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TTT4175 Estimation, Detection and Classification
Assignment no. 2 : Linear models, BLUE, MLE and Bayesian estimators

Problem 1 Linear models and BLUE

1a) We wish to estimate the amplitudes of exponentials in noise. The observed data are

$$x[n] = \sum_{i=1}^{p} A_i r_i^n + w[n], \;\; n = 0, 1, \ldots, N-1 \tag{1}$$

where $w[n]$ is white, Gaussian noise with zero mean and variance $\sigma^2 = 1$. Write down the estimator for the unknown parameters $\{A_i\}$, as well as their covariance matrix. (Hint: Write the answer as compactly as possible, but do not attempt to do any matrix inversions.)

Evaluate your results using $p = 2$, $r_1 = 1$, $r_2 = -1$ and N even.

1b) Computer assignment: We have the following linear model:

$$x(t) = A + Bt + C \sin(2\pi t) + w(t),$$

where $\{A, B, C\}$ are unknown parameters and $w(t)$ is white, Gaussian noise with zero mean and variance $\sigma^2 = 1$. Use the two data files `t.txt` and `x.txt`, which contains the measurement times t_n and observations $x(t_n)$ respectively, and find estimates for the unknown parameters, as well as the Cramer Rao lower bound (CRLB) of the variances of the estimates. Plot $x(t)$ based on your estimated parameters together with the raw data from `x.txt`.

1c) The observed samples $\{x[0], x[1], \ldots, x[N-1]\}$ are drawn iid, from

1. A Laplace distribution

$$p(x[n]; \mu) = \frac{1}{2} e^{-|x[n]-\mu|}$$

2. A Gaussian distribution

$$p(x[n]; \mu) = \frac{1}{\sqrt{2\pi}} e^{-(x[n]-\mu)^2/2}$$

Find the BLUE for the mean μ for both cases. What can you say about the MVU estimator for μ?

Problem 2 Maximum Likelihood estimators

2a) Consider the model

$$x = A + w,$$

where A is an unknown parameter and w is Gaussian noise with zero mean and variance $\sigma^2 = A$

1. Write down the probability density function for the observations, $p(x; A)$.
2. Given N observations $\{x[n]\}$, find the ML estimate for A.
3. Compute the CRLB for the estimation problem.
4. We consider the sample mean estimator for this problem. What is the variance of this estimator?
5. Compare the variance of the sample mean with the CRLB. Does achieve it the lower bound for any finite N? What if $N \to \infty$? What does this mean wrt. the sample mean vs. maximum likelihood estimator performances?

2b) Computer assignment: Let the observations $x[n]$ be given by

$$x = \cos(2\pi f n) + w,$$

where the frequency $f \in (0, 1/2)$ is unknown, and w is white, Gaussian noise with zero mean and variance $\sigma^2 = 1$. Show that for f not too close to 0 or $1/2$ and N sufficiently large, the ML estimate is approximately,

$$\hat{f} = \arg\max_f \sum_{n=0}^{N-1} x[n] \cos(2\pi f n).$$

Hint: Plot $\sum_{n=0}^{N-1} \cos^2(2\pi f n)$ as a function of f for different N.

Next, use the parameters $N = 10$, $f = 0.25$, $\sigma^2 = 0.01$, and plot a few realizations of the function to be maximized.

Use any numerical method to obtain the ML estimate (Hint: The simplest method is a grid search, in which one evaluates the function at a dense set $0 < \Delta f < 2\Delta f < \ldots < 1/2$ and chooses the $f_k = k\Delta f$ that maximizes the function). Do this for $M = 5000$ realizations and plot the histogram of the estimates.

2c) If an efficient estimator exists we always have,

$$\frac{\partial \log p(\mathbf{x}; \theta)}{\partial \theta} = I(\theta)(g(\mathbf{x}) - \theta), \tag{2}$$

Use this to show that the ML approach *always* will find the efficient estimator if it exists.

Problem 3 Bayesian estimators

3a) The posterior pdf for a random variable θ given a single observation x, is given by

$$
\begin{aligned}
p(\theta|x) =& \epsilon\mathcal{N}(x,1) + (1-\epsilon)\mathcal{N}(-x,1) \\
=& \frac{\epsilon}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta-x)^2} + \frac{1-\epsilon}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta+x)^2}.
\end{aligned}
$$

Plot/sketch the pdfs for $\epsilon = 1/2$ and $\epsilon = 3/4$ (assume $x \neq 0$). Find the MAP and Bayes MMSE estimates of θ given a single observation x for each value of ϵ.

3b) Given the posterior pdf

$$
p(\theta|x) = \begin{cases} e^{-(\theta-x)} & \theta \geq x \\ 0 & \theta < x \end{cases},
$$

find the MAP and Bayes MMSE estimators for θ given a single observation x.