NTNU – Trondheim
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Assignment 8

TTK4130 Modeling and Simulation

Problem 1 (Sliding stick, generalized coordinates, Lagrange's equation. 30 %)

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

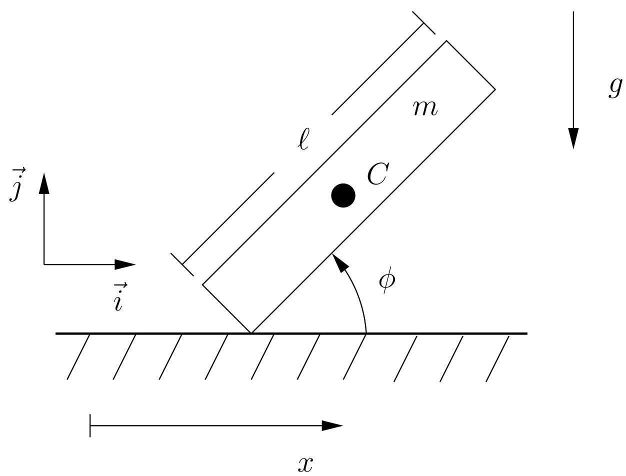

Figure 1: Stick sliding on frictionless surface

(a) Choose appropriate generalized coordinates (the figure should give you some hints).
What are the corresponding generalized (actuator) forces?
Hint: Read section 7.7 in the book.

(b) What are the position, velocity, and angular velocity of the center of mass as function of the chosen generalized coordinates and their derivatives?
Hint: Read section 6.12 in the book.

(c) Express the kinetic and potential energy of the stick as function of the chosen generalized coordinates and their derivatives.
Hint: Read section 8.2 in the book.

(d) Derive the equations of motion for the stick using Lagrange's equation.
Show the details of your calculations.
Hint: Read section 8.2 in the book.

Problem 2 (Robotic manipulator, generalized coordinates, Lagrange's equation, Christoffel symbols. 35 %)
We wish to model a robotic manipulator with the configuration shown in Figure 2.

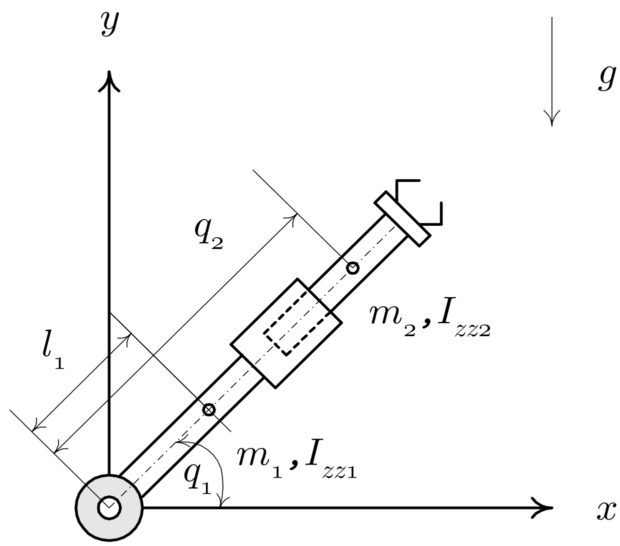

Figure 2: Manipulator

The manipulator has two degrees of freedom, which are represented by the generalized coordinates q_1 and q_2. We will use Lagrange's equation:

$$\frac{d}{d\,t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2 \tag{1}$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy}. \tag{2}$$

Assume that the axes x and y are fixed, i.e. they are the axes of an inertial reference frame. Moreover, the mass and the inertia of the motors are assumed to be neglectable.

The moment of inertia of the first arm is denoted by I_{zz1}, while the moment of inertia of the second arm is denoted by I_{zz2}. Each moment of inertia is referenced to the center of mass of their respective arm. The dots in Figure 2 mark the centers of mass of each arm.

Finally, the arrow marked g illustrates the direction of the acceleration of gravity.

(a) Find the total kinetic energy T of the manipulator, and show that it can be written in the form $T = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$, where $\mathbf{q} = [q_1\,,\, q_2]^T$ and

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{3}$$

Show the details of your calculations.
Hint: Read section 8.2.8 in the book.

(b) Find the potential energy U of the manipulator.
Hint: Read section 8.2 in the book.

(c) Derive the equations of motion for the manipulator using Lagrange's equation.
Show the details of your calculations.
Hint: Read section 8.2 in the book.

(d) Show that the equations of motion found in part (c) can be written as

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}. \tag{4}$$

Explain why several choices are possible for $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$.
Moreover, show that the Christoffel symbol representation of $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{bmatrix}. \tag{5}$$

Finally, find the vector $\mathbf{g}(\mathbf{q})$.
Show the details of your calculations.
Hint: Read section 8.2.8 in the book.

(e) Determine if the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ are symmetric, skew-symmetric or positive definite.

(f) Show that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ has been defined using the Christoffel symbol representation.

(g) Show that the derivative of the energy function $E(\mathbf{q},\dot{\mathbf{q}}) = T(\mathbf{q},\dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{q}}^T \boldsymbol{\tau}\,,$$

where $\boldsymbol{\tau} = [\tau_1\,,\,\tau_2]^T$.
Hint 1: Use the matrix formulations of the different equations.
For example, $T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}(\mathbf{q})^T$.
Hint 2: Use the result from part (f).

Problem 3 (Double inverted pendulum, generalized coordinates, Lagrange's equation. 35 %)
The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 3.

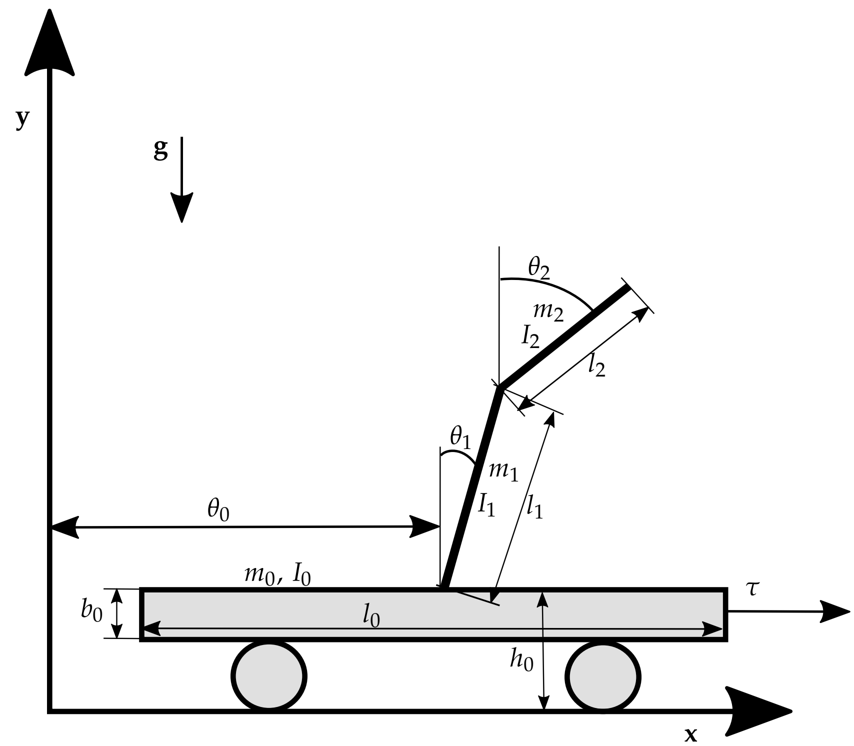

Figure 3: Double inverted pendulum on a cart

The mass of the cart is denoted by m_0, its length by l_0 , its width by b_0 and its height by h_0. The first rod is located above the center of mass of the cart. The length of the first rod is l_1, while the length of the second rod is l_2. Analogously, both rods have a mass and a moment of inertia, which are denoted by m_i and I_i, respectively. Furthermore, the force τ is acting on the cart.

(a) Find the position of the cart and the two rods.

(b) Find the kinetic energy T of the DIPC system.
Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: $\cos(x - y) = \cos x \cos y + \sin x \sin y$

(c) Find the potential energy U of the DIPC system.
Hint: Read section 8.2 in the book.

(d) Derive the equations of motion for the DIPC system. using Lagrange's equation.
Show the details of your calculations.
Hint: Read section 8.2 in the book.