EXAM SOLUTIONS: DETECTION

2a) H_0 : $x \sim p_0(x)$, with $p_0(x) = e^{-x}u(x)$

H_1 : $x \sim p_1(x)$, with $p_1(x) = x e^{-x} u(x)$

NP detector decides H_1 if

$$L(x) = \frac{p_1(x)}{p_0(x)} = \frac{x e^{-x}}{e^{-x}} = x > \lambda$$

P_{FA} = Prob{ decide H_1 when H_0 is true } = Prob$\{ x > \lambda ; H_0 \}$

$$= \int_\lambda^\infty p_0(x)\,dx = \int_\lambda^\infty e^{-x}\,dx = \left[-e^{-x}\right]_\lambda^\infty = e^{-\lambda}$$

$$\Rightarrow \quad \lambda = -\ln P_{FA} = -\ln 0.1 \simeq 2{,}301$$

$\therefore$ Decision rule: Decide H_1 (water is polluted) if

Sample $x(0) > -\ln 0{,}1$.

This will give a P_{FA} of 10%

2b) P_D = Prob{ decide H_1 when H_1 is true } =

$$= \int_\lambda^\infty p_1(x)\,dx = \int_\lambda^\infty x e^{-x}\,dx = \left[-e^{-x}x - e^{-x}\right]_\lambda^\infty$$

$$= \lambda e^{-\lambda} + e^{-\lambda} = (1+\lambda)e^{-\lambda} = (1-\ln P_{FA})\,e^{\ln P_{FA}} =$$

$$= (1-\ln P_{FA})\,P_{FA} \simeq 0{,}33$$

2c) $H_0: \quad x \sim P_0$

$H_1: \quad x \sim P_1$

$P(H_1) = \pi_1 = 0{,}3$ and $P(H_0) = \pi_0 = 1 - \pi_1 = 0{,}7$

Min P_E detector decides H_1 if

$$\frac{P_1(x)}{P_0(x)} = \frac{x e^{-x}}{e^{-x}} > \lambda = \frac{\pi_0}{\pi_1} = \frac{0{,}7}{0{,}3} = \frac{7}{3}$$

$\therefore$ MPE detector decides H_1 (water is polluted) if

$$x(i) > \frac{7}{3}$$

2d)
$$P_E = \pi_0 \,\text{Prob}\{\delta(x) = 1 / H_0\} + \pi_1 \,\text{Prob}\{\delta(x) = 0 / H_1\}$$

$$= \pi_0 \,\text{Prob}\{x > \lambda / H_0\} + \pi_1 \,\text{Prob}\{x < \lambda / H_1\}$$

$$= \pi_0 \int_\lambda^\infty P_0(x)\,dx + \pi_1 \int_0^\lambda P_1(x)\,dx$$

$$= \pi_0 \int_\lambda^\infty e^{-x}\,dx + \pi_1 \int_0^\lambda x e^{-x}\,dx$$

$$= \pi_0 \Big[-e^{-x}\Big]_\lambda^\infty + \pi_1 \Big[-e^{-x}x - e^{-x}\Big]_0^\lambda =$$

$$= \pi_0 e^{-\lambda} + \pi_1 \left(1 - \lambda e^{-\lambda} - e^{-\lambda}\right)$$

$$= \pi_0 e^{-\frac{\pi_0}{\pi_1}} + \pi_1 \left(1 - \frac{\pi_0}{\pi_1} e^{-\frac{\pi_0}{\pi_1}} - e^{-\frac{\pi_0}{\pi_1}}\right) = \pi_1 \left(1 - e^{-\frac{\pi_0}{\pi_1}}\right)$$

$$= \frac{3}{10}\left(1 - e^{-\frac{7}{3}}\right) \approx 0{,}271$$

2e) We can make two types of error

Type I : "decide the water is polluted when it is not"

= "decide H_1 when H_0"

Type II : "decide the water is not polluted when it is"

= "decide H_0 when H_1"

The MPE detector simply minizes the error probability without emphasizing the type of error

The NP detector, on the other hand, allows you to put an upper bound on the probability of Type I error, while maximizing the probability of detecting that the water is polluted, P_D.

$\Rightarrow$ The MPE does not seem attractive from the perspective of the person who shall drink the water. No guarantees on P_D being maximized

$\therefore$ NP detector is more suitable as you can control P_D and P_{FA}. A false alarm is a missed opportunity to drink water, and you are pretty thirsty!

NOTE: Convincing arguments for the use of an MPE detector will not be discarded. However, those arguments should be based on P_D, P_E, P_{FA} and Type I-II errors.