Defining the Error in Solutions to Kepler's equation

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Abstract

This short derivation justifies the use of the error on the function's proximity to zero, ϵ, as a good approximation for the error on the root, δ, in the case of Kepler's equation. The region where this approximation might fail is also shortly discussed.

Let us define the following function as:

$$f(x;e,M) = x - e\sin x - M \tag{1}$$

where $e \in [0,1)$ and $M \in [0,2\pi]$ are parameters and $x \in [0,2\pi]$ is the variable for which we want to find $f(x;e,M) = 0$, which corresponds to *Kepler's equation*.

There are many possible numerical methods for finding the roots, E, given the parameter values. But to be able to properly compare them, we should be concise in what we mean by the approximation error, δ, of the solution, x_0.

$$\delta \equiv |E - x_0| \tag{2}$$

When a root is found, it is straightforward to obtain the error on the value of $f(x_0;e,M)$, we simply substitute x_0 back and see what we get, since:

$$\epsilon \equiv f(x_0;e,M) - 0 = f(x_0;e,M) \tag{3}$$

but equation 3 tells us very little about the error in the estimate of the actual root.

Suppose our algorithm has found a root $x_0 < E$, such that $E = x_0 + \delta$. If we expand $f(x_0;e,M)$ around $x = E$, we get:

$$f(x_0) = f(E) - \delta f'(E) + \frac{\delta^2}{2} f''(E) + \mathcal{O}\left(\delta^3\right) \tag{4}$$

Now, we know that the first term on the RHS of equation 4 is zero. We also defined the LHS as ϵ in equation 3, so we can write:

$$\epsilon = -\delta f'(E) + \frac{\delta^2}{2} f''(E) + \mathcal{O}\left(\delta^3\right) \tag{5}$$

Taking into account the formulas for the derivatives of $f(x;e,M)$, we are able to relate ϵ and δ by E:

$$\epsilon = -\delta\left(1 - e\cos E\right) + \frac{\delta^2}{2} e\sin E + \mathcal{O}\left(\delta^3\right) \tag{6}$$

The term of leading order in δ has a coefficient of magnitude ≈ 1, thus the orders of magnitude of ϵ and δ are roughly the same. Knowing that in most of our uses we work with ϵ much smaller than 10^{-2}, we can confidently neglect the higher order terms in the expansion and claim that:

$$\epsilon \simeq \delta\left(-1 + e\cos E\right) \tag{7}$$

The function $f(x;e,M)$ is monotonous, since $f'(x;e,M) = 1 - e\cos x > 0,\ \forall\, x\ ,\ e \in [0,1)$. This means that:

$$\epsilon = \begin{cases} -|\epsilon|\ , & x_0 < E \\ +|\epsilon|\ , & x_0 > E \end{cases} \tag{8}$$

So, redefining $\epsilon \equiv |f(x_0; e, M)|$:

$$\delta \simeq \frac{\epsilon}{1 - e\cos E} \tag{9}$$

When treating a full orbit, i.e. an array of M values, the extremes of the δ values occur when $\cos E = -1$ and $\cos E = 1$, $\epsilon \approx (1+e)\delta$ and $\epsilon \approx (1-e)\delta$, so $\boxed{\delta \in \left[\frac{\epsilon}{1+e}, \frac{\epsilon}{1-e}\right]}$. This should not be a problem in most situations, but it might lead to larger error in the calculated eccentric anomaly near the pericenter of highly eccentric orbits. For $e = 0.9$, this would amount to a 10-fold increase in δ relative to ϵ, which is significantly higher.