Out: February 4, 2019, 8:00
Deadline: February 17, 2019, 20:00

NTNU – Trondheim
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Assignment 4
TTK4130 Modeling and Simulation

Problem 1 (Use of Matlab/Simulink's ODE solvers. 14 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

Download the file orbit.mdl, which has been uploaded together with this file on Blackboard. Experiment with it in Matlab/Simulink. This file simulates the restricted three-body problem from Ch. 14.1.3 in the book. The parameters in the file are for *Orbit 1* in Table 14.1.

(a) Go to *Simulation* $\rightarrow$ *Model Configuration Parameters* $\rightarrow$ *Variable-step*, and choose the solver ode45 or confirm that it is the default solver. Find what relative tolerances are approximately needed to simulate one, two and three rounds, respectively.

NB: The default stop time in orbit.mdl is for one round.

(b) Select the five-stage explicit Runge-Kutta method (ode5) by going to *Simulation* $\rightarrow$ *Model Configuration Parameters* $\rightarrow$ *Fixed-step*. Find the step length that gives an accurate solution for three rounds.

(c) Explain why the fixed-step method ode5 manages to give a correct simulation for several rounds, while the variable-step method ode45 struggles to do so.

NB: Ch.14.1.3. How does the vector field for the satellite state change when the satellite comes close to the Earth or the Moon? What does this mean for the eigenvalues of the system?

Problem 2 (Taylor expansions, order conditions. 30 %)

The Butcher array for an explicit Runge-Kutta method with two stages is

$$\begin{array}{c|cc} 0 & & \\ c_2 & a_{21} & \\ \hline & b_1 & b_2 \end{array}$$

Hence, the stage computations are

$$\begin{aligned} \mathbf{k_1} &= \mathbf{f}(\mathbf{y}_n, t_n)\,, \\ \mathbf{k_2} &= \mathbf{f}(\mathbf{y}_n + ha_{21}\mathbf{k}_1, t_n + hc_2)\,, \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h(b_1\mathbf{k}_1 + b_2\mathbf{k}_2). \end{aligned}$$

(a) Derive the conditions on b_1, b_2, c_2 and a_{21} for the method to be an explicit Runge-Kutta method of order 2. Moreover, express b_1, b_2, c_2 and a_{21} as a function of one parameter. Remember to state the range of values this parameter can take.

Hint 1: Recall the 1. order Taylor expansion of a function:

$$\mathbf{f}(\mathbf{y} + \Delta, t + \delta) = \mathbf{f}(\mathbf{y}, t) + \frac{\partial \mathbf{f}(\mathbf{y}, t)}{\partial \mathbf{y}}\Delta + \frac{\partial \mathbf{f}(\mathbf{y}, t)}{\partial t}\delta + O(|\Delta|^2) + O(\delta|\Delta|) + O(\delta^2).$$

Hint 2:

$$\frac{\mathbf{f}(\mathbf{y}_n, t_n)}{d\,t} = \frac{\partial \mathbf{f}(\mathbf{y}_n, t_n)}{\partial \mathbf{y}}\frac{d\,\mathbf{y}}{d\,t} + \frac{\partial \mathbf{f}(\mathbf{y}_n, t_n)}{\partial t} = \frac{\partial \mathbf{f}(\mathbf{y}_n, t_n)}{\partial \mathbf{y}}\mathbf{f}(\mathbf{y}_n, t_n) + \frac{\partial \mathbf{f}(\mathbf{y}_n, t_n)}{\partial t}.$$

(b) Find the stability function for this family of Runge-Kutta methods.

What happened to the parameter defined in part (a)? Comment on the results.

Problem 3 (Solver implementation, pneumatic spring. 20 %)

NB: This is a computer exercise, and can therefore be solved in groups of 2 students. If you do so, please write down the name of your group partner in your answer.

Consider the pneumatic spring without damping

$$\ddot{x} + g\left(1 - \left(\frac{x_d}{x}\right)^{\kappa}\right) = 0\,, \tag{1}$$

where $x_d = 1.32$, $\kappa = 2.40$ and $g = 9.81$. Since there is no damping, the physical solution will oscillate around its equilibrium position $x = x_d$.

By defining $y_1 = x$ and $y_2 = \dot{x}$, this system can be written in state-space form as

$$\dot{y}_1 = y_2, \tag{2a}$$

$$\dot{y}_2 = -g\left(1 - \left(\frac{x_d}{y_1}\right)^{\kappa}\right). \tag{2b}$$

(a) Implement the explicit Euler's method or any other explicit two-stage Runge-Kutta method in Matlab. Simulate the pneumatic spring without damping (2) from $t = 0\,\mathrm{s}$ to $t = 10\,\mathrm{s}$, with step length $h = 0.01\,\mathrm{s}$, and initial conditions $y_0 = [2, 0]^T$. Add your Matlab script to your answer, as well as a plot of the position x. Comment on the results.

(b) Implement the implicit Euler's method in Matlab. Simulate (2) with the same parameters and initial conditions as in part (a). Add your Matlab script to your answer, as well as a plot of the position x. Comment on the results.

Hint: In order to solve the nonlinear equation that arises at each iteration, consider using fsolve *from the optimization toolbox, or implement a Newton-type algorithm yourself.*

For example: Define the model

```
f = @(y,t) [ y(2); -g*(1-(x_d/y(1))^K) ];
```

Then, for each iteration, define the equation

$$r(y_{n+1}) = y_n + hf(y_{n+1}, t_{n+1}) - y_{n+1} = 0\,, \tag{3}$$

and solve it by calling fsolve*.*

```
r = @(ynext) (y(:,i) + h*feval(f, ynext, time(i+1)) - ynext);
y(:,i+1) = fsolve(r, y(:,i), opt);
```

In order to obtain accurate results, it is important to set small tolerances for the equation solutions:

```
opt = optimset('Display','off','TolFun',1e-8); % Options for fsolve
```

NB: Using fsolve *for this particular application is not very efficient. In order to speed up the calculations, you may consider to provide the Jacobian of r to* fsolve*.*

(c) Implement the implicit midpoint rule (Gauss method of order 2) in Matlab:

$$y_{n+1} = y_n + hf((y_n + y_{n+1})/2, t_n + h/2). \tag{4}$$

Simulate (2) with the same parameters and initial conditions as in part (a). Add your Matlab script to your answer, as well as a plot of the position x. Comment on the results.

(d) The energy for the system (2) is given by

$$E = \frac{mg}{\kappa - 1}\frac{x_d^{\kappa}}{x^{\kappa-1}} + mgx + \frac{1}{2}m\dot{x}^2 \tag{5}$$

Show that the energy for the actual solution of (2) is constant.

Furthermore, plot the energy for the numerical solutions found in parts (a)-(c). Assume that $m = 200\,\text{kg}$. Comment on the results.

Problem 4 (Differential Algebraic Equations (DAEs), index. 36 %)

Consider the DAEs:

1.

$$
\begin{aligned}
\dot{z}_2(t) &= -z_3(t) + 4z_4(t)^3 && \text{(6a)}\\
\dot{z}_3(t) &= -z_1(t) + 2z_2(t) && \text{(6b)}\\
z_1(t) &= z_4(t)^3 - z_2(t) + q_1(t) && \text{(6c)}\\
z_4(t) &= -z_1(t) + z_3(t) - q_2(t)\,, && \text{(6d)}
\end{aligned}
$$

where q_1 and q_2 are known and sufficiently smooth.

2.

$$
\begin{aligned}
\dot{z}_2(t) &= q_1(t) - z_1(t) && \text{(7a)}\\
\dot{z}_3(t) &= q_2(t) - (1+a)z_2(t) - at(q_1(t) - z_1(t)) && \text{(7b)}\\
q_3(t) &= atz_2(t) + z_3(t)\,, && \text{(7c)}
\end{aligned}
$$

where q_1, q_2 and q_3 are known and sufficiently smooth, and $a \in \mathbb{R}$.

3.

$$
\begin{aligned}
\dot{q}(t) &= v(t) - G^T\eta(t) && \text{(8a)}\\
M\dot{v}(t) &= Fq(t) - G^T\lambda(t) && \text{(8b)}\\
0 &= Gv(t) && \text{(8c)}\\
r(t) &= Gq(t)\,, && \text{(8d)}
\end{aligned}
$$

where q, v, η, λ and r are vectors and M, F and G are matrices of adequate size. Furthermore, M is positive definite and G has full row rank.

4.

$$
\begin{aligned}
m_1\ddot{x}_1(t) &= k(x_2(t) - x_1(t) - x_0) + F(t) && \text{(9a)}\\
m_2\ddot{x}_2(t) &= -k(x_2(t) - x_1(t) - x_0) && \text{(9b)}\\
x_2(t) &= r(t)\,, && \text{(9c)}
\end{aligned}
$$

which corresponds to two carts connected as shown in Figure 5. The force F is applied on the cart at x_1, so that the cart at x_2 follows a desired trajectory r, which is known and sufficiently smooth. Furthermore, assume that m_1, m_2 and k are positive numbers.

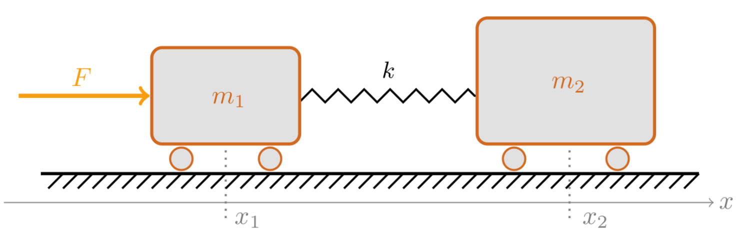

Figure 5: Two carts connected by a spring.

(a) For each DAE, find the differential and algebraic variables, as well as the parameters: Both functions and constants. Justify your answer.
Furthermore, rewrite the system of equations as

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{y}, \mathbf{u}) \tag{10a}$$
$$0 = \mathbf{g}(t, \mathbf{x}, \mathbf{y}, \mathbf{u}), \tag{10b}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{u}$ are the differential, algebraic and parameter variables, respectively.
Hint: For the 3. DAE, start by calculating the index.

(b) Find the index of each DAE.

Problem 5 (Stability functions, linear algebra. Optional)
Consider the following Runge-Kutta methods:

1. Heun's method, which has the Butcher array:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

2. Radau IA of order 3, which has the Butcher array:

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

3. The explicit Runge-Kutta of order 4, which has the Butcher array:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

4. The Lobatto IIIC of order 4, which has the Butcher array:

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

For each of these methods, find the stability function twice by using formulas 14.142 and 14.143 in the course book, respectively. Show the details of your work.

The solutions are:

1. Heun's method:
$$R(s) = 1 + s + \frac{s^2}{2} + \frac{s^3}{6}. \tag{11}$$

2. Radau IA of order 3:
$$R(s) = \frac{1 + \frac{1}{3}s}{1 - \frac{2}{3}s + \frac{1}{6}s^2}. \tag{12}$$

3. The explicit Runge-Kutta of order 4:
$$R(s) = 1 + s + \frac{s^2}{2} + \frac{s^3}{6} + \frac{s^4}{24}. \tag{13}$$

4. The Lobatto IIIC of order 4:

$$R(s) = \frac{1 + \frac{1}{4}s}{1 - \frac{3}{4}s + \frac{1}{4}s^2 - \frac{1}{24}s^3}. \tag{14}$$