NTNU
Norwegian University of
Science and Technology
Department of Electronic Systems

TTT4175 Estimation, Detection and Classification
Project descriptions: Estimation Theory

The two projects described here are both addressing the following problem – we have a complex exponential embedded in white, complex Gaussian noise

$$x(t) = Ae^{i(\omega_0 t+\phi)} + w(t) \quad (1)$$

where there are three unknown parameters that need to be estimated:

- Amplitude $A > 0$.
- Frequency $\omega_o > 0$.
- Phase $-\pi < \phi < \pi$

The problem is of great practical interest, illustrated by the following examples:

- In some communications systems, a carrier signal is modulated by a baseband signal at the transmitter, which in turn is extracted from the carrier signal at the receiver. The carrier signal as observed at the receiver will have the form seen in Equation 1. The frequency ω_0 is unknown because two pieces of hardware will be different due to component variations, the phase is a function of the unknown distance between transmitter and receiver and the amplitude is the attenuation of the signal. To successfully extract the baseband signal, all three parameters describing the received carrier signal must be estimated first.
- In distributed transmit beamforming each node, say a sensor, has its own, independent local oscillator (LO). Due to the variation of the LOs of 10-100 ppm (parts per million), the nodes need to be synchronized to a common frequency reference. In a master-slave architecture, the ”slave” nodes will estimate the frequency sent from a ”master” node and adjust their LOs to synchronize with the other nodes. Again, the model is as in Equation 1.

In what follows we will consider a sampled, bandlimited signal

$$\begin{aligned} x[n] =& x(nT) \\ =& Ae^{i(\omega_0 nT+\phi)} + w(nT) \\ =& Ae^{i(\omega_0 nT+\phi)} + w[n], \end{aligned}$$

with

$$
\begin{aligned}
&w[n] = w_r[n] + i w_i[n] \\
&E[w[n]] = 0 \\
&\text{var}(w_r[n]) = \text{var}(w_i[n]) = \sigma^2 \\
&E[w_r[n] w_i[n]] = 0
\end{aligned}
$$

The two projects are defined as follows:

1. In the first project we investigate the maximum likelihood estimator (MLE) for the frequency and phase estimations problems.
2. In the second project a best linear unbiased estimator (BLUE) is used in the high signal-to-noise range.
3. Both projects will rely on computer simulations and some programming in MATLAB, Python or other languages is needed. Please do not include any code in your final reports.

In both projects we will use the following:

- The sampling rate $F_s = 10^6$ Hz, meaning that $T = 10^{-6}$ sec.
- We will use $\omega_0 = 2\pi f_0$, where $f_0 = 10^5$ Hz.
- Set the phase $\phi = \pi/8$.
- The amplitude $A = 1$ is considered known.
- We have N samples from $n = n_0$ through $n = n_0 + N - 1$. The reason for specifying the initial sampling time n_0 is to simplify the CLRB as will be shown below.
- The signal-to-noise ratio (SNR) is defined as

$$
SNR = \frac{A^2}{2\sigma^2}
$$

The Cramer-Rao bound for unknown phase and frequency (but known amplitude) is given by

$$
\text{var}(\hat{\omega}) \geq \frac{12\sigma^2}{A^2 T^2 N(N^2 - 1)} \tag{2}
$$

$$
\text{var}(\hat{\phi}) \geq \frac{12\sigma^2(n_0^2 N + 2n_0 P + Q)}{A^2 N^2 (N^2 - 1)} \tag{3}
$$

$$
\tag{4}
$$

where

$$P = \frac{N(N-1)}{2}$$
$$Q = \frac{N(N-1)(2N-1)}{6}$$

We see that when setting the first sampling time $n_0 = -P/N$, the matrix becomes diagonal and we can analyse the frequency and phase estimators independently.

Problem 1 Project 1: Maximum likelihood estimator

It can be shown that the maximum likelihood estimate for the frequency and phase given N samples are as follows:

$$\hat{\omega} = \arg\max_{\omega_0} |F(\omega_0)| \tag{5}$$

where

$$F(\omega_0) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-i\omega nT} \tag{6}$$

Given the frequency estimate, we can directly estimate the phase as

$$\hat{\phi} = \angle \left\{ e^{-i\hat{\omega} n_0 T} F(\hat{\omega}) \right\} \tag{7}$$

We define the estimation errors as

$$e_\omega = \omega_0 - \hat{\omega}$$
$$e_\phi = \phi - \hat{\phi}$$

There is no closed form solution to Equation 5. Instead we will use the the fact that $F(\omega_0)$ is a discrete Fourier tranform of the observations. If we take an M-point FFT of the observed signal (zero-padding if necessary), then the mth frequency bin will correspond to the frequency

$$\omega_m = \frac{2\pi m}{MT}.$$

We can then find the MLE by

$$\hat{\omega}_{FFT} = \frac{2\pi m^\star}{MT} \tag{8}$$

where

$$m^\star = \arg\max_m \mathrm{FFT}_M\{\mathbf{x}\}. \tag{9}$$

Here FFT_M denotes an M-point FFT, and $\mathbf{x}$ is the signal vector.

1a) Investigate the performance of the FFT-based MLE by comparing the variance of the frequency and phase estimation errors to the CRLB. Use the following parameters in your simulations:

- Samples number $N = 513$, which gives $n_0 = -256$.
- SNRs (in decibel): -10 through 60, in steps of 10 dB.
- FFT size M: 2^k for $k = 10, 12, 14, 16, 18, 20$. Use zero-padding

1b) Using FFT sizes of 2^{20} in not feasible in practice. Use the estimate for FFT size 2^{10}, and then fine tune the estimate using a numerical search method (Hint: The simplest approach is to use `fminsearch` in MATLAB, or `scipy.optimize.minimize` using Nelder-Mead in Python)

Problem 2 Project 2: BLUE for high SNRs

For some practical applications the MLE approach is too complex with respect to memory use and computational complexity. Instead we use the approximation

$$x[n] \approx Ae^{i(\omega_0 nT + \phi + v[n])} \tag{10}$$

where $v[n]$ is zero mean, iid. noise. The approximation can be shown to be good under high SNRs. Now all the information we need is contained in the phase angle of the observations,

$$\angle x[n]) = \approx \omega_0 nT + \phi + v[n], \tag{11}$$

which we see is a linear system. Since $v[n]$ isn't Gaussian we will be using BLUE.

One practical problem that need to be handled is the problem of *phase unwrapping*. When computing the phase $\angle x[n]$ we will always get some number in the range $[-\pi/2, \pi/2$, which clearly is a problem since we want the phase to grow linearly in n. The solution is to use the MATLAB function `unwrap` (or `numpy.unwrap` in Python). Note that in low SNR conditions this will sometimes fail, which is fine. This is a method that is to be used for high SNRs.

2a) Experiment with the `unwrap` function and plot some of the results for low (-10 dB) and high (30 dB) SNRs.

2b) Compare the BLUE estimator with the CRLB wrt. the frequency and phase error variances for the following parameters.

- Samples number $N = 513$, which gives $n_0 = -256$.
- SNRs (in decibel): -10 through 40, in steps of 10 dB.

2c) An alternatiove approach that doesn't rely on phase unwrapping is based on the difference between two phase estimates

$$\angle x[n+1] - \angle x[n] = \omega_0 T + v[n+1] - v[n]. \tag{12}$$

Here the noise is no longer white between observations. Use the BLUE for colored noise to compute estimates for the same parameters as in the previous problem.

Note that there is no phase estimate direwctly available with this approach. Instead we use the direct ML estimate, which is in closed form when $\hat{\omega}$ is known:

$$\hat{\phi} = \angle \left\{ e^{-i\hat{\omega} n_0 T} F(\hat{\omega}) \right\} \tag{13}$$