

TTK4135 Optimization and Control Spring 2019

Norwegian University of Science and Technology Department of Engineering Cybernetics

Exercise 3
Solution

Problem 1 (35 %) LP and KKT conditions (Exam August 2000)

We consider the following LP on standard form:

$$\min_{x} c^{\top} x \qquad \text{s.t.} \qquad Ax = b, \quad x \ge 0 \tag{1}$$

with $c, x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We know that the KKT conditions for this problem are

$$A^{\top}\lambda^* + s^* = c \tag{2a}$$

$$Ax^* = b \tag{2b}$$

$$x^* \ge 0 \tag{2c}$$

$$s^* > 0 \tag{2d}$$

$$s_i^* x_i^* = 0, \quad i = 1, \dots, n$$
 (2e)

- a The Newton direction is $p_k^N = (-\nabla^2 f_k)^{-1} \nabla f_k$ (see equation (2.15) in the textbook). Here, $f(x) = c^\top x$, so that $\nabla f_k = c$ and $\nabla^2 f_k = 0$. Hence, $(-\nabla^2 f_k)^{-1}$ does not exist, and $p_k^N = (-\nabla^2 f_k)^{-1} \nabla f_k$ is therefore not defined.
- **b** For an optimization problem to be convex, the following must hold:
 - the objective function is convex,
 - the equality constraint functions $c_i(\cdot)$, $i \in \mathcal{E}$ are linear, and
 - the inequality constraint functions $c_i(\cdot)$, $i \in \mathcal{I}$ are concave.

(See page 8 in the textbook.)

Using the definition of convexity (equation (1.4) in the textbook) on the objective function, we have

$$\alpha f(x_1) + (1 - \alpha)f(x_2) = \alpha c^{\top} x_1 + (1 - \alpha)c^{\top} x_2$$

$$= c^{\top} (\alpha x_1 + (1 - \alpha)x_2)$$

$$= f(\alpha x_1 + (1 - \alpha)x_2)$$
(3)

which shows the convexity of the objective function. The equality constraint Ax = b is linear, which means the second requirement is satisfied. The inequality constraint is $x \ge 0$, so c(x) = x must be a concave function if the problem is to be convex. Using the definition of concavity (the opposite of convexity), we have that

$$c(\alpha x + (1 - \alpha)y) = \alpha x + (1 - \alpha)y \tag{4a}$$

and that

$$\alpha c(x) + (1 - \alpha)c(y) = \alpha x + (1 - \alpha)y \tag{4b}$$

That is, the function is both convex and concave at the same time (this is true for all linear functions). Hence, problem (1) is a convex optimization problem since all three requirements are satisfied.

c The dual problem for (1) is defined as

$$\max_{\lambda} b^{\top} \lambda \qquad \text{s.t.} \qquad A^{\top} \lambda \le c \tag{5}$$

We rewrite the dual problem as

$$\min_{\lambda} -b^{\top} \lambda \qquad \text{s.t.} \qquad c - A^{\top} \lambda \ge 0 \tag{6}$$

and define the Lagrangean for the problem as

$$\bar{\mathcal{L}}(\lambda, x) = -b^{\mathsf{T}}\lambda - x^{\mathsf{T}}(c - A^{\mathsf{T}}\lambda) \tag{7}$$

where $x \in \mathbb{R}^n$ are multipliers for the constraints. Differentiating with respect to λ and requiring the derivative to be zero gives

$$\nabla_{\lambda}\bar{\mathcal{L}} = -b + (x^{\mathsf{T}}A^{\mathsf{T}})^{\mathsf{T}} = Ax - b = 0 \tag{8}$$

This, together with the general KKT conditions, gives the following KKT conditions for the dual problem:

$$Ax^* = b (9a)$$

$$A^{\top}\lambda^* \le c \tag{9b}$$

$$x^* > 0 \tag{9c}$$

$$x_i^*(c - A^{\mathsf{T}}\lambda^*)_i = 0, \quad i = 1, \dots, n$$
 (9d)

Defining $s = c - A^{\top} \lambda$ shows that the KKT-conditions for the dual problem (5) equals the KKT-conditions for problem (1).

- **d** The optimal objective $c^{\top}x^*$ of problem (1) and the optimal objective $b^{\top}\lambda^*$ of problem (5) have identical values.
- e A basic feasible point x for problem (1) is defined by the following:
 - A subset $\mathcal{B} \subseteq \{1, \ldots, n\}$ can be defined as containing exactly m indices,
 - $i \notin \mathcal{B} \Rightarrow x_i = 0$,
 - the $m \times m$ matrix B defined by $B = [A_i]_{i \in \mathcal{B}}$, where A_i is the *i*th column of A, is nonsingular.

An equivalent definition is that a basic feasible point x has n-m elements set to zero, and nonzero elements given by the constraint Ax = b.

f The equality constraints Ax = b can be written

$$a_i^{\top} x = b_i, \qquad i \in \mathcal{E}$$
 (10a)

or

$$c_i(x) = a_i^{\top} x - b_i = 0, \qquad i \in \mathcal{E}$$
 (10b)

where a_i^{\top} is the *i*th row of the matrix A. We then have that the gradients of the equality constraints are

$$\nabla c_i(x) = a_i, \qquad i \in \mathcal{E} \tag{11}$$

Since A has full (row) rank, all rows a_i^{\top} are linearly independent. This means that all equality constraint gradients $\nabla c_i(x) = a_i$ are linearly independent.

Problem 2 (40 %) LP

Two reactors, R_I and R_{II} , produce two products A and B. To make 1000 kg of A, 2 hours of R_I and 1 hour of R_{II} are required. To make 1000 kg of B, 1 hour of R_I and 3 hours of R_{II} are required. The order of R_I and R_{II} does not matter. R_I and R_{II} are available for 8 and 15 hours, respectively. The selling price of A is $\frac{3}{2}$ of the selling price of B (i.e., 50 % higher). We want to maximize the total selling price of the two products.

a This problem can be formulated as the following standard-form LP:

min
$$\begin{bmatrix} -3 & -2 & 0 & 0 \end{bmatrix} x$$

s.t. $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$
 $x \ge 0$ (12)

- **b** Figure 1 shows a contour plot with the constraints indicated in the space of x_1 and x_2 .
- **c** By modifying the example file provided to fit this problem, we find the following iteration sequence:

	Itera	ation nu	mber
	1	2	3
x	$\begin{bmatrix} 0 \\ 0 \\ 8 \\ 15 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 0 \\ 0 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 1.8 \\ 4.4 \\ 0 \\ 0 \end{bmatrix}$

From Figure 1 we see that the the solution is at the intersection between the availability constraints $2x_1 + x_2 \le 8$ and $x_1 + 3x_2 \le 15$. The non-negativity constraints are not active.

- **d** The iterations along with iteration numbers are indicated in Figure 1.
- e Looking at the algorithm output (the report given at each iteration) shows that the iteration sequence agrees with the theory in Chapter 13.3.

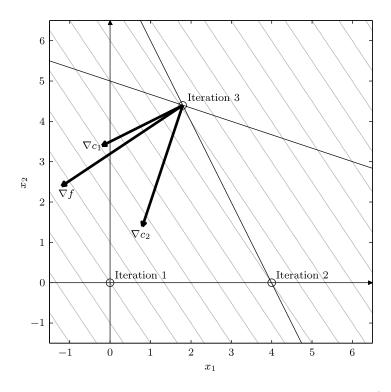


Figure 1: Contour plot with constraints for LP problem (12).

Problem 3 (25 %) QP and KKT Conditions (Exam May 2000)

A quadratic program (QP) can be formulated as

$$\min_{x} \quad q(x) = \frac{1}{2} x^{\top} G x + x^{\top} c \qquad (13a)$$
s.t. $a_i^{\top} x = b_i, \quad i \in \mathcal{E} \qquad (13b)$

$$a_i^{\top} x \ge b_i, \quad i \in \mathcal{I} \qquad (13c)$$

s.t.
$$a_i^{\top} x = b_i, \quad i \in \mathcal{E}$$
 (13b)

$$a_i^{\top} x \ge b_i, \qquad i \in \mathcal{I}$$
 (13c)

where G is a symmetric $n \times n$ matrix, \mathcal{E} and \mathcal{I} are finite sets of indices and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}, \text{ are vectors in } \mathbb{R}^n.$

a The active set $\mathcal{A}(x^*)$ for problem (13) is defined as the set of indices of the constraints for which equality holds at x^* . That is,

$$\mathcal{A}(x^*) = \{ i \in \mathcal{E} \cup \mathcal{I} | a_i^\top x^* = b_i \}$$
(14)

b Defining the Lagrangian for problem (13) as

$$\mathcal{L}(x,\lambda) = q(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

$$= \frac{1}{2} x^{\top} G x + x^{\top} c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^{\top} x - b_i)$$
(15)

From the complementarity condition $\lambda_i^* c_i(x^*) = 0$, $i \in \mathcal{E} \cup \mathcal{I}$, we know that for constraints that are not active at the solution, the corresponding multipliers are zero. Hence, the derivative with respect x of the Lagrangian at the solution can be written as

$$\nabla_{x} \mathcal{L}(x^{*}, \lambda^{*}) = \frac{1}{2} (Gx^{*} + G^{\top}x^{*}) + c - \sum_{i \in \mathcal{A}(x^{*})} \lambda_{i}^{*} a_{i}, \quad G^{\top} = G$$

$$= Gx^{*} + c - \sum_{i \in \mathcal{A}(x^{*})} \lambda_{i}^{*} a_{i}$$
(16)

For all constraints i in the active set $(i \in \mathcal{A}(x^*))$,

$$a_i^{\top} x^* = b_i \tag{17}$$

must hold at the solution. For inequality constraints i that are not active at the solution $(i \in \mathcal{I} \setminus \mathcal{A}(x^*))$,

$$a_i^{\top} x^* \ge b_i \tag{18}$$

must hold. For active inequality constraints i ($i \in \mathcal{I} \cap \mathcal{A}(x^*)$), the multipliers must be non-negative. That is,

$$\lambda_i^* \ge 0 \tag{19}$$

The KKT conditions for problem (13) can now be summarized as

$$Gx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i = 0$$
 (20a)

$$a_i^{\top} x^* = b_i, \quad i \in \mathcal{A}(x^*)$$
 (20b)

$$a_i^{\top} x^* \ge b_i, \quad i \in \mathcal{I} \setminus \mathcal{A}(x^*)$$
 (20c)

$$\lambda_i^* \ge 0, \quad i \in \mathcal{I} \cap \mathcal{A}(x^*)$$
 (20d)