Out: March 4, 2019, 8:00
Deadline: March 24, 2019, 20:00

NTNU – Trondheim
Norwegian University of
Science and Technology

Assignment 8
TTK4130 Modeling and Simulation

Problem 1 (Sliding stick, generalized coordinates, Lagrange's equation. 30 %)

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

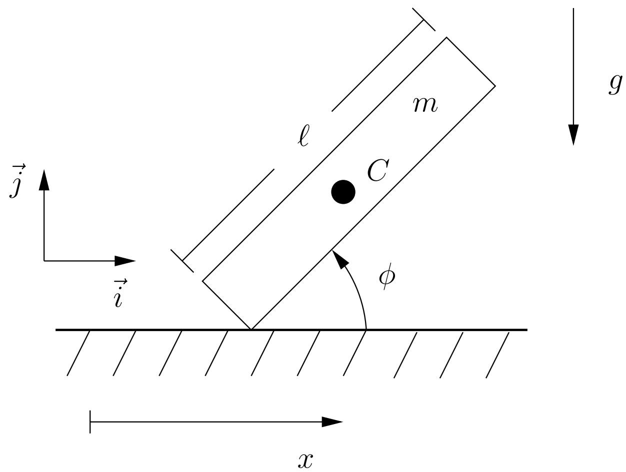

Figure 1: Stick sliding on frictionless surface

(a) Choose appropriate generalized coordinates (the figure should give you some hints).
What are the corresponding generalized (actuator) forces?
Hint: Read section 7.7 in the book.

Solution: A natural choice for generalized coordinates are the horizontal coordinate of the center of mass (denoted by x), and the angle between the stick and the surface (denoted by ϕ). An alternative to x could be the horizontal coordinate of the contact point between the stick and the surface.

There are no generalized (actuator) forces corresponding to these coordinates since the only actuator force, the force of gravity, is conservative.

(b) What are the position, velocity, and angular velocity of the center of mass as function of the chosen generalized coordinates and their derivatives?
Hint: Read section 6.12 in the book.

Solution:

$$\vec{r}_c = x\vec{i} + \frac{\ell}{2}\sin\phi\vec{j}$$
$$\vec{v}_c = \dot{x}\vec{i} + \frac{\ell}{2}\dot{\phi}\cos\phi\vec{j}$$
$$\vec{w}_{ib} = \dot{\phi}\vec{k}$$

(c) Express the kinetic and potential energy of the stick as function of the chosen generalized coordinates and their derivatives.
Hint: Read section 8.2 in the book.

Solution: The kinetic energy for the rigid body is

$$T = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib} = \frac{1}{2} m \left(\dot{x}^2 + \frac{\ell^2}{4} \dot{\phi}^2 \cos^2 \phi \right) + \frac{1}{2} I_z \dot{\phi}^2.$$

The potential energy due to gravity is

$$U = mgz = mg \frac{\ell}{2} \sin \phi.$$

(d) Derive the equations of motion for the stick using Lagrange's equation.
Show the details of your calculations.
Hint: Read section 8.2 in the book.

Solution: Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$\frac{d}{d\,t} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0\,,$$

which reduces to

$$\ddot{x} = 0.$$

The second equation of motion is

$$\frac{d}{d\,t} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0\,,$$

which gives

$$\left(\frac{m\ell^2}{4} \cos^2 \phi + I_z \right) \ddot{\phi} - \frac{m\ell^2}{4} \dot{\phi}^2 \cos \phi \sin \phi + mg \frac{\ell}{2} \cos \phi = 0.$$

Problem 2 (Robotic manipulator, generalized coordinates, Lagrange's equation, Christoffel symbols. 35 %)
We wish to model a robotic manipulator with the configuration shown in Figure 2.

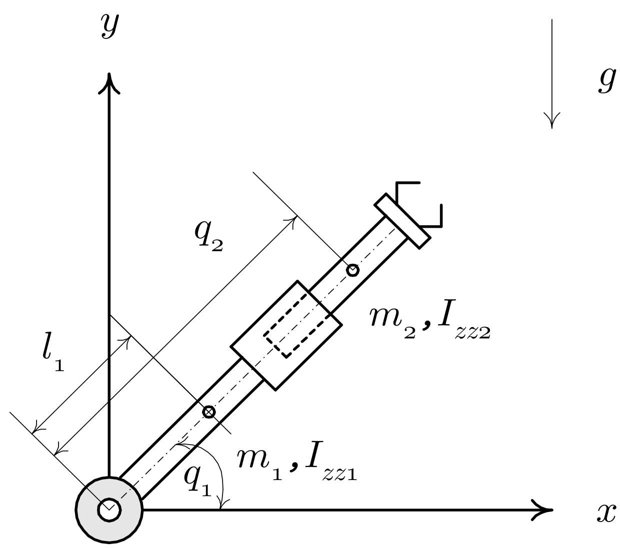

Figure 2: Manipulator

The manipulator has two degrees of freedom, which are represented by the generalized coordinates q_1 and q_2. We will use Lagrange's equation:

$$\frac{d}{d\,t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2 \tag{1}$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy}. \tag{2}$$

Assume that the axes x and y are fixed, i.e. they are the axes of an inertial reference frame. Moreover, the mass and the inertia of the motors are assumed to be neglectable.

The moment of inertia of the first arm is denoted by I_{zz1}, while the moment of inertia of the second arm is denoted by I_{zz2}. Each moment of inertia is referenced to the center of mass of their respective arm. The dots in Figure 2 mark the centers of mass of each arm.

Finally, the arrow marked g illustrates the direction of the acceleration of gravity.

(a) Find the total kinetic energy T of the manipulator, and show that it can be written in the form $T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$, where $\mathbf{q} = [q_1\,,\, q_2]^T$ and

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{3}$$

Show the details of your calculations.

Hint: Read section 8.2.8 in the book.

Solution: The expression for the kinetic energy for each arm is

$$\frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib}.$$

We are only interested in the motion in the xy-plane. Hence, we neglect the velocity in the z-direction, and the angular velocity about the x- and the y-axis.

For the center of mass of each arm, the velocity and the angular velocity about the z-axis decomposed in the inertial reference frame i are:

- Arm 1:

$$\mathbf{r}^i_{c1} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{bmatrix}, \qquad \mathbf{v}^i_{c1} = \begin{bmatrix} -l_1 \sin q_1 \dot{q}_1 \\ l_1 \cos q_1 \dot{q}_1 \end{bmatrix}, \qquad w_{z1} = \dot{q}_1\,,$$
$$\vec{v}_{c1} \cdot \vec{v}_{c1} = \left(\mathbf{v}^i_{c1}\right)^T \mathbf{v}^i_{c1} = l_1^2 \sin^2 q_1 \dot{q}_1^2 + l_1^2 \cos^2 q_1 \dot{q}_1^2 = l_1^2 \dot{q}_1^2.$$

- Arm 2:

$$\mathbf{r}^i_{c2} = \begin{bmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{bmatrix}, \qquad \mathbf{v}^i_{c2} = \begin{bmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{bmatrix}, \qquad w_{z2} = \dot{q}_1\,,$$
$$\begin{aligned}\vec{v}_{c2} \cdot \vec{v}_{c2} = \left(\mathbf{v}^i_{c2}\right)^T \mathbf{v}^i_{c2} &= \dot{q}_2^2 \cos^2 q_1 - \dot{q}_2 \cos q_1 q_2 \sin q_1 \dot{q}_1 + q_2^2 \sin^2 q_1 \dot{q}_1^2 \\ &+ \dot{q}_2^2 \sin^2 q_1 + \dot{q}_2 \sin q_1 q_2 \cos q_1 \dot{q}_1 + q_2^2 \cos^2 q_1 \dot{q}_1^2 = \dot{q}_2^2 + q_2^2 \dot{q}_1^2.\end{aligned}$$

Hence, the kinetic energies of the arms are

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2\,, \\ T_2 &= \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2. \end{aligned}$$

Hence, the total kinetic energy for the system is

$$T = T_1 + T_2.$$

(b) Find the potential energy U of the manipulator.
Hint: Read section 8.2 in the book.

Solution: The potential energies of the arms are

$$\begin{aligned} U_1 &= m_1 g l_1 \sin q_1, \\ U_2 &= m_2 g q_2 \sin q_1. \end{aligned}$$

Hence, the total potential energy is

$$U = U_1 + U_2.$$

(c) Derive the equations of motion for the manipulator using Lagrange's equation.
Show the details of your calculations.
Hint: Read section 8.2 in the book.

Solution: The Lagrangian of the manipulator is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= T_1 + T_2 - U_1 - U_2 \\ &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2 + \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2 - (m_1 l_1 + m_2 q_2) g \sin q_1 \end{aligned}$$

We use Lagrange's equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i\,, \qquad i = 1, 2\,,$$

where

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \dot{q}_1 + I_{zz1}\dot{q}_1 + m_2 q_2^2 \dot{q}_1 + I_{zz2}\dot{q}_1 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \ddot{q}_1 + I_{zz1}\ddot{q}_1 + m_2 q_2^2 \ddot{q}_1 + I_{zz2}\ddot{q}_1 + 2m_2 q_2 \dot{q}_2 \dot{q}_1 \\
\frac{\partial \mathcal{L}}{\partial q_1} &= -\left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \dot{q}_2 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \ddot{q}_2 \\
\frac{\partial \mathcal{L}}{\partial q_2} &= m_2 q_2 \dot{q}_1^2 - m_2 g \sin q_1\,,
\end{aligned}$$

which give the equations of motion:

$$\begin{aligned}
\left(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2\right)\ddot{q}_1 + 2m_2 q_2 \dot{q}_2 \dot{q}_1 + \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 &= \tau_1 \\
m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + m_2 g \sin q_1 &= \tau_2
\end{aligned}$$

Note that τ_1 is the generalized force corresponding to q_1, i.e. a motor torque that produces rotation, while τ_2 is the generalized force corresponding to q_2, i.e. a motor force that generates a translational motion of arm 2.

(d) Show that the equations of motion found in part (c) can be written as

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}. \tag{4}$$

Explain why several choices are possible for $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$.
Moreover, show that the Christoffel symbol representation of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{bmatrix}. \tag{5}$$

Finally, find the vector $\mathbf{g}(\mathbf{q})$.
Show the details of your calculations.
Hint: Read section 8.2.8 in the book.

Solution: Note that the mass matrix $\mathbf{M}(\mathbf{q})$ was defined in part (a).

$$\begin{aligned}
\mathbf{M}(\mathbf{q}) &= \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix} \\
\mathbf{g}(\mathbf{q}) &= \begin{bmatrix} \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\ m_2 g \sin q_1 \end{bmatrix}
\end{aligned}$$

We see that we have a term containing multiplications of derivatives of q_i, i.e. $\dot{q}_1 \dot{q}_2$. This term can be split in two different ways among the factors $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ and $\dot{\mathbf{q}}$. Hence, there at least to different choices for the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$.

We find the Christoffel symbols c_{ijk} by using the formula

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{kj}}{q_i} + \frac{\partial m_{ik}}{q_j} - \frac{\partial m_{ij}}{q_k}\right),$$

which gives

$$\begin{aligned}
c_{111} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_1} + \frac{\partial m_{11}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_1}\right) = 0 \\
c_{112} &= \frac{1}{2}\left(\frac{\partial m_{21}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_2}\right) = -m_2 q_2 \\
c_{122} &= \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_2} - \frac{\partial m_{12}}{\partial q_2}\right) = 0 \\
c_{211} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_1} - \frac{\partial m_{21}}{\partial q_1}\right) = m_2 q_2 \\
c_{222} &= \frac{1}{2}\left(\frac{\partial m_{12}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_1}\right) = 0 \\
c_{222} &= \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_2} + \frac{\partial m_{22}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_2}\right) = 0.
\end{aligned}$$

Note that $c_{ijk} = c_{jik}$ due to the symmetry of $\mathbf{M}(\mathbf{q})$. Consequently, $c_{211} = c_{121}$ and $c_{212} = c_{122}$. We can now find the elements of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ by using the formula

$$c_{ij} = \sum_{i=1}^{n=2} c_{ijk}\dot{q}_i ,$$

which gives

$$\begin{aligned}
c_{11} &= c_{111}\dot{q}_1 + c_{211}\dot{q}_2 = m_2 q_2 \dot{q}_2 \\
c_{12} &= c_{121}\dot{q}_1 + c_{212}\dot{q}_2 = m_2 q_2 \dot{q}_1 \\
c_{21} &= c_{112}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 q_2 \dot{q}_1 \\
c_{22} &= c_{122}\dot{q}_1 + c_{222}\dot{q}_2 = 0.
\end{aligned}$$

Hence,

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{bmatrix}.$$

(e) Determine if the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ are symmetric, skew-symmetric or positive definite.

Solution: The mass matrix $\mathbf{M}(\mathbf{q})$ is

$$\begin{aligned}
&\text{symmetric :} \quad \mathbf{M} = \mathbf{M}^T \quad ; \quad \text{and} \\
&\text{positive definite :} \quad \mathbf{x}^T\mathbf{M}(\mathbf{q})\mathbf{x} > 0 \qquad \forall \mathbf{x} \neq 0
\end{aligned}$$

The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has none of these properties.

(f) Show that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has been defined using the Christoffel symbol representation.

Solution:

$$\dot{\mathbf{M}}(\mathbf{q}) = \begin{bmatrix} 2m_2 q_2 \dot{q}_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \begin{bmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{bmatrix}.$$

Hence,

$$\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \begin{bmatrix} 0 & -2m_2 q_2 \dot{q}_1 \\ 2m_2 q_2 \dot{q}_1 & 0 \end{bmatrix},$$

which is a skew-symmetric matrix.

(g) Show that the derivative of the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T \boldsymbol{\tau},$$

where $\boldsymbol{\tau} = [\tau_1\,,\, \tau_2]^T$.
Hint 1: Use the matrix formulations of the different equations.
For example, $T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ *and* $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}(\mathbf{q})^T$.
Hint 2: Use the result from part (f).

Solution:

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{d}{d\,t}\frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}}. \end{aligned}$$

By inserting (4) and $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}(\mathbf{q})^T$, we obtain that

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{\mathbf{q}}^T \left(\boldsymbol{\tau} - \mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \mathbf{g}\left(\mathbf{q}\right)\right) + \frac{1}{2}\dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}^T(\mathbf{q})\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^T \boldsymbol{\tau} + \frac{1}{2}\dot{\mathbf{q}}^T \left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\dot{\mathbf{q}} = \dot{\mathbf{q}}^T \boldsymbol{\tau}, \end{aligned}$$

where we used that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

Problem 3 (Double inverted pendulum, generalized coordinates, Lagrange's equation. 35 %)

The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 3.

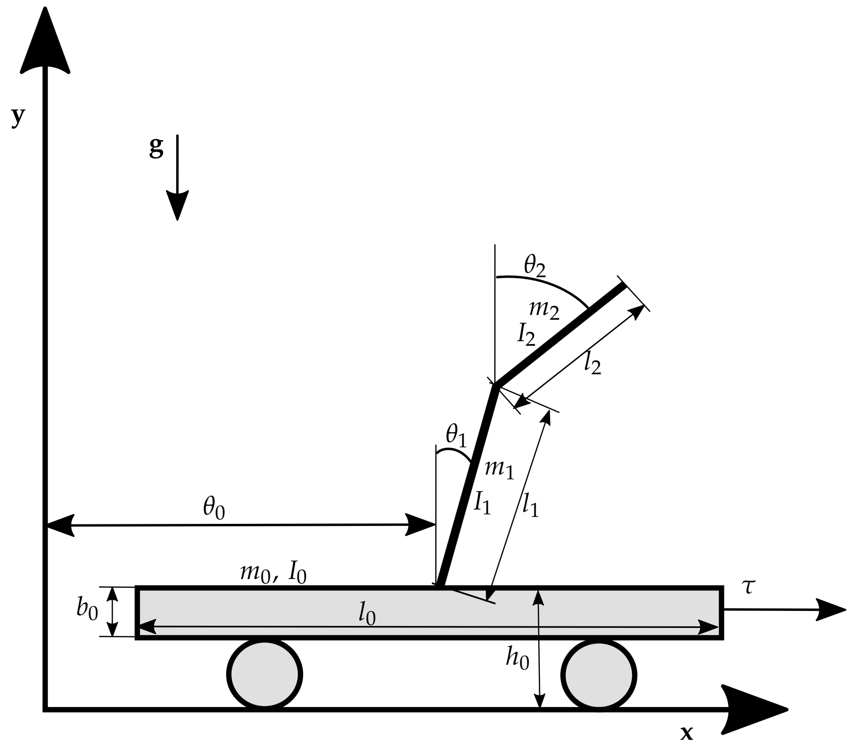

Figure 3: Double inverted pendulum on a cart

The mass of the cart is denoted by m_0, its length by l_0 , its width by b_0 and its height by h_0. The first rod is located above the center of mass of the cart. The length of the first rod is l_1, while the length of the second rod is l_2. Analogously, both rods have a mass and a moment of inertia, which are denoted by m_i and I_i, respectively. Furthermore, the force τ is acting on the cart.

(a) Find the position of the cart and the two rods.

Solution: The position of the cart is given by

$$r_0 = \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix},$$

the position of the centre of mass of the first rod is given by

$$r_1 = \begin{bmatrix} \theta_0 + \frac{1}{2} l_1 \sin\theta_1 \\ h_0 + \frac{1}{2} l_1 \cos\theta_1 \end{bmatrix}$$

and the position of the second rod is given by

$$r_2 = \begin{bmatrix} \theta_0 + l_1 \sin\theta_1 + \frac{1}{2} \sin\theta_2 \\ h_0 + l_1 \cos\theta_1 + \frac{1}{2} l_2 \cos\theta_2 \end{bmatrix}.$$

If the origin of the coordinate system is chosen differently the terms can look slightly different.

(b) Find the kinetic energy T of the DIPC system.
Show the details of your calculations.
Hint 1: Read section 8.2 in the book.
Hint 2: $\cos(x-y) = \cos x \cos y + \sin x \sin y$

Solution: The velocities can be calculated by derivating the expressions for the positions

$$
\begin{aligned}
v_0 &= \begin{bmatrix} \dot{\theta}_0 \\ 0 \end{bmatrix}, \\
v_1 &= \begin{bmatrix} \dot{\theta}_0 + \frac{1}{2} l_1 \dot{\theta}_1 \cos\theta_1 \\ -\frac{1}{2} l_1 \dot{\theta}_1 \sin\theta_1 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} \dot{\theta}_0 + l_1 \dot{\theta}_1 \cos\theta_1 + \frac{1}{2} l_2 \dot{\theta}_2 \cos\theta_2 \\ -l_1 \dot{\theta}_1 \sin\theta_1 - \frac{1}{2} l_2 \dot{\theta}_2 \sin\theta_2 \end{bmatrix}.
\end{aligned}
$$

The kinetic energy is given by

$$T = \sum_i T_i \,,$$

with

$$T_i = \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} I_i \omega_i^T \omega_i.$$

Consequently, the kinetic energy of each body part is given by

$$
\begin{aligned}
T_0 =& \frac{1}{2} m_0 \dot{\theta}_0, \\
T_1 =& \frac{1}{2} m_1 \left(\dot{\theta}_0^2 + l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + \frac{1}{4} l_1^2 \dot{\theta}_1^2 \right) + \frac{1}{2} I_1 \dot{\theta}_1^2, \\
T_2 =& \frac{1}{2} m_2 \left(\dot{\theta}_0^2 + l_1^2 \dot{\theta}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_0 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \\
&+ \frac{1}{2} I_2 \dot{\theta}_2^2.
\end{aligned}
$$

(c) Find the potential energy U of the DIPC system.
Hint: Read section 8.2 in the book.

Solution: The potential energy is given by

$$U = \sum_i U_i \,,$$

where

$$
\begin{aligned}
U_0 &= m_0 g (h_0 - \frac{1}{2} b_0), \\
U_1 &= m_1 g (h_0 + \frac{1}{2} l_1 \cos\theta_1), \\
U_2 &= m_2 g (h_0 + l_1 \cos\theta_1 + \frac{1}{2} \cos\theta_2).
\end{aligned}
$$

If the origin of the coordinate system is chosen differently the terms can look slightly different.

(d) Derive the equations of motion for the DIPC system. using Lagrange's equation.
Show the details of your calculations.

Hint: Read section 8.2 in the book.

Solution: The Lagrange's equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i\,, \tag{6}$$

where L is defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}). \tag{7}$$

The resulting Lagrangian is

$$\begin{aligned} L =& \frac{1}{2}\Bigg[(m_0 + m_1 + m_2)\,\dot{\theta}_0^2 + \left(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\dot{\theta}_1^2 + \\ & \left(\frac{1}{2}m_2 l_2^2 + I_2\right)\dot{\theta}_2^2 + (m_1 l_1 + 2m_2 l_1)\,\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + \\ & m_2 l_2 \dot{\theta}_0\dot{\theta}_2 \cos\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\Bigg] - \\ & \left(\frac{1}{2}m_1 + m_2\right) g l_1 \cos\theta_1 - \frac{1}{2} m_2 g l_2 \cos\theta_2 - (m_0 + m_1 + m_2) h_0 g. \end{aligned} \tag{8}$$

The derivation of the Lagrangian with respect to $\dot{\mathbf{q}}$ is

$$\begin{aligned} &\frac{\partial}{\partial \dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \\ &\begin{bmatrix} (m_0 + m_1 + m_2)\dot{\theta}_0 + \frac{1}{2}\left((m_1 + 2m_2) l_1 \dot{\theta}_1 \cos\theta_1 + m_2 l_2 \dot{\theta}_2 \cos\theta_2\right) \\ (\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1)\dot{\theta}_1 + \frac{1}{2}\left((m_1 + 2m_2) l_1 \dot{\theta}_0 \cos\theta_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) \\ (\frac{1}{2}m_2 l_2^2 + I_2)\dot{\theta}_2 + \frac{1}{2} m_2 l_2 \left(\dot{\theta}_0 \cos\theta_2 + l_1 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right) \end{bmatrix} \end{aligned} \tag{9}$$

The derivation of Eq. 9 with respect to time is

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \\ &\begin{bmatrix} (\sum_i m_i)\ddot{\theta}_0 + \frac{1}{2}\left(l_1(m_1 + 2m_2)(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + m_2 l_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2)\right) \\ (\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1)\ddot{\theta}_1 + \frac{1}{2}\Big(l_1(m_1 + 2m_2)(\ddot{\theta}_0 \cos\theta_1 - \dot{\theta}_0 \dot{\theta}_1 \sin\theta_1) + ... \\ m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right]\Big) \\ (\frac{1}{2}m_2 l_2^2 + I_2)\ddot{\theta}_2 + \frac{1}{2}\Big(m_2 l_2(\ddot{\theta}_0 \cos\theta_2 - \dot{\theta}_0 \dot{\theta}_2 \sin\theta_2) + m_2 l_1 l_2 \big[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - ... \\ \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\big]\Big) \end{bmatrix} \end{aligned} \tag{10}$$

The derivation of the Lagrangian with respect to $\mathbf{q}$ is

$$\frac{\partial}{\partial \mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\begin{bmatrix} 0 \\ -(m_1 + 2m_2) l_1 \dot{\theta}_0 \dot{\theta}_1 \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + ... \\ (m_1 + 2m_2) g l_1 \sin\theta_1 \\ -m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \sin\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 \end{bmatrix}. \tag{11}$$

By putting the results (10) and (11) in (6), we obtain

$$\tau = \left(\sum m_i\right)\ddot{\theta}_0 + \frac{1}{2}\Big(l_1(m_1 + 2m_2 l_1)(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + \\ m_2 l_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2)\Big), \tag{12a}$$

$$0 = \left(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\ddot{\theta}_1 + \frac{1}{2}\Bigg(l_1(m_1 + 2m_2)\ddot{\theta}_0 \cos\theta_1 + \\ m_2 l_1 l_2\left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] - (m_1 + 2m_2) g l_1 \sin\theta_1\Bigg), \tag{12b}$$

$$0 = \left(\frac{1}{2}m_2 l_2^2 + I_2\right)\ddot{\theta}_2 + \\ \frac{1}{2}\left(m_2 l_2 \ddot{\theta}_0 \cos\theta_2 + m_2 l_1 l_2\left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right] - m_2 g l_2 \sin\theta_2\right), \tag{12c}$$

which are the equations of motion for the DIPC system.