Freq. 2016/2017 (11/01/17)

1. a) $\begin{cases} x + 2y + 3z = 1 \\ 2x + 4y + \alpha z = \beta \\ 2x + 4y + 5z = 1 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & \alpha \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & \alpha & \beta \\ 2 & 4 & 5 & 1 \end{array}\right] \xrightarrow[L_3 = L_3 - 2L_1]{L_2 = L_2 - 2L_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & \alpha-6 & \beta-2 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{C_3/C_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & \alpha-6 & 0 & \beta-2 \\ 0 & -1 & 0 & -1 \end{array}\right]$

$-1 + \frac{\beta-2}{(\alpha-6)}$

$\xrightarrow{L_3/L_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & \alpha-6 & 0 & \beta-2 \end{array}\right] \xrightarrow{L_3 = L_3 + (\alpha-6)L_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & \beta-2-\alpha+6 \end{array}\right]$

car = 2 < nº incógnitas → sist. possível simplesmente indeterminado

b) $\alpha = 6 \quad \beta = 2$

$\begin{cases} x + 3z + 2y = 1 \\ -z = -1 \\ 0 = 2-2-6+6 \end{cases} \Leftrightarrow \begin{cases} x = 1 - 3 + 2y \\ z = 1 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2y - 2 \\ z = 1 \\ 0 = 0 \end{cases}$

$CS = \{ (2y-2, y, 1), y \in \mathbb{R} \}$

2. a)

$|B| = -4(-1)^{1+1} \begin{vmatrix} 2 & 8 \\ -4 & 2 \end{vmatrix} + 5(-1)^{1+2} \begin{vmatrix} 2 & 8 \\ \delta & 2 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 2 & 2 \\ \delta & -4 \end{vmatrix}$

$= -4 \times 36 - 5 \times (4 - 8\delta) + 2 \times (-8 - 2\delta)$

$= -144 - 20 + 40\delta - 16 - 4\delta$

$= -180 - 36\delta$

$|B| = 0 \Leftrightarrow -180 - 36\delta = 0 \Leftrightarrow \delta = \frac{180}{36} \Leftrightarrow \delta = 5$

→ para $\delta = 5$, B é singular

b) i. $B^{-1} = \frac{[\text{cof } B]^T}{\det B}$

$\text{cof } B = \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} 2 & 8 \\ -4 & 2 \end{vmatrix} & x & x \\ (-1)^{2+1} \begin{vmatrix} 5 & 2 \\ -4 & 2 \end{vmatrix} & x & x \\ (-1)^{3+1} \begin{vmatrix} 5 & 2 \\ 2 & 8 \end{vmatrix} & x & x \end{bmatrix} = \begin{bmatrix} 4+32 & x & x \\ -(10+8) & x & x \\ 36 & x & x \end{bmatrix} = \begin{bmatrix} 36 & x & x \\ -18 & x & x \\ 36 & x & x \end{bmatrix}$

$\det B = -180$

$$B^{-1} = \begin{bmatrix} \frac{36}{-180} & \frac{-18}{-180} & \frac{36}{-180} \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} & -\frac{1}{5} \\ x & x & x \\ x & x & x \end{bmatrix}$$

ii.

$Bu = C$ $\quad\quad |B| = -180$

$$u_1 = \frac{|B[u_1]|}{|B|} = \frac{\begin{vmatrix} 6 & 5 & 2 \\ 0 & 2 & 8 \\ 4 & -4 & 2 \end{vmatrix}}{-180} =$$

$$= \frac{6\times(-1)^{1+1}\begin{vmatrix} 2 & 8 \\ -4 & 2 \end{vmatrix} + 4(-1)^{3+1}\begin{vmatrix} 5 & 2 \\ 2 & 8 \end{vmatrix}}{-180}$$

$$= \frac{6\times(4+32) + 4(40-4)}{-180} = \frac{6\times 36 + 4\times 36}{}$$

$$= \frac{360}{-180} = -2$$

3. a) $Cv = \lambda v \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -3-1 \\ 0 \\ 1+3 \end{bmatrix} = \begin{bmatrix} -\lambda \\ 0 \\ \lambda \end{bmatrix} \Leftrightarrow \begin{cases} -4 = -\lambda \\ 0 = 0 \\ 4 = \lambda \end{cases} \Leftrightarrow \begin{cases} \lambda = 4 \\ 0 = 0 \\ \lambda = 4 \end{cases}$$

$\rightarrow$ $v(-1, 0, 1)$ é vetor próprio de C, associado ao valor próprio $\lambda = 4$

b) $\det(C-\lambda I)=0$

$\Leftrightarrow \det\left(\begin{bmatrix}3&0&-1\\0&2&0\\-1&0&3\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right)=0$

$\Leftrightarrow \det\left(\begin{bmatrix}3-\lambda&0&-1\\0&2-\lambda&0\\-1&0&3-\lambda\end{bmatrix}\right)=0$

$\Leftrightarrow (2-\lambda)(-1)^{2+2}\begin{vmatrix}(3-\lambda)&-1\\-1&(3-\lambda)\end{vmatrix}=0$

$\Leftrightarrow (2-\lambda)\left[(3-\lambda)^2-1\right]=0$

$\Leftrightarrow 2-\lambda=0 \quad \vee \quad 9-6\lambda+\lambda^2-1=0$

$\Leftrightarrow \lambda=2 \quad \vee \quad \lambda=\frac{6\pm\sqrt{(-6)^2-4\times1\times8}}{2}$

$\Leftrightarrow \lambda=2 \quad \vee \quad \lambda=\frac{6\pm2}{2}$

$\Leftrightarrow \lambda=2 \quad \vee \quad \lambda=4 \quad \vee \quad \lambda=2$

$m.a.(2)=2$

$m.a.(4)=1$

c) $E(2)=?$

$(C-2I)x=0 \Leftrightarrow \begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\left[\begin{array}{ccc|c}1&0&-1&0\\0&0&0&0\\-1&0&1&0\end{array}\right]\xrightarrow[\substack{l_2=l_2\\l_3=l_3+l_1}]{}\left[\begin{array}{ccc|c}1&0&-1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$

car = 1

$\Leftrightarrow \begin{cases}x-z=0\\0=0\\0=0\end{cases} \Leftrightarrow \begin{cases}x=z\\0=0\\0=0\end{cases}$

$m.g.(2)=3-1=2$

$E(2)=\{(z,y,z),\ y,z\in\mathbb{R}\}$

$=\{z(1,0,1)+y(0,1,0),\ y,z\in\mathbb{R}\}$

d) $\boxed{\lambda=2}$

$m.a(\lambda=2)=2$
$m.g.(\lambda=2)=2$ $\Rightarrow$ $m.a(2)=m.g.(2)=2$, logo C é ortogonalmente diagonalizável

→ os vetores associados a $\lambda=2$ são: $u\begin{bmatrix}1\\0\\1\end{bmatrix}$ e $v\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\|u\|=\sqrt{1^2+1^2}=\sqrt{2}\neq 1$

$\|v\|=\sqrt{1}=1$

$u^*=\frac{1}{\|u\|}\times u=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$

$\boxed{\lambda=4}$

$m.a(\lambda=4)=1$

$m.g.(\lambda=4)=3-car(C-4I_3)=3-2=1$

$(C-4I_3)x=0 \Leftrightarrow \begin{bmatrix}-1&0&-1\\0&-2&0\\-1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix}-1&0&-1\\0&-2&0\\-1&0&-1\end{bmatrix}\xrightarrow[L_3=L_3-L_1]{L_2=L_2}\begin{bmatrix}-1&0&-1\\0&-2&0\\0&0&0\end{bmatrix}$ car $=2$

$\Leftrightarrow \begin{cases}-x-z=0\\-2y=0\\0=0\end{cases} \Leftrightarrow \begin{cases}x=-z\\y=0\\0=0\end{cases}$

$E(4)=\{(-z,0,z),\ z\in\mathbb{R}\}$

$=\{z(-1,0,1),\ z\in\mathbb{R}\}$

→ o vetor próprio associado ao valor próprio $\lambda=4$ é $= w=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\|w\|=\sqrt{(-1)^2+1^2}=\sqrt{2}\neq 1$ $w^*=\frac{1}{\|w\|}\times w=\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$

$u=\left[\frac{1}{\sqrt{2}}\ \ 0\ \ \frac{1}{\sqrt{2}}\right]^T$ $v=[0\ \ 1\ \ 0]^T$ $w=\left[-\frac{1}{\sqrt{2}}\ \ 0\ \ \frac{1}{\sqrt{2}}\right]^T$

$C=2\begin{bmatrix}u\end{bmatrix}\begin{bmatrix}u\end{bmatrix}^T+2\begin{bmatrix}v\end{bmatrix}\begin{bmatrix}v\end{bmatrix}^T+4\begin{bmatrix}w\end{bmatrix}\begin{bmatrix}w\end{bmatrix}^T$

4. $A = A^T \rightarrow$ matriz simétrica

$x_1 ; x_2 \rightarrow$ vetores próprios

$\lambda_1 ; \lambda_2 \rightarrow$ valores próprios

$$\lambda_1 x_1 x_1^T = (\lambda_1 x_1)^T x_1 = (Ax)^T x_1 = x_1^T A^T x_1 =$$

$$= (\text{como } A \text{ é simétrica}) = x_1^T A x_1 = x_1^T \lambda_2 x_2 = \lambda_2 x_1^T x_2$$

Assim:

$$\lambda_1 x_1^T x_2 - \lambda_2 x_1^T x_2 = 0 \Leftrightarrow (\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$$\Leftrightarrow x_1^T x_2 = 0 \ (\text{pois } \lambda_1 \neq \lambda_2)$$