Lecture 21: Balance equations – Momentum and energy balances

- Recap balance laws
- The momentum balance
- The energy balance
- (Differential balance laws)

Book: Ch. 11.2, 11.4

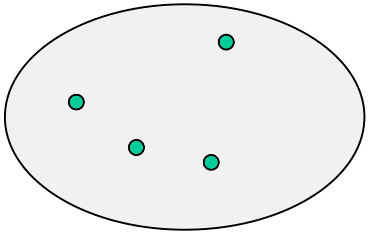
Lecture schedule change

- New lecture: 26.03.2019 10:15-12:00 in S3
- Cancelled lecture 04.04.2019

Process modeling and balance laws

- The balance laws are formulated for «conserved quantities»:
  - Mass (or other quantities that are «equivalent» to mass, such as moles, particles, etc.)
  - Momentum
  - Energy
- Process modeling is done by
  1. formulating the relevant balance laws, and
  2. finding the «closure relations» that is used to determine the flows in a balance law, as function of the state («inventory») of the balance law
- The state («inventory») of a balance law is what is used as a measure for the conserved quantity
  - Such as mass, moles, concentration, level, pressure, … for mass balance,
  - velocity or flows for momentum balance, and
  - temperature for energy balance

The basic physical principles



Consider a volume consisting of a fixed number of fluid particles, with total mass m, total momentum $\vec{p}$ and total energy E. From basic physics (conservation laws), we know the following principles hold:

- Conservation of mass (mass balance):

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0$$

- Newton's second law (momentum balance)

$$\frac{{}^{i}\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}$$

Also holds for angular momentum, $\vec{h} = \vec{r} \times \vec{p}$:

$$\frac{{}^{i}\mathrm{d}}{\mathrm{d}t}\vec{h} = \vec{r} \times \vec{F} = \vec{T}$$

- First law of thermodynamics (conservation of energy, energy balance):

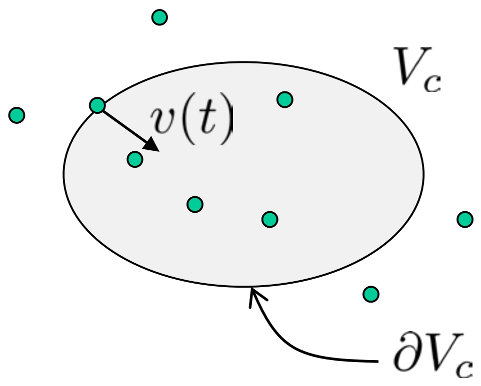
Rate of heat flowing into volume from surroundings

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W}$$

Rate at which work is done by the body at surroundings

The balance laws

- Assume a fixed control volume (of arbitrary size and shape), where fluid flows across the control volume

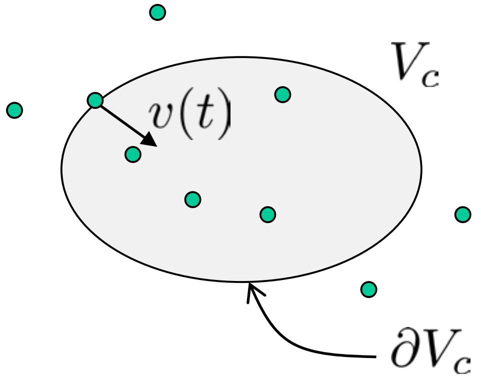


$$B = \iiint_{V_c} \rho\beta(\mathbf{x}, t)\mathrm{d}V$$

- The general integral (macroscopic) balance law for *B* is

$$\frac{\mathrm{d}}{\mathrm{d}t}B = \left\{ \begin{array}{c} \text{transfer of } B \text{ through} \\ \text{surface } \partial V_c \text{ by} \\ \text{fluid flow (convection)} \end{array} \right\} + \left\{ \begin{array}{c} \text{other effects that} \\ \text{transfer } B \text{ into } V_c \\ \text{(indep. of fluid flow)} \end{array} \right\}$$

The integral balance laws



- Mass balance (without reactions/phase transfer)

$$\frac{\mathrm{d}}{\mathrm{d}t}m = \left\{ \begin{array}{c} \text{transfer of mass into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\}$$

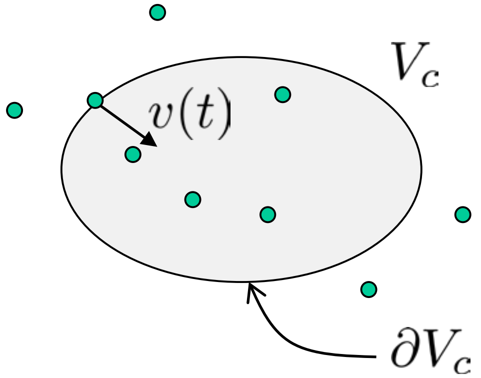
- Momentum (note: momentum is a vector)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \left\{ \begin{array}{c} \text{transfer of momentum into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{generation of momentum} \\ \text{in } V_c \text{ due to forces} \\ \text{acting on } V_c \end{array} \right\}$$

- Energy

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by heat transfer} \\ \text{and by work} \end{array} \right\}$$

The mass balance



- In words

$$\frac{\mathrm{d}}{\mathrm{d}t}m = \left\{ \begin{array}{c} \text{transfer of mass into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\}$$

- Mathematically

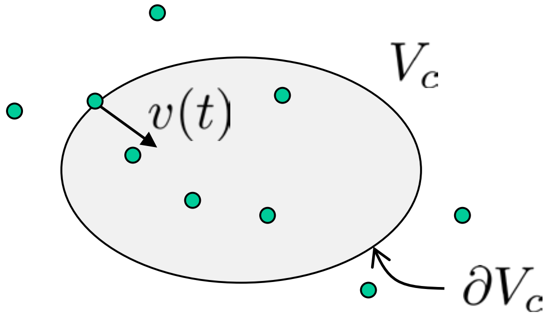
"Convection"

$$\frac{\mathrm{d}}{\mathrm{d}t}m = \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho \mathrm{d}V = -\iint\limits_{\partial V_c} \rho \vec{v} \cdot \vec{n} \mathrm{d}A$$

- Often, we have one (or more) «point inflows» $w_{\text{in},i}$, and outflows $w_{\text{out},i}$. Then mass balance can be formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}m = \sum_i w_{\text{in},i} - \sum_i w_{\text{out},i}$$

The momentum balance



- In words

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \left\{ \begin{array}{c} \text{transfer of momentum into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{generation of momentum} \\ \text{in } V_c \text{ due to forces} \\ \text{acting on } V_c \end{array} \right\}$$

- Mathematically

$$\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p} = \frac{{}^i\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho\vec{v}\mathrm{d}V = -\iint\limits_{\partial V_c} \rho\vec{v}\vec{v}\cdot\vec{n}\mathrm{d}A + \vec{F}^{(r)}$$

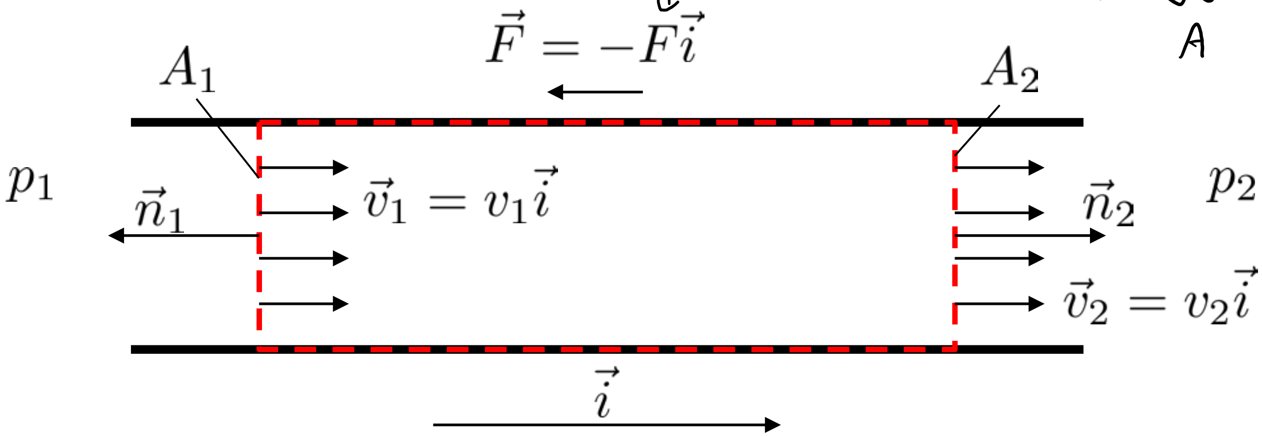
where $\vec{F}^{(r)}$ is resultant force on fluid in control volume

(often: gravity (hydrostatic) and/or friction (hydrodynamic))

Example: Pipeflow

$[\frac{N}{m}]$

$$w = \iint_A \rho v \, dA = \rho \vec{v} A = \rho q$$

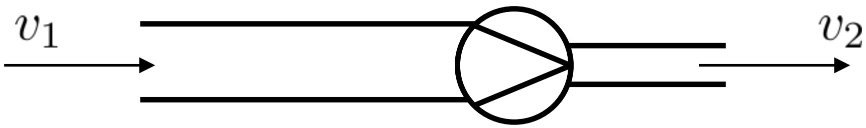


mass balance: $\frac{d}{dt} m = w_1 - w_2$

momentum balance:
$$\frac{d}{dt} \iiint_{V_c} \rho \vec{v} \, dV = \vec{v}_1 \rho v_1 A_1 - \vec{v}_2 \cdot \rho v_2 A_2 + \underbrace{p_1 A_1 - p_2 A_2}_{\text{flow work}} - \underbrace{F}_{\text{shaft work}}$$

$$= v_1 w_1 - v_2 w_2 + p_1 A_1 - p_2 A_2 - F$$

Example: Water-Jet



Assume: steady-state: $\frac{d}{dt} \ldots = 0$

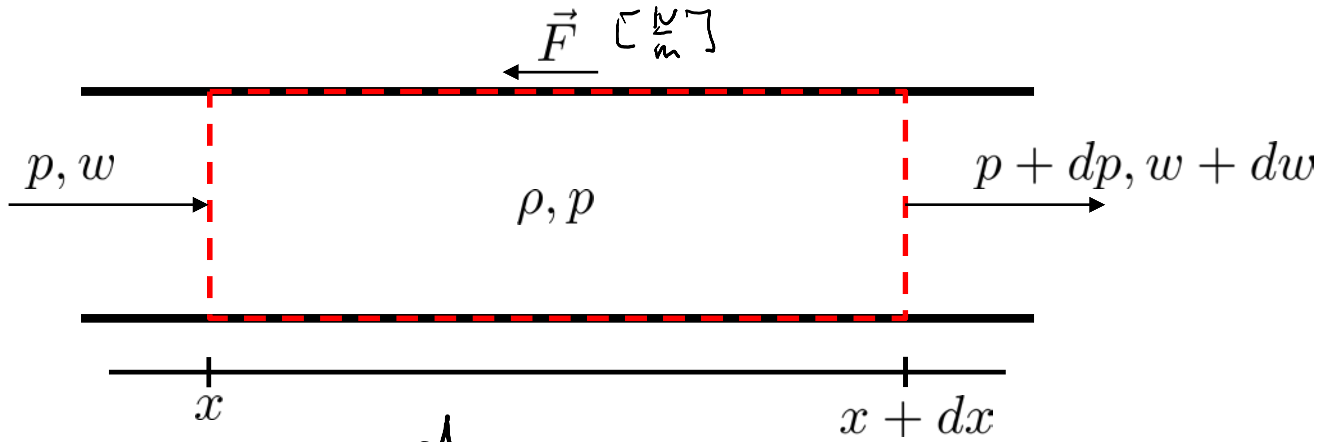
mass balance: $w_1 = w_2 = w$

$$p_1 A_1 \approx p_2 A_2 \approx 0$$

"Thrust": $F = w(v_1 - v_2) \approx -w\,v_2$

Example: Transmission line I

$$w = \iint_A \rho_v \, dA = \rho q$$



mass balance: $\frac{d}{dt} m = w_1 - w_2$

$$A \, dx \, \frac{d\rho}{dt} = w - (w + dw) = -dw$$

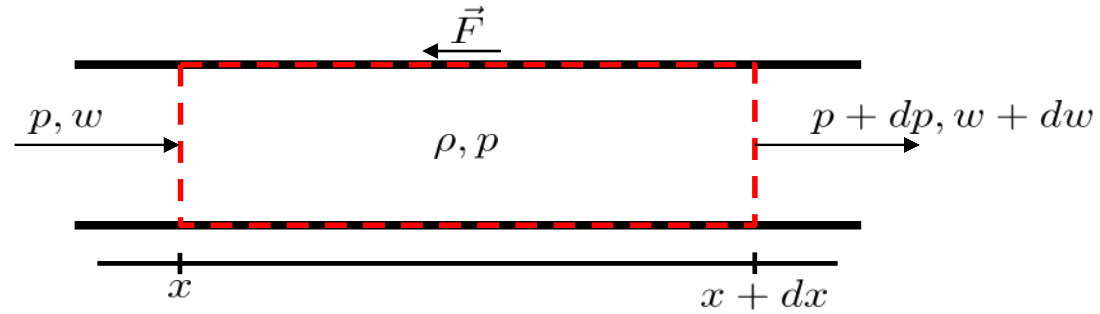
bulk modulus

$$dp = \frac{\beta}{\rho} d\rho$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A} \frac{\partial w}{\partial x} \quad [\text{pde}] \left[\begin{array}{c}\text{differential} \\ \text{balance}\end{array}\right]$$

$$\frac{\partial p}{\partial t} = -\frac{\beta}{\rho A} \frac{\partial w}{\partial x}$$

Example: Transmission line II



momentum balance:

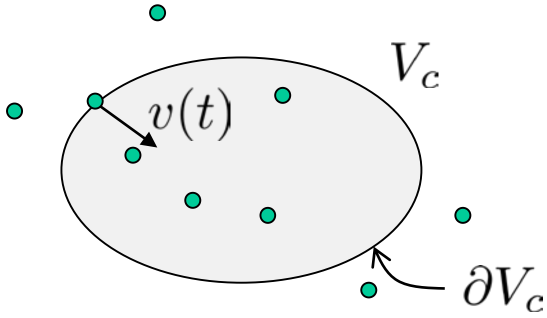
$$\frac{d}{dt}\underbrace{\iiint_{V_c} \rho v \, dV}_{= w\,dx} = A p - A(p + dp) - F\,dx + \iint_{A_1} \rho v^2 \, dA - \iint_{A_2} \rho (v + dv)^2 \, dA$$

$$\longrightarrow \quad \frac{\partial w}{\partial t} = -A \frac{\partial p}{\partial x} - F + \frac{\partial}{\partial x}\left(\rho \iint_{A_1} v^2 \, dA - \iint_{A_2} (v + dv)^2 \, dA \right)$$

Linearize around $v = 0$ and $\rho = \rho_0$

$$\boxed{\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{B}{A} \frac{\partial q}{\partial x} \\ \frac{\partial q}{\partial t} &= -\frac{A}{\rho_0} \frac{\partial p}{\partial x} - \frac{F}{\rho_0} \end{aligned}}$$

The energy balance



- In words

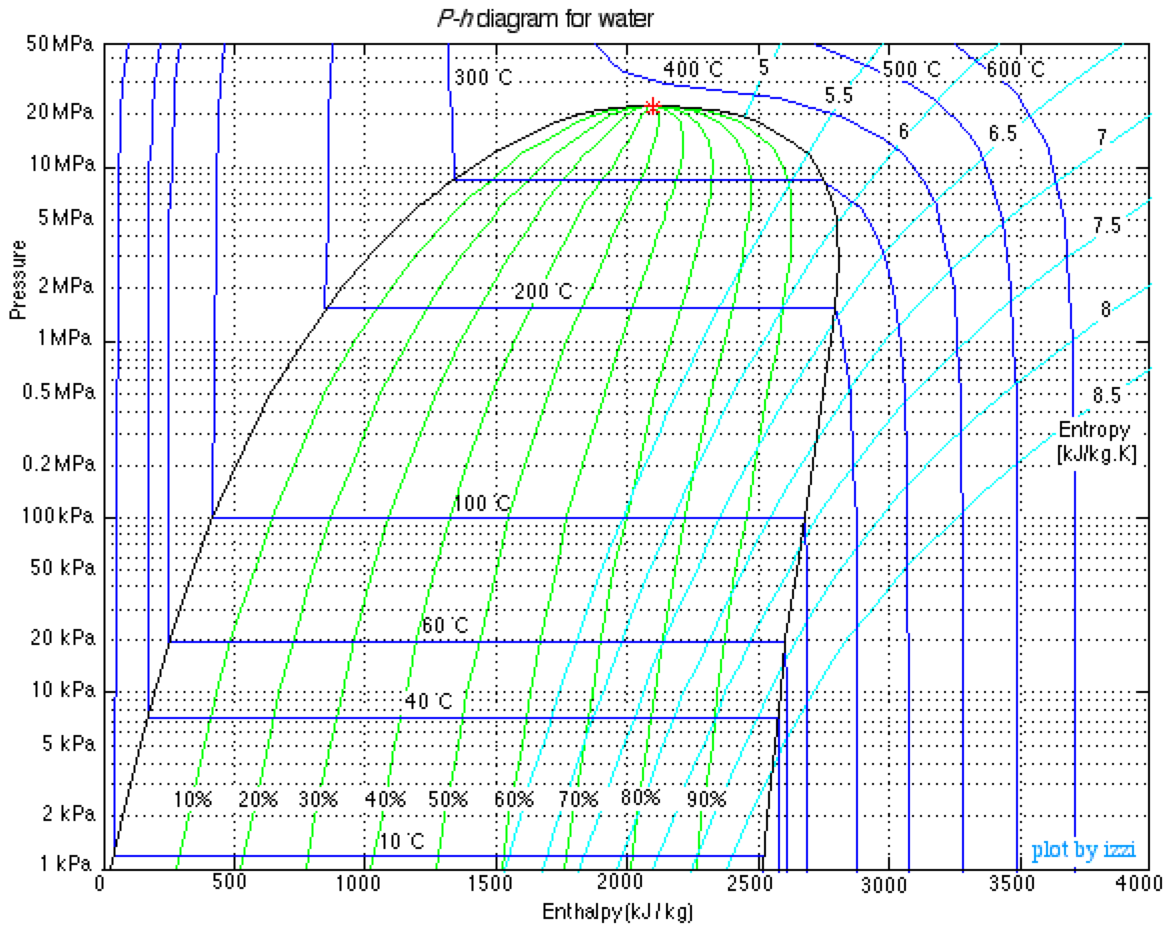
$$\frac{\mathrm{d}}{\mathrm{d}t}E = \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by heat transfer} \\ \text{and by work} \end{array} \right\}$$

- Mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_c} \rho e \mathrm{d}V = \underbrace{- \iint_{\partial V_c} \rho e \vec{v} \cdot \vec{n} \mathrm{d}A}_{\text{Energy flow by convection}} + \dot{Q} - \dot{W}$$

- What is the energy of a fluid?

P-h-diagram for water



Energy

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho e \mathrm{d}V = -\iint\limits_{\partial V_c} \rho e \vec{v} \cdot \vec{n} \mathrm{d}A + \dot{Q} - \dot{W}$$

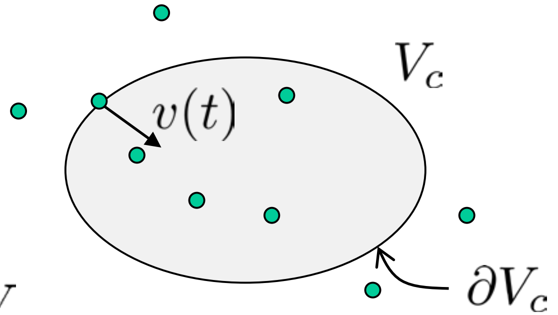
- The energy of a fluid of mass *m*, moving with a velocity *v* at a height *z* in a gravitational field:

$$E = \underbrace{U}_{\substack{\text{internal}\\\text{energy}}} + \underbrace{\frac{1}{2}mv^2}_{\substack{\text{kinetic}\\\text{energy}}} + \underbrace{mgz}_{\substack{\text{potential}\\\text{energy}}}$$

- Specific energy:

$$e = u + \frac{1}{2}v^2 + gz$$

Heat and work flow



$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho e \mathrm{d}V = -\iint\limits_{\partial V_c} \rho e \vec{v}\cdot\vec{n}\mathrm{d}A + \dot{Q} - \dot{W}$$

- Heat flow

$$\dot{Q} = \iint\limits_{\partial V_c} \vec{j}_Q \cdot \vec{n}\mathrm{d}A$$

- Work flow

$$\dot{W} = \underbrace{\iint\limits_{\partial V_c} p\vec{v}\cdot\vec{n}\mathrm{d}A}_{\text{flow work}} + \underbrace{\dot{W}_s}_{\text{shaft work}}$$

Enthalpy

- The energy balance can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho e \mathrm{d}V = -\iint\limits_{\partial V_c} \rho\left(e + \frac{p}{\rho}\right)\vec{v}\cdot\vec{n}\mathrm{d}A - \dot{W}_s + \dot{Q}$$

where the first term on the RHS is convection and flow work

- Define **enthalpy** as

$$h = u + \frac{p}{\rho}$$

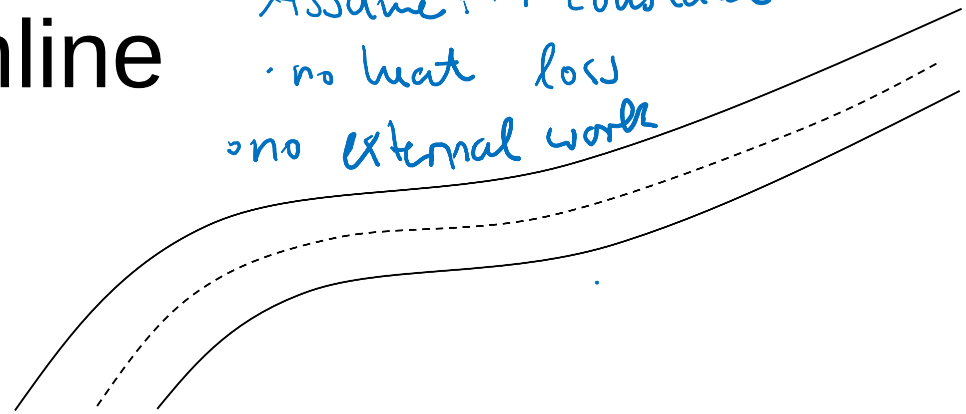
- Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho\left(u + \frac{1}{2}v^2 + gz\right)\mathrm{d}V = -\iint\limits_{\partial V_c} \rho\left(h + \frac{1}{2}v^2 + gz\right)\vec{v}\cdot\vec{n}\mathrm{d}A - \dot{W}_s + \dot{Q}$$

Example: Streamline

Assume: · ρ constant
· no heat loss
· no external work

· steady-state



Mass balance: $w_1 = w_2 = w$

$$\rho v_1 A_1 = \rho v_2 A_2$$

Energy balance: $0 = -\iint_{\partial V_c} \rho e \vec{v} \cdot \vec{n} \, dA - \iint_{\partial V_c} p \vec{v} \cdot \vec{n} \, dA$

$$0 = \rho \left(u_1 + \frac{1}{2} v_1^2 + g z_1 \right) v_1 A_1 - \rho \left(u_2 + \frac{1}{2} v_2^2 + g z_2 \right) v_2 A_2 + p_1 A_1 v_1 - p_2 v_2 A_2$$

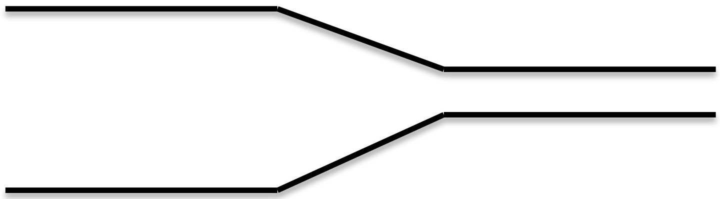
$$= \left(\frac{1}{2} v_1^2 + g z_1 + u_1 - \frac{1}{2} v_2^2 - g z_2 - u_2 + \frac{1}{\rho} p_1 - \frac{1}{\rho} p_2 \right) \cancel{w}$$

$$\Rightarrow \frac{p_1}{\rho} + \frac{1}{2} v_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2 + g z_2$$

$\longrightarrow$ Bernoulli's equation (11.2.6)

Example: Restriction

p_1, A_1, q_1



p_2, A_2, q_2

$$q_i = v_i A_i$$

$$z_1 = z_2 \; ; \; A_1 \gg A_2$$

$$\rightarrow v_1^2 \approx 0$$

Bernoulli's equation

$$v_2^2 = \frac{2}{\rho}(p_1 - p_2)$$

$$v_2 = \sqrt{\frac{2}{\rho}(p_1 - p_2)}$$

$$q_2 = A_2 \sqrt{\frac{2}{\rho}(p_1 - p_2)}$$

→ valve equation

[Cd - friction]

Internal energy and enthalpy

- Specific heat capacities:

$$c_v := \left.\frac{\partial u}{\partial T}\right|_{\text{constant volume}} \qquad c_p := \left.\frac{\partial h}{\partial T}\right|_{\text{constant pressure}}$$

  (found in tables for different fluids, often assumed constant)

- If assumed constant, implies that energy and enthalpy is (linear) function of temperature only:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = c_v \frac{\mathrm{d}T}{\mathrm{d}t} \qquad u(T_2) - u(T_1) = c_v\left(T_2 - T_1\right)$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = c_p \frac{\mathrm{d}T}{\mathrm{d}t} \qquad h(T_2) - h(T_1) = c_p\left(T_2 - T_1\right)$$

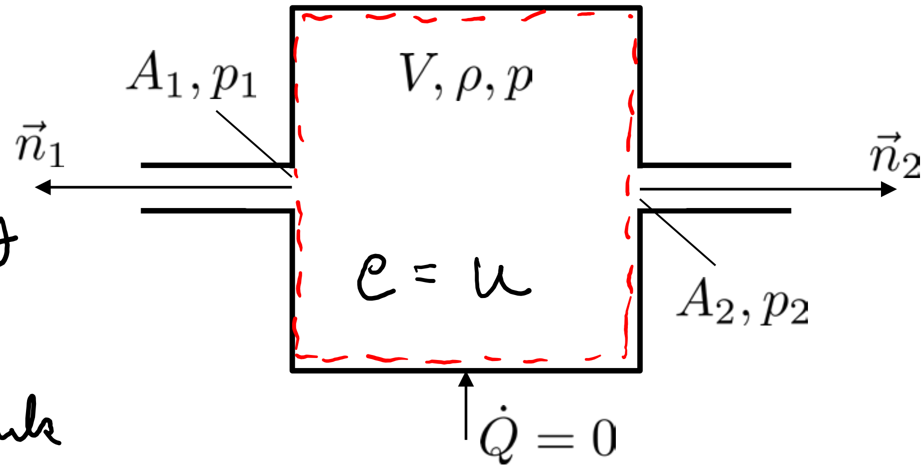
- For ideal gases:

$$c_v = c_p + R$$

- For incompressible fluids (often assumed for liquids):

$$c_v = c_p$$

Example 169



Assume:
- neg. kin. + pot. energy
- no heat flow
- ideally mixed tank

mass balance: $\dot{m} = w_1 - w_2$

energy balance: $d/dt\,(\rho u V) = \rho h_1 v_1 A_1 - \rho h v_2 A_2$

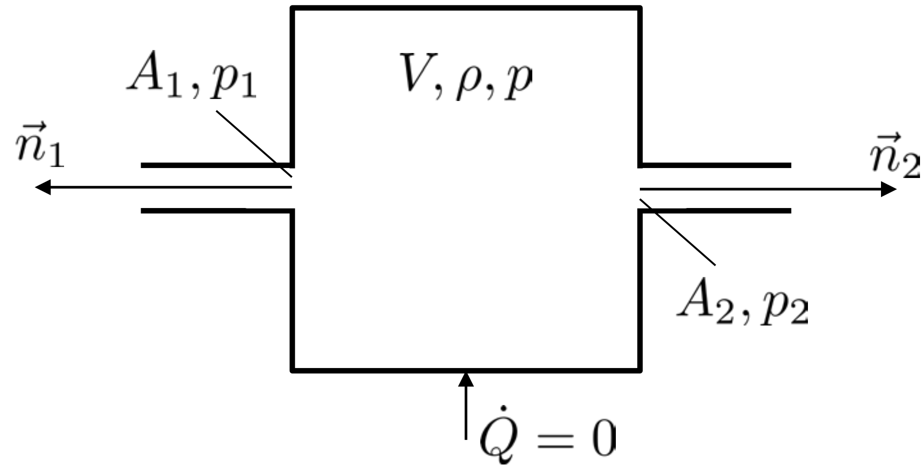
$$d/dt\,(m u) = h_1 w_1 - h w_2$$

$$\dot{m} u + m \dot{u} = h_1 w_1 - h w_2$$

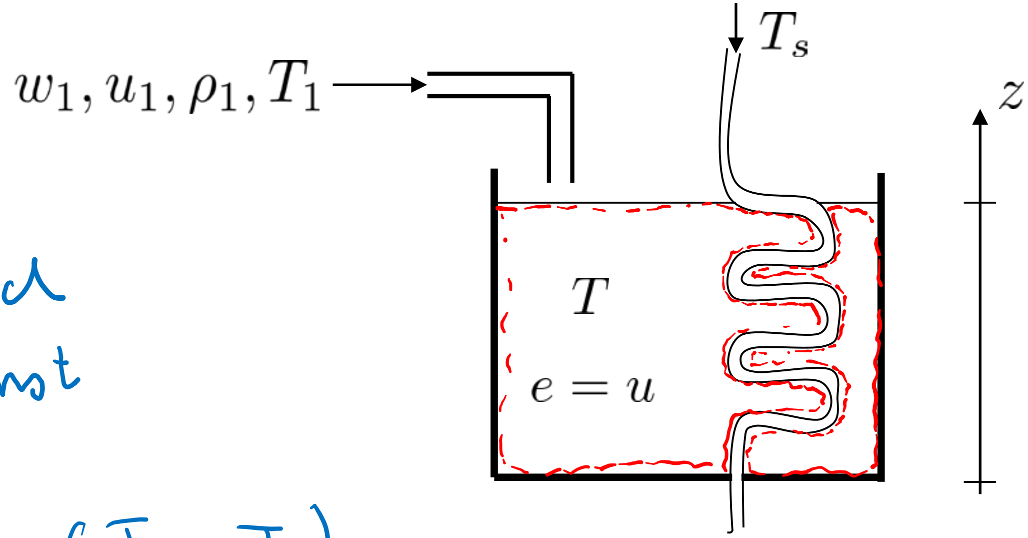
$$m \dot{u} = h_1 w_1 - h w_2 - u (w_1 - w_2)$$

$$\dot{u} = \frac{w_1}{m}(h_1 - u) - \frac{w_2}{m} \underbrace{p/\rho}_{= h - u}$$

Example 169



Example 170

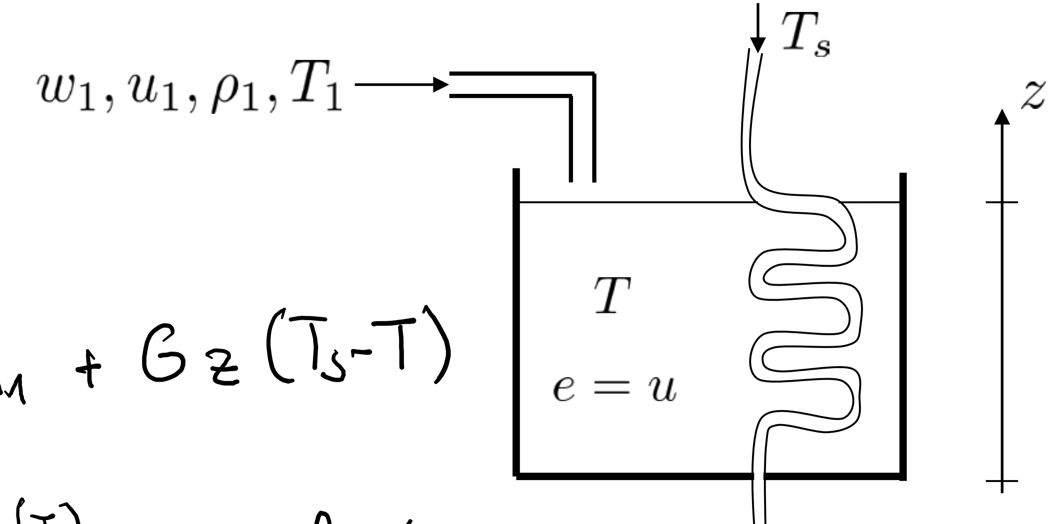


Assume:
- ρ constant
- ideally mixed
- $c_p = c_v = \text{const}$
- $e = u$
- $Q = Gz(T_s - T)$

mass balance: $\frac{d}{dt} m = w_1$

$$\rho A \dot{z} = w_1$$

Example 170



energy balance

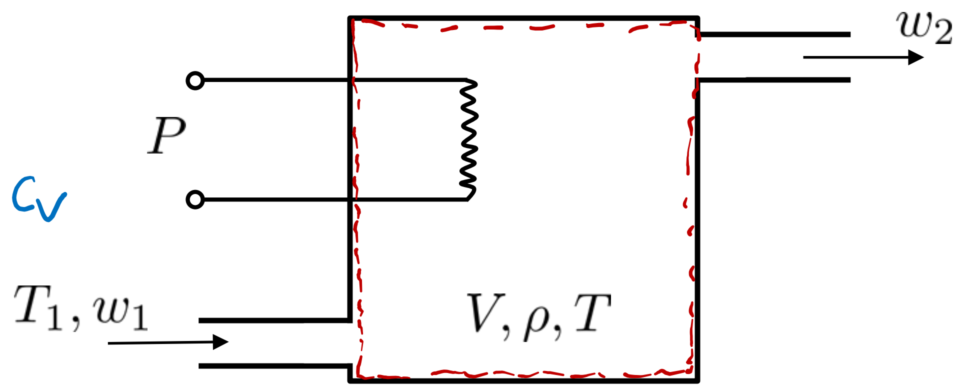
$$\frac{d}{dt}(\rho u V) = w_1 h_1 + Gz(T_s - T)$$

$$u(T)\underbrace{\frac{d}{dt}(\rho V)}_{\dot m = w_1} + \rho V \frac{d}{dt} u(T) = w_1 h_1(T_1) + Gz(T_s - T)$$

$$\rho A z c_p \frac{d}{dt} T = \underbrace{w_1 c_p (T_1 - T)}_{w_1 h(T_1) - w_1 u(T)} + Gz(T_s - T)$$

$$\dot T = \frac{w_1}{\rho A z}(T_1 - T) + \frac{G}{\rho A c_p}(T_s - T)$$

Example 171



- incompressible fluid: $c_p = c_v$
- ideally mixed
- $V = \text{const}$
- $\rho = \text{const.}$

mass balance: $w_1 = w_2 = w$

energy balance: $\frac{d}{dt}(\rho V u(T)) = w h(T_1) - w h(T) + P$

$$u(T) \underbrace{\frac{d}{dt}(\rho V)}_{=0} + \rho V \frac{d}{dt} u(T) = w h_1(T_1) - w h(T) + P$$

$$\rho V c_p \frac{dT}{dt} = w c_p (T_1 - T) + P$$

$$\dot{T} = \frac{w}{m}(T_1 - T) + \frac{P}{c_p \cdot m}$$

Homework

- Read 4.1 - 4.3
- Check Slide 26-30 (differential balance)

Differential mass balance

- Recall the integral mass balance:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_c}\rho \mathrm{d}V}_{} = -\underbrace{\iint_{\partial V_c}\rho\vec{v}\cdot\vec{n}\mathrm{d}A}_{}$$

Mathematics (obvious?)

Divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_c}\rho \mathrm{d}V = \iiint_{V_c}\frac{\partial\rho}{\partial t}\mathrm{d}V$$

$$\iint_{\partial V_c}\rho\vec{v}\cdot\vec{n}\mathrm{d}A = \iiint_{V_c}\vec{\nabla}\cdot(\rho\vec{v})\mathrm{d}V$$

$$(\nabla\cdot v) = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

- That is:

$$\iiint_{V_c}\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})\mathrm{d}V = 0$$

- This must hold for arbitrary control volumes, which implies

rate of change due to moving in a gradient field

local rate of change

$$\boxed{\vec{\nabla}\cdot(\rho\vec{v})+\frac{\partial\rho}{\partial t}=0}$$

Differential mass balance, also called *continuity equation* or *advection equation*

Alternative formulations

- The differential mass balance

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

- From definition of nabla operator, this is the same as

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\rho v_i) = 0, \qquad \mathbf{v} = (v_1, v_2, v_3)^\mathsf{T}$$

- If we introduce the *material derivative*,

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} := \frac{\partial \phi}{\partial t} + \mathbf{v}^\mathsf{T} \nabla \phi = \frac{\partial \phi}{\partial t} + \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i} v_i$$

The material derivative is the derivative following a particle (as opposed to the derivative at a fixed point in space)

and use product rule, we can write

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \vec{v} = 0$$

Differential momentum and energy balances

- Differential momentum balance for inviscid fluid (*Euler's equation*)

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\vec{\nabla} p + \rho \vec{f}, \quad \text{where } \rho \vec{f} \text{ is the mass force (e.g. gravity)}$$

- For viscous (Newtonian) fluids, the differential momentum balance is the famous *Navier-Stokes* equation:

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{v} + \rho \vec{f}$$

- Differential energy balance (for example)

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{1}{2} \vec{v}^2 + u \right) = -\vec{\nabla} \cdot (p \vec{v}) - \vec{\nabla} \cdot \vec{j}_Q + \rho \vec{v} \cdot \vec{f}$$

Computational fluid dynamics

- CFD = solving momentum + mass balances (that is, Navier-Stokes + continuity equation) for different setups

http://physbam.stanford.edu/~fedkiw/

Example of differential energy balances: The heat equation of a solid

- The energy balance:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\vec{v}^2 + u\right) = -\vec{\nabla}\cdot(p\vec{v}) - \vec{\nabla}\cdot\vec{j}_Q + \rho\vec{v}\cdot\vec{f}$$

- Solid: Disregard kinetic and potential energy, no velocity:

$$\rho\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{j}_Q$$

- We need a «closure relation». Here in the form of *Fourier's law*:

$$\vec{j}_Q = -\alpha\vec{\nabla}(\rho c_p T)$$

- Combined with

$$\frac{\partial u}{\partial t} = c_p\frac{\partial T}{\partial t}$$

we get

$$\frac{\partial T}{\partial t} - \alpha\vec{\nabla}\cdot\vec{\nabla}T = 0$$

In one dimension:

$$\frac{\partial T(x,t)}{\partial t} - \alpha\frac{\partial^2 T(x,t)}{\partial x^2} = 0$$