Lecture 10: Rigid body kinematics – vectors, dyadics, rotation matrices

- What is rigid body kinematics?
- Vectors and dyadics
- Rotations

Book: Ch. 6.2, 6.3, 6.4

Kahoot

- https://play.kahoot.it/#/k/5199a4d4-e54b-4f4b-81ea-8c8f1c3170e7

What is rigid body dynamics?

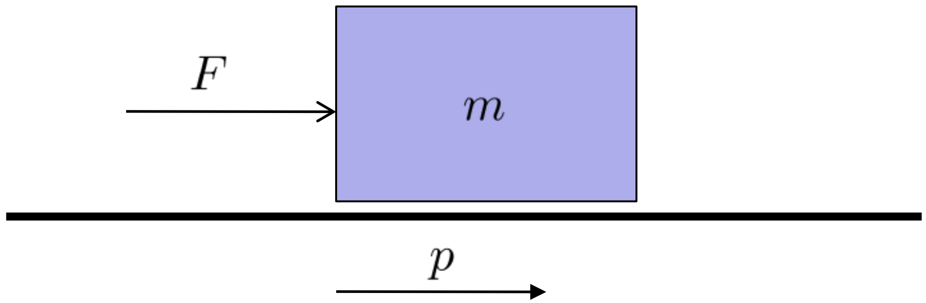
- Rigid body:
  - Wikipedia: "…a rigid body is an idealization of a solid body of finite size in which deformation is neglected."

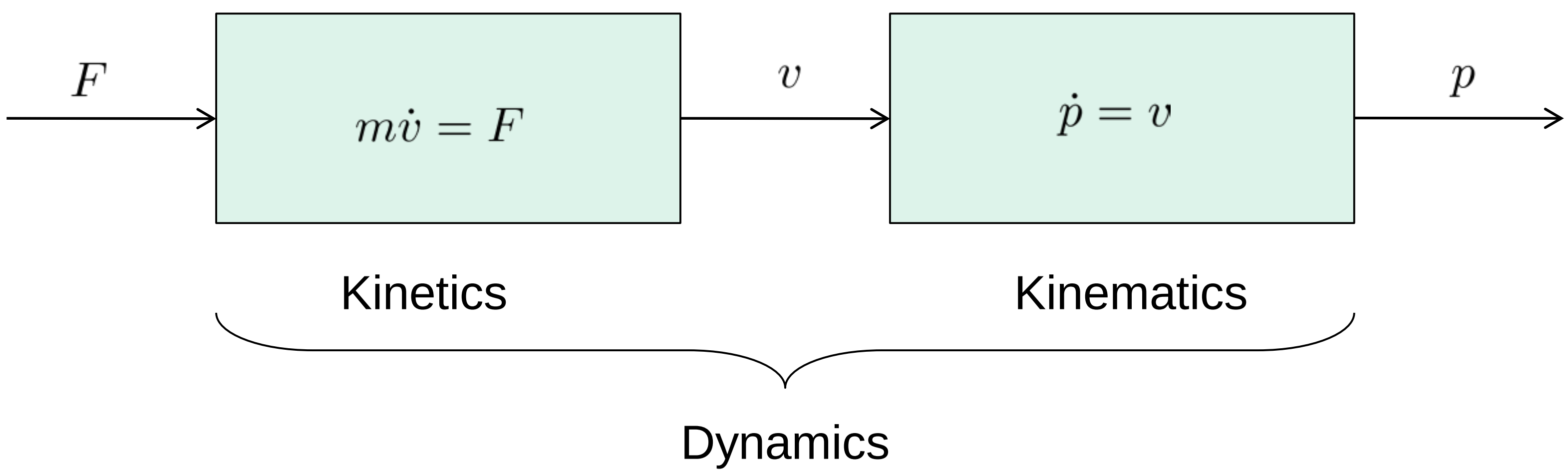
- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: "…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions)."
  - Book: Ch. 6
- Kinetics
  - eb.com: "…the effect of forces and torques on the motion of bodies having mass."
  - Book: Ch. 7, 8.

Remark: Sometimes "dynamics" is used for "kinetics" only

Simplest scalar case





	Kinematics	Kinetics
	Derivatives of position and orientation as function of velocity and angular velocity	Derivatives of velocity and angular velocity as function of applied forces and torques
Translation	1D: $\dot{r} = v$ 3D: $\dot{\mathbf{r}}_c^i = \mathbf{v}_c^i$	1D: $m\dot{v} = F$ 3D: $m\dot{\mathbf{v}}_c^i = \mathbf{F}_{bc}^i$
	$\dot{\mathbf{r}}_c^i = \mathbf{v}_c^i = \mathbf{R}_b^i \mathbf{v}_c^b$	Usually convenient to have forces and velocities in body system: $m\left(\dot{\mathbf{v}}_c^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{v}_c^b\right) = \mathbf{F}_{bc}^b$
Rotation/ orientation	1D: $\dot{\theta} = \omega$ 3D: Depends on parameterization Rotation matrix: $\dot{\mathbf{R}}_b^i = \mathbf{R}_b^i \left(\boldsymbol{\omega}_{ib}^b\right)^\times$ Euler angles: $\dot{\boldsymbol{\phi}} = \mathbf{E}_d^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^b$ Euler parameters: $\dot{\eta} = -\frac{1}{2}\boldsymbol{\epsilon}^\top \boldsymbol{\omega}_{ib}^b$ $\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\eta \mathbf{I} + \boldsymbol{\epsilon}^\times\right)\boldsymbol{\omega}_{ib}^b$	1D: $J\dot{\omega} = T$ 3D: $\mathbf{M}_{b/c}^b \dot{\boldsymbol{\omega}}_{ib}^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b = \mathbf{T}_{bc}^b$

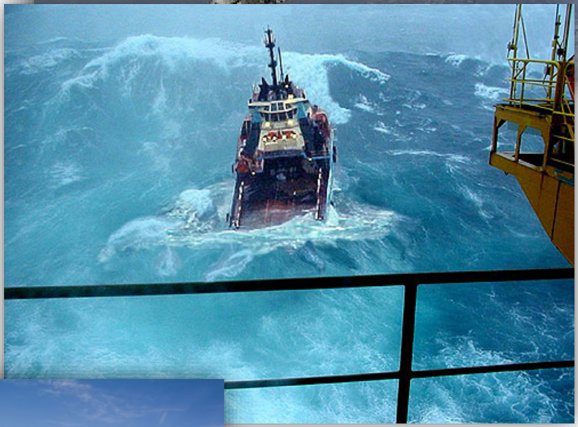
Note! By definition

$$\vec{v}_c := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_c$$

Why do control engineers need to know rigid body kinematics and dynamics?

- Robotics

- Control of marine vessels



- Control of aircraft and satellites

- Control of road vehicles

Resources

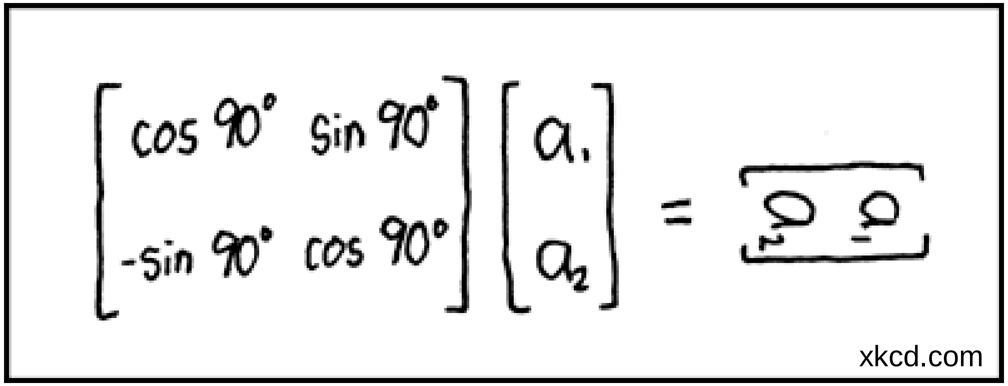
- Rigid body mechanics (often: classical mechanics) is a classical subject, basics developed in 1800s (and earlier) by Newton, **Euler**, Lagrange, ...

- Many resources available online. For example:
  - Leonard Susskind, Stanford: Classical Mechanics
    - https://www.youtube.com/playlist?list=PLA620233B2C4BDD10
  - Walter Levin, MIT: 8.01 Physics I: Classical Mechanics
    - https://www.youtube.com/watch?v=PmJV8CHIqFc
  - Books:
    - Kane & Levison: Dynamics, Theory and Applications
      - Download from http://ecommons.library.cornell.edu/handle/1813/638
    - Goldstein: Classical Mechanics
      - Download from http://www.fisica.net/ebooks/Classical_Mechanics_Goldstein_3ed.pdf

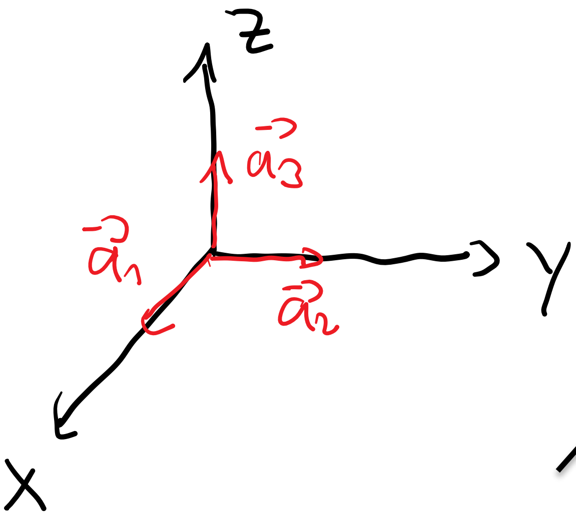
Today: vectors, dyadics, rotations

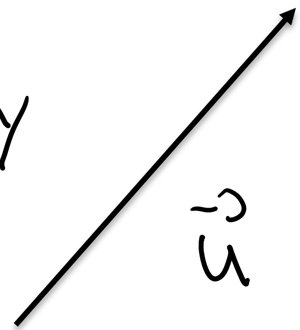
- The rigid bodies live in 3D space, so we need to know about 3D **vectors** and **rotations** to describe positions, attitude and movement.

- Mostly recap!?



Vectors





Vector:

magnitude + direction

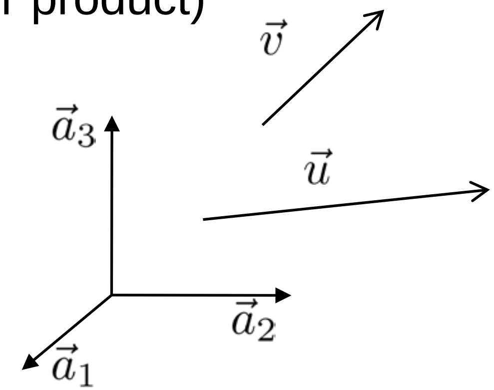
coordinate: $\vec{u}$

coordinate-vector: $\underline{u}$

$$\vec{u} = u_1 \vec{a}_1 + u_2 \vec{a}_2 + u_3 \vec{a}_3$$

$$\underline{u} = \underline{u}^a = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The scalar product (dot product, inner product)



Vectors:

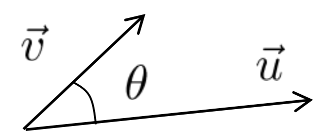
$$\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$$
$$\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$$

Coordinate vectors:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- Definition of scalar product:

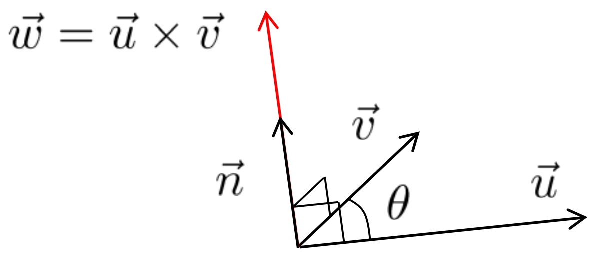
$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta}$$



- Can also be calculated from coordinate-vectors:

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3) \cdot (v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3) \\ &= u_1v_1 + u_2v_2 + u_3v_3 = \mathbf{u}^\mathsf{T}\mathbf{v}\end{aligned}$$

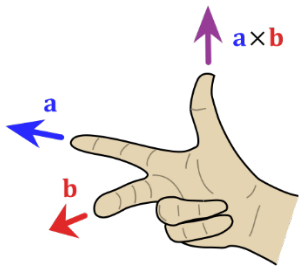
The cross product



- Definition:

$$\boxed{\vec{w} = \vec{u} \times \vec{v} = \vec{n}|\vec{u}||\vec{v}| \sin\theta}$$

- Calculation:



$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\vec{a}_1 - (u_3 v_1 - u_1 v_3)\vec{a}_2 + (u_1 v_2 - u_2 v_1)\vec{a}_3$$

- Introduce *the skew-symmetric form* of vector ***u***

$$\mathbf{u}^{\times} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

- Easy to check that

$$\mathbf{w} = \mathbf{u}^{\times}\mathbf{v} \qquad \Leftrightarrow \qquad \vec{w} = \vec{u} \times \vec{v}$$

Example 78

Fact: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Ans: $\underline{a}^{\times} \underline{b}^{\times} \cdot \underline{c}$

$\underline{a}^{\times}, \underline{b}^{\times}$

$\underline{a}^{\times} \cdot \underline{a}^{\times}$.

Dyadics – Example: Inertia dyadic

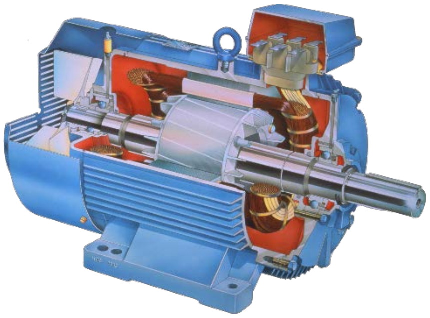
Angular momentum: $\vec{h} = \sum_{i=1}^{3} h_i \vec{a}_i \; ; \; \underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$

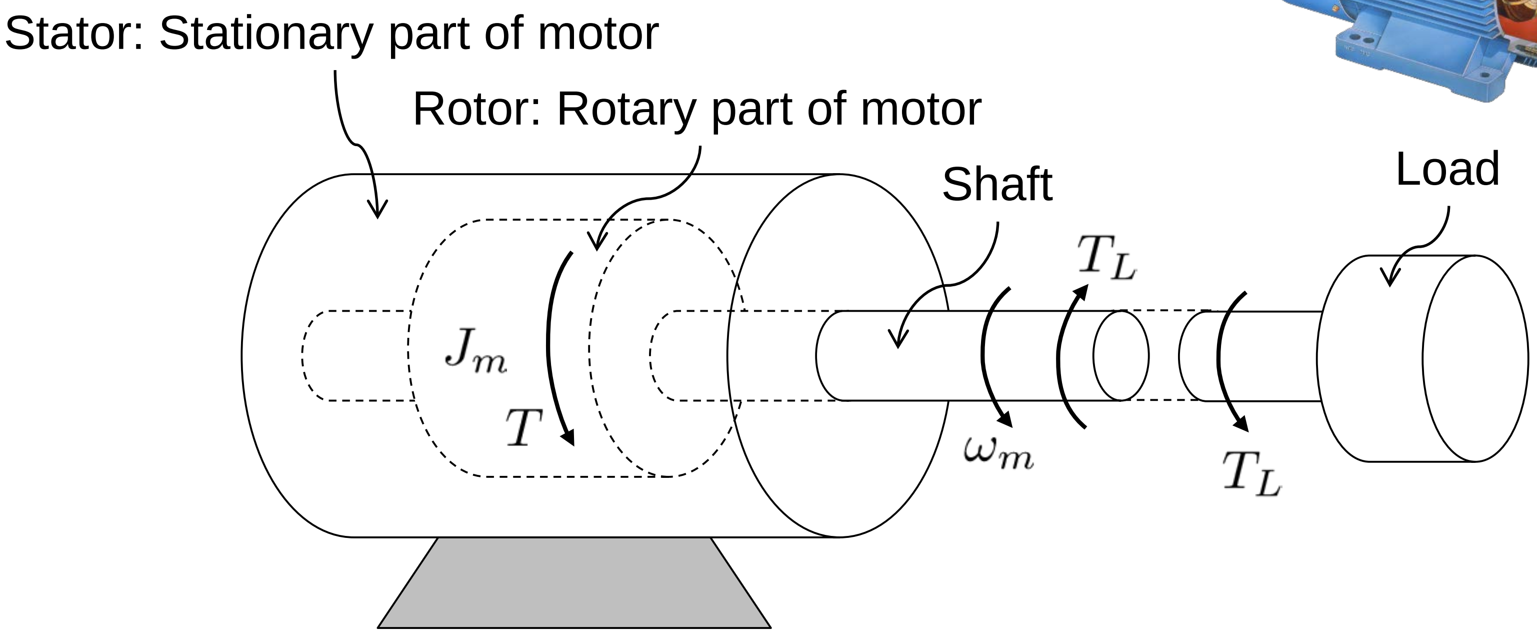
Angular velocity: $\vec{\omega} = \sum_{i=1}^{3} \omega_i \vec{a}_i \; ; \; \underline{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

$$\underline{h} = \underline{M} \cdot \underline{\omega} \qquad h_i = \sum_{j=1}^{3} m_{ij} \omega_j$$

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdot \\ m_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Dyadics: Example Motor





- Equation of motion for motor shaft:

$$J_m \dot{\omega}_m = T - T_L$$

where

- T : Motor torque (set up by some device, e.g. DC motor)
- T_L : Load torque
- J_m : Moment of inertia for rotor and shaft
- ω_m: Angular velocity/motor speed [rad/s, or rev./min]

Define dyadic $\vec{M}$ I

$$\vec{M} = \sum_{i=1}^{3} \sum_{j=1}^{3} m_{ij} \, \vec{a}_i \vec{a}_j$$

$$\vec{M} \cdot \vec{\omega} = \sum_{i=1}^{3} \sum_{j=1}^{3} m_{ij} \, \vec{a}_i \vec{a}_j \cdot \sum_{k=1}^{3} \omega_k \vec{a}_k$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} m_{ij} \, \omega_j \, \vec{a}_i = \vec{h}$$

$$\underbrace{\vec{a}_j \cdot \vec{a}_k}_{= 1 \text{ if } j = k}$$

Define dyadic $\vec{M}$ II

Dvs : $\vec{h} = \vec{M} \cdot \vec{\omega}$

$$\vec{M} \longrightarrow M$$

$$\vec{h} \longrightarrow \underline{h}$$

$$\vec{\omega} \longrightarrow \underline{\omega}$$

coordinate-free

coordinate system given

Example: dyadic product of two vectors

$$\vec{a}_1\vec{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{a}_1\vec{a}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$$

$$\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$$

$$\begin{aligned} \vec{v}\vec{u} = & v_1u_1\vec{a}_1\vec{a}_1 + v_1u_2\vec{a}_1\vec{a}_2 + v_1u_3\vec{a}_1\vec{a}_3 \\ & v_2u_1\vec{a}_2\vec{a}_1 + v_2u_2\vec{a}_2\vec{a}_2 + v_2u_3\vec{a}_2\vec{a}_3 \\ & v_3u_1\vec{a}_3\vec{a}_1 + v_3u_2\vec{a}_3\vec{a}_2 + v_3u_3\vec{a}_3\vec{a}_3 \end{aligned}$$

$$\begin{aligned} \vec{v}\vec{u} = \vec{v}\otimes\vec{u} = & \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \\ = & \begin{pmatrix} v_1u_1 & \dots & \dots \\ v_2u_1 & \dots & \dots \\ v_3u_1 & \dots & \dots \end{pmatrix} \end{aligned}$$

General dyadic $\vec{D} = \sum_i \sum_j d_{ij} \vec{a}_i \vec{a}_j$

$$d_{ij} = \vec{a}_i \cdot \vec{D} \cdot \vec{a}_j \qquad D = \begin{pmatrix} d_{11} & d_{12} & \cdot \\ d_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

pre-multiplication:

$$\vec{w} = \vec{u} \cdot \vec{D} = \sum_k u_k \vec{a}_k \cdot \sum_i \sum_j d_{ij} \vec{a}_i \vec{a}_j$$

$$= \sum_i \underbrace{\sum_j u_i d_{ij}}_{w_j} a_j \quad \rightarrow \vec{c} \cdot (\vec{a}\vec{b}) = (\vec{c} \cdot \vec{a})\vec{b}$$

post-multiplication:

$$\vec{z} = \vec{D} \cdot \vec{u} = \sum_i \sum_j d_{ij} \vec{a}_i \vec{a}_j \cdot \sum_k u_k \vec{a}_k$$

$$= \sum_i \underbrace{\sum_j d_{ij} u_j}_{z_i} \vec{a}_i \quad \rightarrow (\vec{a}\vec{b}) \cdot \vec{c} = \vec{a}(\vec{b} \cdot \vec{c})$$

Shows:

$$\vec{w} = \vec{u} \cdot \vec{D} \iff \underline{w}^T = \underline{u}^T \underline{D}$$

$$\vec{z} = \vec{D} \cdot \vec{u} \iff \underline{z} = \underline{D} \cdot \underline{u}$$

Example: Multiplication with dyadics

$$\vec{I} = \vec{a}_1\vec{a}_1 + \vec{a}_2\vec{a}_2 + \vec{a}_3\vec{a}_3$$

$$\begin{aligned}\vec{I}\vec{v} &= (\vec{a}_1\vec{a}_1 + \vec{a}_2\vec{a}_2 + \vec{a}_3\vec{a}_3)(v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3) \\ &= v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3\end{aligned}$$

- Coordinate-free:

$$\vec{I} \cdot \vec{v} = \vec{v}$$

$$\vec{v} \cdot \vec{I} = \vec{v}$$

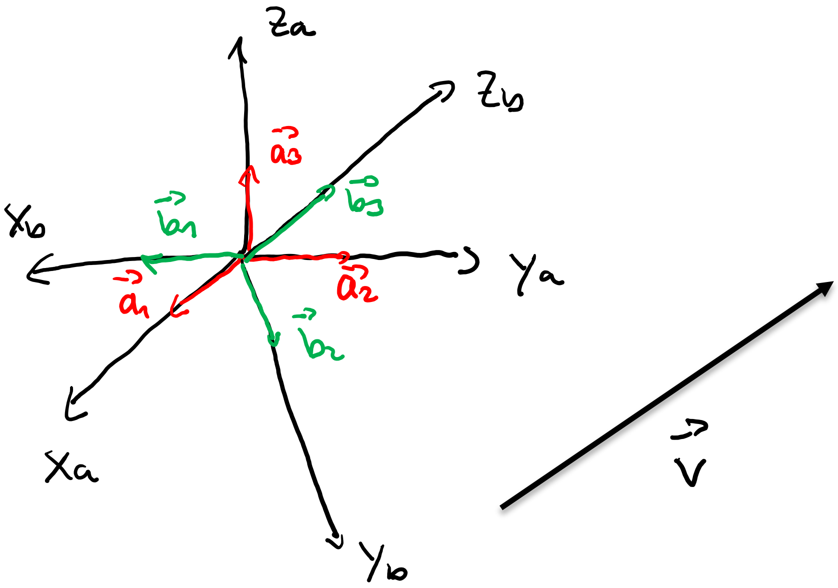
- Coordinate-system given:

$$\mathbf{I}\underline{v} = \underline{v}$$

$$\underline{v}^T\mathbf{I} = \underline{v}^T$$

Example 85: $\vec{u} \times \vec{v} = \vec{u}^{\times} \cdot \vec{v} = \vec{u} \cdot \vec{v}^{\times}$

Rotation matrix I



$$\vec{v} = \sum_i v_i \vec{a}_i$$

$$= \sum_i v_i \vec{b}_i$$

$$v_i^a = \vec{v} \cdot \vec{a}_i$$

$$v_i^b = \vec{v} \cdot \vec{b}_i$$

$$\underline{v}^a = \begin{bmatrix} v_1^a \\ v_2^a \\ v_3^a \end{bmatrix} \qquad \underline{v}^b = \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix}$$

Rotation matrix II

$$v_i^a = \vec{v} \cdot \vec{a}_i = \left(v_1^b \vec{b}_1 + v_2^b \vec{b}_2 + v_3^b \vec{b}_3 \right) \cdot \vec{a}_i$$

$$= \left(\vec{b}_1 \cdot \vec{a}_i \right) v_1^b + \left(\vec{b}_2 \cdot \vec{a}_i \right) v_2^b + \left(\vec{b}_3 \cdot \vec{a}_i \right) v_3^b$$

$$\begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \vec{b}_1 & \vec{a}_1 \vec{b}_2 & \vec{a}_1 \vec{b}_3 \\ \vec{a}_2 \vec{b}_1 & \vec{a}_2 \vec{b}_2 & \vec{a}_2 \vec{b}_3 \\ \vec{a}_3 \vec{b}_1 & \vec{a}_3 \vec{b}_2 & \vec{a}_3 \vec{b}_3 \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix}$$

$$\underset{\sim}{v}^a = R_b^a \, \underline{v}^b$$

R_b^a : "coordinate transformation from b to a"

R_b^a : "Rotation matrix from a to b.

Rotation matrix III –properties

$$R_a^b = (R_b^a)^T$$

$$\underline{v}^b = R_a^b \underline{v}^a = \underbrace{R_a^b R_b^a}_{I} \underline{v}^b$$

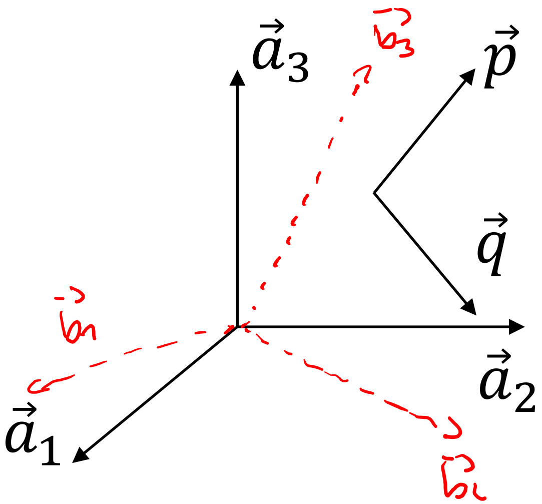
$$R_a^b = (R_b^a)^T = (R_b^a)^{-1}$$

R_b^a is an orthogonal matrix

special orthogonal group

$$SO(3) = \{ R \mid R \in \mathbb{R}^{3\times 3},\ R^T R = I,\ \det(R) = 1 \}$$

Example: Rotation of vectors



Define $\vec{q}$ $\quad \underline{q}^a = R^a_b \, \underline{p}^a$

$$\underline{q}^b = R^b_a \, \underline{q}^a = R^b_a R^a_b \, \underline{p}^a = \underline{p}^a$$

shows : R^a_b rotates $\vec{p}$ to $\vec{q}$

such that $\underline{q}^b = \underline{p}^a$

Example: Rotation matrix

$$\underline{p}^a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{a}_1^a \qquad \underline{q}^a = \mathbf{R}_b^a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{b}_1^a$$

$$\mathbf{R}_b^a = \begin{bmatrix} \underline{b}_1^a & \underline{b}_2^a & \underline{b}_3^a \end{bmatrix}$$

- $\underline{v}^a = \mathbf{R}_b^a \underline{v}^b$: coordinate transformation from b to a
- $\underline{q}^a = \mathbf{R}_b^a \underline{p}^a$: rotation from a to b

Composite rotations

$$\underline{v}^b = R^b_c \underline{v}^c$$

$$\underline{v}^a = R^a_b \underline{v}^b = \underbrace{R^a_b \cdot R^b_c}_{R^a_c} \underline{v}^c$$

$$\underline{v}^a = R^a_c \underline{v}^c$$

$$R^a_d = R^a_b R^b_c R^c_d$$

Coordinate-transformation of dyadics

$$\vec{D} = \sum_i \sum_j d_{ij}^a \vec{a}_i \vec{a}_j, \quad d_{ij}^a = \vec{a}_i \cdot \vec{D} \cdot \vec{a}_j, \quad D^a = \begin{pmatrix} d_{11}^a & d_{12}^a & \cdot \\ d_{12}^a & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\vec{D} = \sum_i \sum_j d_{ij}^b \vec{b}_i \vec{b}_j, \quad d_{ij}^b = \vec{b}_i \cdot \vec{D} \cdot \vec{b}_j, \quad D^b = \begin{pmatrix} d_{11}^b & d_{12}^b & \cdot \\ d_{12}^b & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$D^a \cdot \underline{u}^a = \underline{z}^a = R_b^a \cdot \underline{z}^b = R_b^a \cdot D^b \cdot \underline{u}^b = R_b^a \cdot D^b \cdot R_a^b \underline{u}^a$$

$$\underbrace{D^a = R_b^a \, D^b \, R_a^b}_{\text{similarity transformation}}$$

Examples

dyadic

$$\vec{\omega} = \vec{u} \times \vec{v} = (\vec{u}^{\times}) \cdot \vec{v}$$

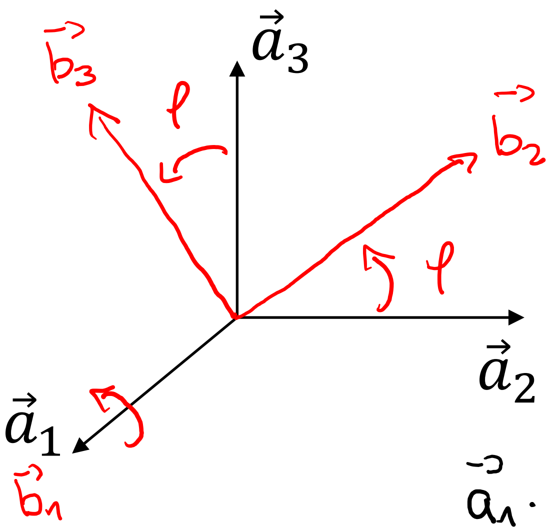
$$\underline{\omega}^a = (\underline{u}^a)^{\times} \underline{v}^a \qquad \underline{\omega}^b = (\underline{u}^b)^{\times} \underline{v}^b \qquad \underline{\omega}^b = \mathbf{R}_a^b \underline{\omega}^a$$

$$\begin{aligned}(\underline{u}^b)^{\times} \underline{v}^b &= \mathbf{R}_a^b (\underline{u}^a)^{\times} \underline{v}^a \\ &= \underbrace{\mathbf{R}_a^b (\underline{u}^a)^{\times} \mathbf{R}_b^a}_{\text{Similarity transformation}} \underline{v}^b\end{aligned}$$

Simple rotations

Scalar product:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$



$\hat{=}$ rotation around fixed axis

$$R_x(\varphi) = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \cdot & \cdot \\ \vec{a}_2 \cdot \vec{b}_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\vec{a}_1 \cdot \vec{b}_1 = 1$$

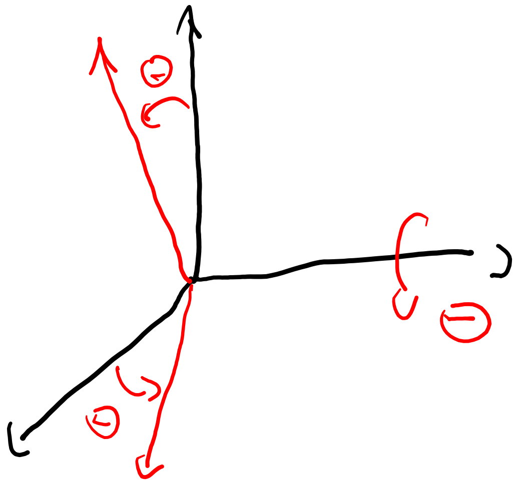
$$\vec{a}_1 \cdot \vec{b}_2 = 0 = \vec{a}_1 \cdot \vec{b}_3 = \vec{a}_3 \cdot \vec{b}_1 = \vec{a}_2 \cdot \vec{b}_1$$

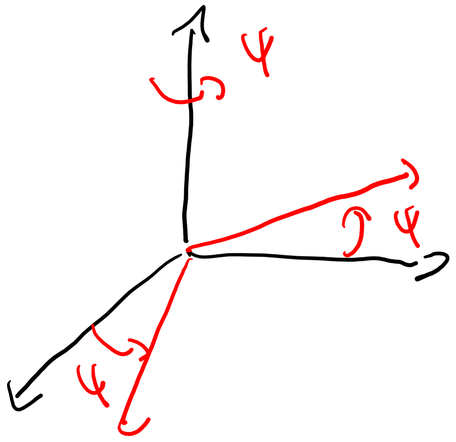
$$\vec{a}_2 \cdot \vec{b}_2 = \cos\varphi = \vec{a}_3 \vec{b}_3$$

$$\vec{a}_3 \cdot \vec{b}_2 = \cos\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi \qquad \vec{a}_2 \cdot \vec{b}_3 = \cos\left(\frac{\pi}{2} + \varphi\right) = -\sin\varphi$$

$$R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & & \cos\varphi \end{bmatrix}$$

Simple rotations II





Homework

- How are the rotation matrices around x-axis, y-axis and z-axis defined?

- What are Euler angles?

- What is the angle-axis description?