Lecture 7: Quadratic programming

- Recap last time – EQPs
- Active set method for solving QPs
  - For medium-sized problems – for large problems, interior point methods may be faster
- Example 16.4

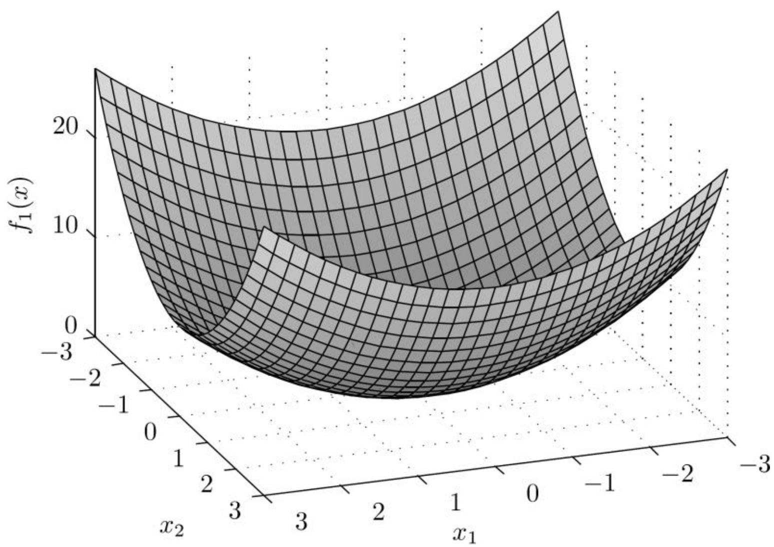
Reference: N&W Ch.15.3-15.5, 16.1-2,4-5

Quadratic programming

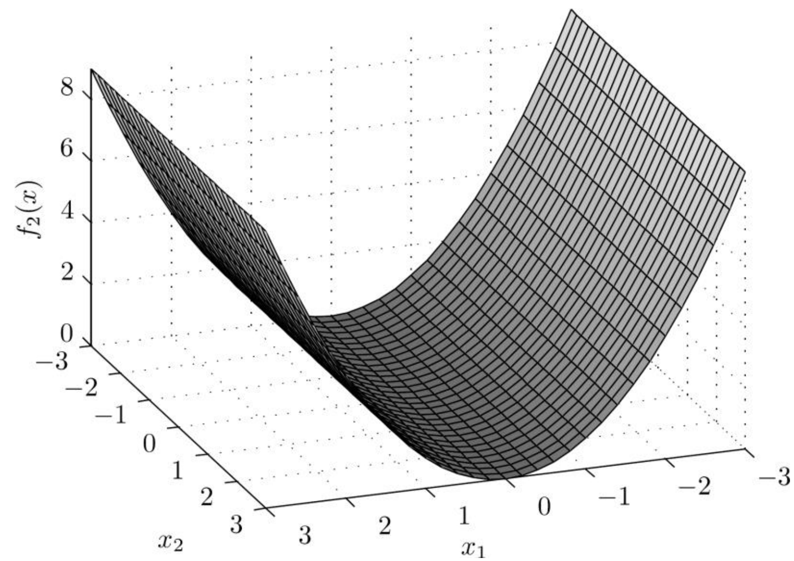
(solving quadratic programs, QPs)

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top G x + c^\top x \quad \text{subject to} \quad \begin{cases} a_i^\top x = b_i, & i \in \mathcal{E} \\ a_i^\top x \geq b_i, & i \in \mathcal{I} \end{cases}$$

- Feasible set convex (as for LPs)
- We say that the QP is (strictly) convex if $G \geq (>)\ 0$



$G > 0$, strictly convex



$G \geq 0$, convex

Important special case:

Equality-constrained QP (EQP)

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{2}x^\top G x + c^\top x$$
$$\text{subject to} \quad Ax = b, \quad A \in \mathbb{R}^{m\times n}$$

Basic assumption:
A full row rank

- KKT-conditions (KKT system, KKT matrix):

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \quad \text{or, if we let } x^* = x + p, \quad \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

- Solvable when $Z^\top G Z > 0$ (columns of *Z* basis for nullspace of *A*):

$$Z^\top G Z > 0 \overset{\text{Lemma 16.1}}{\Rightarrow} K = \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \text{ non-singular}$$
$$\Rightarrow \begin{pmatrix} x^* = x + p \\ \lambda^* \end{pmatrix} \text{ unique solution of KKT system}$$
$$\overset{\text{Theorem 16.2}}{\Rightarrow} x^* \text{ is the unique solution to EQP}$$

- How to solve KKT system (KKT matrix indefinite, but symmetric):
  - Full-space: Symmetric indefinite (LDL) factorization: $P^\top K P = LBL^\top$
  - Reduced space: Use *Ax=b* to eliminate *m* variables. Requires computation of *Z*, basis for nullspace of *A*, which can be costly. Reduced space method can be faster than full-space if *n-m* $\ll$ *n*.

KKT conditions (Theorem 12.1)

Lagrangian: $$\mathcal{L}(x,\lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && \\
c_i(x^*) &= 0, && \forall i \in \mathcal{E}, \\
c_i(x^*) &\geq 0, && \forall i \in \mathcal{I}, \\
\lambda_i^* &\geq 0, && \forall i \in \mathcal{I}, \\
\lambda_i^* c_i(x^*) &= 0, && \forall i \in \mathcal{E} \cup \mathcal{I}.
\end{aligned}$$

(stationarity)

(primal feasibility)

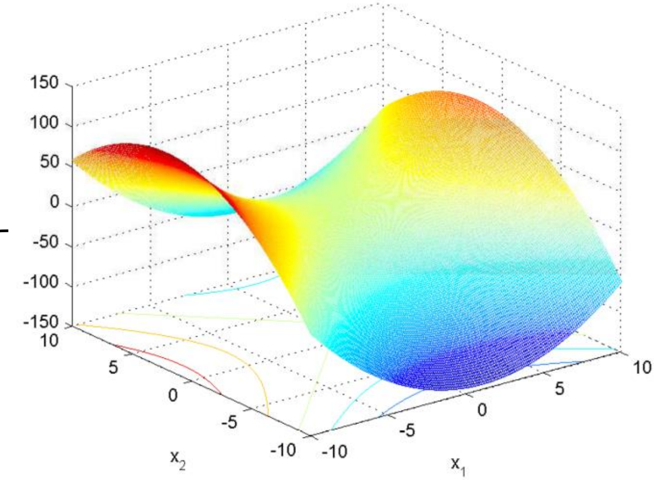
(dual feasibility)

(complementarity condition/ complementary slackness)

Either $\lambda_i^* = 0$ *or* $c_i(x^*) = 0$

(*strict* complimentarity: Only one of them is zero)

Nonconvex QP



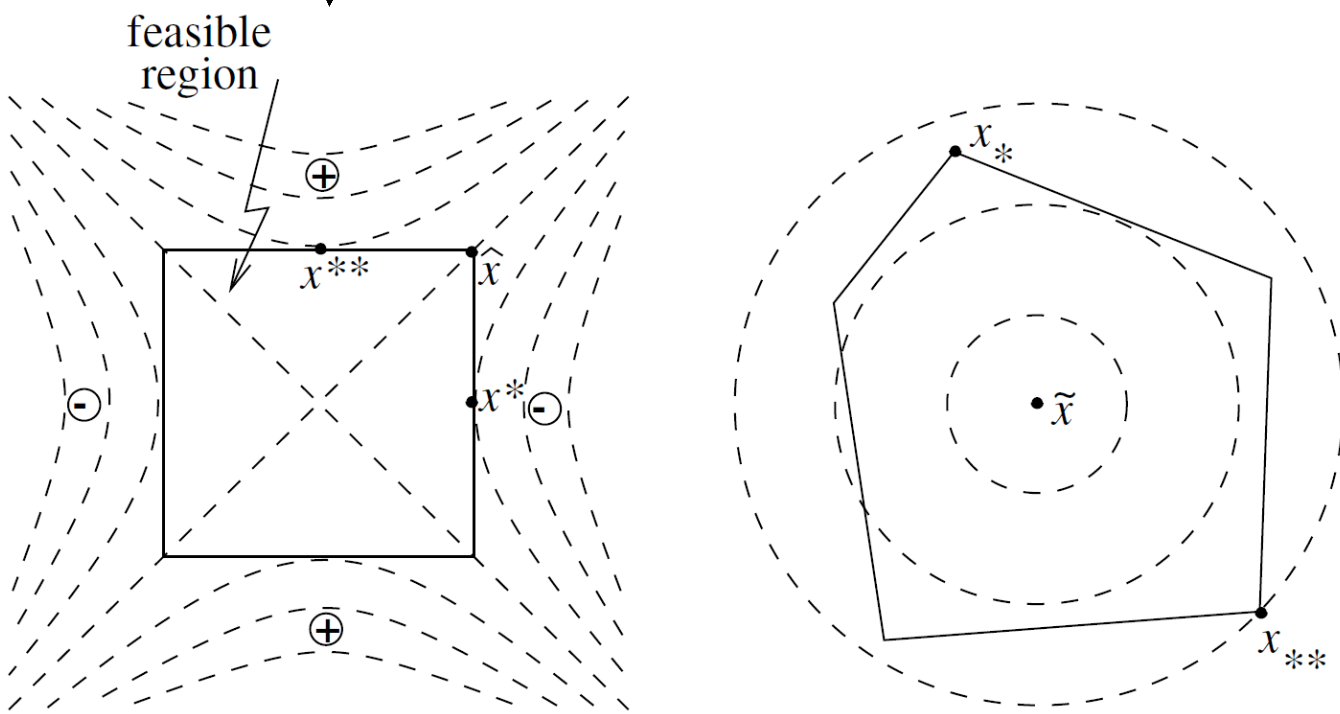


Figure 16.1 in Nocedal & Wright.

Degeneracy

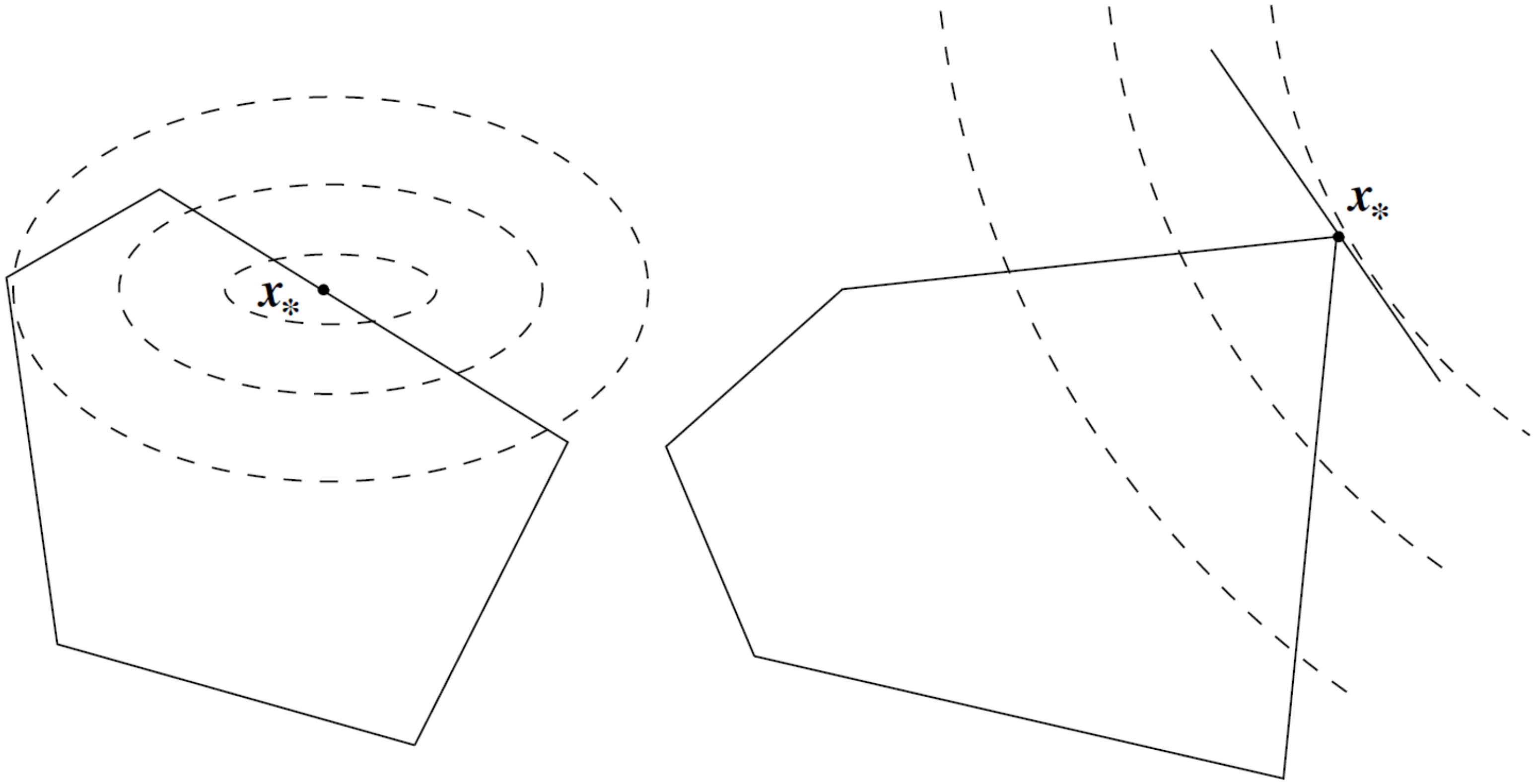


Figure 16.2 in Nocedal & Wright.

General QP problem

$$
\begin{aligned}
\min_x \ & \frac{1}{2} x^\top G x + x^\top c \\
\text{s.t. } & a_i^\top x = b_i, \qquad i \in \mathcal{E} \\
& a_i^\top x \geq b_i, \qquad i \in \mathcal{I}
\end{aligned}
$$

- Lagrangian

$$
\mathcal{L}(x^*, \lambda^*) = \frac{1}{2} x^\top G x + x^\top c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^\top x - b_i)
$$

- KKT conditions

General:

$$
\begin{aligned}
Gx^* + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* a_i = 0 & \\
a_i^\top x^* = b_i, \qquad & i \in \mathcal{E} \\
a_i^\top x^* \geq b_i, \qquad & i \in \mathcal{I} \\
\lambda_i^* \geq 0, \qquad & i \in \mathcal{I} \\
\lambda_i^* (a_i^\top x^* - b_i) = 0, \qquad & i \in \mathcal{E} \cup \mathcal{I}
\end{aligned}
$$

Defined via active set:

$$
\mathcal{A}(x^*) = \mathcal{E} \cup \left\{ i \in \mathcal{I} \middle| a_i^\top x^* = b_i \right\}
$$

$$
\begin{aligned}
Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0 & \\
a_i^\top x^* = b_i, \qquad & i \in \mathcal{A}(x^*) \\
a_i^\top x^* \geq b_i, \qquad & i \in \mathcal{I} \setminus \mathcal{A}(x^*) \\
\lambda_i^* \geq 0, \qquad & i \in \mathcal{A}(x^*) \cap \mathcal{I}
\end{aligned}
$$

Active set method for convex QP

Algorithm 16.3 (Active-Set Method for Convex QP).

Compute a feasible starting point x_0;
Set $\mathcal{W}_0$ to be a subset of the active constraints at x_0;
for $k = 0, 1, 2, \ldots$
  Solve (16.39) to find p_k;
  if $p_k = 0$
    Compute Lagrange multipliers $\hat{\lambda}_i$ that satisfy (16.42),
      with $\hat{\mathcal{W}} = \mathcal{W}_k$;
    if $\hat{\lambda}_i \geq 0$ for all $i \in \mathcal{W}_k \cap \mathcal{I}$
      stop with solution $x^* = x_k$;
    else
      $j \leftarrow \arg\min_{j \in \mathcal{W}_k \cap \mathcal{I}} \hat{\lambda}_j$;
      $x_{k+1} \leftarrow x_k$; $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \backslash \{j\}$;
  else (* $p_k \neq 0$ *)
    Compute α_k from (16.41);
    $x_{k+1} \leftarrow x_k + \alpha_k p_k$;
    if there are blocking constraints
      Obtain $\mathcal{W}_{k+1}$ by adding one of the blocking
        constraints to $\mathcal{W}_k$;
    else
      $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k$;
end (for)

$$\min_p \quad \tfrac{1}{2} p^T G p + g_k^T p \tag{16.39a}$$

$$\text{subject to} \quad a_i^T p = 0, \quad i \in \mathcal{W}_k. \tag{16.39b}$$

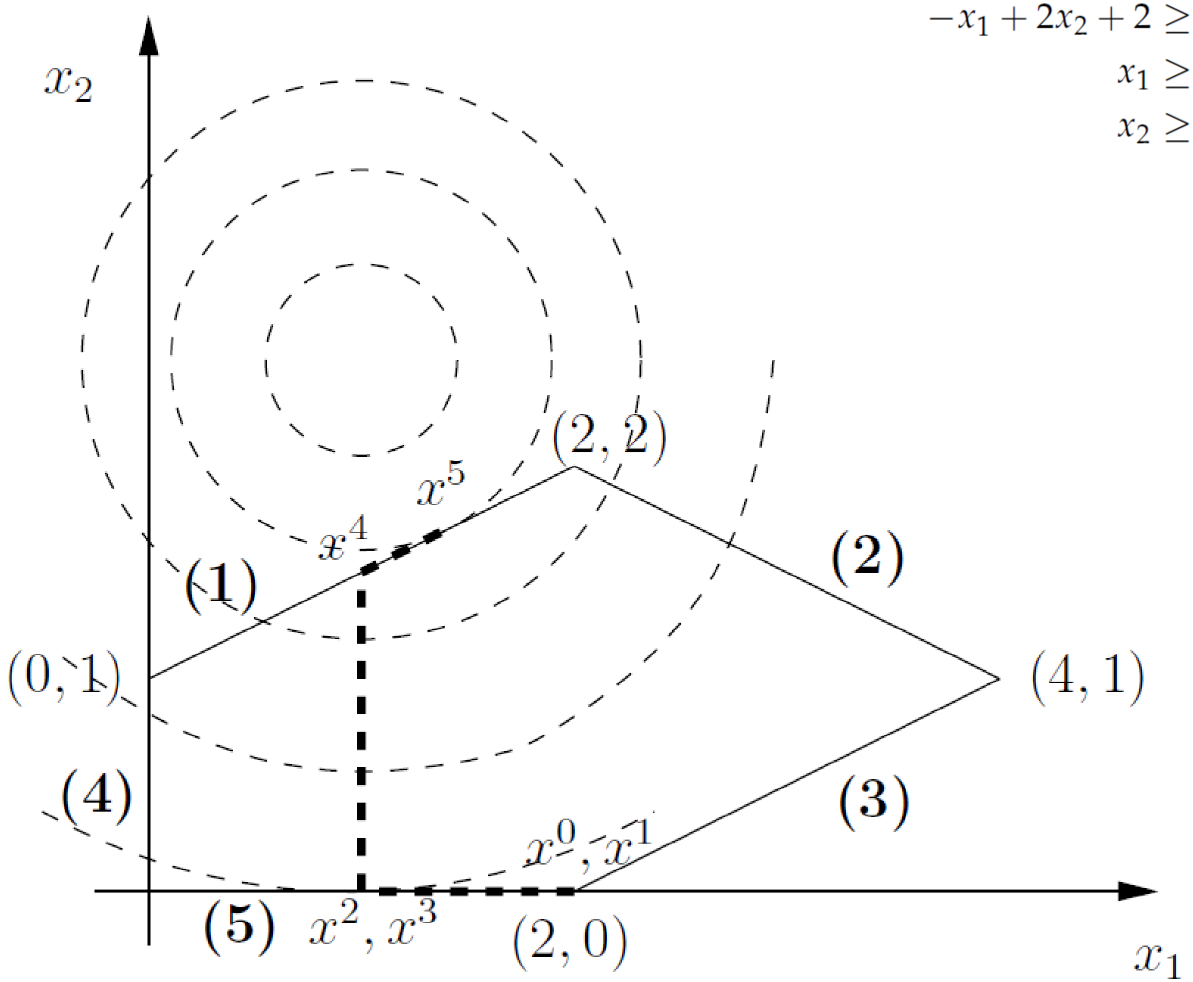
$$\sum_{i \in \hat{\mathcal{W}}} a_i \hat{\lambda}_i = g = G\hat{x} + c, \tag{16.42}$$

$$\alpha_k \stackrel{\text{def}}{=} \min \left(1, \min_{i \notin \mathcal{W}_k,\, a_i^T p_k < 0} \frac{b_i - a_i^T x_k}{a_i^T p_k} \right). \tag{16.41}$$

Example 16.4

$$\min_x q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 + 2 &\geq 0 && (1) \\ -x_1 - 2x_2 + 6 &\geq 0 && (2) \\ -x_1 + 2x_2 + 2 &\geq 0 && (3) \\ x_1 &\geq 0 && (4) \\ x_2 &\geq 0 && (5) \end{aligned}$$



$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_1 = -2$$

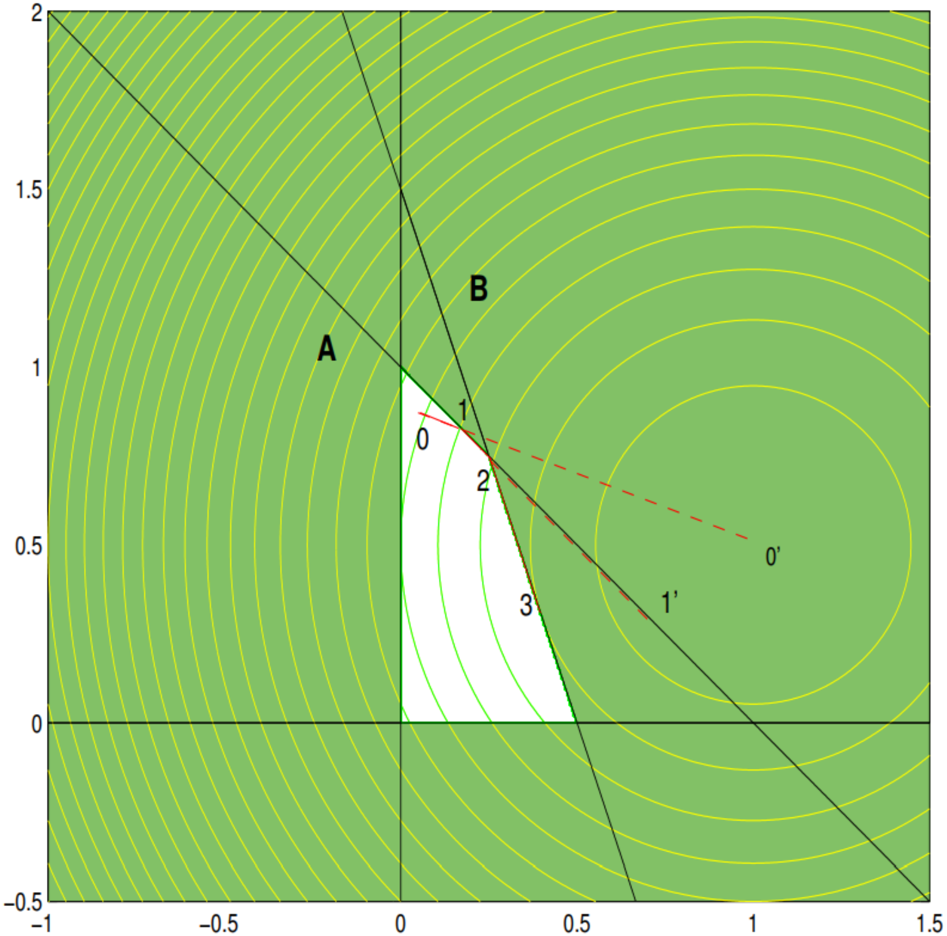
$$a_2 = \begin{bmatrix} -1 & -2 \end{bmatrix}^\mathsf{T}, \quad b_2 = -6$$

$$a_3 = \begin{bmatrix} -1 & 2 \end{bmatrix}^\mathsf{T}, \quad b_3 = -2$$

$$a_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\mathsf{T}, \quad b_4 = 0$$

$$a_5 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\mathsf{T}, \quad b_5 = 0$$

Another example (N. Gould)



$$\min (x_1 - 1)^2 + (x_2 - 0.5)^2$$
$$\text{subject to } x_1 + x_2 \leq 1$$
$$3x_1 + x_2 \leq 1.5$$
$$(x_1, x_2) \geq 0$$

0. Starting point
0'. Unconstrained minimizer
1. Encounter constraint A
1'. Minimizer on constraint A
2. Encounter constraint B, move off constraint A
3. Minimizer on constraint B = required solution

How to find feasible initial point?

- Same way as for LP:
  - Phase I: Define another optimization problem with known feasible initial point, where solution is feasible for original problem.
  - Phase II: Solve original problem.

- Alternative method: “Big M”
  - Relax all constraints; penalize constraint violations in objective