Department of Engineering Cybernetics

Examination paper for TTK4135 Optimization and Control

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skal ha flervalgskjema ☐

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Date Signature

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Exam in TTK4135 Optimization and Control

Monday, May 28th 2018

09:00 – 13:00

Permitted aids (code D): No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (25 %)

Consider the following optimization problem:

$$\begin{aligned} \min_{x} \quad & 4x_1^2 - x_1x_2 + 6x_2^2 + 2x_3^2 + 10x_1 - 2x_2 - 5x_3 \\ \text{subject to} \quad & x_2 + x_3 = 2 \\ & x_2 = \theta \end{aligned}$$

where θ is an unknown constant.

(5 %) (a) Is this is a convex optimization problem? Give reason for your answer.

(2 %) (b) What type of optimization problem is this?

(8 %) (c) Write down the KKT conditions for this problem. The answer should contain numbers.

(6 %) (d) What is the solution x^* to the optimization problem as a function of θ?

(4 %) (e) What are the optimal Lagrangian multipliers λ^* for the constraints, as a function of θ?

Problem 2 (20 %)

Emma is starting a micro brewery, and needs to plan her efforts.

- She is able to produce and sell maximum 100 boxes of beer each day.
- She can work up to 14 hours per day.
- It takes her 1 hour to produce 10 boxes of light beer.
- It takes her 2 hours to produce 10 boxes of dark beer.
- She earns 20 Euros for one box of light beer.
- She earns 30 Euros for one box of dark beer.

She wants to mazimize her profits.

(10 %) (a) Formulate the optimization problem. What type of optimization problem is this?

(10 %) (b) Draw/sketch the feasible region, and the contours of the objective function. What is the optimal solution? It is enough to indicate the optimal solution in the figure, you need not find the exact numbers.

Problem 3 (26 %)

Consider the optimization problem

$$\min_{z \in \mathbb{R}^n} \sum_{j=1}^{n} c_j |z_j| \quad \text{subject to} \quad Az \geq b \tag{1}$$

where $|\cdot|$ is the absolute value, and all c_j are positive.

(4 %) (a) Do the KKT conditions apply to this optimization problem? Explain.

The optimization problem

$$\min_{z^+, z^- \in \mathbb{R}^n} \sum_{j=1}^{n} c_j^\top (z_j^+ + z_j^-) \quad \text{subject to} \quad A(z_j^+ - z_j^-) \geq b, \quad z_j^+ \geq 0, \quad z_j^- \geq 0 \tag{2}$$

is equivalent to (1).

(10 %) (b) Explain ("prove") why these two optimization problems are equivalent. Hint: Let $z_j = z_j^+ - z_j^-$ and $|z_j| = z_j^+ + z_j^-$.

(4 %) (c) Do the KKT conditions apply to (2)? What type of optimization problem is this?

(8 %) (d) Use the above to formulate the open loop dynamic optimization problem

$$\begin{aligned}
\min_{u_0, u_1, x_1, x_2} \quad & \sum_{i=0}^{1} q_i |x_{i+1}| + r_i |u_i| \\
\text{subject to} \quad & x_{i+1} = 1.2 x_i + u_i, \quad i = 0, 1 \\
& |x_i| \leq 1, \quad i = 1, 2 \\
& |u_i| \leq 1, \quad i = 0, 1
\end{aligned}$$

as a linear program (LP). Here, q_i and r_i are positive constants.

Problem 4 (15 %)

Given the nonlinear programming problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) = 0, \tag{3}$$

where $c(x)^\top = [c_1(x), c_2(x), \ldots, c_m(x)]$. Define

$$F(x, \lambda) = \begin{bmatrix} \nabla f(x) - A(x)^\top \lambda \\ c(x) \end{bmatrix} = 0, \tag{4}$$

where $A(x) = \begin{bmatrix} \nabla c_1(x) & \nabla c_2(x) & \ldots & \nabla c_m(x) \end{bmatrix}^\top$.

(10 %) (a) Formulate Newton's method, in simple pseude-code, for solving $F(x, \lambda) = 0$.

(5 %) (b) Explain why, and to what extent, the algorithm in (a) solves (3) (can be answered also if you did not answer (a)).

Problem 5 (14 %)

The optimization problem for infinite horizon linear quadratic control of discrete dynamic systems is given by

$$\min_{z} f^{\infty}(z) = \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^{\top} Q x_{t+1} + \frac{1}{2} u_t^{\top} R u_t \tag{5a}$$

subject to

$$x_{t+1} = Ax_t + Bu_t \tag{5b}$$

$$x_0 = \text{given} \tag{5c}$$

where

$$u_t \in \mathbb{R}^{n_u} \tag{5d}$$

$$x_t \in \mathbb{R}^{n_x} \tag{5e}$$

$$z^{\top} = (u_0^{\top}, x_1^{\top}, u_1^{\top}, x_2^{\top}, \ldots) \tag{5f}$$

(8 %) (a) Assume (A, B) stabilizable. Write down the equations defining the infinite horizon LQ controller. Hint: Theorem 2 in the back may be useful.

(6 %) (b) Draw a block diagram of the output feedback LQ controller (the LQG controller).

Appendix

Part 1 Optimization Problems and Optimality Conditions

A general formulation for constrained optimization problems is

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{A1a}$$
$$\text{s.t.} \quad c_i(x) = 0, \qquad i \in \mathcal{E} \tag{A1b}$$
$$c_i(x) \geq 0, \qquad i \in \mathcal{I} \tag{A1c}$$

where f and the functions c_i are all smooth, differentiable, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices.

The Lagrangian function for the general problem (A1) is

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \tag{A2}$$

The KKT-conditions for (A1) are given by:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \tag{A3a}$$
$$c_i(x^*) = 0, \qquad i \in \mathcal{E} \tag{A3b}$$
$$c_i(x^*) \geq 0, \qquad i \in \mathcal{I} \tag{A3c}$$
$$\lambda_i^* \geq 0, \qquad i \in \mathcal{I} \tag{A3d}$$
$$\lambda_i^* c_i(x^*) = 0, \qquad i \in \mathcal{E} \cup \mathcal{I} \tag{A3e}$$

2nd order (sufficient) conditions for (A1) are given by:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^\top w \geq 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases} \tag{A4}$$

Theorem 1: (Second-Order Sufficient Conditions) *Suppose that for some feasible point* $x^* \in \mathbb{R}^n$ *there is a Lagrange multiplier vector* λ^* *such that the KKT conditions* (A3) *are satisfied. Suppose also that*

$$w^\top \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0. \tag{A5}$$

Then x^* *is a strict local solution for* (A1).

LP problem in standard form:

$$\min_{x} \quad f(x) = c^\top x \tag{A6a}$$

$$\text{s.t.} \quad Ax = b \tag{A6b}$$

$$x \geq 0 \tag{A6c}$$

where $A \in \mathbb{R}^{m \times n}$ and $\operatorname{rank} A = m$.

QP problem in standard form:

$$\min_{x} \quad f(x) = \frac{1}{2} x^\top G x + x^\top c \tag{A7a}$$

$$\text{s.t.} \quad a_i^\top x = b_i, \qquad i \in \mathcal{E} \tag{A7b}$$

$$a_i^\top x \geq b_i, \qquad i \in \mathcal{I} \tag{A7c}$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m \times n}$.

Iterative method:

$$x_{k+1} = x_k + \alpha_k p_k \tag{A8a}$$

$$x_0 \text{ given} \tag{A8b}$$

$$x_k, p_k \in \mathbb{R}^n, \;\; \alpha_k \in \mathbb{R} \tag{A8c}$$

p_k is the search direction and α_k is the line search parameter.

Part 2 Optimal Control

A typical open-loop optimal control problem on the time horizon 0 to N is

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{xt+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{ut} u_t \tag{A9a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, && t = 0, \dots, N-1 && \text{(A9b)} \\
x_0 &= \text{given} && && \text{(A9c)} \\
x^{\text{low}} &\le x_t \le x^{\text{high}}, && t = 1, \dots, N && \text{(A9d)} \\
u^{\text{low}} &\le u_t \le u^{\text{high}}, && t = 0, \dots, N-1 && \text{(A9e)} \\
-\Delta u^{\text{high}} &\le \Delta u_t \le \Delta u^{\text{high}}, && t = 0, \dots, N-1 && \text{(A9f)} \\
Q_t &\succeq 0 && t = 1, \dots, N && \text{(A9g)} \\
R_t &\succeq 0 && t = 0, \dots, N-1 && \text{(A9h)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A9i)} \\
x_t &\in \mathbb{R}^{n_x} && \text{(A9j)} \\
\Delta u_t &= u_t - u_{t-1} && \text{(A9k)} \\
z^\top &= (x_1^\top, \dots, x_N^\top, u_0^\top, \dots, u_{N-1}^\top) && \text{(A9l)}
\end{aligned}$$

The subscript t denotes discrete time sampling instants.

The optimization problem for linear quadratic control of discrete dynamic systems is given by

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t \tag{A10a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t && \text{(A10b)} \\
x_0 &= \text{given} && \text{(A10c)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A10d)} \\
x_t &\in \mathbb{R}^{n_x} && \text{(A10e)} \\
z^\top &= (x_1^\top, \dots, x_N^\top, u_0^\top \dots, u_{N-1}^\top) && \text{(A10f)}
\end{aligned}$$

Theorem 2: *The solution of (A10) with $Q_t \succeq 0$ and $R_t \succ 0$ is given by*

$$u_t = -K_t x_t \tag{A11a}$$

where the feedback gain matrix is derived by

$$\begin{aligned}
K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 && \text{(A11b)} \\
P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 && \text{(A11c)} \\
P_N &= Q_N && && \text{(A11d)}
\end{aligned}$$

Algorithm 18.3 (Line Search SQP Algorithm).

Choose parameters $\eta \in (0, 0.5)$, $\tau \in (0, 1)$, and an initial pair (x_0, λ_0);
Evaluate $f_0, \nabla f_0, c_0, A_0$;
If a quasi-Newton approximation is used, choose an initial $n \times n$ symmetric positive definite Hessian approximation B_0, otherwise compute $\nabla^2_{xx}\mathcal{L}_0$;
repeat until a convergence test is satisfied
  Compute p_k by solving (18.11); let $\hat{\lambda}$ be the corresponding multiplier;
  Set $p_\lambda \leftarrow \hat{\lambda} - \lambda_k$;
  Choose μ_k to satisfy (18.36) with $\sigma = 1$;
  Set $\alpha_k \leftarrow 1$;
  while $\phi_1(x_k + \alpha_k p_k; \mu_k) > \phi_1(x_k; \mu_k) + \eta \alpha_k D_1(\phi(x_k; \mu_k) p_k)$
    Reset $\alpha_k \leftarrow \tau_\alpha \alpha_k$ for some $\tau_\alpha \in (0, \tau]$;
  end (while)
  Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$ and $\lambda_{k+1} \leftarrow \lambda_k + \alpha_k p_\lambda$;
  Evaluate $f_{k+1}, \nabla f_{k+1}, c_{k+1}, A_{k+1}$, (and possibly $\nabla^2_{xx}\mathcal{L}_{k+1}$);
  If a quasi-Newton approximation is used, set
    $s_k \leftarrow \alpha_k p_k$ and $y_k \leftarrow \nabla_x \mathcal{L}(x_{k+1}, \lambda_{k+1}) - \nabla_x \mathcal{L}(x_k, \lambda_{k+1})$,
  and obtain B_{k+1} by updating B_k using a quasi-Newton formula;
end (repeat)