Lecture 19: Rigid body dynamics

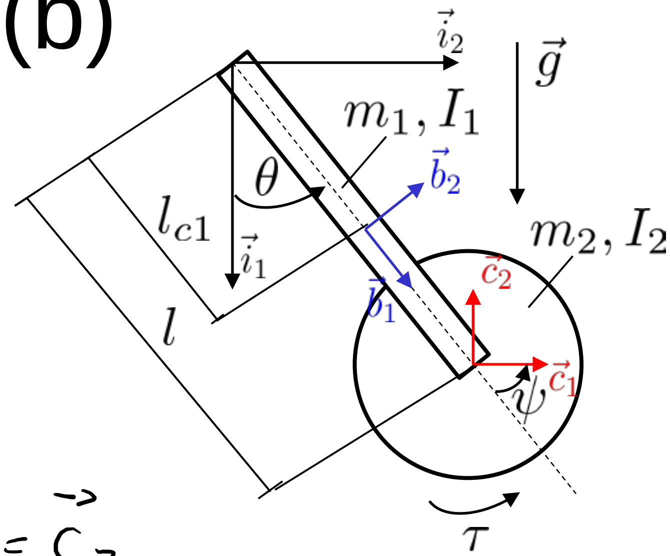
- Block in a pipe example
- Inverted Pendulum example

- Lagrange method of first kind

Gyroscopic pendulum

(Inertia wheel pendulum)

Gyroscopic pendulum (a),(b)



(a) generalised coord. : $[\theta, \psi]$

$q_1 \; ; \; q_2$

generalised forces : $0 \; ; \; \tau$

(b) $\vec{\omega}_{ib} = \dot{\theta}\,\vec{i}_3 = \dot{q}_1 \vec{i}_3$

$\vec{\omega}_{bc} = \dot{q}_2 \vec{b}_3$

$\vec{b}_3 = \vec{i}_3 = \vec{c}_3$

$\vec{\omega}_{ic} = \vec{\omega}_{ib} + \vec{\omega}_{bc} = (\dot{q}_1 + \dot{q}_2)\,\vec{i}_3$

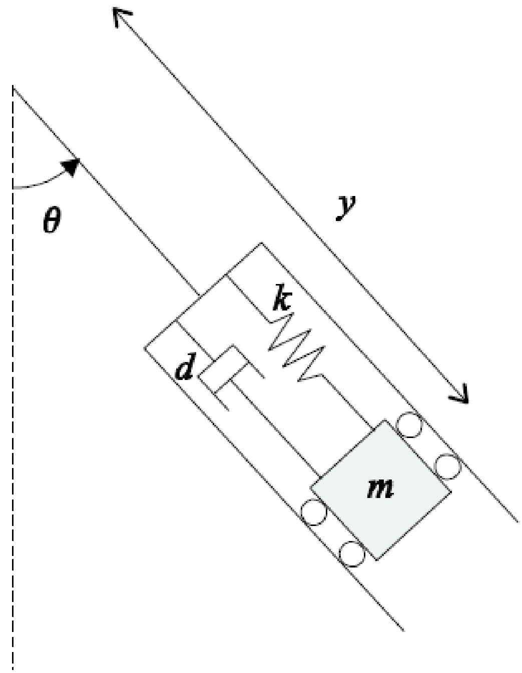


Figure 1: Kloss i rør

Oppgave 3) (15 %)

Figur (1) viser en kloss inne i et rør som svinger om et opphengspunkt. Anta at all masse bortsett fra klossen er neglisjerbar, og at klossens masse er m med massesenter gitt av y som er avtanden mellom massesenteret og opphengspunktet. Videre er fjærkonstanten k og dempekonstanten d. Fjæra er kraftløs når $y = y_0$. Det er ingen friksjon i systemet.

Velg passende generaliserte koordinater $\mathbf{q}$ og bruk Lagranges formulering for å sette opp en matematisk modell.

Block in a pipe I

$$\delta W = F \cdot \delta q$$

Damper: $\delta W = -d\,\dot{q}\,\delta q$

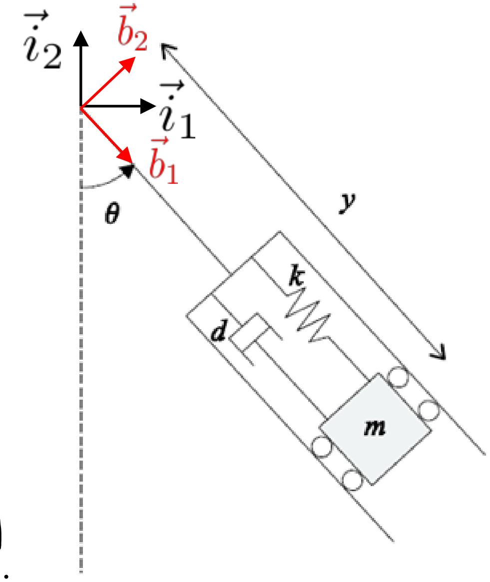


Figure 1: Kloss i rør

generalised force:

$$Q = \frac{\delta W}{\delta q} = -d\,\dot{q}$$

generalised coordinates: θ and y

generalised force: 0 and $-d\,\dot{y}$

damper!

Block in a pipe II

$$r^i = \begin{bmatrix} y \sin\theta \\ -y \cos\theta \end{bmatrix} \qquad v^i = \begin{bmatrix} \dot{y}\sin\theta + y\cos\theta\,\dot{\theta} \\ -\dot{y}\cos\theta + y\sin\theta\,\dot{\theta} \end{bmatrix}$$

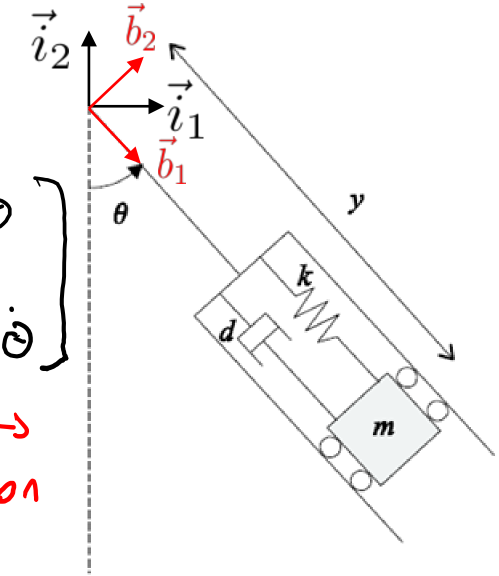


Figure 1: Kloss i rør

$$\vec{r} = y\vec{b}_1 \qquad \vec{v} = \dot{y}\vec{b}_1 + (-\dot{\theta}\vec{b}_3) \times y\vec{b}_1$$

$$= \dot{y}\vec{b}_1 + y\dot{\theta}\vec{b}_2$$

$$\vec{v}\cdot\vec{v} = \dot{y}^2 + y^2\dot{\theta}^2$$

$$T = 1/2\, m\, (v^i)^T v^i = 1/2\, m(\dot{y}^2 + y^2\dot{\theta}^2)$$

$$U = -mgy\cos\theta + 1/2\, k\,(y - y_0)^2$$

Block in a pipe III $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$

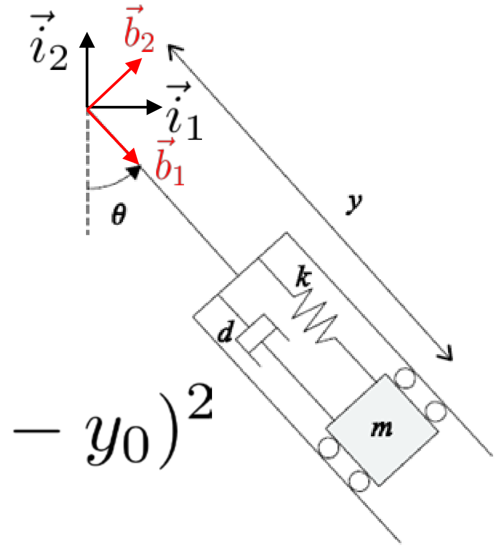


Figure 1: Kloss i rør

$$\mathcal{L} = \mathbf{T} - \mathbf{U}$$

$$= \frac{1}{2}m(\dot{y}^2 + y^2\dot{\theta}^2) + mgy\cos\theta - \frac{1}{2}k(y-y_0)^2$$

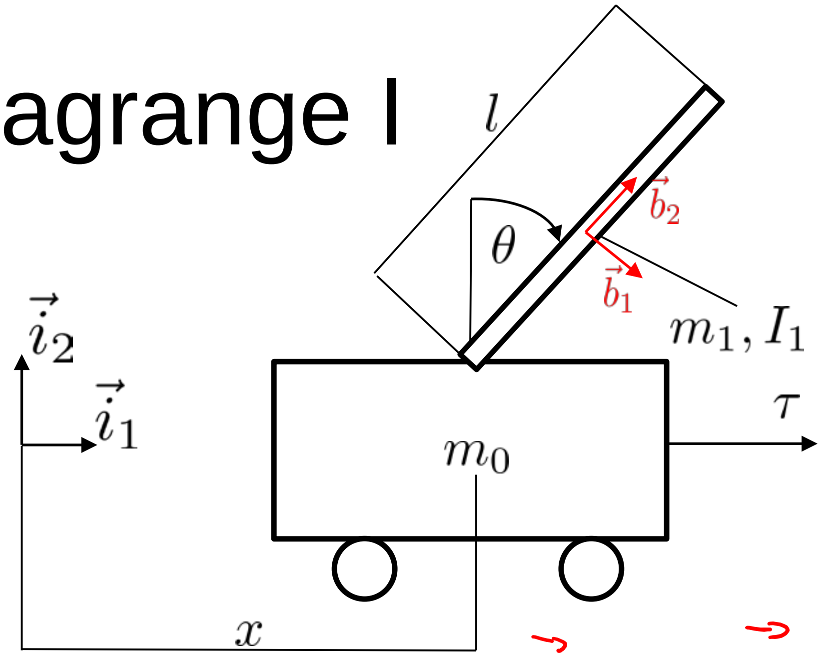
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m y^2 \dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -mg\, y \sin\theta$$

$$\cancel{m} y^{\cancel{2}} \ddot{\theta} + 2\cancel{m}\cancel{y}\dot{y}\dot{\theta} + \cancel{m} g \cancel{y} \sin\theta = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} \qquad \frac{\partial \mathcal{L}}{\partial y} = m y \dot{\theta}^2 + mg\cos\theta - k(y-y_0)$$

$$m\ddot{y} - m y \dot{\theta}^2 - mg\cos\theta + k(y-y_0) = -d\dot{y}$$

Inverted Pendulum – Lagrange I



$$\vec r_0 = x\,\vec i_1 \qquad \vec v_0 = \dot x\,\vec i_1$$

$$\vec r_1 = x\,\vec i_1 + \frac{h}{2}\vec i_2 + \frac{l}{2}\vec b_2$$

$$\vec v_1 = \dot x\,\vec i_1 + \dot\theta\,\frac{l}{2}\vec b_1$$

$$\vec \omega_{ib} = -\dot\theta\,\vec b_3$$

$$T_0 = \frac{1}{2} m_0 \dot x^2$$

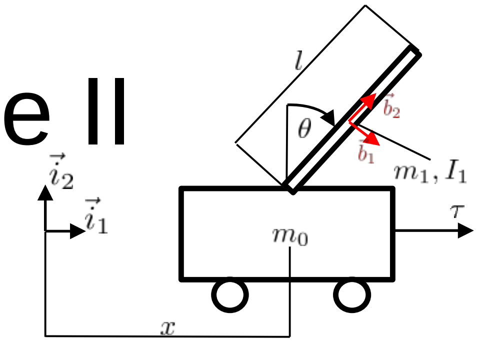
$$T_1 = \frac{1}{2} m_1 \dot x^2 + m_1 \frac{l}{2}\dot\theta\dot x\cos\theta + 1/2\, m_1 \dot\theta^2 \frac{l^2}{4} + \frac{1}{2} I_1 \dot\theta^2$$

$$U_0 = m_0 g \cdot 0$$

$$U_1 = m_1 g\left(\frac{1}{2}h + \frac{1}{2} l\cos\theta\right)$$

$$\mathcal{L} = \Sigma T - \Sigma U$$

Inverted Pendulum – Lagrange II



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$$

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2(m_0 + m_1) + m_1 \tfrac{l}{2}\dot{\theta}\dot{x}\cos\theta + \tfrac{l^2}{8}m_1\dot{\theta}^2 + \tfrac{1}{2}I_1\dot{\theta}^2 - \tfrac{1}{2}m_1 g l \cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_0 + m_1)\dot{x} + m_1 \frac{l}{2}\dot{\theta}\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_1 \frac{l}{2}\dot{\theta}\dot{x}\sin\theta + m_1 \frac{l}{2} g \sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_1 \frac{l}{2}\dot{x}\cos\theta + m_1 \dot{\theta}\frac{l^2}{4} + I_1\dot{\theta}$$

$$\tau = (m_0 + m_1)\ddot{x} - m_1 \frac{l}{2}\dot{\theta}^2 \sin\theta + m_1 \frac{l}{2}\ddot{\theta}\cos\theta$$

$$0 = m_1 \frac{l}{2}\ddot{x}\cos\theta - m_1 \frac{l}{2}\dot{x}\dot{\theta}\sin\theta + m_1 \ddot{\theta}\frac{l^2}{4} + I_1\ddot{\theta} + m_1 \frac{l}{2}\dot{\theta}\dot{x}\sin\theta - m_1 \frac{l}{2} g \sin\theta$$

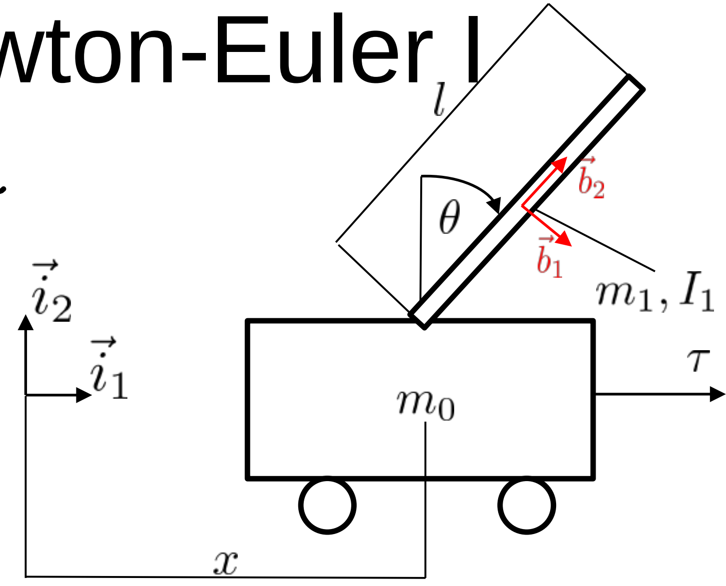
$$= m_1 \frac{l}{2}\ddot{x}\cos\theta + m_1 \ddot{\theta}\frac{l^2}{4} + I_1\ddot{\theta} - m_1 g \frac{l}{2}\sin\theta$$

Inverted Pendulum – Newton-Euler I

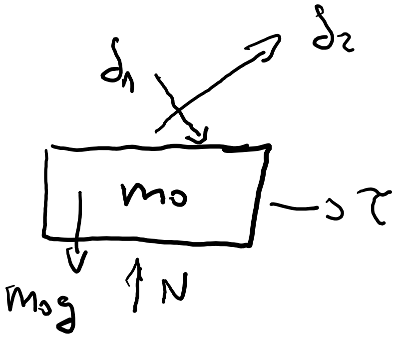
$$\vec{r}_0 = x\,\vec{i}_1 \qquad \vec{r}_1 = x\,\vec{i}_1 + \frac{h}{2}\vec{i}_2 + \frac{l}{2}\vec{b}_2$$

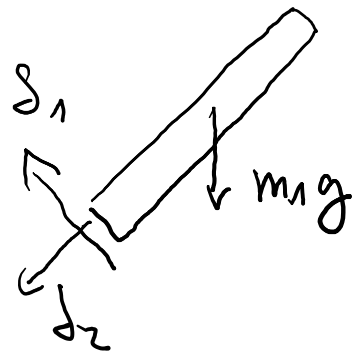
$$\vec{v}_0 = \dot{x}\,\vec{i}_1 \qquad \vec{v}_1 = \dot{x}\,\vec{i}_1 + \dot{\theta}\frac{l}{2}\vec{b}_1$$

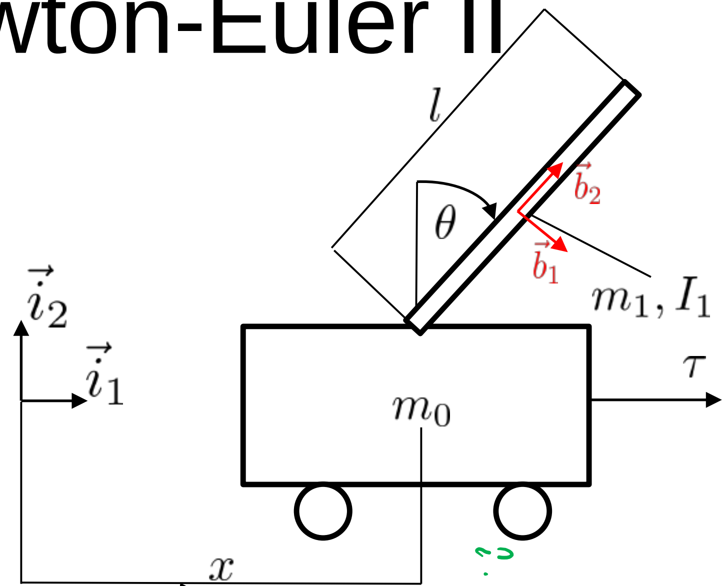
$$\vec{a}_0 = \ddot{x}\,\vec{i}_1 \qquad \vec{a}_1 = \ddot{x}\,\vec{i}_1 + \ddot{\theta}\frac{l}{2}\vec{b}_1 - \dot{\theta}^2\frac{l}{2}\vec{b}_2$$



Inverted Pendulum – Newton-Euler II



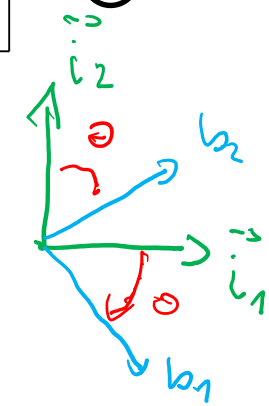




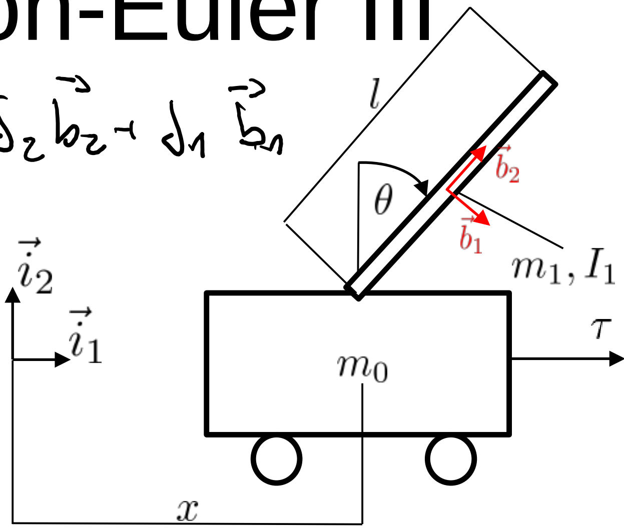
$$\vec{F}_0^{(r)} = \tau \vec{i}_1 + (N - m_0 g)\vec{i}_2 + \delta_2 \vec{b}_2 + \delta_1 \vec{b}_1$$

$$\vec{F}_1^{(r)} = -\delta_2 \vec{b}_2 - \delta_1 \vec{b}_1 - m_1 g \vec{i}_2$$

$$T^{(r)} = -\frac{l}{2}\vec{b}_2 \times \vec{F}_1^{(r)*} = -\frac{l}{2}\delta_1 \vec{b}_3$$



Inverted Pendulum – Newton-Euler III



$$m_0 : \quad m_0 \ddot{x}\, \vec{i}_1 = \tau\, \vec{i}_1 + (N - m_0 g)\, \vec{i}_2 + \delta_2 \vec{b}_2 + \delta_1 \vec{b}_1$$

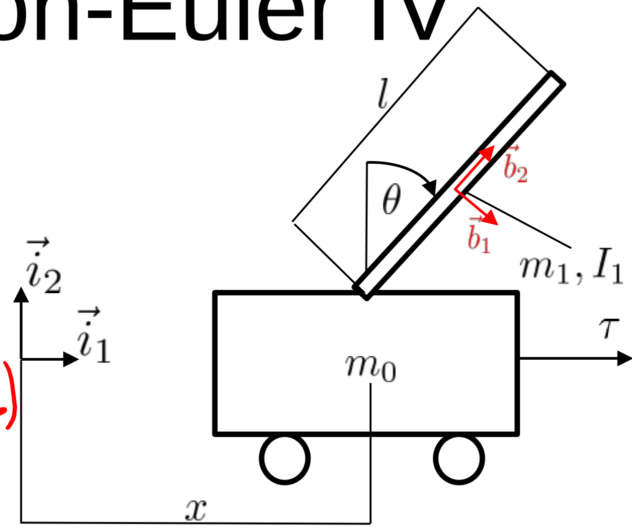
(1) $\cdot\, \vec{i}_1$: $\quad m_0 \ddot{x} = \tau + \delta_2 \sin\theta + \delta_1 \cos\theta$

$$m_1 : \quad m_1 \left(\ddot{x}\, \vec{i}_1 + \ddot{\theta} \frac{l}{2} \vec{b}_1 - \dot{\theta}^2 \frac{l}{2} \vec{b}_2 \right) = -\delta_1 \vec{b}_1 - \delta_2 \vec{b}_2 - m_1 g\, \vec{i}_2$$

(2) $\cdot\, \vec{i}_1$: $\quad m_1 \ddot{x} + m_1 \ddot{\theta} \frac{l}{2} \cos\theta - m_1 \dot{\theta}^2 \frac{l}{2} \sin\theta = -\delta_2 \sin\theta - \delta_1 \cos\theta$

(3) $\cdot\, \vec{b}_1$: $\quad m_1 \ddot{x} \cos\theta + \ddot{\theta} \frac{l}{2} m_1 = -\delta_1 + m_1 g \sin\theta$

Inverted Pendulum – Newton-Euler IV



Torque: (4) $-\frac{l}{2} S_1 = -\ddot{\theta} I_1$

$\vec{T} = \vec{M}_{b/c}\, \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times (\vec{M}_{b/c} \cdot \vec{\omega}_{ib})$

(1)+(2) : $(m_0+m_1)\ddot{x} + m_1 \ddot{\theta} \frac{l}{2} \cos\theta - m_1 \dot{\theta}^2 \frac{l}{2} \sin\theta = \tau$

From (3) : $S_1 = m_1 g \sin\theta - m_1 \ddot{x} \cos\theta - \ddot{\theta} \frac{l}{2} m_1$ (5)

Use (5) in (4) $-\frac{l}{2} m_1 g \sin\theta + \frac{l}{2} m_1 \ddot{x} \cos\theta + \frac{l^2}{4} m_1 \ddot{\theta} = -\ddot{\theta} I_1$

$$0 = m_1 \frac{l}{2} \ddot{x} \cos\theta + \frac{l^2}{4} m_1 \ddot{\theta} + I_1 \ddot{\theta} - m_1 g \frac{l}{2} \sin\theta$$

Lagrange's equation of first kind

constraint equation

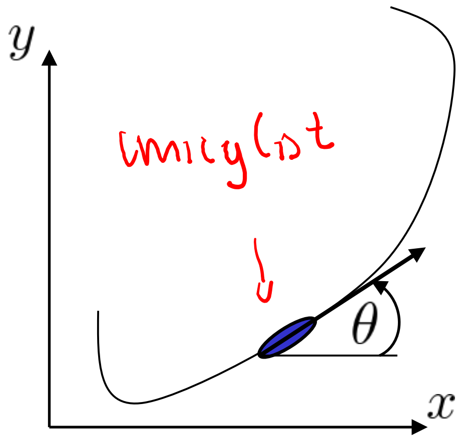
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \underbrace{\lambda_k \frac{\partial f_k}{\partial q_i}} = \tau_i$$

Lagrange multipliers

$\bar{Q} = \lambda_k f_{ki}$: generalised constraining force

- Well suited if constraints contain derivatives (and connot be integrated):
  - Non-holonom constraints

Example: Non-holonomic constraint



- position: (x, y)
- direction: θ
- no constraint in form
  $f(x, y, \theta, t) = 0$
  $\rightarrow$ 3 DoF

But we know the unicyclist moves only in direction θ

$$\dot{y} = \dot{x} \tan\theta$$ [non-holonomic constraint]

Transform holonomic constraints into non-holonomic ones:

$$\dot{F} = \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial t} = 0$$ [holonomic-in-disguise]

Revisit d'Alembert's principle

independent variables

- d'Alembert's principle:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i\right)\delta q_i = 0 \quad i = 1, \ldots, n$$

- Virtual displacement ($dt = 0$): $f_{ki}\delta q_i = 0$ 0

arbitrary function $\lambda_k(t)\, f_{ki} \cdot \delta q_i = 0$

n independ variables

- Possible to add zero to d'Alembert's principle!

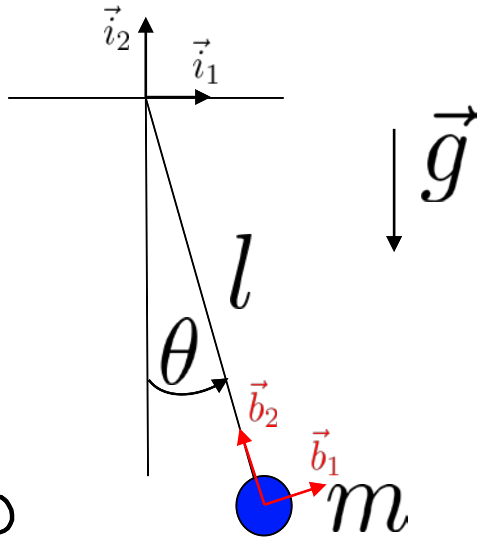
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i - \sum_{k=1}^{m} \lambda_k f_{ki}\right)\delta q_i = 0 \quad i = 1, \ldots, r$$

+ m dependent variables

$$\rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \lambda_k f_{ki} = \tau_i$$

solve r + m equations

Example: Pendulum I



① coordinate: r, Θ

② constraint: $f_1 = r - l = 0$

$$\dot{f}_1 = 1 \cdot \dot{r} = 0$$

$$T = \frac{1}{2} m (r\dot{\Theta})^2$$

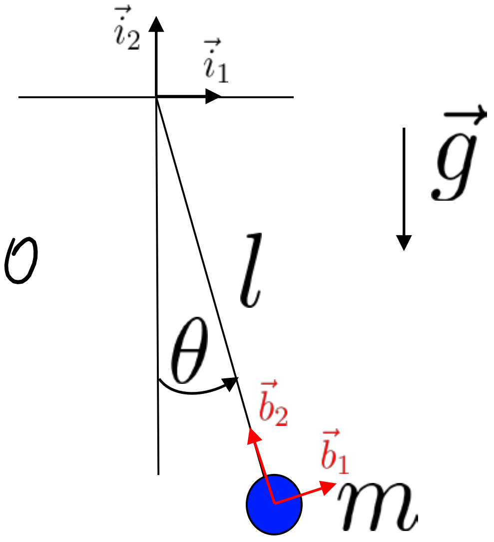
$$U = -mgr\cos\theta$$

$$\mathcal{L} = \frac{1}{2} m r^2 \dot{\Theta}^2 + mgr\cos\Theta$$

$$\mathcal{L}^* = \mathcal{L} + \lambda_1 \cdot f_1$$

Example: Pendulum II $\mathcal{L} = \frac{1}{2} m r^2 \dot{\theta}^2 + m g r \cos\theta$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \tau_i$$



for r: $\rightarrow \quad m r \dot{\theta}^2 + m g \cos\theta - \lambda_1 = 0$

for θ: $\rightarrow \quad m r \ddot{\theta} + m g r \sin\theta = 0$

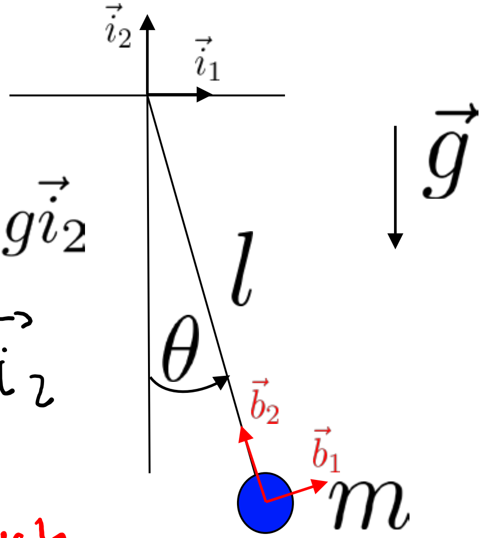
d* for λ_1: $\quad r - l = 0$

$\rightarrow \quad r = l$

$\rightarrow \quad \ddot{\theta} + g/l \sin\theta = 0$

$\lambda_1 = m l \dot{\theta}^2 + m g \cos\theta$

Example: Pendulum – body II



$$m\frac{{}^i d^2}{dt^2}\vec r = \delta \vec b_2 - mg\vec i_2$$

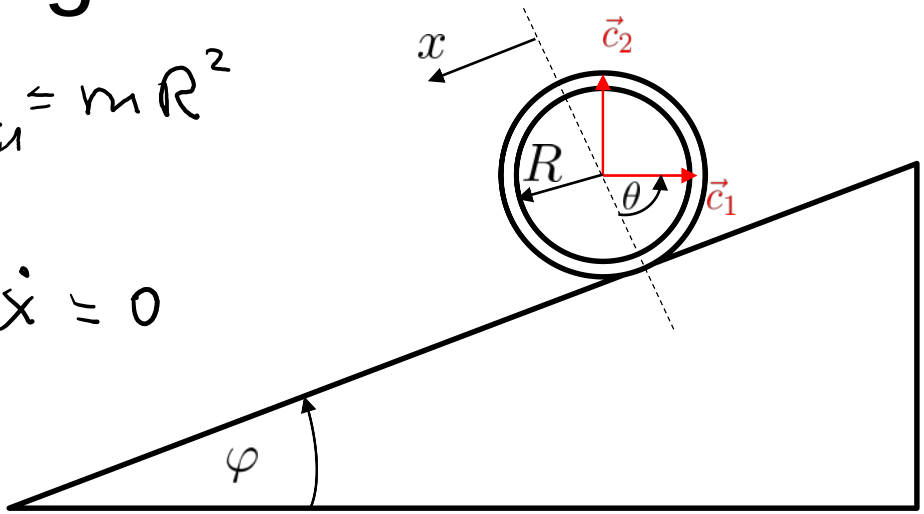
$$m\left(l\ddot\theta\,\vec b_1 + l\dot\theta^2\,\vec b_2\right) = \delta\vec b_2 - mg\,\vec i_2$$

- $\vec b_1$: $\boxed{ml\ddot\theta = -mg\sin\theta}$ ← enough for θ
- $\vec b_2$: $ml\dot\theta^2 = \delta - mg\cos\theta$

← can be used to calculate δ

Hollow cylinder rolling down a hill I

$$I_1 = mR^2$$



cylinder should stay rigid:

$$f_1 = R\dot{\theta} - \dot{x} = 0$$

generalised coordinates:

$$q_1 = x \qquad q_2 = \theta$$

$$f_{1x} = -1 \qquad f_{1\theta} = R$$

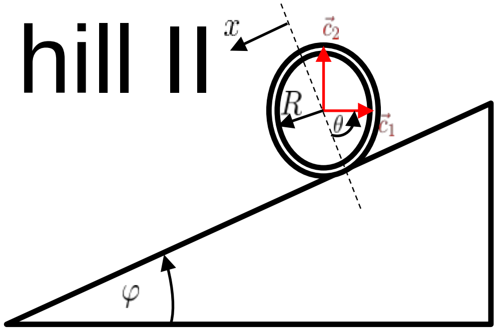
$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_1 \dot{\theta}^2$$

$$U = -mgx \sin\varphi$$

$$\mathcal{L} = 1/2\, m\dot{x}^2 + 1/2\, mR^2 \dot{\theta}^2 + mgx\sin\varphi$$

$$\mathcal{L}^* = \mathcal{L} + \lambda_1 f_1$$

Hollow cylinder rolling down a hill II



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \tau_i$$

$$\mathcal{L} = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} mR^2\dot{\theta} + mgx\sin\varphi$$

for x: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} - \lambda_1 f_{1x} = m\ddot{x} - mg\sin\varphi + \lambda_1 = 0$

for Θ: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} - \lambda_1 f_{1\theta} = mR^2\ddot{\theta} - R\lambda_1 = 0$

$\mathcal{L}^*$ for λ_1: $R\dot{\theta} - \dot{x} = 0$

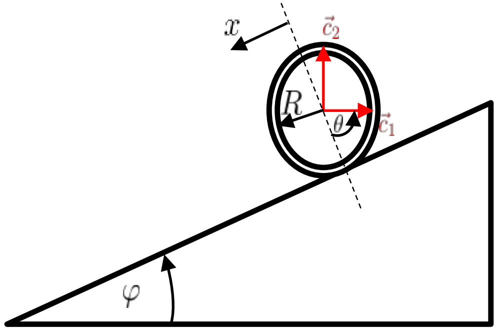
$\rightarrow R\ddot{\theta} - \ddot{x} = 0$

$\rightarrow \lambda_1 = mR\ddot{\theta}$

Hollow cylinder rolling down a hill III

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} - \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \tau_i$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mR^2\dot{\theta} + mgx\sin\varphi$$



$① \Rightarrow \quad mR\ddot{\theta} - mg\sin\varphi + mR\ddot{\theta} = 0$

$$\ddot{\theta} = \frac{1}{2R} g \sin\varphi$$

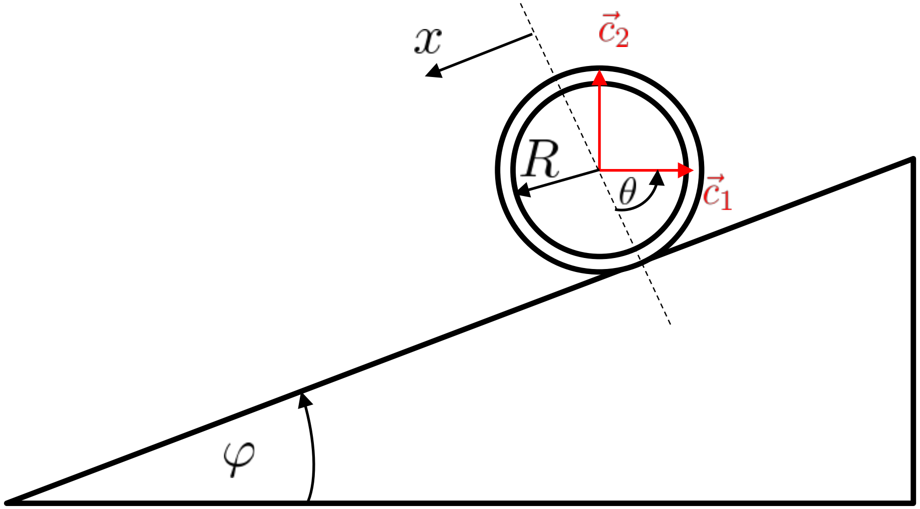
$$\ddot{x} = \frac{1}{2} g \sin\varphi$$

$$\lambda_1 = 1/2\, m g \sin\varphi$$

$$\tilde{Q}_1 = \lambda_1 f_{1x} = -1/2\, mg\sin\varphi$$

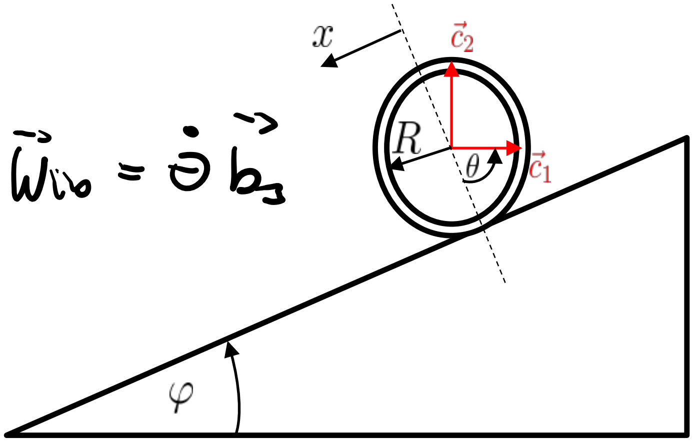
$$\tilde{Q}_2 = \lambda_1 f_{1\theta} = 1/2\, mgR\sin\varphi$$

Homework



- Solve the «hollow cylinder down a hill» task with the Lagrange approach of second kind («normal» Lagrange approach that we used in the lecture)

Hollow cylinder rolling down a hill IV (Lagrange second kind)



$$\dot{x} = v = R\dot{\theta}$$

$$\vec{\omega}_{ib} = \dot{\theta}\,\vec{b}_3$$

generalised coordinate: x

[you could also use θ]

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_1 \left(\frac{\dot{x}}{R}\right)^2 = m\dot{x}^2$$

$$U = -mgx\sin\varphi$$

$$\mathcal{L} = T - U = m\dot{x}^2 + mgx\sin\varphi$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = 2m\dot{x} \qquad \frac{\partial\mathcal{L}}{\partial x} = mg\sin\varphi$$

$$2m\ddot{x} - mg\sin\varphi = 0$$

$$\ddot{x} = \frac{1}{2} g\sin\varphi$$