1ª FREQUÊNCIA 2017

1\. a) $w = \alpha u + \beta v \Leftrightarrow$

$\Leftrightarrow (5,2,0) = \alpha(2,0,1) + \beta(1,2,-2)$

$\Leftrightarrow (5,2,0) = (2\alpha, 0, \alpha) + (\beta, 2\beta, -2\beta)$

$$\Leftrightarrow \begin{cases} 2\alpha + \beta = 5 \\ 2\beta = 2 \\ \alpha - 2\beta = 0 \end{cases} \Leftrightarrow \begin{cases} 2\alpha = 4 \\ \beta = 1 \\ \alpha - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 2 \\ \beta = 1 \\ 2-2=0 \Leftrightarrow 0=0 \text{ P.V} \end{cases}$$

R.: w é combinação linear dos vetores u e v

$\boxed{w = 2u + v}$

b) $\langle u, v\rangle = \langle (2,0,1), (1,2,-2)\rangle = 2\times 1 + 0\times 2 + 1\times(-2) =$

$= 2 - 2 = 0$

$\|u\| = \sqrt{2^2+0^2+1^2} = \sqrt{5}$

$\|v\| = \sqrt{1^2+2^2+(-2)^2} = \sqrt{9} = 3$

c) → para $\{u, v, z\}$ ser um conjunto ortogonal os produtos internos entre os 3 vetores têm de ser todos 0

$u(2, 0, 1)$

$v(1, 2, -2)$

$z(-1, x, 2)$

$\langle u, z\rangle = \langle (2,0,1), (-1,x,2)\rangle$

$= 2\times(-1) + 0\times x + 2$

$= -2 + 2 = 0$ ✓

$\langle v, z\rangle = \langle (1,2,-2)(-1,x,2)\rangle$

$= -1 + 2x - 4$

$\langle v, z\rangle = 0 \Leftrightarrow -1 + 2x - 4 = 0$

$\Leftrightarrow x = \frac{5}{2}$

$$\boxed{z\left(-1, \frac{5}{2}, 2\right)}$$

$\dfrac{\frac{3}{2}z_3}{\frac{2}{1}} = \dfrac{3z_3}{4}$

d) $\rightarrow \left\{ \frac{u}{\|u\|} ; \frac{v}{\|v\|} ; \frac{z}{\|z\|} \right\}$, é um conjunto ortonormal de vetores

$\|u\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$

$\|v\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\|z\| = \sqrt{(-1)^2 + (\frac{5}{2})^2 + 2^2} = \sqrt{\underset{(\times 4)}{1} + \frac{25}{4} + \underset{(\times 4)}{4}} = \sqrt{\frac{4+25+16}{4}}$

$= \frac{\sqrt{45}}{\sqrt{4}} = \frac{\sqrt{45}}{2} = \frac{3\sqrt{5}}{2}$

$\rightarrow \left\{ \frac{u}{\sqrt{5}} ; \frac{v}{3} ; \frac{3\sqrt{5}\, z}{2} \right\}$ é um conjunto ortonormal escrito a partir dos vetores $u ; v ; z$

2. a) $2B^T A + C$

$B^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ $\quad 2B^T = 2\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 4 \\ 0 & 4 & 0 \end{bmatrix}$

$\underset{3\times 3}{2B^T} \underset{3\times 2}{A} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 0+8+6 & 0+20+10 \\ 2+4+12 & 8+10+20 \\ 0+8+0 & 0+20+0 \end{bmatrix} = \begin{bmatrix} 14 & 30 \\ 18 & 38 \\ 8 & 20 \end{bmatrix}$

$2B^T A + C = \begin{bmatrix} 14 & 30 \\ 18 & 38 \\ 8 & 20 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 16 & 31 \\ 18 & 40 \\ 9 & 18 \end{bmatrix}$

b) $R(C) = \left\{ c \in \mathbb{R}^{m\times n} : \sum_{i=1}^{n} x_i \alpha_i \; , \; x_i \in \mathbb{R} \right\}$

$R(C) = \alpha \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = (2\alpha, 0, \alpha) + (\beta, 2\beta, -2\beta) =$

$= (2\alpha + \beta \; , \; 2\beta \; , \; \alpha - 2\beta)$

$\begin{cases} 2\alpha + \beta = 5 \\ 2\beta = 2 \\ 2 - 2\beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 2 \\ \beta = 1 \\ 2 - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 2 \\ \beta = 1 \\ 0 = 0 \end{cases}$ P.V

R.: $w \in R(C)$

c) $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[L_2/L_1]{} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[\substack{L_2=L_2 \\ L_3=L_3-2L_1}]{} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\xrightarrow[L_3=L_3]{} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ car(B) = 3, pq a matriz condensada apresenta 3 pivots ≠s 0;

→ nº pivots = nº incógnitas = nº linhas = nº colunas, sendo estas linearm/ independentes

$B^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

car(B) = car(B^T), logo $N(B^T) = N(B) = \{(0,0,0)\}$

d. $B = I_3 \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_2/L_1]{} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[\substack{L_2=L_2 \\ L_3=L_3-\frac{1}{2}L_1}]{} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & -1/2 & 1 \end{array}\right] \xrightarrow[L_3=L_3-\frac{3}{2}L_2]{} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3/2 & -1/2 & 1 \end{array}\right]$

$\xrightarrow[\substack{L_1=\frac{L_1}{2} \\ L_2=L_2 \\ L_3=-L_3}]{} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 1/2 & -1 \end{array}\right] \xrightarrow[L_1=L_1-L_3]{} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & -3/2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 1/2 & -1 \end{array}\right]$

$\xrightarrow[L_1=L_1-\frac{1}{2}L_2]{} \left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} & \underbrace{\begin{matrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 3/2 & 1/2 & -1 \end{matrix}}_{B^{-1}} \end{array}\right]$ $B^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix}$

3. a) $\begin{cases} x + 3y + 2z = 1 \\ 2x + 5y + \alpha^2 z = 1 \\ x + 5y + \alpha z = \alpha + 1 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & \alpha^2 \\ 1 & 5 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \alpha+1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 5 & \alpha^2 & 1 \\ 1 & 5 & \alpha & \alpha+1 \end{array}\right] \xrightarrow[\substack{L_2=L_2-2L_1 \\ L_3=L_3-L_1}]{} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & \alpha^2-4 & -1 \\ 0 & 2 & \alpha-2 & \alpha \end{array}\right] \xrightarrow[L_3=L_3+2L_2]{} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & \alpha^2-4 & -1 \\ 0 & 0 & 2\alpha^2+\alpha-10 & \alpha+2 \end{array}\right]$

CA

$2\alpha^2 + \alpha - 10 = 0 \Leftrightarrow \alpha = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times (-10)}}{4}$

$\Leftrightarrow \alpha = \frac{-2 \pm \sqrt{44}}{4} \Leftrightarrow \alpha = \frac{-2 + \sqrt{44}}{4} \vee \alpha = \frac{-2 - \sqrt{44}}{4}$

→ se $\alpha = -\frac{2 + \sqrt{44}}{4} \vee \alpha = \frac{-2 - \sqrt{44}}{4}$:

car $(A) = 2 < n = 3$, logo o sist. é possível e simplesmente indeterminado

→ se $\alpha \in \mathbb{R} \setminus \left\{ \frac{-2 - \sqrt{44}}{4} ; \frac{-2 + \sqrt{44}}{4} \right\}$:

car $(A) = 3 = n$, sistema é possível determinado

b) $\boxed{\alpha = 2}$

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + 5y + \alpha^2 z = 1 \\ x + 5y + \alpha z = \alpha + 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 - 3y - 2z \\ 2x + 5y + 4z = 1 \\ x + 5y + 2z = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\qquad} \\ 2(1 - 3y - 2z) + 5y + 4z = 1 \\ (1 - 3y - 2z) + 5y + 2z = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\qquad} \\ 2 - 6y - \cancel{4z} + 5y + \cancel{4z} = 1 \\ 1 - 3y - \cancel{2z} + 5y + \cancel{2z} = 3 \end{cases} \Leftrightarrow \begin{cases} \overline{\qquad} \\ 2 - y = 1 \\ 1 + 2y = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\qquad} \\ +y = +1 \\ 1 + 2 \times 1 = 3 \end{cases} \Leftrightarrow \begin{cases} x = 1 - 3 \times 1 - 2z \\ y = 1 \\ 3 = 3 \end{cases} \Leftrightarrow \begin{cases} x = -2 - 2z \\ y = 1 \\ 3 = 3 \end{cases}$$

$CS = \{(-2 - 2z, 1, z), z \in \mathbb{R}\}$

$= \{z(-4, 1, 1), z \in \mathbb{R}\}$