NTNU – Trondheim
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Assignment 7

TTK4130 Modeling and Simulation

Problem 1 (Mass, center of mass, inertia matrix, the parallel axis theorem. 40 %)

Consider the rectangular right pyramid given by

$$0 \le z \le h\left(1 - \max\left\{\frac{|x|}{r}, \frac{|y|}{s}\right\}\right) \tag{1a}$$

$$-r \le x \le r \tag{1b}$$

$$-s \le y \le s\,, \tag{1c}$$

where h, s and r are positive numbers.

The density of the pyramid is given by $\rho(x, y, z) = ax + by + cz + d$, where a, b, c and d are real numbers such that

$$ch + d > 0 \tag{2a}$$

$$d > r|a| + s|b|. \tag{2b}$$

Furthermore, let b be the frame attached to the pyramid such that $\mathbf{r}^b = [x\,, y\,, z]^T$.

(a) **(Optional)** What do the constraints in (2) mean?
Why are these constraints necessary from a physical point of view?

> **Solution:** The constraints mean that the density is positive at the apex and at the corners of the base of the pyramid.
> Since the pyramid is a convex polytope and the density is an affine function of x, y and z, we conclude that the density is positive on the whole pyramid, which is necessary from a physical perspective.

(b) Verify that the mass of the pyramid is

$$m = \frac{rsh}{3}(4d + ch). \tag{3}$$

Show the details of your calculations.
Hint 1: Read section 6.13. in the book.
Hint 2: Use the integration order $dV = dx\,dy\,dz$.
Hint 3: Use symmetry arguments to reduce the amount of calculations.
Hint 4: Use that

$$\int_0^h \left(1 - \frac{z}{h}\right)^n P(z)\,dz = \frac{h}{n+1}\left(P(0) + \int_0^h \left(1 - \frac{z}{h}\right)^{n+1} P'(z)\,dz\right), \tag{4}$$

where P is a derivable function. In this problem, P will be a polynomial, and the integral on the left side can be calculated recursively. For example,

$$\int_0^h \left(1 - \frac{z}{h}\right)^4 \left(1 - z^2 + z^3\right) dz = \frac{h}{5}\left(1 + \frac{h}{6}\left(0 + \frac{h}{7}\left(-2 + \frac{h}{8}(6 + 0)\right)\right)\right) = \frac{h}{5} - \frac{h^3}{105} + \frac{h^4}{280}. \tag{5}$$

> **Solution:**
> $$m = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} (ax + by + cz + d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} (cz + d)\,dx\,dy\,dz$$
> $$= 4rs \int_0^h \left(1 - \frac{z}{h}\right)^2 (cz + d)\,dz = 4rs\left(\frac{h}{3}\left(d + \frac{h}{4}(c + 0)\right)\right) = \frac{rsh}{3}(4d + ch).$$

(c) Find the center of mass of the pyramid, $\mathbf{r}_c^b = [x_c\,,\, y_c\,,\, z_c]^T$.
Show the details of your calculations.
Hint 1: See hints 1-4 given in part (b).
Hint 2: For $a = b$, $c = 0$ and $r = s$, the expression reduces to $\mathbf{r}_c^b = [\frac{r^2 a}{5d}\,,\, \frac{r^2 a}{5d}\,,\, \frac{h}{4}]^T$.

Solution:

$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} x(ax+by+cz+d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} ax^2\,dx\,dy\,dz$$
$$= \frac{4r^3sa}{3}\int_0^h \left(1-\frac{z}{h}\right)^4 dz = \frac{4r^3sah}{15}$$
$$\Rightarrow \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} y(ax+by+cz+d)\,dx\,dy\,dz = \frac{4rs^3bh}{15}$$
$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} z(ax+by+cz+d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} (cz^2+dz)\,dz$$
$$= 4rs\int_0^h \left(1-\frac{z}{h}\right)^2 (cz^2+dz)\,dz = 4rs\frac{h}{3}\left(0+\frac{h}{4}\left(d+\frac{h}{5}(2c+0)\right)\right) = \frac{rsh^2}{15}(5d+2ch).$$

Hence, the center of mass is

$$\mathbf{r}_c^b = [x_c\ y_c\ z_c]^T = \begin{bmatrix} \frac{4r^2a}{5(4d+ch)} & \frac{4s^2b}{5(4d+ch)} & \frac{h}{5}\frac{5d+2ch}{4d+ch} \end{bmatrix}^T.$$

(d) Find $\mathbf{M}_{b/o}^b$, i.e. find the inertia matrix about the origin, $o = [0\,,\,0\,,\,0]^T$.
Show the details of your calculations.
Hint 1: Read sections 7.3.4-7.3.7 in the book.
Hint 2: See hints 2-4 given in part (b).
Hint 3: For $a = b$, $c = 0$ and $r = s$, the expression reduces to

$$\mathbf{M}_{b/o}^b = \frac{2}{45}\begin{bmatrix} 3r^2h(2r^2+h^2)d & 0 & -r^4h^2a \\ 0 & 3r^2h(2r^2+h^2)d & -r^4h^2a \\ -r^4h^2a & -r^4h^2a & 12r^4hd \end{bmatrix}. \tag{6}$$

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Solution:

$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} x^2(ax+by+cz+d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} x^2(cz+d)\,dx\,dy\,dz$$
$$= \frac{4r^3 s}{3}\int_0^h \left(1-\frac{z}{h}\right)^4 (cz+d)\,dz = \frac{4r^3 s}{3}\frac{h}{5}\left(d+\frac{h}{6}(c+0)\right) = \frac{2r^3 sh}{45}(6d+ch)$$
$$\Rightarrow \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} y^2(ax+by+cz+d)\,dx\,dy\,dz = \frac{2rs^3 h}{45}(6d+ch)$$
$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} xy(ax+by+cz+d)\,dx\,dy\,dz = 0$$
$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} z^2(ax+by+cz+d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} (cz^3+dz^2)\,dx\,dy\,dz$$
$$= 4rs\int_0^h \left(1-\frac{z}{h}\right)^2 (cz^3+dz^2)\,dz = 4rs\frac{h}{3}\left(0+\frac{h}{4}\left(0+\frac{h}{5}\left(2d+\frac{h}{6}(6c+0)\right)\right)\right) = \frac{rsh^3}{15}(2d+ch)$$
$$\int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} xz(ax+by+cz+d)\,dx\,dy\,dz = \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} ax^2 z\,dx\,dy\,dz$$
$$= \frac{4r^3 sa}{3}\int_0^h \left(1-\frac{z}{h}\right)^4 z\,dz = \frac{4r^3 sa}{3}\frac{h}{5}\left(0+\frac{h}{6}(1+0)\right) = \frac{2}{45}r^3 sh^2 a$$
$$\Rightarrow \int_0^h \int_{-s(1-\frac{z}{h})}^{s(1-\frac{z}{h})} \int_{-r(1-\frac{z}{h})}^{r(1-\frac{z}{h})} yz(ax+by+cz+d)\,dx\,dy\,dz = \frac{2}{45}rs^3 h^2 b.$$

Hence, the inertia matrix about the origin is

$$\mathbf{M}_{b/o}^b = \frac{1}{45}\begin{bmatrix} 2rs^3h(6d+ch)+3rsh^3(2d+ch) & 0 & -2r^3sh^2a \\ 0 & 2r^3sh(6d+ch)+3rsh^3(2d+ch) & -2rs^3h^2b \\ -2r^3sh^2a & -2rs^3h^2b & 2rs(r^2+s^2)h(6d+ch) \end{bmatrix}.$$

(e) Assume that $h=r=s$, $a=b>0$, $c=0$ and $d=3ah$.
Prove that the inertia matrix about the center of mass is

$$\mathbf{M}_{b/c}^b = \frac{h^6 a}{900}\begin{bmatrix} 839 & 16 & 20 \\ 16 & 839 & 20 \\ 20 & 20 & 1408 \end{bmatrix}. \quad (7)$$

Show the details of your calculations.
Hint 1: Use the parallel axis theorem (section 7.4.7 in the book).
Hint 2: See hint 2 given in part (c) and hint 3 given in part (d).

Solution: For $h=r=s$, $a=b>0$, $c=0$ and $d=3ah$, we have that

$$m = 4h^4 a$$
$$\mathbf{r}_c^b = \begin{bmatrix} \frac{h}{15} & \frac{h}{15} & \frac{h}{4} \end{bmatrix}^T$$
$$\mathbf{M}_{b/o}^b = \frac{2}{45}\begin{bmatrix} 27h^6a & 0 & -h^6a \\ 0 & 27h^6a & -h^6a \\ -h^6a & -h^6a & 36h^6a \end{bmatrix}.$$

Hence,

$$
\begin{aligned}
\mathbf{M}^b_{b/c} &= \mathbf{M}^b_{b/o} + m \left(\mathbf{r}^b_c\right)^\times \left(\mathbf{r}^b_c\right)^\times \\
&= \frac{2}{45}\begin{bmatrix} 27 & 0 & -1 \\ 0 & 27 & -1 \\ -1 & -1 & 36 \end{bmatrix} h^6 a - \frac{1}{900}\begin{bmatrix} 241 & -16 & -60 \\ -16 & 241 & -60 \\ -60 & -60 & 32 \end{bmatrix} h^6 a \\
&= \frac{h^6 a}{900}\begin{bmatrix} 839 & 16 & 20 \\ 16 & 839 & 20 \\ 20 & 20 & 1408 \end{bmatrix}.
\end{aligned}
$$

Problem 2 (Kinematic and dynamic modeling of a model sounding rocket. 60 %)

In this problem, we will model the kinematics and dynamics of TTK4130's model sounding rocket, which goes under the name of ”Quick Lunch”.

This model rocket starts its journey attached to a launch ramp on the ground. Here the rocket motor is ignited and the generated thrust causes the rocket to move. The launch ramp stabilizes the rocket, while it wins velocity and momentum. After a short time, the rocket leaves the ramp. The motor will still generate thrust, but will eventually burn out. The rocket will then lose gradually speed and will flatten out. After reaching its apogee, a parachute is deployed.

The phases of the rocket's flight are illustrated in Figure (1). The models derived in this problem will be valid for the first three phases, which are launch, power flight and coasting.

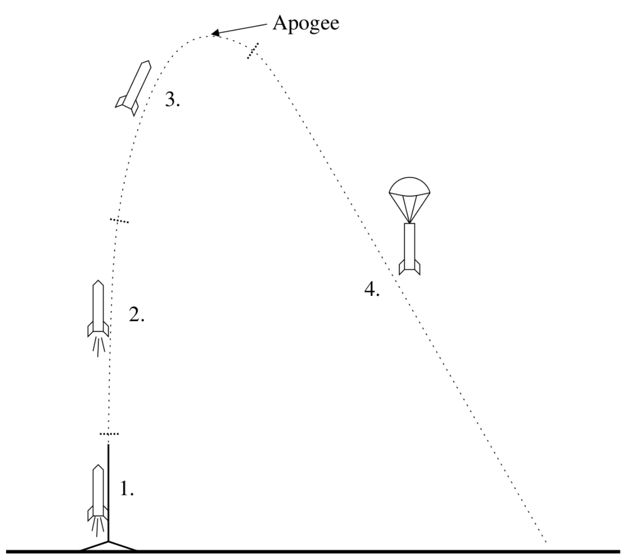


Figure 1: The basic phases of a typical model rocket flight: 1. Launch, 2. Powered flight, 3. Coasting and 4. Recovery. Source: OpenRocket documentation.

The forces acting on the rocket are the gravity **G**, the motor thrust **T**, the drag **D** and the normal force **N**, also known as lift. These forces are shown in Figure (2.a).

The gravity acts on the center of mass, while the motor thrust acts on the tail of the rocket along its longitudinal axis.

The drag and lift forces constitute the aerodynamical force, and are the most complicated part to model since they depend strongly and non-linearly on the air velocity and the angle of attack. The air velocity is the linear velocity of the rocket respect to the wind, while the angle of attack is the angle between the longitudinal axis of the rocket and the air velocity. Both the air velocity, v, and the angle attack, α, are shown in Figure (2.a-c). Furthermore, the drag and lift forces act on the center of pressure, which depends on the geometry of the rocket and on the angle of attack.

The center of mass, CG, and the center of pressure, CP, are show in Figures (2) and (3).

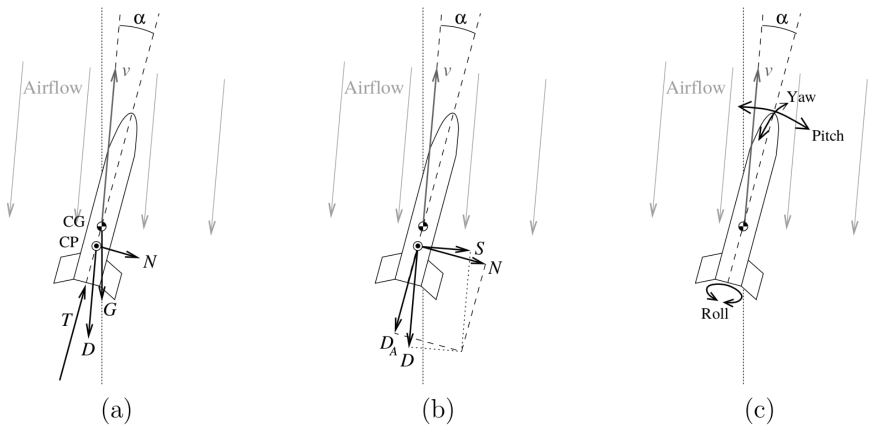


Figure 2: (a) Forces acting on a rocket in free flight: Gravity **G**, motor thrust **T**, drag **D** and normal force **N**. (b) Perpendicular component pairs of the total aerodynamical force: Normal force **N** and axial drag $\mathbf{D}_A$; side force **S** and drag **D**. (c) The pitch θ, yaw ψ and roll ϕ directions of a model rocket. Source: OpenRocket documentation.

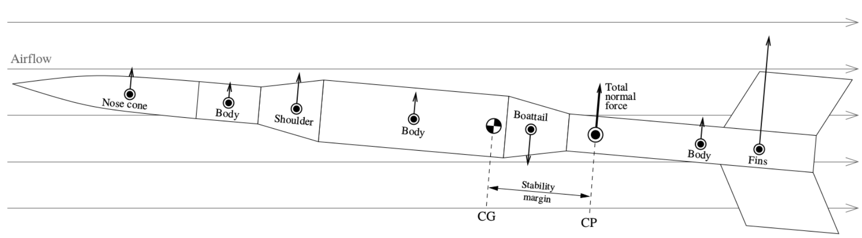


Figure 3: Normal forces produced by the rocket components. Source: OpenRocket documentation.

(a) **(Optional)** As one can see in Figures (2) and (3), the center of mass is closer to the nose of the rocket than the center of pressure. This is necessary in order to achieve a stable flight.

Explain why this is the case. A short physical explanation is more than sufficient, i.e. no formulas are needed. You may consider to add a sketch to illustrate your answer.

Hint: What effect do the force moments have on the angle of attack?

Solution: If the center of mass is closer to the nose than the center of pressure, the moments generated by the aerodynamic forces will by restoring moments (similar to a spring force), and will try to reduce the angle of attack to zero.

On the other hand, if the center of mass is closer to the tail than the center of pressure, the aerodynamic moments will destabilize the rocket, i.e. as soon as the angle of attack is non-zero, the moments will make it diverge. This is extremely dangerous because the rocket will spin about its center of mass as soon as it leaves the launch ramp.

The stability relation between the center of mass and center of pressure of a rocket is similar to the one between the center of mass and the center of buoyancy of a ship.

The position and orientation of the rocket will be specified with respect to the standard *NED*-frame, which is denoted by $\{n\}$. The x- and y-axes of the *NED*-frame point to the North and East, respectively, while the z-axis points down into the ground in order to complete the right-handed reference system. For convenience, we place the origin of the *NED*-frame at the base of the launch ramp, where the tail of the rocket is also located. Furthermore, we assume that the *NED*-frame is inertial, which is a reasonable assumption for small model rockets that neither fly very fast nor cover large distances.

The *NED*-frame is transformed to the local rocket frame $\{r\}$ by

1. A rotation by θ (pitch) degrees about the y-axis of the NED frame.
2. A rotation by ψ (yaw) degrees about the z-axis of the obtained frame.
3. A rotation by ϕ (roll) degrees about the x-axis of the obtained frame.
4. A translation along the x-axis of the obtained frame that moves the origin from the tail to the nose of the rocket.

Furthermore, the x-axis of the rocket frame is parallel to the longitudinal axis of the rocket, and its orientation is chosen such that the positive direction goes from the tail to the nose.

The Euler-angles $\Theta = [\theta\,,\,\psi\,,\,\phi]^T$ are depicted in Figure (2.c).

(b) Let L be the length of the rocket, i.e. the distance between its nose and its tail.
Find the homogeneous transformation matrix $\mathbf{T}_r^n = \mathbf{T}_r^n(\Theta, L)$.
Hint: See assignment 6 problem 1.h. Is $R_r^n = R_b^a$ or is $R_r^n = R_a^b$?

Solution:

$$\begin{aligned}\mathbf{R}_r^n &= \mathbf{R}_y(\theta)\mathbf{R}_z(\psi)\mathbf{R}_x(\phi) = \\ &= \begin{bmatrix} \cos\theta\cos\psi & \sin\theta\sin\phi - \cos\theta\sin\psi\cos\phi & \sin\theta\cos\phi + \cos\theta\sin\psi\sin\phi \\ \sin\psi & \cos\psi\cos\phi & -\cos\psi\sin\phi \\ -\sin\theta\cos\psi & \cos\theta\sin\phi + \sin\theta\sin\psi\cos\phi & \cos\theta\cos\phi - \sin\theta\sin\psi\sin\phi \end{bmatrix} \\ \mathbf{r}_{nr}^n &= \mathbf{R}_r^n L\mathbf{e}_1.\end{aligned}$$

Hence,

$$\mathbf{T}_r^n = \begin{bmatrix} \cos\theta\cos\psi & \sin\theta\sin\phi - \cos\theta\sin\psi\cos\phi & \sin\theta\cos\phi + \cos\theta\sin\psi\sin\phi & L\cos\theta\cos\psi \\ \sin\psi & \cos\psi\cos\phi & -\cos\psi\sin\phi & L\sin\psi \\ -\sin\theta\cos\psi & \cos\theta\sin\phi + \sin\theta\sin\psi\cos\phi & \cos\theta\cos\phi - \sin\theta\sin\psi\sin\phi & -L\sin\theta\cos\psi \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will now find the kinematic differential equations for the rocket.

(c) Show that

$$\dot{\mathbf{r}}^n = \mathbf{R}_r^n \mathbf{v}_{nr}^r\,, \tag{8}$$

where $\mathbf{r}^n$ is the position of the rocket nose in the *NED*-frame and $\mathbf{v}_{nr}^r$ is the velocity of the rocket frame respect to the *NED*-frame decomposed in the rocket frame.

Solution: Equation (8) follows directly from the observation $\dot{\mathbf{r}}^n = \mathbf{v}_{nr}^n$.

(d) Find the matrix $\mathbf{E}_r = \mathbf{E}_r(\Theta)$ such that

$$\mathbf{E}_r(\Theta)\dot{\Theta} = \omega_{nr}^r\,, \tag{9}$$

where ω_{nr}^r is the angular velocity of the rocket frame respect to the NED-frame decomposed in the rocket frame.

For which values of Θ is $\mathbf{E}_r(\Theta)$ non-singular? Find $\mathbf{E}_r(\Theta)^{-1}$.

Why is the y-z-x convention for the Euler-angles more convenient than the classical z-y-x convention for this particular application?

Hint 1: Read sections 6.8.4 and 6.9.4 in the book.

Hint 2: What are the singularities of the classical z-y-x convention and of the new y-z-x convention?

Solution: We define $\mathbf{R}_a^n = \mathbf{R}_y(\theta)$, $\mathbf{R}_b^a = \mathbf{R}_z(\psi)$ and $\mathbf{R}_r^b = \mathbf{R}_x(\phi)$. Hence,

$$\begin{aligned}
\omega_{nr}^r &= \omega_{na}^r + \omega_{ab}^r + \omega_{br}^r \\
&= \mathbf{R}_b^r \mathbf{R}_a^b \omega_{na}^a + \mathbf{R}_b^r \omega_{ab}^b + \omega_{br}^r \\
&= \mathbf{R}_x(-\phi)\mathbf{R}_z(-\psi) \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + \mathbf{R}_x(-\phi) \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \sin\psi \\ \cos\phi\cos\psi \\ -\sin\phi\cos\psi \end{bmatrix} \dot\theta + \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix} \dot\psi + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot\phi = \underbrace{\begin{bmatrix} \sin\psi & 0 & 1 \\ \cos\phi\cos\psi & \sin\phi & 0 \\ -\sin\phi\cos\psi & \cos\phi & 0 \end{bmatrix}}_{\mathbf{E}_r(\Theta)}.
\end{aligned}$$

It is immediate to verify that $\det \mathbf{E}_r(\Theta) = \cos\psi$. Hence, $\mathbf{E}_r(\Theta)$ is non-singular if and only if $\psi \neq \pm\frac{\pi}{2}$. In such case,

$$\mathbf{E}_r(\Theta)^{-1} = \frac{1}{\cos\psi} \begin{bmatrix} 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\cos\psi & \cos\phi\cos\psi \\ \cos\psi & -\cos\phi\sin\psi & \sin\phi\sin\psi \end{bmatrix}.$$

A pitch angle of around $\pm\frac{\pi}{2}$ is a lot more probable than a yaw angle of around $\pm\frac{\pi}{2}$. Therefore, it is convenient to use the y-z-x convention for this particular application.

Note that a roll angle of around $\pm\frac{\pi}{2}$ is very likely to happen.

We will now find the dynamic differential equations for the rocket.

Assume that the rocket is a rigid body and that it is symmetric about its longitudinal axis.

Let m be the mass of the rocket and let $\mathbf{M}_{r/o}^r$ be its moment of inertia about the origin of the r frame.

Furthermore, let $\mathbf{r}_g^r = [x_g(t)\,,\,0\,,\,0]^T$ and $\mathbf{r}_p^r = \mathbf{r}_p^r(\alpha)$ be the positions of the center of mass and center of pressure of the rocket, respectively.

(e) If T is the magnitude of the thrust, find the expression for $\mathbf{T}^r = \mathbf{T}^r(T)$.

Moreover, find the expression for $\mathbf{G}^r = \mathbf{G}^r(\Theta, m, g)$, where g is the acceleration of gravity.

Finally, show that

$$\begin{bmatrix} m\mathbf{I} & (m(\mathbf{r}_g^r)^\times)^T \\ (m(\mathbf{r}_g^r)^\times)^T & \mathbf{M}_{r/o}^r \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}}_{nr}^r \\ \dot\omega_{nr}^r \end{bmatrix} + \begin{bmatrix} m(\omega_{nr}^r)^\times((\omega_{nr}^r)^\times \mathbf{r}_g^r + \mathbf{v}_{nr}^r) \\ (\omega_{nr}^r)^\times \mathbf{M}_{r/o}^r \omega_{nr}^r + m(\mathbf{r}_g^r)^\times(\omega_{nr}^r)^\times \mathbf{v}_{nr}^r \end{bmatrix} = \begin{bmatrix} \mathbf{T}^r + \mathbf{G}^r + \mathbf{D}^r + \mathbf{N}^r \\ (\mathbf{r}_p^r)^\times(\mathbf{D}^r + \mathbf{N}^r) \end{bmatrix} \tag{10}$$

and write down the differential equations for $\mathbf{r}^n$, Θ, $\mathbf{v}_{nr}^r$ and ω_{nr}^r in matrix form.

How many differential equations and how many degrees of freedom are they?

Hint: Use the results from section 7.3.8 in the book.

Solution:

$$\mathbf{T}^r = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{G}^r = \mathbf{R}_n^r \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = mg \begin{bmatrix} -\sin\theta\cos\psi \\ \cos\theta\sin\phi + \sin\theta\sin\psi\cos\phi \\ \cos\theta\cos\phi - \sin\theta\sin\psi\sin\phi \end{bmatrix}.$$

Equation (10) follows directly from equations (7.97) and (7.98) in the book, and from the fact that the thrust and the gravity force do not generate a moment on the rocket.

There are 12 differential equations in total for 6 degrees of freedom:

$$\dot{\mathbf{r}}^n = \mathbf{R}_r^n \mathbf{v}_{nr}^r$$

$$\dot{\Theta} = \mathbf{E}_r(\Theta)^{-1}\omega_{nr}^r$$

$$\begin{bmatrix} \dot{\mathbf{v}}_{nr}^r \\ \dot{\omega}_{nr}^r \end{bmatrix} = \begin{bmatrix} m\mathbf{I} & (m(\mathbf{r}_g^r)^\times)^T \\ (m(\mathbf{r}_g^r)^\times)^T & \mathbf{M}_{r/o}^r \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{T}^r + \mathbf{G}^r + \mathbf{D}^r + \mathbf{N}^r - m(\omega_{nr}^r)^\times((\omega_{nr}^r)^\times \mathbf{r}_g^r + \mathbf{v}_{nr}^r) \\ (\mathbf{r}_p^r)^\times(\mathbf{D}^r + \mathbf{N}^r) - (\omega_{nr}^r)^\times \mathbf{M}_{r/o}^r \omega_{nr}^r + m(\mathbf{r}_g^r)^\times(\omega_{nr}^r)^\times \mathbf{v}_{nr}^r \end{bmatrix}.$$

The model we have derived is not accurate for the launch phase of the flight, since the launch ramp constrains the movement of the rocket.

(f) Explain how you would modify the model derived in parts (c)-(e) in order to take these constraints into consideration.

More precisely, explain how you would implement the transition from the launch phase to the powered flight phase (see Figure (1)): What kind of solver would you use? How would you detect the transition from one phase to the other?

Hint: Read section 14.8.2 and 14.8.3 in the book.

Solution: During the launch phase there is only one degree of freedom because the rocket can only move in the direction of the launch ramp.

Hence, $\Theta(t) = \Theta(0)$, $\omega_{nr}^r(t) = \mathbf{0}$ and $\mathbf{v}_{nr}^r(t)$ is parallel to the launch ramp direction, i.e.

$$(\mathbf{v}_{nr}^r(t))^T\mathbf{e}_2 = 0 \qquad \text{and} \qquad (\mathbf{v}_{nr}^r(t))^T\mathbf{e}_3 = 0.$$

These algebraic constraints together with the differential equations found in part (d) give rise to a differential algebraic equation.

Since the moment the rocket leaves the launch ramp is given by the position of the rocket, and not by a predetermined moment in time, an integration method with event-detection, such as a continuous Runge-Kutta method, has to be used in order to achieve a smooth transition between the launch and the powered flight phase.

Note that the rocket is still on the launch ramp if and only if

$$(\mathbf{r}^r(t))^T\mathbf{e}_1 \leq L_r + L\,,$$

where L_r is the distance between the nose of the rocket and the attachment to the launch ramp.

Hence, the event detection can be formulated as the following zero crossing problem

$$g(\mathbf{x}, t) = (\mathbf{r}^r(t))^T\mathbf{e}_1 - L_r - L = 0\,,$$

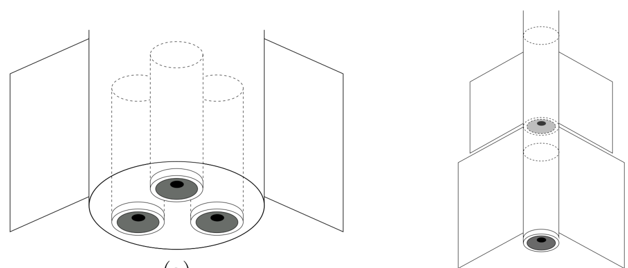
where $\mathbf{x}$ is the state of the rocket.

The rocket motor makes up a large part of the rocket's mass, and its propellant is consumed very rapidly during flight. Therefore a correct modeling of the mass, center of mass and moment of inertia of the rocket as a function of time is necessary to achieve a realistic simulation of the rocket trajectory.

There are many types of rocket motors as shown in Figure (4). In this particular case, the rocket is equipped with only one solid-propellant motor.

For simplicity, we assume that the solid-propellant is contained in a cylindrical chamber, and that after ignition, the propellant is consumed uniformly from the axis to the walls of this cylinder, i.e. the consumed propellant volume is also a cylinder with the same axis as the one of the chamber.

Assume that the axis of the propellant chamber is perfectly aligned with the longitudinal axis of the rocket, and that the bottom of the cylinder is located at the tail of the rocket.



(a) (b)

Figure 4: (a) A cluster rocket motor. (b) A two-staged rocket motor. Source: OpenRocket documentation.

(g) Let H and R be the height and radius of the propellant chamber, respectively.
Moreover, let $m_p = m_p(t)$ be the mass of the propellant, and let ρ be the propellant density, which is assumed constant.
Find the center of mass of the propellant $\mathbf{r}^r_{p,c} = \mathbf{r}^r_{p,c}(t)$, and show that the inertia matrix of the propellant about its center of mass is

$$\mathbf{M}^r_{p,r/c} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad , \quad \text{where} \tag{11a}$$

$$I_{xx} = m_p \left(R^2 - \frac{m_p}{2\pi\rho H} \right) \tag{11b}$$

$$I_{yy} = I_{zz} = m_p \left(\frac{H^2}{12} + \frac{R^2}{2} - \frac{m_p}{4\pi\rho H} \right). \tag{11c}$$

Finally, find the the inertia matrix of the propellant about the nose of the rocket, $\mathbf{M}^r_{p,r/0}$.

Hint 1: Use symmetry arguments and cylindrical coordinates.

Hint 2: Let let $r = r(t)$ be the radius of the cylinder that represents the consumed propellant volume, and express r as a function of H, R, m_p and ρ.

Hint 3: Use the parallel axis theorem.

Solution: Because of the symmetry of the propellant volume at all times, and due to the location of the propellant chamber, we conclude that

$$\mathbf{r}^r_{p,c} = \begin{bmatrix} -L + \frac{H}{2} \\ 0 \\ 0 \end{bmatrix}$$

and that the non-diagonal terms of $\mathbf{M}^r_{p,r/c}$ have to be zero, and that $I_{yy} = I_{zz}$.

Since $m = \pi\rho H(R^2 - r^2)$, it follows that $r = \sqrt{R^2 - \frac{m}{\pi\rho H}}$. Hence,

$$
\begin{aligned}
I_{xx} &= \iiint \rho(y^2 + z^2)\, dy\, dz\, dx = \iiint \rho r^3\, dr\, d\theta\, dx = \frac{\pi\rho H}{2}(R^4 - r^4) \\
&= \frac{\pi\rho H}{2}(R^2 - r^2)(R^2 + r^2) = m_p \left(R^2 - \frac{m_p}{2\pi\rho H} \right) \\
I_{yy} &= \iiint \rho \left(\left(x - \frac{H}{2} \right)^2 + z^2 \right) dy\, dz\, dx = \rho \iiint \left(x - \frac{H}{2} \right)^2 r\, dr\, d\theta\, dx + \rho \iiint r^3 \sin^2\theta\, dr\, d\theta\, dx \\
&= \rho \frac{1}{2}(R^2 - r^2) 2\pi \frac{H^3}{12} + \rho \frac{1}{4}(R^4 - r^4)\pi H = m_p \left(\frac{H^2}{12} + \frac{R^2}{2} - \frac{m_p}{4\pi\rho H} \right).
\end{aligned}
$$

Finally, we use the parallel axis theorem to find $\mathbf{M}^r_{p,r/0}$:

$$
\begin{aligned}
\mathbf{M}^r_{p,r/0} &= \mathbf{M}^r_{p,r/c} + m_p \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(L - \frac{H}{2}\right)^2 & 0 \\ 0 & 0 & \left(L - \frac{H}{2}\right)^2 \end{bmatrix} \\
&= m_p \begin{bmatrix} R^2 - \frac{m_p}{2\pi\rho H} & 0 & 0 \\ 0 & \frac{H^2}{12} + \frac{R^2}{2} - \frac{m_p}{4\pi\rho H} + \left(L - \frac{H}{2}\right)^2 & 0 \\ 0 & 0 & \frac{H^2}{12} + \frac{R^2}{2} - \frac{m_p}{4\pi\rho H} + \left(L - \frac{H}{2}\right)^2 \end{bmatrix}.
\end{aligned}
$$

Due to the complicated geometry of the rocket and the different densities of its parts, the mass, center of mass and inertia matrix is calculated using CAD programs or other software for the case when the rocket is fully loaded with propellant. Hence, these values are only valid for right before take-off, i.e. for $t = 0$.

We will now find the mass, center of mass and inertia matrix for the rocket during flight.

(h) Find the mass of the rocket $m(t)$ as a function of $m(0)$, $m_p(t)$ and $m_p(0)$.

Since the center of mass of the rocket and of the propellant have the form $\mathbf{r}^b_g = [x_g(t)\,,\,0\,,\,0]^T$ and $\mathbf{r}^b_{p,g} = [x_{p,g}(t)\,,\,0\,,\,0]^T$, respectively, it is sufficient to find $x_g(t)$.

Express $x_g(t)$ as a function of $x_g(0)$, $x_{p,g}(t)$, $x_{p,g}(0)$, $m(t)$, $m(0)$, $m_p(t)$ and $m_p(0)$.

Finally, express $\mathbf{M}^b_{b/o}(t)$ as a function of $\mathbf{M}^b_{b/o}(0)$, $\mathbf{M}^b_{p,b/o}(t)$ and $\mathbf{M}^b_{p,b/o}(0)$.

Hint: The integrals over the volume of the rocket without the volume of the propellant chamber do not change with time. In symbols,

$$
\iiint_{V_r(t)} [\cdot]\, dV - \iiint_{V_p(t)} [\cdot]\, dV = \iiint_{V_r(0)} [\cdot]\, dV - \iiint_{V_p(0)} [\cdot]\, dV\,, \tag{12}
$$

where $V_r(t)$ and $V_p(t)$ are the volume of the rocket and of the propellant at time t, respectively.

Solution: Since $m(t) - m_p(t) = m(0) - m_p(0)$, then

$$
m(t) = m(0) + m_p(t) - m_p(0).
$$

For the x-coordinate of the center of mass, we have that

$$\begin{aligned}
&\iiint_{V_r(t)} x\rho\, dV - \iiint_{V_p(t)} x\rho\, dV = \iiint_{V_r(0)} x\rho\, dV - \iiint_{V_p(0)} x\rho\, dV \\
&m(t)x_g(t) - m_p(t)x_{p,g}(t) = m(0)x_g(0) - m_p(0)x_{p,g}(0) \\
&x_g(t) = \frac{m(0)x_g(0) + m_p(t)x_{p,g}(t) - m_p(0)x_{p,g}(0)}{m(t)}.
\end{aligned}$$

Finally, for the inertia matrix, we exploit the fact that the origin of the rocket frame does not change with time. Hence, we have that

$$\begin{aligned}
&\iiint_{V_r(t)} - \left(\mathbf{r}^b\right)^\times \left(\mathbf{r}^b\right)^\times dV - \iiint_{V_p(t)} - \left(\mathbf{r}^b\right)^\times \left(\mathbf{r}^b\right)^\times dV \\
&= \iiint_{V_r(0)} - \left(\mathbf{r}^b\right)^\times \left(\mathbf{r}^b\right)^\times dV - \iiint_{V_p(0)} - \left(\mathbf{r}^b\right)^\times \left(\mathbf{r}^b\right)^\times dV \\
&\mathbf{M}^b_{b/o}(t) - \mathbf{M}^b_{p,b/o}(t) = \mathbf{M}^b_{b/o}(0) - \mathbf{M}^b_{p,b/o}(0) \\
&\mathbf{M}^b_{b/o}(t) = \mathbf{M}^b_{b/o}(0) + \mathbf{M}^b_{p,b/o}(t) - \mathbf{M}^b_{p,b/o}(0).
\end{aligned}$$