Out: January 14, 2019, 8:00
Deadline: January 27, 2019, 20:00

NTNU – Trondheim
Norwegian University of Science and Technology

Assignment 1
TTK4130 Modeling and Simulation

About this assignment

The objective of this assignment is to give an introduction to the Modelica language, as well as to recapitulate some results and techniques that have been learned in previous courses, and that are useful -if not necessary- for solving exercises in this course.

Problem 1 handles about solving a simple differential equation with Modelica, and will serve as an introduction to this modeling language. We recommend to use the program Dymola for this exercise. However, any modeling program based on Modelica can be used, as for example openModelica.

Problems 2, 3 and 4 constitute the recapitulation part of this assignment, where the keywords are: equilibrium points, linearization, stability of linear systems, eigenvalues and eigenvectors, Jordan canonical form and linear ODEs.

Problem 1 (The Modelica language, simulation. 15 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

The differential equation

$$\dot{x} = -3x + 17\,, \quad x(0) = -2\,,$$

can be represented in Modelica by the following model:

```
model FirstOrder  "A linear 1. order diff. eq."
  // Parameters and variables
  parameter Real a = -3  "Growth rate";
  parameter Real b = 17  "Steady-state value";
  Real x(start= -2)  "State";
equation
  // The differential equation
  der(x) = a*x + b  "1. order diff. eq.";
end FirstOrder;
```

(a) What are the keywords *model*, *equation*, *end*, *parameter*, *Real*, *start*, *der* in Modelica? What are they used for?

(b) What are the quoted texts and the texts behind two slashes called? What are they used for?

(c) Implement the Modelica code above and simulate it as it is. Add a plot with the obtained values for x to your answer.

NB: The .mo file for this model has been uploaded together with this file.

(d) Would you obtain the same results if you replaced the keyword *parameter* with *constant* in the code above? What is the difference between these keywords?

(e) Change the simulation time, number of intervals and solver used in the simulation, as well as the parameters and initial condition of the model to the values of your choosing. Resimulate without modifying the model code.

Explain which changes you made, and add a plot with the obtained values for x.

(f) Change the growth rate parameter to a positive value. Simulate the system with an explicit (e.g. ”euler”) and an implicit solver (e.g. ”dassl”). All other simulation and model parameter values should be the same for both simulations, and they should verify the following conditions:

- There are between 20 and 100 simulation intervals per second.
- The product between the growth rate and the simulation time lies between 5 and 20.

Explain which changes you made, and add a plot with the values for x obtained from each solver.

(g) **(Optional)** Implement and simulate a Modelica model that solves the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

for the parameter values and initial conditions of your choosing.

Add your Modelica model and a plot with the obtained results for x and $\dot{x}$ to your answer. Explain the simulation setup you used.

Problem 2 (Equilibrium points, linearization, stability. 30 %)

Consider the systems

1.
$$\dot{x} = \begin{cases} -x - \frac{y}{\ln\sqrt{x^2+y^2}} & ,[x,y] \neq [0,0] \\ 0 & ,[x,y] = [0,0] \end{cases} \qquad \dot{y} = \begin{cases} -y + \frac{x}{\ln\sqrt{x^2+y^2}} & ,[x,y] \neq [0,0] \\ 0 & ,[x,y] = [0,0] \end{cases}$$

NB: The vector field is continuously differentiable.

2.
$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad \dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

3.
$$\dot{x} = \left(\frac{y}{1+2y+y^2} - d\right)x \qquad \dot{y} = d(4-y) - \frac{2.5xy}{1+2y+y^2},$$

where $a, b, d > 0$ are model parameters. For each system, do the following exercises:

(a) Find all equilibrium points of the system, and parameterize them as a function of the model parameters, if any.

(b) Linearize the system around each of its equilibrium points. Determine whether the linearized systems are stable, asymptotically stable or unstable.

(c) Based on the results from the previous part, what can be concluded about the stability of the equilibrium points of the original non-linear system?

Problem 3 (Eigenvectors, Jordan canonical form, linear ODEs. 35 %)

Consider the matrices

1. $\begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & -8 & -2 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 1 & 2 & 2 \\ 1 & 1 & -1 & -1 \\ -2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 5 \end{bmatrix}$

(a) For each matrix, find all complex eigenvalues and eigenvectors.

(b) **(Optional)** For at least two of the matrices, find the real Jordan canonical form, J, and an associated similarity transformation matrix, i.e., a matrix P such that $P^{-1}AP = J$, where A is the considered matrix.

NB: Notes on the Jordan canonical form have been uploaded together with this file.

Consider now the system

$$\begin{aligned}
\dot{x} &= x + 3z + 4y \\
\dot{y} &= -4y - 3z - x \\
\dot{z} &= -3z - 2y + x + 32u\,,
\end{aligned}$$

where $u(t)$ is a known function.

(c) Assume $u \equiv 0$. Determine whether the linear system is stable, asymptotically stable or unstable.

(d) **(Optional)** Find $[x(t), y(t), z(t)]^T$, $t \geq 0$ for $[x(0), y(0), z(0)]^T = [1, 2, -4]^T$ and $u(t) = te^{-2t}$.

Problem 4 (Modeling, linearization. 20 %)

An iron ball of radius R and mass m is lifted by a magnet with a coil of N turns and a current i around a core of length l_c and cross section $A = \pi R^2$. The vertical position of the ball is z, which is positive in the downwards direction. The flux ϕ flows through the iron core, then over the air gap, through the ball, and finally along the return path through the open air as shown in Figure 1. The magnetomotive force on the ball is Ni, which can be expressed as

$$Ni = \phi\left(\mathcal{R}_a + \mathcal{R}_c + \mathcal{R}_b + \mathcal{R}_r\right) \tag{1}$$

where

$$\mathcal{R}_a = \frac{z}{A\mu_0} \tag{2}$$

is the reluctance of the air gap, and $\mathcal{R}_c$, $\mathcal{R}_b$ and $\mathcal{R}_r$ are the reluctances of the core, ball and return path, respectively. The reluctances $\mathcal{R}_c$ and $\mathcal{R}_b$ are negligible, and $\mathcal{R}_r$ may be assumed to be constant as the total return path will not change significantly as the ball moves.

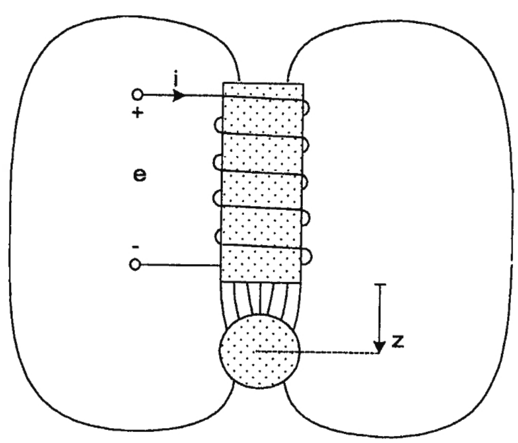

Figure 1: Magnetic levitation experiment

(a) Let the length of the return path be denoted z_0 (assumed constant), and assume a relationship such as (2) for $\mathcal{R}_r$. Write up the total magnetomotive force Ni.

Based on the above, we can calculate the inductance

$$L(z) = \frac{N\phi}{i} = \frac{N^2 A\mu_0}{z + z_0}. \tag{5}$$

From this, the magnetic force on the ball can be found from

$$F = \frac{i^2}{2} \frac{\partial L(z)}{\partial z}. \tag{6}$$

(b) Use Newton's second law to find the equation of motion for the ball.

(c) Linearize about a constant position z_d (and a corresponding constant current input, i_d).