F_a and F_f as a function of time.

Comment on the results, and compare with the results obtained for Coulomb's friction model.

NB: There is a typo in (5.23) in the book.

Solution: This friction model can be implemented in several ways. For example, one can reuse the framework from part b.: The "else" If Action Subsystem is shown in Figure 6. In the "if" If Action Subsystem, the limit of the saturation block was changed from $\pm F_c$ to $\pm F_s$.

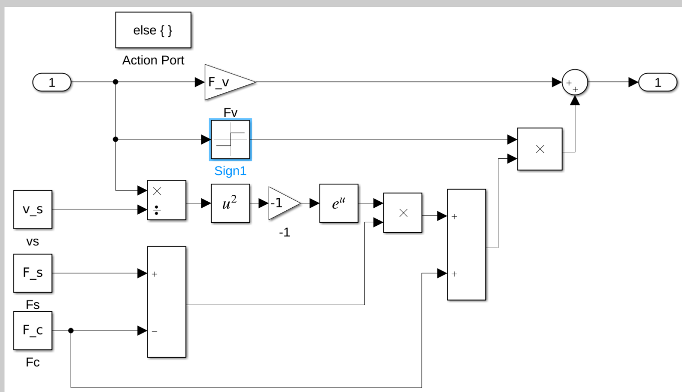
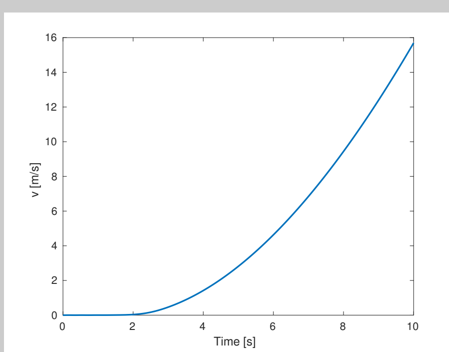
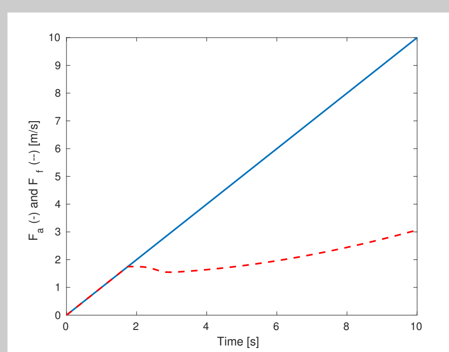


Figure 6: The If Action Subsystem for Stribeck friction.

A plot of velocity and forces is shown in Figure 7. We observe that in the sticking region, the friction force is equal to the applied force due to Karnopp's model, while in the sliding region we have an initial larger sticking force, which then reduces before it increases again due to the viscous friction for higher velocities.



(a) Velocity



(b) F_a (blue, whole) and F_f (red, dashed).

Figure 7: Velocity and forces for Karnopp's model with Stribeck friction.

(e) Finally, we will implement the LuGre dynamic friction model:

$$\dot{z} = v - \sigma_0 \frac{|v|}{g(v)} z \tag{14a}$$

$$g(v) = F_c + (F_s - F_c)e^{-(v/v_s)^2} \tag{14b}$$

$$F_f = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v\,, \tag{14c}$$

where z represents a small displacement in the stick-zone and σ_0 represents the "spring-stiffness" of the asperities of the surface. Set $\sigma_0 = 750$, $\sigma_1 = 0$ and $\sigma_2 = F_v$.

Simulate with $k_a = 1$ and $v(0) = 0$.

Add a figure with the Simulink block diagrams that implement LuGre model to your answer. Moreover, enclose plots of the velocity v and the forces F_a and F_f as a function of time.

Comment on the results, and compare with the results obtained in part d.

Furthermore, play around with the parameters σ_0 and σ_1, and comment on the model behaviour.

Hint: Do you obtain an oscillatory response? If yes, how would you modify σ_0 or σ_1 to get rid of this?

NB: There is a typo in (5.43) in the book.

Solution: The implementation of the dynamic LuGre friction model is shown in Figure 8. In this model, we do not have to worry about discontinuities and events, and can use both variable-step and fixed-step solvers. Note, however, that the model can become stiff if too large σ_0 and σ_1 are used. Hence, we might have to use small step lengths in fixed-step solvers, and variable-step solvers may need long time to simulate.

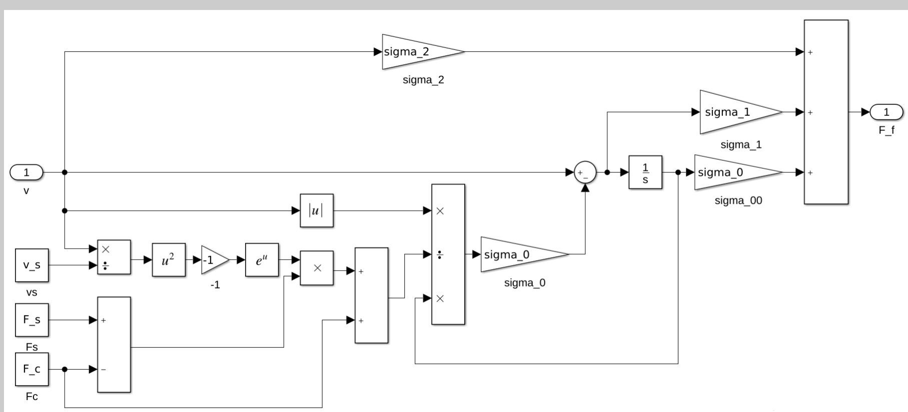
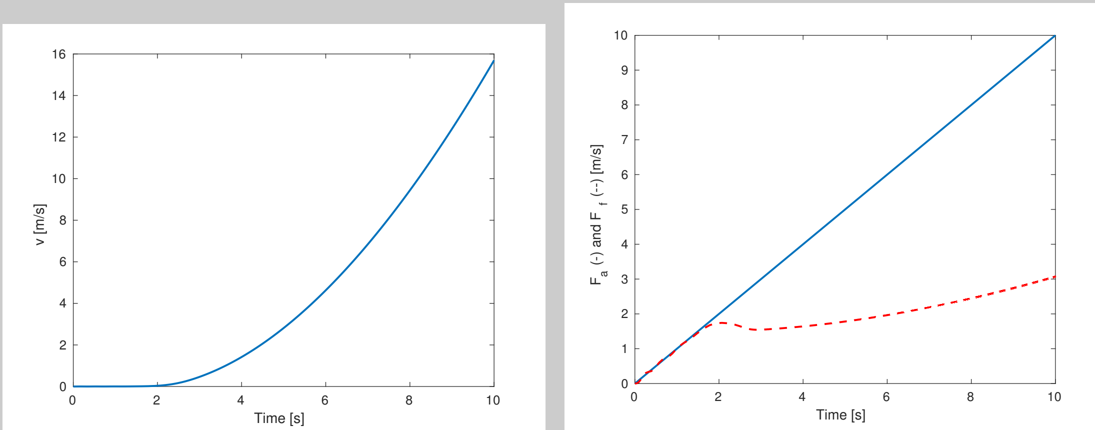


Figure 8: Simulink implementation of the LuGre friction model.

As can be seen in Figure 9, the results are similar to those obtained in part d. (Figure 7). However, if we look closer, we observe that we have small oscillations in the friction force in the sticking region, which will lead to small oscillations in the velocity, which again might cause unphysical drift (integral of velocity). These oscillations can be reduced by using a smaller σ_0, but this will increase the "time constant", and may therefore not be a good solution. Increasing σ_1 is another way to reduce oscillations in the sticking region, but may introduce significant stiffness in the model.



(a) Velocity (b) F_a (blue, whole) and F_f (red, dashed).

Figure 9: Velocity and forces, LuGre friction model with Stribeck friction.