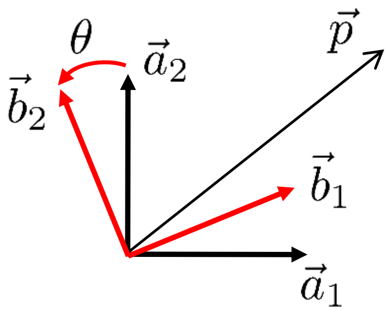
Kahoot

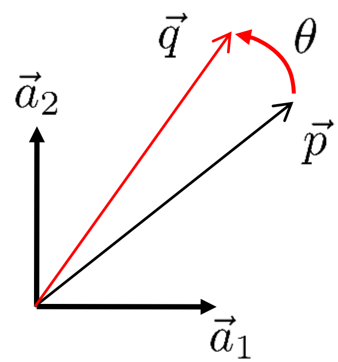
- https://play.kahoot.it/#/k/8c1f768d-76cf-40e4-8163-ea279354e62a

Rotation vs transformation (same, again)

- A coordinate vector may change either as a result of a rotation of a coordinate system (a **coordinate transformation**) or a rotation of the vector itself (a **rotation**).

- That is, a rotation from *a* to *b* can be interpreted in two ways:





$\mathbf{p}^b = \mathbf{R}_a^b \mathbf{p}^a$ (or $\mathbf{p}^a = \mathbf{R}_b^a \mathbf{p}^b$)

$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$ such that $\mathbf{q}^b = \mathbf{p}^a$

- That is, the matrix $\mathbf{R}_b^a$ rotates from *a* to *b*, but transforms from *b* to *a*!

- (Sometimes these two interpretations of the rotations originating from a rotation matrix are called passive vs active transformations, or alias vs alibi transformations)

Lecture 12: Rigid body kinematics – Rotations, angular velocity

Representations of rotation

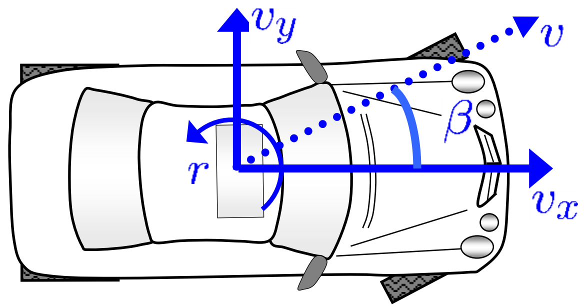
- Rotation matrices
- Euler angles
- 3-parameter specification of rotations
  - Roll-pitch-yaw
- Angle-axis, Euler-parameters
  - 4-parameter specification of rotations

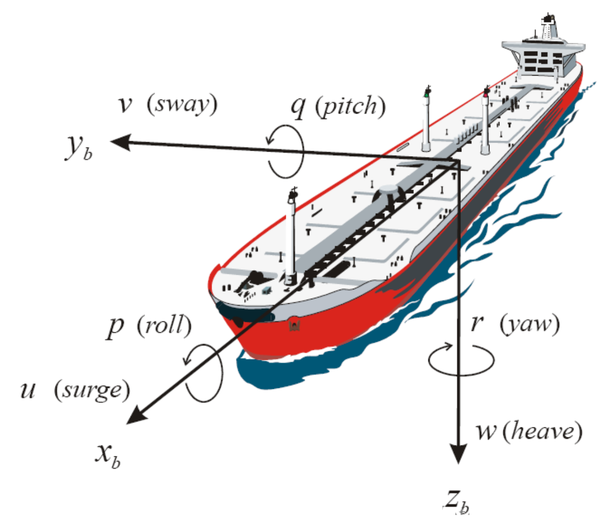
- Angular velocity

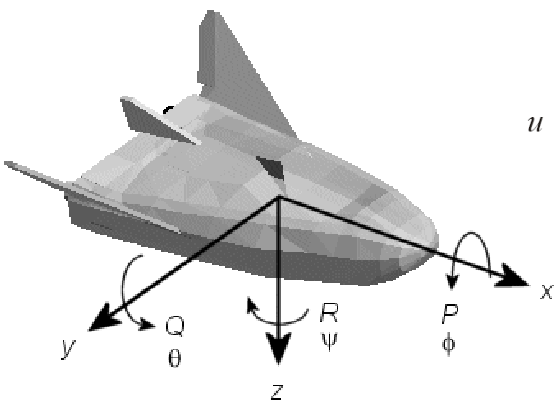
Book: Ch. 6.6, 6.7, 6.8

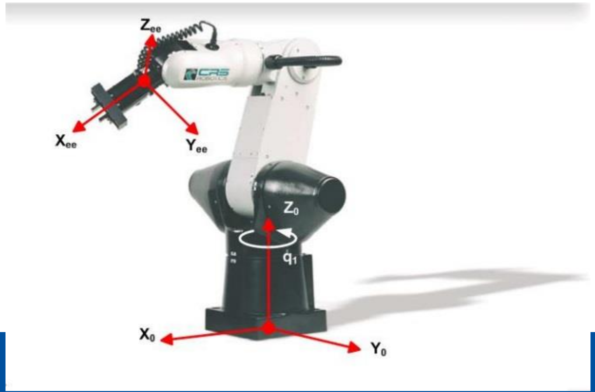
Why rotation matrices?

- Rotation matrices are used to describe rotations and orientations of rigid bodies
  - Road vehicles
  - Marine vessels
  - Airplanes, satellites
  - Robotics







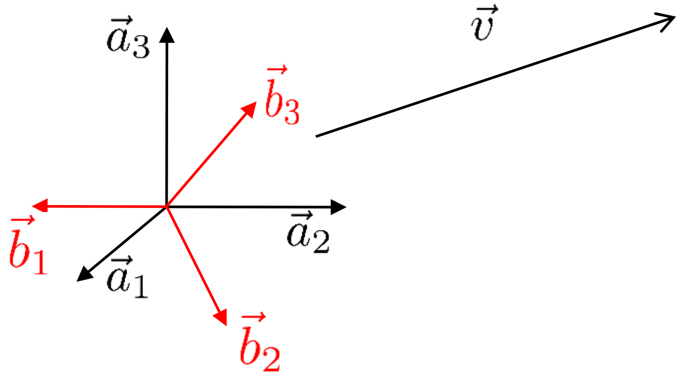


Rotation matrices

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- Transform a coordinate vector from *b* to *a*

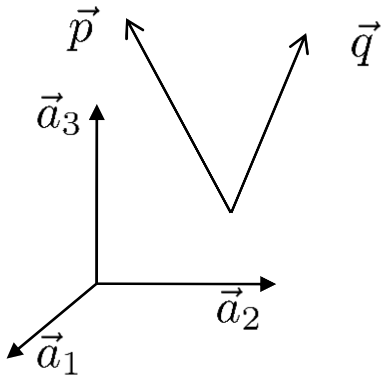
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- Rotate a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

such that $\mathbf{q}^b = \mathbf{p}^a$.



Representations of rotations

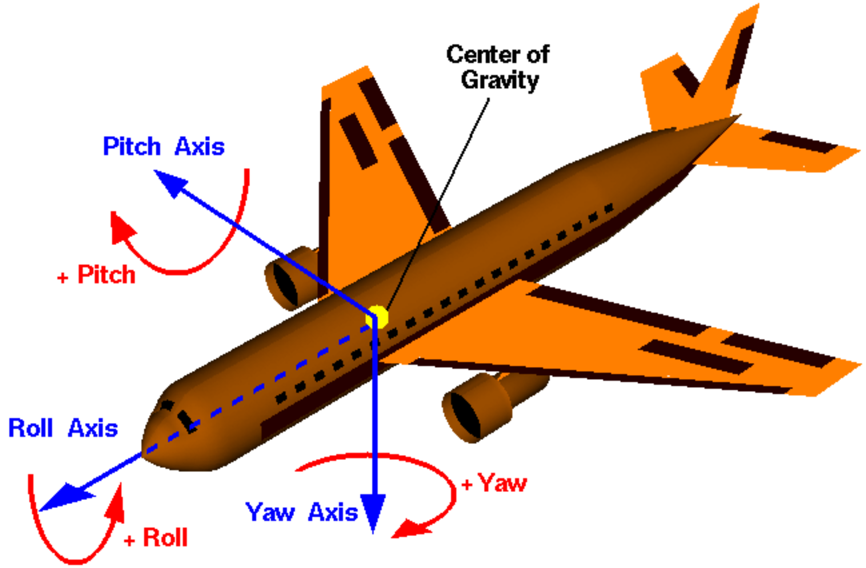
- Rotation matrix
  - Simple, but over-parameterized (9 parameters)

Euler's Theorem:
"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common: Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this later)

- Angle-axis, Euler-parameters
  - 4-parameters are used
  - No singularity problems

Euler-angles: Roll-pitch-yaw



- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

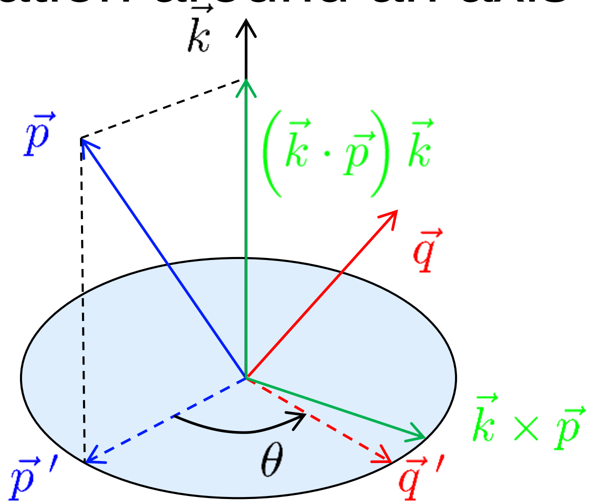
$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Rotation of vectors based on angle-axis representation

- Angle-axis: All rotations can be represented as a simple rotation around an axis

Somewhat different derivation of the rotation dyadic. Compare p. 228 in book.



$$\vec{p}\,' = \vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \vec{q} - \left(\vec{k}\cdot\vec{q}\right)\vec{k} = \vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \cos\theta\,\vec{p}\,' + \sin\theta\,\vec{k}\times\vec{p}$$

$$\vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k} = \cos\theta\,\left(\vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}\right) + \sin\theta\,\vec{k}\times\vec{p}$$

$$\boxed{\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}}$$

Angle-axis rotation dyadic, rotation matrix

- Rotation θ about an axis $\vec{k}$

$$\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\,\vec{k}\left(\vec{k}\cdot\vec{p}\right)$$

- Angle-axis rotation by a dyadic

$$\vec{q} = \Big(\underbrace{\cos\theta\,\vec{I} + \sin\theta\,\vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}}\Big)\cdot\vec{p}$$

$$\vec{q} = \vec{R}_{\vec{k},\theta}\cdot\vec{p}$$

- Angle-axis rotation matrix

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\,\mathbf{I} + \sin\theta\,(\mathbf{k}^a)^{\times} + (1-\cos\theta)\,\mathbf{k}^a(\mathbf{k}^a)^{\mathsf{T}}}$$

- Alternative expression (using $\mathbf{k}^a = \mathbf{k}$ and $\mathbf{k}^{\times}\mathbf{k}^{\times} = \mathbf{k}(\mathbf{k})^{\mathsf{T}} - \mathbf{I}$):

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{I} + \sin\theta\,\mathbf{k}^{\times} + (1-\cos\theta)\,\mathbf{k}^{\times}\mathbf{k}^{\times}}$$

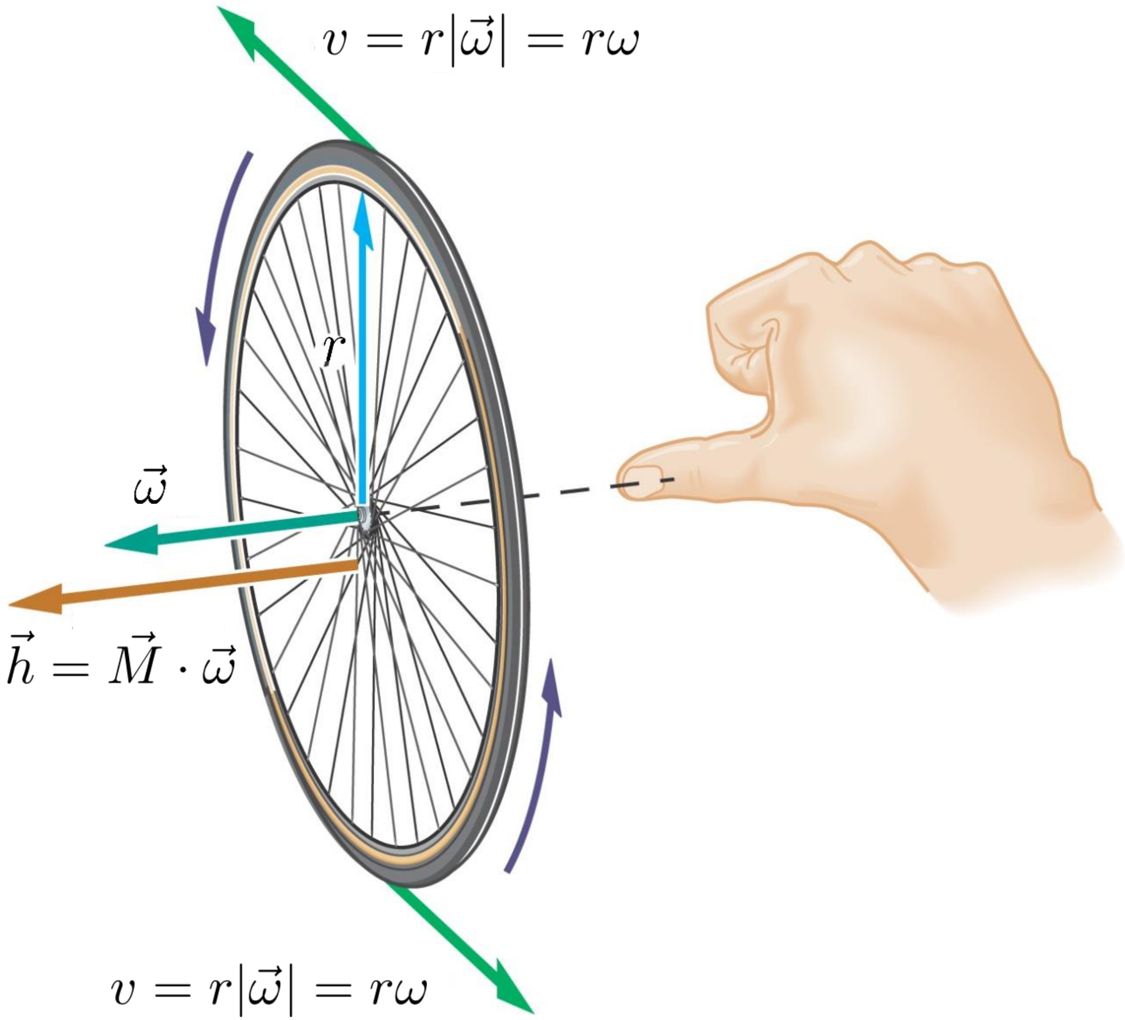
Use of Euler parameters

- ABB robots use Euler parameters (quaternions) internally in the robot control program
  - and Euler angles “externally”

- In Modelica.multibody, one can use either rotation matrices or Euler parameters (quaternions)

- Euler parameters (quaternions) often used in “advanced control” of robots, satellites, etc.

Angular velocity



Kinematic differential equations

- Translation: $\underline{v} \rightarrow \underline{r}$: $$\dot{\underline{r}} = \underline{v}$$

- Rotation: $\underline{\omega}_{ab}^{a} \rightarrow \mathbf{R}_{b}^{a}$: $$\dot{\mathbf{R}}_{b}^{a} = ?$$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler angle} \qquad \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler parameter} \qquad \begin{aligned} \dot{\eta} &= ? \\ \dot{\underline{\varepsilon}} &= ? \end{aligned}$$

Definition angular velocity I

R_b^a: orthogonal → $R_b^a (R_b^a)^T =$

Definition angular velocity II

Angular velocity for simple rotation I

$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$[\underline{\omega}_x(\dot{\varphi})]^\times = \dot{\mathbf{R}}_x(\varphi)\mathbf{R}_x(\varphi)^T$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\varphi & -\cos\varphi \\ 0 & \cos\varphi & -\sin\varphi \end{pmatrix} \dot{\varphi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \dot{\varphi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\varphi} \\ 0 & \dot{\varphi} & 0 \end{pmatrix}$$

Angular velocity for simple rotation II

For angle-axis parameterisation

$$\mathbf{R}_b^a = \mathbf{R}_{k,\theta} = \mathbf{I} + \underline{k}^\times \sin\theta + \underline{k}^\times \underline{k}^\times (1 - \cos\theta)$$

- Assume $\underline{k}$ is constant:

$$(\underline{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^T$$

$$= \dot{\theta} \left(\underline{k}^\times \cos\theta + \underline{k}^\times \underline{k}^\times \sin\theta \right)$$

$$\left(\mathbf{I} - \underline{k}^\times \sin\theta + \underline{k}^\times \underline{k}^\times (1 - \cos\theta) \right)$$

$$[\text{use:} \underline{k}^\times \underline{k}^\times \underline{k}^\times = \underline{k}^\times (\underline{k}\underline{k}^T - \underline{k}^T \underline{k}\mathbf{I}) = -\underline{k}^\times]$$

$$= \ldots$$

$$= \dot{\theta} \underline{k}^\times$$

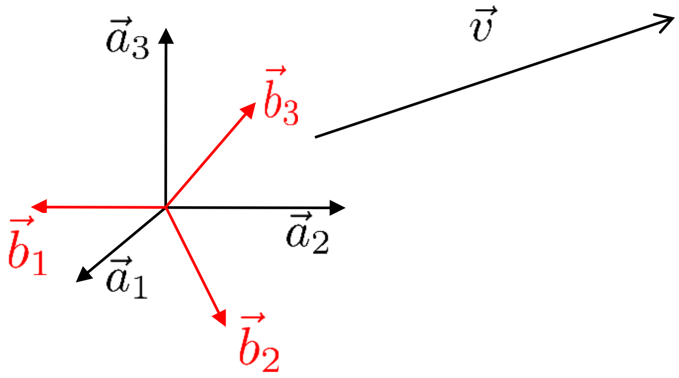
$$\underline{\omega}_{ab}^a = \dot{\theta} \underline{k}$$

$$\vec{\omega}_{ab} = \dot{\theta} \vec{k}$$

Composite rotations

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$$

Differentiation of coordinate vector



$$\underline{u}^a = \begin{pmatrix} u_1^a \\ u_2^a \\ u_3^a \end{pmatrix} \qquad \underline{u}^b = \begin{pmatrix} u_1^b \\ u_2^b \\ u_3^b \end{pmatrix}$$

Differentiation of coordinate-free vector

$$\frac{d}{dt}\vec{u} = ?$$

Kinematic differential equations

- Translation: $\underline{v} \to \underline{r}$: $\qquad \dot{\underline{r}} = \underline{v}$

- Rotation: $\underline{\omega}_{ab}^{a} \to \mathbf{R}_b^a$: $\qquad \dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^{a})^{\times}\mathbf{R}_b^a$

$$\underline{\omega}_{ab}^{a} \to \text{Euler angle} \qquad \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

$$\underline{\omega}_{ab}^{a} \to \text{Euler parameter} \qquad \begin{aligned} \dot{\eta} &= ? \\ \dot{\underline{\varepsilon}} &= ? \end{aligned}$$

Kinematic differential equation of Euler angles I

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

- If $\theta = 90^o$:
  - $\vec{a}_3$ is parallel to $\vec{c}_1$ (ψ and ϕ have the same axis)
  - Angular velocity components along $\vec{a}_3 \times \vec{b}_2$ (evtl. $\vec{c}_1 \times \vec{b}_2$) cannot be described
  - Singularity of the Euler angles

Kinematic differential equation of Euler angles II

Kinematic differential equation of Euler angles III

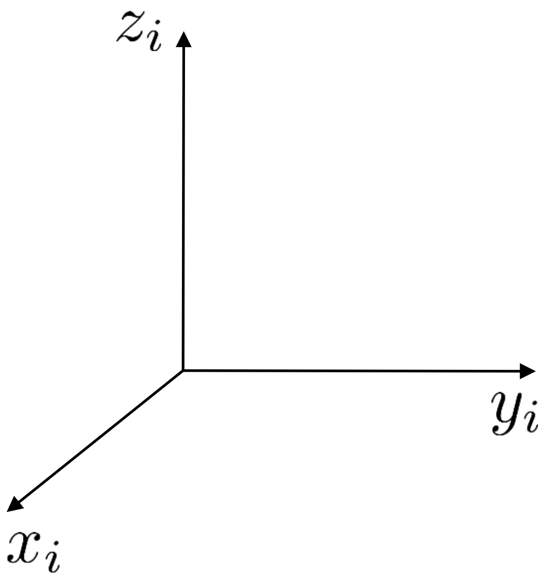
Kinematic differential equation of Euler parameter

$$\mathbf{R}_b^a = \mathbf{R}(\eta, \underline{\varepsilon}) \qquad \dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^a)^\times \mathbf{R}_b^a$$

- It can be derived (quaternion algebra p. 248)

$$\dot{\eta} = -\frac{1}{2}\underline{\varepsilon}^T \underline{\omega}_{ab}^a$$
$$\dot{\underline{\varepsilon}} = \frac{1}{2}(\eta \mathbf{I} - \underline{\varepsilon}^\times)\underline{\omega}_{ab}^a$$

Kinematics of rigid body I

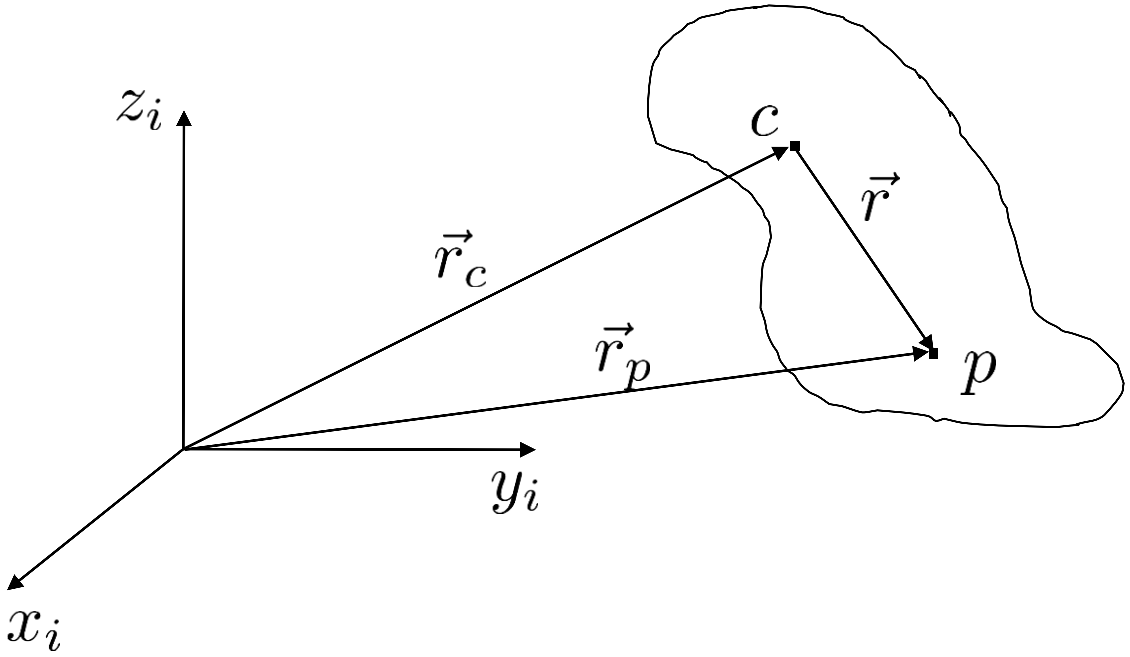


Kinematics of rigid body II

Kinematics of rigid body III

$$\vec{a}_c = \vec{a}_o + \frac{{}^b d^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r}_g + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)$$

Center of mass



What is rigid body dynamics?

- Rigid body:
  - Wikipedia: "…a rigid body is an idealization of a solid body of finite size in which deformation is neglected."

- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: "…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions)."
  - Book: Ch. 6
- Kinetics
  - eb.com: "…the effect of forces and torques on the motion of bodies having mass."
  - Book: Ch. 7, 8.

Remark: Sometimes "dynamics" is used for "kinetics" only

Homework

- Derive $[\omega_y(\dot{\theta})]^\times$ and $[\omega_z(\dot{\psi})]^\times$ from $R_y(\theta)$ and $R_z(\psi)$, respectively.
- Derive w_{ad}^b for the Euler angles using the roll-pitch-yaw case (check 6.9.4). *Think good about the order and direction of transformations.*
- Read 6.12
- Read 7.1-7.2