Computational Methods Assignment 9 - Shooting Method

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1 Application

The differential equation was solved by creating a function that took the value of $\hat{y_2}(0)$ (guess) and returned the difference between the calculated value of $\hat{y_1}(10)$ and the given boundary condition $y_1(10) = 0$. Within this function, the system of ordinary differential equations was integrated using the Runge-Kutta 45 method from assignment 5 for the supplied guess $\hat{y_2}(0)$. This was then minimised using Brent's method from assignment 8, with suitable bracketing used to estimate the root of this function. The root was calculated as -6.68559 in 23 iterations of Brent's method from the starting bracket of $[-10, 0]$. This root was then inserted into the original differential equation as a boundary condition to solve using the Runge-Kutta 45 method. This required 154 steps and 2254 function evaluations, with the results for $(x, y_1(x))$ with $(x, y_2(x))$, and $(y_1(x), y_2(x))$ plotted in Fig 1 and Fig 2 respectively.

2 Conditions on the use of Brent's Method

Brent's method should always work for implementing the shooting method for systems of $N \leq 3$ differential equations, because the problem can always be adjusted so that we only need to guess one of the starting values. For $N = 1$ the case, it is trivial. For $N = 2$, it is either the case that there is a condition at either boundary, or that both conditions are at one boundary. For the former case, only one of the unknown boundary conditions can be guessed, and the procedure may start from either boundary. For the latter case, the shooting method need not be used and the solution can be found my integrating from either boundary to the other. For $N = 3$, excluding the case where the shooting method is not even needed (all three BC's at one boundary), then the distribution of boundary conditions at either side will always have one boundary with just a single unknown boundary condition. So the side with just one boundary condition can be selected and the shooting method can be implemented to obtain the two unknowns at the other boundary, using the comparision of the single known value on this other boundary to what was calculated from the ODE solver, to determine convergence. It is important that Brent's method is only supplied one guess to bracket and converge towards, and that there are not multiple-dimensional roots.

Brent's method will work for $N = 2$ and $N = 3$ as discussed above, for boundary to boundary shooting method, however will fail for shooting from/to and interior point. This is because shooting from/to an interior point required two guesses to be supplied, a guess for the ODE solving method to be used from the interior point to each of the two exterior boundaries, as it is essentially two parallel shooting problems. This means that Brent's method cannot be used in this case, as there are two free variables to try and vary to minimise the deviation of the calculated boundary values from their true values. Since two guesses are always required for every case of shooting to/from an interior point, then for any N, Brent's method will fail.

Sometimes in life, the sick and twisted hand of fate can inflict the most horrifying situation of having to implement the shooting method from/to inside an interior point which renders Brent's method a pathetic, powerless sack of pointers. Apart from impaling ones laptop on a nearby flagpole and crying oneself to sleep, a possible solution is to use a multi-dimensional root finding method such as the multi-dimensional form of the Newton-Raphson, which can take more than one values and minimise the function (which takes the guesses, solves the equations, and give the difference between calculation and true boundary values). Alternatively, there are several iterative relaxation methods which can be used to determine the interior values by iterating from an initial guess, given arbitrary combinations of boundary conditions.

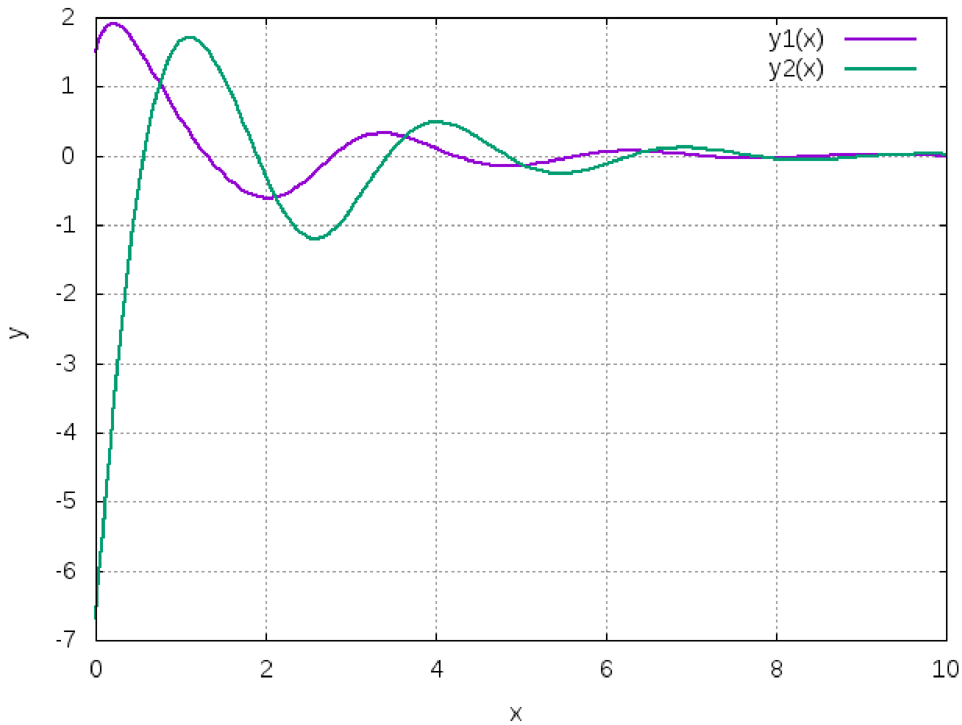

Figure 1: $(x, y_1(x))$ (Purple) and $(x, y_2(x))$ (Green), for the specified set of differential equations and boundary conditions solved using the shooting method.

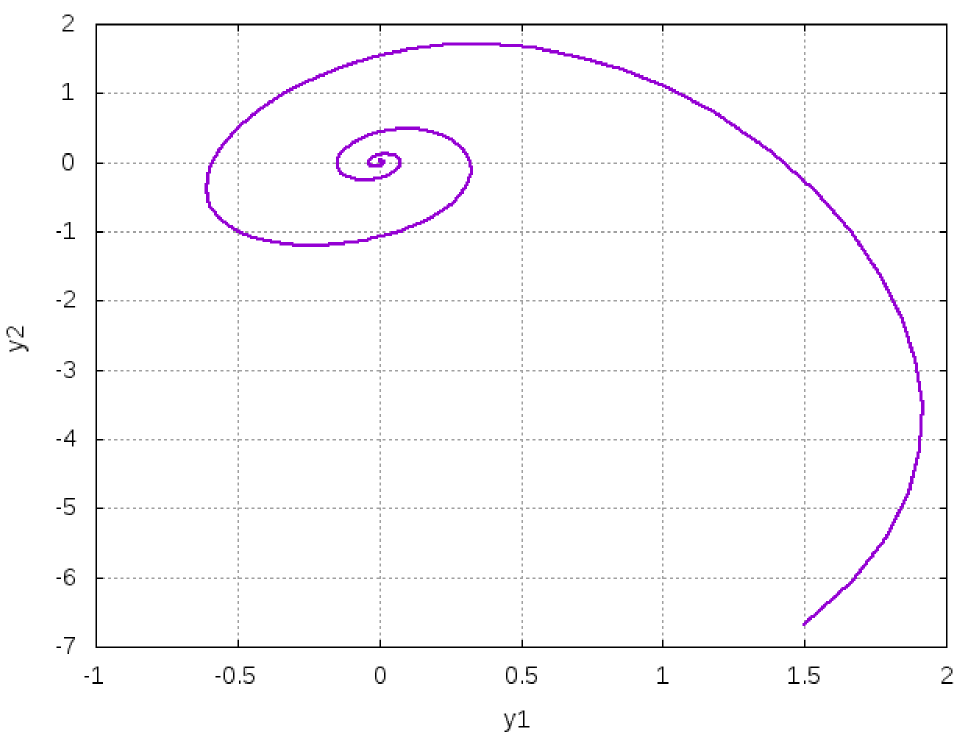

Figure 2: $(y_1(x), y_2(x))$ for the specified set of differential equations and boundary conditions solved using the shooting method.