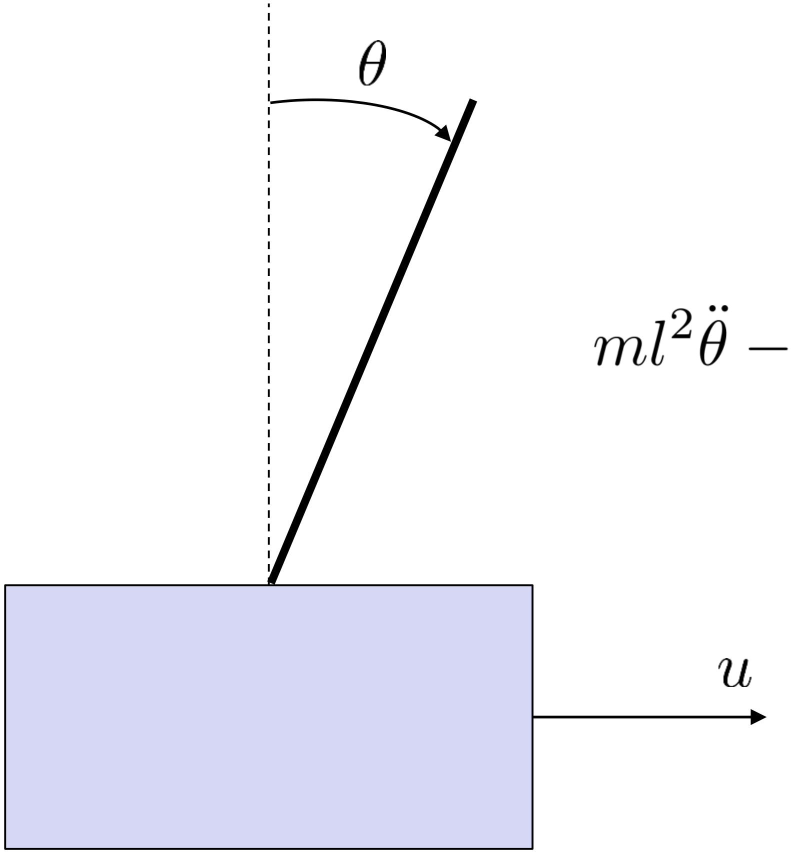
Lecture 2:

- Model types (E1.1-1.3,E2.1-2.2)
  - State space models, transfer functions
  - Linear models, nonlinear models

Example: “Stick balancing”



$$ml^2\ddot{\theta} - mg\sin\theta = -mlu\cos\theta$$

Example: “Stick balancing”

$$ml^2\ddot{\theta} - mg\sin\theta = -mlu\cos\theta$$

1.3 Transfer functions

Linear time invariant model (LTI)

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}$$

System matrix

Control matrix

$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

Output matrix

Feed-Forward matrix

Laplace notation

$$\underline{x}(s) = \mathcal{L}\{\underline{x}(t)\}$$

$$\underline{u}(s) = \mathcal{L}\{\underline{u}(t)\}$$

$$\underline{y}(s) = \mathcal{L}\{\underline{y}(t)\}$$

$$\mathcal{L}\{\underline{\dot{x}}(t)\} = s\mathcal{L}\{\underline{x}(t)\} - \underbrace{x(t=0)}_{\text{Assume } = 0}$$

Transform LTI system

$$\underline{\dot{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \qquad \underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

Rational transfer function

$$\frac{y(s)}{u(s)} = H(s)$$

- Rational transfer function if it can be expressed as:

$$H(s) = K\frac{P(s)}{Q(s)}$$
$$= K\frac{(s+z_1)\dots(s+z_m)}{(s+p_1)\dots(s+p_n)}$$

- m: zeros ; n: poles

Partial differential equations

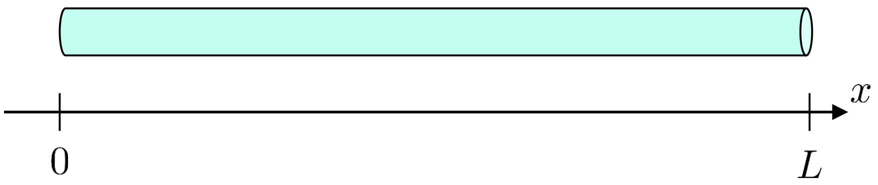
- Partial differential equations (pde) lead to irrational transfer functions

- They can be approximated by rational transfer functions with infinitely order

  → infinit dimension system

- Example: Transport equation/advection /wave equation

PDE – Example I



PDE – Example II

Lecture 3: Energy functions and passivity

Using ”energy” as a concept for characterizing system behavior

- Energy functions (aka Lyapunov functions)
  - If the ”internal energy” of a system decreases, the system is stable
  - ”Introvert” (not concerned with surroundings)
- Passivity
  - Does a system produce ”energy” to its surroundings?
  - ”Extrovert” (mainly concerned with surroundings, via inputs and outputs)
- The above concepts are connected via storage functions (next time)

Book: E2.3, E2.4

Energy function

- The system: $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$

- Assume we have a function $V(x,t) \geq 0$, which describes the «energy» of the system

- The derivative of the energy function $V(x,t)$ is

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t)$$

- If we have $\dot{V} \leq 0$

  → Energy of the system decreases monotonically

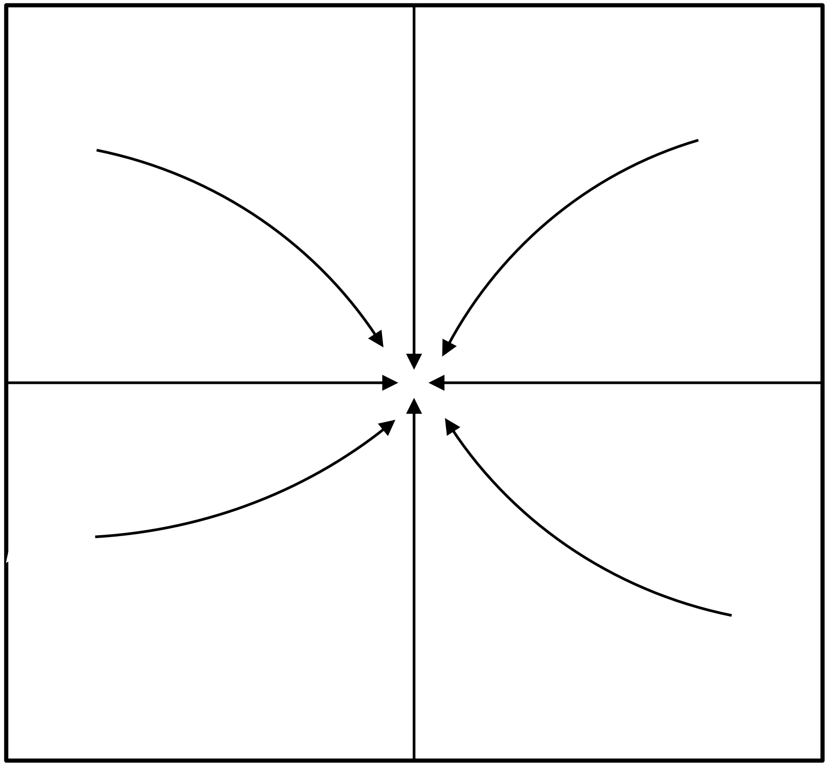
  → stability

Stable system

- Equilibrium point is stable if for any possible $\varepsilon > 0$ radius around the steady state point a region with the radius δ exist, such that for all initial values $|x_0 - x_e| < \delta$ the solution $x(t)$ fullfils for all $t > t_0$ the following condition:

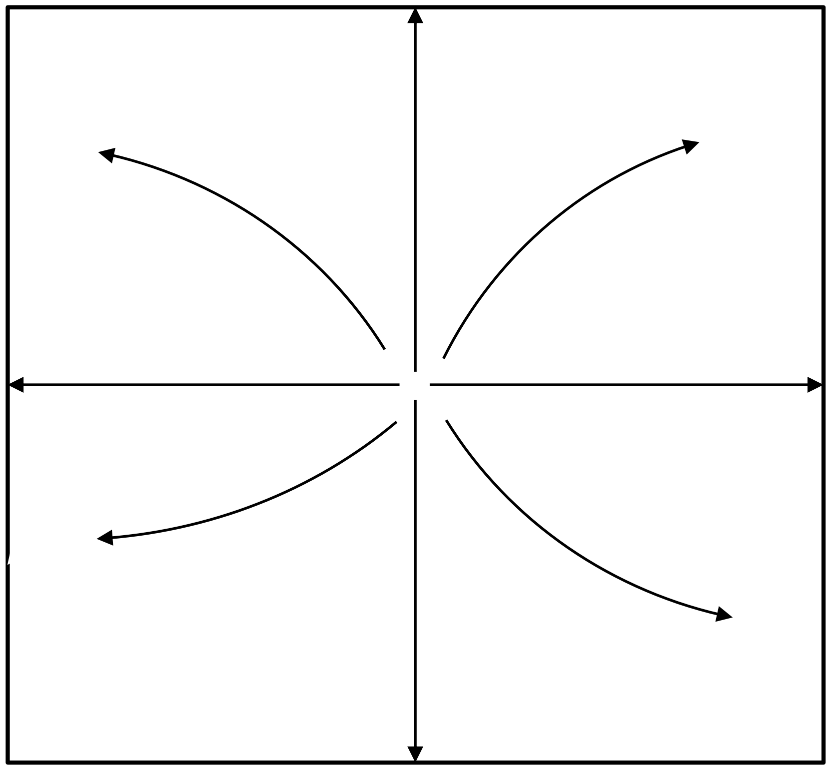
$$|x(t) - x_e| < \varepsilon$$

Phase diagram for system with real Eigenvalues



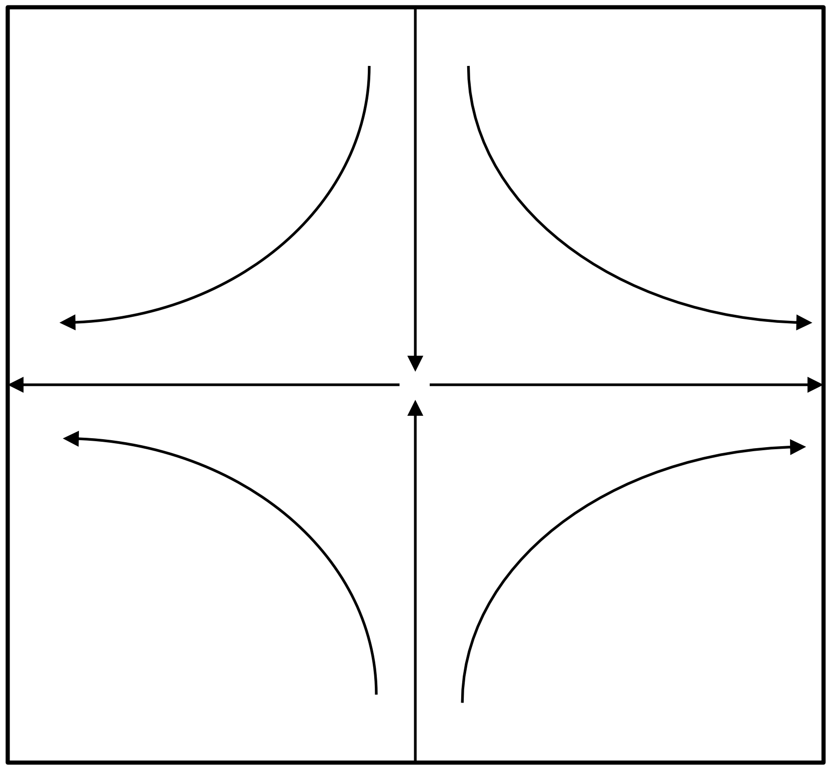
Stable

Phase diagram for system with real Eigenvalues



Unstable

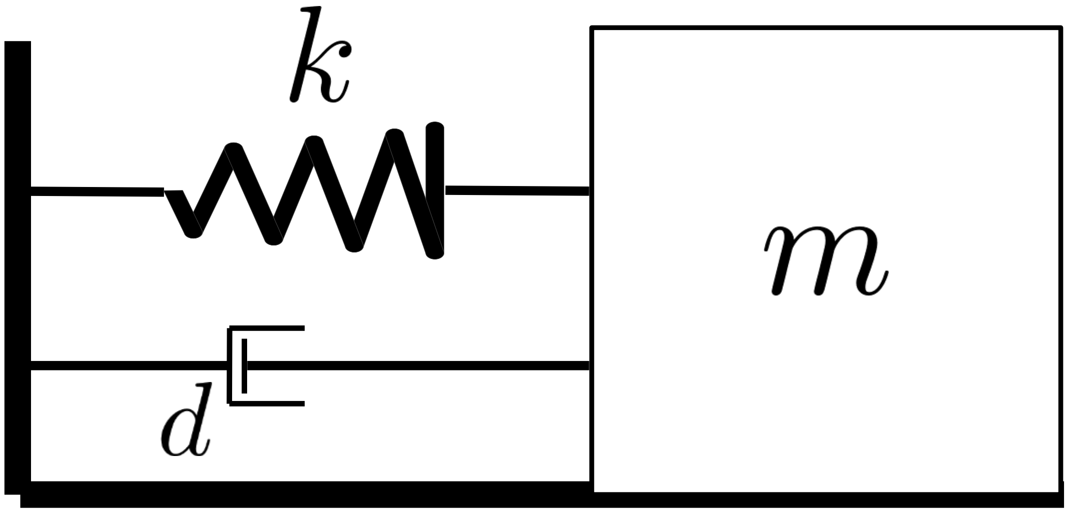
Phase diagram for system with real Eigenvalues



Saddle → unstable

Mass-spring-damper I

$$m\ddot{x} + d\dot{x} + kx = 0$$

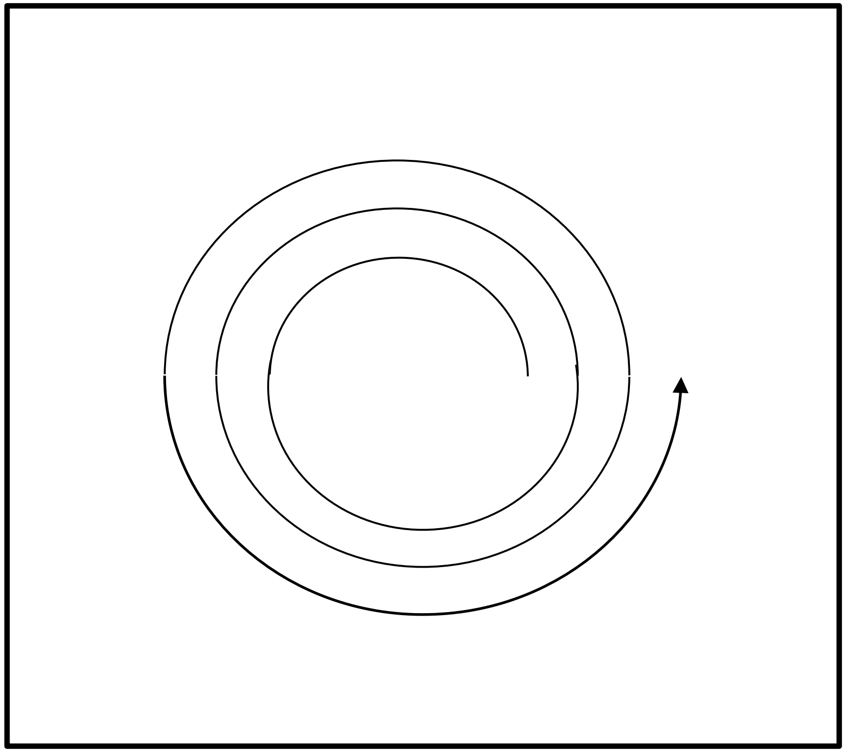


Mass-spring-damper II $m\ddot{x} + d\dot{x} + kx = 0$

Mass-spring-damper III $m\ddot{x} + d\dot{x} + kx = 0$

Mass-spring-damper

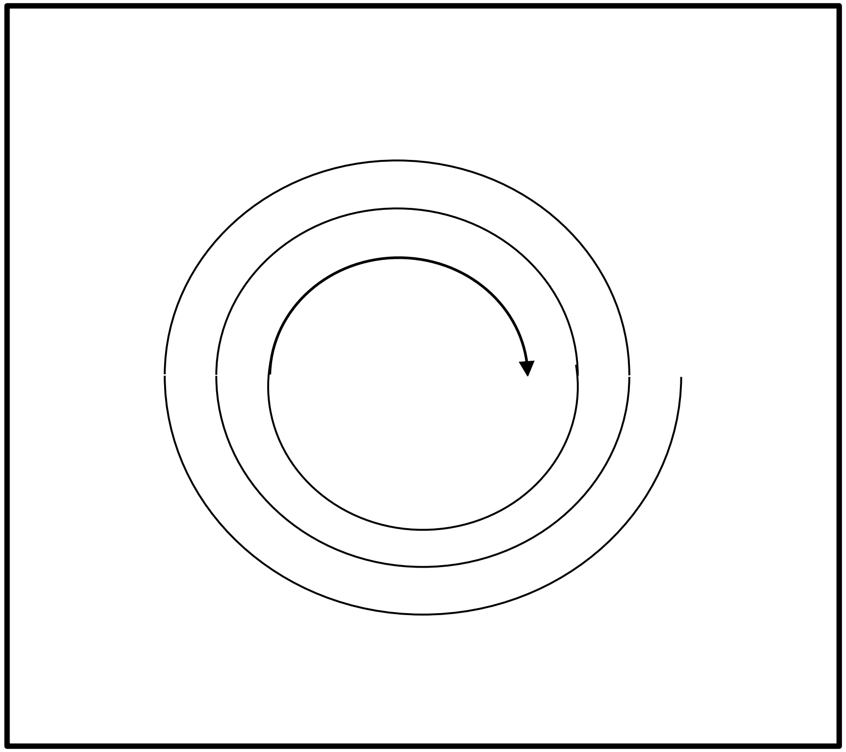
$$\lambda_{1,2} = u \pm iv$$



unstable $u > 0$

Mass-spring-damper

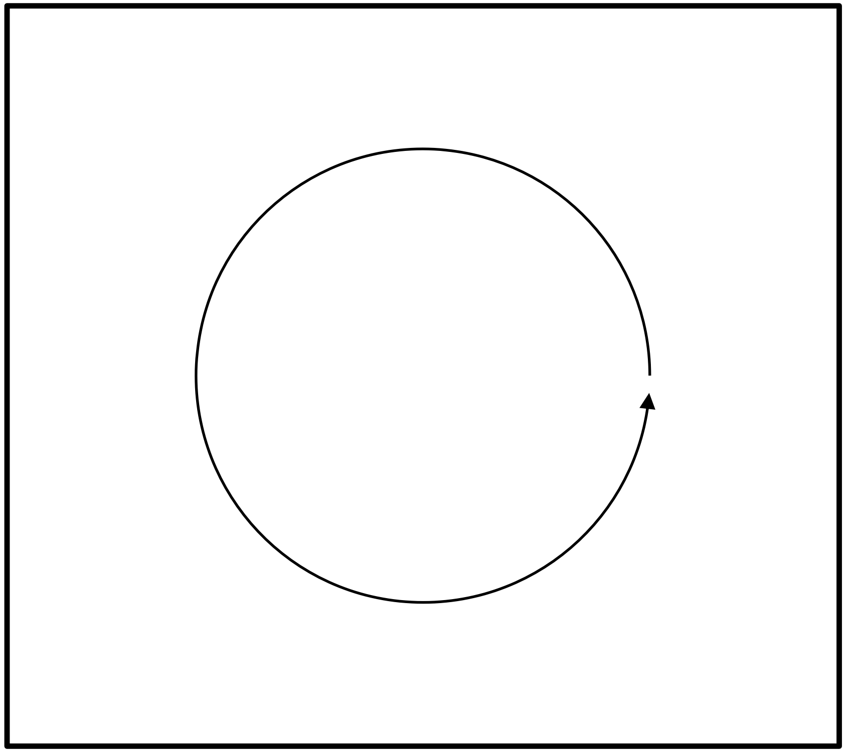
$$\lambda_{1,2} = u \pm iv$$



stable $\quad u < 0$

Mass-spring-damper

$$\lambda_{1,2} = u \pm iv$$

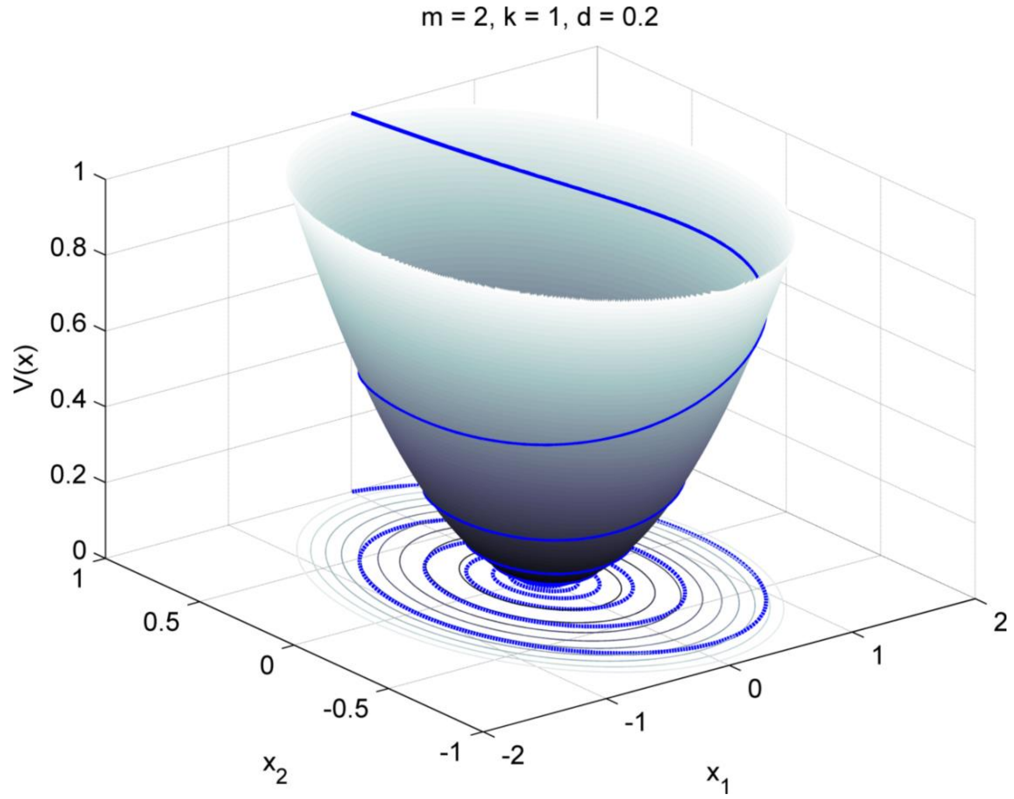


centre $\qquad u = 0$

Mass-spring-damper

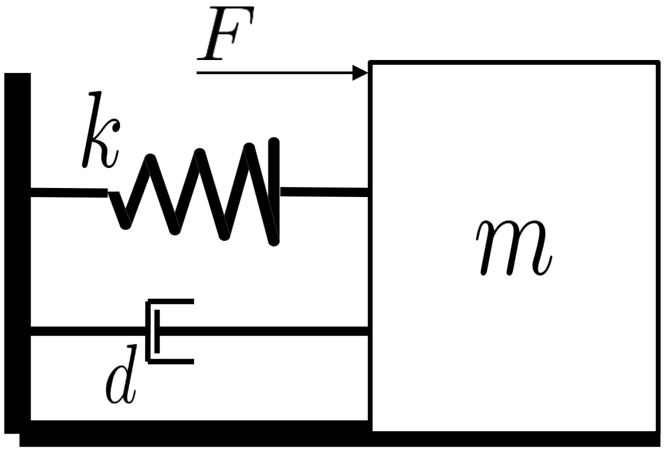
$$m\ddot{x} + d\dot{x} + kx = 0$$

$$V(x) = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$$



Mass-spring-damper with force

$$m\ddot{x} + d\dot{x} + kx = F$$



$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{F}{m} - \frac{k}{m}x_1 - \frac{d}{m}x_2 \end{aligned}$$

General: Energy-based controller design

$$\left.\begin{aligned} \dot{x} &= f(x,u,t) \\ \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t) \end{aligned}\right\}$$

Choose a controller that

$$\dot{V} \leq 0$$

Why learn about passivity? Preview...

- Say you have several systems (or models), and you want to interconnect them
  - For instance, a process and a controller, or a motor and a load, or two buffer tanks in series, ...
  - Will the interconnection be stable?

- Bad news: The interconnection of stable systems is not necessarily stable

- **Good news: The interconnection of passive systems is passive (and therefore stable)!**

Homework (recommended)

- Derive the derivative of the energy function of the mass-spring-damper system with force
  - $\dot{V} = -dx_2^2 + Fx_2$

- Read section 2.4.1, 2.4.2, 2.4.3 in the book
  - Try to proof passivity of the transfer-function:

$$H(s) = \frac{1}{1 + Ts}$$

by first transfering the function to the time domain