Lecture 13: Rigid body kinematics – Kinematic differential equations

- Brief recap of representations of rotation
  - Rotation matrices (6.4)
  - Euler angles (6.5)
    - 3-parameter representation of rotations
    - Roll-pitch-yaw
  - Angle-axis, Euler-parameters (6.6, 6.7)
    - 4-parameter representation of rotations
  - Angular velocity (6.8)
  - Kinematic differential equations

- Today:
  - Rigid body kinematics: Configuration
  - Newton Euler equation

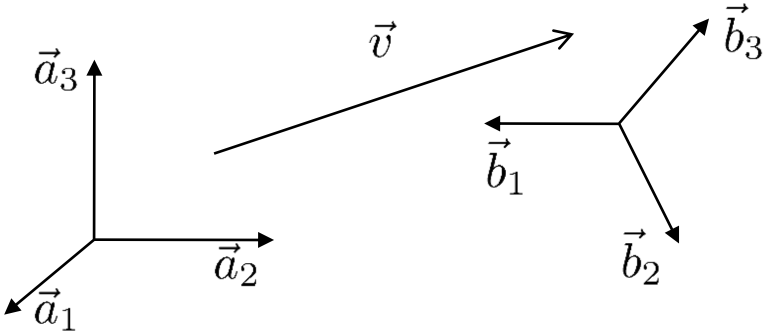
Book: Ch. 6.9, 6.12, 6.13, 7.1

Rotation matrices

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- **Transform** a coordinate vector from *b* to *a*

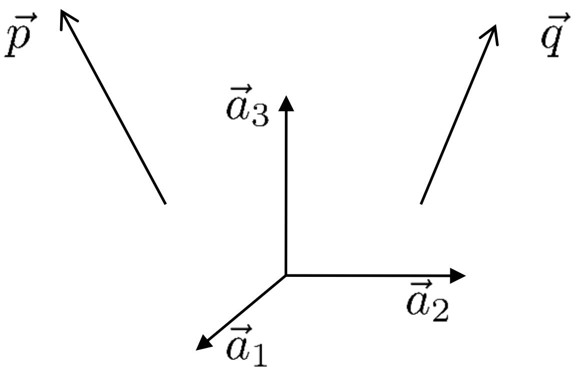
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- **Rotate** a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

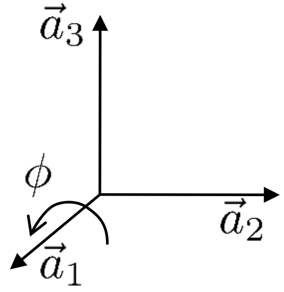
such that $\mathbf{q}^b = \mathbf{p}^a$.



Simple rotations

- Simple rotation = rotation about an axis
- Example: Rotation matrix for rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



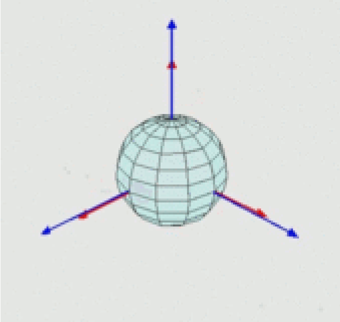
Representations of rotations

- Rotation matrix
  - Easy to use, but not to visualize (also over-parameterized, 9 parameters)

Euler's Theorem:

"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

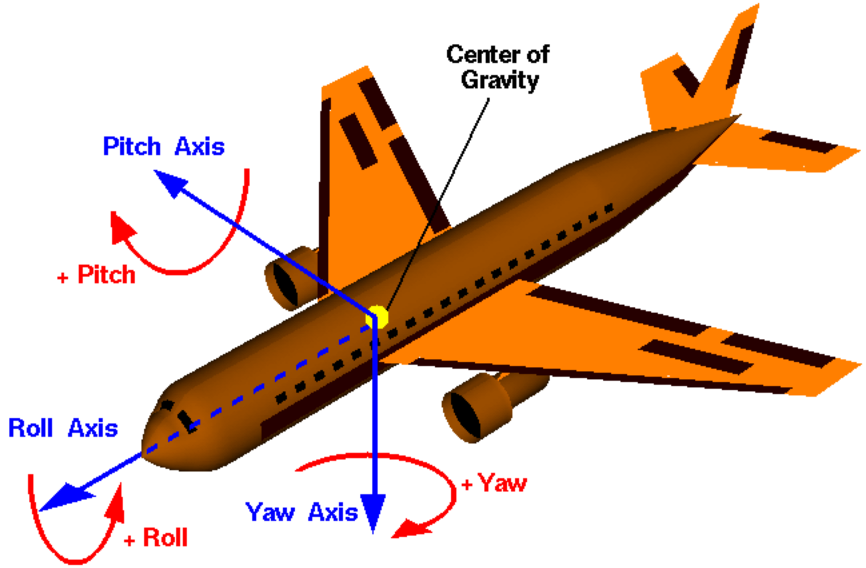
- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common(?): Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this today)



Source: Wikipedia

- Angle-axis, Euler-parameters
  - 4-parameter representations of rotations
  - No singularity problems

Euler-angles: Roll-pitch-yaw

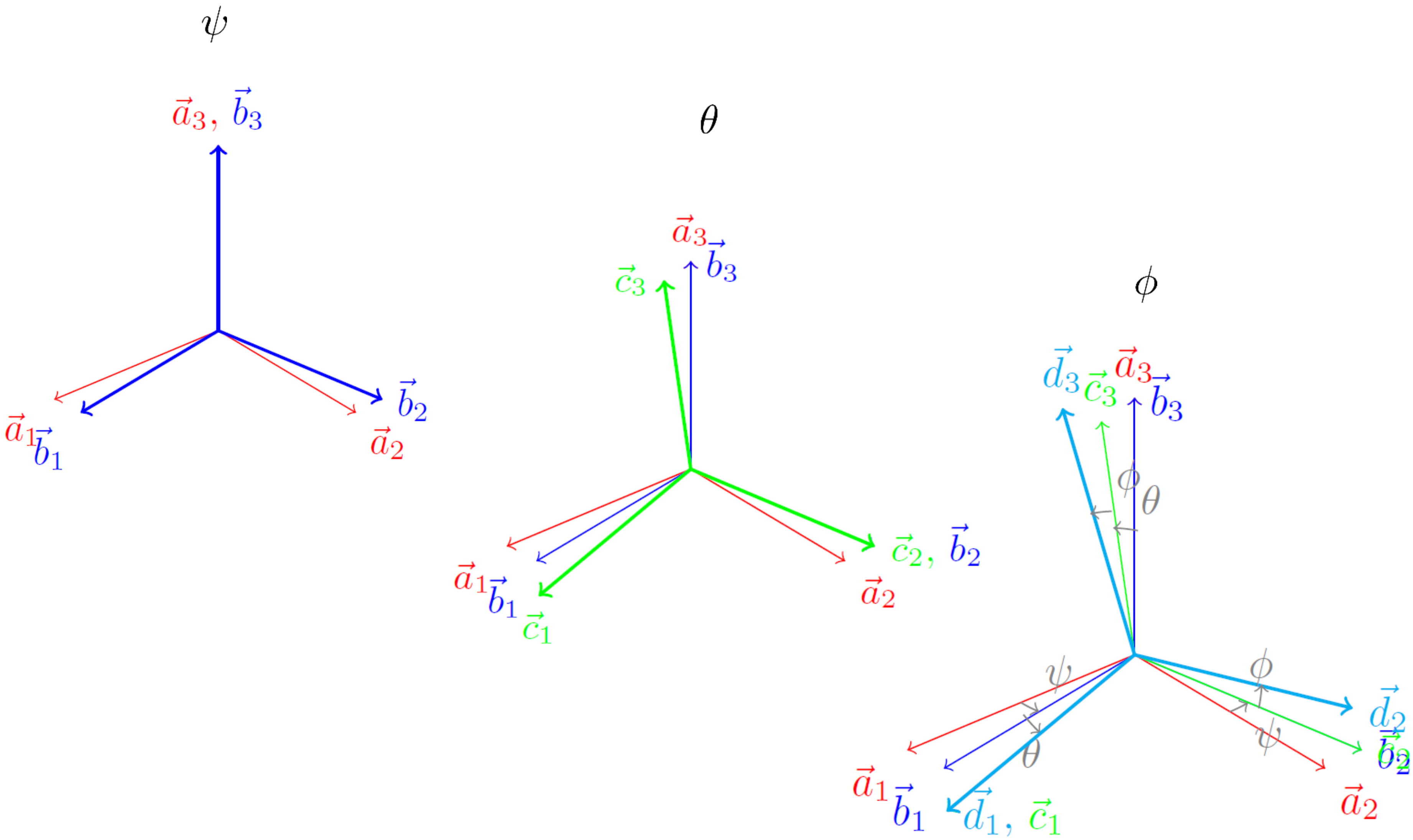


- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

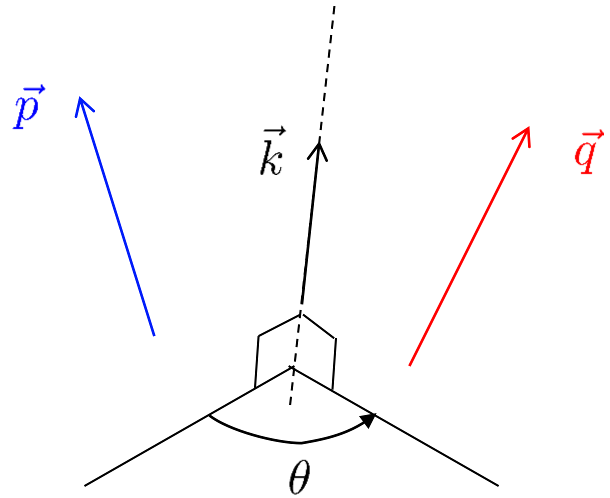
$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Euler angles



Angle-axis representation of rotations

All rotations can be represented as a simple rotation around an axis



- Angle-axis parameters:
  - Coordinate free: $\vec{k}$, θ

$$\vec{q} = \Big(\underbrace{\cos\theta\, \vec{I} + \sin\theta\, \vec{k}^{\times} + (1-\cos\theta)\, \vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}} \Big) \cdot \vec{p}$$

  - With coordinates: $\mathbf{k}^a$, θ

$$\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\, \mathbf{I} + \sin\theta\, (\mathbf{k}^a)^{\times} + (1-\cos\theta)\, \mathbf{k}^a (\mathbf{k}^a)^{\mathsf{T}}$$

Euler parameters

- Euler parameters are closely related to angle-axis:
  - Coordinate-free:

$$\eta = \cos\frac{\theta}{2}$$
$$\vec{\epsilon} = \vec{k}\sin\frac{\theta}{2}$$

With coordinates:

$$\eta = \cos\frac{\theta}{2}$$
$$\boldsymbol{\epsilon} = \mathbf{k}\sin\frac{\theta}{2}$$

- Rotation matrix (on coordinate form):

$$\mathbf{R}(\eta, \boldsymbol{\epsilon}) = \mathbf{I} + 2\eta\boldsymbol{\epsilon}^{\times} + 2\boldsymbol{\epsilon}^{\times}\boldsymbol{\epsilon}^{\times}$$

- Much used, since:
  - Compact, singularity-free representation of orientation
  - No trigonometric terms in expression for rotation matrix
  - $\eta^2 + \vec{\epsilon}\cdot\vec{\epsilon} = 1$: Easy to normalize (avoid roundoff errors)
    - Rotation matrices may tend to become non-orthogonal when simulated
  - Euler parameters are *(unit) quaternions*:
    - Quaternions are generalized complex numbers
    - Can use algebra of quaternions for calculations and analysis

Derivatives of rotations

- Derivative of position $\mathbf{r}$ is velocity, $\dot{\mathbf{r}} = \mathbf{v}$.

- Derivative of rotation matrix $\mathbf{R}_b^a$ is $\dot{\mathbf{R}}_b^a$. What is this?

- Seems natural that a concept of angular velocity should be involved, but how?

- What are derivatives of representations of rotations?
  - Derivatives of Euler angles? Euler parameters?
  - These are the kinematic differential equations!

Angular velocity

- The rotation matrix is orthogonal:

$$\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} = \mathbf{I}$$

- Differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}\right] = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} + \mathbf{R}_b^a \left(\dot{\mathbf{R}}_b^a\right)^\mathsf{T} = \mathbf{0}$$

- If we define $\mathbf{S} = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$, this says that $\mathbf{S} + \mathbf{S}^\mathsf{T} = \mathbf{0}$ which means that $\mathbf{S}$ is skew symmetric.

$$\mathbf{S} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \left(\boldsymbol{\omega}_{ab}^a\right)^\times$$

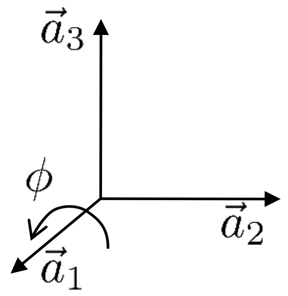
- The vector $\boldsymbol{\omega}_{ab}^a$ defined by $\left(\boldsymbol{\omega}_{ab}^a\right)^\times = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$ is the angular velocity of frame *b* relative to frame *a* (decomposed in *a*)

- The equation $\dot{\mathbf{R}}_b^a = \left(\boldsymbol{\omega}_{ab}^a\right)^\times \mathbf{R}_b^a$ is the kinematic differential equation for rotation matrices

Angular velocity of simple rotations

- Rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



- We calculate $(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^\mathsf{T}$:

$$\dot{\mathbf{R}}_{x,\phi} (\mathbf{R}_{x,\phi})^\mathsf{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{pmatrix} \dot{\phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{pmatrix}$$

- That is:

$$\boldsymbol{\omega}_x = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}$$

- Similar for rotations around *y*- and *z*-axis: $\boldsymbol{\omega}_y = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}, \quad \boldsymbol{\omega}_z = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

- Angle-axis representations (constant axis):

$$\boldsymbol{\omega}_{ab}^a = \dot{\theta}\mathbf{k}^a$$

Composite rotations

- Given
  - composite rotation $\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$, and
  - individual angular velocities $\boldsymbol{\omega}_{ab}^a$, $\boldsymbol{\omega}_{bc}^b$, and $\boldsymbol{\omega}_{cd}^c$

  How to calculate the composite angular velocity $\boldsymbol{\omega}_{ad}^a$?

- It can be shown (easy, see book p. 241) that

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd}$$

- On coordinate form:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a$$

- So:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \mathbf{R}_b^a \boldsymbol{\omega}_{bc}^b + \mathbf{R}_b^a \mathbf{R}_c^b \boldsymbol{\omega}_{cd}^c$$

Kinematic differential equation of Euler angles

$$\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c = \mathbf{R}_{z,\psi} \mathbf{R}_{y,\theta} \mathbf{R}_{x,\phi}$$

$$\vec{\boldsymbol{\omega}}_{ad} = \vec{\boldsymbol{\omega}}_{ab} + \vec{\boldsymbol{\omega}}_{bc} + \vec{\boldsymbol{\omega}}_{cd} \ = \dot{\psi}\vec{\boldsymbol{a}}_3 + \dot{\theta}\vec{\boldsymbol{b}}_2 + \dot{\phi}\vec{\boldsymbol{c}}_1$$

$$\underline{\boldsymbol{\omega}}_{ad}^a = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_z(\psi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathbf{R}_z(\psi)\mathbf{R}_y(\theta) \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}$$

$$= \mathbf{E}_a(\underline{\mathbf{\Phi}})\underline{\dot{\mathbf{\Phi}}} \qquad\qquad \underline{\mathbf{\Phi}} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$\underline{\dot{\mathbf{\Phi}}} = \mathbf{E}_a^{-1}(\underline{\mathbf{\Phi}})\underline{\boldsymbol{\omega}_{\boldsymbol{ad}}^{\boldsymbol{a}}} \qquad\qquad \theta \neq 90^o$$

Kinematic differential equation of Euler parameter

$$\mathbf{R}_b^a = \mathbf{R}(\eta, \underline{\varepsilon}) \qquad \dot{\mathbf{R}}_b^a = (\underline{\omega}_{ab}^a)^\times \mathbf{R}_b^a$$

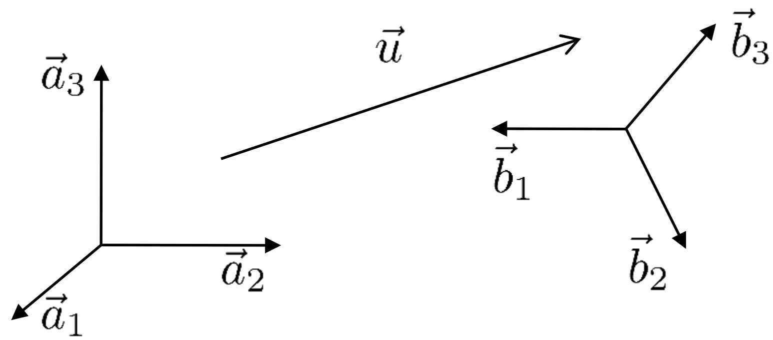
- It can be derived (quaternion algebra p. 248)

$$\dot{\eta} = -\frac{1}{2}\underline{\varepsilon}^T \underline{\omega}_{ab}^a$$
$$\dot{\underline{\varepsilon}} = \frac{1}{2}(\eta \mathbf{I} - \underline{\varepsilon}^\times)\underline{\omega}_{ab}^a$$

Differentiation of vectors (6.8.5, 6.8.6)

- Coordinate representation:

$$\mathbf{u}^a = \mathbf{R}^a_b \mathbf{u}^b$$



- Differentiation:

$$\dot{\mathbf{u}}^a = \mathbf{R}^a_b \dot{\mathbf{u}}^b + \dot{\mathbf{R}}^a_b \mathbf{u}^b$$

$$\dot{\mathbf{R}}^a_b = \mathbf{R}^a_b \left(\boldsymbol{\omega}^b_{ab}\right)^\times$$

$$\boxed{\dot{\mathbf{u}}^a = \mathbf{R}^a_b \left[\dot{\mathbf{u}}^b + \left(\boldsymbol{\omega}^b_{ab}\right)^\times \mathbf{u}^b\right]}$$

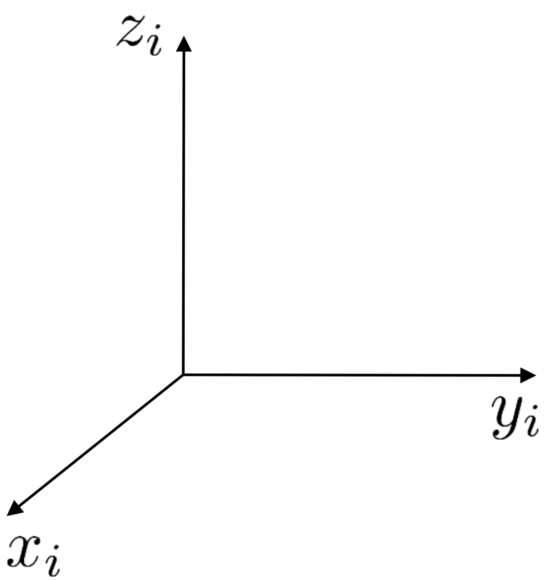
Note! Generally,

$$\dot{\mathbf{u}}^a \neq \mathbf{R}^a_b \dot{\mathbf{u}}^b$$

- On vector form:

$$\boxed{\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u}}$$

Kinematics of rigid body I

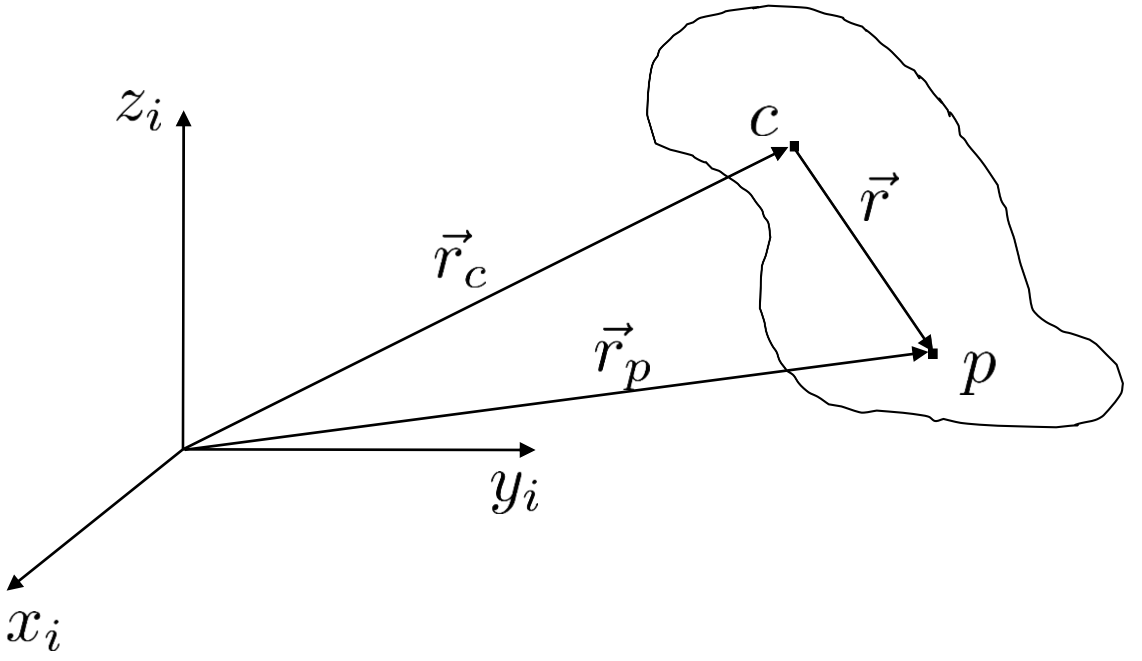


Kinematics of rigid body II

Kinematics of rigid body III

$$\vec{a}_p = \vec{a}_o + \frac{{}^b d^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r}_g + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)$$

Center of mass



What is rigid body dynamics?

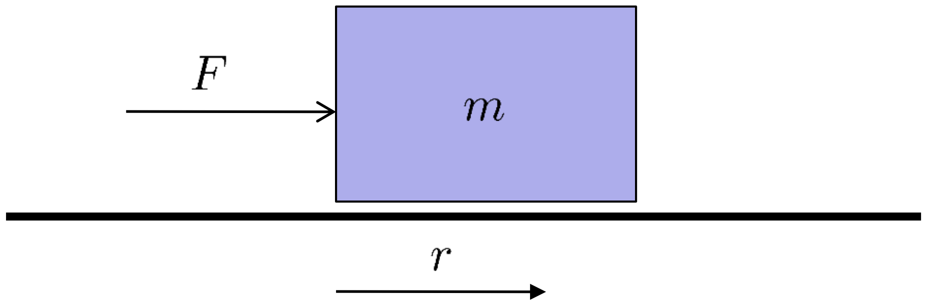
- Rigid body:
  - Wikipedia: "…a rigid body is an idealization of a solid body of finite size in which deformation is neglected."

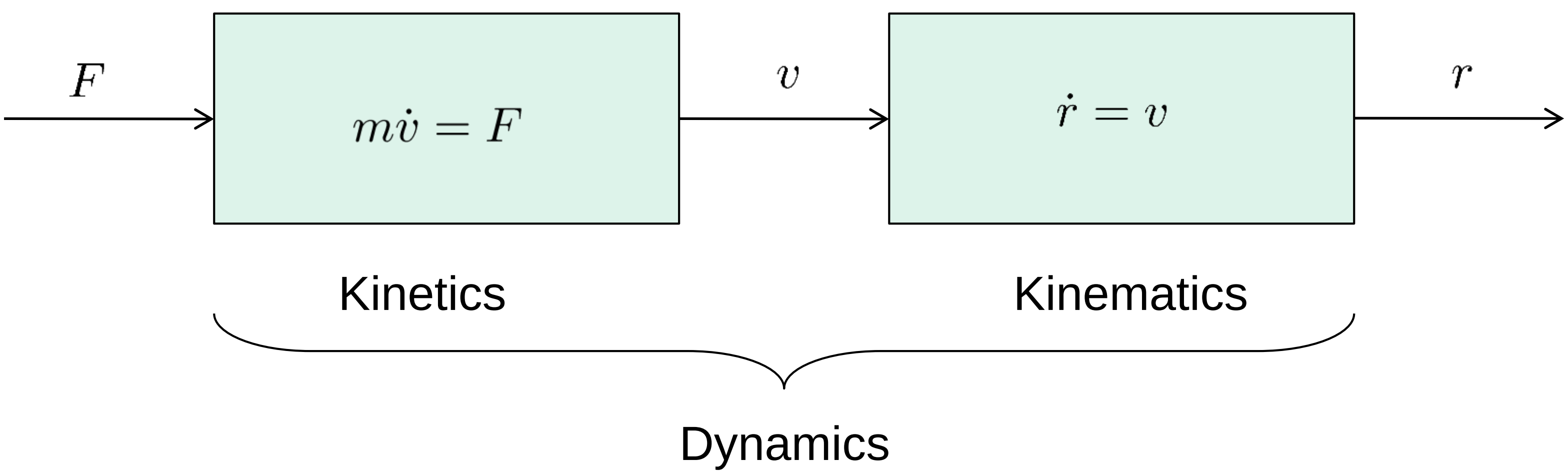
- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: "…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions)."
  - Book: Ch. 6
- Kinetics
  - eb.com: "…the effect of forces and torques on the motion of bodies having mass."
  - Book: Ch. 7, 8.

Remark: Sometimes "dynamics" is used for "kinetics" only

Simplest scalar case

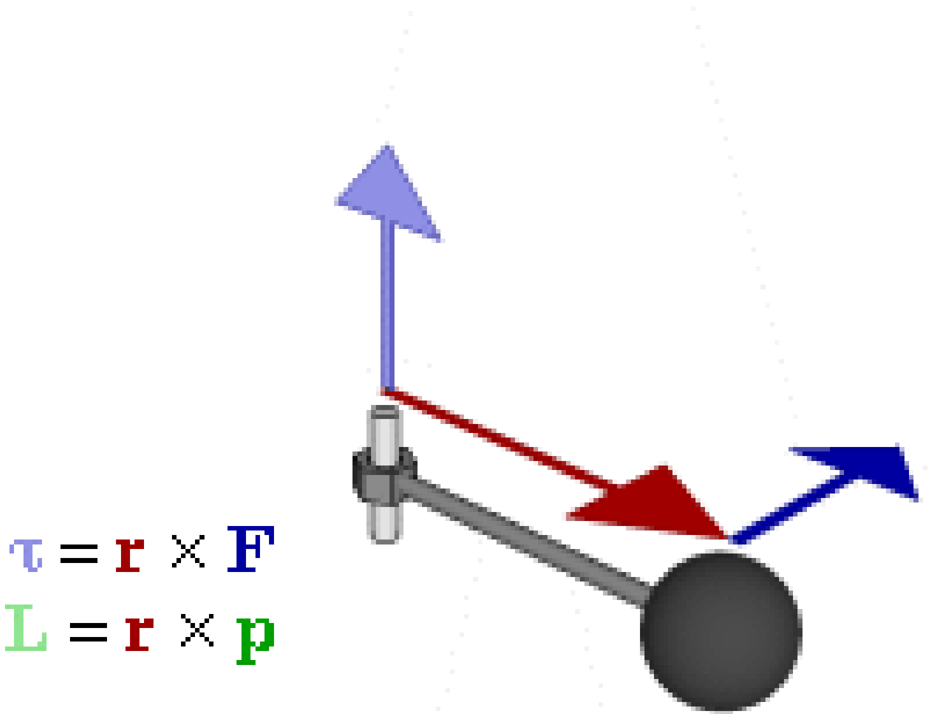




Newton-Euler equation of motion I

Newton-Euler equation of motion II

Torque, and linear/angular momentum



Source: Wikipedia

- Book:
  - Torque: $\vec{N}, \vec{T}$
  - Angular momentum: $\vec{h}$

Euler's 2nd law of motion I

Euler’s 2nd law of motion II

Euler’s 2nd law of motion III

Newton-Euler for center of mass

- That shows:
  - Newton's laws can be formulated for the center of mass

$$\frac{{}^i d}{dt}\vec{p}_c = \vec{F}_{bc}$$

$$\frac{{}^i d}{dt}\vec{h}_{b/c} = \vec{T}_{bc}$$

Angular momentum

$$\vec{h}_{b/c} = \int_b \vec{r} \times \vec{v}_p dm$$

Euler’s 2nd law of motion about CoM

EoM with reference of CoM

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)$$

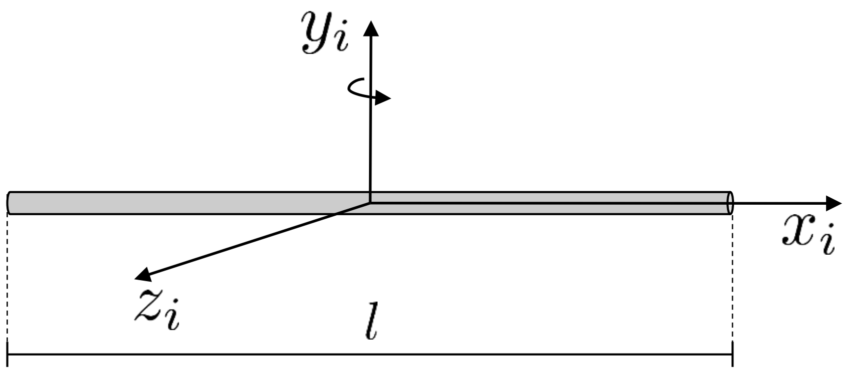
Inertia dyadic I

$$\vec{M}_{b/c} = -\int_b \vec{r}^{\times} \cdot \vec{r}^{\times}$$

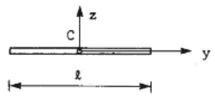
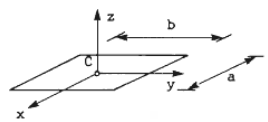
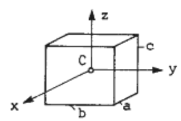
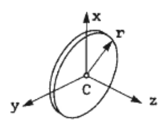
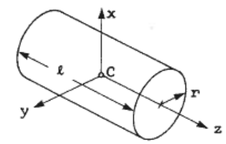
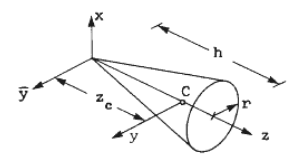
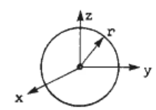
Inertia dyadic II

$$\vec{M}_{b/c} = -\int_b \vec{r}^{\times} \cdot \vec{r}^{\times}$$

Example: Slender beam



Finding moments of inertia

Homogen slank stav 	$I_z = \frac{1}{12} m l^2$ $I_{\bar z} = \frac{1}{3} m l^2$
Tynn rektangulær plate 	$I_z = \frac{1}{12} m\,(a^2 + b^2)$ $I_x = \frac{1}{12} m b^2$ $I_y = \frac{1}{12} m a^2$
Rektangulært prisme 	$I_z = \frac{1}{12} m\,(a^2 + b^2)$
Tynn sirkulær skive 	$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{4} m r^2$
Sirkulær sylinder 	$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{12} m (3r^2 + l^2)$
Tynt sylinderskall	$I_z = m r^2$ $I_x = I_y = \frac{1}{2} m r^2 + \frac{1}{12} m l^2$
Rett sirkulær kjegle 	$I_z = \frac{1}{10} m r^2$ $I_y = \frac{3}{20} m r^2 + \frac{3}{80} m h^2$ $I_{\bar y} = \frac{3}{20} m r^2 + \frac{3}{5} m h^2$ $z_c = 3h/4$
Kule 	$I_C = \frac{2}{5} m r^2$
Kuleskall	$I_C = \frac{2}{3} m r^2$

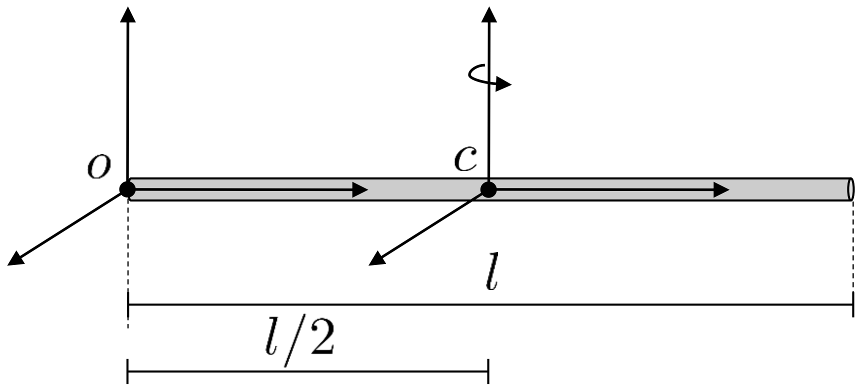
From F. Irgens, Dynamikk

- http://en.wikipedia.org/wiki/List_of_moment_of_inertia_tensors
- For other/general rigid bodies (vessels/planes/etc.), computer programs can find moments of inertia

Parallel axis theorem

$$\vec{M}_{b/o} = \vec{M}_{b/c} - m(\underline{r}_g^b)^\times(\underline{r}_g^b)^\times$$
$$= \vec{M}_{b/c} + m\left[(\underline{r}_g^b)^T\underline{r}_g^b\mathbf{I} - \underline{r}_g^b(\underline{r}_g^b)^T\right]$$

Example:



Kahoot

- https://play.kahoot.it/#/k/4152faff-75ee-49ea-bb9e-b4c79dd85785