

Lecture 17: Lagrangian mechanics (Lagrange's equation of motion)

Electrical motor (passivity)

Newton's law (for particles, or Newton-Euler EoM for rigid bodies) in combination with

- d'Alembert's principle
- Generalized coordinates

gives Lagrange's equations of motion

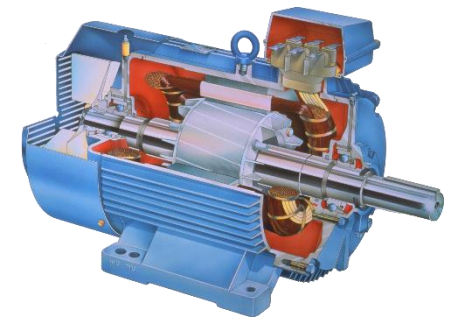
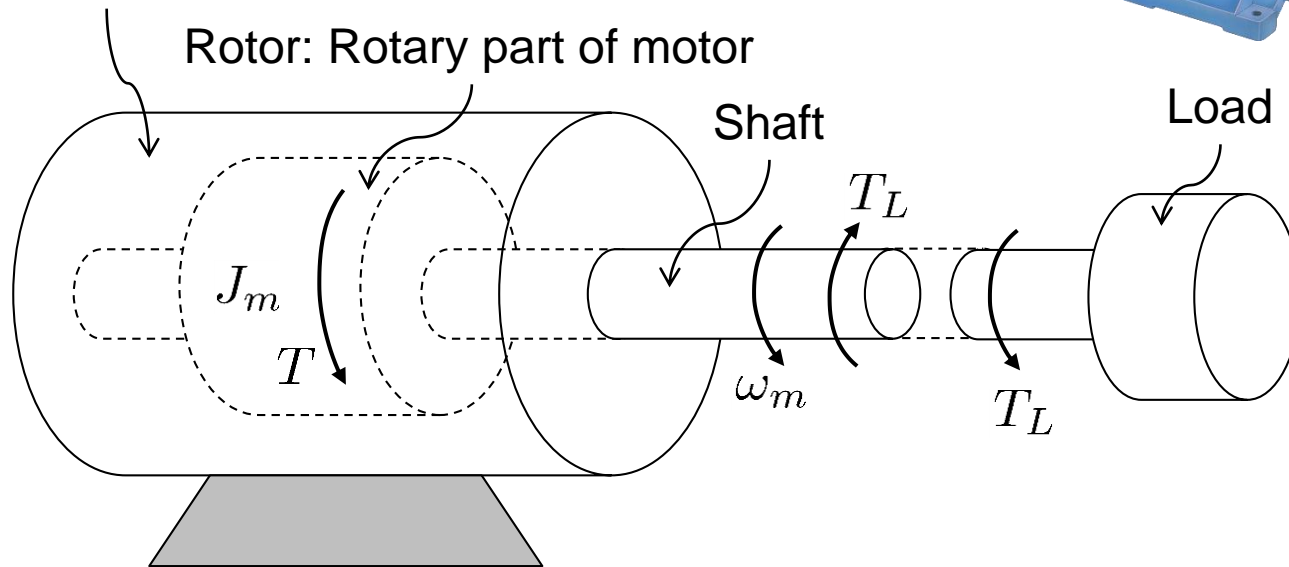
- Brief examples

Book: Ch. 3.3, 7.7, 8.2

Motors

Stator: Stationary part of motor

Rotor: Rotary part of motor



- Equation of motion for motor shaft:

$$J_m \dot{\omega}_m = T - T_L$$

where

- T : Motor torque (set up by some device, e.g. DC motor)
- T_L : Load torque
- J_m : Moment of inertia for rotor and shaft
- ω_m : Angular velocity/motor speed [rad/s, or rev./min]

Gears

Rotational gear
(cogwheel)



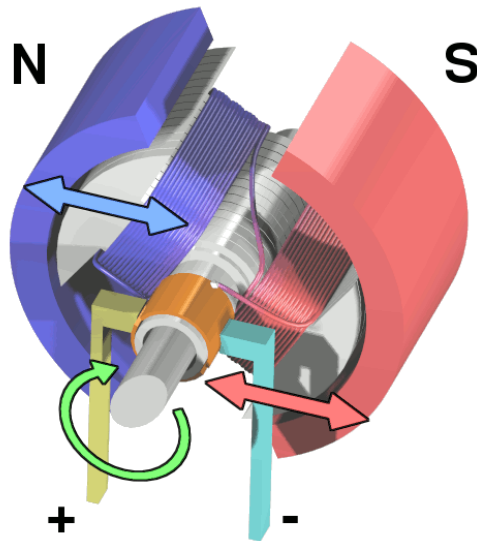
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Translational gear
(rack and pinion)

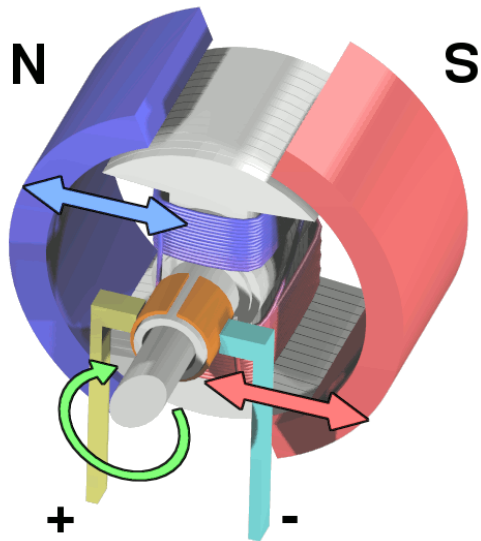


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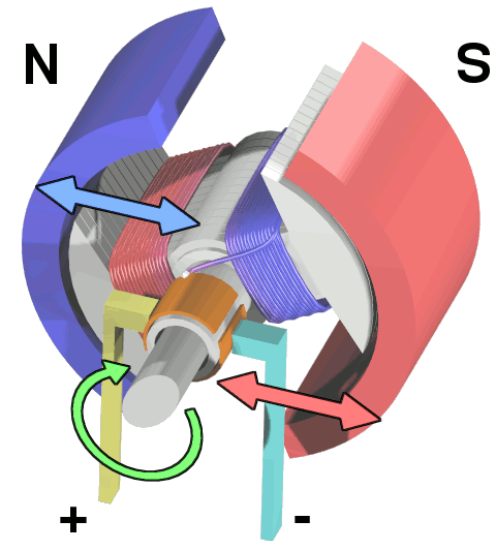
A simple DC electric motor



A simple DC electric motor. When the coil is powered, a magnetic field is generated around the armature. The left side of the armature is pushed away from the left magnet and drawn toward the right, causing rotation.



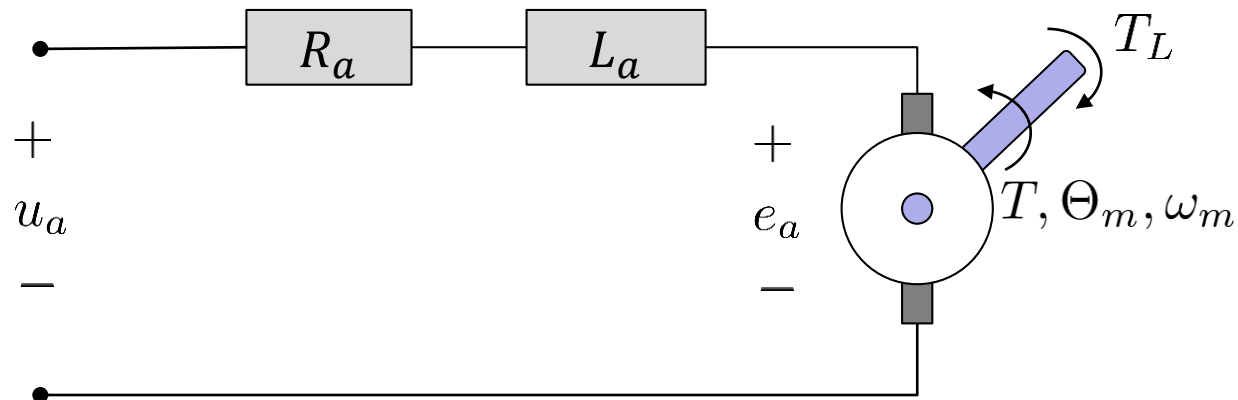
The armature continues to rotate.



When the armature becomes horizontally aligned, the commutator reverses the direction of current through the coil, reversing the magnetic field. The process then repeats.

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Armature circuit I



Check rational TFs for PRness

Theorem: A rational, proper transfer function $H(s)$ is positive real (and hence passive) if and only if

1. $H(s)$ has no poles in $\text{Re}[s] > 0$.
2. $\text{Re}[H(j\omega)] \geq 0$ for all $\omega \in [-\infty, \infty]$ such that $j\omega$ is not a pole of $H(s)$.
3. If $j\omega_0$ is a pole of $H(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is,

$$\text{Res}_{s=j\omega_0} H(s) = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)H(s) > 0.$$

Transfer function of current controlled DC motor

$$\frac{i_a}{u_a}(s) = \frac{J_m}{K_T K_E} \frac{s}{1 + T_m s + T_m T_a s^2}$$

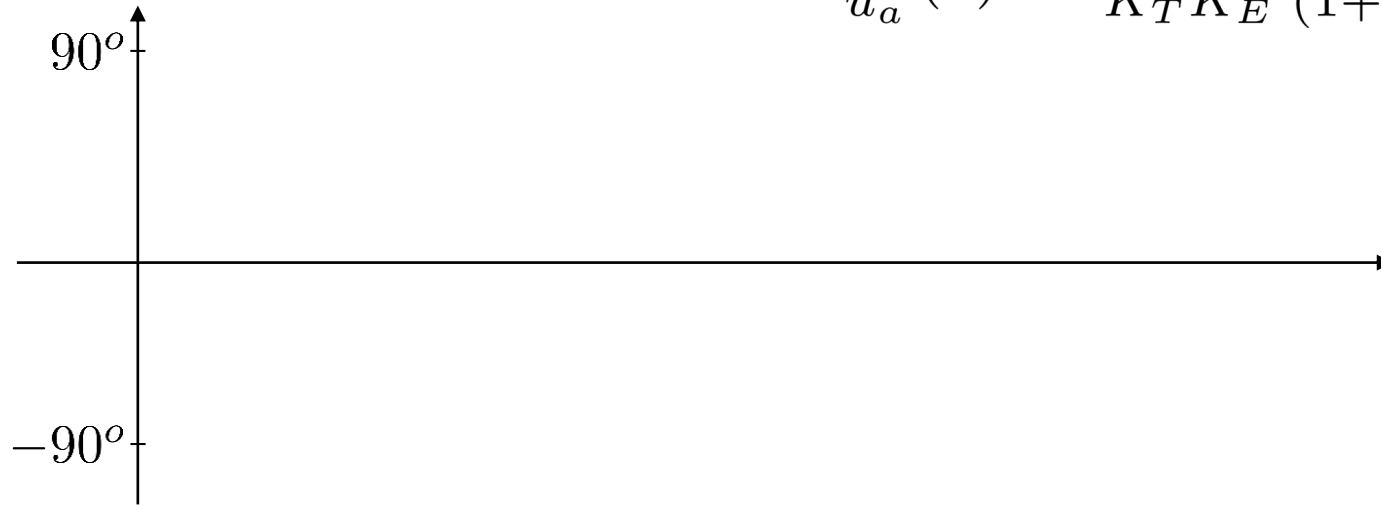
$$T_m = \frac{J_m R_a}{K_E K_T}$$

$$T_a = \frac{L_a}{R_a}$$

$$\frac{i_a}{u_a}(s) \approx \frac{J_m}{K_T K_E} \frac{s}{(1 + T_m s)(1 + T_a s)}$$

Passivity current controlled DC motor

$$\frac{i_a}{u_a}(s) \approx \frac{J_m}{K_T K_E} \frac{s}{(1+T_m s)(1+T_a s)}$$



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Newton-Euler equations of motion

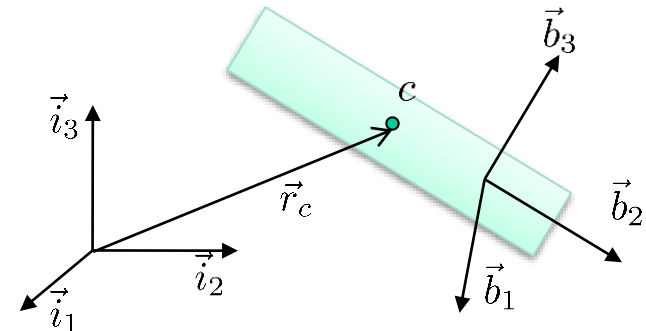
- Newton's law (for particle k)

$$m_k \vec{a}_k = \vec{F}^{(r)}$$

- Newton-Euler EoM for rigid bodies:
 - Integrate Newton's law over body, define center of mass
 - Define torque/moment and angular momentum to handle forces that give rotation about center of mass
 - Define inertia dyadic/matrix

$$\vec{F}_{bc} = m \vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib} \right)$$



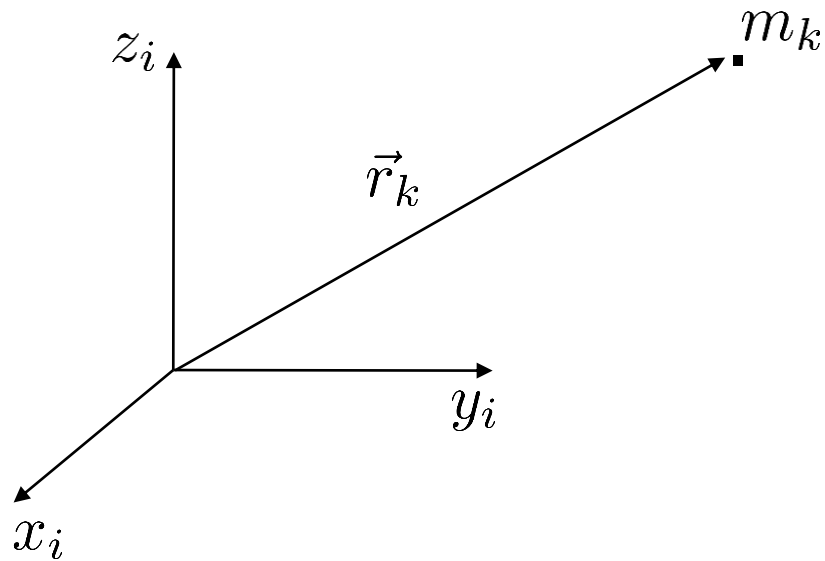
(Here: Referenced to center of mass)

- Implemented in e.g. Dymola (Modelica.Multibody library)

Types of forces

- Two types of forces:
 - Active forces
 - Forces of constraints

Generalized coordinates



Virtual displacement

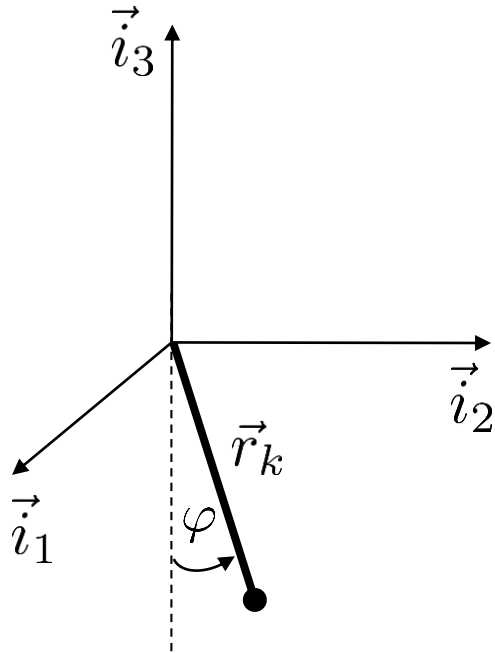
$$\delta \vec{r}_k = \sum_{i=1}^n \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i$$

d'Alembert's principle I

d'Alembert's principle II

$$\delta \vec{r}_k = \sum_{i=1}^n \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i$$

Example: Generalized coordinates on Pendulum



Lagrange EoM (for a particle) – preliminary

$$\vec{v}_k = \frac{{}^i d}{dt} \vec{r}_k = \sum_{i=1}^n \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i + \frac{\partial \vec{r}_k}{\partial t}$$

$$\frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \frac{\partial}{\partial q_i} \frac{{}^i d}{dt} \vec{r}_k = \frac{{}^i d}{dt} \frac{\partial \vec{r}_k}{\partial q_i}$$

Lagrange EoM (for a particle) I

$$T = \sum_{k=1}^N \frac{1}{2} m_k \vec{v}_k \vec{v}_k$$

Lagrange EoM (for a particle) II

$$\sum_{i=1}^n \delta q_i \sum_{k=1}^N \frac{\partial \vec{r}_k}{\partial q_i} \left(m_k \frac{d^2}{dt^2} \vec{r}_k - \vec{F}_k \right) = 0$$

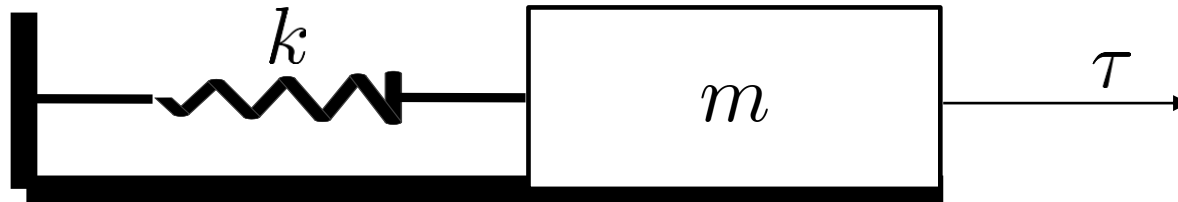
Generalised force of q_i

- Assume potential field $U(\underline{q})$ that gives force: $-\frac{\partial U}{\partial q_i}$
- Assume «generalised actuator force» τ_i

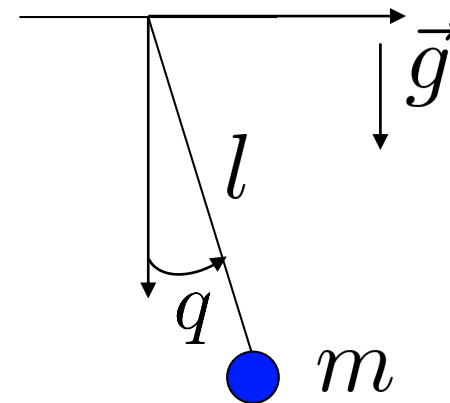
Lagrangian $\mathcal{L}(\underline{q}, \underline{\dot{q}}, t) = \mathbf{T}(\underline{q}, \underline{\dot{q}}, t) - \mathbf{U}(\underline{q})$

Generalised force Q_i

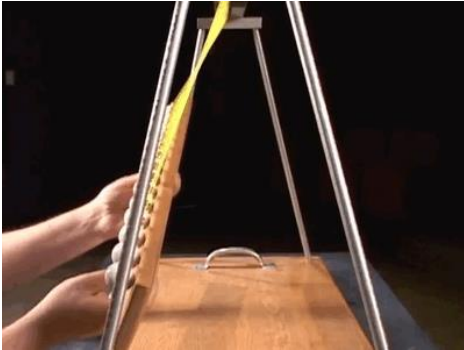
Example: Mass-Spring system



Example: Pendulum



Is there a fundamental difference?



Pendulum



Lab helicopter



Quadrotor



Satellite

- Newton-Euler or Lagrange?
 - Newton-Euler can (of course) be used for everything, but if you are calculating by hand/symbolically, it is far easier to use Lagrange when you have constrained motion (forces of constraint)

Lagrange vs Newton-Euler

Newton-Euler

- Vectors
- Forces and moments
- Does not eliminate forces of constraints:
 - Obtains solutions for all forces and kinematic variables
 - "Inefficient" (large DAE models)
- More general
 - Large systems can be handled, but for some configurations tricks are needed
 - Used in advanced modeling software

Lagrange

- Algebraic
- Energy
- Eliminates forces of constraints
 - Solutions only for generalized coordinates (and forces)
 - "Efficient" (smaller ODE models)
- Less general
 - Need independent generalized coordinates
 - Difficult to automate for large/complex problems

Robotic manipulator 8.2.8

Homework

- Derive the EoM of the pendulum with help of the Newton-Euler approach using only the inertial frame, only the body frame; and using the Lagrange approach with generalised coordinates:

