Lecture 13: Rigid body kinematics – Kinematic differential equations

- Brief recap of representations of rotation
  - Rotation matrices (6.4)
  - Euler angles (6.5)
    - 3-parameter representation of rotations
    - Roll-pitch-yaw
  - Angle-axis, Euler-parameters (6.6, 6.7)
    - 4-parameter representation of rotations
  - Angular velocity (6.8)
  - Kinematic differential equations

- Today:
  - Rigid body kinematics: Configuration
  - Newton Euler equation

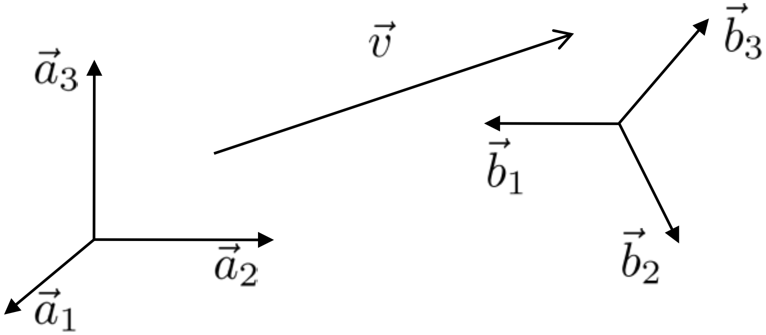
Book: Ch. 6.9, 6.12, 6.13, 7.1

Rotation matrices

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- **Transform** a coordinate vector from *b* to *a*

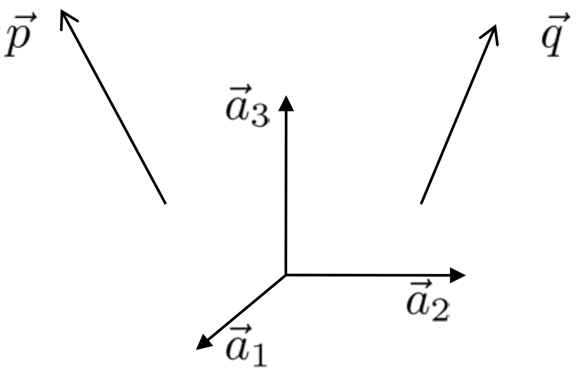
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- **Rotate** a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

  such that $\mathbf{q}^b = \mathbf{p}^a$.

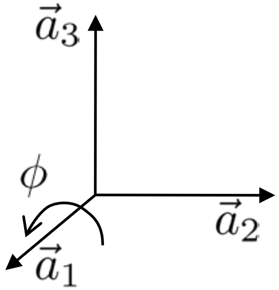


Simple rotations

- Simple rotation = rotation about an axis

- Example: Rotation matrix for rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



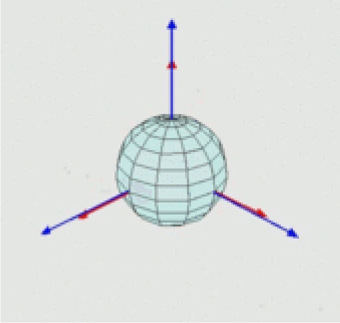
Representations of rotations

- Rotation matrix
  - Easy to use, but not to visualize (also over-parameterized, 9 parameters)

Euler's Theorem:

> "Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

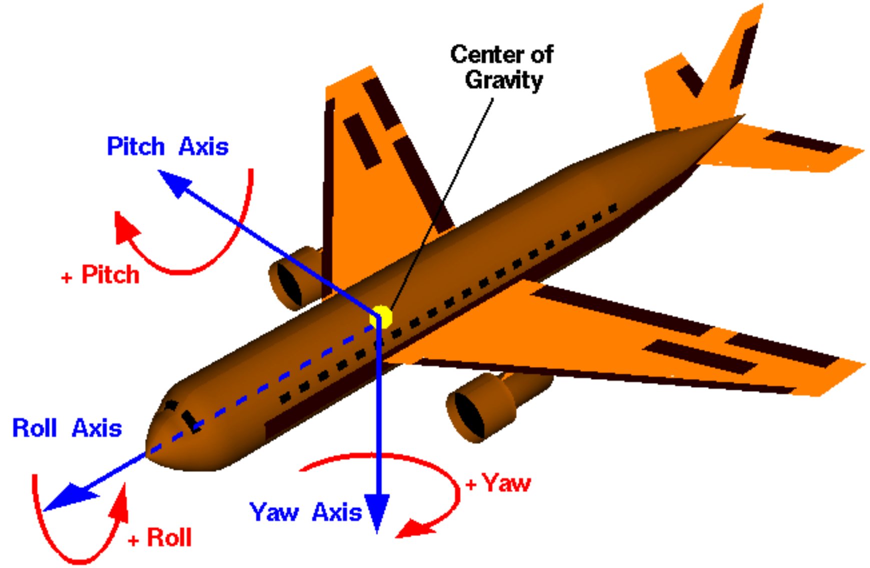
- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common(?): Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this today)



Source: Wikipedia

- Angle-axis, Euler-parameters
  - 4-parameter representations of rotations
  - No singularity problems

Euler-angles: Roll-pitch-yaw

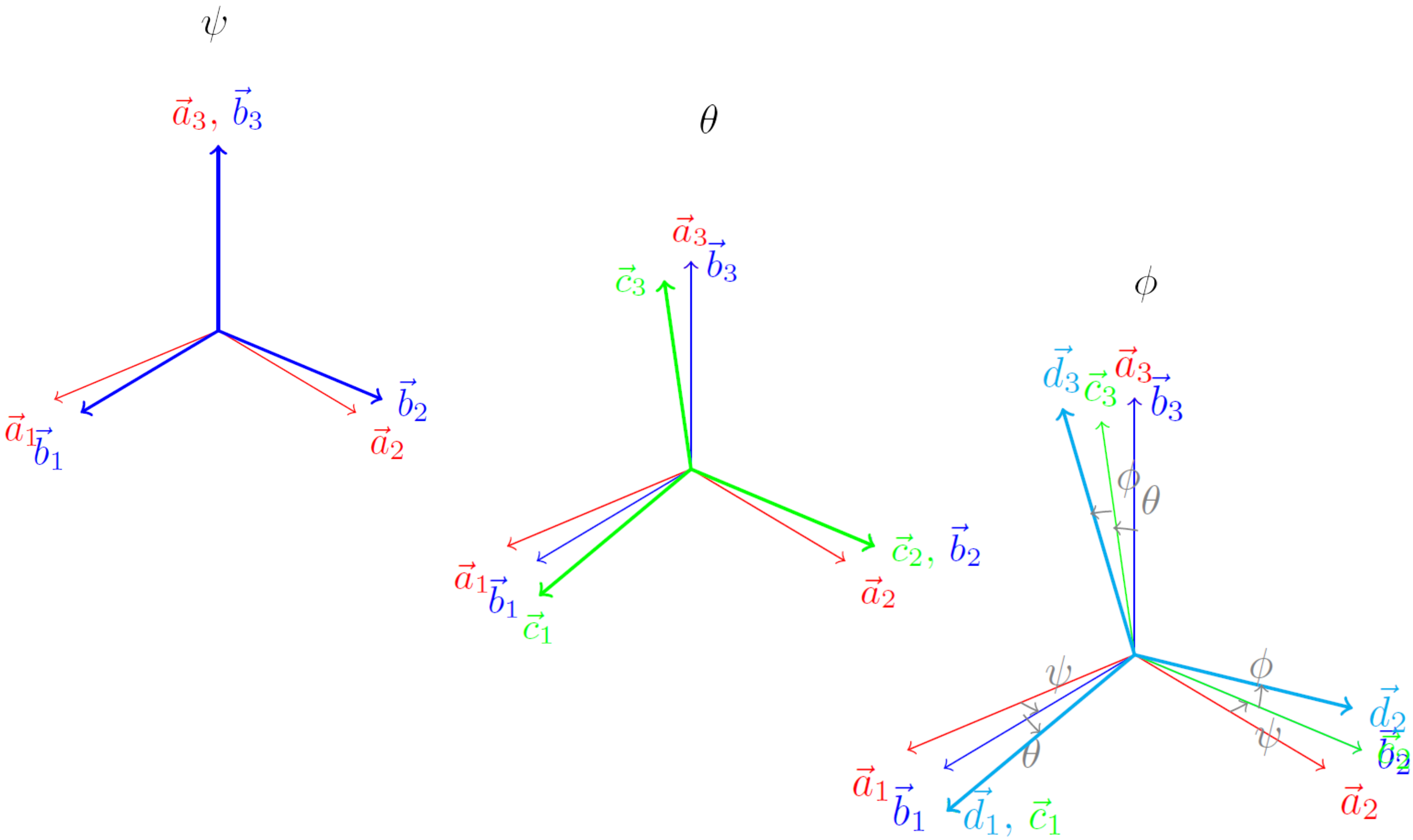


- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

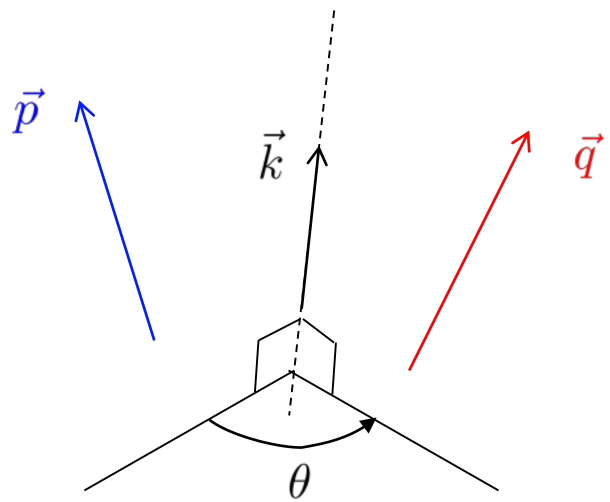
$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Euler angles



Angle-axis representation of rotations

All rotations can be represented as a simple rotation around an axis



- Angle-axis parameters:
  - Coordinate free: $\vec{k}$, θ

$$\vec{q} = \Big(\underbrace{\cos\theta\,\vec{I} + \sin\theta\,\vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}} \Big) \cdot \vec{p}$$

  - With coordinates: $\mathbf{k}^a$, θ

$$\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\,\mathbf{I} + \sin\theta\,(\mathbf{k}^a)^{\times} + (1-\cos\theta)\,\mathbf{k}^a(\mathbf{k}^a)^{\mathsf{T}}$$

Euler parameters

- Euler parameters are closely related to angle-axis:
  - Coordinate-free:

$$\eta = \cos\frac{\theta}{2}$$
$$\vec{\epsilon} = \vec{k}\sin\frac{\theta}{2}$$

With coordinates:

$$\eta = \cos\frac{\theta}{2}$$
$$\boldsymbol{\epsilon} = \mathbf{k}\sin\frac{\theta}{2}$$

- Rotation matrix (on coordinate form):

$$\mathbf{R}(\eta, \boldsymbol{\epsilon}) = \mathbf{I} + 2\eta\boldsymbol{\epsilon}^{\times} + 2\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times}$$

- Much used, since:
  - Compact, singularity-free representation of orientation
  - No trigonometric terms in expression for rotation matrix
  - $\eta^2 + \vec{\epsilon}\cdot\vec{\epsilon} = 1$: Easy to normalize (avoid roundoff errors)
    - Rotation matrices may tend to become non-orthogonal when simulated
  - Euler parameters are *(unit) quaternions*:
    - Quaternions are generalized complex numbers
    - Can use algebra of quaternions for calculations and analysis

Derivatives of rotations

- Derivative of position $\mathbf{r}$ is velocity, $\dot{\mathbf{r}} = \mathbf{v}$.
- Derivative of rotation matrix $\mathbf{R}_b^a$ is $\dot{\mathbf{R}}_b^a$. What is this?
- Seems natural that a concept of angular velocity should be involved, but how?
- What are derivatives of representations of rotations?
  - Derivatives of Euler angles? Euler parameters?
  - These are the kinematic differential equations!

Angular velocity

- The rotation matrix is orthogonal:

$$\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} = \mathbf{I}$$

- Differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}\right] = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} + \mathbf{R}_b^a \left(\dot{\mathbf{R}}_b^a\right)^\mathsf{T} = \mathbf{0}$$

- If we define $\mathbf{S} = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$, this says that $\mathbf{S} + \mathbf{S}^\mathsf{T} = \mathbf{0}$ which means that $\mathbf{S}$ is skew symmetric.

$$\mathbf{S} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \left(\boldsymbol{\omega}_{ab}^a\right)^\times$$

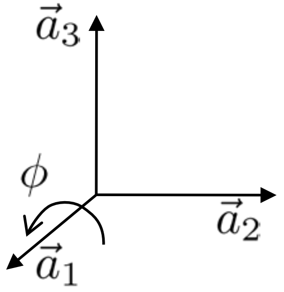
- The vector $\boldsymbol{\omega}_{ab}^a$ defined by $\left(\boldsymbol{\omega}_{ab}^a\right)^\times = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$ is the angular velocity of frame *b* relative to frame *a* (decomposed in *a*)

- The equation $\dot{\mathbf{R}}_b^a = \left(\boldsymbol{\omega}_{ab}^a\right)^\times \mathbf{R}_b^a$ is the kinematic differential equation for rotation matrices

Angular velocity of simple rotations

- Rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



- We calculate $\left(\boldsymbol{\omega}_{ab}^{a}\right)^{\times} = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^{\mathsf{T}}$:

$$\dot{\mathbf{R}}_{x,\phi}\left(\mathbf{R}_{x,\phi}\right)^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{pmatrix} \dot{\phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{pmatrix}$$

- That is:

$$\boldsymbol{\omega}_x = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}$$

- Similar for rotations around *y*- and *z*-axis: $\boldsymbol{\omega}_y = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}, \quad \boldsymbol{\omega}_z = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

- Angle-axis representations (constant axis):

$$\boldsymbol{\omega}_{ab}^{a} = \dot{\theta}\mathbf{k}^{a}$$

Composite rotations

- Given
  - composite rotation $\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$, and
  - individual angular velocities $\boldsymbol{\omega}_{ab}^a$, $\boldsymbol{\omega}_{bc}^b$, and $\boldsymbol{\omega}_{cd}^c$

  How to calculate the composite angular velocity $\boldsymbol{\omega}_{ad}^a$?

- It can be shown (easy, see book p. 241) that

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd}$$

- On coordinate form:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a$$

- So:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \mathbf{R}_b^a \boldsymbol{\omega}_{bc}^b + \mathbf{R}_b^a \mathbf{R}_c^b \boldsymbol{\omega}_{cd}^c$$

Kinematic differential equation of Euler angles

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c = \mathbf{R}_{z,\psi} \mathbf{R}_{y,\theta} \mathbf{R}_{x,\phi}$$

$$\vec{\boldsymbol{\omega}}_{ad} = \vec{\boldsymbol{\omega}}_{ab} + \vec{\boldsymbol{\omega}}_{bc} + \vec{\boldsymbol{\omega}}_{cd} \ = \dot{\psi} \vec{\boldsymbol{a}}_3 + \dot{\theta} \vec{\boldsymbol{b}}_2 + \dot{\phi} \vec{\boldsymbol{c}}_1$$

$$\underline{\boldsymbol{\omega}}_{ad}^a = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_z(\psi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathbf{R}_z(\psi) \mathbf{R}_y(\theta) \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}$$

$$= \mathbf{E}_a(\underline{\boldsymbol{\Phi}}) \dot{\underline{\boldsymbol{\Phi}}} \qquad \underline{\boldsymbol{\Phi}} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$\dot{\underline{\boldsymbol{\Phi}}} = \mathbf{E}_a^{-1}(\underline{\boldsymbol{\Phi}}) \underline{\boldsymbol{\omega}_{\boldsymbol{ad}}^{\boldsymbol{a}}} \qquad \theta \neq 90^o$$

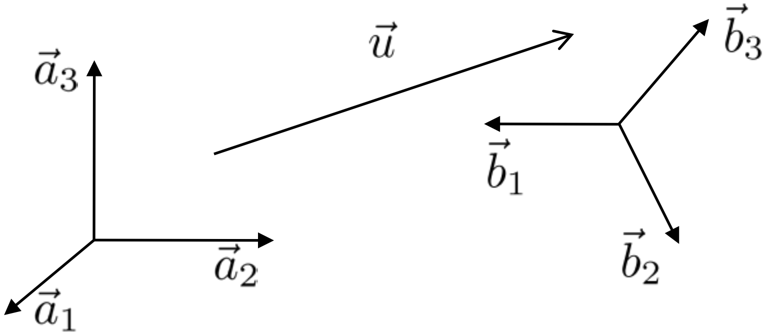
Kinematic differential equation of Euler parameter

$$\mathbf{R}_b^a = \mathbf{R}(\eta, \underline{\varepsilon}) \qquad \dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^a)^\times \mathbf{R}_b^a$$

- It can be derived (quaternion algebra p. 248)

$$\dot{\eta} = -\frac{1}{2}\underline{\varepsilon}^T \underline{\omega}_{ab}^a$$
$$\dot{\underline{\varepsilon}} = \frac{1}{2}(\eta \mathbf{I} - \underline{\varepsilon}^\times)\underline{\omega}_{ab}^a$$

Differentiation of vectors (6.8.5, 6.8.6)



- Coordinate representation:

$$\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b$$

- Differentiation:

$$\dot{\mathbf{u}}^a = \mathbf{R}_b^a \dot{\mathbf{u}}^b + \dot{\mathbf{R}}_b^a \mathbf{u}^b$$

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \left(\boldsymbol{\omega}_{ab}^b\right)^\times$$

$$\boxed{\dot{\mathbf{u}}^a = \mathbf{R}_b^a \left[\dot{\mathbf{u}}^b + \left(\boldsymbol{\omega}_{ab}^b\right)^\times \mathbf{u}^b\right]}$$

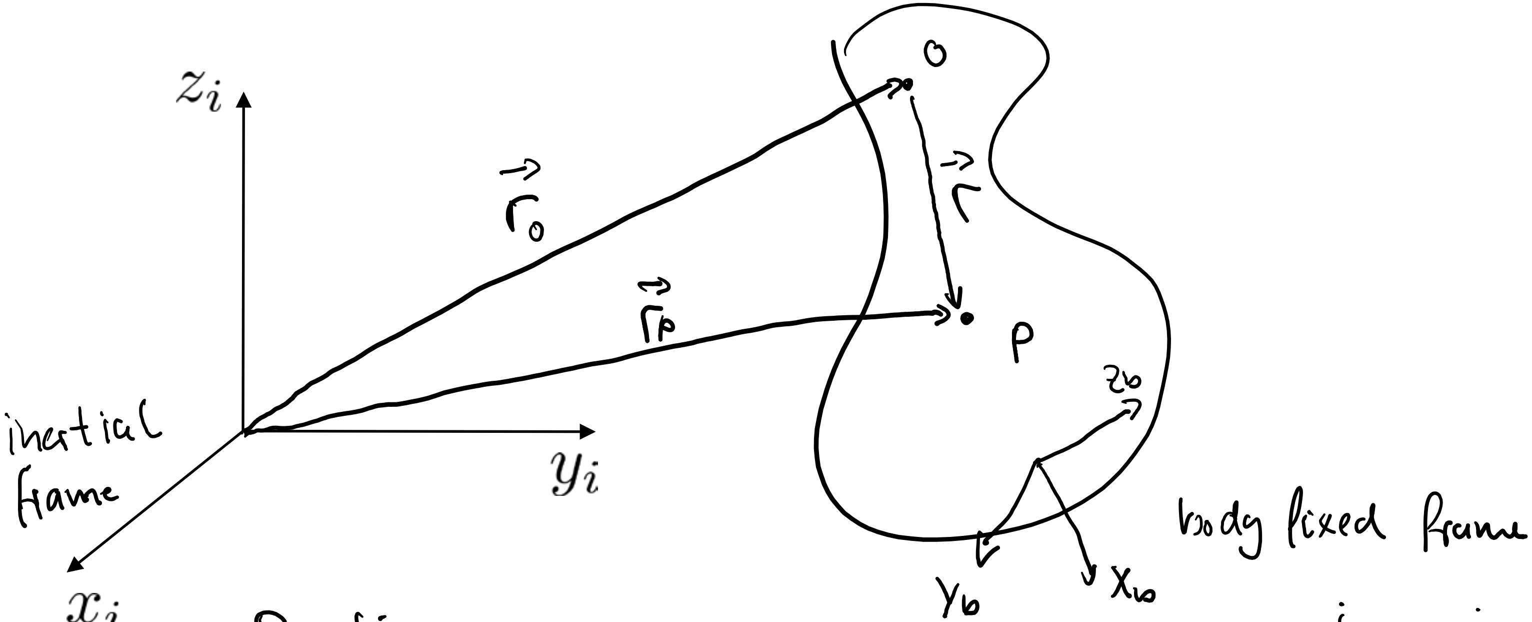
Note! Generally,

$$\dot{\mathbf{u}}^a \neq \mathbf{R}_b^a \dot{\mathbf{u}}^b$$

- On vector form:

$$\boxed{\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u}}$$

Kinematics of rigid body I



Position: $\vec{r}_p = \vec{r}_o + \vec{r}$ $\qquad \underline{r}^i = R^i_b\, \underline{r}^b$

Velocity: $\vec{v}_o = \frac{{}^i d}{dt}\vec{r}_o \qquad \vec{v}_p = \frac{{}^i d}{dt}\vec{r}_p$

$$\vec{v}_p = \frac{{}^i d}{dt}\vec{r}_p = \frac{{}^i d}{dt}\vec{r}_o + \frac{{}^i d}{dt}\vec{r}$$

$$= \vec{v}_o + \frac{{}^b d}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r}$$

Kinematics of rigid body II

$$\vec{a}_o = \frac{{}^i d^2}{dt^2}\vec{r}_o \qquad \vec{a}_p = \frac{{}^i d^2}{dt^2}\vec{r}_p$$

$$\vec{\alpha}_{ib} = \frac{{}^i d}{dt}\vec{\omega}_{ib} = \frac{{}^b d}{dt}\vec{\omega}_{ib} + \cancel{\vec{\omega}_{ib}\times\vec{\omega}_{ib}}$$

$$\frac{{}^i d^2}{dt^2}\vec{r}_p = \frac{{}^i d^2}{dt^2}\vec{r}_o + \frac{{}^i d^2}{dt^2}\vec{r}$$

$$\vec{a}_p = \vec{a}_o + \frac{{}^i d}{dt}\left(\frac{{}^i d}{dt}\vec{r}\right)$$

$$= \vec{a}_o + \frac{{}^i d}{dt}\left(\frac{{}^b d}{dt}\vec{r} + \vec{\omega}_{ib}\times\vec{r}\right)$$

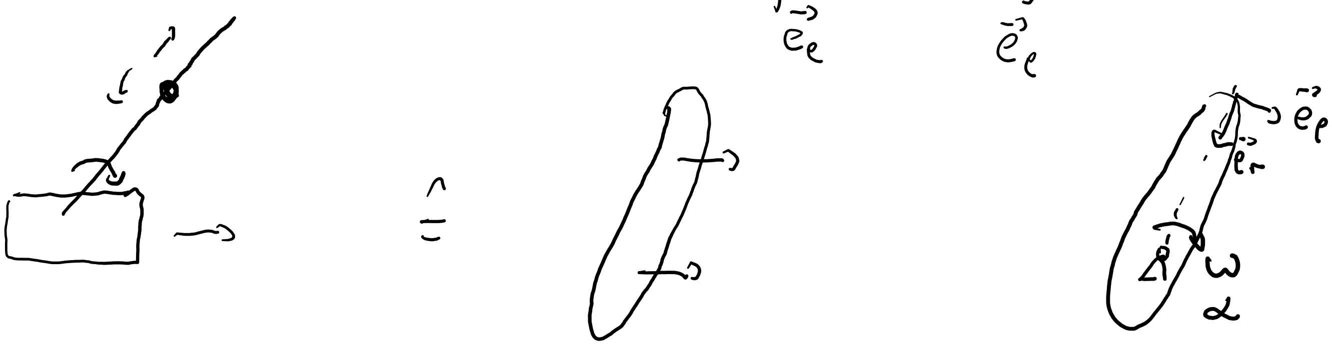
$$= a_o + \frac{{}^b d}{dt}\frac{{}^b d}{dt}\vec{r} + \vec{\omega}_{ib}\times\frac{{}^b d}{dt}\vec{r} + \frac{{}^b d}{dt}\vec{\omega}_{ib}\times\vec{r} +$$

$$\vec{\omega}_{ib}\times\left(\vec{\omega}_{ib}\times\vec{r}\right) + \vec{\omega}_{ib}\times\frac{{}^b d}{dt}\vec{r}$$

Kinematics of rigid body III

$$\vec{a}_p = \vec{a}_o + \frac{{}^b d^2}{dt^2}\vec{r} + \underbrace{2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r}}_{\text{Coriolis}} + \underbrace{\vec{\alpha}_{ib} \times \vec{r}_g}_{\text{transvesal}} + \underbrace{\vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)}_{\text{centripetal}}$$

$\vec{e}_r$ $\quad$ $\vec{e}_\ell$ $\quad$ $\vec{e}_\ell$ $\quad$ $-\vec{e}_r$



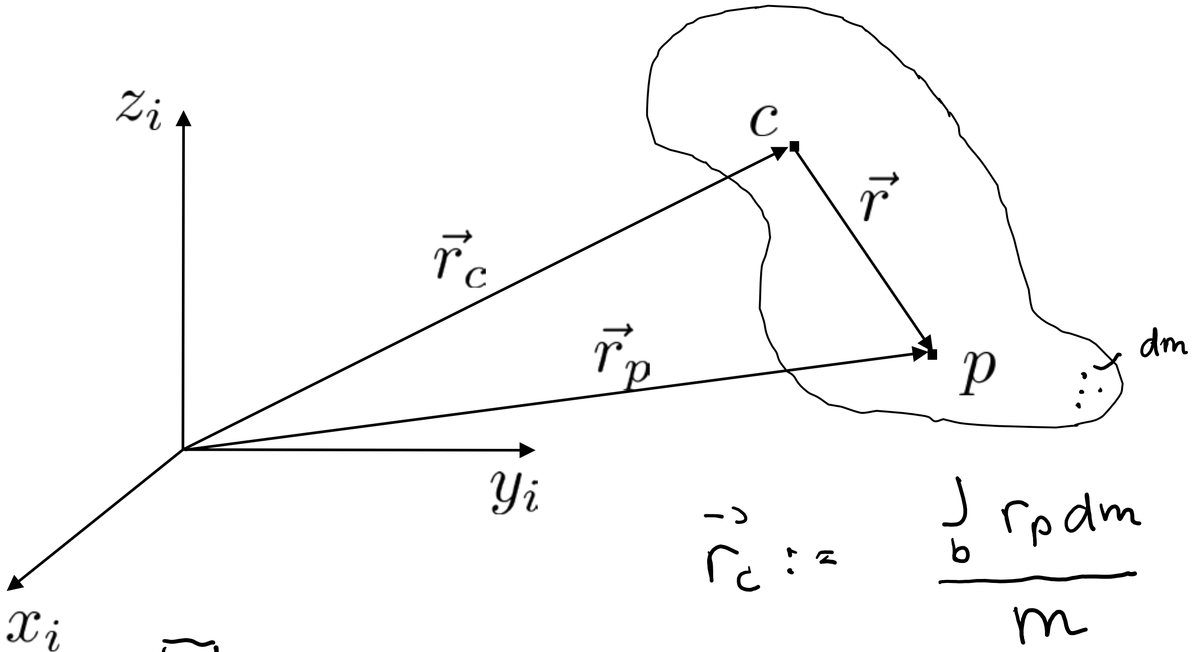
if $\vec{r}$ is fixed

$$\frac{{}^b d}{dt}\vec{r} = 0$$

$$\vec{v}_p = \vec{v}_o + \vec{\omega}_{ib} \times \vec{r}$$

$$\vec{a}_p = \vec{a}_o + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})$$

Center of mass

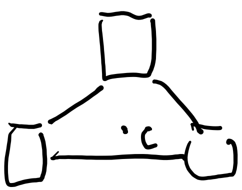


$$\vec{r} = \vec{r}_p - \vec{r}_c$$

$$m = \int_b dm$$

$$\vec{r}_c := \frac{\int_b r_p \, dm}{m}$$

$$\vec{a}_c := \frac{\int_b \vec{a}_p \, dm}{m}$$



$$\vec{v}_c := \frac{\int_b \vec{v}_p \, dm}{m}$$

$$\int_b r \, dm = \int_b r_p \, dm - m\vec{r}_c = m\vec{r}_c - m\vec{r}_c = 0$$

What is rigid body dynamics?

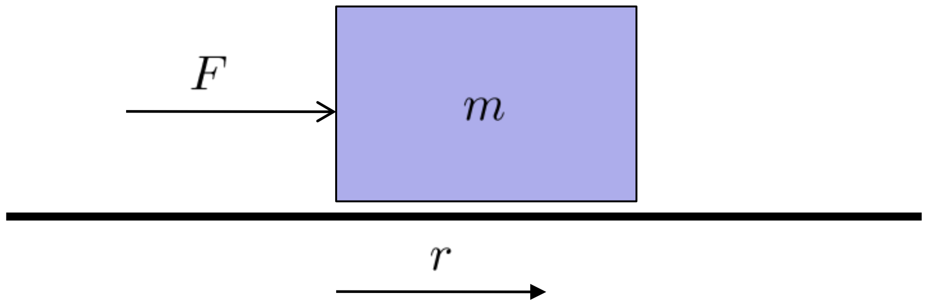
- Rigid body:
  - Wikipedia: "…a rigid body is an idealization of a solid body of finite size in which deformation is neglected."

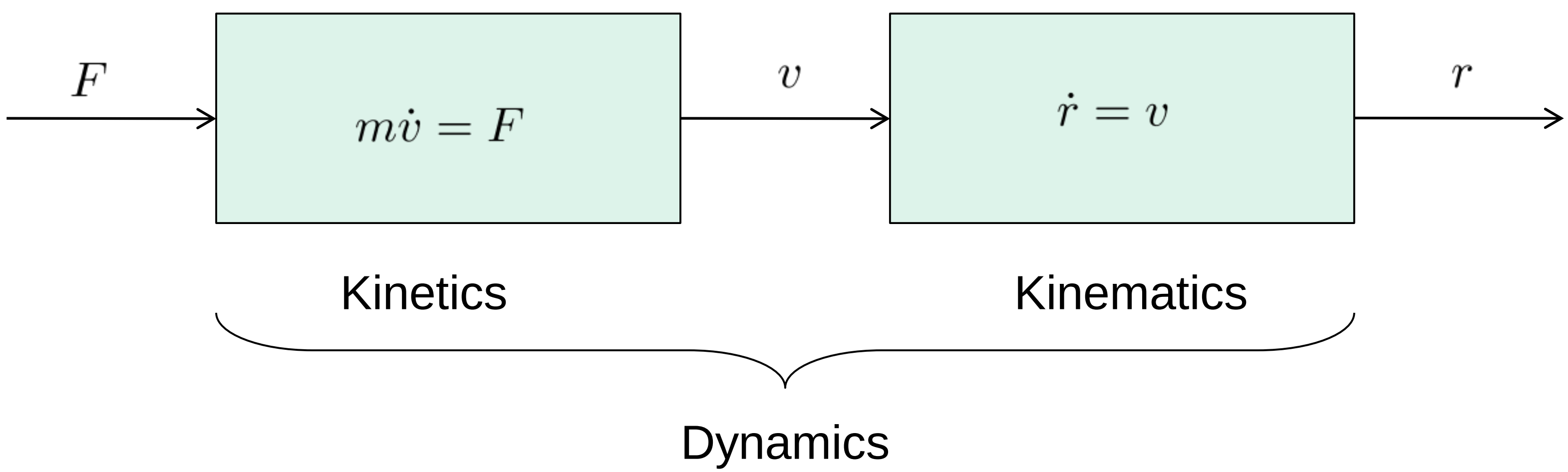
- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: "…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions)."
  - Book: Ch. 6
- Kinetics
  - eb.com: "…the effect of forces and torques on the motion of bodies having mass."
  - Book: Ch. 7, 8.

Remark: Sometimes "dynamics" is used for "kinetics" only

Simplest scalar case





Newton-Euler equation of motion I

Newton: For a particle:

$$dm\,\vec{a}_p = \vec{f}_p = \vec{f}_{p,ext} + \vec{f}_{p,int}$$

$$m\,\vec{a}_c = \underbrace{\int_b \vec{f}_{p,ext}\,dm}_{:= \vec{F}_{bc}} + \underbrace{\int_b \vec{f}_{p,int}\,dm}_{=0}$$

← resultant force

Newton's law for center of mass

$$m\,\vec{a}_c = \vec{F}_{bc}$$

"resulting force" on the center of mass (CoM)

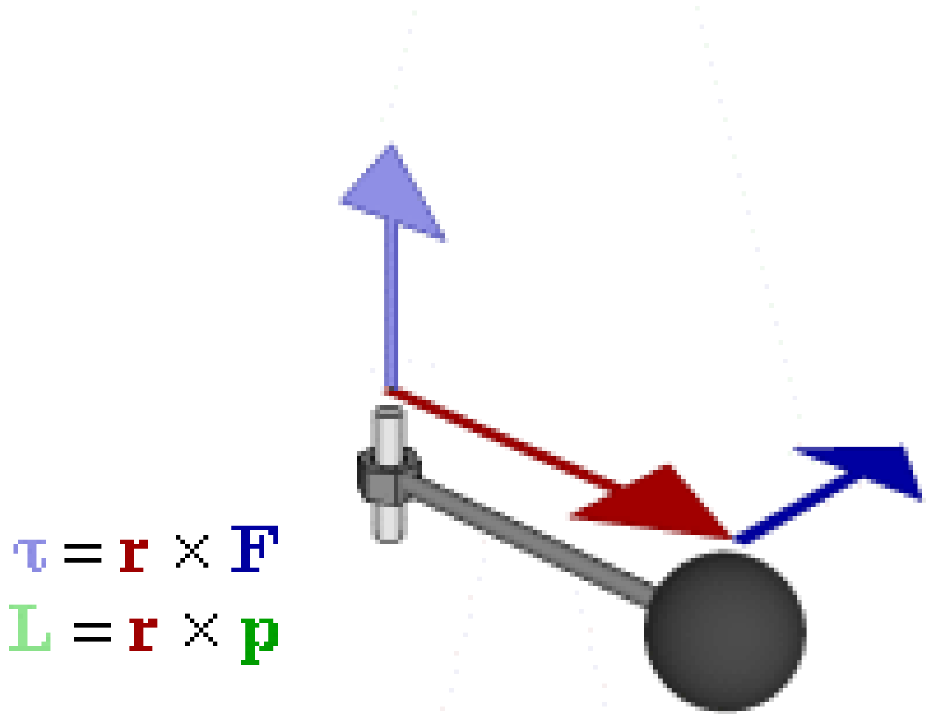
Newton-Euler equation of motion II

alternatively : $\vec{p}_c := m\vec{v}_c$ $\quad \frac{d}{dt}\vec{p}_c = F_{bc}$

Define angular momentum of a particle

$$\vec{h}_p = \vec{r}_p \times \vec{p}_p \qquad \vec{p}_p = dm\,\vec{v}_p$$

Torque, and linear/angular momentum



Source: Wikipedia

- Book:
  - Torque: $\vec{N}, \vec{T}$
  - Angular momentum: $\vec{h}$

Euler's 2nd law of motion I

torque for a particle: $\vec{\tau}_p = \vec{r}_p \times \vec{f}_p$

$$\frac{{}^i d}{dt} \vec{h}_p = \underbrace{\frac{{}^i d}{dt} \vec{r}_p \times \vec{p}_p}_{=0} + \vec{r}_p \times \frac{d}{dt} \vec{p}_p := \vec{\tau}_p$$

Integrate (sum) over the mass

$$\int_b \frac{{}^i d}{dt} \vec{h}_p = \int_b \vec{\tau}_p$$

$$① \quad \frac{{}^i d}{dt} \int_b \vec{r}_p \times \vec{v}_p \, dm = \int_b \vec{r}_p \times \vec{f}_p \quad ②$$

Euler's 2nd law of motion II

1

$$= \frac{{}^i d}{dt} \int_b (\vec{r} + \vec{r}_c) \times \vec{v}_p \, dm$$

$$= \frac{{}^i d}{dt} \int_b \vec{r} \times \vec{v}_p \, dm + \int_b \vec{r}_c \times \frac{{}^i d}{dt} \vec{p}_p + \underbrace{\int_b \frac{{}^i d}{dt} \vec{r}_c \times \vec{p}_p}_{}$$

$$\frac{{}^i d}{dt} \vec{r}_c \times \int_b \vec{v}_p \, dm = \frac{{}^i d}{dt} \vec{r}_c \times \vec{v}_c \cdot m = 0$$

$$= \frac{{}^i d}{dt} \int_b \vec{r} \times \vec{v}_p \, dm + \int_b \vec{r}_c \times \vec{f}_p$$

2

$$= \int_b (\vec{r} + \vec{r}_c) \times \vec{f}_p = \int_b \vec{r} \times \vec{f}_p + \int_b \vec{r}_c \times \vec{f}_p$$

Euler's 2nd law of motion III

Therefore

$$\frac{{}^i d}{dt} \underbrace{\int_b \vec{r} \times \vec{v}_p \, dm}_{=:\,\vec{h}_{b/c}} = \underbrace{\int_b \vec{r} \times \dot{\vec{f}}_p}_{\vec{T}_{b/c}} = \int_b \vec{r} \times f_{p,ext}$$

$\vec{h}_{b/c}$: angular momentum about CoM

$T_{b/c}$: torque about CoM

Newton-Euler for center of mass

- That shows:
  - Newton's laws can be formulated for the center of mass

$$\frac{{}^i d}{dt}\vec{p}_c = \vec{F}_{bc}$$ (linear moments)

$$\frac{{}^i d}{dt}\vec{h}_{b/c} = \vec{T}_{bc}$$ (rotation)

Angular momentum

$$\vec{h}_{b/c} = \int_b \vec{r} \times \vec{v}_p dm$$

$\vec{v}_p = \vec{v}_c + \vec{\omega}_{ib} \times \vec{r}$

$$= \int_b \vec{r} \times (\vec{v}_c + \vec{\omega}_{ib} \times \vec{r}) dm$$

$$= \underbrace{\int_b \vec{r} dm}_{=0} \times \vec{v}_c + \int_b \vec{r} \times (\vec{\omega}_{ib} \times \vec{r}) dm$$

$$= - \int_b \vec{r} \times (\vec{r} \times \vec{\omega}_{ib}) dm$$

$$= \underbrace{- \int_b \vec{r}^{\times} \cdot \vec{r}^{\times} dm}_{\vec{M}_{b/c} \text{ : inertia dyadic}} \cdot \vec{\omega}_{ib}$$

$$= \vec{M}_{b/c} \cdot \vec{\omega}_{ib}$$

Euler's 2nd law of motion about CoM

$$\vec{T}_{bc} = \frac{{}^{i}d}{dt}\vec{h}_{bc} = \frac{{}^{i}d}{dt}\left(\vec{M}_{bc}\cdot\vec{\omega}_{ib}\right)$$

$$= \frac{{}^{b}d}{dt}\left(\vec{M}_{bc}\cdot\vec{\omega}_{ib}\right) + \vec{\omega}_{ib}\times\left(\vec{M}_{bc}\cdot\vec{\omega}_{ib}\right)$$

$$= \vec{M}_{bc}\cdot\vec{\alpha}_{ib} + \vec{\omega}_{ib}\times\left(\vec{M}_{bc}\cdot\vec{\omega}_{ib}\right)$$

$$\vec{\alpha}_{ib} = \frac{{}^{b}d}{dt}\vec{\omega}_{ib} = \frac{{}^{i}d}{dt}\vec{\omega}_{ib}$$

$$\frac{{}^{b}d}{dt}\vec{M}_{bc} = 0$$

EoM with reference of CoM

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib} \right)$$

Inertia dyadic I

$$\vec{M}_{b/c} = -\int_b \vec{r}^{\times} \cdot \vec{r}^{\times} \; dm$$

$$= \int_b \left[\vec{r} \cdot \vec{r} \, \mathrm{I} - \vec{r}\,\vec{r} \right] dm$$

Evaluate in frame

$$\vec{M}_{b/c} = \sum_{i=1}^{3} \sum_{j=1}^{3} m_{ij}^{b} \, b_i \, b_j \qquad m_{ij}^{b} : \text{constant}$$

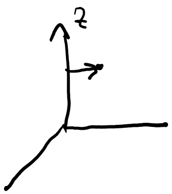
matrix form: $M_{b/c}^{b} = \begin{bmatrix} m_{11}^{b} & \cdot & \cdot \\ m_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

Inertia dyadic II

$$\vec{M}_{b/c} = -\int_b \vec{r}^{\times} \cdot \vec{r}^{\times} \, dm$$

$$M^b_{b/c} = \int_b \left[(\underline{r}^b)^T \cdot \underline{r}^b \, \underline{I} - \underline{r}^b (\underline{r}^b)^T \right] dm \qquad r^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

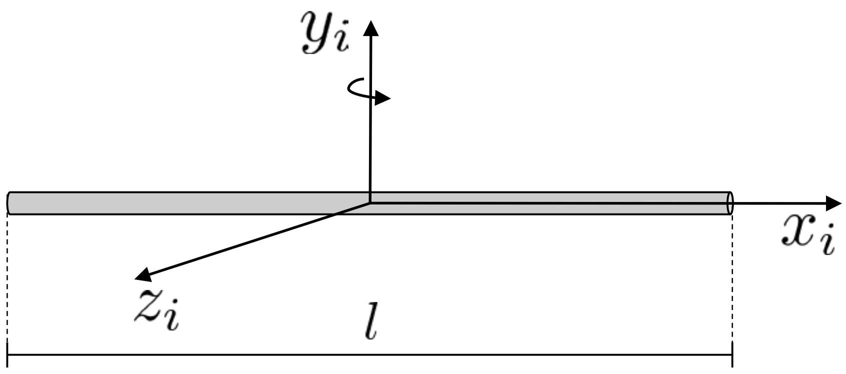
$$= \int_b \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$



$M^b_{b/c}$ constant

$M^i_{b/c} = R^i_b \, M^b_{b/c} \, R^b_i$ not constant

Example: Slender beam



Homework

- Try to derive the moment of inertia for the slender beam (slide 34) using the information on slide 33.
- Derive the acceleration of the point *p* on a rigid body with the help of the body fixed frame and the CoM.

- Read 7.3

Kahoot

- https://play.kahoot.it/#/k/4152faff-75ee-49ea-bb9e-b4c79dd85785