Norwegian University of
Science and Technology
Department of Electronic Systems

TTT4175 Estimation, Detection and Classification
Assignment no. 1 : MVU and CRLB

Problem 1 MVU estimators

1a) We want to measure the distance between ourselves and a distant object using radar reflections. We transmit a radar pulse at time $t = 0$ and receives an echo at some time $t = T + w$, where w is a measurement error having a Gaussian distribution with zero mean and variance σ^2. The speed of light is $c = 3 \cdot 10^8$ m/s.

Assuming that you have a single measurement – what should σ^2 be if we want to be 99% sure that the distance error is less than one meter?

Now assume that $\sigma = 10^{-8}$. How many observations do you now need to be 99% sure that the measurement error is less than one meter?

1b) We have N iid. observations, $[x[0], x[1], \ldots, x[n-1]]$ from a Gaussian distribution,

$$x \sim \mathcal{N}(0, \sigma^2), \tag{1}$$

where σ^2 is unknown. We want to use the estimator

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]. \tag{2}$$

Is this estimator unbiased? What is the variance of the estimator and what happens when $N \to \infty$? (Hint: $E\{x^4\} = 3\sigma^4$)

1c) We have a constant signal A embedded in iid. noise w,

$$x = A + w, \tag{3}$$

where $w \sim \mathcal{N}(0, \sigma^2)$. We want to estimate the power, $\theta = A^2$, of the constant signal, and want to use the following estimator,

$$\hat{\theta} = \left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \hat{A}^2. \tag{4}$$

This estimator is as we see the square of the sample mean, which we know is an MVU estimator. Check if the estimator $\hat{\theta}$ is biased or not. What happens when the number of observations $N \to \infty$?

Problem 2 The Cramer-Rao Lower Bound

2a) You are analysing a wireless communications protocol where the information packets arrive as in a Poisson process at times $[t_0, t_1, \ldots]$. You are interested in the distribution of the waiting times $\delta_i = t_i - t_{i-1}$ between packets, which you know is distributed as an exponential distribution with parameter β,

$$p(\delta; \beta) = \frac{1}{\beta} e^{-\delta/\beta} \tag{5}$$

i. Given N observations of waiting times, compute the CRLB.

ii. We suggest using the estimator $\hat{\beta} = \bar{\delta}$, where $\bar{\delta}$ is the sample mean. What is the variance of $\hat{\beta}$, and is it efficient?

An alternative parameterization of the exponential distribution is

$$p(\delta; \lambda) = \lambda e^{-\lambda\delta}. \tag{6}$$

Compute the CRLB for an estimator $\hat{\lambda}$. Does an efficient estimator exist in this case?

2b) In wireless communications one often models random channels as having a *complex Gaussian distribution*, $H = h_R + j h_Q$, where

$$h_R \sim \mathcal{N}(0, \sigma^2)$$
$$h_Q \sim \mathcal{N}(0, \sigma^2)$$

We are often just interested in the absolute value of the channel, the channel gain, $R = |H| = \sqrt{h_R^2 + h_Q^2}$, which turns out to have a Rayleigh distribution with parameter σ,

$$p(r; \sigma^2) = \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)} \tag{7}$$

Given N observations $[r[0], r[1], \ldots, r[N-1]]$, find the CRLB and the efficient estimator for the parameter $\alpha = \sigma^2$ if it exists.

2c) Computer assigment: In many practical applications, our measurements will be corrupted by "shot noise". This noise occurs rarely, but has very large magnitudes when it do. Assume that we have a model for a DC component in additive noise

$$x = A + w, \tag{8}$$

where the noise no longer is Gaussian, but is instead distributed as

$$w \sim (1 - \varepsilon)\mathcal{N}(0, \sigma^2) + \varepsilon\mathcal{S}(\gamma). \tag{9}$$

This means that w is a sampled from a Gaussian distribution with probability $1 - \varepsilon$, but it can also be a sample from another "shot noise"-distribution with probability ε.

If we use the sample mean in this case, the estimate is likeliy to be very biased. Instead we want to use the *median* estimator, where the median of a set of obervations is the middle observation in the sorted set.

Write a small simulation using your favorite programming language, and compute the variance of the median as an estimator of A (ignore the shot noise for now). Let $A = 1$ in your simulations, and let σ^2 range from 1 through 10^{-3}. Assume that you are using $N = 1000$ observations in your estimates. Plot the variance of the median estimator together with the CRLB for the sample mean estimator. Is the result what you expected?

Now set $\varepsilon = 10^{-2}$ and assume that the shot noise is generated by taking the absolute value of Gaussian noise with variance $\sigma_s^2 = 20$. Use simulations to compare the variance of the sample mean- and median estimator for a range of σ^2.