# Lecture 8: Stability, Padé approximations

- Stability of Runga-Kutta methods
  - **A-stability**
  - Aliasing
  - **L-stability**
  - Padé approximations
  - (Nonlinear stability: AN-, B- and algebraic stability)

Book: 14.6

# Explicit Runge-Kutta (ERK) methods

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- ERK:

$$
\begin{aligned}
k_1 &= f(y_n, t_n) \\
k_2 &= f(y_n + h a_{21} k_1, t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{31} k_1 + a_{32} k_2), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1} k_1 + a_{\sigma,2} k_2 + \ldots + a_{\sigma,\sigma-1} k_{\sigma-1}), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\mathsf{T}
\end{array}
\qquad
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}
$$

# Recap: Test system, stability function

- One step method (typically: Runge-Kutta):

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Apply it to scalar test system:

$$\dot{y} = \lambda y$$

- We get:

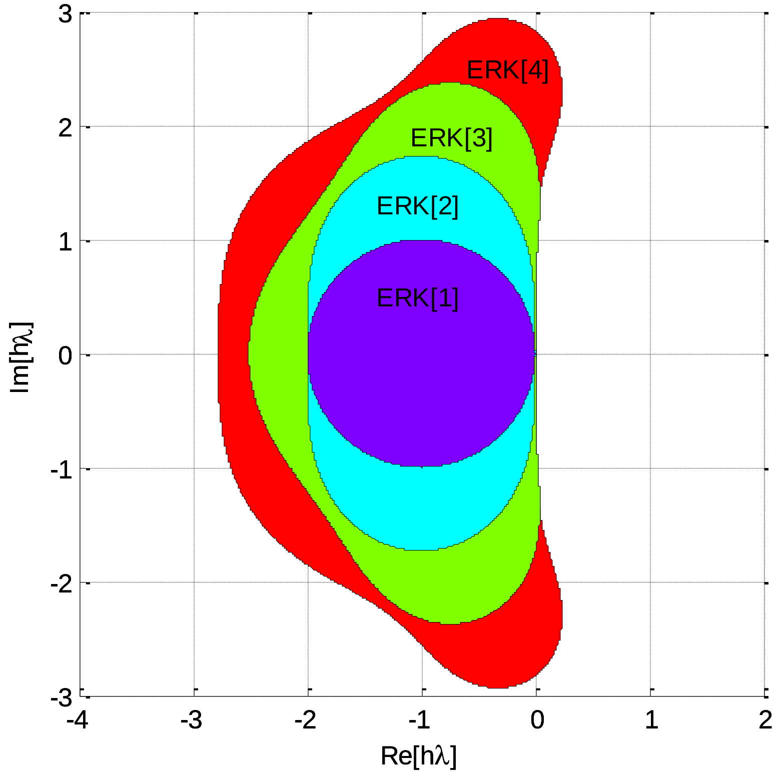
$$y_{n+1} = R(h\lambda) y_n$$

where $R(h\lambda)$ is stability function

- The method is stable (for test system!) if

$$|R(h\lambda)| \leq 1$$

# Stability regions for ERK methods



# Implicit Runge-Kutta (IRK) methods

- IVP: $$\dot{y} = f(y, t), \quad y(0) = y_0$$
- IRK:

$$\begin{aligned}
k_1 &= f(y_n + h(a_{1,1}k_1 + a_{1,2}k_2 + \ldots + a_{1,\sigma}k_\sigma), t_n + c_1 h) \\
k_2 &= f(y_n + h(a_{2,1}k_1 + a_{2,2}k_2 + \ldots + a_{2,\sigma}k_\sigma), t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{3,1}k_1 + a_{3,2}k_2 + \ldots + a_{3,\sigma}k_\sigma), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1}k_1 + a_{\sigma,2}k_2 + \ldots + a_{\sigma,\sigma}k_\sigma), t_n + c_\sigma h)
\end{aligned}$$

$$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)$$

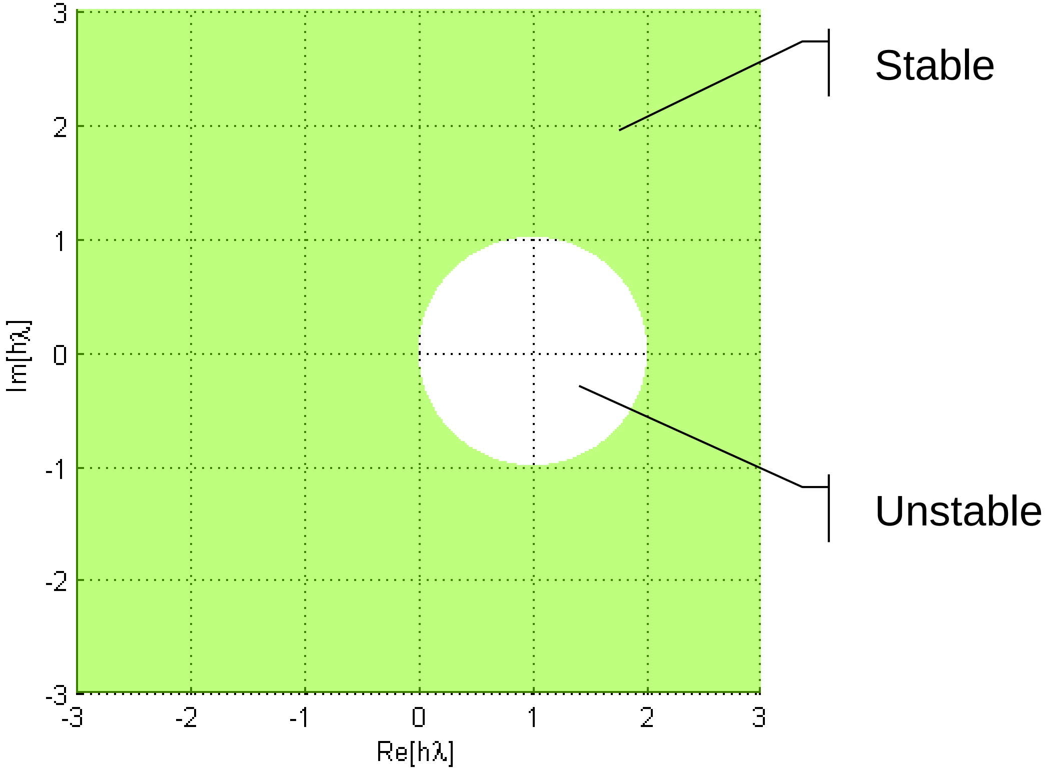
- Butcher array:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^{\mathsf{T}} \end{array}$$

$$\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1,\sigma-1} & a_{1,\sigma} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2,\sigma-1} & a_{2,\sigma} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & a_{\sigma,\sigma} \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}$$

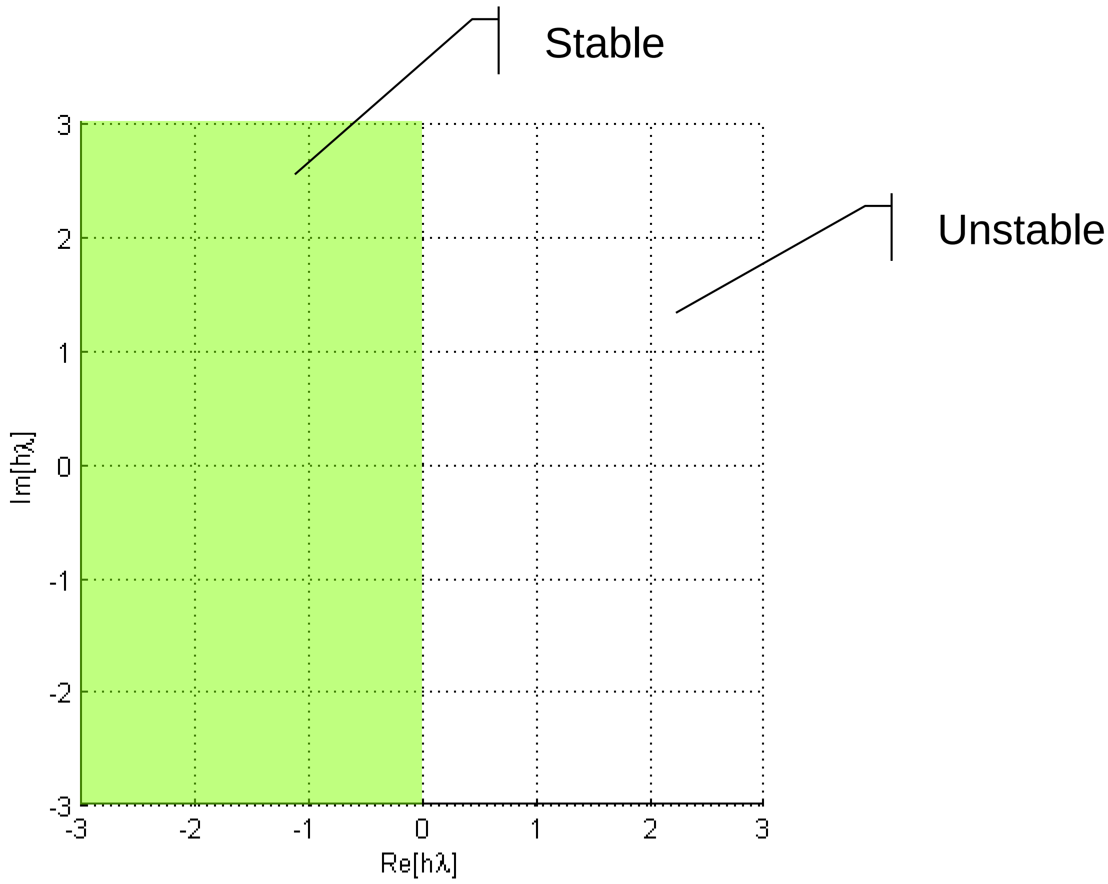
# Stability regions for implicit methods

- Implicit Euler



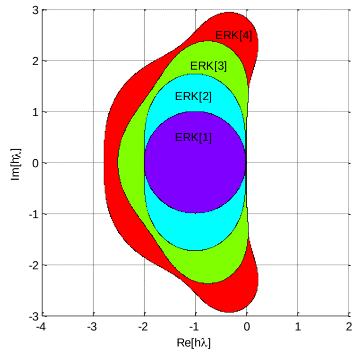
# Stability regions for implicit methods

- Trapezoidal rule and implicit midpoint rule:

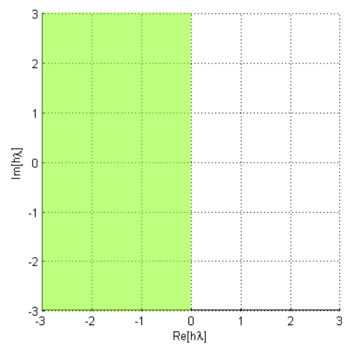


# Why use IRK methods?

- IRK methods are much more complex (since we have to solve a set of nonlinear equations for each step) than ERK methods, so why use them?
  - Accuracy? Stability?




- Not because of accuracy
  - Even if an IRK method may have higher accuracy for a given number of stages, it is easy and cheap to achieve same accuracy for ERK by increasing the number of stages




- It's because of the much larger stability region!

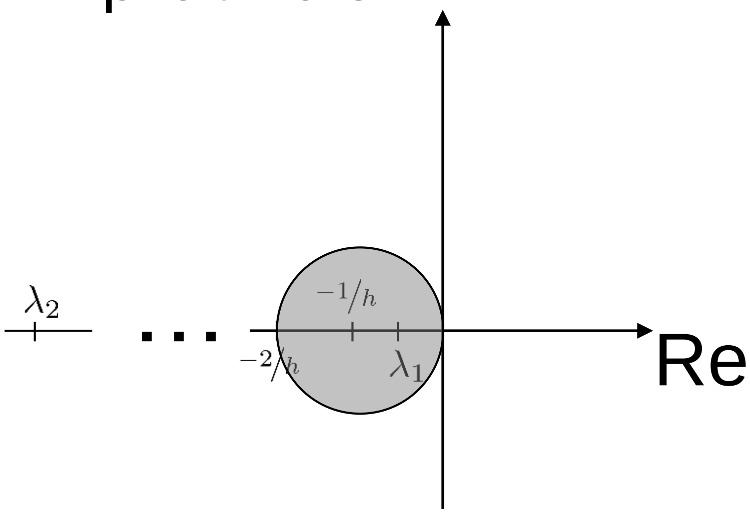
- When is this important?
  - **Stiff systems!**

# Stiff system

$$\dot{y}_1 = \lambda_1 y_1$$
$$\dot{y}_2 = \lambda_2 y_2$$

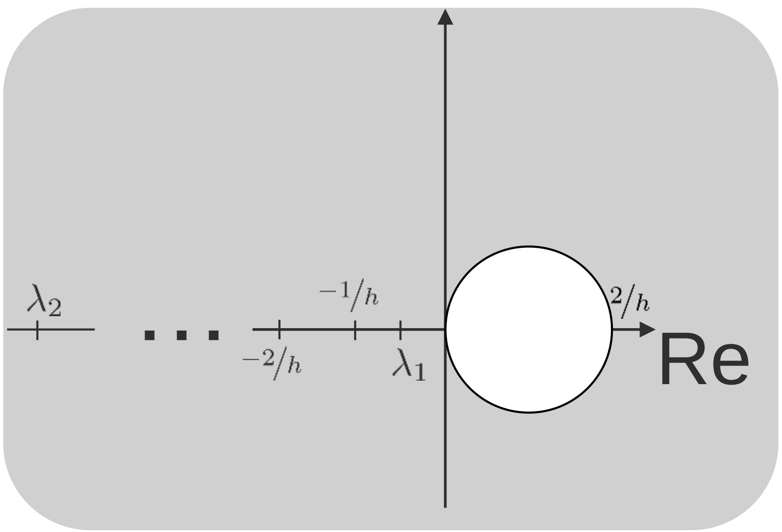
$$\lambda_2 << \lambda_1 < 0$$

Explicit Euler




must choose a small *h* for stability

Implicit Euler

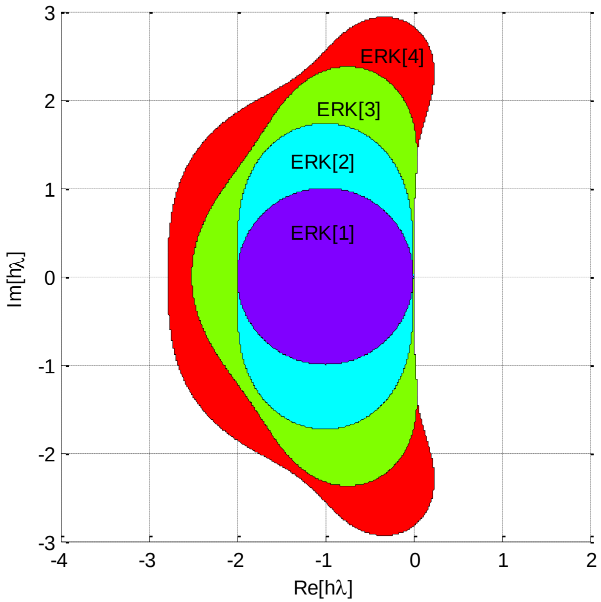



can choose all *h* for stability

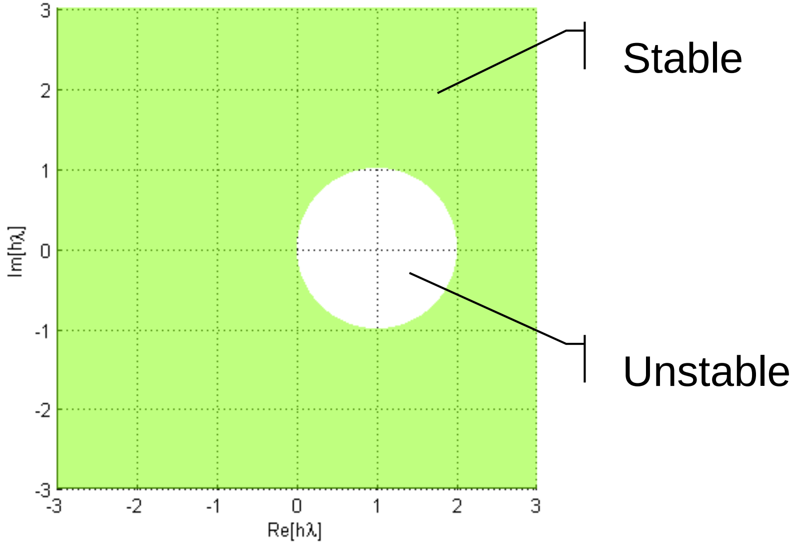
# Example 221

# ERK vs IRK stability regions

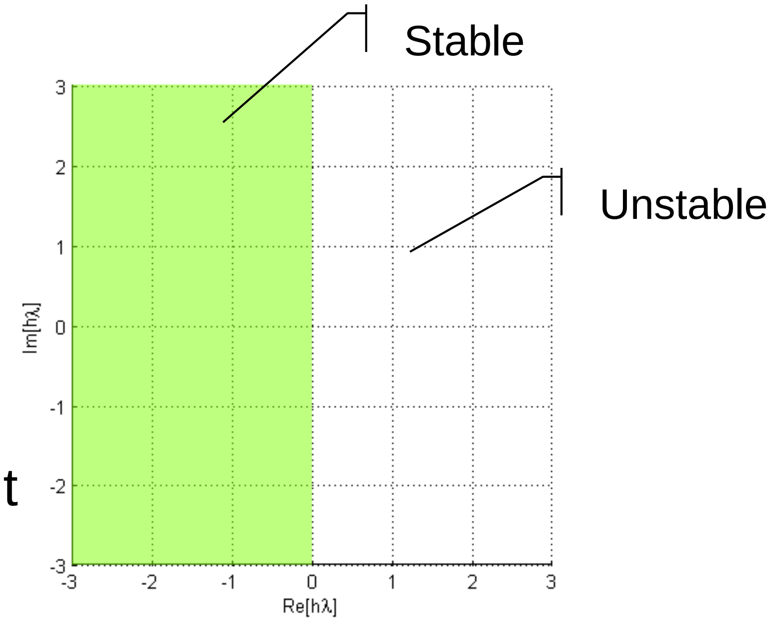
## ERK-methods




## Implicit Euler




## Trapezoidal/ Implicit Midpoint



# Aliasing I

# Aliasing II

- Assume:

Test system I

$$\dot{y} = \lambda y \ ; \ \ \lambda = \sigma + j\omega$$

$$y_L(t_n; t_{n+1}) = e^{h\lambda} y_n$$

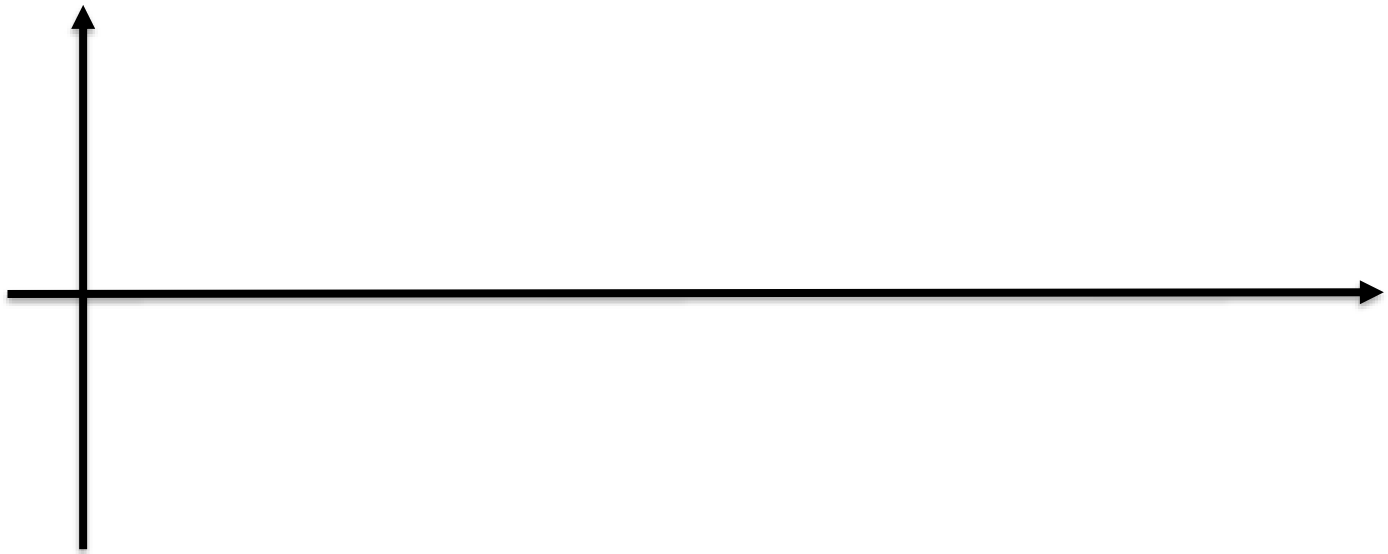
Test system II

$$\dot{y} = \mu y$$

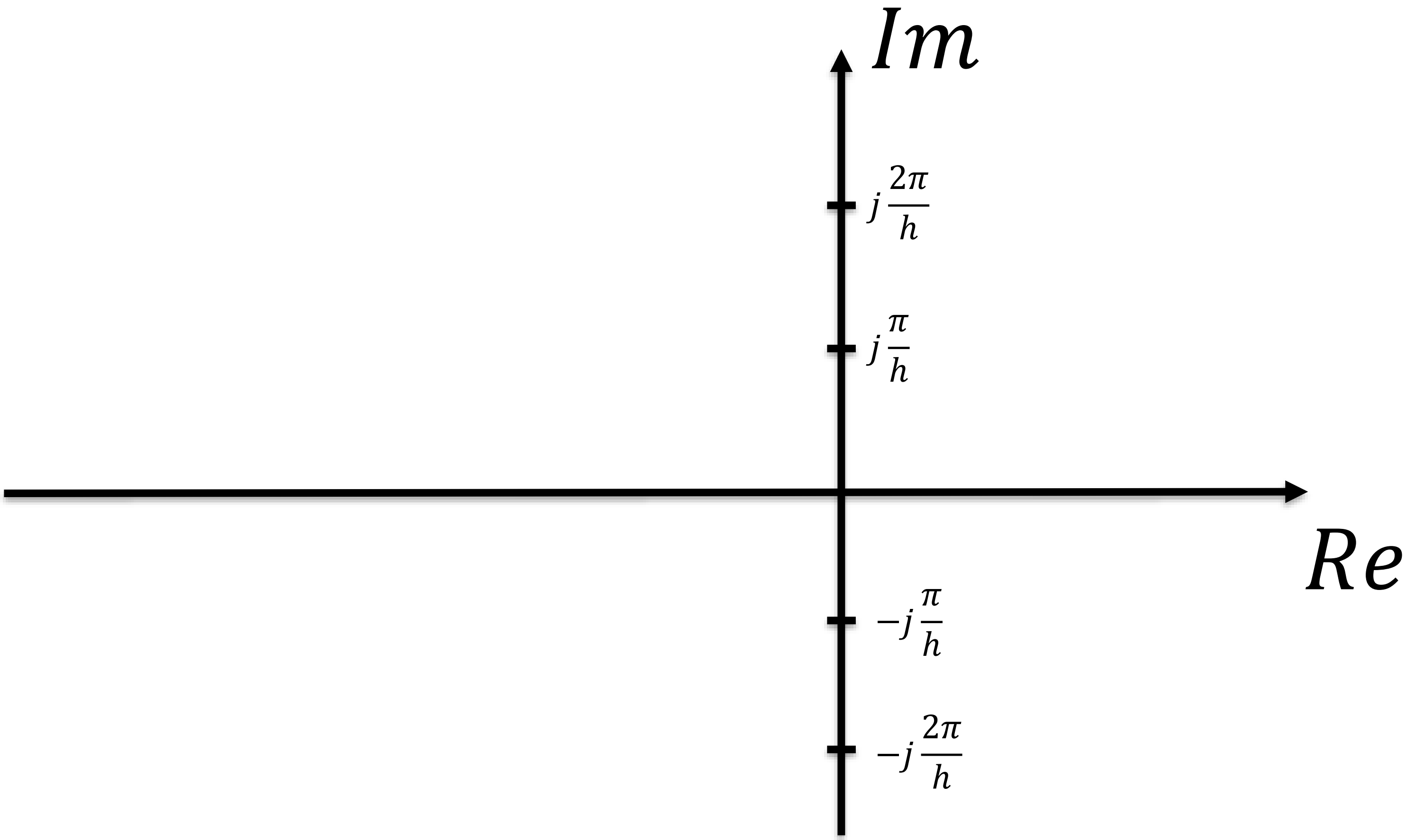
$$y_L(t_n; t_{n+1}) = e^{h\mu} y_n$$

When are these two systems the same?

# Visualisation I

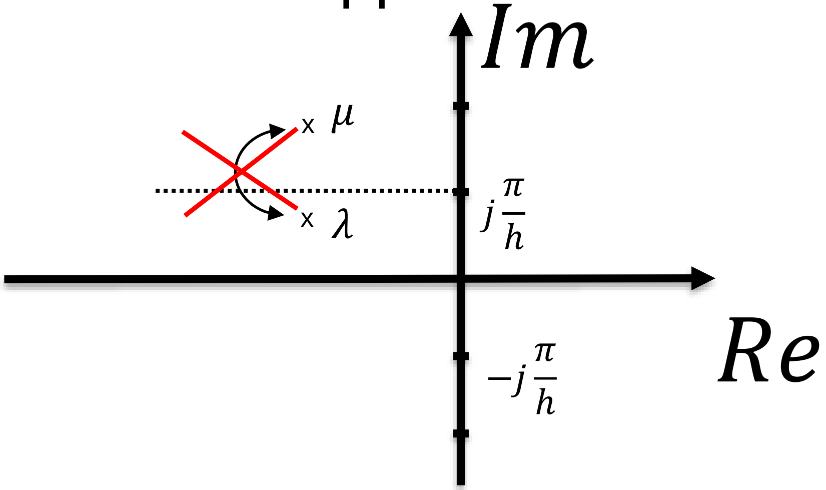


# Visualisation II



# L-stability I

- Assume $\mu$ eigenvalues and we want to suppress:



# L-stability II

# Example – L-stability

# Padé-approximation I

$$
\begin{tikzcd}
& \text{Test system: } \dot{y} = \lambda y \arrow[dl, "\text{exact}"'] \arrow[dr, "\text{RK}"] & \\
y_L(t_n; t_{n+1}) = e^{h\lambda} y_n & & y_{n+1} = R(h\lambda) y_n
\end{tikzcd}
$$

«good» methods fulfill $e^s \approx R(s)$ $\quad s = h\lambda$

# Padé-approximation II

**Def:** Padé approximation $P_m^k(s)$ of $e^s$ is the rational function:

$$P_m^k(s) = \frac{1 + \beta_1 s + \cdots + \beta_k s^k}{1 + \gamma_1 s + \cdots + \gamma_m s^m},$$

which approximates $e^s$ the best given $k$ and $m$

# Padé approximations to $e^s$

| $m \backslash k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{1}{1}$ | $\frac{1+s}{1}$ | $\frac{1+s+\frac{1}{2}s^2}{1}$ | $\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$ |
| 1 | $\frac{1}{1-s}$ | $\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$ | $\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$ | $\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$ |
| 2 | $\frac{1}{1-s+\frac{1}{2}s^2}$ | $\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$ | $\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$ | $\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$ |
| 3 | $\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$ | $\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$ | $\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$ | $\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$ |

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

# Padé approximations to $e^s$

| $m \backslash k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{1}{1}$ | $\frac{1+s}{1}$ | $\frac{1+s+\frac{1}{2}s^2}{1}$ | $\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$ |
| 1 | $\frac{1}{1-s}$ | $\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$ | $\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$ | $\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$ |
| 2 | $\frac{1}{1-s+\frac{1}{2}s^2}$ | $\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$ | $\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$ | $\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$ |
| 3 | $\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$ | $\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$ | $\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$ | $\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$ |

L-stable L-stable A-stable

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

# Padé approximations as stability function

1. Assume $k \leq m \leq k + 2$

   In that case $\left|P_m^k(s)\right| \leq 1$ if $Re[s] \leq 0$

2. .$|P_m^m(j\omega)| = 1$ if $\omega \to \infty$

3. Assume $m > k$

   In that case $\left|P_m^k(s)\right| \to 0$ if $\omega \to \infty$

One step methods with stability function $R(s) = P_m^k(s)$ are:

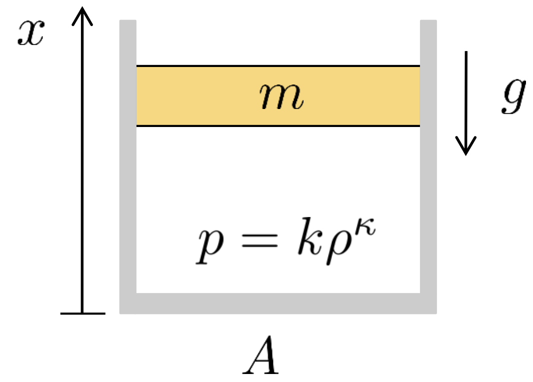
- A-stable if $k = m$
- L-stable if $m = k + 1$ or $m = k + 2$

# Pneumatic spring example, again

- Model from Newton's 2nd law:

$$\ddot{x} + g(1 - x^{-\kappa}) = 0$$

  "mass-spring-damper with nonlinear spring"



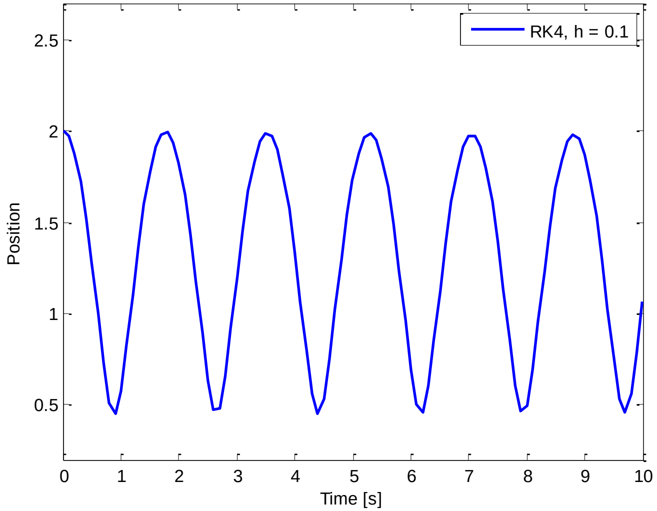

- On state-space form $\dot{y} = f(y, t)$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -g(1 - y_1^{-\kappa}) \end{pmatrix}$$

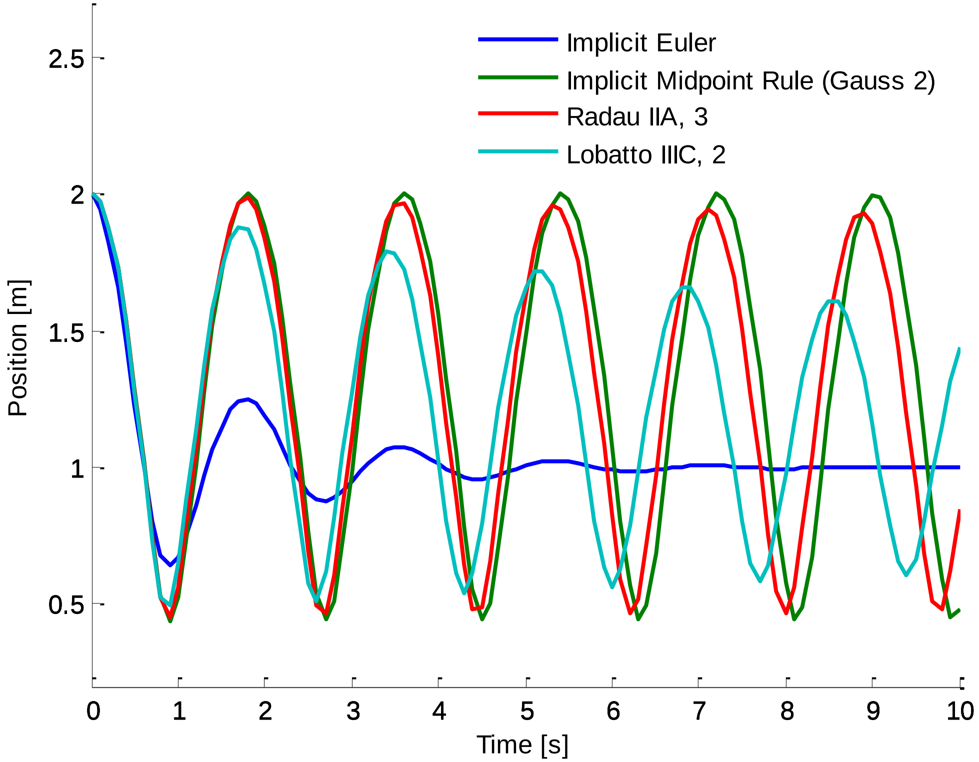
- Linearization about equilibrium:

$$\frac{\partial f}{\partial y} = \begin{pmatrix} 0 & 1 \\ -g\kappa & 0 \end{pmatrix}, \qquad \lambda_{1,2} = \pm j\omega_0, \quad \omega_0 = \sqrt{g\kappa} \approx 3.7$$

# Simulation




h = 0.5 (stability limit for RK4)



# Padé approximations to $e^s$

| $m \backslash k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{1}{1}$ | $\frac{1+s}{1}$ | $\frac{1+s+\frac{1}{2}s^2}{1}$ | $\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$ |
| 1 | $\frac{1}{1-s}$ | $\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$ | $\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$ | $\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$ |
| 2 | $\frac{1}{1-s+\frac{1}{2}s^2}$ | $\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$ | $\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$ | $\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$ |
| 3 | $\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$ | $\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$ | $\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$ | $\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$ |

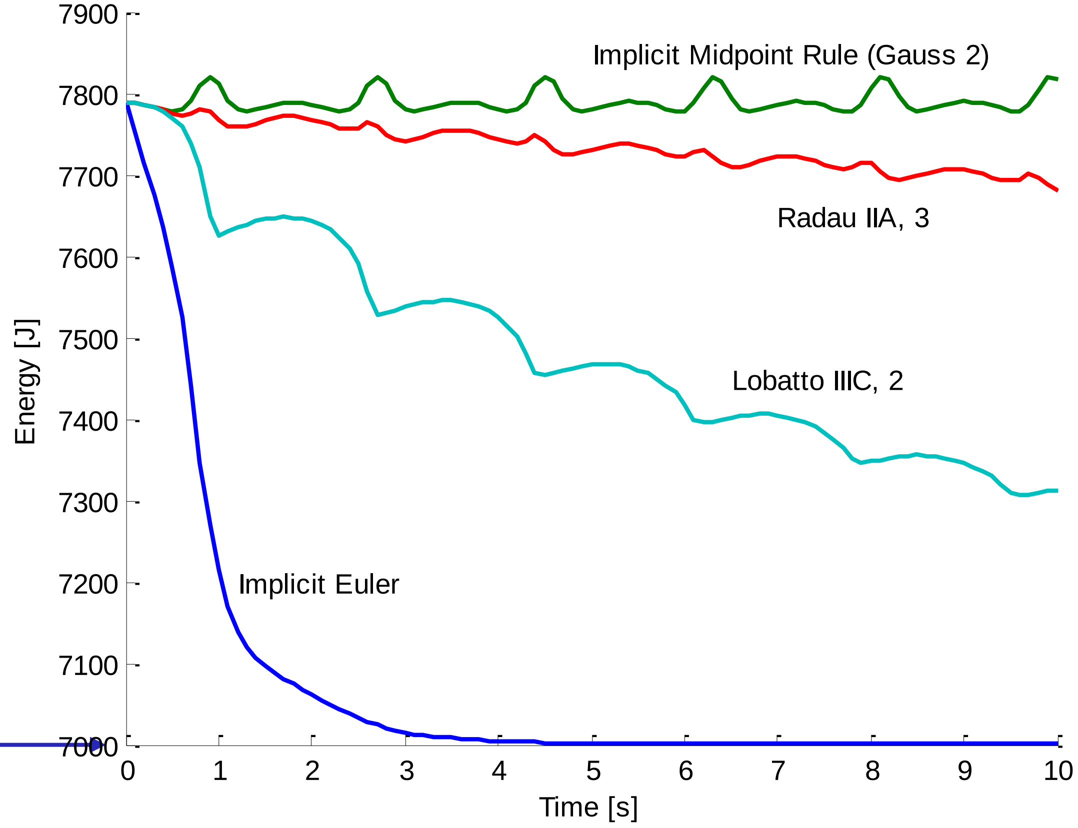
L-stable

L-stable

A-stable

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

# Energy



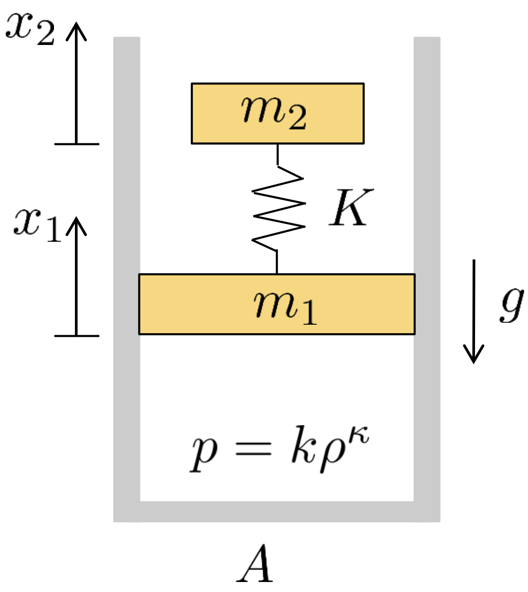

Equilibrium energy

# Pneumatic spring with resonant load

- Equations of motion (Newton's law):

$$\ddot{x}_1 + g\left(1 - \frac{m_1 + m_2}{m_1} x_1^{-\kappa}\right) + \frac{\omega_2^2}{2}(x_1 - x_2) = 0$$

$$\ddot{x}_2 + g + \frac{\omega_2^2}{2}(x_2 - x_1) = 0$$




- Linearization around equilibrium:

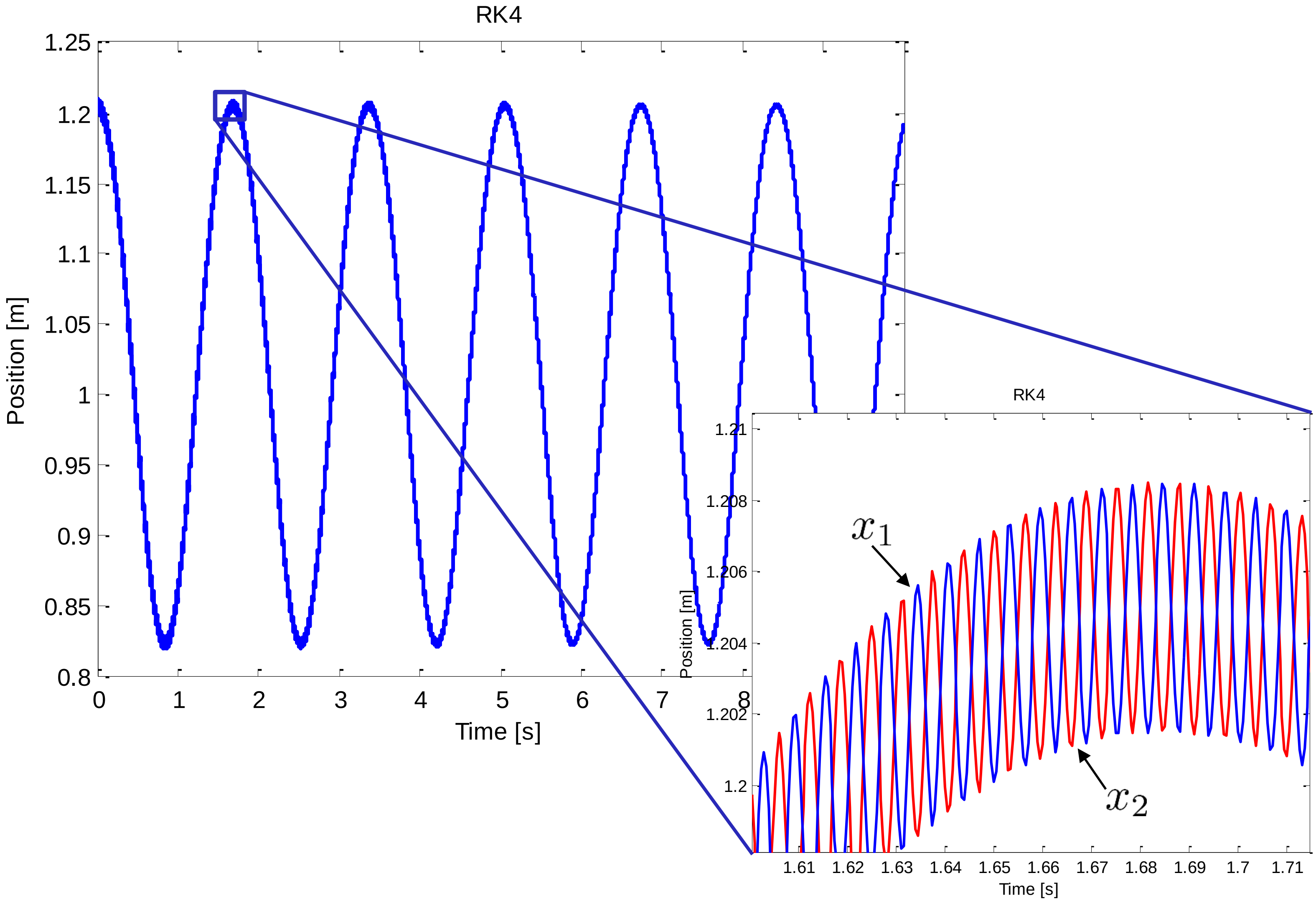
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -g\frac{m_1+m_2}{m_1}\kappa(x_1^*)^{-(\kappa-1)} - \frac{\omega_2^2}{2} & 0 & \frac{\omega_2^2}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\omega_2^2}{2} & 0 & -\frac{\omega_2^2}{2} & 0 \end{pmatrix}$$

- Eigenvalues:

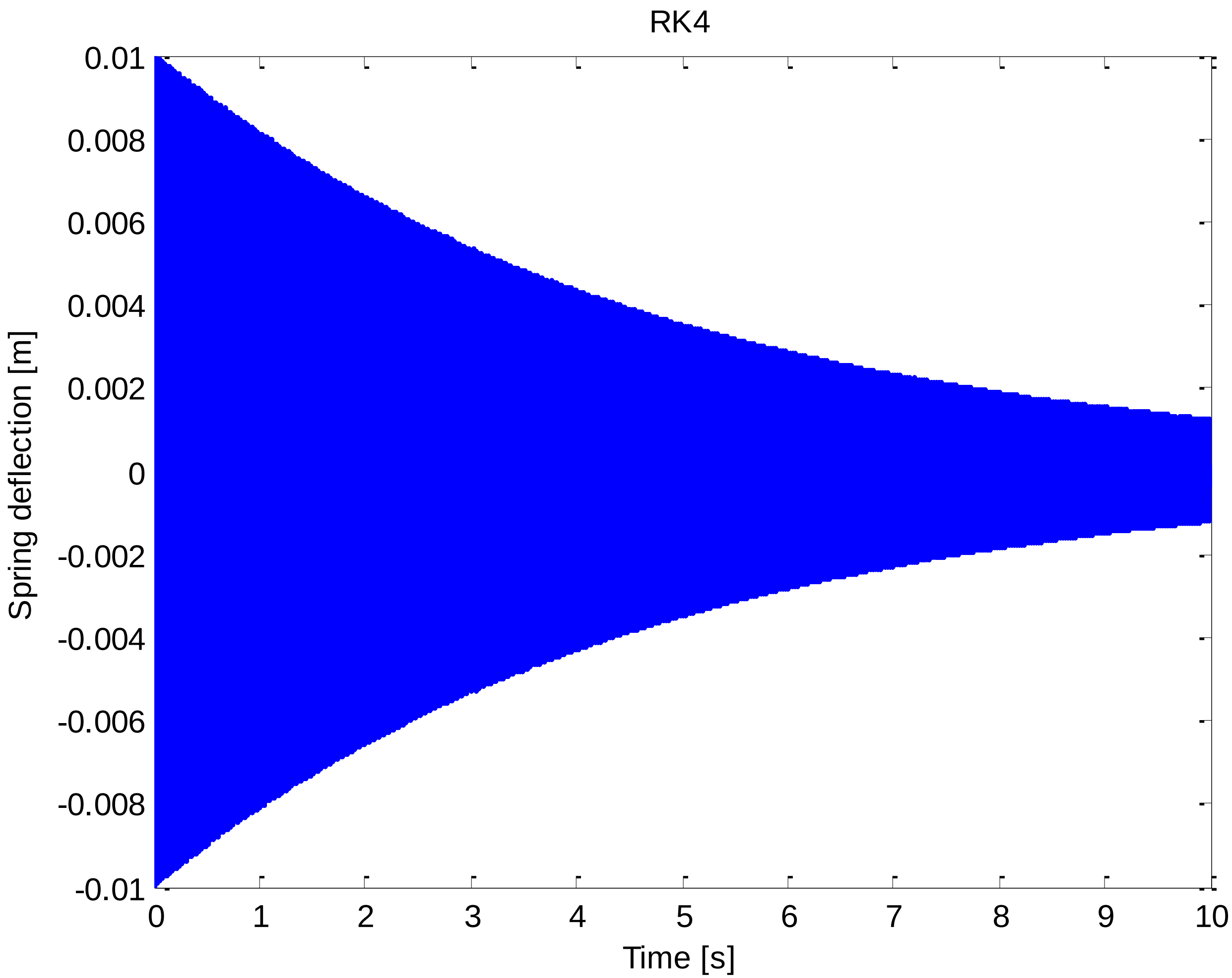
$$\lambda_{1,2} = \pm j\omega_1, \quad \omega_1 = 3.7\ \mathrm{rad/s}$$
$$\lambda_{3,4} = \pm j\omega_2, \quad \omega_2 = 1000\ \mathrm{rad/s}$$

# Position of the two masses

RK4 with time step h = 0.0005

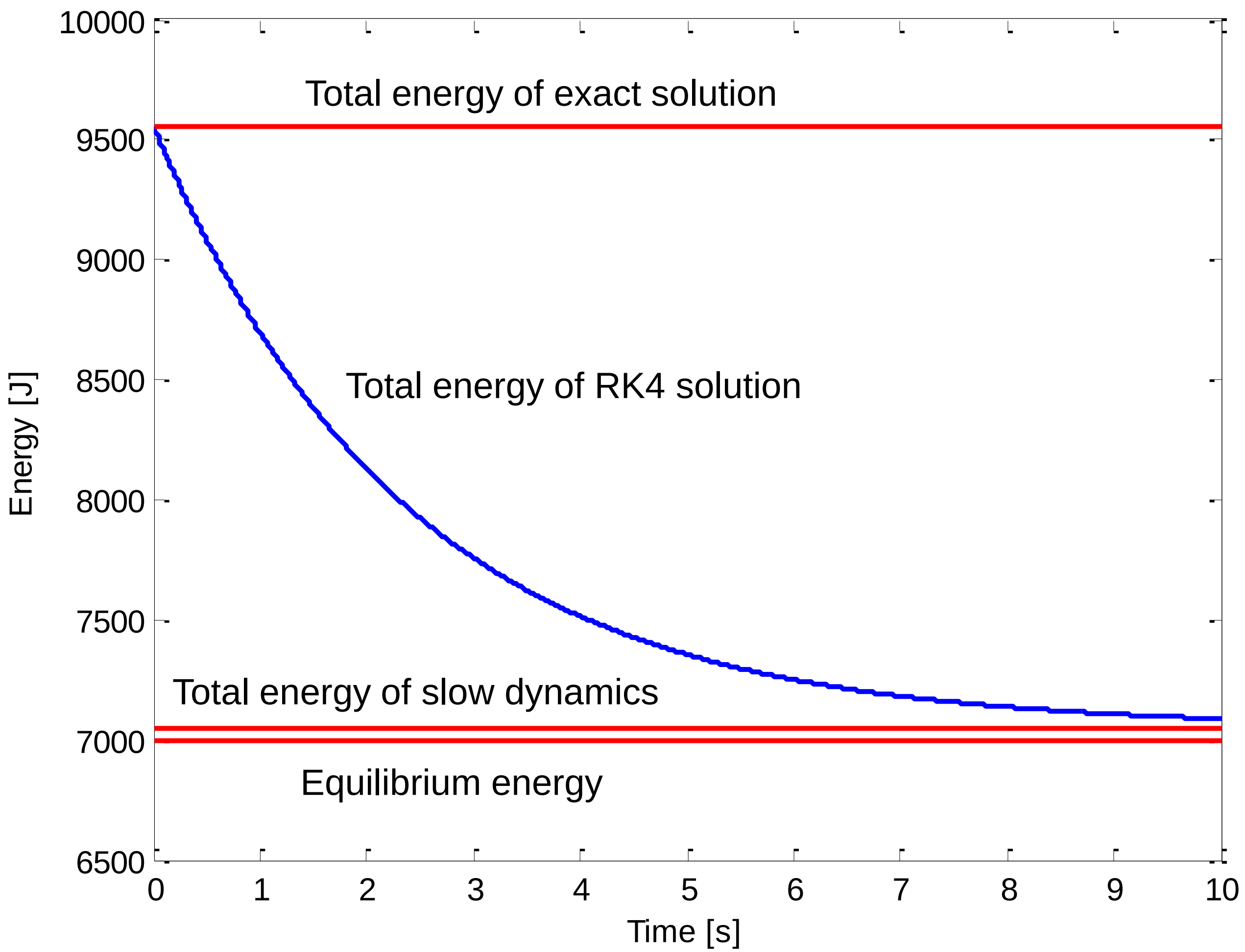


# Spring deflection, RK4 with h = 0.0005




- Oscillation is lightly damped by integration method

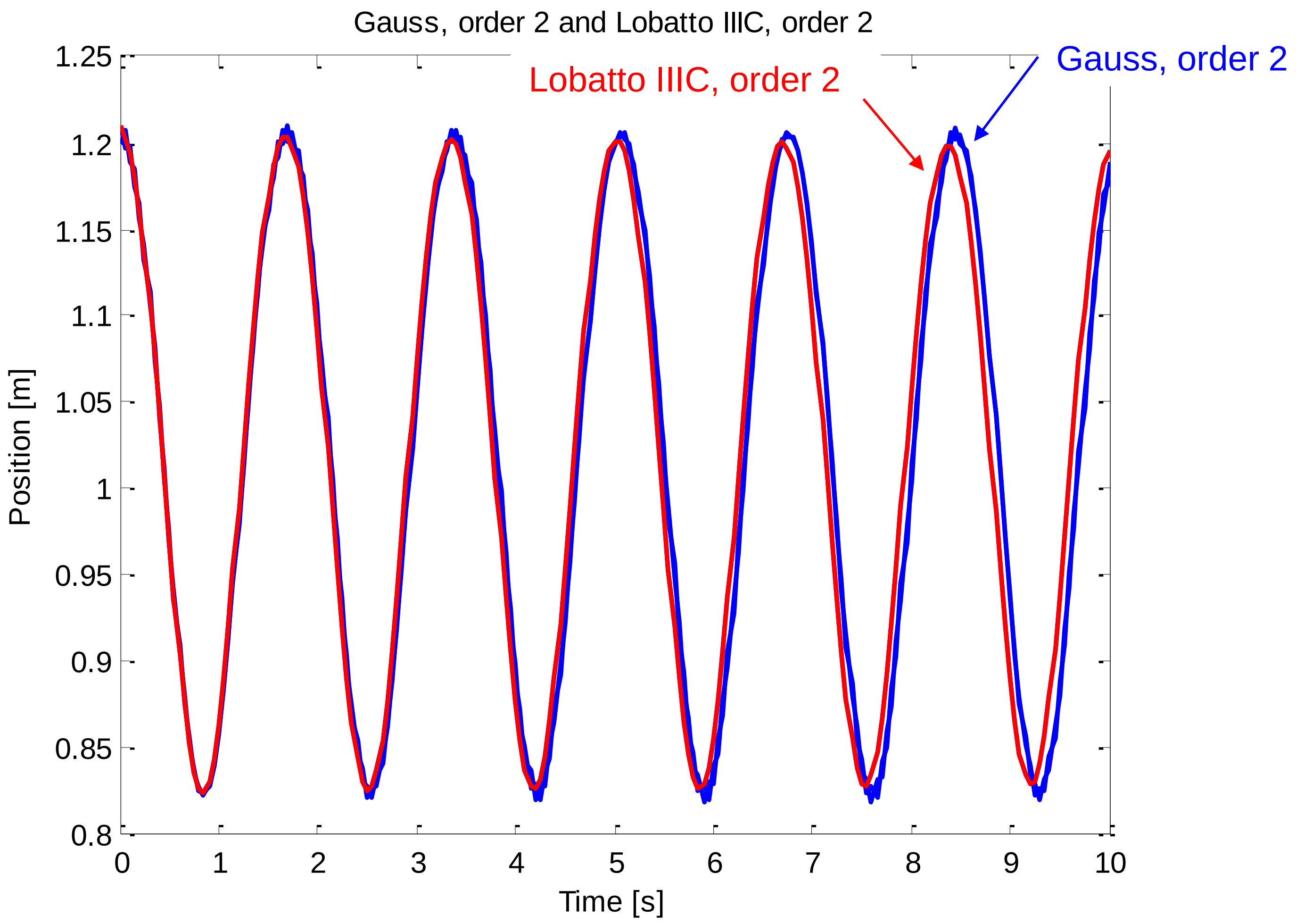
# Energy of RK4 solution, h = 0.0005




- Energy related to fast dynamics slowly damped out

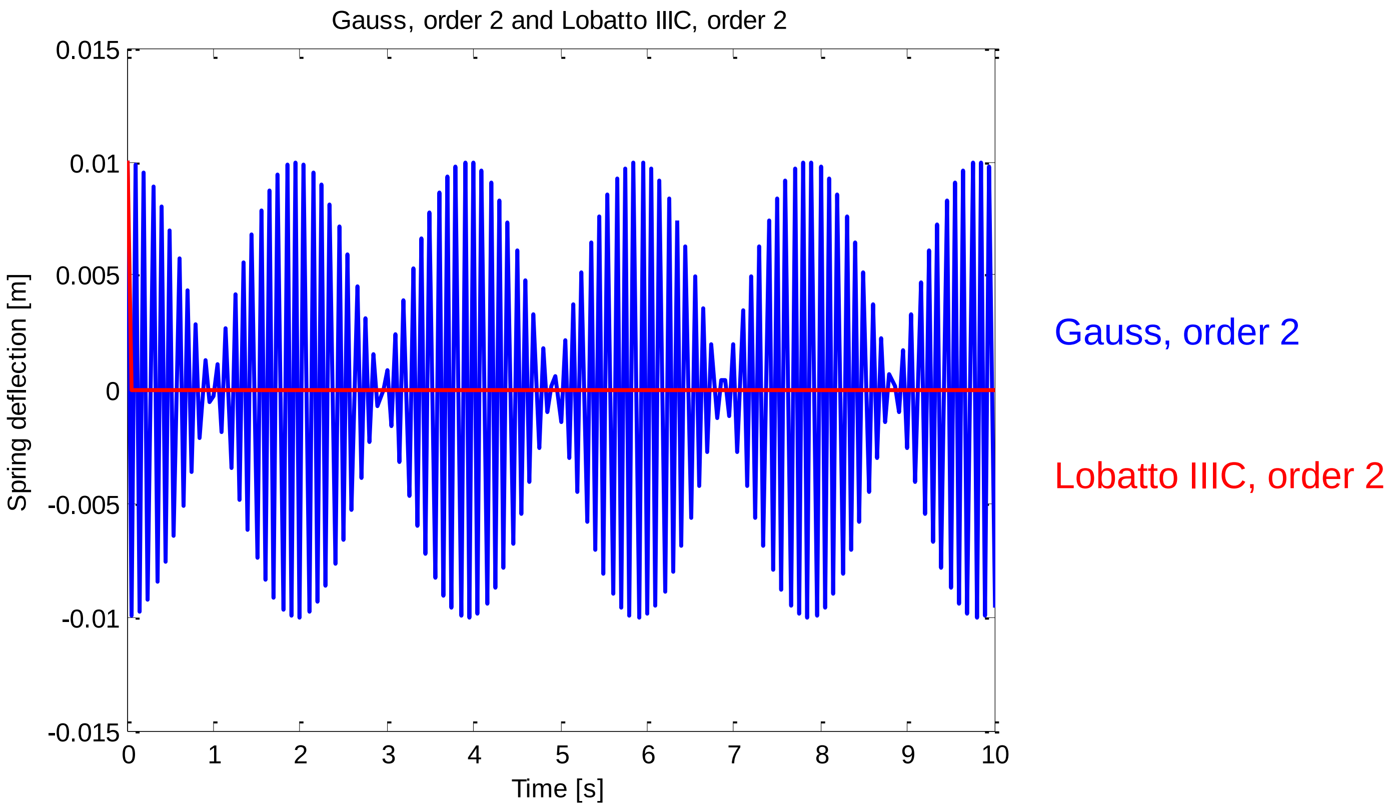
# Position of the two masses

Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05



# Spring deflection

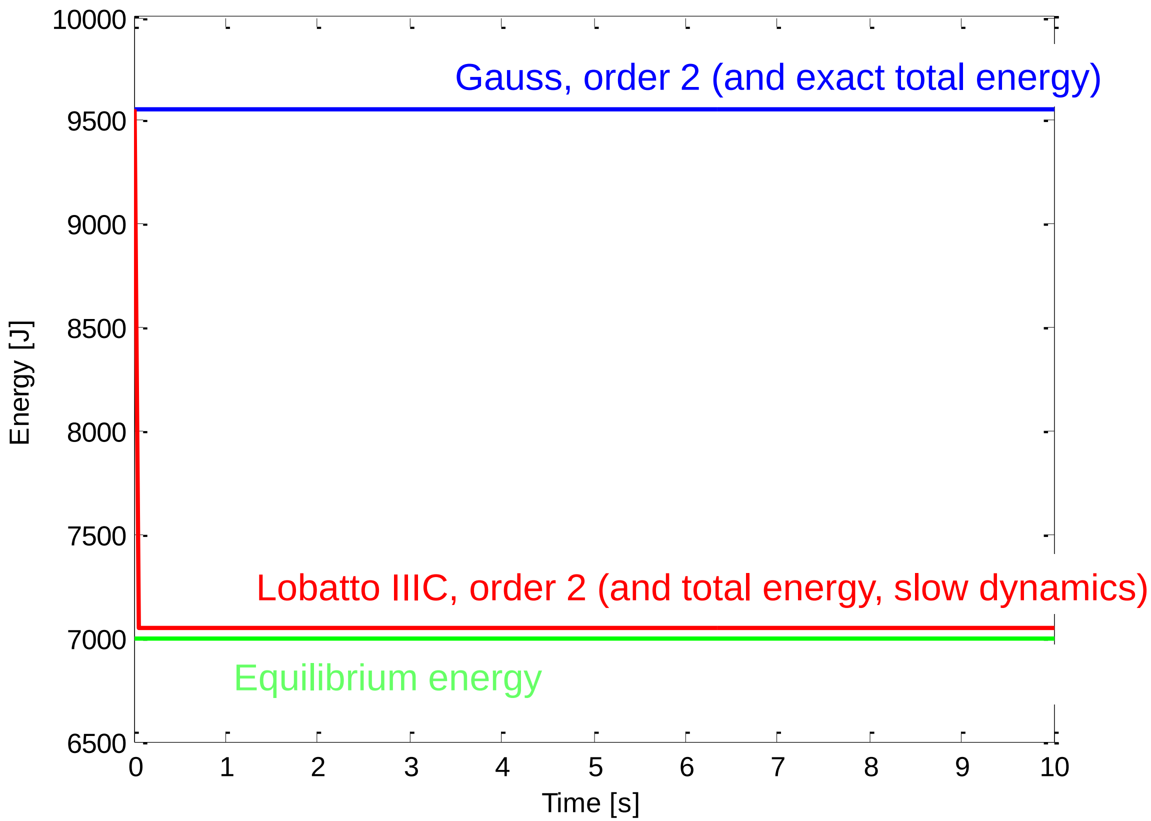
## Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05




- Gauss method gives no damping, but shifts fast dynamics and energy to frequencies below Nyquist frequency, $\omega_N = \frac{\pi}{h} = \frac{\pi}{0.05} = 62.8$
- Lobatto IIIC dampens out fast dynamics in one step

# Total energies

Gauss, order 2 and Lobatto IIIC, order 2, h = 0.05




- Gauss does not dampen energies at all (same as exact total energy)
- Lobatto IIIC dampens out energy associated with fast dynamics in very few steps, to the energy of slow dynamics

# Modelica

- Replacable / redeclare
- Choices

- Check out :
  - http://book.xogeny.com/components/architectures/replaceable/

# Homework

- Check out the Modelica Circuit example (uploaded on Blackboard).
  - Look at the structure replaceable / redeclare
  - Read: http://book.xogeny.com/components/architectures/replaceable/
- Read 14.7.

# Kahoot

- https://play.kahoot.it/#/k/694ab821-e4e0-421a-a5d6-0d297fd2cf1c

# Self-study section

# Example: ”Lambert’s problem”

- IVP:

$$\dot{u} = \frac{1}{100} - (\frac{1}{100} + u + v)(1 + (u + 1000)(u + 1)), \quad u(0) = 0$$
$$\dot{v} = \frac{1}{100} - (\frac{1}{100} + u + v)(1 + v^2), \quad v(0) = 0$$

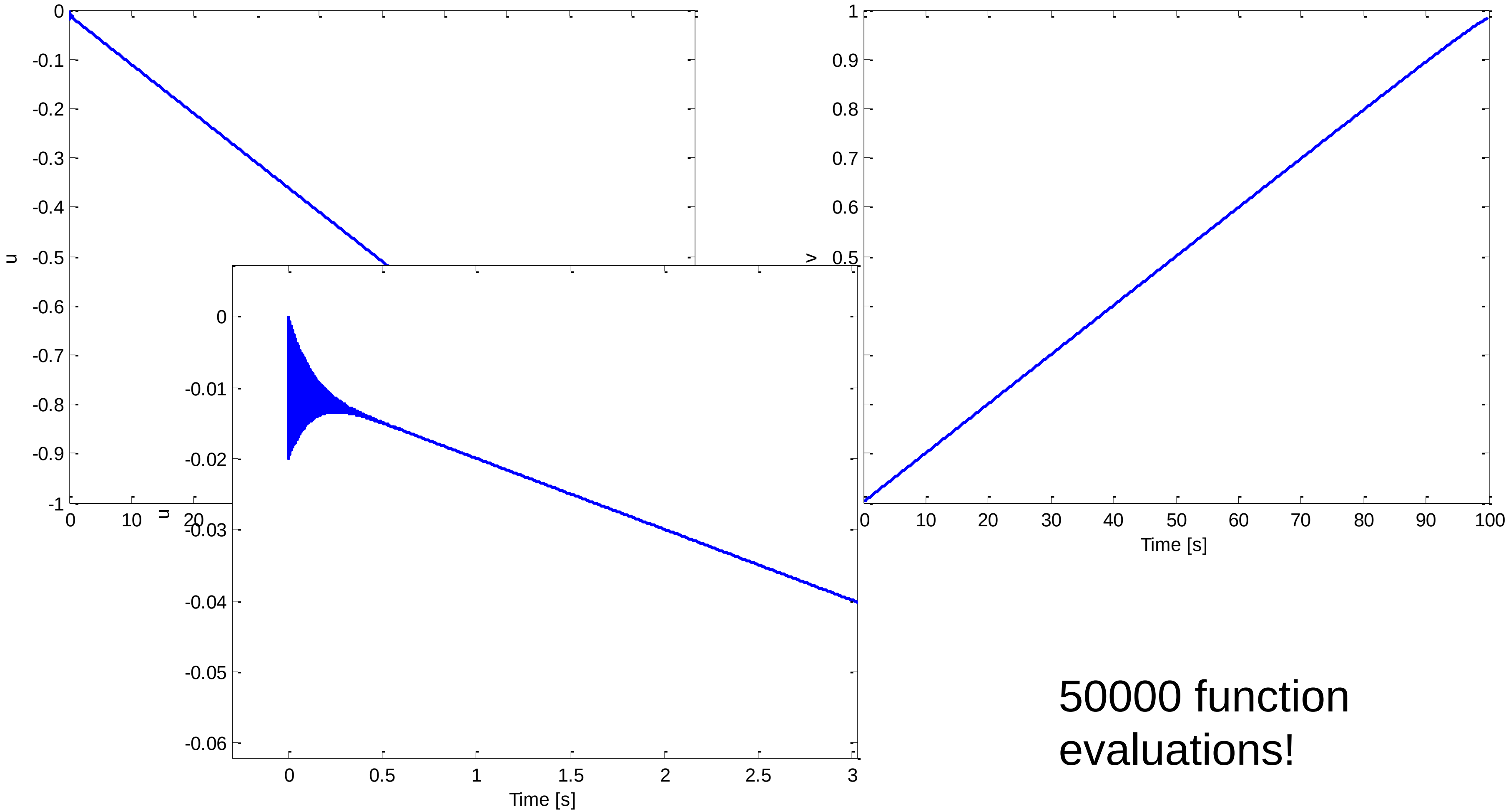
- Task: Simulate from $t$ = 0 s til $t$ = 100 s

- Eigenvalues:

$$(u, v) = (0, 0) \Rightarrow \lambda_1 \approx -1000, \lambda_2 \approx -0.01$$
$$(u, v) = (-.5, .5) \Rightarrow \lambda_1 \approx -500, \lambda_2 \approx -0.03$$
$$(u, v) = (-1, 1) \Rightarrow \lambda_1 \approx -11, \lambda_2 \approx -1$$

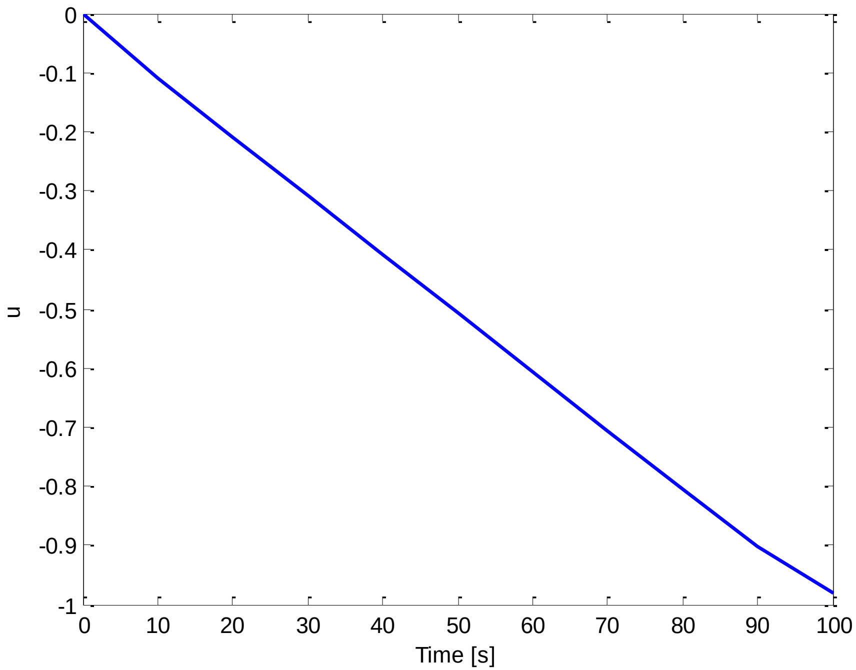
# Using Euler (explicit), h = 0.002

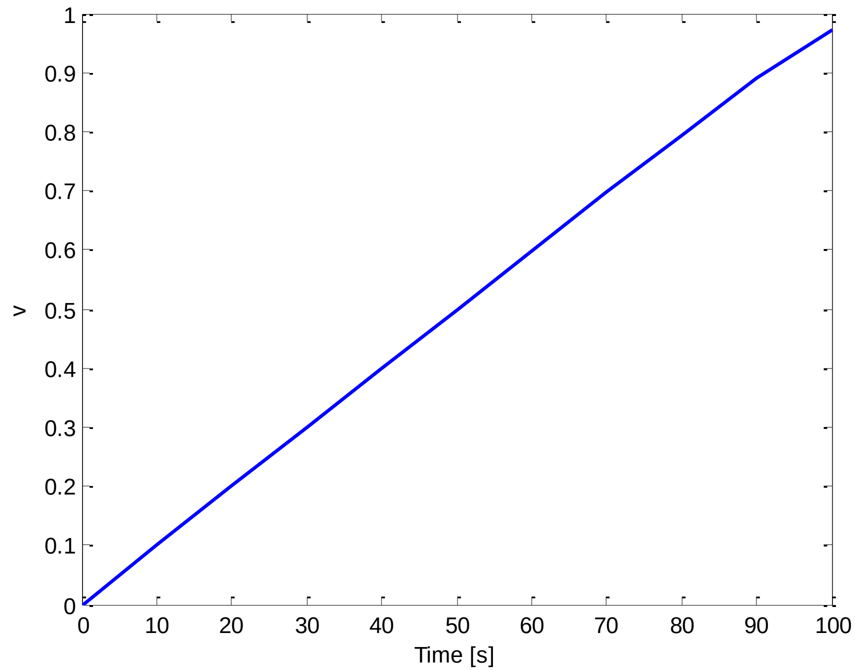
$$h|\lambda| \leq 2$$




50000 function evaluations!

# Attempt 3: Euler implicit, h = 10

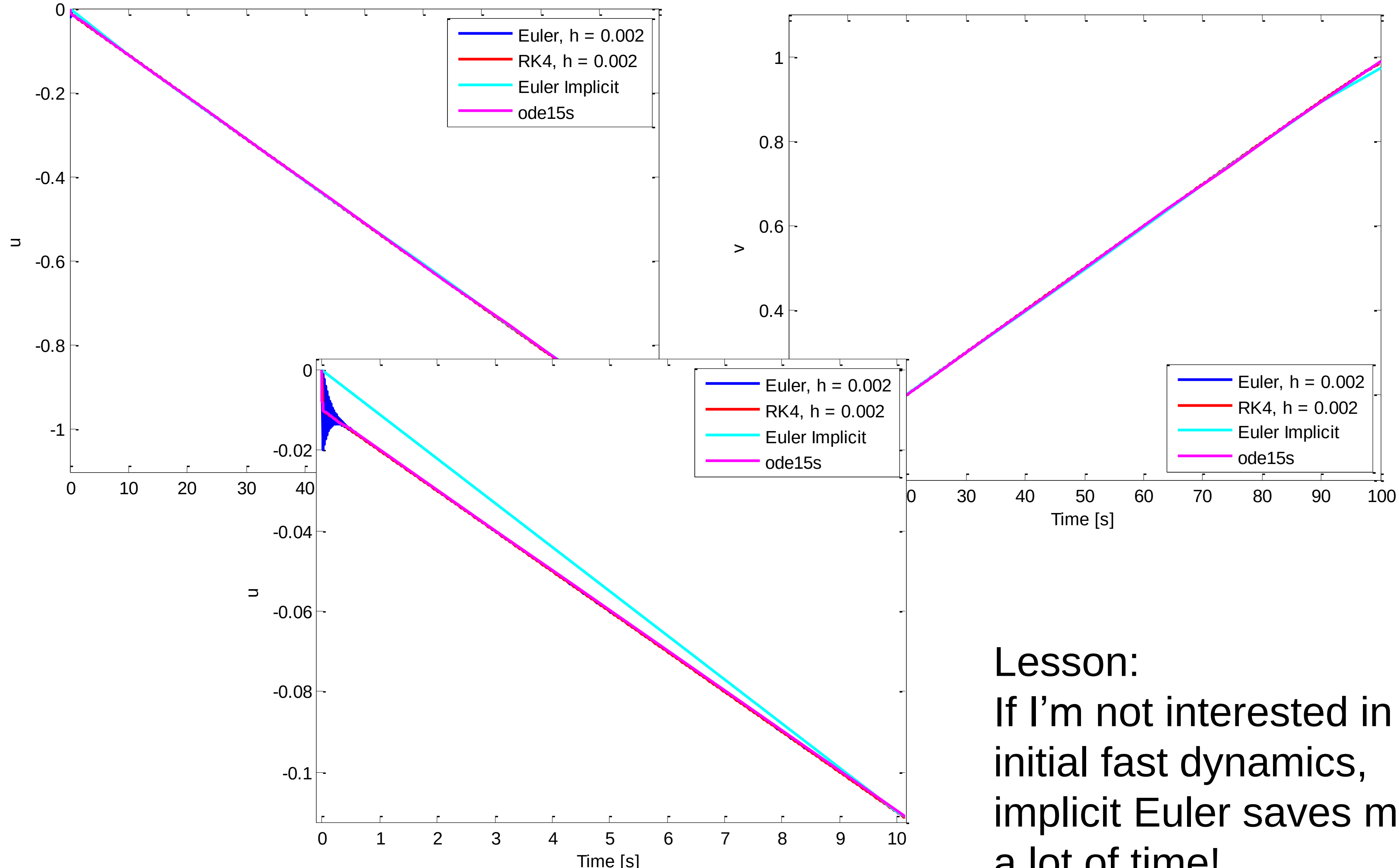



149 function evaluations!

(dependent on solution algorithm)

# Comparisons




Lesson:
If I'm not interested in initial fast dynamics, implicit Euler saves me a lot of time!