TTT4275 Summary Detection
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Lecturer: Magne Hallstein Johnsen,

IES, NTNU

Intro to detection - 1

- Detection of (**rare**) events $s(n)$ based on noisy observations $x(n)$ $n = 0, \ldots, N-1$

- Binary hypothesis

$$
\begin{aligned}
H_1 &: x(n) = s(n) + w(n) \\
H_0 &: x(n) = w(n)
\end{aligned}
$$

- Some practical issues not included in this course :
  - Multiple (more than two) hypotheses
  - How to estimate time window; i.e. $n = 0$ and $n = N-1$
  - How often to detect?

Intro to detection - 2

- Assuming random $w(n)$, i.e. a density $p(w)$ the hypotheses must also have distributions $p(x, H_1)$ and $p(x, H_0)$. Thus we will focus on **statistically** based detection

- Often the densities have a parametric form, $p(x, H_i) = p(x, \theta_i)$, $i = 0, 1$

- Then we can recast the detection to a socalled *simple test* ·

$$\begin{aligned} H_1 &: \quad \theta = \theta_1 \\ H_0 &: \quad \theta = \theta_0 \end{aligned}$$

- In general we have $p(x, \theta_i) = p(x/\theta_i)p(\theta_i)$ $i = 0, 1$

- In most cases θ_i is unknown but deterministic, thus we can drop the priors $p(\theta_i)$

Intro to detection - 3

- In detection we have two true regions H_1, H_0 and two decision regions Ω_1, Ω_0

- Thus we have four different detection outcomes
  - Correct detection : $P_D = P(x \in \Omega_1 / H_1)$
  - Correct rejection : $P_D = P(x \in \Omega_0 / H_0)$
  - Missed detection : $P_M = P(x \in \Omega_0 / H_1)$
  - False accept/alarm : $P_{FA} = P(x \in \Omega_1 / H_0)$

- The two types of errors do mutually conflict; decreasing one increases the other

- Performance is usually given as a function of P_{FA} with y-axis being
  - P_D (Receiver Operating Characteristics - ROC)
  - P_M (Detection Error Tradeoff - DET) where equal error rate (EER) is shown

The Likelihood Ratio Test (LRT) - 1

- Given hypothesis distributions $p(x, H_i) = p(x, \theta_i)$ $i = 0, 1$ the socalled **"most powerful"** test is the LRT

$$L(x) = \frac{p(x/H_1)}{p(x/H_0)} \lessgtr \lambda \tag{1}$$

- The ratio (i.e distributions) is problem dependent while the threshold λ is mainly dependent of the choice of detection method.

- The simplest method is the maximum likelihood test, i.e. $\lambda = 1$

- The most general method is called the Bayes risk where

$$\lambda = \frac{P_0 C_{10}}{P_1 C_{01}} \tag{2}$$

  where P_i $i = 0, 1$ is the hypothesis priors while C_{01} and C_{10} are respectively the costs of miss and false alarm. These two costs are usually chosen by the developer.

The Likelihood Ratio Test (LRT) - 2

- In for instance medical applications the misses usually are more costly than false alarms.
- In other applications the two types of errors are equally costly, i.e $C_{01} = C_{10}$. Using the Bayes law $p(x/H_i)P_i = P(H_i/x)p(x)$ we then can reformulate the LRT to

$$L(x) = \frac{P(H_1/x)}{P(H_0/x)} \lessgtr 1 \tag{3}$$

- This is called the Maximum A Posteriori (MAP) detector and corresponds to the detector with minimum number of errors.
- In some applications there is a required maximum value on one of the error types.
- The Neyman-Pearson (NP) detector assumes a fixed P_{FA}. From this the threshold and thus the P_M is found.
- Instead we can start with a fixed P_M and derive the threshold and P_{FA}

Different detection cases

- The LR treshold λ is mostly dependent on choice of detection method

- The LR is dependent on the problem case

- In this course we will assume gaussian noise $p(w) = N(0, \sigma^2)$, i.e.

$$H_0 \; : \; x(n) = w(n) \quad n = 0, \ldots, N-1$$

- We will investigate the following cases for H_1 :

  - Constant in noise $x(n) = A + w(n)$

  - Random signal in noise $x(n) = s(n) + w(n)$ where $p(s) = N/A, \sigma_s^2)$

  - Deterministic sequence in noise $x(n) = s(n) + w(n)$

Detection of constant in gaussian noise

- Defining the log lilkelihood ratio test

$$LL(\mathbf{x}) = log[p(\mathbf{x}/H_1) - log[p(\mathbf{x}/H_0) \lessgtr log(\lambda)$$

- Using the independence assumption $p(\mathbf{x}) = \prod_{n=0}^{N-1} p(x(n))$ we get

$$\begin{aligned} LL(\mathbf{x})) &= \frac{NA}{\sigma^2}z - \frac{NA^2}{2\sigma^2} \lessgtr log(\lambda) \Rightarrow \\ z &\lessgtr \frac{A\sigma^2}{N}log(\lambda) + \frac{A}{2} = \eta \end{aligned} \quad (4)$$

where $z = T(x) = \frac{1}{N}\sum x(n)$ is the sample mean

- Note that eq. 4 is an equivalent test for the LRT. In general the term $z = T(x)$ is called a sufficient statistic

- The false alarm is then given by

$$P_{FA} = \int_{\eta}^{\infty} p(z/H_0)dz = \int_{\eta}^{\infty} N(0, \sigma^2/N)dz = Q(\frac{\eta\sqrt{N}}{\sigma}) \quad (5)$$

Detecting a random variable

- Now we have $H_1 \;:\; x(n) = s(n) + w(n) \;\; n = 0, \ldots, N-1$

  where s is a random variable with density $p(s) = N(A, \sigma_s^2)$

- This leads to the distributions $p(x/H_0) = N(0, \sigma^2)$ and

  $p(x/H_1) = N(A, \sigma_x^2)$ where $\sigma_x^2 = \sigma_s^2 + \sigma^2$

- Deriving the test for the sufficient statistics we get

$$z = T(\mathbf{x}) = \sigma_s^2 \bar{x}_{sp} + 2A\sigma^2 \bar{x}_{sm} \lessgtr \sigma^2 A^2 + \sigma_s^2 \sigma_x^2 [log(\frac{\sigma_x^2}{\sigma^2}) + \frac{2}{N} log(\lambda)] \quad (6)$$

  where $\bar{x}_{sp} = \sum_n x^2(n)/N$ (power estimate) and $\bar{x}_{sm} = \sum_n x(n)/N$ (sample mean)

- z does not have a simple density, thus P_{FA} and P_M are not easily derived

- For the case $A = 0$ we have a power/energy detector; i.e. $z = \bar{x}_{sp}$

Detecting a deterministic sequence

- The hypothesis densities are $p(x(n)/H_1) = N(s(n), \sigma^2)$ and $p(x(n)/H_0) = N(0, \sigma^2)$

- Deriving $LLRT(\mathbf{x})$ we end up with

$$z = T(\mathbf{x}) = \sum_n x(n)s(n) \lesseqgtr 2\sigma^2 log(\lambda) + E_s = \eta \tag{7}$$

where $E_s = \sum_n s^2(n)$

- This detector is called a correlator and/or a matched filter

- We showed that $p(z/H_0) = N(0, \sigma^2 E_s)$ and $p(z/H_1) = N(E_s, \sigma^2 E_s)$

- Thus the false alarm is given by

$$P_{FA} = \int_\eta^\infty p(z/H_0)dz = \int_\eta^\infty N(0, \sigma^2 E_s)dz = Q(\frac{\eta}{\sqrt{E_s}\sigma}) \tag{8}$$

Generalized LLRT

- The value of the constant A in the H_1 hypothesis is usually not known

- We measure $\mathbf{x} = [x(n),\ n = 0, \ldots, N-1]$ but we do not know the mean A of $p(x/H_1) = N(A, \sigma^2)$

- What about using the estimate $\hat{A} = \sum_n x(n)/N$?

- Problem is that we do not know if we have the case H_1 (estimate is good) or H_0 (estimate is wrong)

- The estimator gives $H_1 : \hat{A} = A + q(n)$ or $H_0 : \hat{A} = q(n)$ where $p(q) = N(0, \sigma^2/N)$

- If we know the sign of A we can can set up a treshold η based on $P(x/H_0) = P(q/H_0) = \eta << 1$.

- Another option is to use the absolute value $|x(n)|$, however $p(|x|)$ is not Gaussian.