# Example 13.1 in Nocedal & Wright

Tor Aksel N. Heirung

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**Abstract**

This note goes through Example 13.1 in detail, highlighting Procedure 13.1 and some of the theory from Chapter 13.3. The example from the textbook is included at the end; errors are marked with red boxes.

## Optimization Problem

Let us first state the linear optimization problem as

$$\begin{aligned} \min \quad & -4x_1 - 2x_2 && \text{(1a)} \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 && \text{(1b)} \\ & 2x_1 + (1/2)x_2 \leq 8 && \text{(1c)} \\ & x \geq 0 && \text{(1d)} \end{aligned}$$

This problem is illustrated in Figure 1.

By adding the slack variables $x_3$ and $x_4$ to constraints (1b) and (1c), respectively, we can write problem (1) in standard form,

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{3}$$

## Iteration 1

We start with the basis $\mathcal{B} = \{3, 4\}$, which means we must have $\mathcal{N} = \{1, 2\}$. By the definition of nonbasic variables, this means we have

$$x_{\mathrm{N}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

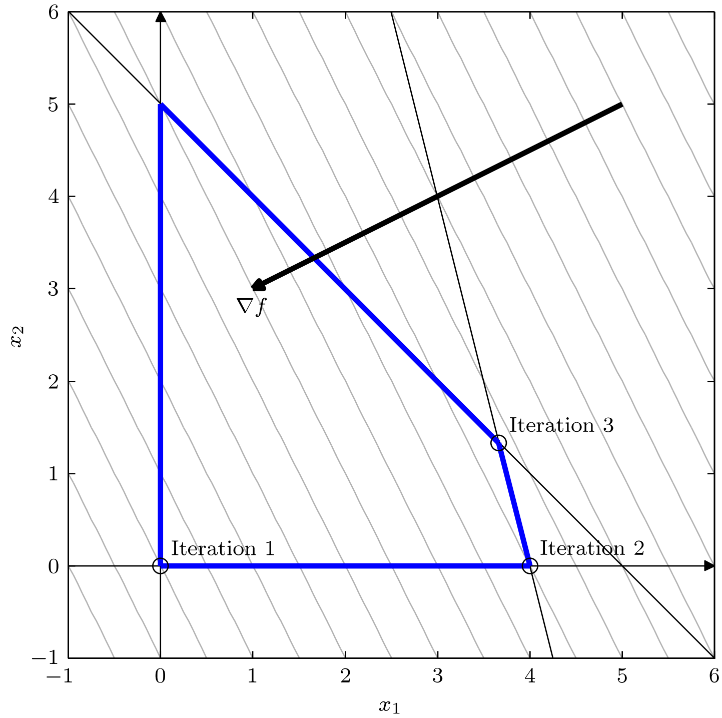


Figure 1: Illustration of problem (1), with all constraints included. The feasible area is the blue polytope. The gradient of the objective function is indicated and all iterations are marked.

Furthermore, the matrix $B$ contains the 3rd and 4th columns of $A$; that is,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

Similarly, $N$ contains the 1st and 2nd columns of $A$:

$$N = \begin{bmatrix} 1 & 1 \\ 2 & \frac{1}{2} \end{bmatrix} \tag{6}$$

We can now calculate the values of $x_\mathrm{B}$, $\lambda$ and $s_\mathrm{N}$:

$$x_\mathrm{B} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{7a}$$

$$\lambda = B^{-\top} c_\mathrm{B} = B^{-\top} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7b}$$

$$s_\mathrm{N} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = c_\mathrm{N} - N^\top \lambda = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - N^\top \lambda = \begin{bmatrix} -4 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \tag{7c}$$

Notice that the value of value of $x_\mathrm{B}$ can be anticipated by looking at $Ax = b$ with $x_1 = x_2 = 0$. The value of $\lambda$ can also be predicted, since none of the inequality constraints (1b)-(1c) are active at this point. We also note that the objective function value at this point is $c^\top x = 0$.

The smallest element in $s_\mathrm{N}$ is $s_1 = -4$. We then set the index of the variable entering the basis to 1, that is, $q = 1$ and $x_q = x_1$ will enter the basis. Furthermore, $A_q = A_1$, so we use the first column of $A$ to calculate the direction $d$:

$$d = B^{-1}A_q = B^{-1}A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8}$$

The entering variable, $x_q = x_1$, will get the value $x_q^+ = x_1^+$ at the next point/basis. This value can also be interpreted as a form of step length in the direction $d$. We calculate this value with the minimum ratio test:

$$x_q^+ = x_1^+ = \min_{\substack{i \\ d_i > 0}} \left\{ \frac{(x_\mathrm{B})_i}{d_i} \right\} = \min \left\{ \frac{(x_\mathrm{B})_1}{d_1}, \frac{(x_\mathrm{B})_2}{d_2} \right\} =$$
$$\min \left\{ \frac{x_3}{d_1}, \frac{x_4}{d_2} \right\} = \min \left\{ \frac{5}{1}, \frac{8}{2} \right\} = \min \{5, 4\} = 4 \tag{9}$$

This value corresponds to the 2nd element of the basis, and hence $p = 2$ (the minimizing $i$). The 2nd element of the basis vector is $x_4$, so this element will leave the basis.

Using the direction $d$ and the "step length" $x_1^+$, we now find the new value of the current basis vector:

$$x_\mathrm{B}^+ = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}^+ = x_\mathrm{B} - dx_1^+ = \begin{bmatrix} 5 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} 4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

Notice that $x_4$ goes to zero, while $x_3$ does not. This is exactly as anticipated, as $x_4$ is to leave the basis, and by definition must be zero at the new point; $x_3$ will stay in the basis and can not go to zero. (If $x_3$ had gone to zero, the new basis would have been degenerate.) Since $x_1^+ = 4$, we know that the vector of nonbasic variables at the new point must be

$$x_\mathrm{N}^+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{11}$$

Since $x_4$ is leaving the basis and $x_1$ is entering, we update the basis from $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ to $\mathcal{B} = \{3, 1\}$ and $\mathcal{N} = \{4, 2\}$.

# Iteration 2

With $\mathcal{B} = \{3, 1\}$ and $\mathcal{N} = \{4, 2\}$, the vector of nonbasic variables must be

$$x_\mathrm{N} = \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

while the matrices $B$, $B^{-1}$ and $N$ are

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \tag{13a}$$

$$N = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \tag{13b}$$

With these matrices, the vectors $x_B$, $\lambda$ and $s_N$ become

$$x_B = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} = B^{-1}b = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{14a}$$

$$\lambda = B^{-\top}c_B = B^{-\top}\begin{bmatrix} c_3 \\ c_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{14b}$$

$$s_N = \begin{bmatrix} s_4 \\ s_2 \end{bmatrix} = c_N - N^\top\lambda = \begin{bmatrix} c_4 \\ c_2 \end{bmatrix} - N^\top\lambda = \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{14c}$$

Notice that the value of $x_B$ did not have to be calculated. We knew from equation (10) that we would have $x_3 = 1$, and we know from equation (9) that $x_1 = 4$. The objective function value at this point is $c^\top x = -16$, a decrease of 16 from the previous value (0). Notice that this matches $s_q x_q^+ = s_1 x_1^+ = -4 \cdot 4 = -16$ (see equation (13.24) in the textbook). Also notice that $\lambda_2$ has become nonzero, as we have hit the second inequality constraint (1c).

The only negative component of $s_N$ is $s_2 = -1$. Hence, $q = 2$, which means that $x_2$ will enter the basis. The direction $d$ is then calculated using the 2nd column of $A$:

$$d = B^{-1}A_q = B^{-1}A_2 = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} \tag{15}$$

Using the minimum ratio test to find $x_q^+ = x_2^+$ gives

$$x_q^+ = x_2^+ = \min_{\substack{i \\ d_i > 0}} \left\{ \frac{(x_B)_i}{d_i} \right\} = \min \left\{ \frac{(x_B)_1}{d_1}, \frac{(x_B)_2}{d_2} \right\} =$$
$$\min \left\{ \frac{x_3}{d_1}, \frac{x_1}{d_2} \right\} \min \left\{ \frac{1}{3/4}, \frac{4}{1/4} \right\} = \min \left\{ \frac{4}{3}, 16 \right\} = \frac{4}{3} \tag{16}$$

This value corresponds to the 1st element of the basis, and hence $p = 1$ (the minimizing $i$). The 1st element of the basis vector is $x_3$, so this element will leave the basis.

With $d$ and $x_2^+$ the new value of the current basis vector becomes

$$x_B^+ = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}^+ = x_B - dx_2^+ = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}\frac{4}{3} = \begin{bmatrix} 0 \\ \frac{11}{3} \end{bmatrix} \tag{17}$$

We see that as anticipated, $x_3$, which is leaving the basis, is zero at the next point, whereas $x_1$, which stays in the basis, stays nonzero. Since we have $x_2^+ = \frac{4}{3}$, the new value of the current nonbasic vector must be

$$x_N^+ = \begin{bmatrix} x_4 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} 0 \\ \frac{4}{3} \end{bmatrix} \tag{18}$$

Now, $s_q x_q^+ = s_2 x_2^+ = -1 \cdot \frac{4}{3} = -\frac{4}{3}$, indicating that the objective function value will be $-16 - \frac{4}{3} = -\frac{52}{3}$ at the next point.

Since $x_3$ is leaving the basis and $x_2$ is entering, we update the basis from $\mathcal{B} = \{3, 1\}$ and $\mathcal{N} = \{4, 2\}$ to $\mathcal{B} = \{2, 1\}$ and $\mathcal{N} = \{4, 3\}$.

## Iteration 3

With $\mathcal{B} = \{2, 1\}$ and $\mathcal{N} = \{4, 3\}$ the vector of nonbasic variables must be

$$x_{\mathrm{N}} = \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

while $B$, $B^{-1}$ and $N$ are

$$B = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 2 \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{2}{3} \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{20a}$$

$$N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{20b}$$

The vectors $x_{\mathrm{B}}$, $\lambda$ and $s_{\mathrm{N}}$ then become

$$x_{\mathrm{B}} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = B^{-1}b = \frac{2}{3} \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{11}{3} \end{bmatrix} \tag{21a}$$

$$\lambda = B^{-\top} c_{\mathrm{B}} = B^{-\top} \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} \tag{21b}$$

$$s_{\mathrm{N}} = \begin{bmatrix} s_4 \\ s_3 \end{bmatrix} = c_{\mathrm{N}} - N^\top \lambda = \begin{bmatrix} c_4 \\ c_3 \end{bmatrix} - N^\top \lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \end{bmatrix} \tag{21c}$$

Again, we could have predicted the value of $x_B$ from equations (16) and (17). Notice that since both inequality constraints (1b)-(1c)are active at this point, both elements of $\lambda$ are nonzero.

The objective function value at this point is $c^\top x = -\frac{52}{3}$, as predicted above. Since both elements of $s_{\mathrm{N}}$ are positive, we can conclude that the solution to the LP is found at

$$x^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^* = \begin{bmatrix} \frac{11}{3} \\ \frac{4}{3} \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

# The Example in Nocedal and Wright

❑ **EXAMPLE 13.1**

Consider the problem

$$\begin{aligned} \min\ -4x_1 - 2x_2 \quad & \text{subject to} \\ x_1 + x_2 + x_3 &= 5, \\ 2x_1 + (1/2)x_2 + x_4 &= 8, \\ x &\geq 0. \end{aligned}$$

Suppose we start with the basis $\mathcal{B} = \{3, 4\}$, for which we have

$$x_{\mathrm{B}} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad s_{\mathrm{N}} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \boxed{-3} \\ -2 \end{bmatrix},$$

and an objective value of $c^T x = 0$. Since both elements of $s_{\mathrm{N}}$ are negative, we could choose either 1 or 2 to be the entering variable. Suppose we choose $q = 1$. We obtain $d = (1, 2)^T$, so we cannot (yet) conclude that the problem is unbounded. By performing the ratio calculation, we find that $p = 2$ (corresponding to the index 4) and $x_1^+ = 4$. We update the basic and nonbasic index sets to $\mathcal{B} = \{3, 1\}$ and $\mathcal{N} = \{4, 2\}$, and move to the next iteration.

At the second iteration, we have

$$x_{\mathrm{B}} = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 \\ \boxed{-3/2} \end{bmatrix}, \quad s_{\mathrm{N}} = \begin{bmatrix} s_4 \\ s_2 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 \\ -5/4 \end{bmatrix}},$$

with an objective value of $\boxed{-12.}$ We see that $s_{\mathrm{N}}$ has one negative component, corresponding to the index $q = 2$, so we select this index to enter the basis. We obtain $d = \boxed{(3/2, -1/2)^T}$, so again we do not detect unboundedness. Continuing, we find that the maximum value of $x_2^+$ is $4/3$, and that $p = 1$, which indicates that index 3 will leave the basis $\mathcal{B}$. We update the index sets to $\mathcal{B} = \{2, 1\}$ and $\mathcal{N} = \{4, 3\}$ and continue.

At the start of the third iteration, we have

$$x_{\mathrm{B}} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 11/3 \end{bmatrix}, \quad \lambda = \boxed{\begin{bmatrix} -5/3 \\ -2/3 \end{bmatrix}}, \quad s_{\mathrm{N}} = \begin{bmatrix} s_4 \\ s_3 \end{bmatrix} = \boxed{\begin{bmatrix} 7/3 \\ 5/3 \end{bmatrix}},$$

with an objective value of $c^T x = \boxed{-41/3.}$ We see that $s_{\mathrm{N}} \geq 0$, so the optimality test is satisfied, and we terminate.

❐