Final Exam 2018
Estimation theory
Solutions

1a) We have

$$x[n] = A\cos(2\pi f n) + \omega[n]$$

where

$$\omega[n] \sim \mathcal{N}(0, \sigma^2)$$

Then

$$x[n] - A\cos(2\pi f n) \sim \mathcal{N}(0, \sigma^2)$$

$$\Rightarrow \quad p(x[n]) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x[n] - A\cos(2\pi f n))^2}$$

1b) If we can do the factorization

$$\frac{d}{dA} \log p(\mathbf{x}; A) = I(A)\left(g(x) - A\right)$$

then $g(x)$ is an efficient estimator of A (and hence MVU), and

$$\operatorname{var}(\hat{A}) \geq \left(I(A)\right)^{-1}$$

is the CRLB

$$\frac{d}{dA} \log p(\mathbf{x}; A) =$$

$$= \frac{d}{dA} \log \prod_{n=0}^{N-1} p(x[n]; A)$$

$$= \frac{d}{dA} \sum_{n=0}^{N-1} \log P(x[n]; A)$$

$$= \frac{d}{dA} \sum_{n=0}^{N-1} -\frac{1}{2} \log 2\pi\sigma^2 - \frac{1}{2\sigma^2} \left(x[n] - A\cos(2\pi f n)\right)^2$$

$$= \sum_{n=0}^{N-1} \sigma^{-2} \left(x[n] - A\cos(2\pi f n)\right) \cos(2\pi f n)$$

$$= \sigma^{-2} \sum_{n=0}^{N-1} x[n] \cos(2\pi f n) - A \sum_{n=0}^{N-1} \cos^2(2\pi f n)$$

$$= \sigma^{-2} \sum_{n=0}^{N-1} \cos^2(2\pi f n) \left(\frac{\sum_{n=0}^{N-1} x[n] \cos(2\pi f n)}{\sum_{n=0}^{N-1} \cos^2(2\pi f n)} - A \right)$$

$$= I(A) \left(g(x) - A \right)$$

with

$$\hat{A} = g(x) = \frac{\sum_{n=0}^{N-1} x[n] \cos(2\pi f n)}{\sum_{n=0}^{N-1} \cos^2(2\pi f n)}$$

and

$$I(A) = \frac{\sum_{n=0}^{N-1} \cos^2(2\pi f n)}{\sigma^2}$$

$$\Rightarrow \text{CRLB}: \operatorname{var}(\hat{A}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \cos^2(2\pi f n)}$$

1c) Amplified signal:

$$y[n] = GA\cos(2\pi f n) + w[n]$$

$$\approx GA\cos(2\pi f n) + Gu[n] + v[n]$$

Here $w[n]$ is the "true noise" which is non-Gaussian due to the distortion of the non-linear amplifier. Setting

$$w[n] \approx Gu[n] + v[n]$$

is a good approximation if the distortion is very small. However, since $w[n]$ is non-Gaussian, we do <u>not</u> have a linear model, and so we use BLUE.

1d) The BLUE for a scalar is

$$\hat{A} = \frac{s^T C^{-1} x}{s^T C^{-1} s}$$

where s is a vector defined by

$$E\{x[n]\} = s_n A$$

and

$$C = E\{(x - E[x])(x - E[x])\}$$

In our case we have

$$E[x[n]] = E[GA\cos(2\pi n f) + w[n]]$$

$$= G\cos(2\pi n f) \cdot A$$

$$\Rightarrow s_n = G\cos(2\pi f n)$$

$$C = E\{w \cdot w^T\}$$

$$\approx E\{(Gu + v)(Gu + v)^T\}$$

$$= (G^2 \sigma_u^2 + \sigma_v^2) \cdot I$$

Thus yields

$$\hat{A} = \frac{s^T C^{-1} x}{s^T C^{-1} s}$$

$$= \frac{s^T x}{s^T s}$$

$$= \frac{\sum_{n=0}^{N-1} \cos(2\pi f n) \cdot x[n]}{G \sum_{n=0}^{N-1} \cos^2(2\pi f n)}$$

The variance of the BLUE is

$$\mathrm{Var}(\hat{A}) \geq (s^T C^{-1} s)^{-1}$$

$$= \left((G^2 \sigma_u^2 + \sigma_v^2)^{-1} \sum_{n=0}^{N-1} G^2 \cos^2(2\pi f n) \right)^{-1}$$

$$= \frac{G^2 \sigma_u^2 + \sigma_v^2}{G^2 \sum \cos^2(2\pi f n)}$$

1e) The Bayes MSE estimate is

$$\hat{A}_{Bmse} = \underset{A'}{\operatorname{argmin}} \; E_{x,A} \{ (A - A')^2 \}$$

$$= \underset{A'}{\operatorname{argmin}} \int (A - A')^2 \, P(A, x) \, dA \, dx$$

$$= \int A \cdot P(A \mid x) \, dA$$

To find this estimate we must first find

$$P(A \mid x) = \frac{P(x \mid A) \, P(A)}{P(x)}$$

and then solve the integral, possibly using numerical methods