Lecture 17: Lagrangian mechanics (Lagrange's equation of motion)

Electrical motor (passivity)

Newton's law (for particles, or Newton-Euler EoM for rigid bodies) in combination with

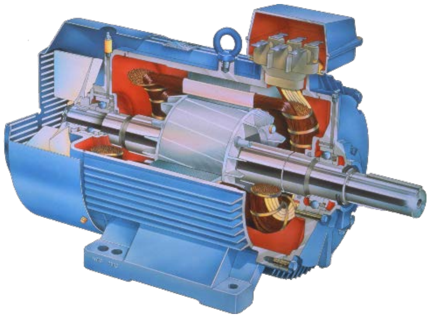
- d'Alembert's principle
- Generalized coordinates

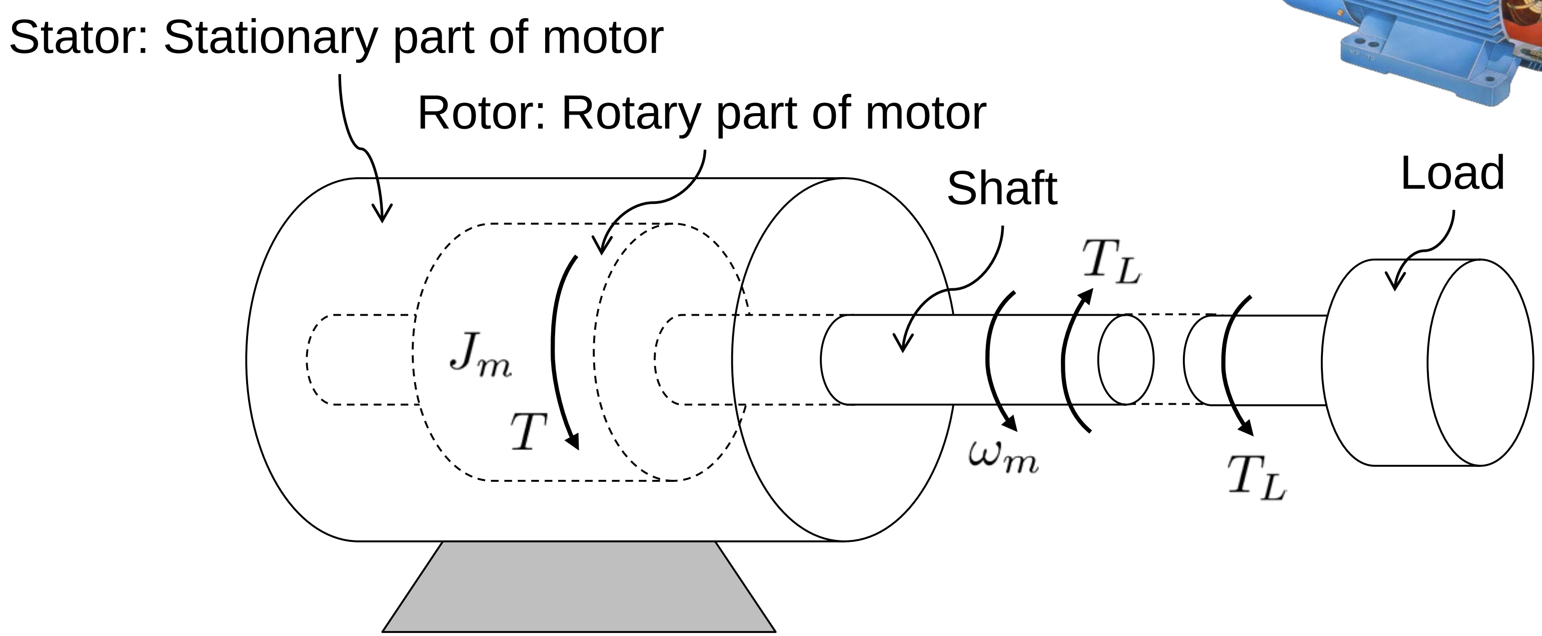
gives Lagrange's equations of motion

- Brief examples

Book: Ch. 3.3, 7.7, 8.2

Motors





- Equation of motion for motor shaft:

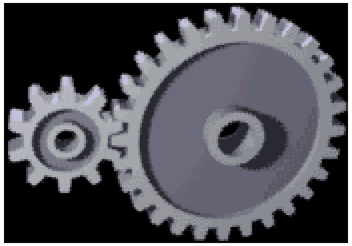
$$J_m \dot{\omega}_m = T - T_L$$

where

- T : Motor torque (set up by some device, e.g. DC motor)
- T_L : Load torque
- J_m : Moment of inertia for rotor and shaft
- ω_m: Angular velocity/motor speed [rad/s, or rev./min]

Gears

Rotational gear
(cogwheel)



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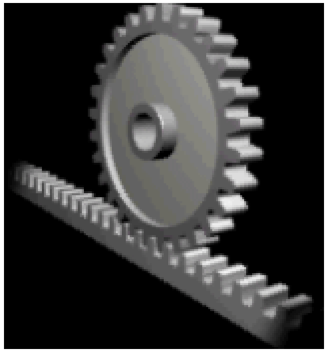
Translational gear
(rack and pinion)



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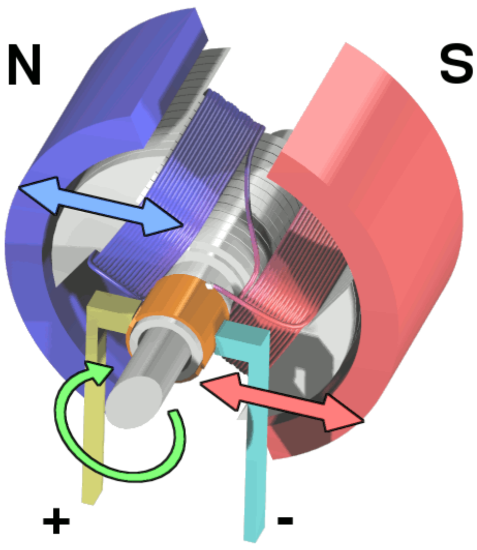
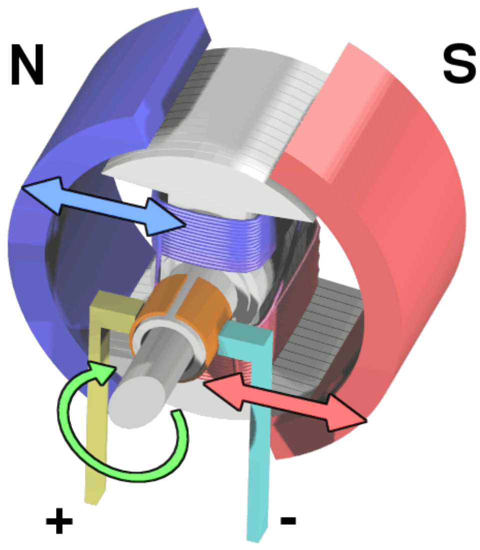
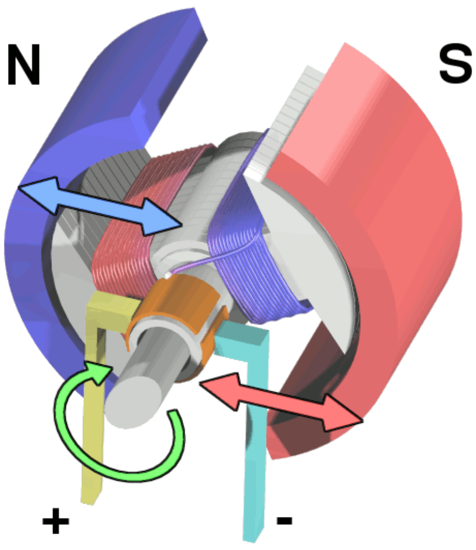
A simple DC electric motor







A simple DC electric motor. When the coil is powered, a magnetic field is generated around the armature. The left side of the armature is pushed away from the left magnet and drawn toward the right, causing rotation.

The armature continues to rotate.

When the armature becomes horizontally aligned, the commutator reverses the direction of current through the coil, reversing the magnetic field. The process then repeats.

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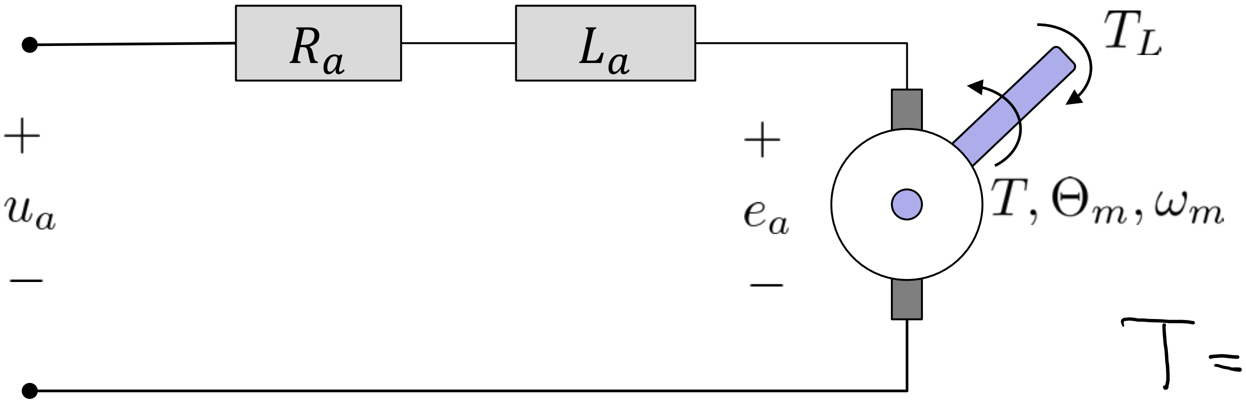
Armature circuit I



$T = K_T \cdot i$

Kirchhof's 2nd law

$$u_a = R_a \cdot i_a + L_a \frac{di_a}{dt} + e_a$$

motor equation

$$J_m \dot{\omega}_m = T - T_L$$

Check rational TFs for PRness

Theorem: A rational, proper transfer function $H(s)$ is positive real (and hence passive) if and only if

1. $H(s)$ has no poles in $\mathrm{Re}[s] > 0$.
2. $\mathrm{Re}[\mathrm{H}(j\omega)] \geq 0$ for all $\omega \in [-\infty, \infty]$ such that $j\omega$ is not a pole of $H(s)$.
3. If $j\omega_0$ is a pole of $H(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is,

$$\operatorname*{Res}_{s=j\omega_0} H(s) = \lim_{s \to j\omega_0} (s - j\omega_0) H(j\omega) > 0.$$

Transfer function of current controlled DC motor

$$\frac{i_a}{u_a}(s) = \frac{J_m}{K_T K_E} \frac{s}{1+T_m s+T_m T_a s^2}$$

$$T_m = \frac{J_m R_a}{K_E K_T} \qquad T_a = \frac{L_a}{R_a}$$

mechanical time constant

electrical time constant

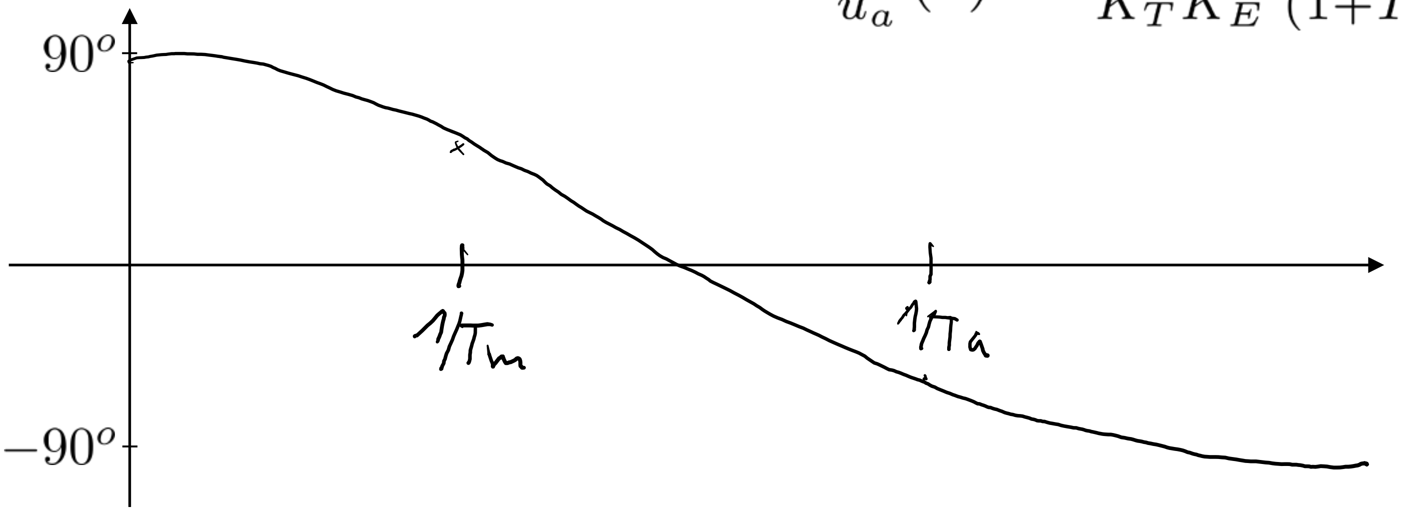
with assumption $T_a << T_m$

$$\frac{i_a}{u_a}(s) \approx \frac{J_m}{K_T K_E} \frac{s}{(1+T_m s)(1+T_a s)}$$

Passivity current controlled DC motor

$$\frac{i_a}{u_a}(s) \approx \frac{J_m}{K_T K_E} \frac{s}{(1+T_m s)(1+T_a s)}$$

1



2

1. Pols $p_1 = -1/T_a$ $p_2 = -1/T_m$

2. $$\frac{j\omega(1 - T_m j\omega)(1 - T_a j\omega)}{(1+\omega^2 T_m^2)(1+\omega^2 T_a^2)} = \frac{j\omega(1 - (T_a+T_m)j\omega - T_a^2)}{(1+\omega^2 T_m^2)(1+\omega^2 T_a^2)}$$

$$\mathrm{Re} \quad \frac{\omega^2 (T_a + T_m)}{(1+\omega^2 T_m^2)(1+\omega^2 T_a^2)} \geq 0$$

3. –

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Newton's law (for particles, or Newton-Euler EoM for rigid bodies) in combination with

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Newton-Euler equations of motion

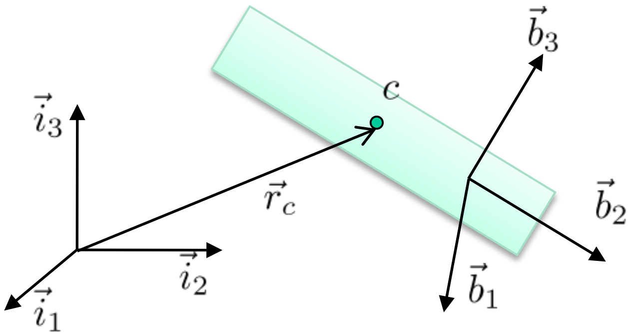
- Newton's law (for particle *k*)

$$m_k \vec{a}_k = \vec{F}^{(r)}$$

- Newton-Euler EoM for rigid bodies:
  - Integrate Newton's law over body, define center of mass
  - Define torque/moment and angular momentum to handle forces that give rotation about center of mass
  - Define inertia dyadic/matrix

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)$$



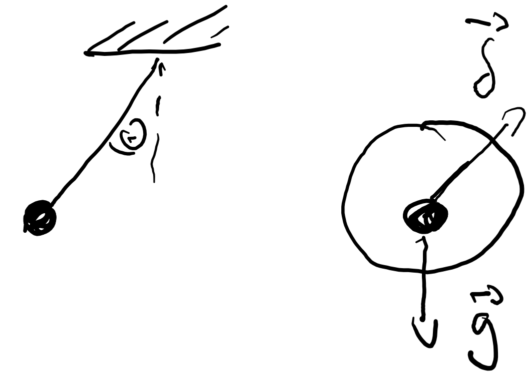
(Here: Referenced to center of mass)

- Implemented in e.g. Dymola (Modelica.Multibody library)

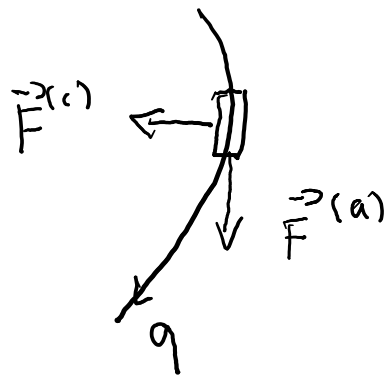
Types of forces

- Two types of forces:
  - Active forces
  - Forces of constraints

Example I

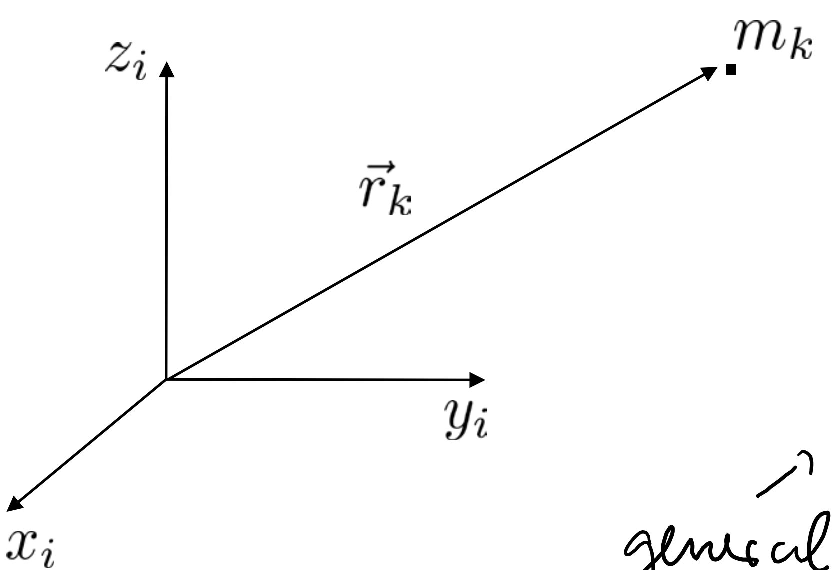


Example II :

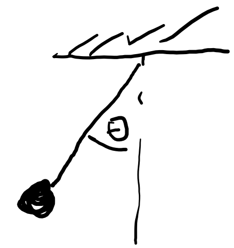


$$\vec F^{(r)} = \vec F^{(c)} + \vec F^{(a)}$$
$$= \vec F^{(c)} + \vec F$$

Generalized coordinates



$$\vec{r}_k = \vec{r}_k(q(t), t)$$



$q_1, \ldots, q_n$

generalised coordinates

$$\vec{r}_k = \vec{r}_k(\theta(t), t)$$

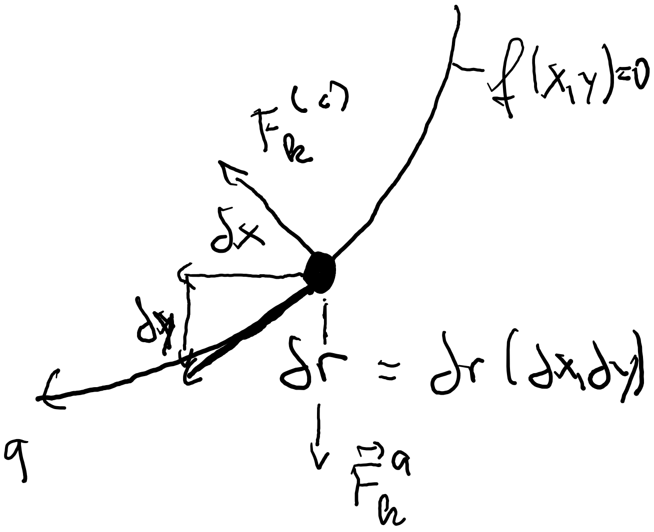
$$V_k = \frac{d}{dt}\vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_k}{\partial t}$$

motion along the free axis

rotation of coordinate system

Virtual displacement

$$\delta\vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i}\delta q_i$$



n independent virtual displacements $\delta\vec{r}_k$

→ system has "n degrees of freedom"

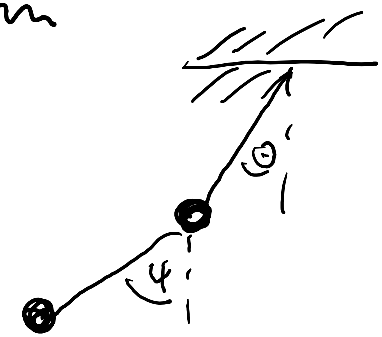
$$A(q)\cdot\dot{q} = 0$$

linear constraints

d'Alembert's principle I

N particles moving independently

-> 3N degrees of freedom



-> Constraints reduce degrees of freedom

Forces of constraints should statisfy the principle of virtual work

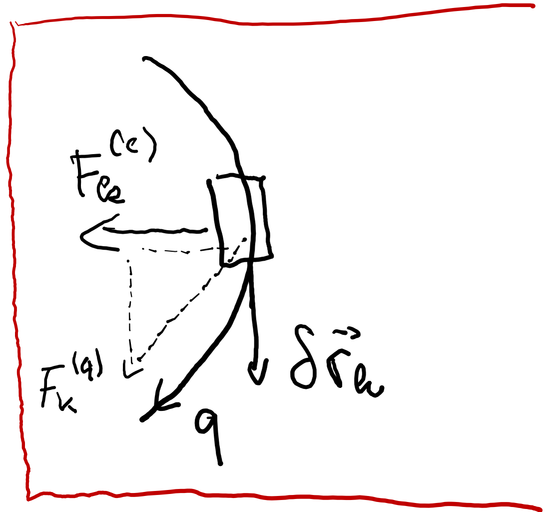
$$\sum_{k=1}^{N} \delta \vec{r}_k \cdot F_k^{(c)} = 0$$

d'Alembert's principle II

$$\delta\vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i}\delta q_i$$

Newton's law: $m_k \frac{d^2}{dt^2}\vec{r}_k = \vec{F}_k^{(r)}$

$= \vec{F}_k + \vec{F}_k^{(c)}$



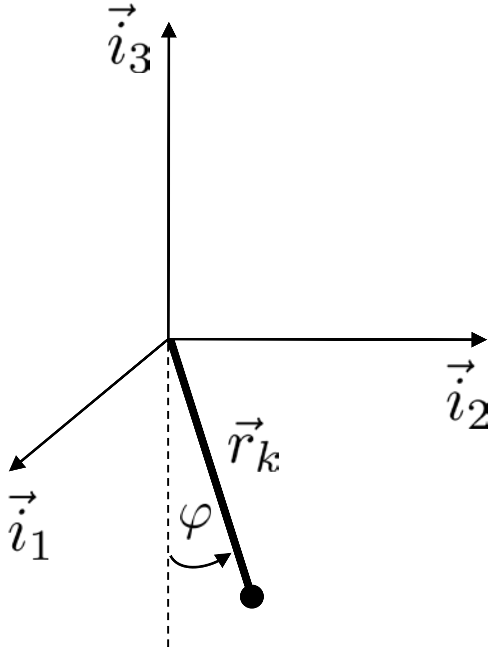
$$\vec{F}_k^{(c)} \cdot \delta\vec{r}_k = \left[m_k \frac{d^2}{dt^2}\vec{r}_k - \vec{F}_k\right]\delta\vec{r}_k = 0$$

$$\sum_{k=1}^{N} \delta\vec{r}_k \left(m_k \frac{d^2}{dt^2}\vec{r}_k - \vec{F}_k\right) = 0$$

$$\sum_{i=1}^{n} \delta q_i \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\left(m_k \frac{d^2}{dt^2}\vec{r}_k - \vec{F}_k\right) = 0$$

$$\Rightarrow \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\left(m_k \frac{d^2}{dt^2}\vec{r}_k - \vec{F}_k\right) = 0 \qquad i=1,\dots,n$$

Example: Generalized coordinates on Pendulum



$$\vec{r}_k = x_k \vec{i}_1 + y_k \vec{i}_2 + z_k \vec{i}_3$$

not independent

$$\underline{r}_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} 0 \\ l \sin\varphi \\ -l\cos\varphi \end{bmatrix}$$

generalized coordinate: φ

$$\vec{r}_k = \vec{r}_k(\varphi)$$

In general: $\vec{r}_k = \vec{r}_k(q_1, q_2, \dots, q_n, t)$

$$= \vec{r}_k(q, t)$$

generalized coordinates

"parametrisation of DoF"

Lagrange EoM (for a particle) – preliminary

$$\vec{v}_k = \frac{{}^i d}{dt}\vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_k}{\partial t}$$

$$\frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{\partial \vec{v}_k}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{{}^i d}{dt}\vec{r}_k = \frac{{}^i d}{dt}\frac{\partial \vec{r}_k}{\partial q_i}$$

Lagrange EoM (for a particle) I

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k \vec{v}_k \vec{v}_k$$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{N} m_k \vec{v}_k \cdot \frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \sum_{k=1}^{N} m_k \vec{v}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{\partial T}{\partial q_i} = \sum_{k=1}^{N} m_k \vec{v}_k \cdot \frac{\partial \vec{v}_k}{\partial q_i} = \sum_{k=1}^{N} m_k \vec{v}_k \cdot \frac{d}{dt} \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{N} \frac{d}{dt} \left(m_k \vec{v}_k \frac{\partial \vec{r}_k}{\partial q_i} \right)$$

$$= \sum_{k=1}^{N} \left(m_k \vec{a}_k \frac{\partial \vec{r}_k}{\partial q_i} + m_k \vec{v}_k \frac{d}{dt} \frac{\partial \vec{r}_k}{\partial q_i} \right)$$

Lagrange EoM (for a particle) II

$$\sum_{i=1}^{n} \delta q_i \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\left(m_k \frac{{}^i d^2}{dt^2}\vec{r}_k - \vec{F}_k\right) = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} m_k \vec{a}_k + \frac{\partial T}{\partial q_i}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q_i} - \underbrace{\sum_{k=1}^{N} \frac{\partial r_k}{\partial q_i} \vec{F}_k}_{Q_i} = 0 \qquad i = 1, \ldots, n$$

Q_i ← generalised force of q_i

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad i = 1, \ldots, n$$

Generalised force of q_i

- Assume potential field $U(\underline{q})$ that gives force: $-\frac{\partial U}{\partial q_i}$

Example: Gravity: $U = mgh$; $-\frac{\partial U}{\partial q_i} = -mg$

- Assume «generalised actuator force» τ_i (belongs to q_i)

$$\Rightarrow \quad Q_i = -\frac{\partial U}{\partial q_i} + \tau_i$$

Lagrange EoM

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = \tau_i$$

Lagrangian $\mathcal{L}(\underline{q}, \dot{\underline{q}}, t) = \mathbf{T}(\underline{q}, \dot{\underline{q}}, t) - \mathbf{U}(\underline{q})$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \qquad i = 1, \ldots, n$$

This hold also for rigid bodies:

$$T(\underline{q}, \dot{\underline{q}}) = 1/2\, m\, \vec{v}_c \cdot \vec{v}_c + 1/2\, \vec{\omega}_{ib} \cdot \vec{M}_{bc} \cdot \vec{\omega}_{ib}$$

$$U(\underline{q}) = m g h(\underline{q})$$

$$\vec{v}_c(\underline{q}) \; ; \; \vec{\omega}_{ib}(\underline{q}) \; ; \; \vec{M}_{bc}(\underline{q})$$

Generalised force Q_i

$$P = \sum_{k=1}^{N} \vec{V}_k \cdot \vec{F}_k = \sum_{k=1}^{N} \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} \dot{q}_i \cdot \vec{F}_k$$

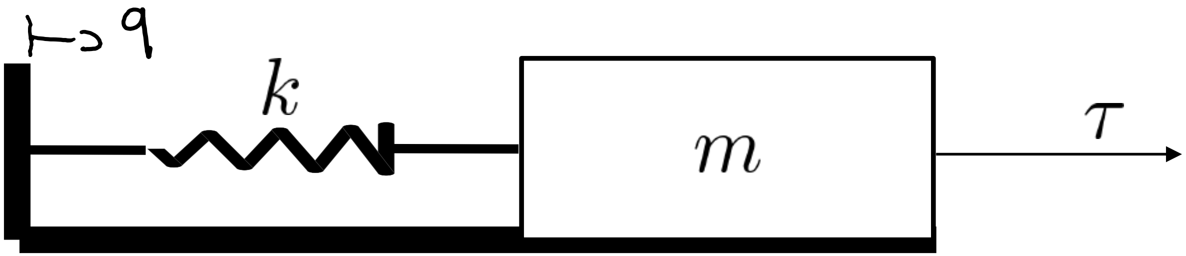
$$= \sum_{i=1}^{n} \dot{q}_i \underbrace{\sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \vec{F}_k}_{Q_i}$$

$$= \sum_{i=1}^{n} \dot{q}_i Q_i$$

q_i position $\qquad Q$: force.

q_i angle $\qquad Q$: torque

Example: Mass-Spring system



$$T = \frac{1}{2} m \dot{q}^2 \qquad U = \frac{1}{2} k (q - q_0)^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k (q - q_0)^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m \dot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -k (q - q_0)$$

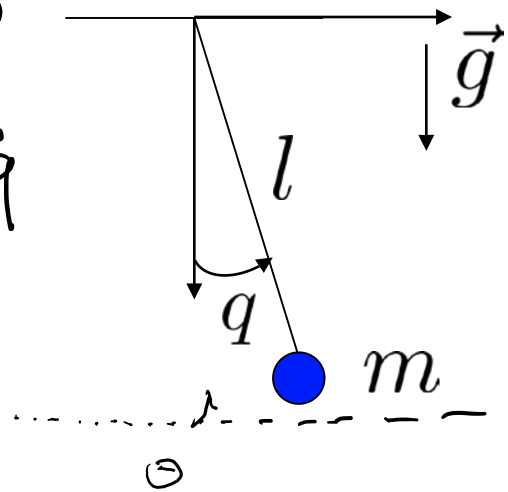
$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = m \ddot{q} + k (q - q_0) = \tau$$

Newton-Euler: $m \ddot{q} = \sum F$

$$= \tau - k (q - q_0)$$

Example: Pendulum

$\Pi_1 = 0$



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{q}^2 \qquad v = l \dot{q}$$

$$U = -mgl\cos q \qquad [mgl(1-\cos q)]$$

$\Pi_2 = 0$

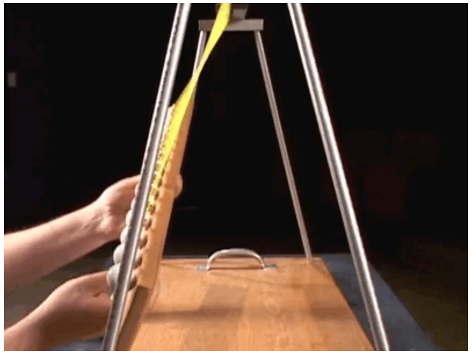
$$\mathcal{L} = T - U$$
$$= \frac{1}{2} m l^2 \dot{q}^2 + mgl\cos q$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m l^2 \dot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -mgl\sin q$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = m l^2 \ddot{q} + mgl\sin q = 0$$
$$\ddot{q} + \frac{g}{l}\sin q = 0$$

"small deflection" $\approx \quad \ddot{q} + \underbrace{g/l}_{\omega_0^2}\, q = 0. \qquad \omega_0 = \sqrt{g/l}$

Is there a fundamental difference?


Pendulum


Lab helicopter


Quadrotor


Satellite

- Newton-Euler or Lagrange?
  - Newton-Euler can (of course) be used for everything, but if you are calculating by hand/symbolically, it is far easier to use Lagrange when you have constrained motion (forces of constraint)

Lagrange vs Newton-Euler

Newton-Euler

- Vectors
- Forces and moments
- Does not eliminate forces of constraints:
  - Obtains solutions for all forces and kinematic variables
  - ”Inefficient” (large DAE models)
- More general
  - Large systems can be handled, but for some configurations tricks are needed
  - Used in advanced modeling software

Lagrange

- Algebraic
- Energy
- Eliminates forces of constraints
  - Solutions only for generalized coordinates (and forces)
  - ”Efficient” (smaller ODE models)
- Less general
  - Need independent generalized coordinates
  - Difficult to automate for large/complex problems

Robotic manipulator 8.2.8

Homework

- Derive the EoM of the pendulum with help of the Newton-Euler approach using only the inertial frame, only the body frame; and using the Lagrange approach with generalised coordinates: