

Lecture 4: Passivity

Passivity (E2.4)

- Positive Real (PR) transfer functions
- Passivity and storage functions

Energy function

- The system: $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$
- Assume we have a function $V(x, t) \geq 0$, which describes the «energy» of the system

- The derivative of the energy function $V(x, t)$ is

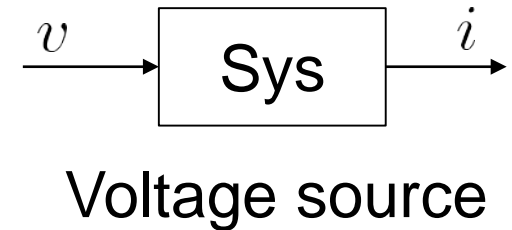
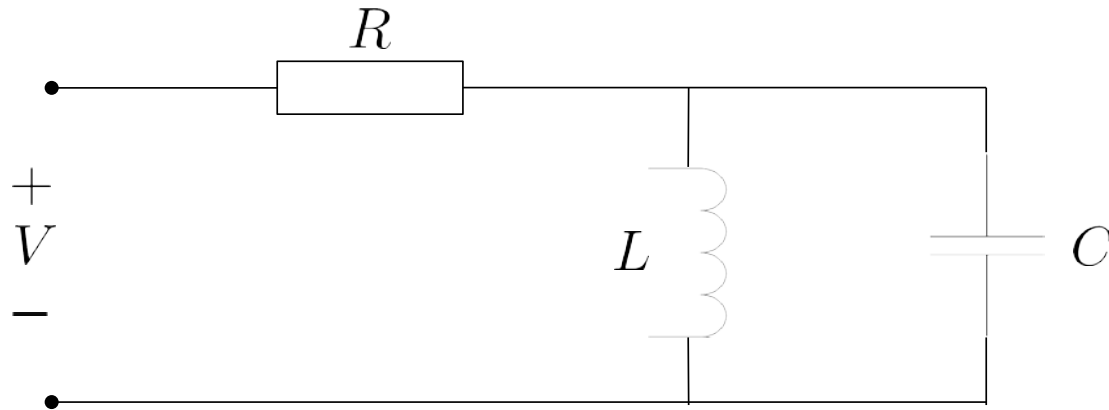
$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u, t)$$

- If we have $\dot{V} \leq 0$
 - Energy of the system decreases monotonically
 - stability

Why learn about passivity? Preview...

- Say you have several systems (or models), and you want to interconnect them
 - For instance, a process and a controller, or a motor and a load, or two buffer tanks in series, ...
 - Will the interconnection be stable?
- Bad news: The interconnection of stable systems is not necessarily stable
- **Good news: The interconnection of passive systems is passive (and therefore stable)!**

Passivity: Circuit example

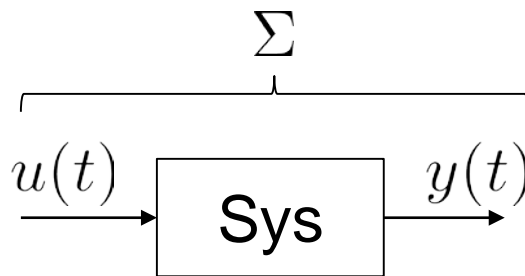


$$P(t) = i(t)v(t)$$

$P(t) > 0$: given to the system

$P(t) < 0$: goes out of the system

Definition of passivity



- A system Σ is passive if there exist $E_0 > 0$ such that for all control time histories u and all $t \geq 0$ the following holds:

$$\int_0^T y(t)u(t)dt \geq -E_0$$

Passivity



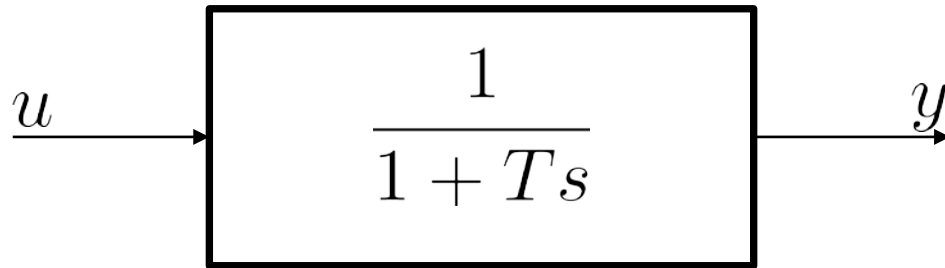
- A system with input u and output y is passive if

$$\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$$

for all $t \geq 0$, for all input trajectories.

- If the product yu has power as unit, then if (for all u)
 - $\int_0^t y(\tau)u(\tau)d\tau \geq 0$: Energy is absorbed within the system, nothing delivered to the outside
 - $\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$: Some energy can be delivered to the outside, limited (typically) by the initial energy in the system.
 - $\int_0^t y(\tau)u(\tau)d\tau \rightarrow -\infty$: There is an inexhaustible energy source in the system. Not passive!

Example: Passivity



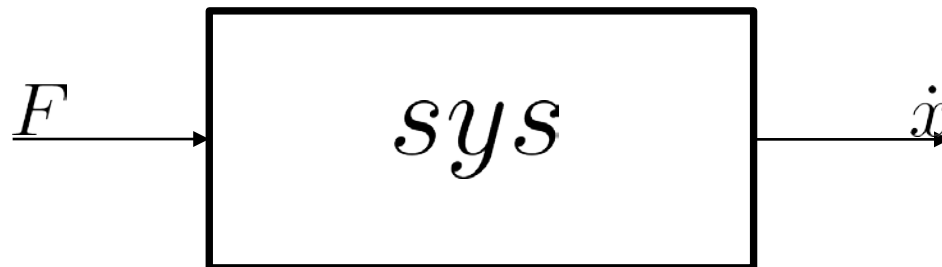
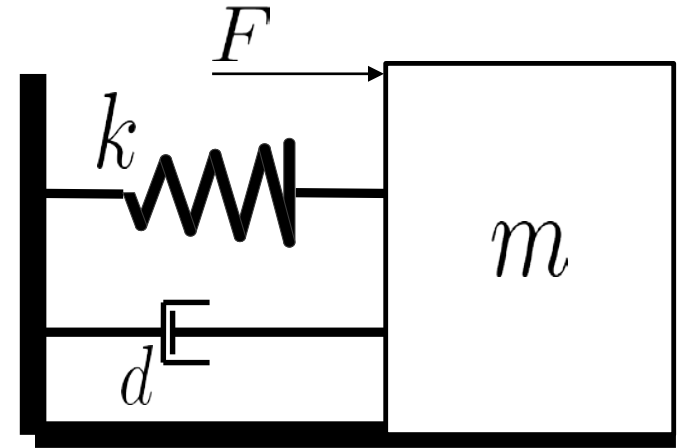
$$\dot{y} = -\frac{1}{T}y + \frac{1}{T}u$$

$$u = T\dot{y} + y$$

$$\begin{aligned} \int_{t_0}^t y(\tau)u(\tau)d\tau &= \int_{t_0}^t y(\tau) \left[T y(\tau)\dot{\tau} + y(\tau) \right] d\tau \\ &= T \int_{t_0}^t y\dot{y}d\tau + \int_{t_0}^t y^2d\tau \\ &= \frac{T}{2} [y^2(t) - y^2(0)] + \int_{t_0}^t y^2d\tau \\ &\geq -\frac{T}{2}y^2(0) = -E_0 \end{aligned}$$

Example: Mass-Spring-damper I

$$m\ddot{x} + d\dot{x} + kx = F$$



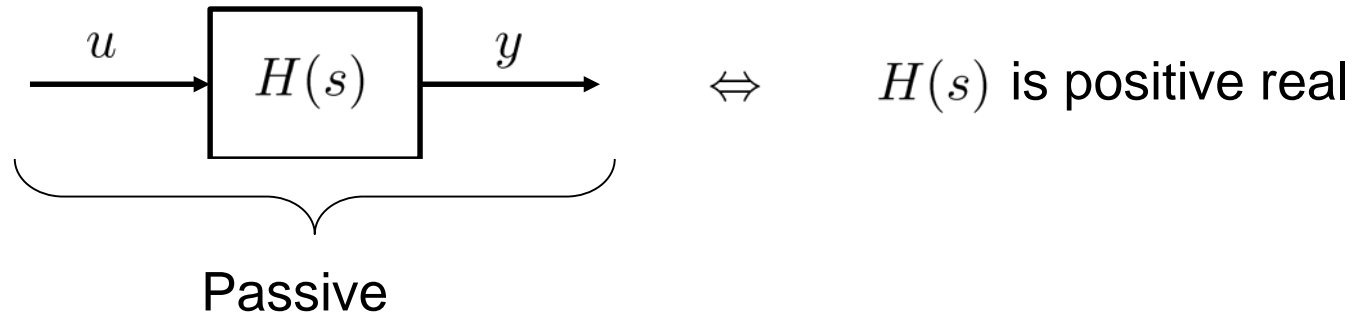
Example: Mass-Spring-damper II

$$m\ddot{x} + d\dot{x} + kx = F$$

Example: Mass-Spring-damper III

$$m\ddot{x} + d\dot{x} + kx = F$$

Positive real transfer functions



Definition: The transfer function $H(s)$ (rational or irrational) is positive real if

1. $H(s)$ analytic in $\text{Re}[s] > 0$.
2. $H(s)$ is real for all positive and real s .
3. $\text{Re}[H(s)] \geq 0$ for all $\text{Re}[s] > 0$.

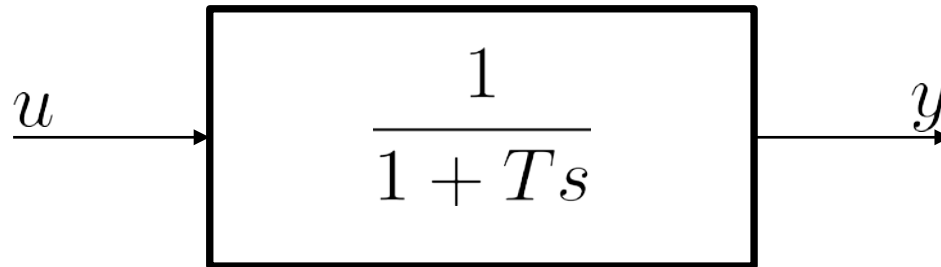
Check rational TFs for PRness

Theorem: A rational, proper transfer function $H(s)$ is positive real (and hence passive) if and only if

1. $H(s)$ has no poles in $\text{Re}[s] > 0$.
2. $\text{Re}[H(j\omega)] \geq 0$ for all $\omega \in [-\infty, \infty]$ such that $j\omega$ is not a pole of $H(s)$.
3. If $j\omega_0$ is a pole of $H(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is,

$$\text{Res}_{s=j\omega_0} H(s) = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)H(s) > 0.$$

Example: Positive Real



1. Pol: $s = -\frac{1}{T} \quad \text{Re}[s] \leq 0$

2. $H(jw) = \frac{1}{1 + Tjw} = \frac{1 - jwT}{1 + (wT)^2}$

$$\text{Re}[H(jw)] = \frac{1}{1 + (Tw)^2} \geq 0$$

3. Ok, since no pols on imaginary axis

Example 32 – $H(s)$ positive real?

$$H(s) = K \frac{(s + z_1)(s + z_2) \dots}{s(s + p_1)(s + p_2) \dots}$$

- Assume: $\operatorname{Re}[p_i] > 0$ and $\operatorname{Re}[z_i] > 0$
- Hint: Poles/zeros come all in complex conjugated pairs: $(\alpha + j\beta)(\alpha - j\beta) = \alpha^2 + \beta^2$

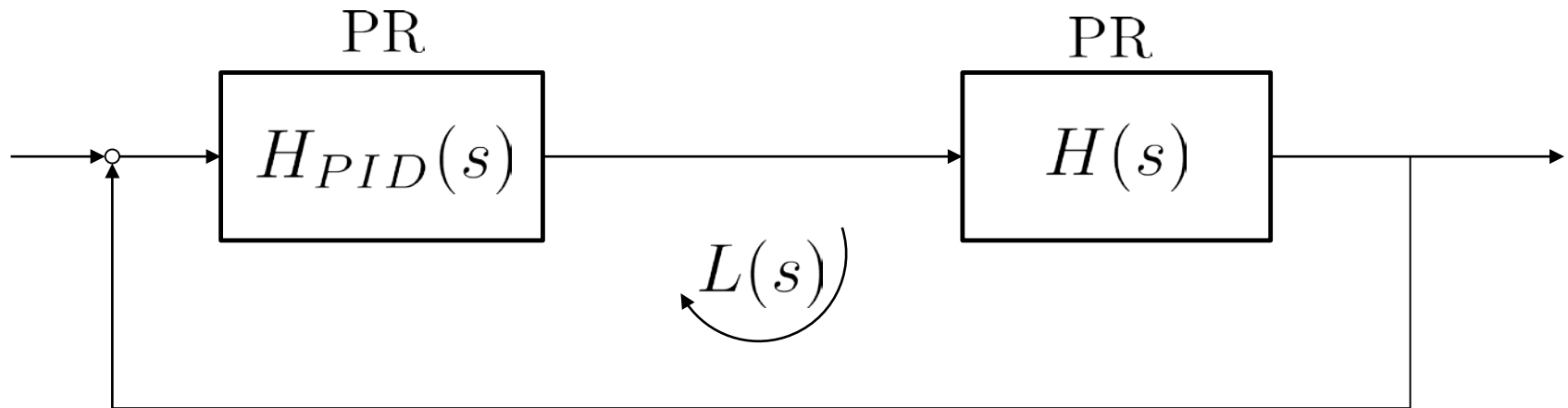
PID controller I

$$H_{PID}(s) = K_p \frac{1 + T_i s}{T_i s} \frac{1 + T_d s}{1 + \alpha T_d s}$$

$$K_p > 0, \quad T_i > 0, \quad T_d > 0, \quad T_d < T_i, \quad 0 \leq \alpha \leq 1$$

PID controller II

Why is this useful?

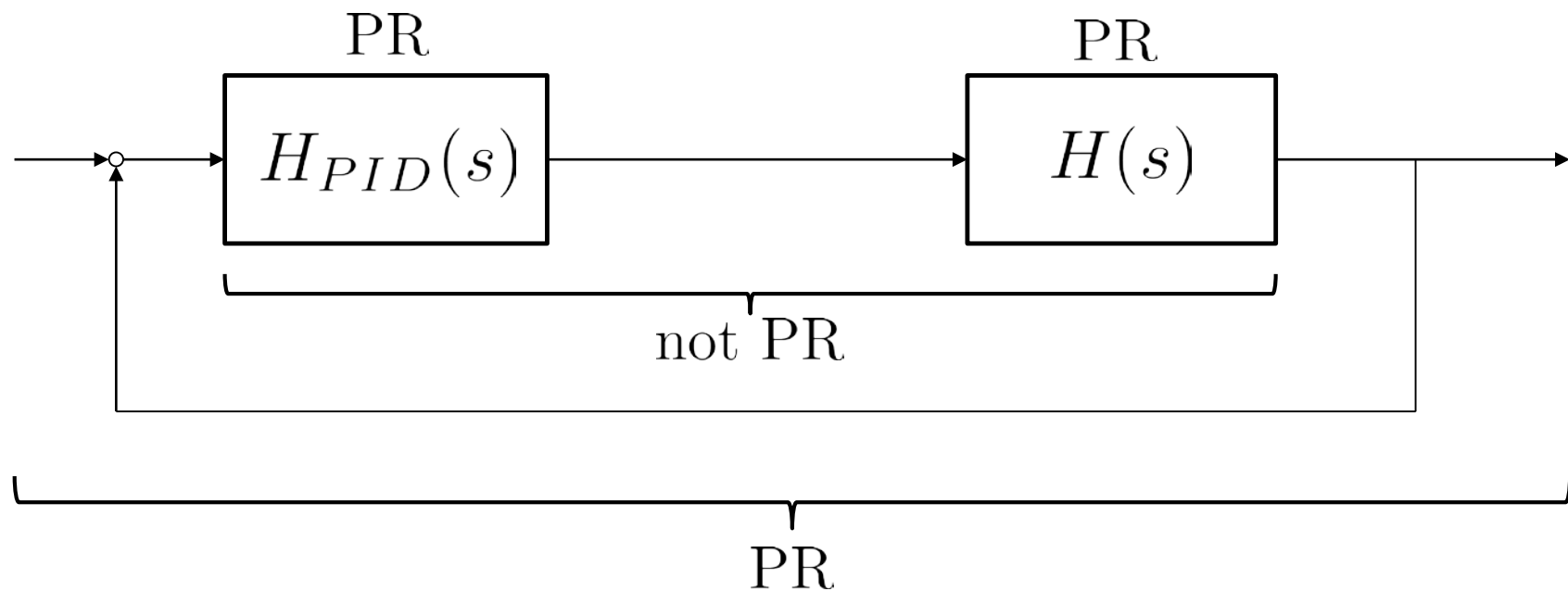


$$L(s) = H_{PID}H(s)$$

$$\angle L(j\omega) = \angle H_{PID}(j\omega) + \angle H(j\omega)$$

$$\rightarrow |\angle L(j\omega)| \leq 180^\circ$$

In addition:



Storage function I

- We can proof passivity via the storage function
- Consider the system:

$$\dot{x} = f(x, u)$$

$$y = h(x)$$

- Assume we have a storage function $V(x) \geq 0$ and a dissipation function $g(x) \geq 0$
- Such that the time derivative for all control inputs u is:

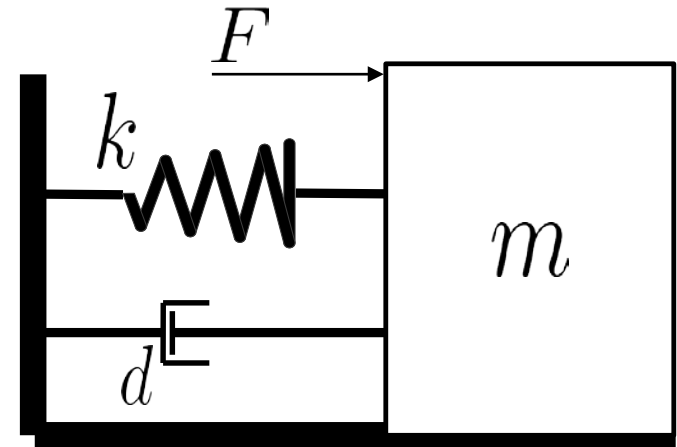
$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) = u^T y - g(x)$$

→ System with input u and output y is passiv

Storage function II

Example: Storage function

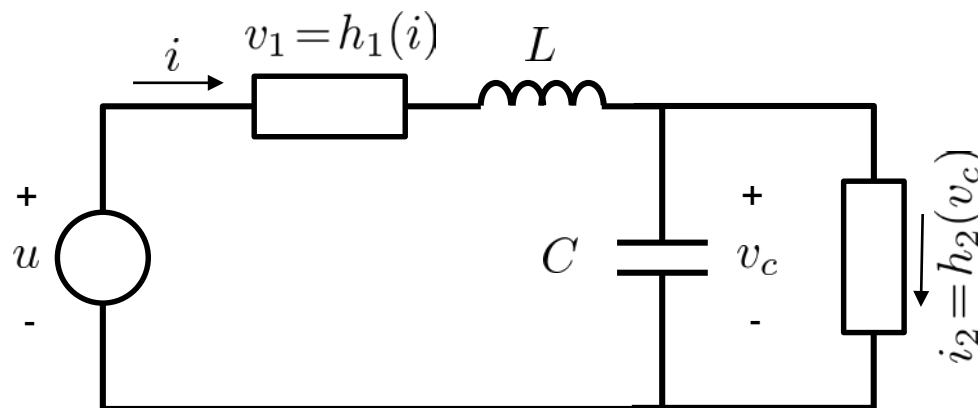
$$m\ddot{x} + d\dot{x} + kx = F$$



$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \geq 0$$

$$\begin{aligned} \dot{V} &= F\dot{x} - d\dot{x}^2 \\ &= uy - \underbrace{g(x)}_{\geq 0} \end{aligned}$$

Example storage functions



- States: $x_1 = i$, $x_2 = v_c$
- Model (Kirchoff's laws):

$$L\dot{x}_1 = u - h_1(x_1) - x_2$$

$$C\dot{x}_2 = x_1 - h_2(x_2)$$
- Output&input: $y = i$, $u = u$
- Nonlinear resistors fulfilling

$$x_i h_i(x_i) \geq 0$$

- Storage (energy) function:

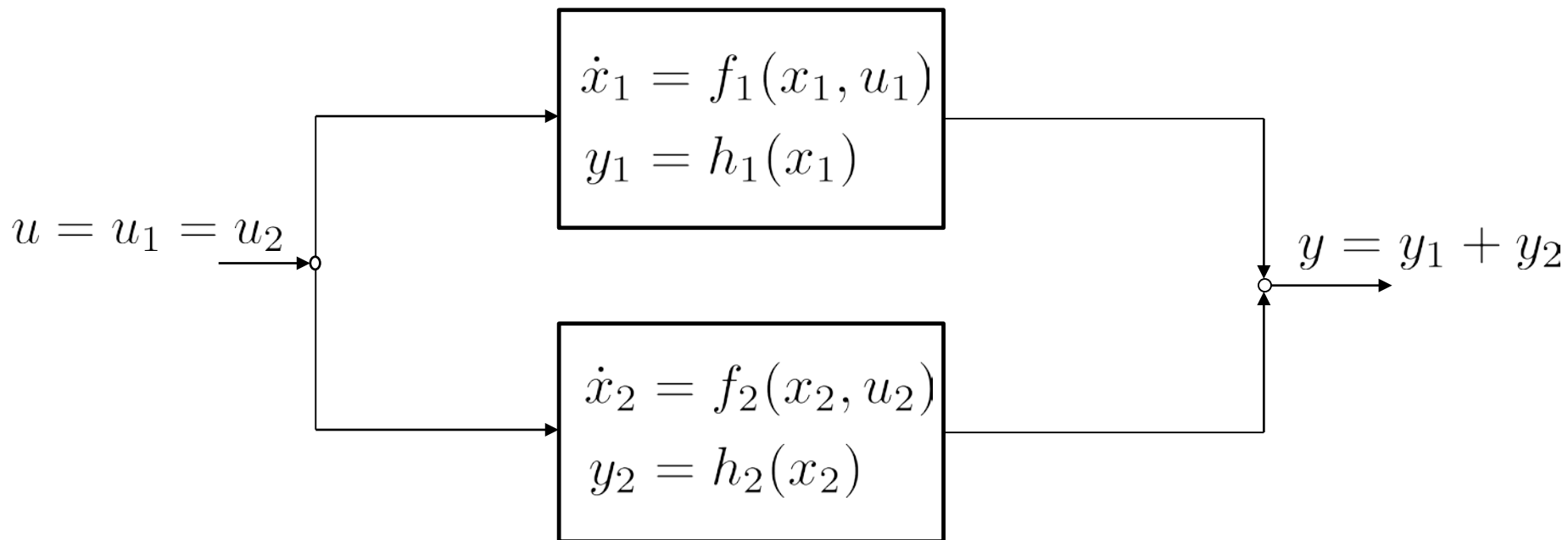
$$V(\mathbf{x}) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$$

- Differentiate:

$$\begin{aligned}\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\ &= x_1(u - h_1(x_1) - x_2) + x_2(x_1 - h_2(x_2)) \\ &= yu - x_1h_1(x_1) - x_2h(x_2)\end{aligned}$$

- Passive!

Connection of passive systems – parallel I

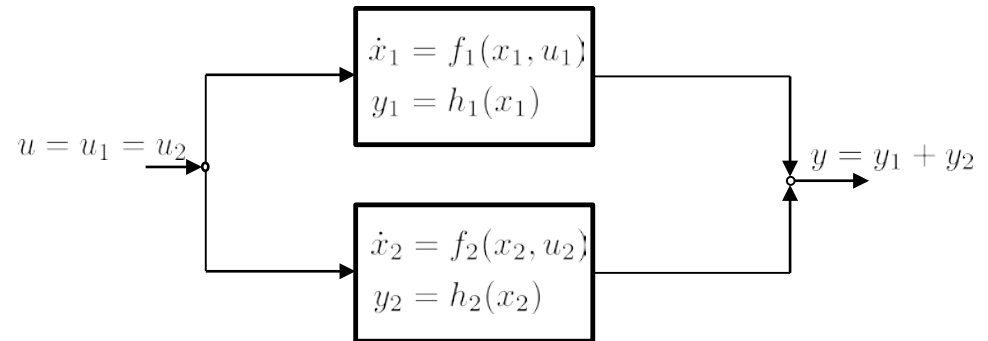


- Assume both systems passive with storage functions:

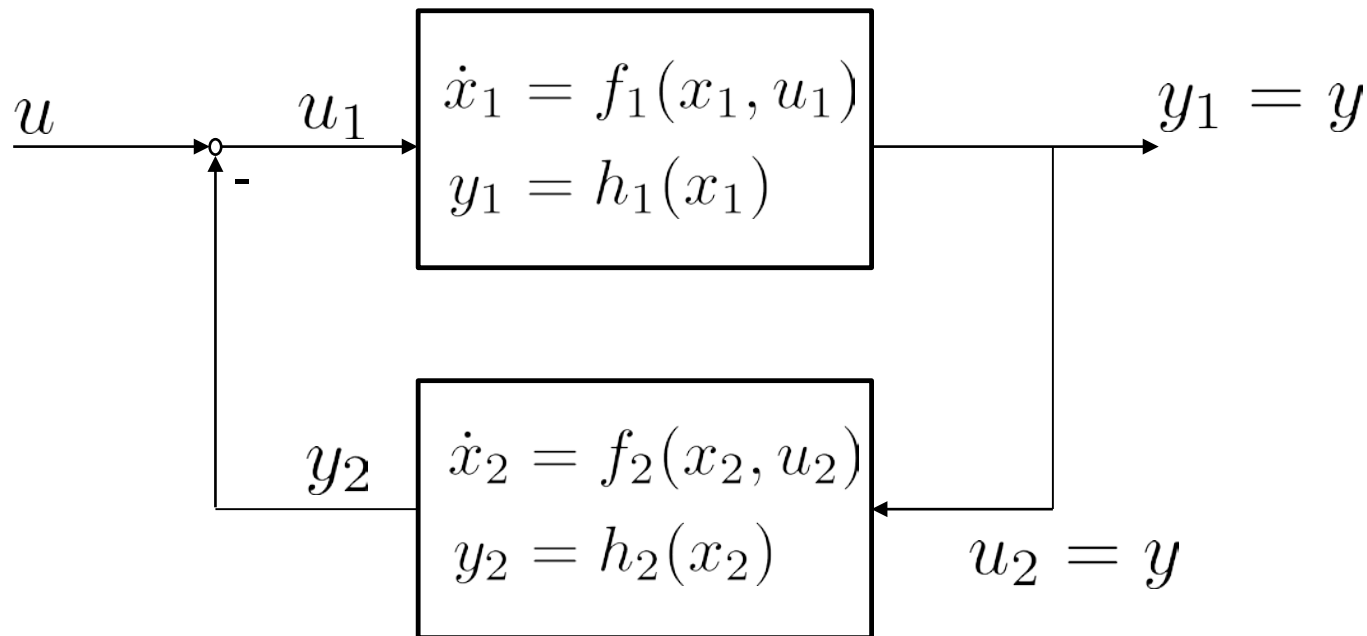
$$V_i \geq 0; \quad g_i \geq 0$$

$$\dot{V}_i = \frac{\partial V}{\partial x_i} f_i(x_i, u_i) \leq u_i^T y_i - g(x_i) \quad i = 1, 2$$

Connection of passive systems – parallel II



Connection of passive systems - feedback

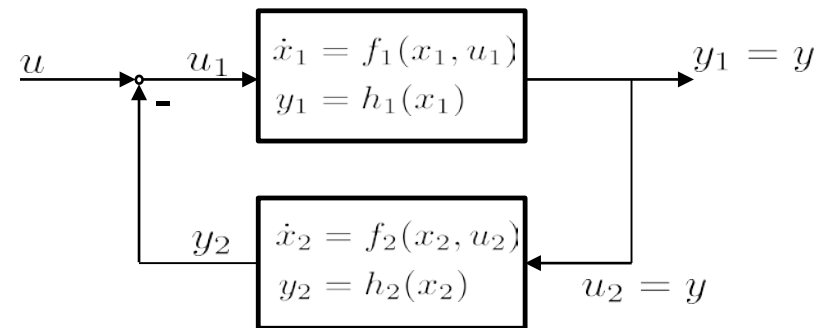


- Assume both systems passive with storage functions:

$$V_i \geq 0; \quad g_i \geq 0$$

$$\dot{V}_i = \frac{\partial V}{\partial x_i} f_i(x_i, u_i) \leq u_i^T y_i - g(x_i) \quad i = 1, 2$$

Connection of passive systems - feedback



Kahoot

- <https://play.kahoot.it/#/k/c452fe59-cad5-4f8a-ba94-475d2a5569b6>

Why learn about simulation methods?

1. You will need to implement your own solvers
 - What solver fits my problem, what time-step should I choose?
 - Primarily: Explicit solvers

2. You will need to make qualified choices of solvers when using advanced modeling software
 - What solver fits my problem, choice of accuracy?
 - Typically: Implicit solvers with varying time-steps

 - Examples:
 - Simulink: Three-body problem, satellite in combined moon and earth gravity field (orbit.mdl, ode45 vs ode1 (Euler))
 - Dymola

Initial value problem

Computational error

Method: One step method

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Order of a one step method:

A method is of order p if p is the smallest integer such that:

$$e_{n+1} = O(h^{p+1})$$

Homework

- Try to proof that

$$H(s) = K \frac{(s + z_1)(s + z_2) \dots}{s(s + p_1)(s + p_2) \dots}$$

is a positive Real

- Find out how a loudspeaker works.
- Read Section 14.1.