Lecture 11: Linear Quadratic (LQ) Control

- Recap: Model Predictive Control (MPC)
- LQ; a special case of MPC with a very attractive solution
  - (MPC sometimes called "constrained LQ")

- Finite horizon LQ control
- Infinite horizon LQ control

Reference: B&H Ch. 4.3-4.4

Linear MPC; open loop dynamic optimization

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{xt+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{ut} u_t + \frac{1}{2} \Delta u_t^\top S_t \Delta u_t$$

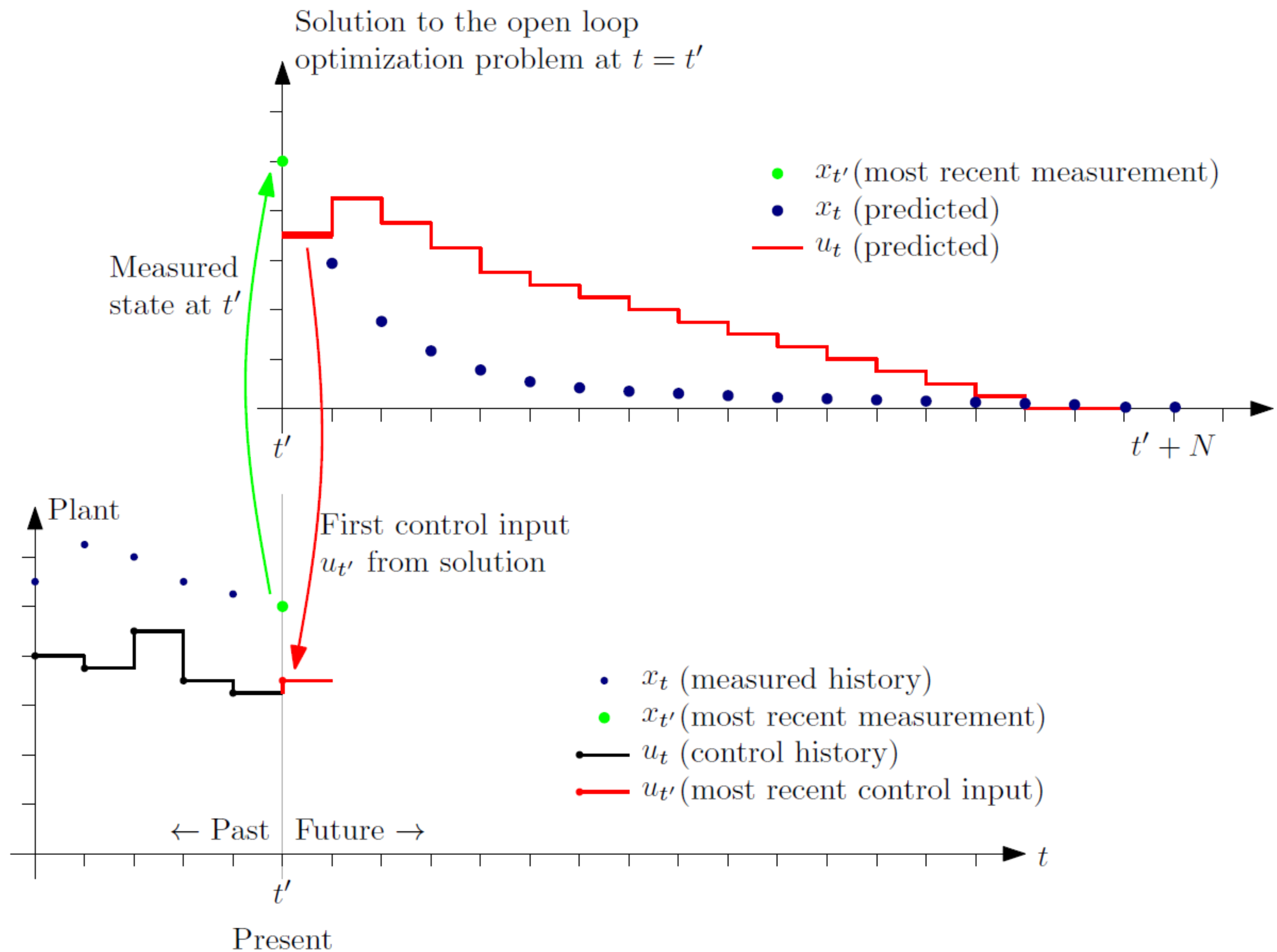
subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\} \\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\} \\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
-\Delta u^{\text{high}} &\leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
Q_t &\succeq 0 \quad t = \{1, \dots, N\} \\
R_t &\succeq 0 \quad t = \{0, \dots, N-1\} \\
S_t &\succeq 0 \quad t = \{0, \dots, N-1\}
\end{aligned}$$

where

$$\begin{aligned}
x_0 &\text{ and } u_{-1} \text{ is given} \\
\Delta u_t &:= u_t - u_{t-1} \\
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u)
\end{aligned}$$

Model predictive control principle



Necessary conditions (Ch. 12, N&W)

- Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

Theorem 12.1 (First-Order Necessary Conditions).

Suppose that x^* is a local solution of (12.1), that the functions f and c_i in (12.1) are continuously differentiable, and that the LICQ holds at x^*. Then there is a Lagrange multiplier vector λ^*, with components λ_i^*, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied at (x^*, λ^*)

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \tag{12.34a}$$

$$c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E}, \tag{12.34b}$$

$$c_i(x^*) \geq 0, \quad \text{for all } i \in \mathcal{I}, \tag{12.34c}$$

$$\lambda_i^* \geq 0, \quad \text{for all } i \in \mathcal{I}, \tag{12.34d}$$

$$\lambda_i^* c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{12.34e}$$

Second-order conditions

Theorem 12.6 (Second-Order Sufficient Conditions).

Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions (12.34) are satisfied. Suppose also that

$$w^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) w > 0, \quad \text{for all } w \in \mathcal{C}(x^*, \lambda^*), w \neq 0. \tag{12.65}$$

Then x^* is a strict local solution for (12.1).

- Critical directions:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{E}, \\ \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0, \\ \nabla c_i(x^*)^T w \geq 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0. \end{cases} \tag{12.53}$$

- The critical directions are the “allowed” directions where it is not clear from KKT-conditions whether the objective will decrease or increase

Thm 16.4: For convex QP, KKT is sufficient

- From N&W, p. 464:

KKT conditions

For convex QP, when G is positive semidefinite, the conditions (16.37) are in fact sufficient for x^* to be a global solution, as we now prove.

Theorem 16.4.

If x^* satisfies the conditions (16.37) for some λ_i^*, $i \in \mathcal{A}(x^*)$, and G is positive semidefinite, then x^* is a global solution of (16.1).

- That is, since the solution of the Riccati equation implies the KKT conditions are fulfilled, Thm 16.4 means that Riccati equation gives the global solution
  - Side-remark: It is, in fact, the *unique* global solution. If G is positive definite (implied by Q positive definite), this follows from the proof of Thm 16.4. If Q positive semidefinite, further arguments are necessary (for instance using Thm 12.6 as in the note).

- Finite horizon LQ controller

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t \\
\text{subject to} \quad x_{t+1} = A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\
x_0 = \text{given} \\
Q_t \succeq 0 \quad t = 1, \dots, N \\
R_t \succ 0 \quad t = 0, \dots, N-1
\end{aligned}
$$

where

$$
\begin{aligned}
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u)
\end{aligned}
$$

- State feedback solution

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$
\begin{aligned}
K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 \\
P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0, \dots, N-1 \\
P_N &= Q_N
\end{aligned}
$$

(discrete) Riccati equation

Linear quadratic control

- The optimal solution to LQ control is a linear, time-varying state feedback:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

- Note that the gain matrix K_t is independent of the states. It can therefore be computed in advance (knowing A_t, B_t, Q_t, R_t)

- The matrix (difference) equation

$$\begin{aligned} P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

  is called the (discrete) *Riccati equation*

- Note that the "boundary condition" is given at the end of the horizon, and the P_t -matrices must be found iterating backwards in time

Example

$$\min \sum_{t=0}^{10} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r\, u_t^2$$

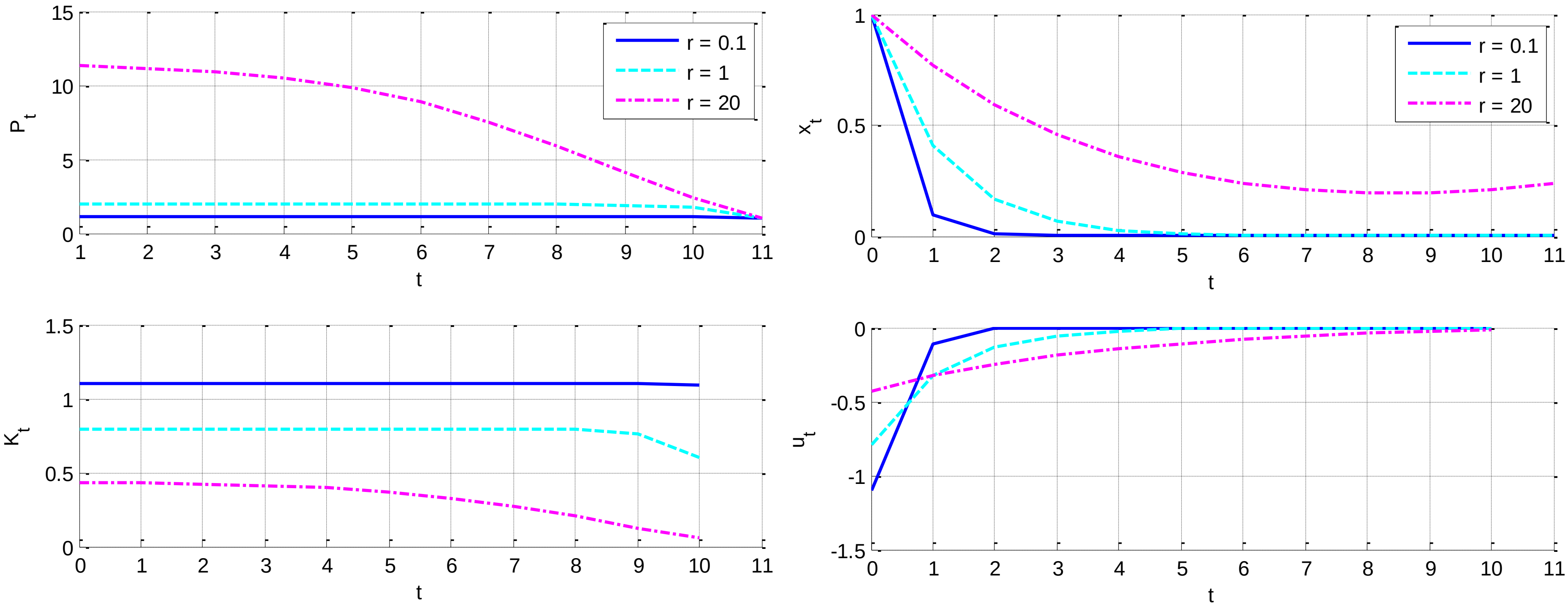
$$\text{s.t.} \quad x_{t+1} = 1.2 x_t + u_t, \quad t = 0, 1, \ldots, 10$$

LQ solution

$$P_t = 1 + \frac{1.44 r P_{t+1}}{P_{t+1} + r}, \quad t = 10, \ldots, 1$$

$$P_{11} = 1$$

$$K_t = 1.2 \frac{P_{t+1}}{P_{t+1} + r}, \quad t = 0, \ldots, 10$$



MPC vs LQ

- The difference between finite horizon LQ and MPC open loop problem is constraints

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} \; & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.:} \quad & x_{t+1} = A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\
& x_0 = \text{given}
\end{aligned}
$$

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} \; & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.:} \quad & x_{t+1} = A_t x_t + B_t u_t, \quad t = 0, \dots, N-1 \\
& x^{\text{low}} \le x_t \le x^{\text{high}}, \quad t = 1, \dots, N \\
& u^{\text{low}} \le u_t \le u^{\text{high}}, \quad t = 0, \dots, N-1 \\
& x_0 = \text{given}
\end{aligned}
$$

- LQ open loop solution:

$$
u_t = -K_t x_t, \quad t = 0, \dots, N-1 \quad \text{where} \begin{cases} K_t & = R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad t = 0, \dots, N-1 \\ P_t & = Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad t = 0, \dots, N-1 \\ P_N & = Q_N \end{cases}
$$

- That is: The MPC control law when there are no constraints, is

$$
u_t = -K_0 x_t
$$

Previous example (MPC optimality implies stability?)

$$\min \sum_{t=0}^{1} \frac{1}{2} x_{t+1}^2 + \frac{1}{2} r\, u_t^2$$

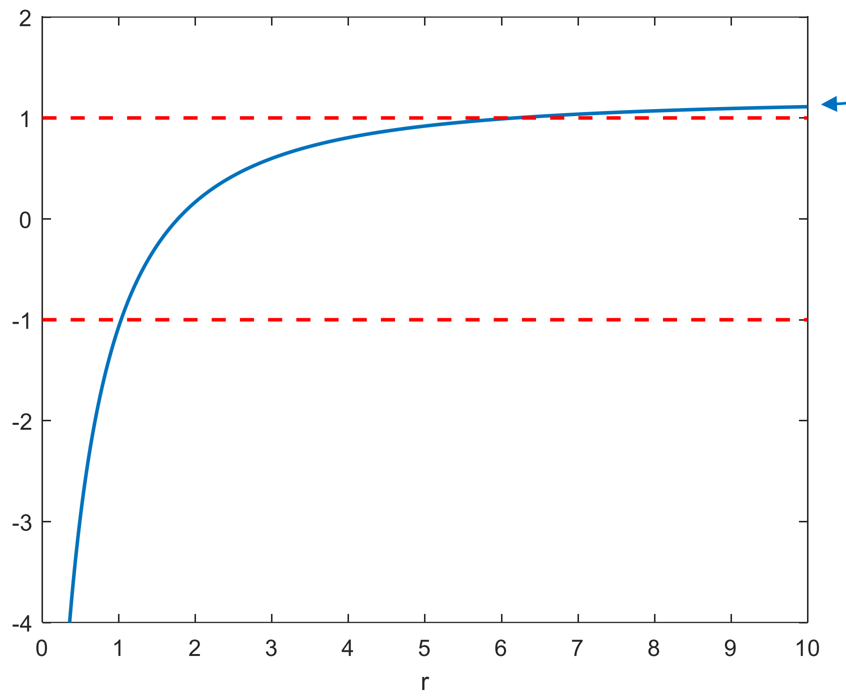
$$\text{s.t.} \quad x_{t+1} = 1.2 x_t + u_t, \quad t = 0, 1$$

MPC solution

$$u_t = -\frac{1.2 + 2.64r}{1 + 3.2r + r^2} x_t$$

MPC closed loop

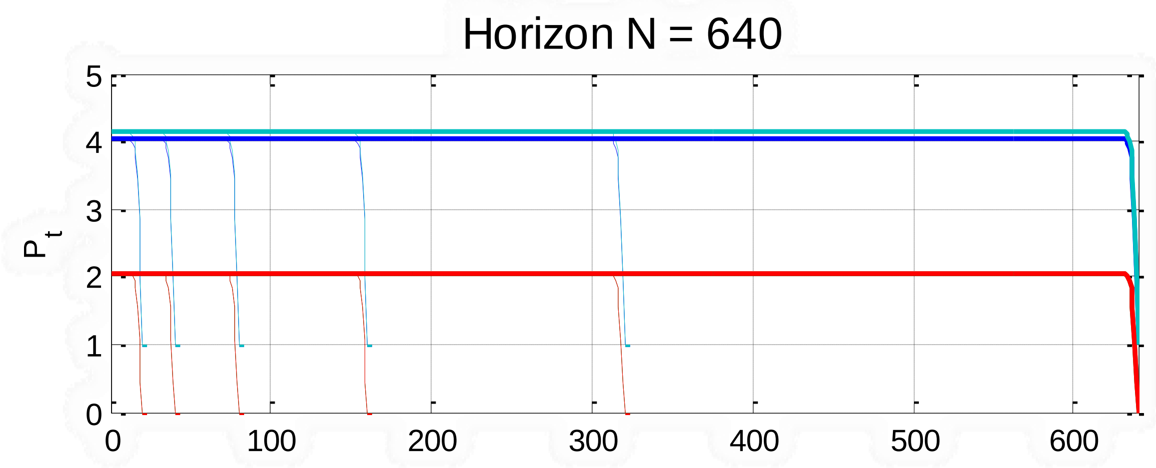
$$x_{t+1} = \underbrace{\left(1.2 - \frac{1.2 + 2.64r}{1 + 3.2r + r^2}\right)} x_t$$



Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$



Controllability vs stabilizability
Observability vs detectability

- Stabilizable: All unstable modes are controllable (that is: all uncontrollable modes are stable)

- Detectability: All unstable modes are observable (that is: all unobservable modes are stable)

- Controllability implies stabilizability
- Observability implies detectability

Riccati equations

- Discrete-time Riccati equation in the note (and lecture)

$$P_t = Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad P_N = Q_N$$

- However, another, equivalent, form is found in other sources:

$$P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (R_t + B_t^\top P_{t+1} B_t)^{-1} B_t^\top P_{t+1} A_t, \quad P_N = Q_N$$

- The latter is more numerically stable due to “enforced symmetry”

- The trick used to get the different formulas is the “Matrix Inversion Lemma” (a very useful Lemma in control theory, optimization, …)

- Discrete-time Algebraic Riccati equation (DARE) in the note (and lecture)

$$P = Q + A^\top P(I + BR^{-1}B^\top P)^{-1} A$$

- Other form (e.g. Matlab)

$$P = Q + A^\top PA - A^\top PB(R + B^\top PB)^{-1} B^\top PA$$

  – Note: This is a quadratic equation with two solutions. The one we want is the positive definite solution (the “stabilizing” solution).

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>> help dare
dare  Solve discrete-time algebraic Riccati equations.

   [X,L,G] = dare(A,B,Q,R,S,E) computes the unique
             stabilizing solution X of the discrete-time
             algebraic Riccati equation
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