Lecture 11: Rigid body kinematics – the rotation matrix

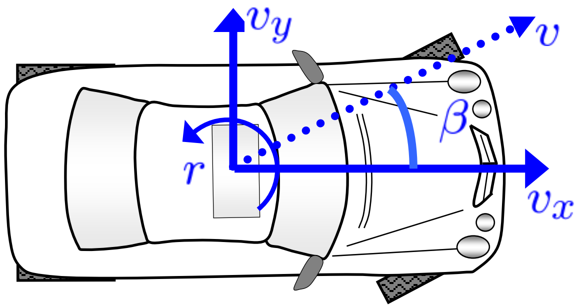
- What are rotation matrices used for?

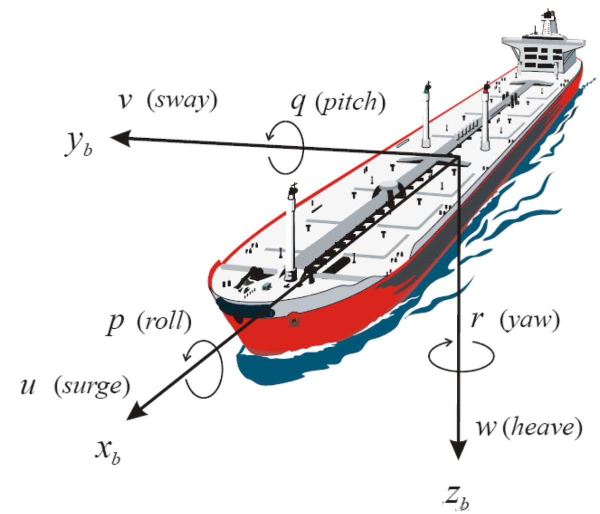
- Rotation matrices
  - Composite rotations, simple rotations
  - Homogenous transformation matrices
- Euler angles
  - 3-parameter specification of rotations
  - Roll-pitch-yaw
- Angle-axis, Euler-parameters
  - 4-parameter specification of rotations

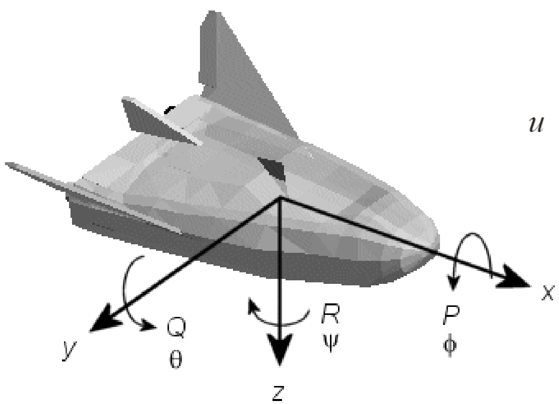
Book: Ch. 6.4, 6.5, 6.6

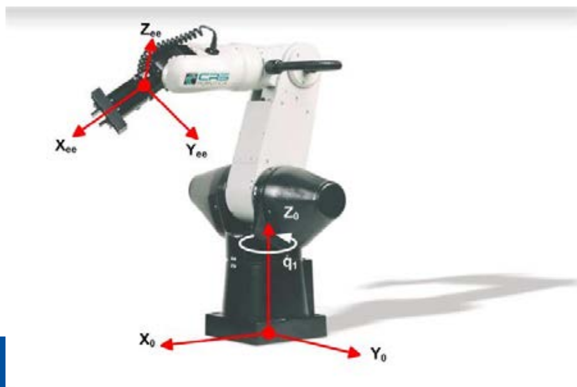
Why rotation matrices?

- Rotation matrices are used to describe rotations and orientations of rigid bodies
  - Road vehicles
  - Marine vessels
  - Airplanes, satellites
  - Robotics

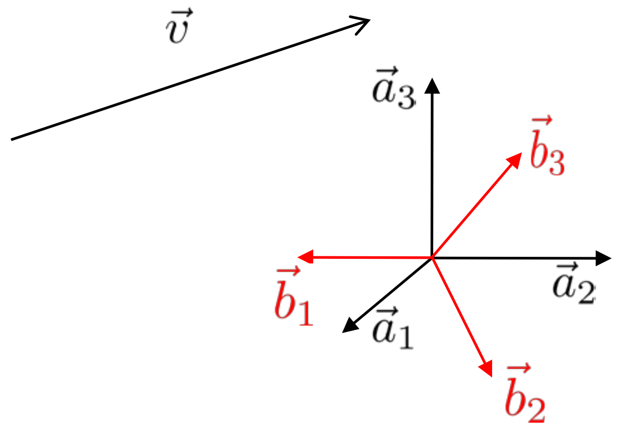








Rotation matrices



- The vector $\vec{v}$ can be written as

$$\vec{v} = \sum_{j=1}^{3} v_j^a \vec{a}_j \qquad \text{or} \qquad \vec{v} = \sum_{j=1}^{3} v_j^b \vec{b}_j$$

- These must be the same:

$$\sum_{j=1}^{3} v_j^a \vec{a}_j = \sum_{j=1}^{3} v_j^b \vec{b}_j$$

- Scalar product with $\vec{a}_i$ on both sides:

$$\sum_{j=1}^{3} v_j^a \vec{a}_j \cdot \vec{a}_i = \sum_{j=1}^{3} v_j^b \vec{b}_j \cdot \vec{a}_i \quad \Rightarrow \quad v_i^a = \sum_{j=1}^{3} v_j^b \vec{a}_i \cdot \vec{b}_j$$

- Gives

$$\mathbf{v}^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \mathbf{R}_b^a \mathbf{v}^b$$

Rotation matrices, properties

- We have shown

$$\mathbf{v}^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \mathbf{R}_b^a \mathbf{v}^b$$

- Switching *a* and *b*, we obtain

$$\mathbf{v}^b = \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \begin{pmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \vec{b}_1 \cdot \vec{a}_3 \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \vec{b}_2 \cdot \vec{a}_3 \\ \vec{b}_3 \cdot \vec{a}_1 & \vec{b}_3 \cdot \vec{a}_2 & \vec{b}_3 \cdot \vec{a}_3 \end{pmatrix} \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \mathbf{R}_a^b \mathbf{v}^a$$

- We see that $\mathbf{R}_a^b = (\mathbf{R}_b^a)^\mathsf{T}$

- From $\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b = \mathbf{R}_b^a \mathbf{R}_a^b \mathbf{v}^a$, we see that $\mathbf{R}_b^a \mathbf{R}_a^b = \mathbf{I}$

$$\boxed{\mathbf{R}_a^b = (\mathbf{R}_b^a)^\mathsf{T} = (\mathbf{R}_b^a)^{-1}}$$

The set of rotation matrices

For a matrix $\mathbf{R}$ to be a rotation matrix:

- The matrix must be orthogonal:

$$\mathbf{R}\mathbf{R}^{\mathsf{T}} = \mathbf{I}$$

- The determinant must be one

$$\det \mathbf{R} = 1$$

- The set of these matrices has a name: SO(3), or Special Orthogonal group of order 3:

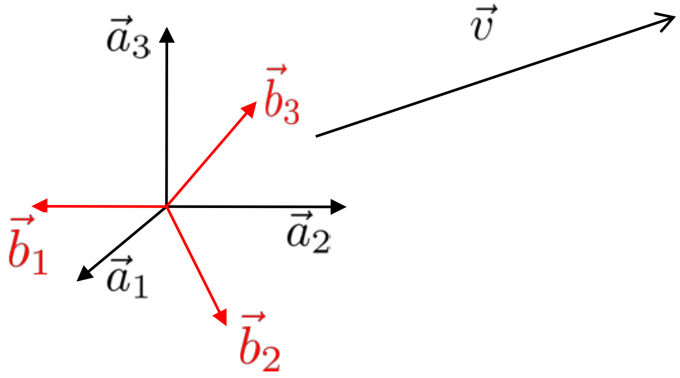
$$\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I},\ \det \mathbf{R} = 1\}$$

Rotation matrices

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- **Transform** a coordinate vector from *b* to *a*

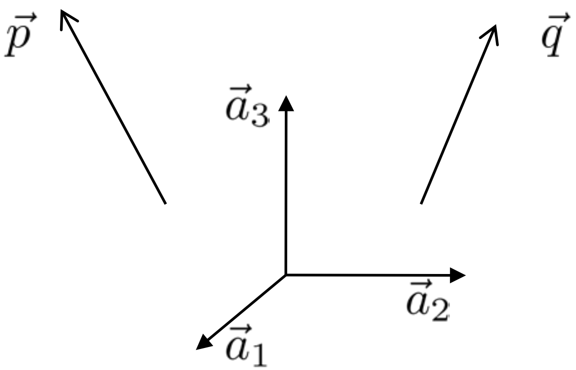
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- **Rotate** a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

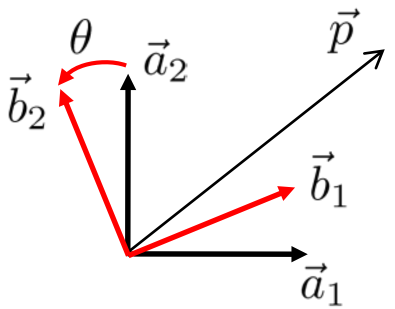
such that $\mathbf{q}^b = \mathbf{p}^a$.

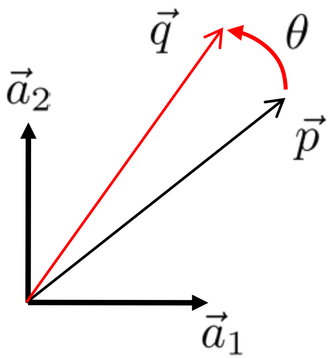


Rotation vs transformation (same, again)

- A coordinate vector may change either as a result of a rotation of a coordinate system (a **coordinate transformation**) or a rotation of the vector itself (a **rotation**).

- That is, a rotation from *a* to *b* can be interpreted in two ways:





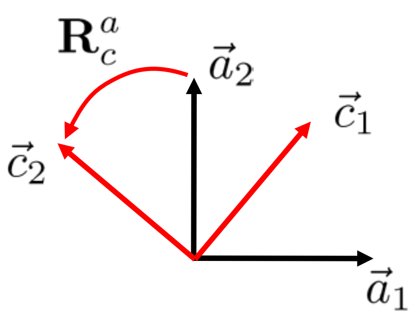
$\mathbf{p}^b = \mathbf{R}_a^b \mathbf{p}^a$ (or $\mathbf{p}^a = \mathbf{R}_b^a \mathbf{p}^b$)

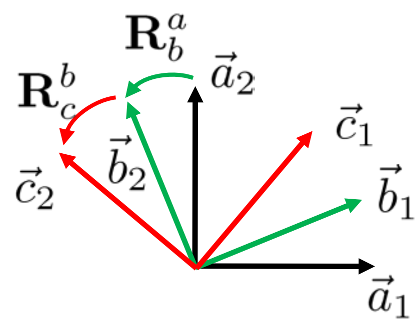
$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$ such that $\mathbf{q}^b = \mathbf{p}^a$

- That is, the matrix $\mathbf{R}_b^a$ rotates from *a* to *b*, but transforms from *b* to *a*!

- (Sometimes these two interpretations of the rotations originating from a rotation matrix are called passive vs active transformations, or alias vs alibi transformations)

Composite rotations





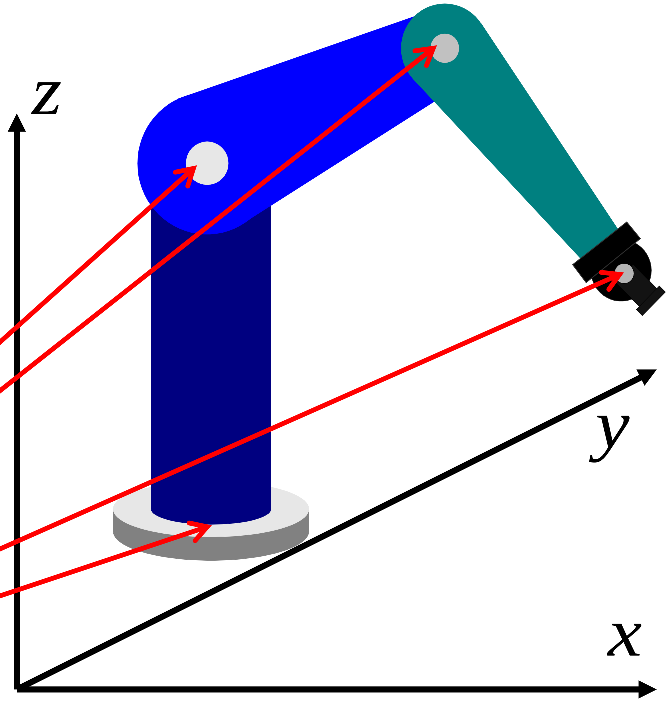
$$\mathbf{v}^a = \mathbf{R}^a_c \mathbf{v}^c$$

$$\mathbf{v}^b = \mathbf{R}^b_c \mathbf{v}^c$$
$$\mathbf{v}^a = \mathbf{R}^a_b \mathbf{v}^b = \mathbf{R}^a_b \mathbf{R}^b_c \mathbf{v}^c$$

$$\boxed{\mathbf{R}^a_c = \mathbf{R}^a_b \mathbf{R}^b_c}$$

(and $\mathbf{R}^a_d = \mathbf{R}^a_b \mathbf{R}^b_c \mathbf{R}^c_d$, etc.)

Kinematics in robotics

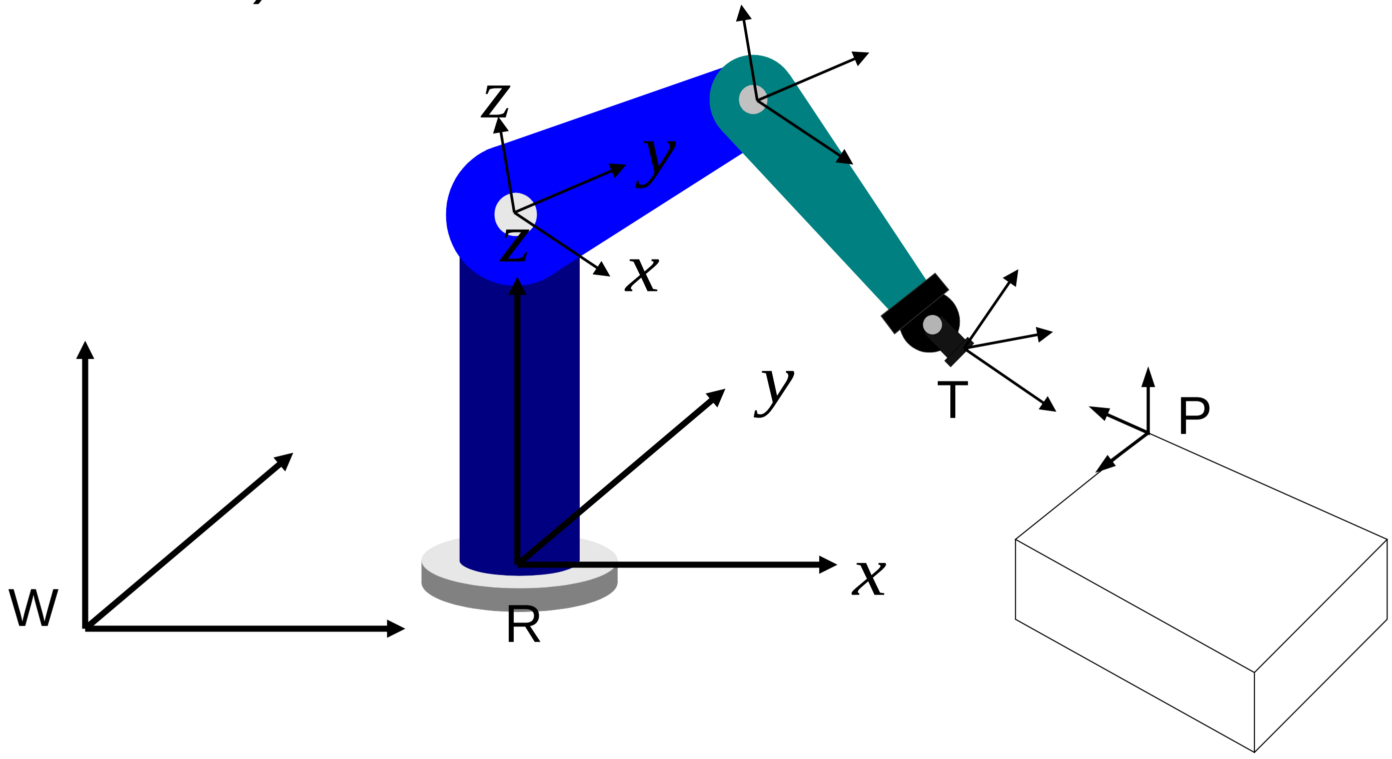


- Forward kinematics
  - Given joint variables
    $$q = (q_1, q_2, q_3, \ldots, q_n)$$
  - What are end-effector position and orientation?

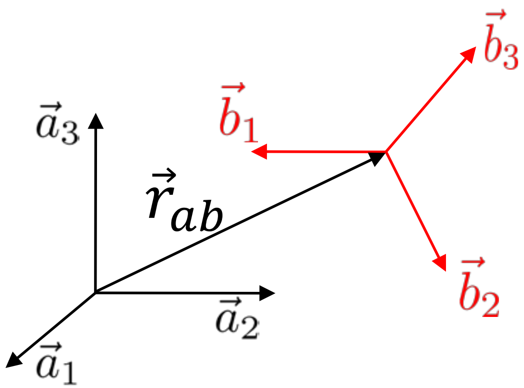
- Inverse kinematics
  - Given (desired) end-effector position and orientation.
  - What are the corresponding joint variables?

Coordinate systems in robotics

- World frame
- Joint frame
- Tool (end-effector) frame



Homogenous transformation matrices I



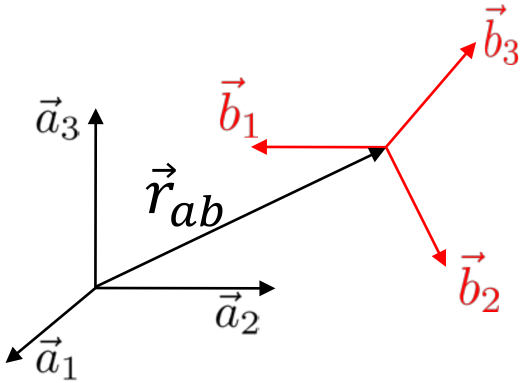
Orientation (R_b^a) and position ($\vec{r}_{ab}$) of b relative to a

$$T_b^a = \begin{bmatrix} R_b^a & \underline{r}_{ab}^a \\ [0\ 0\ 0] & 1 \end{bmatrix} \in SE(3)$$

$$SE(3): \left\{ T \;\middle|\; T = \begin{bmatrix} R & r \\ 0^T & 1 \end{bmatrix},\ R \in SO(3);\ r \in \mathbb{R}^3 \right\}$$

special euclidean group of dim 3

Homogenous transformation matrices II



$$\begin{pmatrix} R_b^a & \underline{r}_{ab}^a \\ 0\;0\;0 & 1 \end{pmatrix} \begin{pmatrix} R_a^b & \underline{r}_{ba}^b \\ 0\;0\;0 & 1 \end{pmatrix} =$$

$$= \begin{bmatrix} I_{3\times3} & R_b^a \underline{r}_{ba}^b + \underline{r}_{ab}^a \\ 0\;0\;0 & 1 \end{bmatrix} =$$

$$\underline{r}_{ba}^a + \underline{r}_{ab}^a$$

$$= -\underline{r}_{ab}^a + \underline{r}_{ab}^a$$

$$= 0$$

$$= \underline{I}_{4\times4}$$

$$\Rightarrow \left(T_b^a\right)^{-1} = \begin{pmatrix} R_a^b & \underline{r}_{ba}^b \\ 0\;0\;0 & 1 \end{pmatrix} = T_a^b$$

Composite homogenous transformation

$$T_b^a \cdot T_c^b = \begin{pmatrix} R_b^a & \underline{r}_{ab}^a \\ 0^T & 1 \end{pmatrix} \cdot \begin{pmatrix} R_c^b & \underline{r}_{bc}^b \\ 0^T & 1 \end{pmatrix}$$

$$= \begin{pmatrix} R_b^a R_c^b & R_b^a \underline{r}_{bc}^b + \underline{r}_{ab}^a \\ 0^T & 1 \end{pmatrix}$$

$$= \begin{pmatrix} R_c^a & \underline{r}_{ac}^a \\ 0^T & 1 \end{pmatrix}$$

$$\Rightarrow \quad T_b^a \cdot T_c^b = T_c^a$$

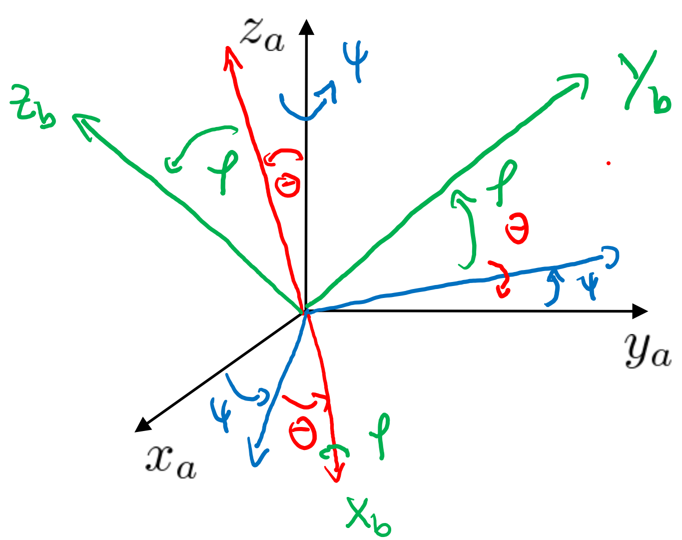
Euler angles

most common:	roll	pitch	yaw
Angle	φ	θ	ψ
axis	x	y	z

order: coordinate transf. $\bullet \longrightarrow$

rotation $\longleftarrow \bullet$

$$R_b^a = R_z(\psi) \cdot R_y(\theta)\, R_x(\varphi)$$



Angle-axis parameterisation I

it can be shown: R_b^a orthogonal $\rightarrow \lambda(R_b^a) = 1$

$\rightarrow$ it is possible to find (eigen-) vector $\underline{k}$ such that

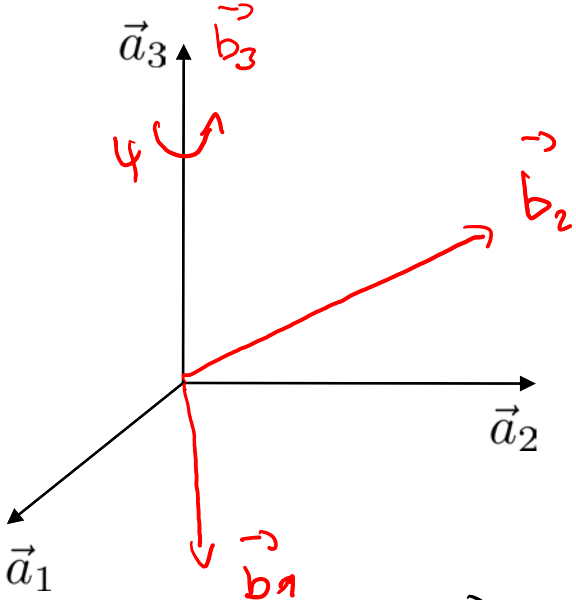
$$R_b^a \underline{k} = \underline{k} \qquad (\text{choose } \underline{k}^T \underline{k} = 1)$$

$$\underline{k} = \underline{k}^b \qquad \underline{k}^a = R_b^a \underline{k}^b = \underline{k}^b$$

$\vec{k}$ has the same representation

in a and b

Example: Angle-axis parameterisation



$\vec{k} = \vec{a}_3 = \vec{b}_3$

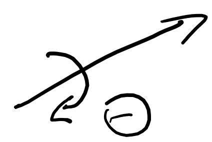
$$\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_z(\psi)\,\underline{k} = \underline{k}$$

$$\begin{pmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{k}$ is a rotation axis

All rotations can be described by vector $\vec{k}$ and angle Θ



4 parameters + 1 constraint ($\vec{k} \cdot \vec{k} = 1$)

Representations of rotations

- Rotation matrix
  - Simple, but over-parameterized (9 parameters)

Euler's Theorem:
"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

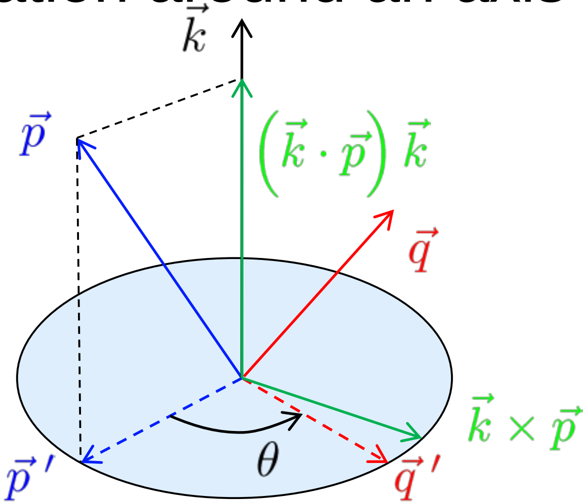
- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common: Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this later)

- Angle-axis, Euler-parameters
  - 4-parameters are used
  - No singularity problems

Rotation of vectors based on angle-axis representation I

- Angle-axis: All rotations can be represented as a simple rotation around an axis

Somewhat different derivation of the rotation dyadic. Compare p. 228 in book.



$$\vec{p}\,' = \vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \vec{q} - \left(\vec{k}\cdot\vec{q}\right)\vec{k} = \vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \cos\theta\,\vec{p}\,' + \sin\theta\,\vec{k}\times\vec{p}$$

$$\vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k} = \cos\theta\,\left(\vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}\right) + \sin\theta\,\vec{k}\times\vec{p}$$

$$\boxed{\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}}$$

Rotation of vectors based on angle-axis representation II

$$\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$= \underbrace{\left[\cos\theta\,\vec{I} + \sin\theta\,\vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\,\vec{k}\right]}_{\vec{R}_{k,\theta}}\vec{p}$$

$$\vec{q} = \vec{R}_{k,\theta}\cdot\vec{p}$$

$$R_b^a = R_{k,\theta} = \cos\theta\, I + \sin\theta\,(\underline{k}^a)^{\times} + (1-\cos\theta)\,\underline{k}^a\cdot(\underline{k}^a)^T$$

$$\underline{k}^{\times}\underline{k}^{\times} = \underline{k}\,k^T - k^T k\, I = \underline{k}\,k^T - \underline{I}$$

$$R_{k,\theta} = I + \underline{k}^{\times}\sin\theta + \underline{k}^{\times}k^{\times}(1-\cos\theta)$$

Compare with simple rotation

Check : $\vec{k} = \vec{a}_3$ $\qquad \underline{k}^a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

choose angle-axis Rotation matrix (previous slide)

$\rightarrow \underline{R_{z,\varphi}}$

Inverse: $(R^a_b)^{-1} = (R^a_b)^T = R_{k,-\theta} = R_{-k,\theta}$

Euler parameter

angle-axis : $\vec{k}$, θ

Euler parameter : $\eta = \cos\theta/2$; $\vec{\varepsilon} = \vec{k}\sin\theta/2$

[Quaternions : $p = \begin{pmatrix} \eta \\ \underline{\varepsilon} \end{pmatrix}$]

$$\eta^2 + \vec{\varepsilon}\cdot\vec{\varepsilon} = \cos^2\theta/2 + \vec{k}\cdot\vec{k}$$

Rotation matrix $R_e(\eta, \underline{\varepsilon})$

$$\underline{k}\sin\theta = 2\,\underline{k}\sin\theta/2\cos\theta/2$$ [$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$]

$$\underline{k}^\times\sin\theta = 2\eta\,\underline{\varepsilon}^\times$$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} - 1$

$= 1 - 2\sin^2\frac{\theta}{2}$ $\Leftrightarrow$

$$(1-\cos\theta)\,\underline{k}^\times\underline{k}^\times = 2\sin^2\theta/2\,\underline{k}^\times\underline{k}^\times = 2\,\underline{\varepsilon}^\times\underline{\varepsilon}^\times$$

$$\Rightarrow\quad R_e(\eta, \underline{\varepsilon}) = I + 2\eta\,\underline{\varepsilon}^\times + 2\,\underline{\varepsilon}^\times\underline{\varepsilon}^\times$$

$$R_e(-\eta, -\underline{\varepsilon}) = R_e(\eta, \underline{\varepsilon}) \;;\quad R_e(\eta, \underline{\varepsilon})^T = R_e(\eta, -\underline{\varepsilon})$$

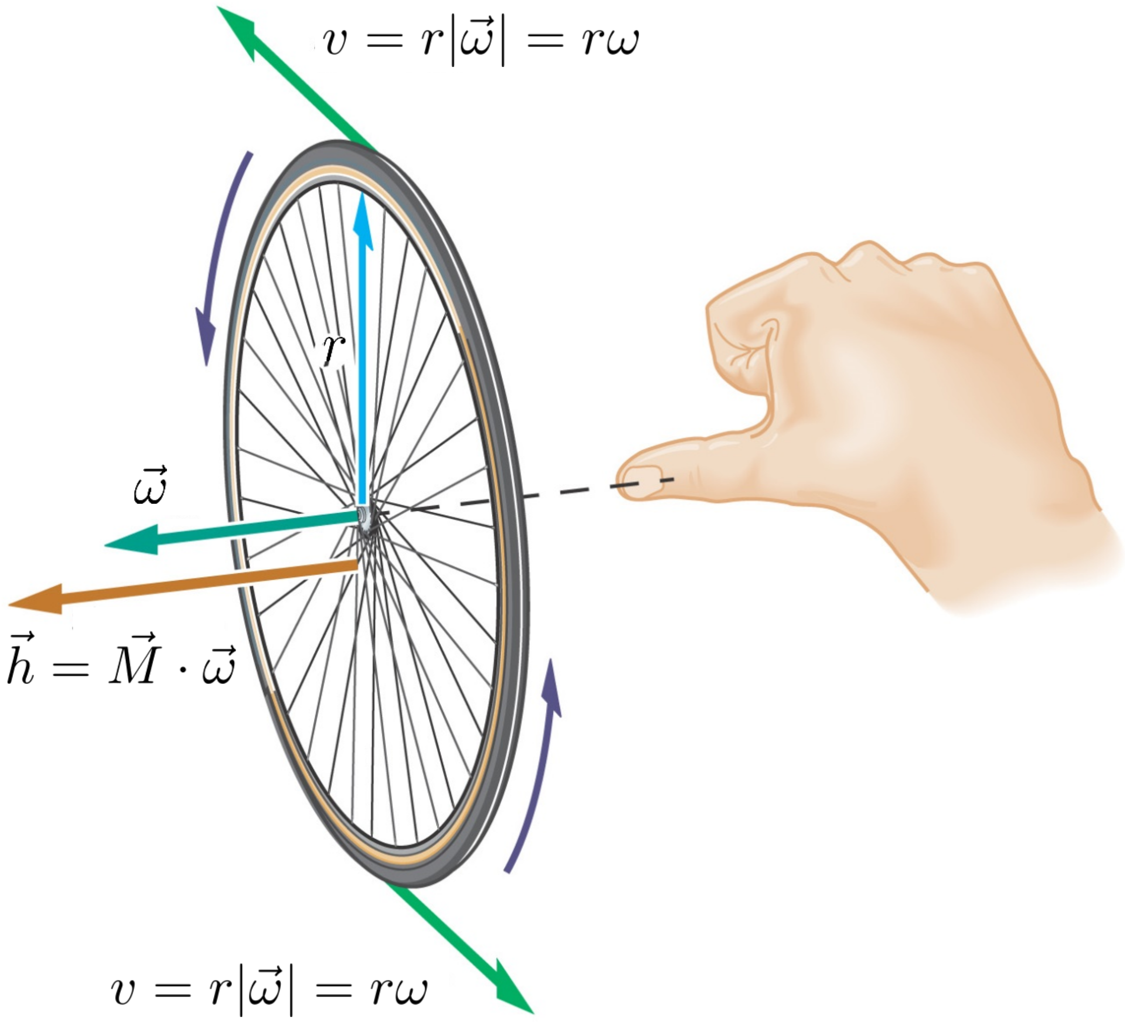
Use of Euler parameters

- ABB robots use Euler parameters (quaternions) internally in the robot control program
  - and Euler angles “externally”

- In Modelica.multibody, one can use either rotation matrices or Euler parameters (quaternions)

- Euler parameters (quaternions) often used in “advanced control” of robots, satellites, etc.

Angular velocity



Kinematic differential equations

- Translation: $\underline{v} \rightarrow \underline{r}$: $\qquad \dot{\underline{r}} = \underline{v}$

- Rotation: $\underline{\omega}_{ab}^{a} \rightarrow \mathbf{R}_{b}^{a}$: $\qquad \dot{\mathbf{R}}_{b}^{a} = ?$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler angle} \qquad \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

$$\underline{\omega}_{ab}^{a} \rightarrow \text{Euler parameter} \qquad \begin{array}{l} \dot{\eta} = ? \\ \dot{\underline{\varepsilon}} = ? \end{array}$$

Homework

- Derive rotation matrix of the angle axis representation assuming $\underline{k}_1 = [1,0,0]^T$ and $\underline{k}_2 = [0,1,0]^T$.
- Draw the coordinate systems (three) of the rotation using the classical Euler angles $[R_z(\psi)R_y(\theta)R_z(\phi)]$.

- How is the angular velocity defined; and how is it connected to the different representations of rotation (check: 6.8)?

Kahoot

- https://play.kahoot.it/#/k/8c1f768d-76cf-40e4-8163-ea279354e62a