# Introduction to Differential-Algebraic Equations (DAEs)

Sébastien Gros
NTNU, Eng. Cybernetics

Guest Lecture for Modeling and Simulation
28$^{\text{th}}$ of January 2019

# Objectives of the lecture

**Learn the basics of DAEs**

- ✓ understand what a DAE is
- ✓ identify the different forms of DAEs
- ✓ understand why there are "easy" and "hard" DAEs
- ✓ introduction to the differential index

*Required background: calculus, analysis, linear algebra, basics on ODEs*

**Please interrupt me for questions/discussions!**

## Why do we use DAEs ?

# Why do we use DAEs ?

**Large systems made of many subsystems**:

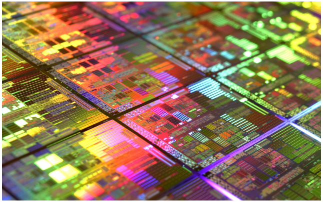
- Describe each subsystems independently (differential equations)
- Write balance equations, flows, etc... (algebraic equations)

# Why do we use DAEs ?

**Large systems made of many subsystems**:

- Describe each subsystems independently (differential equations)
- Write balance equations, flows, etc... (algebraic equations)

**E.g.**

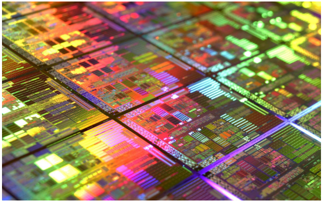


# Why do we use DAEs ?

**Large systems made of many subsystems**:

- Describe each subsystems independently (differential equations)
- Write balance equations, flows, etc... (algebraic equations)

**E.g.**



**Mechanical/Robotic systems**:

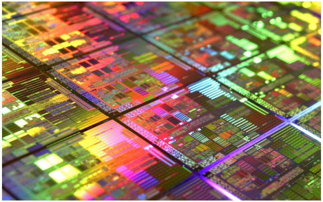
- DAE models can be much simpler than ODE models
- Modelling procedure is often easier using DAEs
- Often yields more "natural" representations of the system

# Why do we use DAEs ?

**Large systems made of many subsystems**:

- Describe each subsystems independently (differential equations)
- Write balance equations, flows, etc... (algebraic equations)

**E.g.**



**Mechanical/Robotic systems**:

- DAE models can be much simpler than ODE models
- Modelling procedure is often easier using DAEs
- Often yields more "natural" representations of the system

## Simulating ODEs

Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

**State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$**

# Simulating ODEs

Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

**State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$**

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads

# Simulating ODEs

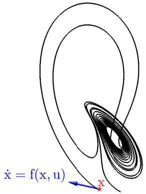
Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

**State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$**

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads



## Simulating ODEs

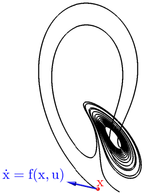
Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

**State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$**

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads




**Implicit ODEs**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$$

...simulation similarly requires one to be able to compute $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$

# Simulating ODEs

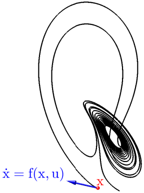
Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

**State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$**

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads




**Implicit ODEs**

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$$

...simulation similarly requires one to be able to compute $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$

**When can we do that?**

## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

## More on Implicit ODEs - Newton

How to “solve” an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**

# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$

## More on Implicit ODEs - Newton

How to “solve” an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead.

## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead.

## More on Implicit ODEs - Newton

How to “solve” an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\left\|\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)\right\| \geq \mathrm{Tol}$ **do**

Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$

# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

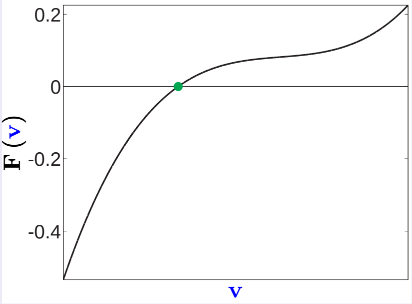
Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

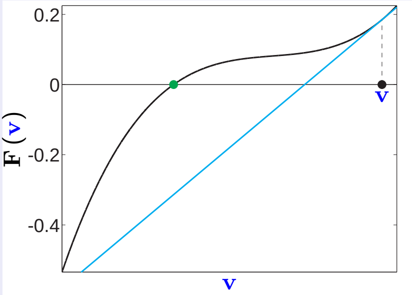
Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

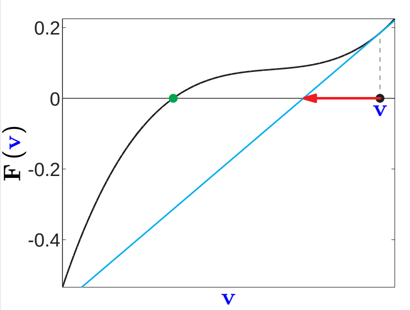
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$

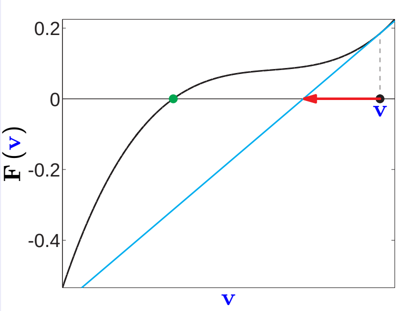
**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$
**while** $\|\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)\| \geq \mathrm{Tol}$ **do**
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$
**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

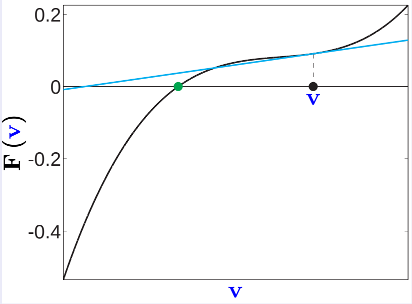
**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$
**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$
**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

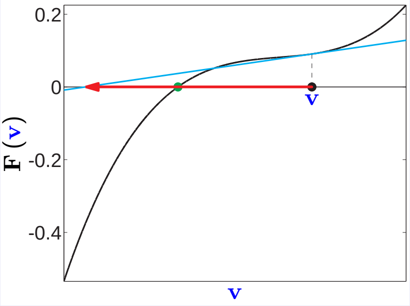
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha \Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

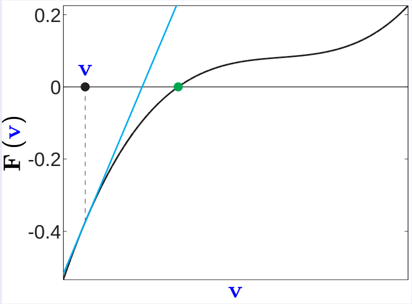
**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$
**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$
**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

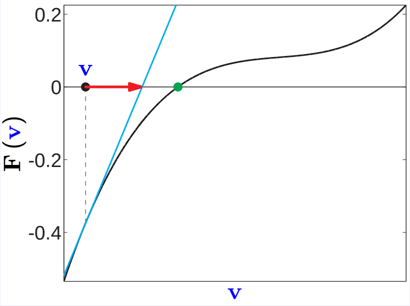
Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)\| \geq \mathrm{Tol}$ **do**

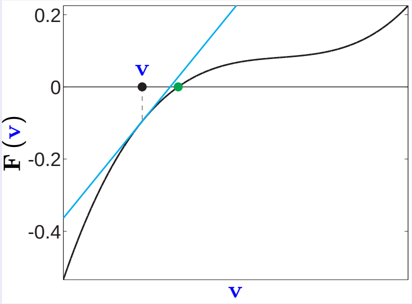
Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**

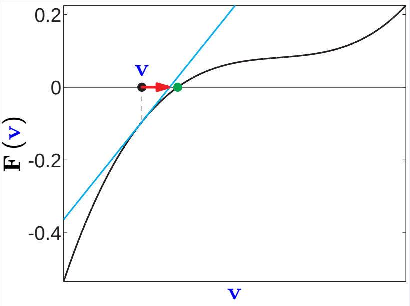
Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$

**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$

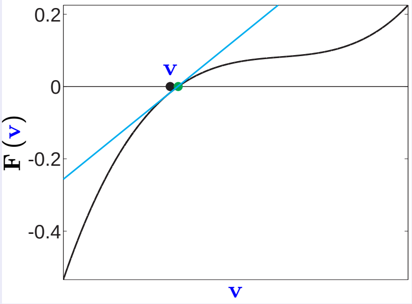
**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$
**while** $\|\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)\| \geq \mathrm{Tol}$ **do**
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right)}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha \Delta\mathbf{v}$
**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



## More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$

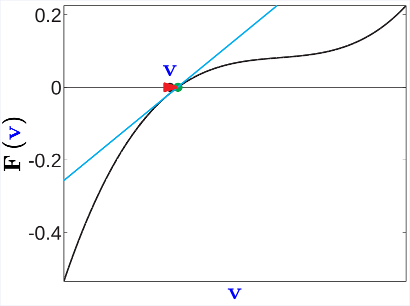
**Algorithm:** Newton method

**Input**: Guess $\mathbf{v}$, tolerance $\mathrm{Tol}$
**while** $\|\mathbf{F}(\mathbf{v}, \mathbf{w})\| \geq \mathrm{Tol}$ **do**
  Solve for $\Delta\mathbf{v}$:

$$\frac{\partial \mathbf{F}(\mathbf{v}, \mathbf{w})}{\partial \mathbf{v}} \Delta\mathbf{v} + \mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$$

  Update: $\mathbf{v} \leftarrow \mathbf{v} - \alpha\Delta\mathbf{v}$
**return** $\mathbf{v}$

For some $0 < \alpha \leq 1$



# More on Implicit ODEs - Newton

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) **implicitly**
- In order to get $\dot{\mathbf{x}}$ from $\mathbf{x}, \mathbf{u}$, we need to **solve** (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. $e^{\dot{x}} + \dot{x}^3 - u = 0$

  Use numerical methods instead. **What is that?**

**Newton** solves $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$

**Algorithm:** Newton method

**Input**: Guess $\dot{\mathbf{x}}$, tolerance $\mathrm{Tol}$

**while** $\|\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right)\| \geq \mathrm{Tol}$ **do**

  Solve for $\Delta\dot{\mathbf{x}}$:

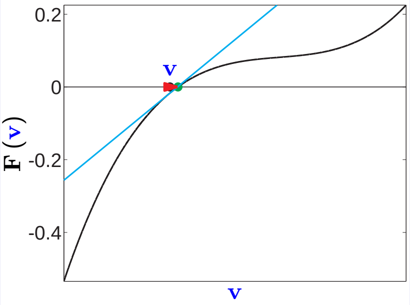
$$\frac{\partial \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right)}{\partial \dot{\mathbf{x}}} \Delta\dot{\mathbf{x}} + \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$$

  Update: $\dot{\mathbf{x}} \leftarrow \dot{\mathbf{x}} - \alpha\Delta\dot{\mathbf{x}}$

**return** $\dot{\mathbf{x}}$

For some $0 < \alpha \leq 1$

**Does this always work? Kinda...**



# Solving ODEs - Implicit Function Theorem

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{3}$$

# Solving ODEs - Implicit Function Theorem

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

## Solving ODEs - Implicit Function Theorem

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \qquad (3)$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

## Solving ODEs - Implicit Function Theorem

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

... this is about solving equations.

# Solving ODEs - Implicit Function Theorem

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

... this is about solving equations.

Generally: when can we solve e.g. $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$?

## Solving ODEs - Implicit Function Theorem

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

... this is about solving equations.

Generally: when can we solve e.g. $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$?

**I**mplicit **F**unction **T**heorem says we can <u>if</u> $\frac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

## Solving ODEs - Implicit Function Theorem

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

... this is about solving equations.

Generally: when can we solve e.g. $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$?

**I**mplicit **F**unction **T**heorem says we can if $\frac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

Special case: linear equations

$\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = A\mathbf{v} + \mathbf{w} = 0$ is solvable for $\mathbf{v}$ if $\frac{\partial \mathbf{F}}{\partial \mathbf{v}} = A$ is full rank

# Solving ODEs - Implicit Function Theorem

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{3}$$

Can we necessarily solve (3) for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$ ?

... this is about solving equations.

Specifically: when can we solve $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ for $\dot{\mathbf{x}}$?

**I**mplicit **F**unction **T**heorem says we can <u>if</u> $\dfrac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is full rank (at $\mathbf{x}, \mathbf{u}$ given)

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \qquad (4)$$

State is

$$\mathbf{x} = \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}\right]$$

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \quad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where $\dot{\mathbf{x}}_2$ does not appear in, hence (4) does not deliver $\dot{\mathbf{x}}$...

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[ \begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array} \right] = 0 \quad (4)$$

State is

$$\mathbf{x} = \left[ \begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array} \right]$$

where $\dot{\mathbf{x}}_2$ does not appear in, hence (4) does not deliver $\dot{\mathbf{x}}$...

Jacobian reads as

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right]$$

$\rightarrow$ (4) is a DAE

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where $\dot{\mathbf{x}}_2$ does not appear in, hence (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian reads as

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \quad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where $\dot{\mathbf{x}}_2$ does not appear in, hence (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian reads as

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

Actually (4) yields

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1$$
$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

## What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \qquad (4)$$

State is

$$\mathbf{x} = \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}\right]$$

where $\dot{\mathbf{x}}_2$ does not appear in, hence (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian reads as

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$$

$\rightarrow$ (4) is a DAE

Actually (4) yields

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1$$
$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

In applications, DAEs are most often differential equations where **some states derivatives do not appear** as in e.g. (4)

## What are DAEs?

### Definition

An implicit differential equation $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[ \begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array} \right] = 0$$

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[ \begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array} \right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array}\right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

**This is a DAE even though both $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ appear**

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array}\right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

has solution

$$\mathbf{x}_1 = -\dot{u}$$
$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

DAE well defined only for $u$ continuous!

**This is a DAE even though both $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ appear**

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

# What are DAEs?

## Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[ \begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\, \dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{array} \right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{array} \right]$$

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix}$$

For $\mathbf{x}_2(0) = \mathbf{x}_1(0)$, has solution

$$\dot{\mathbf{x}}_1 = u - \mathbf{x}_1$$
$$\mathbf{x}_2 = \mathbf{x}_1$$

Is it a DAE or an ODE?

# What are DAEs?

**Definition**

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix}$$

For $\mathbf{x}_2(0) = \mathbf{x}_1(0)$, has solution

$$\dot{\mathbf{x}}_1 = u - \mathbf{x}_1$$
$$\mathbf{x}_2 = \mathbf{x}_1$$

Is it a DAE or an ODE? It can be both!

**A differential equations can be both an ODE & DAE, even jump back-and-forth. Avoided in practice, i.e. we like $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ having fixed rank**

# A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

# A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \text{ st}\dot{\text{a}}\text{tes}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

## A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \ \dot{\text{states}}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

- Fully-Implicit DAEs

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

# A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \text{ states}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

- Fully-Implicit DAEs

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

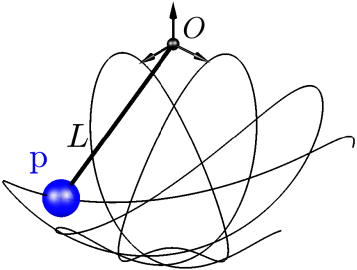
- Semi-explicit DAEs

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\longrightarrow$ "explicit ODE + algebraic equations"

# DAEs in Mechanics - A small example

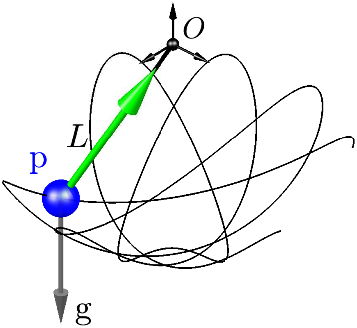
Pendulum simulation




- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass

# DAEs in Mechanics - A small example

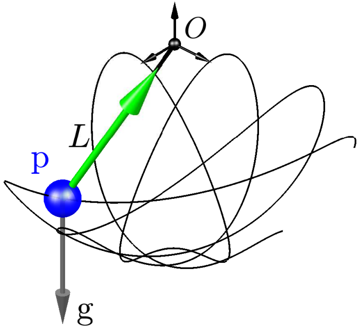
Pendulum simulation




- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$

# DAEs in Mechanics - A small example

Pendulum simulation



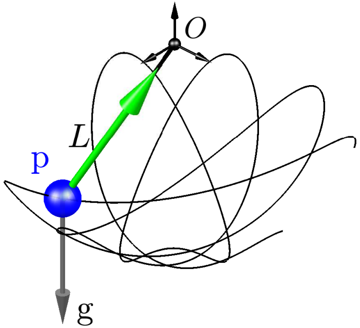

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

where $\vec{\mathbf{g}}$ is gravity.

# DAEs in Mechanics - A small example

Pendulum simulation



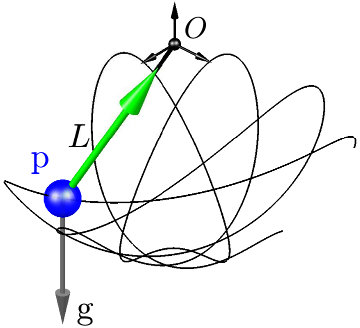

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

  where $\vec{\mathbf{g}}$ is gravity.
- Direction of force $\mathbf{F}$ is $-\mathbf{p}$.
- What magnitude?

# DAEs in Mechanics - A small example

Pendulum simulation




- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

where $\vec{\mathbf{g}}$ is gravity.

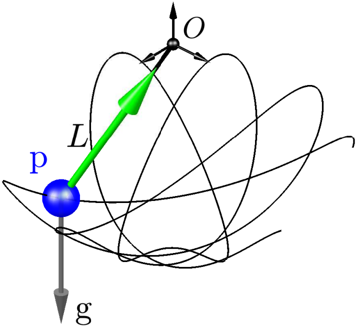
- Direction of force $\mathbf{F}$ is $-\mathbf{p}$.
- What magnitude? Define

$$\mathbf{F} = -z\mathbf{p}$$

where $z \in \mathbb{R}$

# DAEs in Mechanics - A small example

Pendulum simulation




**Semi-explicit DAE:**

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

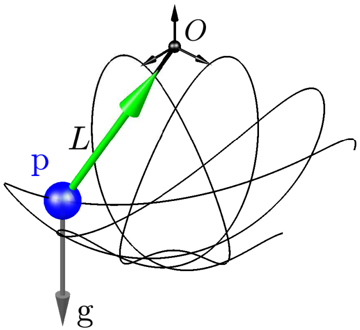
  where $\vec{\mathbf{g}}$ is gravity.
- Direction of force $\mathbf{F}$ is $-\mathbf{p}$.
- What magnitude? Define

$$\mathbf{F} = -z\mathbf{p}$$

  where $z \in \mathbb{R}$

# DAEs in Mechanics - A small example

Pendulum simulation




**Semi-explicit DAE:**

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

where $\vec{\mathbf{g}}$ is gravity.

- Direction of force $\mathbf{F}$ is $-\mathbf{p}$.
- What magnitude? Define

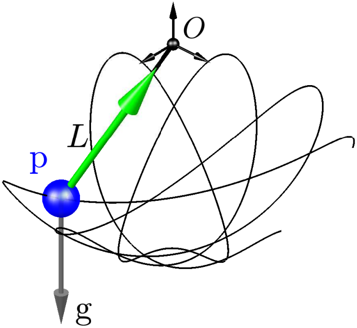
$$\mathbf{F} = -z\mathbf{p}$$

where $z \in \mathbb{R}$

- Algebraic variable $z$ "adjusts" $\mathbf{F}$ to keep $\mathbf{p}$ at distance $L$ from $O$

## DAEs in Mechanics - A small example

Pendulum simulation




**Semi-explicit DAE:**

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Cable force $\mathbf{F}$ maintains $\mathbf{p}$ at a distance $L$ from $O$
- Motion (Newton's law):

$$\ddot{\mathbf{p}} = \vec{\mathbf{g}} + \mathbf{F}$$

  where $\vec{\mathbf{g}}$ is gravity.
- Direction of force $\mathbf{F}$ is $-\mathbf{p}$.
- What magnitude? Define

$$\mathbf{F} = -z\mathbf{p}$$

  where $z \in \mathbb{R}$

- Algebraic variable $z$ "adjusts" $\mathbf{F}$ to keep $\mathbf{p}$ at distance $L$ from $O$
- DAE must hold this specification as a constraint

## Conversion semi-explicit $\leftrightarrow$ fully-implicit

| Semi-explicit DAEs | | Fully-Implicit DAEs |
|---|---|---|
| $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$<br>$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$ | $\longrightarrow$ | |
| | $\longleftarrow$ | $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$ |

## Conversion semi-explicit $\leftrightarrow$ fully-implicit

| Semi-explicit DAEs | | Fully-Implicit DAEs |
|---|---|---|
| $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$<br>$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$ | $\longrightarrow$ | $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = \left[ \begin{array}{c} \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{array} \right] = 0$ |
| | $\longleftarrow$ | $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$ |

# Conversion semi-explicit $\leftrightarrow$ fully-implicit

| Semi-explicit DAEs | | Fully-Implicit DAEs |
|---|---|---|
| $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u})$<br>$0 = \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u})$ | $\longrightarrow$ | $\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u}) \\ \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u}) \end{bmatrix} = 0$ |
| $\dot{\mathbf{x}} = \mathbf{v}$<br>$0 = \mathbf{F}(\mathbf{v},\mathbf{z},\mathbf{x},\mathbf{u})$ | $\longleftarrow$ | $\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = 0$ |

# Conversion semi-explicit $\leftrightarrow$ fully-implicit

| Semi-explicit DAEs | | Fully-Implicit DAEs |
|---|---|---|
| $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$<br>$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$ | $\longrightarrow$ | $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{bmatrix} = 0$ |
| $\dot{\mathbf{x}} = \mathbf{v}$<br>$0 = \mathbf{F}(\mathbf{v}, \mathbf{z}, \mathbf{x}, \mathbf{u})$ | $\longleftarrow$ | $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$ |

Fully-implicit or semi-explicit DAEs are not *really* different.

- we like semi-explicit DAEs for their neat structure
- fully-implicit $\rightarrow$ semi-explicit adds variables $\mathbf{v}$, can be counter-productive

# Simulating a DAE

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

**What does it mean to be able to simulate a DAE "easily"?**

# Simulating a DAE

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

**What does it mean to be able to simulate a DAE "easily"?**

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

## Simulating a DAE

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

**What does it mean to be able to simulate a DAE "easily"?**

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

# Simulating a DAE

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

**What does it mean to be able to simulate a DAE "easily"?**

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

As long as the DAE delivers $\dot{\mathbf{x}}, \mathbf{z}$, simulating is "just" about following $\dot{\mathbf{x}}$.

# Simulating a DAE

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

**What does it mean to be able to simulate a DAE "easily"?**

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

As long as the DAE delivers $\dot{\mathbf{x}}, \mathbf{z}$, simulating is "just" about following $\dot{\mathbf{x}}$.

When can the algebraic equation

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\mathbf{z}$?

When can the DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$?

## "Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

## "Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

### Reminder - Solving equations

Generally: when can we solve e.g. $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$?

**I**mplicit **F**unction **T**heorem says we can if $\dfrac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

## "Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

### Reminder - Solving equations

Generally: when can we solve e.g. $\mathbf{F}(\mathbf{v}, \mathbf{w}) = 0$ for $\mathbf{v}$?

**I**mplicit **F**unction **T**heorem says we can if $\dfrac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

### Semi-explicit DAE case

- Getting $\dot{\mathbf{x}}$ from the first equation is trivial
- **I**mplicit **F**unction **T**heorem says that we can solve $\mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) = 0$ for $\mathbf{z}$

if (square) Jacobian $\dfrac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank

# "Easy" DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

## "Easy" DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

### Reminder - Solving equations

Generally: when can we solve e.g. $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$?

**I**mplicit **F**unction **T**heorem says we can <u>if</u> $\dfrac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

# "Easy" DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$ ?

## Reminder - Solving equations

Generally: when can we solve e.g. $\mathbf{F}\left(\mathbf{v}, \mathbf{w}\right) = 0$ for $\mathbf{v}$?

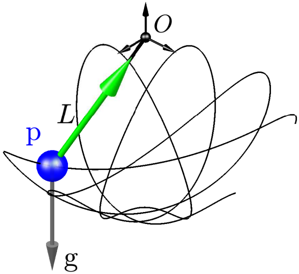
**I**mplicit **F**unction **T**heorem says we can if $\dfrac{\partial \mathbf{F}}{\partial \mathbf{v}}$ is full rank (at $\mathbf{w}$ given)

## Fully-Implicit DAE case

**I**mplicit **F**unction **T**heorem says that we can solve $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$ for $\dot{\mathbf{x}}$, $\mathbf{z}$

if (square) Jacobian $\left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{array}\right]$ is full rank (it is square)

# Back to our mechanical example




**Semi-explicit DAE:**

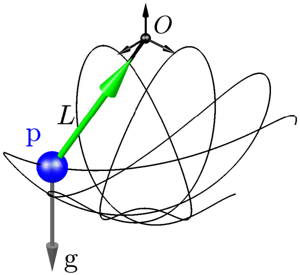
$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \tag{6b}$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

## Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

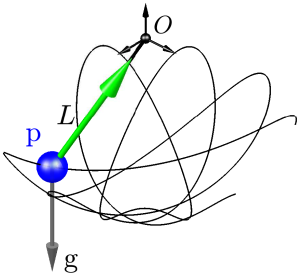
$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \tag{6b}$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

# Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \tag{6b}$$

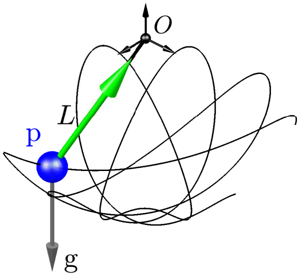
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$

# Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{array}{l}\dot{\mathbf{p}} = \mathbf{v} \\ \dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}\end{array}\right\} \equiv \mathbf{f} \qquad (6a)$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \qquad (6b)$$

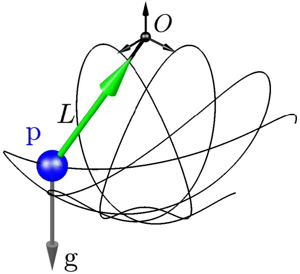
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, $z$ does not enter in $\mathbf{g}$!

# Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \tag{6b}$$

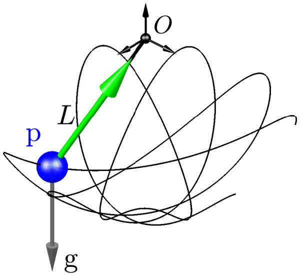
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, $z$ does not enter in $\mathbf{g}$!
- (6b) cannot be solved for $z$...

# Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\end{aligned}\right\} \equiv \mathbf{f} \qquad (6a)$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \qquad (6b)$$

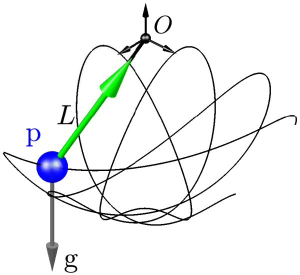
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, $z$ does not enter in $\mathbf{g}$!
- (6b) cannot be solved for $z$...
- (6a) delivers $\dot{\mathbf{x}}$ if we know $z$...

# Back to our mechanical example




**Semi-explicit DAE:**

$$\left.\begin{array}{l}\dot{\mathbf{p}} = \mathbf{v} \\ \dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}\end{array}\right\} \equiv \mathbf{f} \qquad (6a)$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \qquad (6b)$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

**Is that an "easy" DAE?**

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, $z$ does not enter in $\mathbf{g}$!
- (6b) cannot be solved for $z$...
- (6a) delivers $\dot{\mathbf{x}}$ if we know $z$...

**It is not an "easy" DAE.**

## "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

## "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

... but they (may) deliver well-defined trajectories nonetheless!!

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

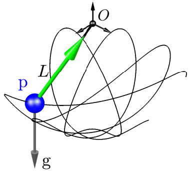
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

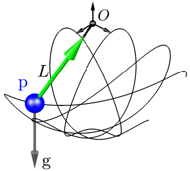
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



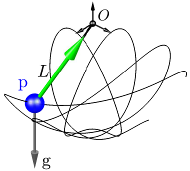

**Some insights**

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




**Some insights**

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds

## "Hard" DAEs

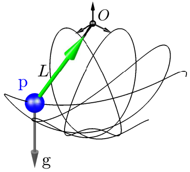
**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




### Some insights

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}}$$

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

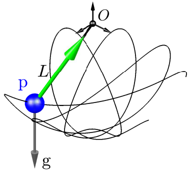
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




**Some insights**

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}}$$

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

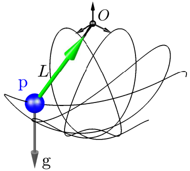
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




**Some insights**

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p}$$

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

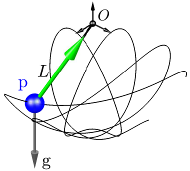
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




## Some insights

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

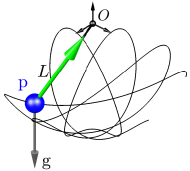
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




## Some insights

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects $z$ to (7c) via **2 integrations**

# "Hard" DAEs

**Definition** (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

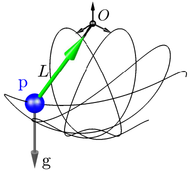
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




## Some insights

- $z$ "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects $z$ to (7c) via **2 integrations**

- How to reconnect $z$ to (7c) *algebraically*? (equation we could "solve for $z$"...)

## “Hard” DAEs

**Definition** (informal): “hard” DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

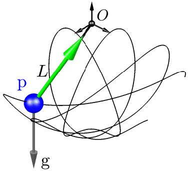
... but they (may) deliver well-defined trajectories nonetheless!!

**Example**: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$




### Some insights

- $z$ “adjusts” $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of “influence”:

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects $z$ to (7c) via **2 integrations**

- How to reconnect $z$ to (7c) *algebraically*? (equation we could “solve for $z$”...)

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ twice on (7c) to “rewind” the chain to $z$

# Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to $z$

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

# Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to $z$

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

# Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to $z$

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v} \end{aligned}$$

# Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to $z$

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v} \end{aligned}$$

**Yields "easy" DAE:**

$$\dot{\mathbf{p}} = \mathbf{v} \tag{8a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{8b}$$

$$z = \frac{\mathbf{p}^\top \vec{\mathbf{g}} + \mathbf{v}^\top \mathbf{v}}{\mathbf{p}^\top \mathbf{p}} \tag{8c}$$

as (8c) delivers $z$

# Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to $z$

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \mathbf{v} \end{aligned} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v} \end{aligned}$$

**Yields "easy" DAE:**

$$\dot{\mathbf{p}} = \mathbf{v} \tag{8a}$$

$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p} \tag{8b}$$

$$z = \frac{\mathbf{p}^\top \vec{\mathbf{g}} + \mathbf{v}^\top \mathbf{v}}{\mathbf{p}^\top \mathbf{p}} \tag{8c}$$

as (8c) delivers $z$

$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$ **One more $\frac{\mathrm{d}}{\mathrm{d}t}$ on** (8c) **turns** (8) **into an ODE**

# Differential Index

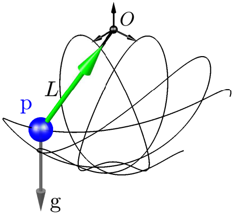
**Definition**

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

# Differential Index

**Definition**

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE




**Pendulum example**

**Index 3**

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

Physical model
"Hard" DAE

**Index 1**

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \vec{\mathbf{g}} - z\mathbf{p}$$
$$0 = 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v}$$

$$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

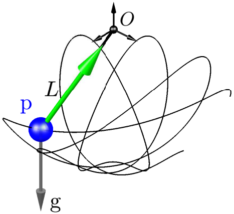
**ODE**

2 time-differentiations
$\rightarrow$ "Easy" DAE

# Differential Index

**Definition**

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE




**Pendulum example**

**Index 3**

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned}$$

Physical model
"Hard" DAE

$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$

**Index 1**

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= 2\mathbf{p}^\top\dot{\mathbf{v}} + 2\mathbf{v}^\top\mathbf{v}\end{aligned}$$

2 time-differentiations
→ "Easy" DAE

$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$ **ODE**

The transformation **index-n** $\xrightarrow{\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}}$ **index-1** is called **index reduction**

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\dot{\mathbf{z}} = -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

## Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\dot{\mathbf{z}} = -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

## Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\dot{\mathbf{z}} = -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

# Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

**Index-1 semi-explicit DAE:**

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\dot{\mathbf{z}} = -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

where $\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}$ must be full rank.

Delivers $\ddot{\mathbf{x}}$ and $\dot{\mathbf{z}}$.

# Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

# Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

**Index-1 semi-explicit DAE:**

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\
0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)
\end{aligned}
$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by $1^{\text{st}}$ equations
- DAE can be "easily" simulated

# Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

**Index-1 semi-explicit DAE:**

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})\\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by $1^{\text{st}}$ equations
- DAE can be "easily" simulated

# Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

**Index-1 semi-explicit DAE:**

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by $1^{\text{st}}$ equations
- DAE can be "easily" simulated

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

# Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

**Index-1 semi-explicit DAE:**

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by $1^{\text{st}}$ equations
- DAE can be "easily" simulated

**Index-1 fully-implicit DAE:**

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

where $\left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{array}\right]$ is full rank.

Then

- IFT: $\mathbf{F} = 0$ can be solved for $\dot{\mathbf{x}}, \mathbf{z}$
- DAE can be "easily" simulated

# Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

# Index reduction and consistency conditions

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned}$$

## Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned} \quad \longrightarrow \quad \begin{aligned}&\dot{\mathbf{p}} = \mathbf{v}\\ &\begin{bmatrix} I & \mathbf{p}\\ \mathbf{p}^\top & 0\end{bmatrix}\begin{bmatrix}\dot{\mathbf{v}}\\ z\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{g}}\\ -\mathbf{v}^\top\mathbf{v}\end{bmatrix}\end{aligned}$$

# Index reduction and consistency conditions

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned} \quad \longrightarrow \quad \begin{gathered} \dot{\mathbf{p}} = \mathbf{v} \\ \begin{bmatrix} I & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ z \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{g}} \\ -\mathbf{v}^\top \mathbf{v} \end{bmatrix} \end{gathered}$$

Original model imposes

$$\mathbf{p}^\top \mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top \mathbf{p}\right) = 0 \tag{10}$$

**Does** (10) $\Rightarrow$ (9) **??** I.e. does index-1 model match original model?

# Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned} \longrightarrow \begin{gathered}\dot{\mathbf{p}} = \mathbf{v}\\ \begin{bmatrix} I & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{v}} \\ z \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{g}} \\ -\mathbf{v}^\top\mathbf{v} \end{bmatrix}\end{gathered}$$

Original model imposes

$$\mathbf{p}^\top\mathbf{p} = L^2 \qquad (9)$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top\mathbf{p}\right) = 0 \qquad (10)$$

**Does** (10) $\Rightarrow$ (9) **??** I.e. does index-1 model match original model?

Let's write $c(t) = \mathbf{p}(t)^\top\mathbf{p}(t) - L^2$

## Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned} \quad \longrightarrow \quad \begin{gathered}\dot{\mathbf{p}} = \mathbf{v}\\ \begin{bmatrix} I & \mathbf{p}\\ \mathbf{p}^\top & 0\end{bmatrix}\begin{bmatrix}\dot{\mathbf{v}}\\ z\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{g}}\\ -\mathbf{v}^\top\mathbf{v}\end{bmatrix}\end{gathered}$$

Original model imposes

$$\mathbf{p}^\top\mathbf{p} = L^2 \qquad (9)$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top\mathbf{p}\right) = 0 \qquad (10)$$

**Does** (10) $\Rightarrow$ (9) **??** I.e. does index-1 model match original model?

Let's write $c(t) = \mathbf{p}(t)^\top\mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c}(t) = 0 \quad \Rightarrow \quad c(t) = c(0) + \dot{c}(0)t$$

## Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned} \quad \longrightarrow \quad \begin{gathered}\dot{\mathbf{p}} = \mathbf{v}\\ \begin{bmatrix} I & \mathbf{p}\\ \mathbf{p}^\top & 0\end{bmatrix}\begin{bmatrix}\dot{\mathbf{v}}\\ z\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{g}}\\ -\mathbf{v}^\top\mathbf{v}\end{bmatrix}\end{gathered}$$

Original model imposes

$$\mathbf{p}^\top\mathbf{p} = L^2 \qquad (9)$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top\mathbf{p}\right) = 0 \qquad (10)$$

**Does** (10) $\Rightarrow$ (9) **??** I.e. does index-1 model match original model?

Let's write $c(t) = \mathbf{p}(t)^\top\mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c}(t) = 0 \quad \Rightarrow \quad c(t) = c(0) + \dot{c}(0)t$$

Hence index-1 model imposes $c(t) = 0$ for all $t$ iff:

$$c(0) = 0 \qquad \dot{c}(0) = 0$$

# Index reduction and consistency conditions

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= \vec{\mathbf{g}} - z\mathbf{p}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned} \quad \longrightarrow \quad \begin{aligned}&\dot{\mathbf{p}} = \mathbf{v}\\ &\begin{bmatrix} I & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{v}} \\ z \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{g}} \\ -\mathbf{v}^\top\mathbf{v} \end{bmatrix}\end{aligned}$$

Original model imposes

$$\mathbf{p}^\top\mathbf{p} = L^2 \qquad (9)$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top\mathbf{p}\right) = 0 \qquad (10)$$

**Does** (10) $\Rightarrow$ (9) **??** I.e. does index-1 model match original model?

Let's write $c(t) = \mathbf{p}(t)^\top\mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c}(t) = 0 \quad \Rightarrow \quad c(t) = c(0) + \dot{c}(0)t$$

Hence index-1 model imposes $c(t) = 0$ for all $t$ iff:

$$c(0) = 0 \qquad \dot{c}(0) = 0$$

**These are called consistency conditions. Must be satisfied by $\mathbf{x}(0)$.**

## Wrap-up: what did we discuss?

- DAEs are differential equations that do not deliver the *entire state derivatives*, e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

  does not deliver $\dot{\mathbf{z}}$

- Some DAEs are ambiguous, often avoided in practice
- Conversion

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0 \qquad \longleftrightarrow \qquad \begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

  is always possible but not always beneficial

- There are “easy” and “hard” DAEs:

| DAE | Index | Solvability |
|---|---|---|
| “Easy” | 1 | equations readily provide $\dot{\mathbf{x}}, \mathbf{z}$ |
| “Hard” | $> 1$ | equations do not readily provide $\dot{\mathbf{x}}, \mathbf{z}$ |

- High-index DAEs can be transformed to low-index ones

# What's beyond this lecture?

- Consistency conditions for high-index DAEs
- Tikhonov theorem: how DAEs approximate stiff ODEs?
- Numerical methods for DAEs: how to simulate them efficiently?
- Numerical methods for high-index DAEs: how to bypass index-reduction?