Lecture 14: Newton-Euler equations of motion

- Rigid body kinetics (Newton-Euler equations of motion)
  - Newton’s law
  - Angular momentum
  - Inertia dyadic

Book: Ch. 7.3

What is rigid body dynamics?

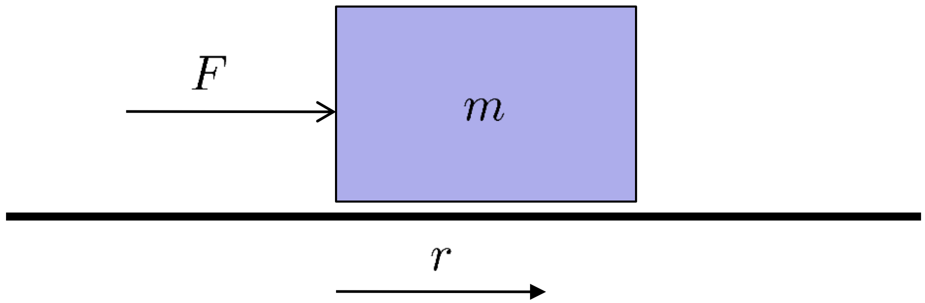
- Rigid body:
  - Wikipedia: "…a rigid body is an idealization of a solid body of finite size in which deformation is neglected."

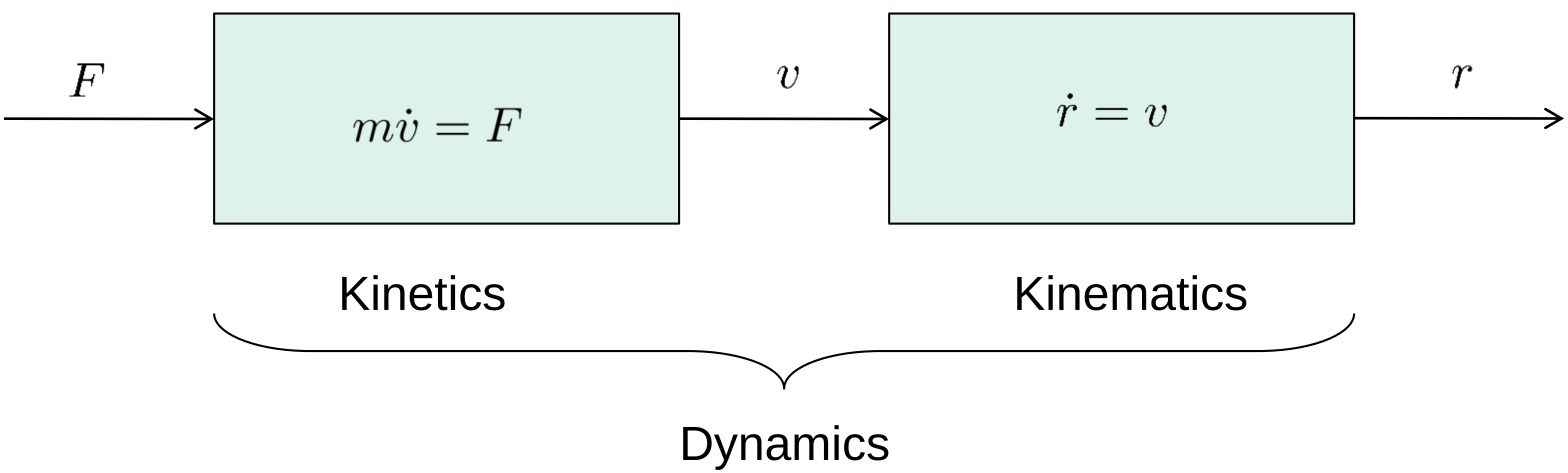
- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: "…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions)."
  - Book: Ch. 6
- Kinetics
  - eb.com: "…the effect of forces and torques on the motion of bodies having mass."
  - Book: Ch. 7, 8.

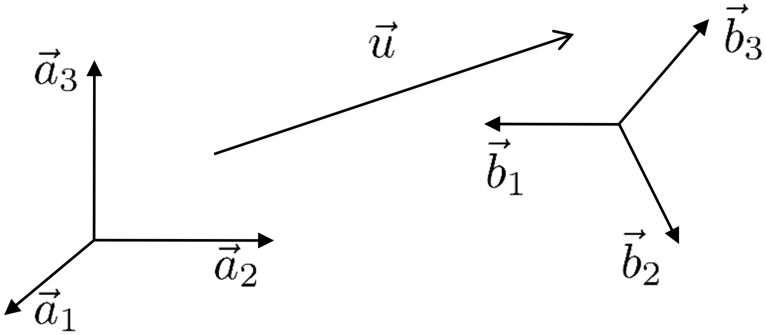
Remark: Sometimes "dynamics" is used for "kinetics" only

Simplest scalar case





Differentiations of vectors (6.8.5, 6.8.6)



- Coordinate representation:

$$\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b$$

- Differentiation:

$$\dot{\mathbf{u}}^a = \mathbf{R}_b^a \dot{\mathbf{u}}^b + \dot{\mathbf{R}}_b^a \mathbf{u}^b$$

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \left(\boldsymbol{\omega}_{ab}^b\right)^\times$$

$$\boxed{\dot{\mathbf{u}}^a = \mathbf{R}_b^a \left[\dot{\mathbf{u}}^b + \left(\boldsymbol{\omega}_{ab}^b\right)^\times \mathbf{u}^b\right]}$$

Note! Generally,

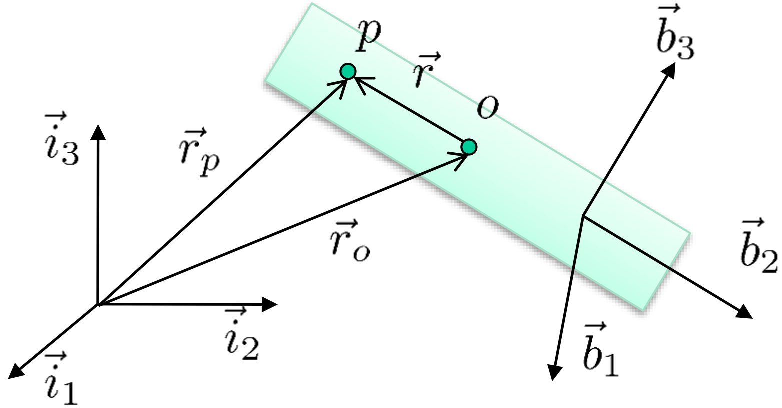
$$\dot{\mathbf{u}}^a \neq \mathbf{R}_b^a \dot{\mathbf{u}}^b$$

- On vector form:

$$\boxed{\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u}}$$

Rigid body kinematics

- Velocities and accelerations (Ch. 6.12)



$$\vec{v}_o := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_o, \quad \vec{v}_p := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_p$$

$$\vec{a}_o := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_o, \quad \vec{a}_p := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_p$$

$$\vec{\alpha}_{ib} := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ib} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ib}$$

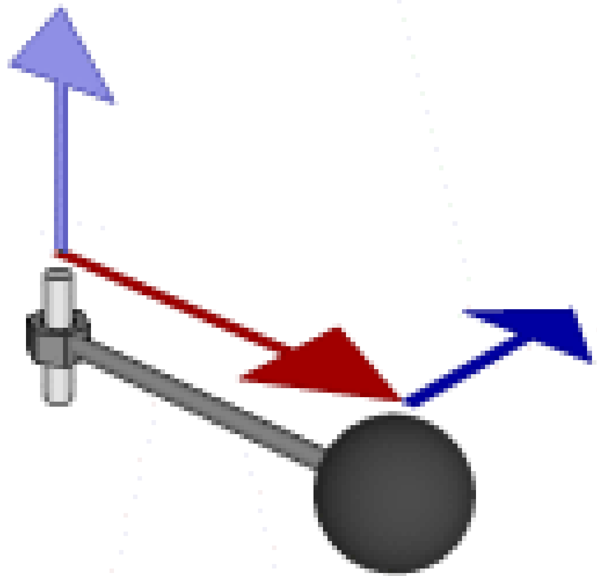
$$\begin{aligned}\vec{v}_p &= \vec{v}_o + \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} \\ &= \vec{v}_o + \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ &= \vec{v}_o + \vec{\omega}_{ib} \times \vec{r}, \quad \vec{r} \text{ fixed.}\end{aligned}$$

$$\boxed{\vec{a}_p = \vec{a}_o + \frac{{}^b\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})}$$

$$\boxed{\vec{a}_p = \vec{a}_o + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}), \quad \vec{r} \text{ fixed.}}$$

Torque, and linear/angular momentum

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$



Source: Wikipedia

- Book:
  - Torque: $\vec{N}, \vec{T}$
  - Angular momentum: $\vec{h}$

EoM with reference of CoM

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib} \right)$$

Inertia dyadic I

$$\vec{M}_{b/c} = -\int_b \vec{r}^{\times} \cdot \vec{r}^{\times}$$

Inertia matrix

- Found for each rigid body by calculating

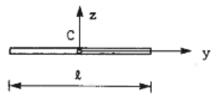
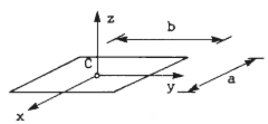
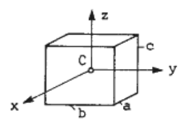
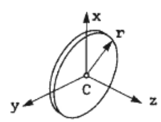
$$M_{b/c}^{b} = \int_b (\mathbf{r}^b)^\mathsf{T}\mathbf{r}^b I - \mathbf{r}^b(\mathbf{r}^b)^\mathsf{T}\mathrm{d}m = \int_b \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \mathrm{d}m$$

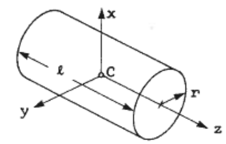
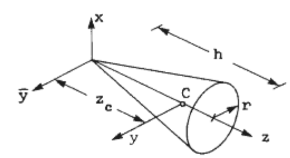
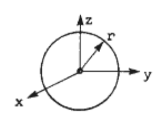
- Constant in body-fixed coordinate system!
- Not constant in inertial coordinate system

$$M_{b/c}^{i} = R_b^i M_{b/c}^{b} (R_b^i)^\mathsf{T}$$

- Books and wikipedia have tables for common geometries, otherwise computer programs calculates, or can be calculated/identified based on experiments
- Typically, axis in body-system chosen as body symmetri axis, giving zeros in inertia matrix. If symmetric about all axis, the inertia matrix becomes diagonal.

Finding moments of inertia

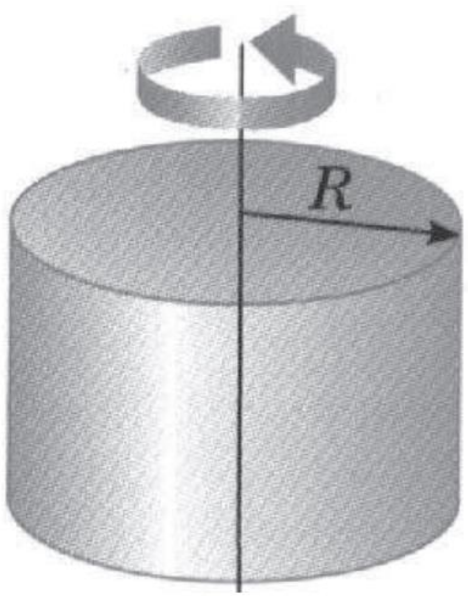
Homogen slank stav		$I_z = \frac{1}{12} m l^2$ $I_{\bar z} = \frac{1}{3} m l^2$
Tynn rektangulær plate		$I_z = \frac{1}{12} m (a^2 + b^2)$ $I_x = \frac{1}{12} m b^2$ $I_y = \frac{1}{12} m a^2$
Rektangulært prisme		$I_z = \frac{1}{12} m (a^2 + b^2)$
Tynn sirkulær skive		$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{4} m r^2$

Sirkulær sylinder		$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{12} m (3r^2 + l^2)$
Tynt sylinderskall		$I_z = m r^2$ $I_x = I_y = \frac{1}{2} m r^2 + \frac{1}{12} m l^2$
Rett sirkulær kjegle		$I_z = \frac{1}{10} m r^2$ $I_y = \frac{3}{20} m r^2 + \frac{3}{80} m h^2$ $I_{\bar y} = \frac{3}{20} m r^2 + \frac{3}{5} m h^2$ $z_c = 3h/4$
Kule		$I_C = \frac{2}{5} m r^2$
Kuleskall		$I_C = \frac{2}{3} m r^2$

From F. Irgens, Dynamikk

- http://en.wikipedia.org/wiki/List_of_moment_of_inertia_tensors
- For other/general rigid bodies (vessels/planes/etc.), computer programs can find moments of inertia

Inertia matrix, examples



Homogeneous Disk

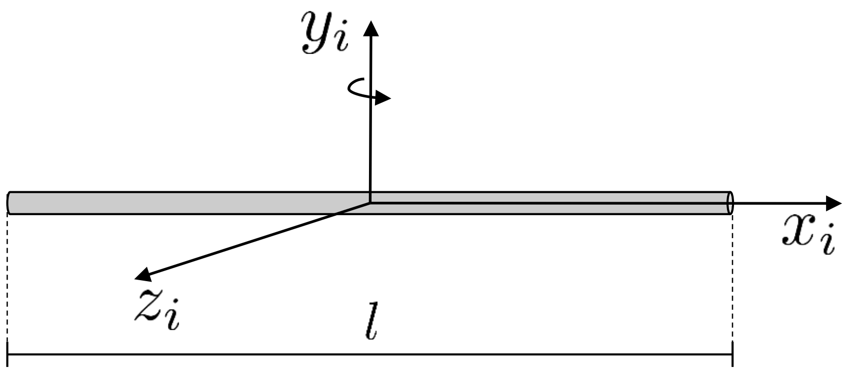
$$I_{disk} = \frac{1}{4}mr^2 \begin{bmatrix} 1 + \frac{1}{3}\frac{h^2}{r^2} & 0 & 0 \\ 0 & 1 + \frac{1}{3}\frac{h^2}{r^2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

F/A-18

$$I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} kslug - ft^2$$

1 slug = 14.6 kg
1 ft = 0.304 m

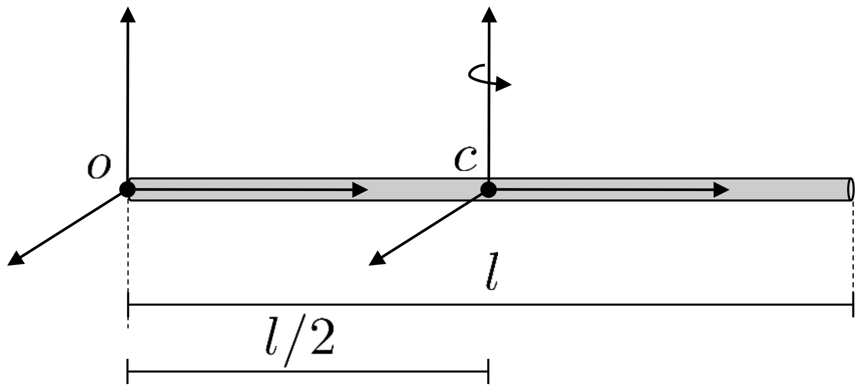
Example: Slender beam



Parallel axis theorem

$$\vec{M}_{b/o} = \vec{M}_{b/c} - m(\underline{r}_g^b)^\times(\underline{r}_g^b)^\times$$
$$= \vec{M}_{b/c} + m\left[(\underline{r}_g^b)^T\underline{r}_g^b\mathbf{I} - \underline{r}_g^b(\underline{r}_g^b)^T\right]$$

Example:



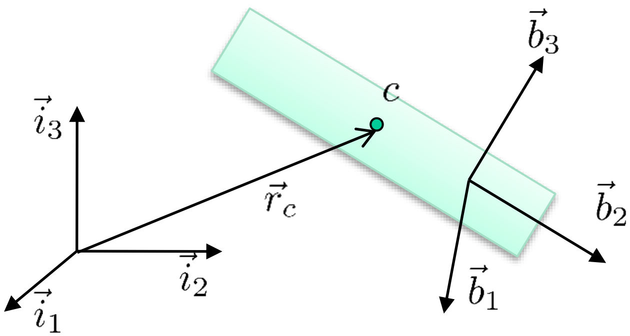
Summary: EoM rigid body kinetics I

Summary: EoM rigid body kinetics II

Newton-Euler EoM

- Referenced to center of mass (CoM):

$$\boxed{\begin{aligned}\vec{F}_{bc} &= m\vec{a}_c \\ \vec{T}_{bc} &= \vec{M}_{b/c}\cdot\vec{\alpha}_{ib} + \vec{\omega}_{ib}\times\left(\vec{M}_{b/c}\cdot\vec{\omega}_{ib}\right)\end{aligned}}$$



- Sometimes convenient to have them referenced to other point *o*:
  - Forces and moments in *o*:

$$\vec{F}_{bo} = \vec{F}_{bc}$$

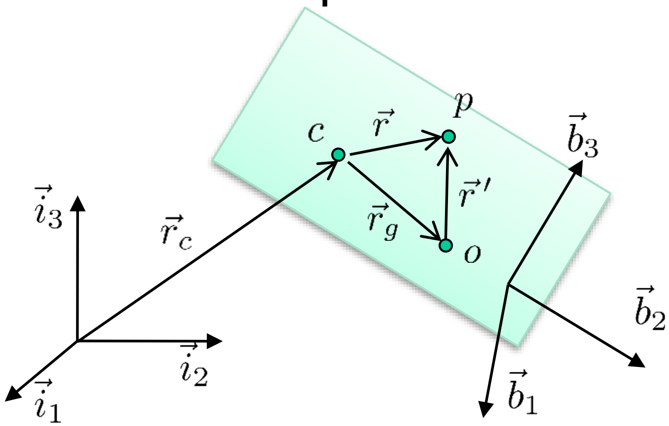
$$\vec{T}_{bo} = \vec{T}_{bc} + \vec{r}_g \times \vec{F}_{bc}$$

  - Use

$$\vec{a}_c = \vec{a}_o + \vec{\alpha}_{ib}\times\vec{r}_g + \vec{\omega}_{ib}\times(\vec{\omega}_{ib}\times\vec{r}_g)$$

  - Define

$$\vec{M}_{b/o} := -\int_b (\vec{r}\,')^\times(\vec{r}\,')^\times \mathrm{d}m$$



$$\boxed{\begin{aligned}\vec{F}_{bo} &= m\left(\vec{a}_o + \vec{\alpha}_{ib}\times\vec{r}_g + \vec{\omega}_{ib}\times(\vec{\omega}_{ib}\times\vec{r}_g)\right) \\ \vec{T}_{bo} &= \vec{r}_g\times\vec{a}_o + \vec{M}_{b/o}\cdot\vec{\alpha}_{ib} + \vec{\omega}_{ib}\times\left(\vec{M}_{b/o}\cdot\vec{\omega}_{ib}\right)\end{aligned}}$$

  - Useful when CoM changes – no need to recalculate inertia matrix – still need to know CoM

Traits of Newton-Euler EoM

(and a preview: Lagrange EoM)

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)$$

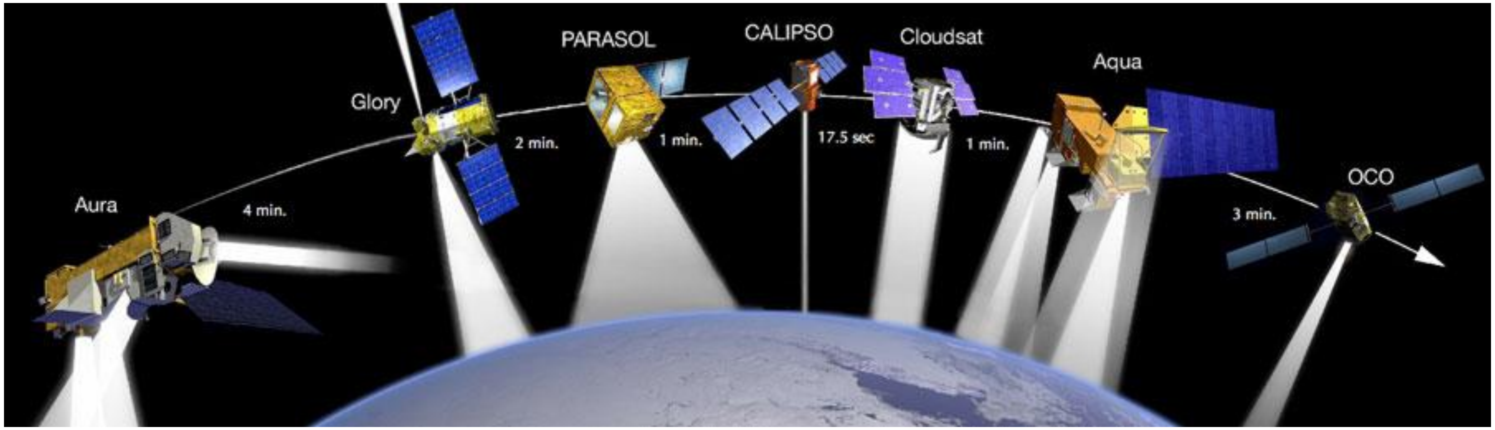
Newton-Euler EoM:

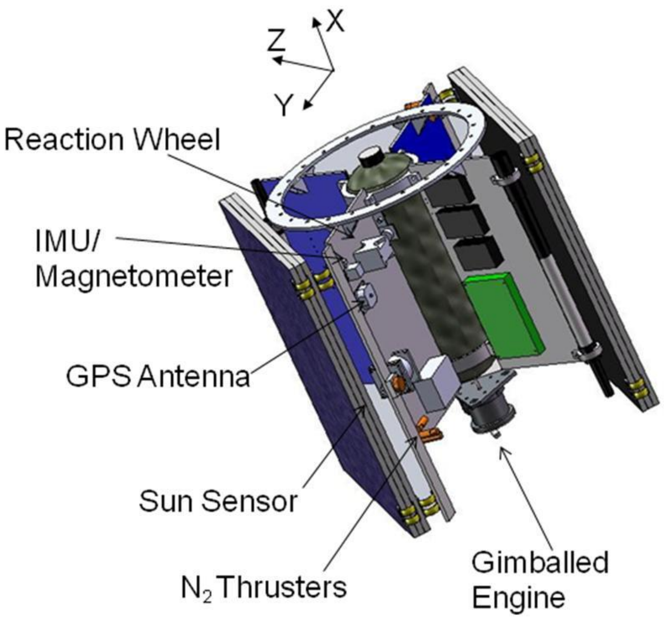
- Involves working with vectors
  - Lagrange: Algebraic manipulations
- Forces and moments are central
  - Lagrange: Energy and work are central
- All forces in the system must be considered
  - Lagrange: Forces of constraint are implicitly eliminated with the use of generalized coordinates (and generalized forces)
- Somewhat complicated to use by hand, but can be implemented in computer systems
  - Lagrange: Easier to do by hand, not suitable for complex systems

- d'Alembert's principle: Elimination of forces of constraint (Ch. 7.7)
  - Can simplify application of Newton-Euler EoM
    - Kane's EoM (Ch. 7.8, 7.9)
  - Starting point for Lagrange EoM (Ch. 8.2)

	Kinematics	Kinetics
	Derivatives of position and orientation as function of velocity and angular velocity	Derivatives of velocity and angular velocity as function of applied forces and torques

Satellite attitude dynamics





http://satellite.mit.edu/

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib} \right)$$

Example: Satellite I

- Assume the body-fixed frame is chosen such that

$$M^b_{b/c} = \begin{pmatrix} m_{12} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}$$

$$\begin{pmatrix} m_{12} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} m_{12} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

Example: Satellite II

Airplane EoM (from book about airplane dynamics)

Force equations

$$X - mgS_\theta = m(\dot{u} + qw - rv)$$
$$Y + mgC_\theta S_\Phi = m(\dot{v} + ru - pw)$$
$$Z + mgC_\theta C_\Phi = m(\dot{w} + pv - qu)$$

$$m\left(\dot{\mathbf{v}}_c^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{v}_c^b\right) = \mathbf{F}_{bc}^b$$

Moment equations

$$L = I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq$$
$$M = I_y\dot{q} + rp(I_x - I_z) + I_{xz}(p^2 - r^2)$$
$$N = -I_{xz}\dot{p} + I_z\dot{r} + pq(I_y - I_x) + I_{xz}qr$$

$$\mathbf{M}_{b/c}^b\dot{\boldsymbol{\omega}}_{ib}^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{M}_{b/c}^b\boldsymbol{\omega}_{ib}^b = \mathbf{T}_{bc}^b$$

Body angular velocities in terms of Euler angles and Euler rates

$$p = \dot{\Phi} - \dot{\psi}S_\theta$$
$$q = \dot{\theta}C_\Phi + \dot{\psi}C_\theta S_\Phi$$
$$r = \dot{\psi}C_\theta C_\Phi - \dot{\theta}S_\Phi$$

Euler rates in terms of Euler angles and body angular velocities

$$\dot{\theta} = qC_\Phi - rS_\Phi$$
$$\dot{\Phi} = p + qS_\Phi T_\theta + rC_\Phi T_\theta$$
$$\dot{\psi} = (qS_\Phi + rC_\Phi)\sec\theta$$

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_d^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^b$$

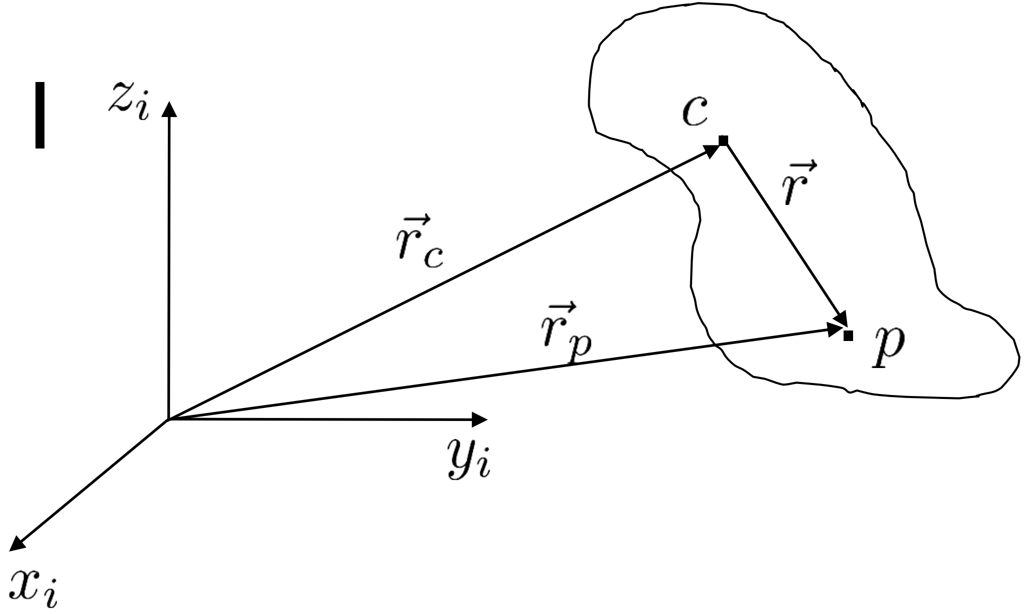
Velocity of aircraft in the fixed frame in terms of Euler angles and body velocity components

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} C_\theta C_\psi & S_\Phi S_\theta C_\psi - C_\Phi S_\psi & C_\Phi S_\theta C_\psi + S_\Phi S_\psi \\ C_\theta S_\psi & S_\Phi S_\theta S_\psi + C_\Phi C_\psi & C_\Phi S_\theta S_\psi - S_\Phi C_\psi \\ -S_\theta & S_\Phi C_\theta & C_\Phi C_\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\dot{\mathbf{r}}_c^i = \mathbf{v}_c^i = \mathbf{R}_b^i\mathbf{v}_c^b$$

Flight Stability and Automatic Control
R. C. Nelson, 2nd Edition

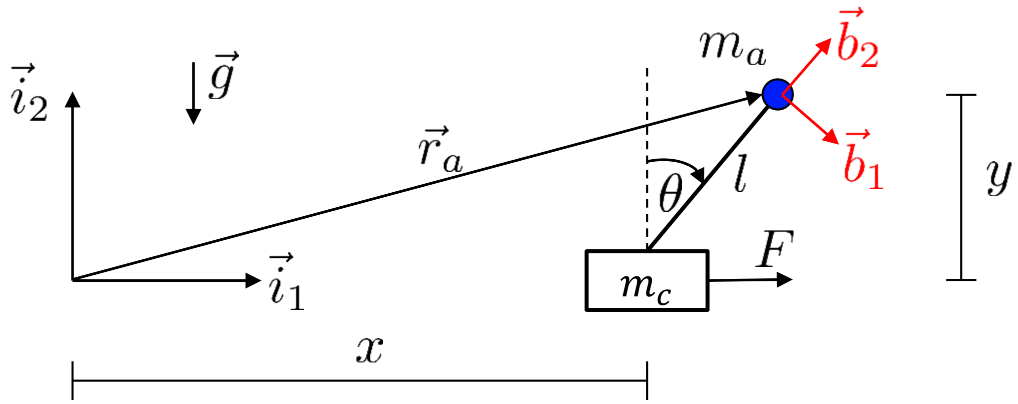
Kinematic energy I

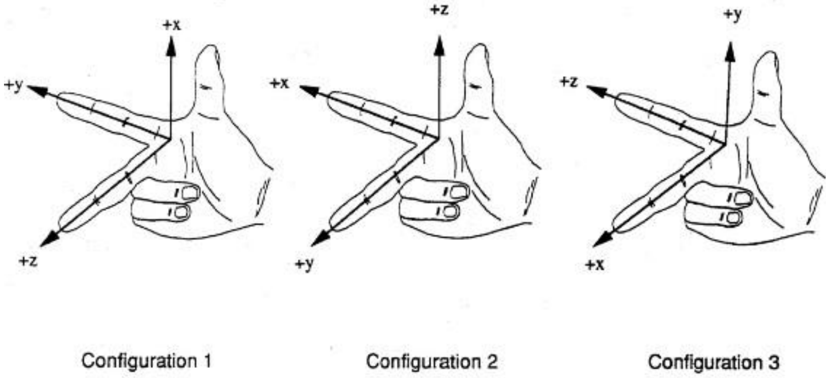


Kinematic energy II

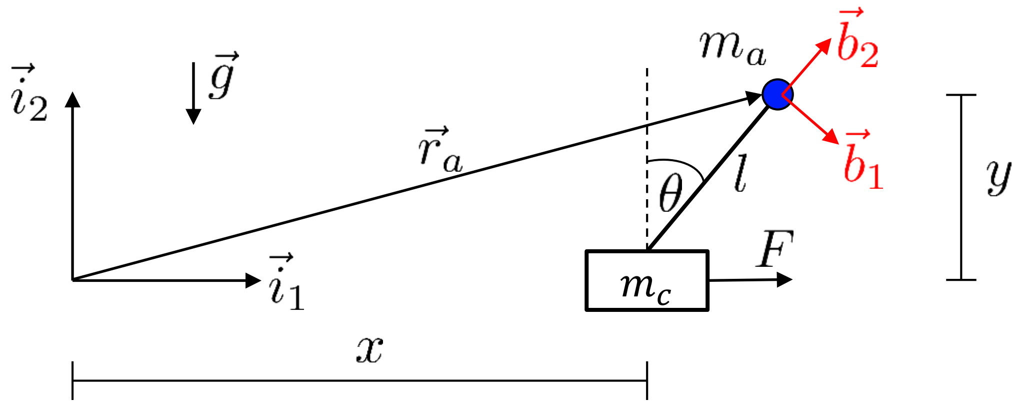
$$K = \frac{1}{2} m (\underline{v}_c^b)^T \underline{v}_c^b + \frac{1}{2} (\underline{\omega}_c^b)^T M_{b/c}^b \underline{\omega}_c^b$$

Example: Inverted pendulum

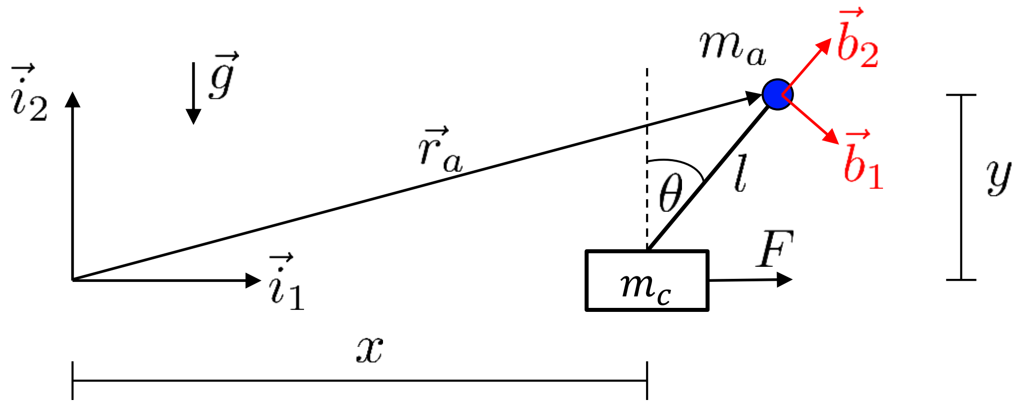




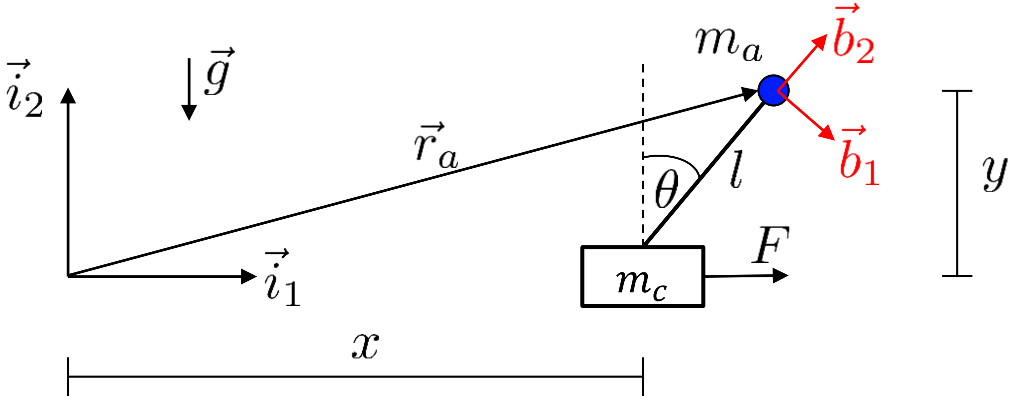
Example: Inverted pendulum - kinematics



Example: Inverted pendulum – kinetics I

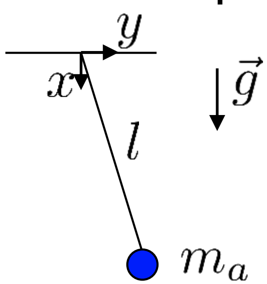


Example: Inverted pendulum – kinetics II

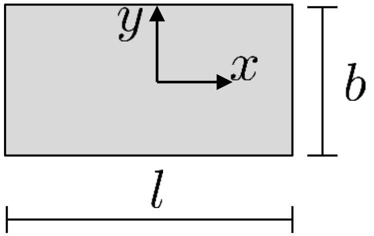


Homework

- Find the equation of motion of a pendulum using Newton's law:



- Find the moment of inertia of a rectangular plate



- Try to find the acceleration of the inverted pendulum (slide 26) using only the inertial frame (check your result by transforming the acceleration to the body frame)

- Read 5.1-5.3