Lecture 4: Passivity

Passivity (E2.4)

- Positive Real (PR) transfer functions
- Passivity and storage functions

Energy function

- The system: $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$

- Assume we have a function $V(x,t) \geq 0$, which describes the «energy» of the system

- The derivative of the energy function $V(x,t)$ is

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t)$$

- If we have $\dot{V} \leq 0$

  → Energy of the system decreases monotonically

  → stability

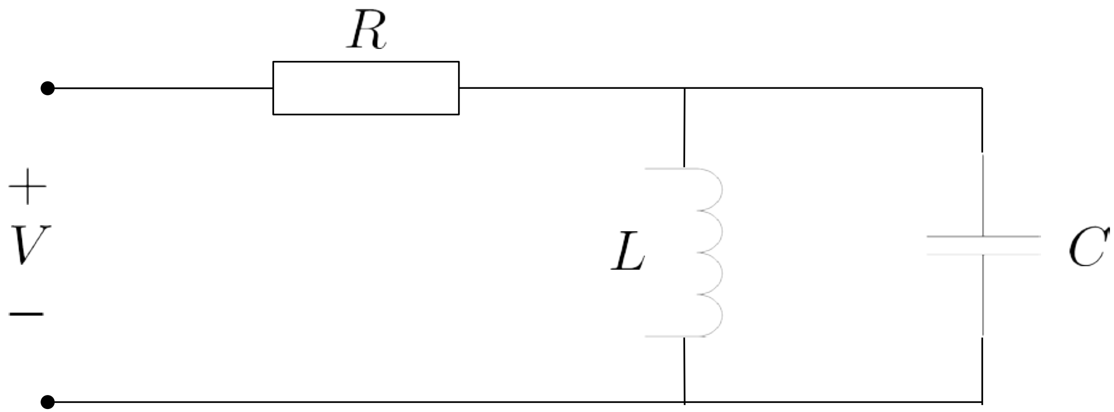
Why learn about passivity? Preview...

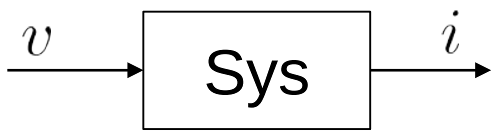
- Say you have several systems (or models), and you want to interconnect them
  - For instance, a process and a controller, or a motor and a load, or two buffer tanks in series, ...
  - Will the interconnection be stable?

- Bad news: The interconnection of stable systems is not necessarily stable

- **Good news: The interconnection of passive systems is passive (and therefore stable)!**

Passivity: Circuit example





Voltage source

$$P(t) = i(t)v(t)$$

$P(t) > 0$: given to the system

$P(t) < 0$: goes out of the system

total energy:

$$E(t) = E(t_0) + \int_{t_0}^{t} P(\tau)\, d\tau$$

$E(t) > 0$: energy stored or absorbed by system

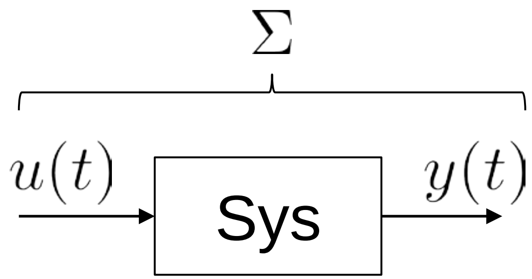
$E(t) < 0$: energy produced to the environment

passive system $E(t) > 0$

$$\rightarrow \int_{t_0}^{t} P(\tau)\, d\tau \geq -E_0(t_0)$$

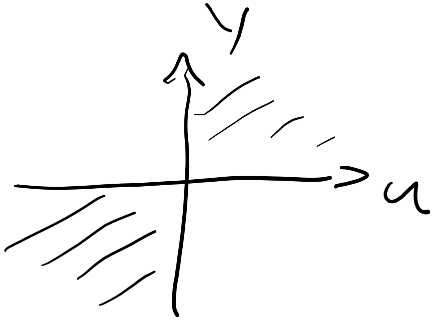
$$\rightarrow \int_{t_0}^{t} i(\tau)\, v(\tau)\, d\tau \geq -E(t_0)$$

Definition of passivity



- A system Σ is passive if there exist $E_0 > 0$ such that for all control time histories u and all $t \geq 0$ the following holds:

$$\int_0^T y(t)u(t)dt \geq -E_0$$



e.g. $\int_0^T y(t)\,u(t)\,dt \geq 0$

Passivity

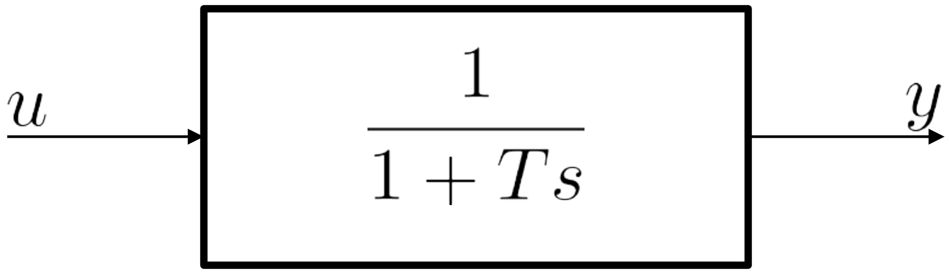
- A system with input *u* and output *y* is passive if

$$\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$$

for all $t \geq 0$, for all input trajectories.

- If the product yu has power as unit, then if (for all *u*)
  - $\int_0^t y(\tau)u(\tau)d\tau \geq 0$: Energy is absorbed within the system, nothing delivered to the outside
  - $\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$: Some energy can be delivered to the outside, limited (typically) by the initial energy in the system.
  - $\int_0^t y(\tau)u(\tau)d\tau \to -\infty$: There is an inexhaustible energy source in the system. Not passive!

Example: Passivity



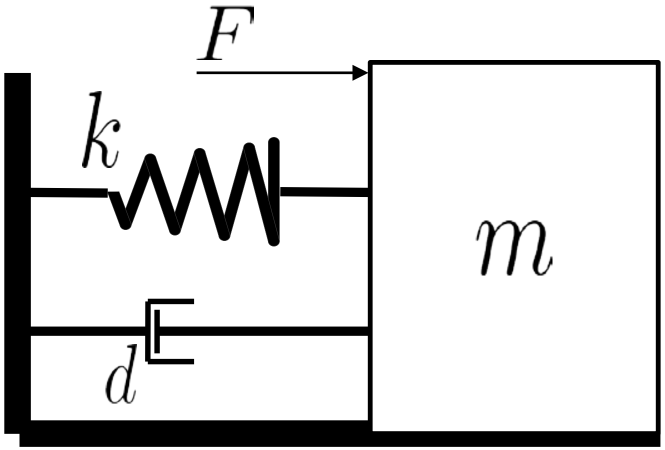
$$\dot{y} = -\frac{1}{T}y + \frac{1}{T}u$$

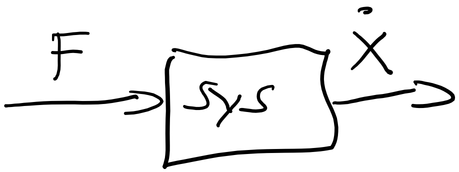
$$u = T\dot{y} + y$$

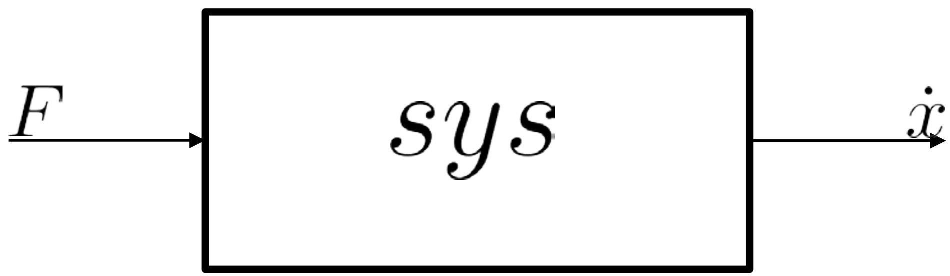
$$\begin{aligned}
\int_{t_0}^{t} y(\tau)u(\tau)d\tau &= \int_{t_0}^{t} y(\tau)\left[T\dot{y}(\tau) + y(\tau)\right] d\tau \\
&= T\int_{t_0}^{t} y\dot{y}d\tau + \int_{t_0}^{t} y^2 d\tau \\
&= \frac{T}{2}\left[y^2(t) - y^2(0)\right] + \int_{t_0}^{t} y^2 d\tau \\
&\geq -\frac{T}{2}y^2(0) = -E_0
\end{aligned}$$

Example: Mass-Spring-damper I

$$m\ddot{x} + d\dot{x} + kx = F$$







Example: Mass-Spring-damper II

$$m\ddot{x} + d\dot{x} + kx = F$$

$$\int_{t_0}^{t} F\dot{x}\,d\tau = \int_{t_0}^{t} \dot{x}\,(m\ddot{x} + d\dot{x} + kx)\,d\tau$$

$$= m\int_{t_0}^{t} \dot{x}\,\underbrace{\ddot{x}\,d\tau}_{d\dot{x}} + d\int_{t_0}^{t} \dot{x}^2\,d\tau + k\int_{t_0}^{t} x\,\underbrace{\dot{x}\,d\tau}_{dx}$$

$$\ddot{x} = \frac{d\dot{x}}{d\tau} \qquad \dot{x} = \frac{dx}{d\tau}$$

$$= m\int_{\dot{x}(t_0)}^{\dot{x}(t)} \dot{x}\,d\dot{x} + d\int_{t_0}^{t} \dot{x}^2\,d\tau + k\int_{x(t_0)}^{x(t)} x\,dx$$

$$= \tfrac{1}{2}m\,[\underbrace{\dot{x}^2(t)}_{\geq 0} - \dot{x}^2(t_0)] + \underbrace{d\int_{t_0}^{t} \dot{x}^2\,d\tau}_{\geq 0} + \tfrac{1}{2}k\,[\underbrace{x^2(t)}_{\geq 0} - x^2(t_0)]$$

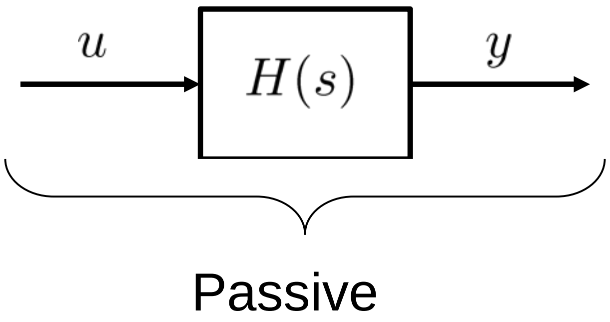
$$\geq -\tfrac{1}{2}m\,\dot{x}^2(t_0) - \tfrac{1}{2}k\,x^2(t_0) =: -E_0$$

"initial stored energy" $\rightarrow$ passive

Example: Mass-Spring-damper III

$$m\ddot{x} + d\dot{x} + kx = F$$

Positive real transfer functions



$\Leftrightarrow$ $H(s)$ is positive real

Definition: The transfer function $H(s)$ (rational or irrational) is positive real if

1. $H(s)$ analytic in $\mathrm{Re}[s]>0$.
2. $H(s)$ is real for all positive and real s.
3. $\mathrm{Re}[H(s)] \geq 0$ for all $\mathrm{Re}[s]>0$.

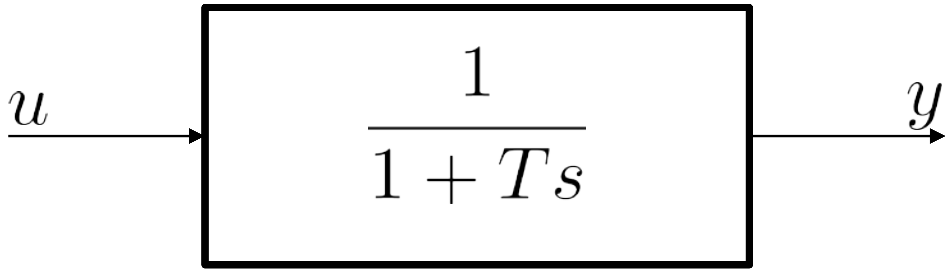
Check rational TFs for PRness

Theorem: A rational, proper transfer function $H(s)$ is positive real (and hence passive) if and only if

1. $H(s)$ has no poles in $\mathrm{Re}[s] > 0$.
2. $\mathrm{Re}[\mathrm{H}(j\omega)] \geq 0$ for all $\omega \in [-\infty, \infty]$ such that $j\omega$ is not a pole of $H(s)$.
3. If $j\omega_0$ is a pole of $H(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is,

$$\operatorname*{Res}_{s=j\omega_0} H(s) = \lim_{s \to j\omega_0} (s - j\omega_0) H(j\omega) > 0.$$

Example: Positive Real



1. Pol: $s = -\frac{1}{T}$ $\qquad Re[s] \leq 0$

2. $H(jw) = \frac{1}{1+Tjw} = \frac{1-jwT}{1+(wT)^2}$

$$Re[H(jw)] = \frac{1}{1+(Tw)^2} \geq 0$$

3. Ok, since no pols on imaginary axis

$\rightarrow$ PR $\rightarrow$ passive

Example 32 – H(s) positive real?

$$H(s) = K\frac{(s+z_1)(s+z_2)\dots}{s(s+p_1)(s+p_2)\dots}$$

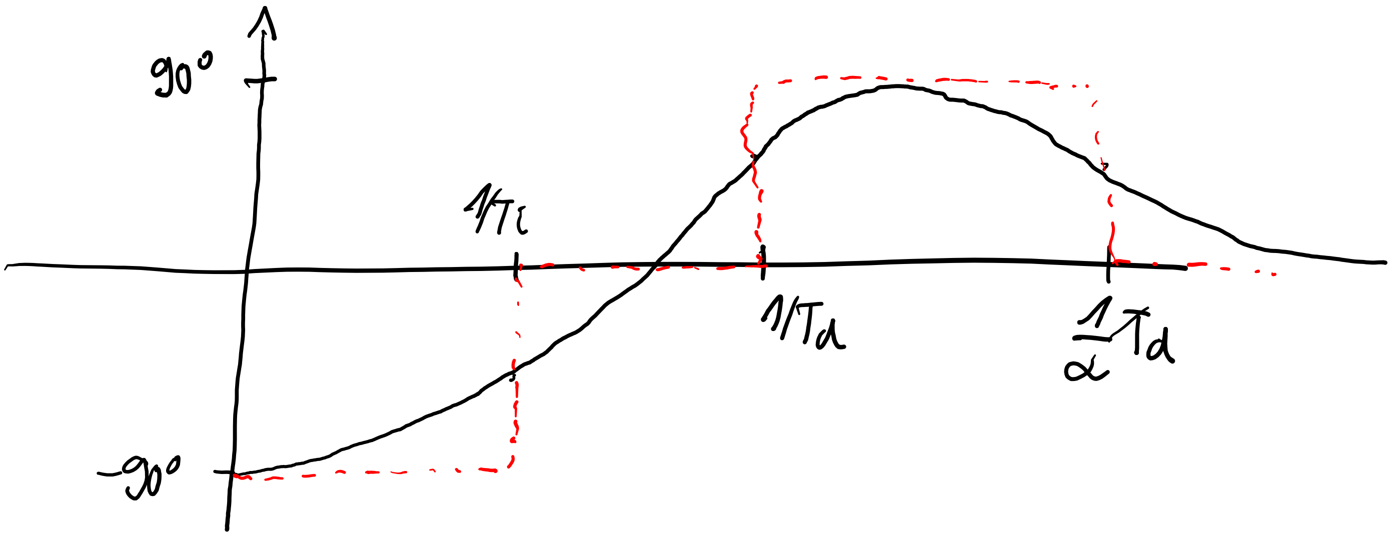
- Assume: $Re[p_i] > 0$ and $Re[z_i] > 0$
- Hint: Poles/zeros come all in complex conjugated pairs: $(\alpha + j\beta)(\alpha - j\beta) = \alpha^2 + \beta^2$

PID controller I

$$H_{PID}(s) = K_p \frac{1 + T_i s}{T_i s} \frac{1 + T_d s}{1 + \alpha T_d s}$$

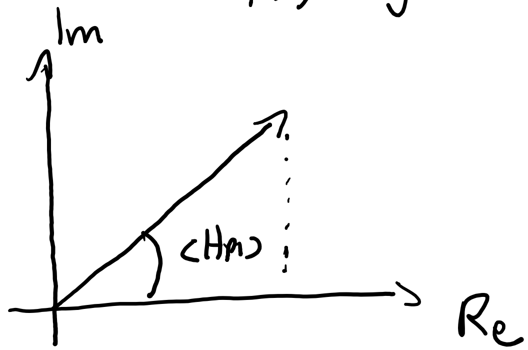
$$K_p > 0, \quad T_i > 0, \quad T_d > 0, \quad T_d < T_i, \quad 0 \leq \alpha \leq 1$$

Phase diagram



PID controller II

$$-90^\circ \le H_{PID}(j\omega) \le 90^\circ$$



$$\Downarrow$$

$$Re\left[H_{PID}(j\omega)\right] \ge 0$$

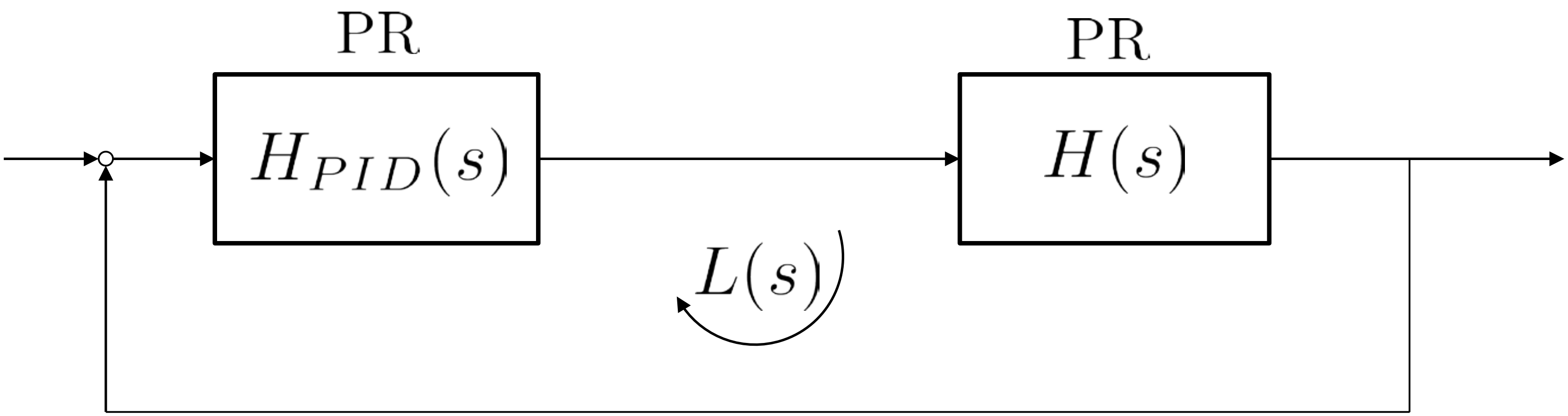
$$\Downarrow$$

$H_{PID}(s)$ is PR

$$\Downarrow$$

H_{PID} is passive

Why is this useful?

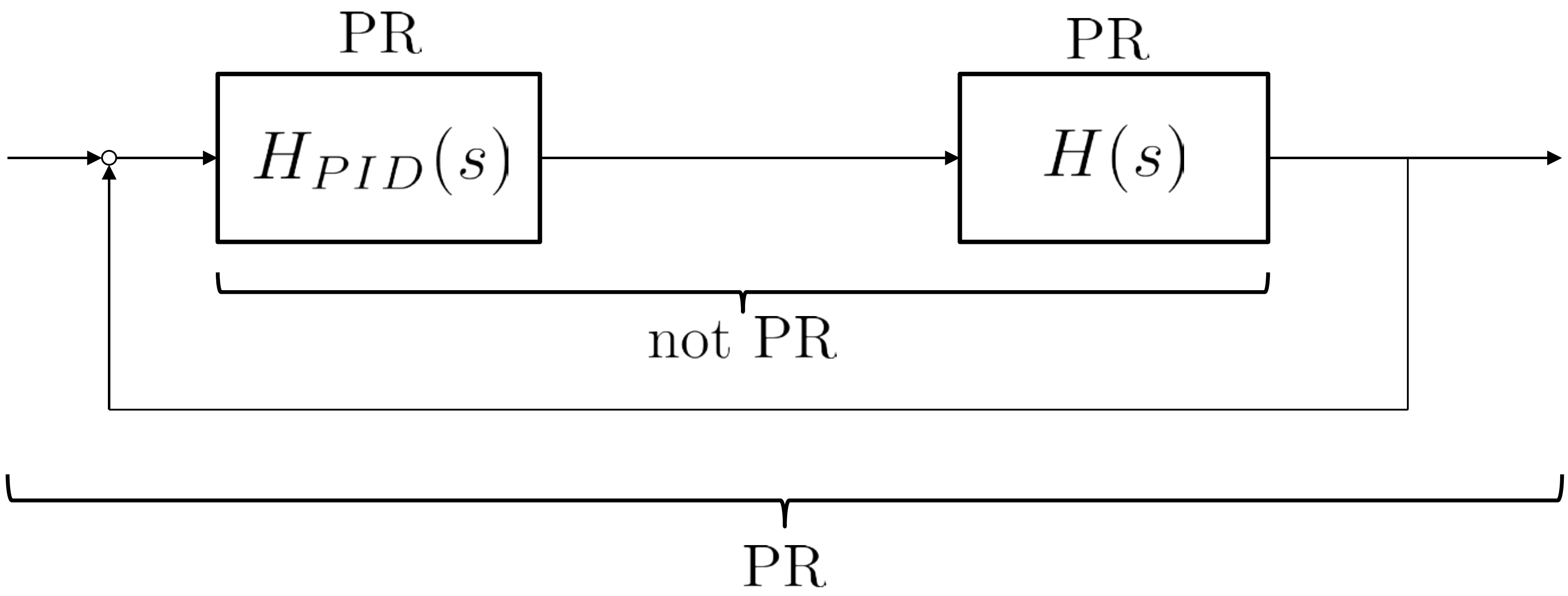


$$L(s) = H_{PID}H(s)$$

$$\angle L(jw) = \angle H_{PID}(jw) + \angle H(jw)$$

→ $|\angle L(jw)| \leq 180^o$

In addition:



Storage function I

- We can proof passivity via the storage function
- Consider the system:

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

- Assume we have a storage function $V(x) \geq 0$ and a dissipation function $g(x) \geq 0$
- Such that the time derivative for all control inputs u is:

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) = u^T y - g(x)$$

→ System with input u and output y is passiv

Storage function II

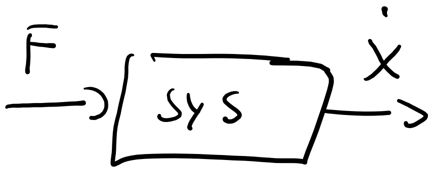
proof:

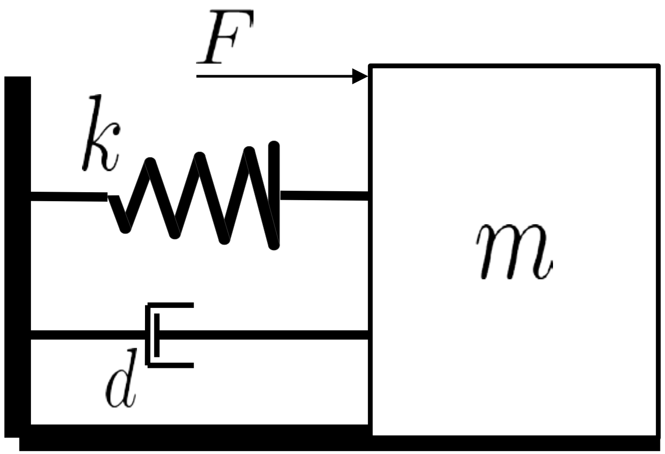
$$\int_{t_0}^{t} y^T u \, d\tau = \overbrace{V[x(t)]}^{\geq 0} - V[x(t_0)] + \underbrace{\int_0^T g[x(t)]\, dt}_{\geq 0}$$

$$\geq - \underbrace{V[x(t_0)]}_{E_0}$$

Example: Storage function

$$m\ddot{x} + d\dot{x} + kx = F$$



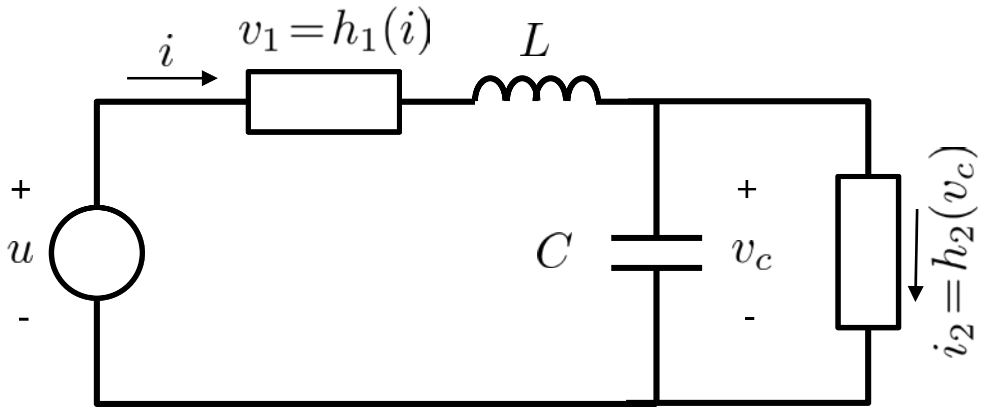


$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \geq 0$$

$$\dot{V} = F\dot{x} - d\dot{x}^2$$

$$= uy - \underbrace{g(x)}_{\geq 0} \longrightarrow$$ passive

Example storage functions



- States: $x_1 = i, \quad x_2 = v_c$
- Model (Kirchoff's laws):

$$L\dot{x}_1 = u - h_1(x_1) - x_2$$
$$C\dot{x}_2 = x_1 - h_2(x_2)$$

- Output&input: $y = i, \quad u = u$
- Nonlinear resistors fulfilling

$$x_i h_i(x_i) \geq 0$$

- Storage (energy) function:

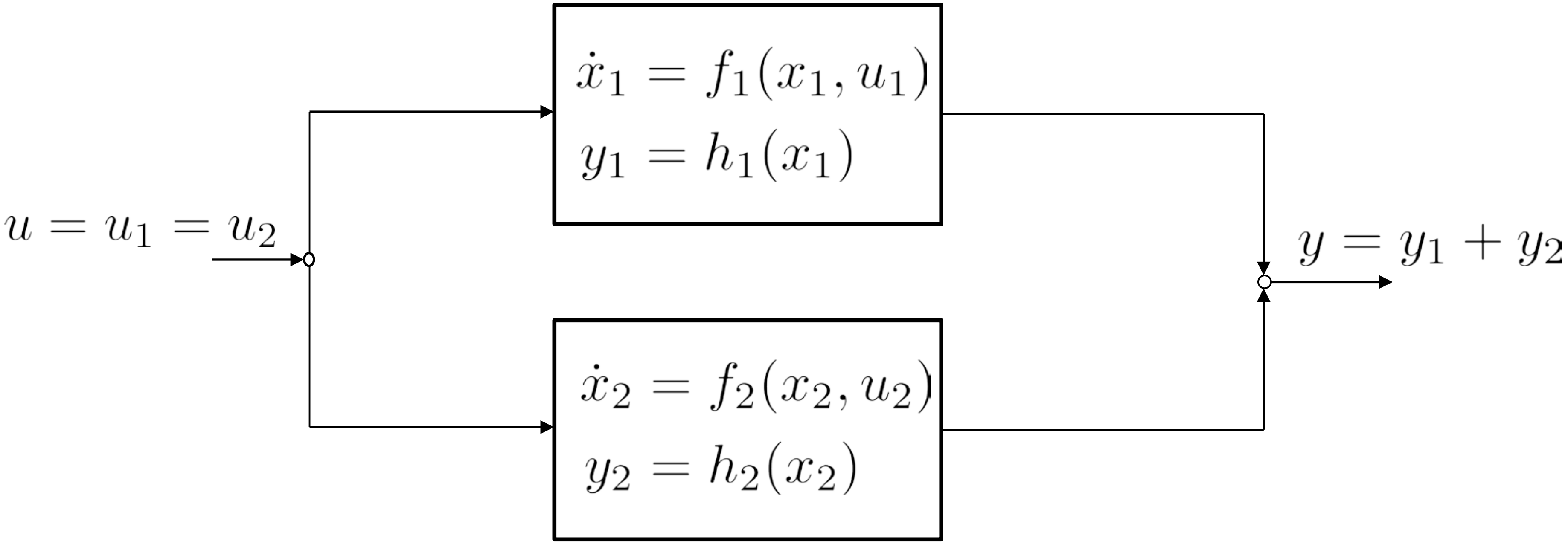
$$V(\mathbf{x}) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$$

- Differentiate:

$$\begin{aligned} \dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\ &= x_1(u - h_1(x_1) - x_2) + x_2(x_1 - h_2(x_2)) \\ &= yu - \underbrace{(x_1 h_1(x_1) + x_2 h_2(x_2))}_{g(x)} \end{aligned}$$

- Passive!

Connection of passive systems – parallel I

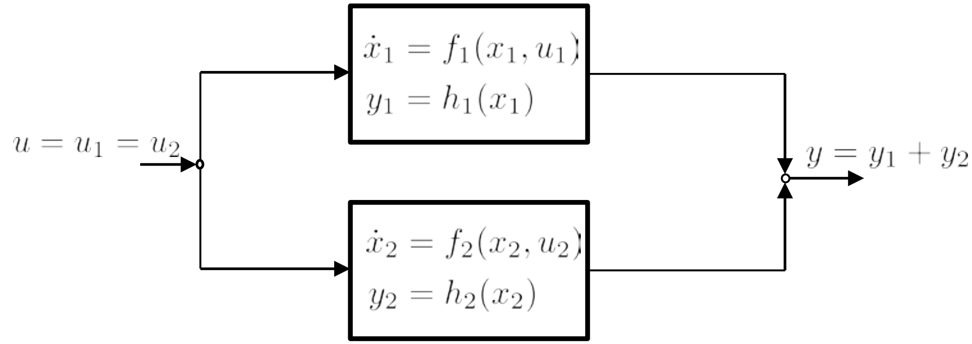


- Assume both systems passive with storage functions:

$$V_i \geq 0; \quad g_i \geq 0$$

$$\dot{V}_i = \frac{\partial V}{\partial x_i} f_i(x_i, u_i) \leq u_i^T y_i - g(x_i) \quad i = 1, 2$$

Connection of passive systems – parallel II (2.4.15)



$$V = V_1 + V_2 \geq 0$$

$$g = g_1 + g_2 \geq 0$$

$$\dot{V} = \dot{V}_1 + \dot{V}_2$$

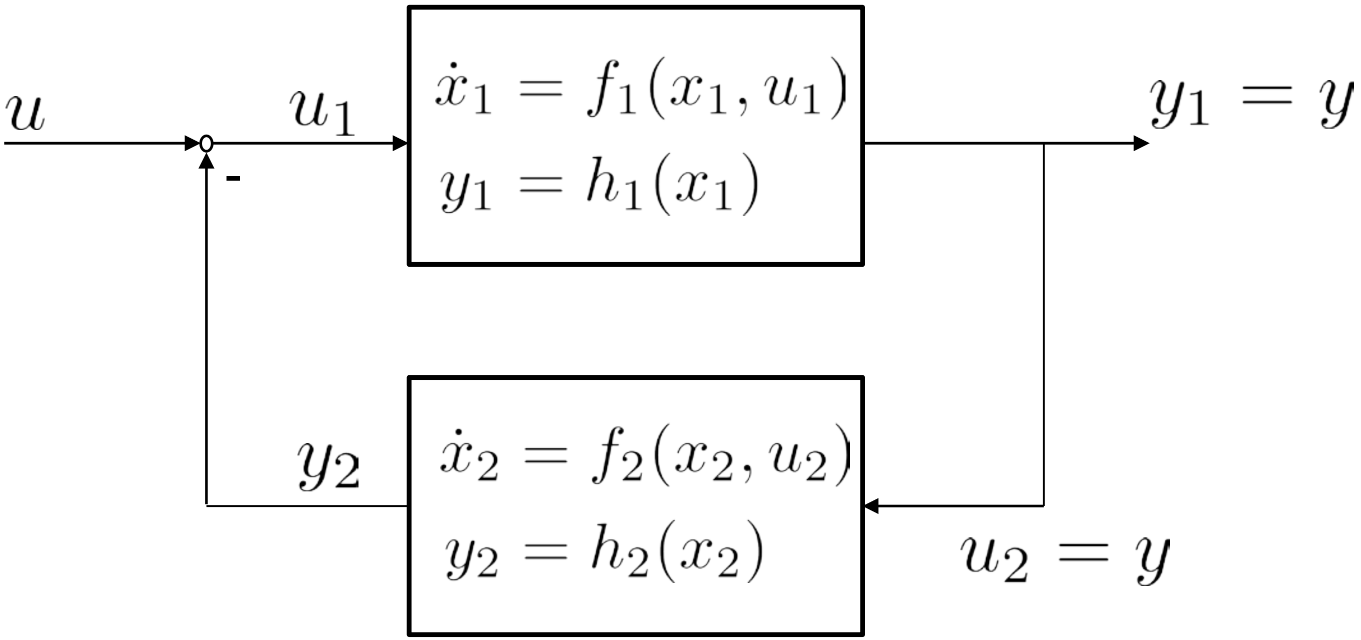
$$\leq u_1^T y_1 - g_1(x_1) + u_2^T y_2 - g_2(x_2)$$

$$= u^T (y_1 + y_2) - g(x)$$

$$= u^T y - g(x)$$

$\rightarrow$ passive

Connection of passive systems - feedback

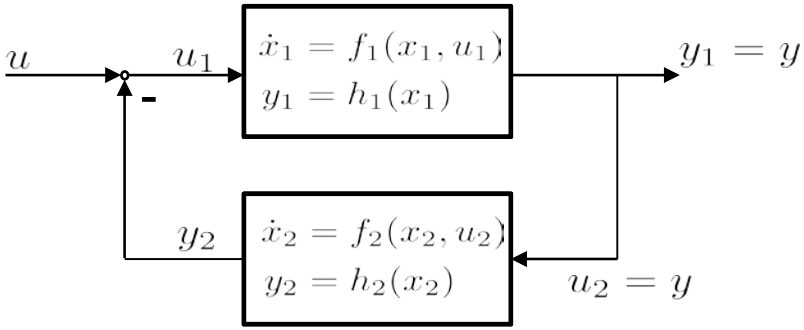


- Assume both systems passive with storage functions:

$$V_i \geq 0; \quad g_i \geq 0$$

$$\dot{V}_i = \frac{\partial V}{\partial x_i} f_i(x_i, u_i) \leq u_i^T y_i - g(x_i) \quad i = 1, 2$$

Connection of passive systems - feedback



$$V = V_1 + V_2 \geq 0 \qquad g = g_1 + g_2 \geq 0$$

$$\dot{V} = \dot{V}_1 + \dot{V}_2$$

$$\leq u_1^T y_1 - g_1(x_1) + u_2^T y_2 - g_2(x_2)$$

$$= (u - y_2)^T y + y^T y_2 - g(x)$$

$$= u^T y - g(x)$$

$\rightarrow$ passive

Kahoot

- https://play.kahoot.it/#/k/c452fe59-cad5-4f8a-ba94-475d2a5569b6

Why learn about simulation methods?

1. You will need to implement your own solvers
   - What solver fits my problem, what time-step should I choose?
   - Primarily: Explicit solvers

2. You will need to make qualified choices of solvers when using advanced modeling software
   - What solver fits my problem, choice of accuracy?
   - Typically: Implicit solvers with varying time-steps

   - Examples:
     - Simulink: Three-body problem, satellite in combined moon and earth gravity field (orbit.mdl, ode45 vs ode1 (Euler))
     - Dymola

Initial value problem

Computational error

Method: One step method

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Order of a one step method:

A method is of order *p* if *p* is the smallest integer such that:

$$e_{n+1} = O(h^{p+1})$$

Homework

- Try to proof that

$$H(s) = K\frac{(s+z_1)(s+z_2)\dots}{s(s+p_1)(s+p_2)\dots}$$

  is a positive Real

- Find out how a loudspeaker works.
- Read Section 14.1.