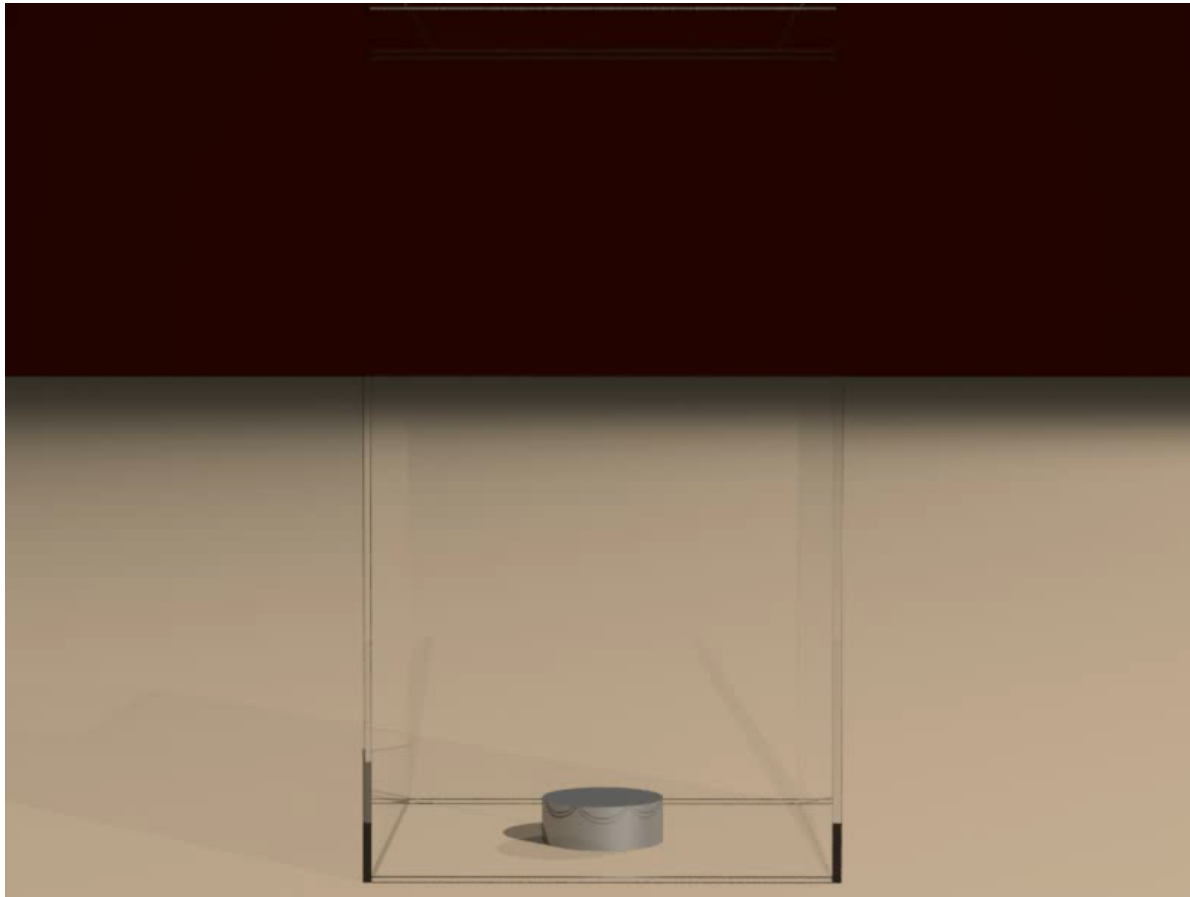# Lecture 24: Process modeling & balance laws

- **Balance laws**
  - Differential balances
  - Material derivative

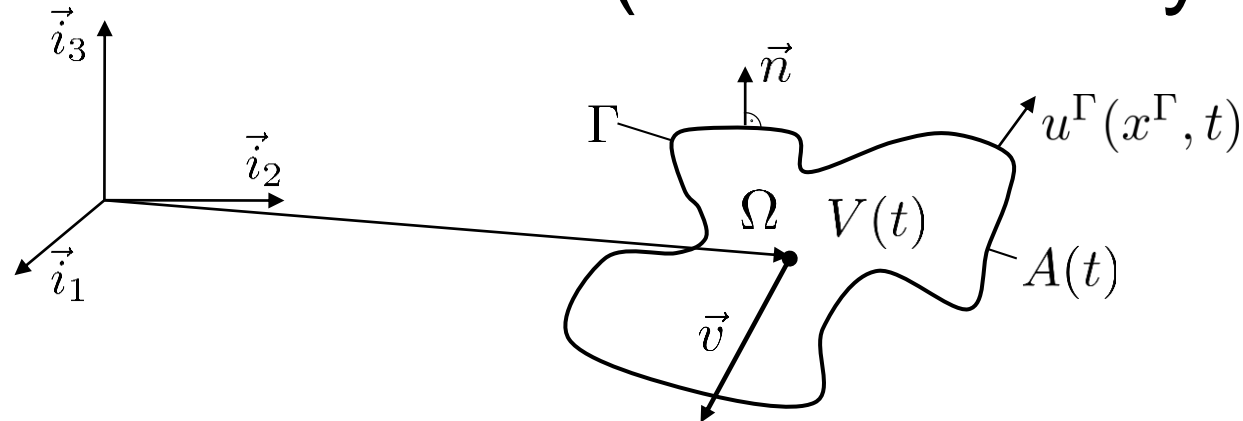Book: 10.4, 11.1-11.4

Computational fluid dynamics

- CFD = solving momentum + mass balances (that is, Navier-Stokes + continuity equation) for different setups

http://physbam.stanford.edu/~fedkiw/

Control volume (infinitesimally small)

- How can the extensive property Ψ change in the domain Ω ?

General integral balance

- Arbitrary size: In order to get a local character we have to introduce properties with local characteristics

Derivation of differential balance I

- Goal: Find a derivative of the local density ψ instead of the total amount of extensive property Ψ

$$\frac{d}{dt} \int_\Omega \psi dV = - \int_\Gamma \phi_\psi \underline{n} dA + \int_\Omega \sigma_\psi dV$$

→ change differentiation and integration

- Use: Reynold's transport theorem:

$$\frac{d}{dt} \int_\Omega \psi dV = \int_\Omega \frac{\partial \psi}{\partial t} dV + \int_\Gamma \psi \underline{u}^\Gamma \underline{n} dA$$

Derivation of differential balance II

$$\int_\Omega \frac{\partial \psi}{\partial t} dV = -\int_\Gamma \psi \underline{u}^\Gamma \underline{n} dA - \int_\Gamma \phi_\psi \underline{n} dA + \int_\Omega \sigma_\psi dV$$

- Finally we have to transfer the surface integrals into volume integrals to get the same integration domain
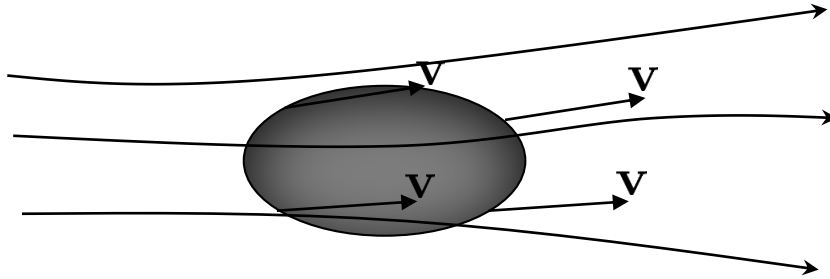
- Use: Divergence theorem

Derivation of differential balance III

$$\int_\Omega \frac{\partial \psi}{\partial t} dV = - \int_\Gamma \psi \underline{u}^\Gamma \underline{n} dA - \int_\Gamma \phi_\psi \underline{n} dA + \int_\Omega \sigma_\psi dV$$

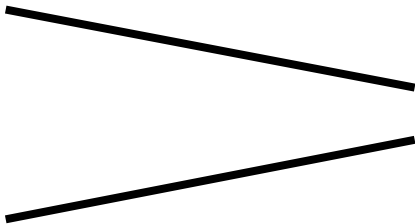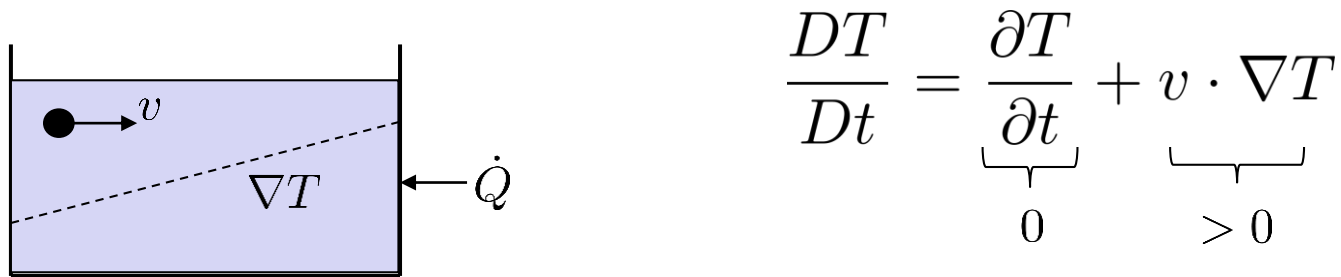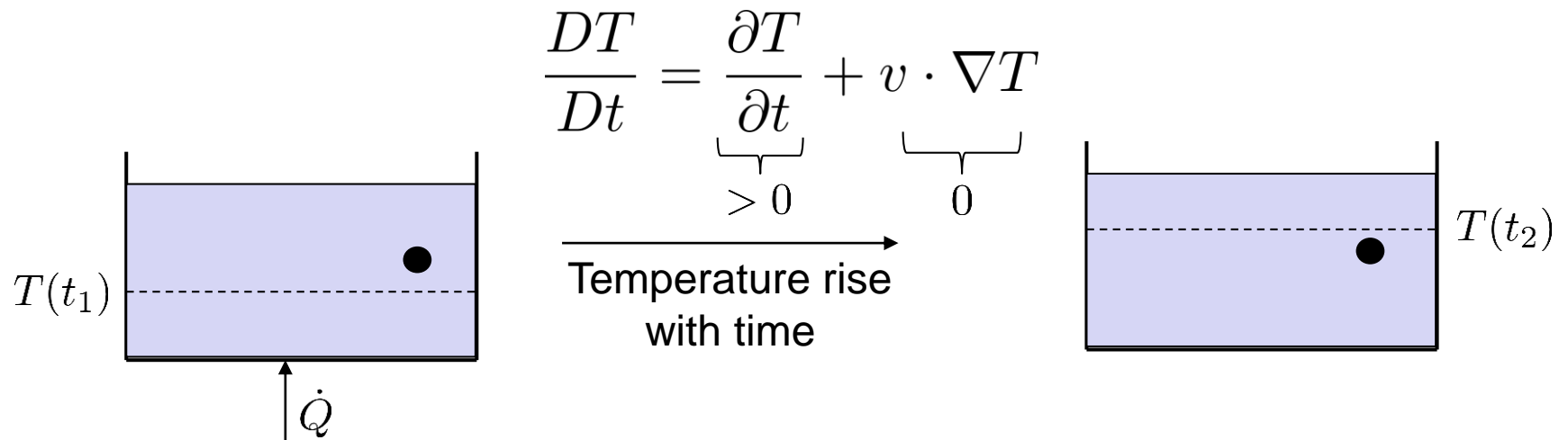- Apply divergence theorem:

Mass balance
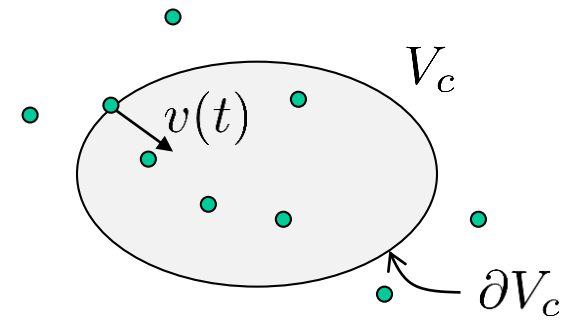
"Moving with the flow"

- The boundary of the element moves with the bulk velocity

- Property change:
  - Unsteady flow
  - Motion through a gradient of the property

- Example:

Example: Material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{> 0} + \underbrace{v \cdot \nabla T}_{0}$$

$T(t_1)$

$\dot{Q}$

Temperature rise with time

$T(t_2)$

v

∇T

$\dot{Q}$

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{0} + \underbrace{v \cdot \nabla T}_{> 0}$$

The momentum balance I

- In words

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \left\{ \begin{array}{c} \text{transfer of momentum into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{generation of momentum} \\ \text{in } V_c \text{ due to forces} \\ \text{acting on } V_c \end{array} \right\}$$

- Mathematically

$$\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p} = \frac{{}^i\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_c} \rho\vec{v}\mathrm{d}V = -\iint\limits_{\partial V_c} \rho\vec{v}\vec{v}\cdot\vec{n}\mathrm{d}A + \vec{F}^{(r)}$$

where $\vec{F}^{(r)}$ is resultant force on fluid in control volume

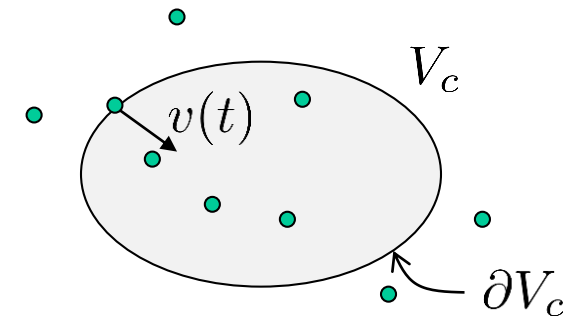(often: gravity (hydrostatic) and/or friction (hydrodynamic))

The momentum balance II

$$\frac{d}{dt}\int_\Omega \rho\underline{v}dV = -\int_\Gamma \rho\underline{v}\cdot\underline{v}\cdot\underline{n}dA + \int_\Omega BdV + \int_\Gamma \underline{n}\cdot\underline{\sigma}dA$$

$$\vdots$$

$$\frac{\partial\rho\underline{v}}{\partial t} + \nabla\cdot\underline{v}\rho\underline{v} - \nabla\underline{\sigma}$$

The energy balance I

V_c

$v(t)$

∂V_c

- In words

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by fluid flow} \\ \text{across surface } \partial V_c \end{array} \right\} + \left\{ \begin{array}{c} \text{transfer of energy into} \\ V_c \text{ by heat transfer} \\ \text{and by work} \end{array} \right\}$$
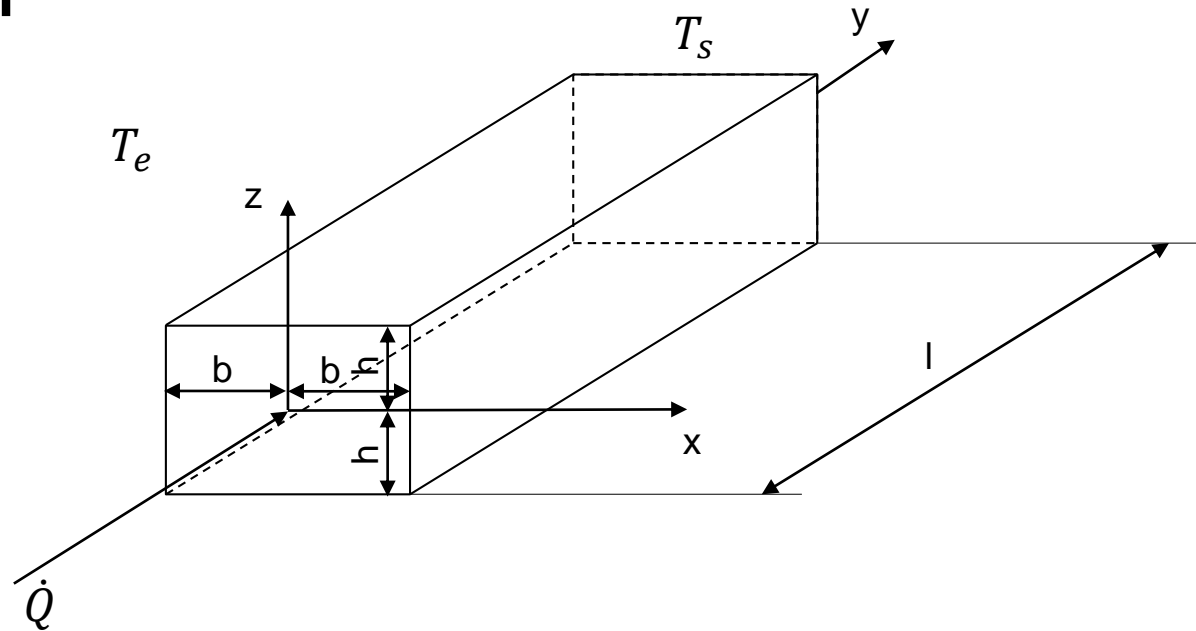
- Mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho e \, \mathrm{d}V = \underbrace{-\iint\limits_{\partial V_c} \rho e \vec{v}\cdot\vec{n}\,\mathrm{d}A}_{\text{Energy flow by convection}} + \dot{Q} - \dot{W}$$
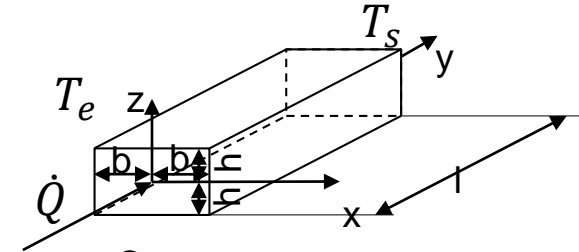
The energy balance II

$$\frac{d}{dt} \int_\Omega \rho e \, dV = - \int_\Gamma \rho e \cdot \underline{v} \cdot \underline{n} \, dA + \frac{dQ}{dt} - \frac{dW}{dt}$$
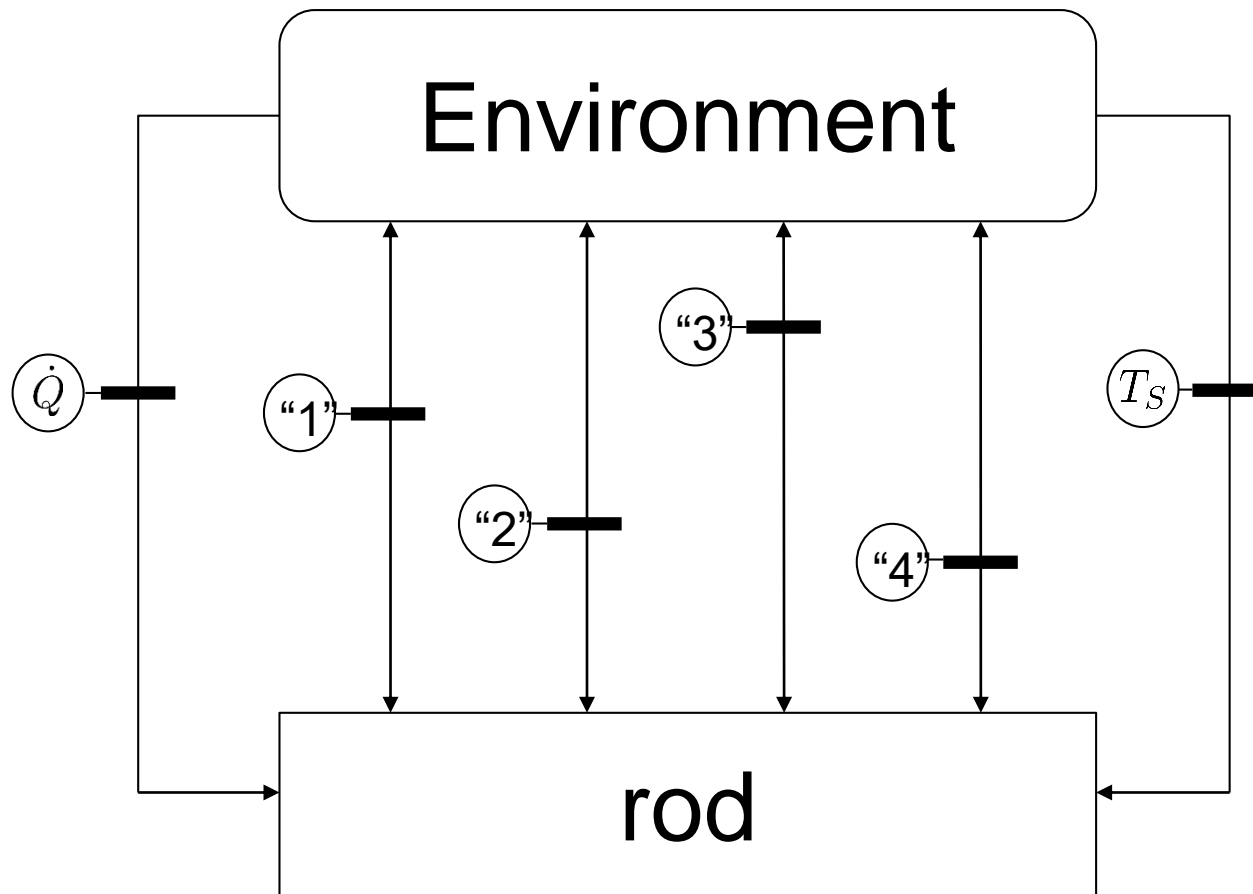
Example – heated bar

- At all sidewalls there is heat exchange with the environment (T_e, heat exchange coefficient α)

- At the front side there is a heat flux $\dot{Q}$

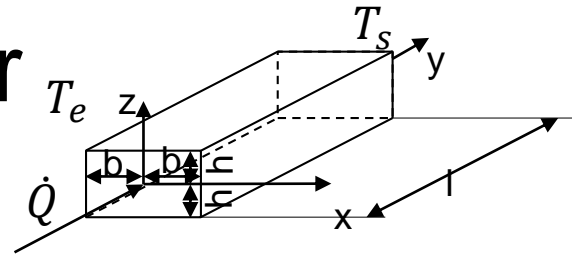- At the back side there is a constant temperature T_s
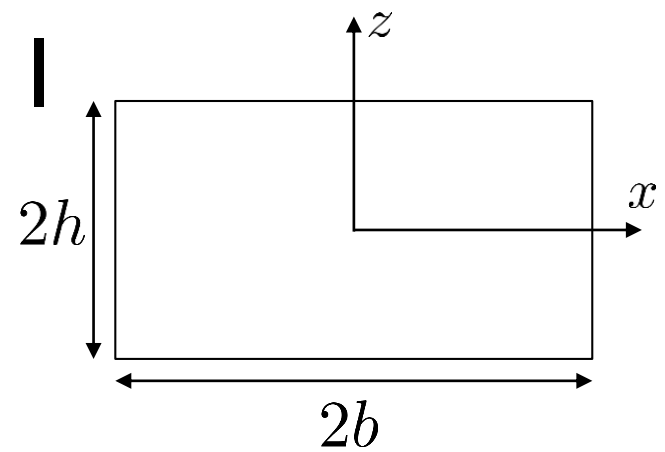
Abstraction of process
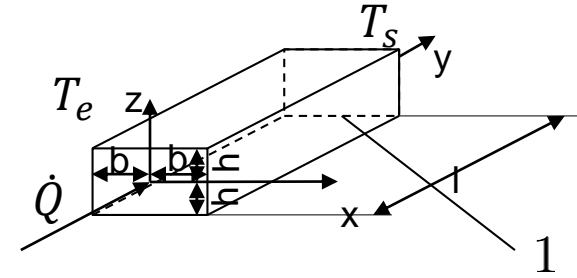
- How many interaction with the environment?

Energy balance – heated bar

Solve differential balance I

Boundary conditions

Solve differential balance II (pde)