# Linear Algebra
## and Its Applications

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CHAPTER 9

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INTRODUCTORY EXAMPLE

## The Berlin Airlift

After World War II, the city of Berlin was an "island" surrounded by the Soviet zone of occupied Germany. The city was divided into four sections, with the British, French, and Americans having jurisdiction over West Berlin and the Soviets over East Berlin. But the Russians were eager for the other three nations to abandon Berlin. After months of harassment, on June 24, 1948, they imposed a blockade on West Berlin, cutting off all access by land and rail. With a civilian population of about 2.5 million people, the isolated western sectors became dependent on reserve stocks and airlift replacements.

Four days later, the first American planes landed in Berlin with supplies of food, and "Operation Vittles" had begun. At first the airlift seemed doomed to failure because the needs of the city were overwhelming. The Russians had cut off all electricity and coal shipments, and the city was literally under siege. But the Western Allies responded by flying in thousands of tons of food, coal, medicine, and other supplies on a daily basis. In May 1949, Stalin relented, and the blockade was lifted. The airlift, however, continued for another four months.

The Berlin Airlift was unbelievably successful in using relatively few aircraft to deliver an enormous amount of supplies. The design and conduct of this operation required intensive planning and calculations, which led to the theoretical development of linear programming, and the invention of the simplex method by George Dantzig. The potential of this new tool was quickly recognized by business and industry, where it is now used to allocate resources, plan production, schedule workers, organize investment portfolios, formulate marketing strategies, and perform many other tasks involving optimization.

There are many situations in business, politics, economics, military strategy, and other areas where one tries to optimize a certain benefit. This may involve maximizing a profit or the payoff in a contest or minimizing a cost or other loss. This chapter presents two mathematical models that deal with optimization problems.[1] The fundamental results in both cases depend on properties of convex sets and hyperplanes. Section 9.1 introduces the theory of games and develops strategies based on probability. Sections 9.2–9.4 explore techniques of linear programming and use them to solve a variety of problems, including matrix games larger than those in Section 9.1.

## 9.1 MATRIX GAMES

The theory of games analyzes competitive phenomena and seeks to provide a basis for rational decision-making. Its growing importance was highlighted in 1994 when the Nobel Prize in Economics was awarded to John Harsanyi, John Nash, and Reinhard Selten, for their pioneering work in the theory of noncooperative games.[2]

The games in this section are **matrix games** whose various outcomes are listed in a payoff matrix. Two players in a game compete according to a fixed set of rules. Player $R$ (for *row*) has a choice of $m$ possible moves (or choices of action), and player $C$ (for *column*) has $n$ moves. By convention, the **payoff matrix** $A = [a_{ij}]$ lists the amounts that the **row** player $R$ wins **from** player $C$, depending on the choices $R$ and $C$ make. Entry $a_{ij}$ shows the amount $R$ wins when $R$ chooses action $i$ and $C$ chooses action $j$. A negative value for $a_{ij}$ indicates a loss for $R$, the amount $R$ has to pay to $C$. The games are often called **two-person zero-sum games** because the algebraic sum of the amounts gained by $R$ and $C$ is zero.

**EXAMPLE 1** Each player has a supply of pennies, nickels, and dimes. At a given signal, both players display (or "play") one coin. If the displayed coins are not the same, then the player showing the higher-valued coin gets to keep both. If they are both pennies or both nickels, then player $C$ keeps both; but if they are both dimes, then player $R$ keeps them. Construct a payoff matrix, using $p$ for display of a penny, $n$ for a nickel, and $d$ for a dime.

**Solution** Each player has three choices, $p$, $n$, and $d$, so the payoff matrix is $3 \times 3$:

$$\begin{array}{cc} & \text{Player } C \\ & \begin{array}{ccc} p & n & d \end{array} \\ \text{Player } R \quad \begin{array}{c} p \\ n \\ d \end{array} & \begin{bmatrix} \phantom{p} & \phantom{n} & \phantom{d} \\ & & \\ & & \end{bmatrix} \end{array}$$

[1] I am indebted to my brother, Dr. Steven R. Lay, for designing and writing most of this chapter and class testing it at Lee University. I have also class tested it and made a few changes/additions. It works well, and the students enjoyed it. However, I would appreciate feedback from anyone who uses this, faculty or students.

[2] The popular 2002 movie, *A Beautiful Mind*, tells a poignant story of the life of John Nash.

Consider a row for $R$ and fill in what $R$ receives (or pays), depending on the choice $C$ makes. First, suppose $R$ plays a penny. If $C$ also plays a penny, $R$ loses 1 cent, because the coins match. The (1, 1) entry is –1. If $C$ plays either a nickel or a dime, $R$ also loses 1 cent, because $C$ displays the higher-valued coin. This information goes in row 1:

$$\begin{array}{cc} & \text{Player } C \\ & \begin{array}{ccc} p & n & d \end{array} \\ \text{Player } R \quad \begin{array}{c} p \\ n \\ d \end{array} & \begin{bmatrix} -1 & -1 & -1 \\ & & \\ & & \end{bmatrix} \end{array}$$

Next, suppose $R$ plays a nickel. If $C$ plays a penny, $R$ wins the penny. Otherwise, $R$ loses the nickel, because either $C$ matches the nickel or shows the higher-value dime. Finally, when $R$ plays a dime, $R$ gains either a penny or a nickel, whichever is shown by $C$, because $R$'s dime is of higher value. Also, when both players display a dime, $R$ wins the dime from $C$ because of the special rule for that case.

$$\begin{array}{cc} & \text{Player } C \\ & \begin{array}{ccc} p & n & d \end{array} \\ \text{Player } R \quad \begin{array}{c} p \\ n \\ d \end{array} & \begin{bmatrix} -1 & -1 & -1 \\ 1 & -5 & -5 \\ 1 & 5 & 10 \end{bmatrix} \end{array}$$

By looking at the payoff matrix in Example 1, the players discover that some plays are better than others. Both players know that $R$ is likely to choose a row that has positive entries, while $C$ is likely to choose a column that has negative entries (a payment from $R$ to $C$). Player $R$ notes that every entry in row 3 is positive and chooses to play a dime. No matter what $C$ may do, the worst that can happen to $R$ is to win a penny. Player $C$ notes that every column contains a positive entry and therefore $C$ cannot be certain of winning anything. So player $C$ chooses to play a penny, which will minimize the potential loss.

From a mathematical point of view, what has each player done? Player $R$ has found the minimum of each row (the worst that could happen for that play) and has chosen the row for which this minimum is largest. (See Fig. 1.) That is, $R$ has computed

$$\max_i \left[ \min_j a_{ij} \right]$$

$$\begin{array}{cccl} & \text{Player } C & \text{Row minima} & \\ \text{Player } R & \begin{bmatrix} -1 & -1 & -1 \\ 1 & -5 & -5 \\ 1 & 5 & 10 \end{bmatrix} & \begin{array}{c} -1 \\ -5 \\ 1 \end{array} & \begin{array}{l} \\ \\ \leftarrow \text{Max of the minima} \end{array} \\ \text{Column maxima} & \begin{array}{ccc} 1 & 5 & 10 \end{array} & & \\ & \text{Min of the maxima} & & \end{array}$$

**FIGURE 1**

Observe that for $C$, a large positive payment to $R$ is worse than a small positive payment. Thus $C$ has found the maximum of each column (the worst that can happen to $C$ for that play) and has chosen the column for which this maximum is smallest. Player $C$ has found

$$\min_j \left[\max_i a_{ij}\right]$$

For this payoff matrix $[a_{ij}]$,

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = 1$$

**DEFINITION** If the payoff matrix of a matrix game contains an entry $a_{ij}$ that is both the minimum of row $i$ and the maximum of column $j$, then $a_{ij}$ is called a **saddle point**.

In Example 1, the entry $a_{31}$ is a saddle point for the payoff matrix. As long as both players continue to seek their best advantage, player $R$ will always display a dime (row 3) and player $C$ will always display a penny (column 1). Some games may have more than one saddle point.

The situation is not quite so simple in the next example.

**EXAMPLE 2** Again suppose that each player has a supply of pennies, nickels, and dimes to play, but this time the payoff matrix is given as follows:

$$
\begin{array}{rcl}
 & \text{Player } C & \\
 & \begin{array}{ccc} \;\;p\;\; & \;\;n\;\; & \;\;d\;\; \end{array} & \text{Row minima} \\
\text{Player } R \;\; \begin{array}{c} p \\ n \\ d \end{array} & \begin{bmatrix} 10 & -5 & 5 \\ 1 & 1 & -1 \\ 0 & -10 & -5 \end{bmatrix} & \begin{array}{l} -5 \\ -1 \leftarrow \text{Max of the minima} \\ -10 \end{array} \\
\text{Column maxima} & \begin{array}{ccc} 10 & 1 & 5 \end{array} & \\
 & \uparrow \text{Min of the maxima} &
\end{array}
$$

If player $R$ reasons as in the first example and looks at the row minima, $R$ will choose to play a nickel, thereby maximizing the minimum gain (in this case a loss of 1). Player $C$, looking at the column maxima (the greatest payment to $R$), will also select a nickel to minimize the loss to $R$.

Thus, as the game begins, $R$ and $C$ both continue to play a nickel. After a while, however, $C$ begins to reason, "If $R$ is going to play a nickel, then I'll play a dime so that I can win a penny." However, when $C$ starts to play a dime repeatedly, $R$ begins to reason, "If $C$ is going to play a dime, then I'll play a penny so that I can win a nickel." Once $R$

has done this, $C$ switches to a nickel (to win a nickel) and then $R$ starts playing a nickel . . . and so on. It seems that neither player can develop a winning strategy.

Mathematically speaking, the payoff matrix for the game in Example 2 does not have a saddle point. Indeed,

$$\max_i \min_j a_{ij} = -1$$

while

$$\min_j \max_i a_{ij} = 1$$

This means that neither player can play the same coin repeatedly and be assured of optimizing the winnings. In fact, any predictable strategy can be countered by the opponent. But is it possible to formulate some combination of plays that over the long run will produce an optimal return? The answer is *yes* (as Theorem 3 later will show), when each move is made at random, but with a certain probability attached to each possible choice.

Here is a way to imagine how player $R$ could develop a strategy for playing a matrix game. Suppose that $R$ has a device consisting of a horizontal metal arrow whose center of gravity is supported on a vertical rod in the middle of a flat circular region. The region is cut into pie-shaped sectors, one for each of the rows in the payoff matrix. Player $R$ gives the arrow an initial spin and waits for it to come to rest. The position of the arrowhead at rest determines one play for $R$ in the matrix game.

If the area of the circle is taken as 1 unit, then the areas of the various sectors sum to 1; and these areas give the relative frequencies, or *probabilities*, of selecting the various plays in the matrix game, when the game is played many times. For instance, if there are five sectors of equal area and if the arrow is spun many times, player $R$ will select each of the five plays about $1/5$ of the time. This strategy is specified by the vector in $\mathbb{R}^5$ whose entries all equal $1/5$. If the five sectors of the circle are unequal in size, then in the long run some game plays will be chosen more frequently than the others. The corresponding strategy for $R$ is specified by a vector in $\mathbb{R}^5$ that lists the areas of the five sectors.

**DEFINITIONS**

A **probability vector** in $\mathbb{R}^m$ is the set of all $\mathbf{x}$ in $\mathbb{R}^m$ whose entries are nonnegative and sum to one. Such an $\mathbf{x}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad x_i \geq 0 \text{ for } i = 1, \ldots, m \text{ and } \sum_{i=1}^{m} x_i = 1$$

Let $A$ be an $m \times n$ payoff matrix for a game. The **strategy space** for player $R$ is the set of all probability vectors in $\mathbb{R}^m$, and the **strategy space** for player $C$ is the set of all probability vectors in $\mathbb{R}^n$. A point in a strategy space is called a **strategy**. If one entry in a strategy is 1 (and the other entries are zeros), the strategy is called a **pure strategy**.

The pure strategies in $\mathbb{R}^m$ are the standard basis vectors for $\mathbb{R}^m$, $\mathbf{e}_1, \ldots, \mathbf{e}_m$. In general, each strategy $\mathbf{x}$ is a linear combination, $x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m$, of these pure strategies with nonnegative weights that sum to one.[3]

Suppose now that $R$ and $C$ are playing the $m \times n$ matrix game $A = [a_{ij}]$, where $a_{ij}$ is the entry in the $i$th row and the $j$th column of $A$. There are $mn$ possible outcomes of the game, depending on the row $R$ chooses and the column $C$ chooses. Suppose $R$ uses strategy $\mathbf{x}$ and $C$ uses strategy $\mathbf{y}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Since $R$ plays the first row with probability $x_1$ and $C$ plays the first column with probability $y_1$ and since their choices are made independently, it can be shown that the probability is $x_1y_1$ that $R$ chooses the first row *and* $C$ chooses the first column. Over the course of many games, the expected payoff to $R$ for this outcome is $a_{11}x_1y_1$ for one game. A similar computation holds for each possible pair of choices that $R$ and $C$ can make. The sum of the expected payoffs to $R$ over all possible pairs of choices is called the **expected payoff**, $E(\mathbf{x}, \mathbf{y})$, of the game to player $R$ for strategies $\mathbf{x}$ and $\mathbf{y}$. That is,

$$E(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = \mathbf{x}^T A \mathbf{y}$$

Roughly speaking, the number $E(\mathbf{x}, \mathbf{y})$ is the average amount that $C$ will pay to $R$ per game, when $R$ and $C$ play a large number of games using the strategies $\mathbf{x}$ and $\mathbf{y}$, respectively.

Let $X$ denote the strategy space for $R$ and $Y$ the strategy space for $C$. If $R$ were to choose a particular strategy, say $\tilde{\mathbf{x}}$, and if $C$ were to discover this strategy, then $C$ would certainly choose $\mathbf{y}$ to minimize

$$E(\tilde{\mathbf{x}}, \mathbf{y}) = \tilde{\mathbf{x}}^T A \mathbf{y}$$

The **value** of using strategy $\tilde{\mathbf{x}}$ is the number $v(\tilde{\mathbf{x}})$ defined by

$$v(\tilde{\mathbf{x}}) = \min_{\mathbf{y} \in Y} E(\tilde{\mathbf{x}}, \mathbf{y}) = \min_{\mathbf{y} \in Y} \tilde{\mathbf{x}}^T A \mathbf{y} \tag{1}$$

Since $\tilde{\mathbf{x}}^T A$ is a $1 \times n$ matrix, the mapping $\mathbf{y} \mapsto E(\tilde{\mathbf{x}}, \mathbf{y}) = \tilde{\mathbf{x}}^T A \mathbf{y}$ is a linear functional on the probability space $Y$. From this, it can be shown that $E(\tilde{\mathbf{x}}, \mathbf{y})$ attains its minimum when $\mathbf{y}$ is one of the pure strategies, $\mathbf{e}_1, \ldots, \mathbf{e}_n$, for $C$.[4]

Recall that $A\mathbf{e}_j$ is the $j$th column of the matrix $A$, usually denoted by $\mathbf{a}_j$. Since the minimum in (1) is attained when $\mathbf{y} = \mathbf{e}_j$ for some $j$, (1) may be written, with $\mathbf{x}$ in place

---

[3] More precisely, each strategy is a convex combination of the set of pure strategies—that is, a point in the convex hull of the set of standard basis vectors. This fact connects the theory of convex sets to the study of matrix games. The strategy space for $R$ is an $(m-1)$-dimensional simplex in $\mathbb{R}^m$, and the strategy space for $C$ is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$. See Sections 8.3 and 8.5 for definitions.

[4] A linear functional on $Y$ is a linear transformation from $Y$ into $\mathbb{R}$. The pure strategies are the extreme points of the strategy space for a player. The stated result follows directly from Theorem 16 in Section 8.5.

of $\tilde{\mathbf{x}}$, as

$$v(\mathbf{x}) = \min_j E(\mathbf{x}, \mathbf{e}_j) = \min_j \mathbf{x}^T A \mathbf{e}_j = \min_j \mathbf{x}^T \mathbf{a}_j = \min_j \mathbf{x} \cdot \mathbf{a}_j \tag{2}$$

That is, $v(\mathbf{x})$ is the minimum of the inner product of $\mathbf{x}$ with each of the columns of $A$. The goal of $R$ is to choose $\mathbf{x}$ to maximize $v(\mathbf{x})$.

**DEFINITION**

The number $v_R$, defined by

$$v_R = \max_{\mathbf{x}\in X} v(\mathbf{x}) = \max_{\mathbf{x}\in X} \min_{\mathbf{y}\in Y} E(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x}\in X} \min_j \mathbf{x} \cdot \mathbf{a}_j$$

with the notation as described above, is called the **value of the game to row player** $R$. A strategy $\hat{\mathbf{x}}$ for $R$ is called **optimal** if $v(\hat{\mathbf{x}}) = v_R$.

Of course, $E(\mathbf{x}, \mathbf{y})$ may exceed $v_R$ for some $\mathbf{x}$ and $\mathbf{y}$ if $C$ plays poorly. Thus, $\hat{\mathbf{x}}$ is optimal for $R$ if $E(\hat{\mathbf{x}}, \mathbf{y}) \geq v_R$ for all $\mathbf{y} \in Y$. This value $v_R$ can be thought of as the most that player $R$ can be *sure* to receive from $C$, independent of what player $C$ may do.

A similar analysis for player $C$, using the pure strategies for $\mathbf{x}$, shows that a particular strategy $\mathbf{y}$ will have a value $v(\mathbf{y})$ given by

$$v(\mathbf{y}) = \max_{\mathbf{x}\in X} E(\mathbf{x}, \mathbf{y}) = \max_i E(\mathbf{e}_i, \mathbf{y}) = \max_i \text{row}_i(A)\mathbf{y} \tag{3}$$

because $\mathbf{e}_i^T A = \text{row}_i(A)$. That is, the value of strategy $\mathbf{y}$ to $C$ is the maximum of the inner product of $\mathbf{y}$ with each of the rows of $A$. The number $v_C$, defined by

$$v_C = \min_{\mathbf{y}\in Y} v(\mathbf{y}) = \min_{\mathbf{y}\in Y} \max_i \text{row}_i(A)\mathbf{y}$$

is called the **value of the game to** $C$. This is the least that $C$ will have to lose regardless of what $R$ may do. A strategy $\hat{\mathbf{y}}$ for $C$ is called **optimal** if $v(\hat{\mathbf{y}}) = v_C$. Equivalently, $\hat{\mathbf{y}}$ is optimal if $E(\mathbf{x}, \hat{\mathbf{y}}) \leq v_C$ for all $\mathbf{x}$ in $X$.

**THEOREM 1**

In any matrix game, $v_R \leq v_C$.

PROOF For any $\mathbf{x}$ in $X$, the definition $v(\mathbf{x}) = \min_{\mathbf{y}\in Y} E(\mathbf{x}, \mathbf{y})$ implies that $v(\mathbf{x}) \leq E(\mathbf{x}, \mathbf{y})$ for each $\mathbf{y}$ in $Y$. Also, since $v(\mathbf{y})$ is the maximum of $E(\mathbf{x}, \mathbf{y})$ over all $\mathbf{x}$, $v(\mathbf{y}) \geq E(\mathbf{x}, \mathbf{y})$ for each individual $\mathbf{x}$. These two inequalities show that

$$v(\mathbf{x}) \leq E(\mathbf{x}, \mathbf{y}) \leq v(\mathbf{y})$$

for all $\mathbf{x} \in X$ and for all $\mathbf{y} \in Y$. For any fixed $\mathbf{y}$, the left inequality above implies that $\max_{\mathbf{x}\in X} v(\mathbf{x}) \leq E(\mathbf{x}, \mathbf{y})$. Similarly, for each $\mathbf{x}$, $E(\mathbf{x}, \mathbf{y}) \leq \min_{\mathbf{y}\in Y} v(\mathbf{y})$. Thus,

$$\max_{\mathbf{x}\in X} v(\mathbf{x}) \leq \min_{\mathbf{y}\in Y} v(\mathbf{y})$$

which proves the theorem. ∎

**EXAMPLE 3** Let $A = \begin{bmatrix} 10 & -5 & 5 \\ 1 & 1 & -1 \\ 0 & -10 & -5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$, where $A$ comes from Example 2. Compute $E(\mathbf{x}, \mathbf{y})$ and verify that this number lies between $v(\mathbf{x})$ and $v(\mathbf{y})$.

**Solution** Compute

$$E(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 10 & -5 & 5 \\ 1 & 1 & -1 \\ 0 & -10 & -5 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{15}{4} \\ 0 \\ -5 \end{bmatrix} = -\frac{5}{16}$$

Next, from (2), $v(\mathbf{x})$ is the minimum of $E(\mathbf{x}, \mathbf{e}_j)$ for $1 \le j \le 3$. So compute

$$E(\mathbf{x}, \mathbf{e}_1) = \tfrac{10}{4} + \tfrac{1}{2} + 0 = 3$$
$$E(\mathbf{x}, \mathbf{e}_2) = -\tfrac{5}{4} + \tfrac{1}{2} - \tfrac{10}{4} = -\tfrac{13}{4}$$
$$E(\mathbf{x}, \mathbf{e}_3) = \tfrac{5}{4} - \tfrac{1}{2} - \tfrac{5}{4} = -\tfrac{1}{2}$$

Then $v(\mathbf{x}) = \min\left\{3, -\frac{13}{4}, -\frac{1}{2}\right\} = -\frac{13}{4} < -\frac{5}{16} = E(\mathbf{x}, \mathbf{y})$. Similarly, $E(\mathbf{e}_1, \mathbf{y}) = \frac{15}{4}$, $E(\mathbf{e}_2, \mathbf{y}) = 0$, and $E(\mathbf{e}_3, \mathbf{y}) = -5$, and so $v(\mathbf{y}) = \max\left\{\frac{15}{4}, 0, -5\right\} = \frac{15}{4}$. Thus $E(\mathbf{x}, \mathbf{y}) \le v(\mathbf{y})$, as expected.

In Theorem 1, the proof that $v_R \le v_C$ was simple. A fundamental result in game theory is that $v_R = v_C$, but this is not easy to prove. The first proof by John von Neumann in 1928 was technically difficult. Perhaps the best-known proof depends strongly on certain properties of convex sets and hyperplanes. It appeared in the classic 1944 book *Theory of Games and Economic Behavior*, by von Neumann and Oskar Morgenstern.[5]

**THEOREM 2**

**Minimax Theorem**

In any matrix game, $v_R = v_C$. That is,

$$\max_{\mathbf{x} \in X} \min_{\mathbf{y} \in Y} E(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{y} \in Y} \max_{\mathbf{x} \in X} E(\mathbf{x}, \mathbf{y})$$

**DEFINITION**

The common value $v = v_R = v_C$ is called the **value of the game**. Any pair of optimal strategies $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is called a **solution** to the game.

When $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a solution to the game, $v_R = v(\hat{\mathbf{x}}) \le E(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \le v(\hat{\mathbf{y}}) = v_C$, which shows that $E(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = v$.

[5] More precisely, the proof involves finding a hyperplane that strictly separates the origin $\mathbf{0}$ from the convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{e}_1, \ldots, \mathbf{e}_m\}$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of $A$ and $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are the standard basis vectors in $\mathbb{R}^m$. The details are in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), pp. 159–163.

The next theorem is the main theoretical result of this section. A proof can be based either on the Minimax Theorem or on the theory of linear programming (in Section 9.4).[6]

**THEOREM 3** **Fundamental Theorem for Matrix Games**

In any matrix game, there are always optimal strategies. That is, every matrix game has a solution.

## 2 x *n* Matrix Games

When a game matrix $A$ has 2 rows and $n$ columns, an optimal row strategy and $v_R$ are fairly easy to compute. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

The objective of player $R$ is to choose $\mathbf{x}$ in $\mathbb{R}^2$ to maximize $v(\mathbf{x})$. Since $\mathbf{x}$ has only two entries, the probability space $X$ for $R$ may be parameterized by a variable $t$, with a typical $\mathbf{x}$ in $X$ having the form $\mathbf{x}(t) = \begin{bmatrix} 1-t \\ t \end{bmatrix}$ for $0 \le t \le 1$. From formula (2), $v(\mathbf{x}(t))$ is the minimum of the inner product of $\mathbf{x}(t)$ with each of the columns of $A$. That is,

$$\begin{aligned} v(\mathbf{x}(t)) &= \min\left\{ \mathbf{x}(t)^T \begin{bmatrix} a_{1j} \\ a_{2j} \end{bmatrix} : j = 1, \dots, n \right\} \\ &= \min\left\{ a_{1j}(1-t) + a_{2j}t : j = 1, \dots, n \right\} \end{aligned} \tag{4}$$

Thus $v(\mathbf{x}(t))$ is the minimum value of $n$ linear functions of $t$. When these functions are graphed on one coordinate system for $0 \le t \le 1$, the graph of $z = v(\mathbf{x}(t))$ as a function of $t$ becomes evident, and the maximum value of $v(\mathbf{x}(t))$ is easy to find. The process is illustrated best by an example.

**EXAMPLE 4** Consider the game whose payoff matrix is

$$A = \begin{bmatrix} 1 & 5 & 3 & 6 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

a. On a $t$-$z$ coordinate system, sketch the four lines $z = a_{1j}(1-t) + a_{2j}t$ for $0 \le t \le 1$, and darken the line segments that correspond to the graph of $z = v(\mathbf{x}(t))$, from (4).

[6] The proof based on the Minimax Theorem goes as follows: The function $v(\mathbf{x})$ is continuous on the compact set $X$, so there exists a point $\hat{\mathbf{x}}$ in $X$ such that

$$v(\hat{\mathbf{x}}) = \max_{\mathbf{x} \in X} v(\mathbf{x}) = v_R$$

Similarly, there exists $\hat{\mathbf{y}}$ in $Y$ such that

$$v(\hat{\mathbf{y}}) = \min_{\mathbf{y} \in Y} v(\mathbf{y}) = v_C$$

According to the Minimax Theorem, $v_R = v_C = v$.

b. Identify the highest point $M = (t, z)$ on the graph of $v(\mathbf{x}(t))$. The $z$-coordinate of $M$ is the value $v_R$ of the game for $R$, and the $t$-coordinate determines an optimal strategy $\hat{\mathbf{x}}(t)$ for $R$.

**Solution**

a. The four lines are

$$\begin{aligned} z &= 1\cdot(1-t) + 4\cdot t = \phantom{-}3t + 1 \\ z &= 5\cdot(1-t) + 0\cdot t = -5t + 5 \\ z &= 3\cdot(1-t) + 1\cdot t = -2t + 3 \\ z &= 6\cdot(1-t) + 2\cdot t = -4t + 6 \end{aligned}$$

See Fig. 2. Notice that the line $z = a_{1j}\cdot(1-t) + a_{2j}\cdot t$ goes through the points $(0, a_{1j})$ and $(1, a_{2j})$. For instance, the line $z = 6\cdot(1-t) + 2\cdot t$ for column 4 goes through the points $(0, 6)$ and $(1, 2)$. The heavy polygonal path in Fig. 2 represents $v(\mathbf{x})$ as a function of $t$, because the $z$-coordinate of a point on this path is the minimum of the corresponding $z$-coordinates of points on the four lines in Fig. 2.

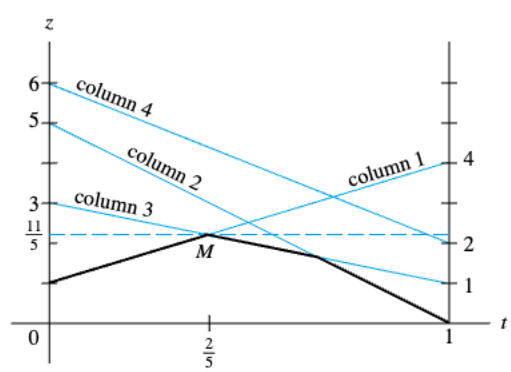



**FIGURE 2**

b. The highest point, $M$, on the graph of $v(\mathbf{x})$ is the intersection of the lines corresponding to the first and third columns of $A$. The coordinates of $M$ are $\left(\frac{2}{5}, \frac{11}{5}\right)$.[7] The value of the game for $R$ is $\frac{11}{5}$. This value is attained at $t = \frac{2}{5}$, so the optimal strategy for $R$ is $\hat{\mathbf{x}} = \begin{bmatrix} 1 - \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$.

For any $2\times n$ matrix game, Example 4 illustrates the method for finding an optimal solution for player $R$. Theorem 3 guarantees that there also exists an optimal strategy for

[7] Solve the equations for columns 1 and 3 simultaneously:

$$\left.\begin{array}{l} \text{(column 1) } z = 3t + 1 \\ \text{(column 3) } z = -2t + 3 \end{array}\right\} \Rightarrow t = \frac{2}{5}, z = \frac{11}{5}$$

player $C$, and the value of the game is the same for $C$ as for $R$. With this value available, an analysis of the graphical solution for $R$, as in Fig. 2, will reveal how to produce an optimal strategy $\hat{\mathbf{y}}$ for $C$. The next theorem supplies the key information about $\hat{\mathbf{y}}$.

**THEOREM 4** Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ be optimal strategies for an $m \times n$ matrix game whose value is $v$, and suppose that

$$\hat{\mathbf{x}} = \hat{x}_1\mathbf{e}_1 + \cdots + \hat{x}_m\mathbf{e}_m \quad \text{in } \mathbb{R}^m \tag{5}$$

Then $\hat{\mathbf{y}}$ is a convex combination of the pure strategies $\mathbf{e}_j$ in $\mathbb{R}^n$ for which $E(\hat{\mathbf{x}}, \mathbf{e}_j) = v$. In addition, $\hat{\mathbf{y}}$ satisfies the equation

$$E(\mathbf{e}_i, \hat{\mathbf{y}}) = v \tag{6}$$

for each $i$ such that $\hat{x}_i \neq 0$.

PROOF Write $\hat{\mathbf{y}} = \hat{y}_1\mathbf{e}_1 + \cdots + \hat{y}_n\mathbf{e}_n$ in $\mathbb{R}^n$, and note that $v = E(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = v(\hat{\mathbf{x}}) \leq E(\hat{\mathbf{x}}, \mathbf{e}_j)$ for $j = 1, \ldots, n$. So there exist nonnegative numbers $\varepsilon_j$ such that

$$E(\hat{\mathbf{x}}, \mathbf{e}_j) = v + \varepsilon_j \quad (j = 1, \ldots, n)$$

Then

$$\begin{aligned} v = E(\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= E(\hat{\mathbf{x}}, \hat{y}_1\mathbf{e}_1 + \cdots + \hat{y}_n\mathbf{e}_n) \\ &= \sum_{j=1}^{n} \hat{y}_j E(\hat{\mathbf{x}}, \mathbf{e}_j) = \sum_{j=1}^{n} \hat{y}_j (v + \varepsilon_j) \\ &= v + \sum_{j=1}^{n} \hat{y}_j \varepsilon_j \end{aligned}$$

because the $\hat{y}_j$ sum to one. This equality is possible only if $\hat{y}_j = 0$ whenever $\varepsilon_j > 0$. Thus $\hat{\mathbf{y}}$ is a linear combination of the $\mathbf{e}_j$ for which $\varepsilon_j = 0$. For such $j$, $E(\hat{\mathbf{x}}, \mathbf{e}_j) = v$.

Next, observe that $E(\mathbf{e}_i, \hat{\mathbf{y}}) \leq v(\hat{\mathbf{y}}) = E(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for $i = 1, \ldots, m$. So there exist nonnegative numbers $\delta_i$ such that

$$E(\mathbf{e}_i, \hat{\mathbf{y}}) + \delta_i = v \quad (i = 1, \ldots, m) \tag{7}$$

Then, using (5) gives

$$\begin{aligned} v = E(\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= \sum_{i=1}^{m} \hat{x}_i E(\mathbf{e}_i, \hat{\mathbf{y}}) \\ &= \sum_{i=1}^{m} \hat{x}_i (v - \delta_i) = v - \sum_{i=1}^{m} \hat{x}_i \delta_i \end{aligned}$$

since the $\hat{x}_i$ sum to one. This equality is possible only if $\delta_i = 0$ when $\hat{x}_i \neq 0$. By (7), $E(\mathbf{e}_i, \hat{\mathbf{y}}) = v$ for each $i$ such that $\hat{x}_i \neq 0$. ∎

**EXAMPLE 5** The value of the game in Example 4 is $\frac{11}{5}$, attained when $\hat{\mathbf{x}} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$. Use this fact to find an optimal strategy for the column player $C$.

**Solution** The $z$-coordinate of the maximum point $M$ in Fig. 2 is the value of the game, and the $t$-coordinate identifies the optimal strategy $\mathbf{x}(\frac{2}{5}) = \hat{\mathbf{x}}$. Recall that the $z$-coordinates of the lines in Fig. 2 represent $E(\mathbf{x}(t), \mathbf{e}_j)$ for $j = 1, \ldots, 4$. Only the lines for columns 1 and 3 pass through the point $M$, which means that

$$E(\hat{\mathbf{x}}, \mathbf{e}_1) = \tfrac{11}{5} \qquad \text{and} \qquad E(\hat{\mathbf{x}}, \mathbf{e}_3) = \tfrac{11}{5}$$

while $E(\hat{\mathbf{x}}, \mathbf{e}_2)$ and $E(\hat{\mathbf{x}}, \mathbf{e}_4)$ are greater than $\frac{11}{5}$. By Theorem 4, the optimal column strategy $\hat{\mathbf{y}}$ for $C$ is a linear combination of the pure strategies $\mathbf{e}_1$ and $\mathbf{e}_3$ in $\mathbb{R}^2$. Thus, $\hat{\mathbf{y}}$ has the form

$$\hat{\mathbf{y}} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ c_3 \\ 0 \end{bmatrix}$$

where $c_1 + c_3 = 1$. Since both coordinates of the optimal $\hat{\mathbf{x}}$ are nonzero, Theorem 4 shows that $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{11}{5}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{11}{5}$. Each condition, by itself, determines $\hat{\mathbf{y}}$. For example,

$$E(\mathbf{e}_1, \hat{\mathbf{y}}) = \mathbf{e}_1^T A \hat{\mathbf{y}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 & 2 \\ 1 & 5 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \\ c_3 \\ 0 \end{bmatrix} = 4c_1 + c_3 = \tfrac{11}{5}$$

Substitute $c_3 = 1 - c_1$, and obtain $4c_1 + (1 - c_1) = \frac{11}{5}$, $c_1 = \frac{2}{5}$ and $c_3 = \frac{3}{5}$. The optimal strategy for $C$ is $\hat{\mathbf{y}} = \begin{bmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \\ 0 \end{bmatrix}$.

## Reducing the Size of a Game

The general $m \times n$ matrix game can be solved using linear programming techniques, and Section 9.4 describes one method for doing this. In some cases, however, a matrix game can be reduced to a "smaller" game whose matrix has only two rows. If this happens, the graphical method of Examples 4 and 5 is available.

**DEFINITION** Given $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$, with entries $a_i$ and $b_i$, respectively, vector $\mathbf{a}$ is said to **dominate** vector $\mathbf{b}$ if $a_i \geq b_i$ for all $i = 1, \ldots, n$ and $a_i > b_i$ for at least one $i$. If $\mathbf{a}$ dominates $\mathbf{b}$, then $\mathbf{b}$ is said to be **recessive** to $\mathbf{a}$.

Suppose that in the matrix game $A$, row $r$ dominates row $s$. This means that for $R$ the pure strategy of choosing row $r$ is at least as good as the pure strategy of choosing row $s$, no matter what $C$ may choose, and for some choice by $C$, $r$ is better than $s$. It follows that the recessive row $s$ (the "smaller" one) can be ignored by $R$ without hurting $R$'s expected payoff. A similar analysis applies to the columns of $A$, in which case the dominating "larger" column is ignored. These observations are summarized in the following theorem.

**THEOREM 5** Let $A$ be an $m \times n$ matrix game. If row $s$ in the matrix $A$ is recessive to some other row, then let $A_1$ be the $(m-1) \times n$ matrix obtained by deleting row $s$ from $A$. Similarly, if column $t$ of matrix $A$ dominates some other column, let $A_2$ be the $m \times (n-1)$ matrix obtained by deleting column $t$ from $A$. In either case, any optimal strategy of the reduced matrix game $A_1$ or $A_2$ will determine an optimal strategy for $A$.

**EXAMPLE 6** Use the process described in Theorem 5 to reduce the following matrix game to a smaller size. Then find the value of the game and optimal strategies for both players in the original game.

$$A = \begin{bmatrix} 7 & 1 & 6 & 7 \\ 8 & 3 & 1 & 0 \\ 4 & 5 & 3 & 3 \end{bmatrix}$$

**Solution** Since the first column dominates the third, player $C$ will never want to use the first pure strategy. So delete column 1 and obtain

$$\begin{bmatrix} * & 1 & 6 & 7 \\ * & 3 & 1 & 0 \\ * & 5 & 3 & 3 \end{bmatrix}$$

In this matrix, row 2 is recessive to row 3. Delete row 2 and obtain

$$\begin{bmatrix} * & 1 & 6 & 7 \\ * & * & * & * \\ * & 5 & 3 & 3 \end{bmatrix}$$

This reduced $2 \times 3$ matrix can be reduced further by dropping the last column, since it dominates column 2. Thus, the original matrix game $A$ has been reduced to

$$B = \begin{bmatrix} 1 & 6 \\ 5 & 3 \end{bmatrix} \quad \text{when } A = \begin{bmatrix} 7 & \mathbf{1} & \mathbf{6} & 7 \\ 8 & 3 & 1 & 0 \\ 4 & \mathbf{5} & \mathbf{3} & 3 \end{bmatrix} \tag{8}$$

and any optimal strategy for $B$ will produce an optimal strategy for $A$, with zeros as entries corresponding to deleted rows or columns.

A quick check of matrix $B$ shows that the game has no saddle point (because 3 is the max of the row minima and 5 is the min of the column maxima). So the graphical solution method is needed. Figure 3 shows the lines corresponding to the two columns of $B$, whose equations are $z = 4t + 1$ and $z = -3t + 6$. They intersect where $t = \frac{5}{7}$; the

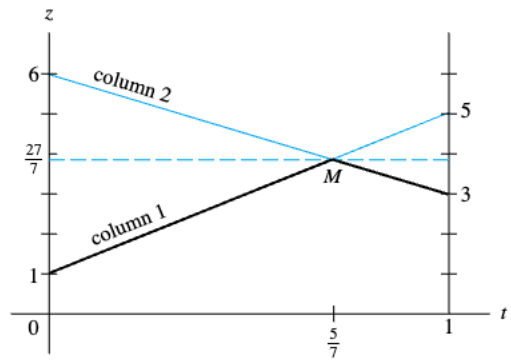



**FIGURE 3**

value of the game is $\frac{27}{7}$, and the optimal row strategy for matrix $B$ is

$$\hat{\mathbf{x}} = \mathbf{x}(\tfrac{5}{7}) = \begin{bmatrix} 1 - \frac{5}{7} \\ \frac{5}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{5}{7} \end{bmatrix}$$

Since the game has no saddle point, the optimal column strategy must be a linear combination of the two pure strategies. Set $\hat{\mathbf{y}} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, and use the second part of Theorem 4 to write

$$\tfrac{27}{7} = E(\mathbf{e}_1, \hat{\mathbf{y}}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 + 6c_2 = (1 - c_2) + 6c_2$$

Solving gives $5c_2 = \frac{20}{7}$, $c_2 = \frac{4}{7}$, and $c_1 = 1 - c_2 = \frac{3}{7}$. Thus $\hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$. As a check, compute $E(\mathbf{e}_2, \hat{\mathbf{y}}) = 5(\frac{3}{7}) + 3(\frac{4}{7}) = \frac{27}{7} = v$.

The final step is to construct the solution for matrix $A$ from the solution for matrix $B$ (given by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ above). Look at the matrices in (8) to see where the extra zeros go. The row and column strategies for $A$ are, respectively,

$$\hat{\mathbf{x}} = \begin{bmatrix} \frac{2}{7} \\ 0 \\ \frac{5}{7} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{3}{7} \\ \frac{4}{7} \\ 0 \end{bmatrix}$$

## PRACTICE PROBLEM

Find the optimal strategies and the value of the matrix game

$$\begin{bmatrix} -3 & 4 & 1 & 3 \\ 2 & 2 & -1 & 0 \\ 1 & 5 & 2 & 3 \end{bmatrix}$$

## 9.1 EXERCISES

In Exercises 1–4, write the payoff matrix for each game.

**1.** Player $R$ has a supply of dimes and quarters. Player $R$ chooses one of the coins, and player $C$ must guess which coin $R$ has chosen. If the guess is correct, $C$ takes the coin. If the guess is incorrect, $C$ gives $R$ an amount equal to $R$'s chosen coin.

**2.** Players $R$ and $C$ each show one, two, or three fingers. If the total number $N$ of fingers shown is even, then $C$ pays $N$ dollars to $R$. If $N$ is odd, $R$ pays $N$ dollars to $C$.

**3.** In the traditional Japanese children's game *janken* (or "stone, scissors, paper"), at a given signal, each of two players shows either no fingers (stone), two fingers (scissors), or all five (paper). Stone beats scissors, scissors beats paper, and paper beats stone. In the case of a tie, there is no payoff. In the case of a win, the winner collects 5 yen. (On December 10, 2004, Fox Sports broadcast the 2004 Rock Paper Scissors World Championships. See www.worldrps.com.)

**4.** Player $R$ has three cards: a red 3, a red 6, and a black 7. Player $C$ has two cards: a red 4 and a black 9. They each show one of their cards. If the cards are the same color, $R$ receives the larger of the two numbers. If the cards are of different colors, $C$ receives the sum of the two numbers.

Find all saddle points for the matrix games in Exercises 5–8.

**5.** $\begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix}$

**6.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$

**7.** $\begin{bmatrix} 5 & 3 & 4 & 3 \\ -2 & 1 & -5 & 2 \\ 4 & 3 & 7 & 3 \end{bmatrix}$

**8.** $\begin{bmatrix} -2 & 4 & 1 & -1 \\ 3 & 5 & 2 & 2 \\ 1 & -3 & 0 & 2 \end{bmatrix}$

**9.** Let $M$ be the matrix game having payoff matrix $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix}$. Find $E(\mathbf{x}, \mathbf{y})$, $v(\mathbf{x})$, and $v(\mathbf{y})$ when $\mathbf{x}$ and $\mathbf{y}$ have the given values.

a. $\mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

b. $\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

**10.** Let $M$ be the matrix game having payoff matrix $\begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & -2 & 0 \\ 1 & -2 & 2 & 1 \end{bmatrix}$. Find $E(\mathbf{x}, \mathbf{y})$, $v(\mathbf{x})$, and $v(\mathbf{y})$ when $\mathbf{x}$ and $\mathbf{y}$ have the given values.

a. $\mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 0 \\ \frac{1}{4} \end{bmatrix}$

b. $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

In Exercises 11–18, find the optimal row and column strategies and the value of each matrix game.

**11.** $\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$

**12.** $\begin{bmatrix} 2 & -2 \\ -3 & 6 \end{bmatrix}$

**13.** $\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

**14.** $\begin{bmatrix} 3 & 5 & 3 & 2 \\ -1 & 9 & 1 & 8 \end{bmatrix}$

**15.** $\begin{bmatrix} 4 & 6 & 2 & 0 \\ 1 & 3 & 2 & 5 \end{bmatrix}$

**16.** $\begin{bmatrix} 5 & -1 & 1 \\ 4 & 2 & 3 \\ -2 & -3 & 1 \end{bmatrix}$

**17.** $\begin{bmatrix} 0 & 1 & -1 & 4 & 3 \\ 1 & -1 & 3 & -1 & -3 \\ 2 & -1 & 4 & 0 & -2 \\ -1 & 0 & -2 & 2 & 1 \end{bmatrix}$

**18.** $\begin{bmatrix} 6 & 4 & 5 & 5 \\ 0 & 4 & 2 & 7 \\ 6 & 3 & 5 & 2 \\ 2 & 5 & 3 & 7 \end{bmatrix}$

**19.** A certain army is engaged in guerrilla warfare. It has two ways of getting supplies to its troops: it can send a convoy up the river road or it can send a convoy overland through the jungle. On a given day, the guerrillas can watch only one of the two roads. If the convoy goes along the river and the guerrillas are there, the convoy will have to turn back and 4 army soldiers will be lost. If the convoy goes overland and encounters the guerrillas, half the supplies will get through, but 7 army soldiers will be lost. Each day a supply convoy travels one of the roads, and if the guerrillas are watching the other road, the convoy gets through with no losses. Set up and solve the following as matrix games, with $R$ being the army.

a. What is the optimal strategy for the army if it wants to maximize the amount of supplies it gets to its troops? What is the optimal strategy for the guerrillas if they want to prevent the most supplies from getting through? If these strategies are followed, what portion of the supplies gets through?

b. What is the optimal strategy for the army if it wants to minimize its casualties? What is the optimal strategy for the guerrillas if they want to inflict maximum losses on the army? If these strategies are followed, what portion of the supplies gets through?

**20.** Suppose in Exercise 19 that whenever the convoy goes overland two soldiers are lost to land mines, whether they are attacked or not. Thus, if the army encounters the guerrillas, there will be 9 casualties. If it does not encounter the guerrillas, there will be 2 casualties.

a. Find the optimal strategies for the army and the guerrillas with respect to the number of army casualties.

b. In part (a), what is the "value" of the game? What does this represent in terms of the troops?

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

**21.** a. The payoff matrix for a matrix game indicates what $R$ wins for each combination of moves.

b. With a pure strategy, a player makes the same choice each time the game is played.

c. The value $v(\mathbf{x})$ of a particular strategy $\mathbf{x}$ to player $R$ is equal to the maximum of the inner product of $\mathbf{x}$ with each of the columns of the payoff matrix.

d. The Minimax Theorem says that every matrix game has a solution.

e. If row $s$ is recessive to some other row in payoff matrix $A$, then row $s$ will not be used (that is, have probability zero) in some optimal strategy for (row) player $R$.

**22.** a. If $a_{ij}$ is a saddle point, then $a_{ij}$ is the smallest entry in row $i$ and the largest entry in column $j$.

b. Each pure strategy is an optimal strategy.

c. The value $v_R$ of the game to player $R$ is the maximum of the values of the various possible strategies for $R$.

d. The Fundamental Theorem for Matrix Games shows how to solve every matrix game.

e. If column $t$ dominates some other column in a payoff matrix $A$, then column $t$ will not be used (that is, have probability zero) in some optimal strategy for (column) player $C$.

**23.** Find the optimal strategies and the value of the game in Example 2.

**24.** Bill and Wayne are playing a game in which each player has a choice of two colors: red or blue. The payoff matrix with Bill as the row player is given below.

$$\begin{array}{c c} & \begin{array}{cc} \text{red} & \text{blue} \end{array} \\ \begin{array}{c} \text{red} \\ \text{blue} \end{array} & \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \end{array}$$

For example, this means that if both people choose red, then Bill pays Wayne one unit.

a. Using the same payoffs for Bill and Wayne, write the matrix that shows the winnings with Wayne as the row player.

b. If $A$ is the matrix with Bill as the row player, write your answer to (a) in terms of $A$.

**25.** Consider the matrix game $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $A$ has no saddle point.

a. Find a formula for the optimal strategies $\hat{\mathbf{x}}$ for $R$ and $\hat{\mathbf{y}}$ for $C$. What is the value of the game?

b. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and let $\alpha$ and $\beta$ be real numbers with $\alpha \neq 0$. Use your answer in part (a) to show that the optimal strategies for the matrix game $B = \alpha A + \beta J$ are the same as for $A$. In particular, note that the optimal strategies for $A$ and $A + \beta J$ are the same.

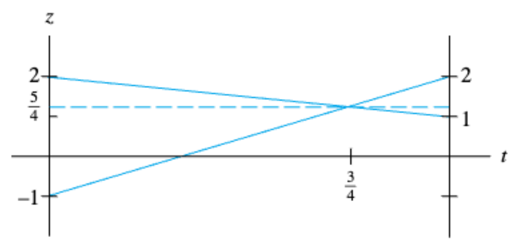
**26.** Let $A$ be a matrix game having value $v$. Find an example to show that $E(\mathbf{x}, \mathbf{y}) = v$ does not necessarily imply that $\mathbf{x}$ and $\mathbf{y}$ are optimal strategies.

## Solution to Practice Problem

The first row is recessive to the third row, so the first row may be eliminated. The second and fourth columns dominate the first and third columns, respectively. Deletion of the second and fourth columns leaves the matrix $B$:

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{when } A = \begin{bmatrix} -3 & 4 & 1 & 3 \\ 2 & 2 & -1 & 0 \\ 1 & 5 & 2 & 3 \end{bmatrix}$$

The game for $B$ has no saddle point, but a graphical analysis will work. The two columns of $B$ determine the two lines shown below, whose equations are $2\cdot(1-t)+1\cdot t$ and $z = -1\cdot(1-t)+2\cdot t$.




These lines intersect at the point $\left(\frac{3}{4}, \frac{5}{4}\right)$. The value of the game is $\frac{5}{4}$, and the optimal row strategy for the matrix game $B$ is

$$\hat{\mathbf{x}} = \mathbf{x}\left(\tfrac{3}{4}\right) = \begin{bmatrix} 1-\frac{3}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

By Theorem 4, the optimal column strategy, $\hat{\mathbf{y}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, satisfies two equations $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{5}{4}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{5}{4}$, because $\hat{\mathbf{x}}$ is a linear combination of both $\mathbf{e}_1$ and $\mathbf{e}_2$. Each of these equations determines $\hat{\mathbf{y}}$. For example,

$$\frac{5}{4} = E(\mathbf{e}_1, \hat{\mathbf{y}}) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 2c_1 - c_2 = 2c_1 - (1-c_1) = 3c_1 - 1$$

Thus, $c_1 = \frac{3}{4}$, and so $c_2 = \frac{1}{4}$, and $\hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$. As a check, compute

$$E(\mathbf{e}_2, \hat{\mathbf{y}}) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{5}{4}$$

This solves the game for $B$. The optimal row strategy $\hat{\mathbf{x}}$ for $A$ needs a 0 in the first entry (for the deleted first row); the optimal column strategy $\hat{\mathbf{y}}$ for $A$ needs 0's in entries 2 and 4 (for the two deleted columns). Thus

$$\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{y}} = \begin{bmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{4} \\ 0 \end{bmatrix}$$

## 9.2 LINEAR PROGRAMMING—GEOMETRIC METHOD

Since the 1950s, the variety and size of industrial linear programming problems have grown along with the dramatic increase in computing power. Still, at their core, linear programming problems have a concise mathematical description, discussed in this section. The final example in the section presents a geometric view of linear programming that is important for visualizing the algebraic approach needed for larger problems.

Generally speaking, a linear programming problem involves a system of linear inequalities in variables $x_1, \ldots, x_n$ and a linear functional $f$ from $\mathbb{R}^n$ into $\mathbb{R}$. The system typically has many free variables, and the problem is to find a solution $\mathbf{x}$ that maximizes or minimizes $f(\mathbf{x})$.

**EXAMPLE 1** The Shady-Lane grass seed company blends two types of seed mixtures, EverGreen and QuickGreen. Each bag of EverGreen contains 3 pounds of fescue seed, 1 pound of rye seed, and 1 pound of bluegrass. Each bag of QuickGreen contains 2 pounds of fescue, 2 pounds of rye, and 1 pound of bluegrass. The company has 1200 pounds of fescue seed, 800 pounds of rye seed, and 450 pounds of bluegrass available to put into its mixtures. The company makes a profit of \$2 on each bag of EverGreen and \$3 on each bag of QuickGreen that it produces. Set up the mathematical problem that determines the number of bags of each mixture that Shady-Lane should make in order to maximize its profit.

**Solution** The phrase "maximize . . . profit" identifies the goal or objective of the problem. The first step, then, is to create a formula for the profit. Begin by naming the quantities that can vary. Let $x_1$ be the number of bags of EverGreen and $x_2$ the number of bags of QuickGreen that are produced. Since the profit on each bag of EverGreen is \$2 and the profit on each bag of QuickGreen is \$3, the total profit (in dollars) is

$$2x_1 + 3x_2 \qquad \text{(profit function)}$$

The next step is to write inequalities or equalities that $x_1$ and $x_2$ must satisfy, one for each of the ingredients that are in limited supply. Notice that each bag of EverGreen requires 3 pounds of fescue seed and each bag of QuickGreen requires 2 pounds of fescue seed. So the total amount of fescue required is $3x_1 + 2x_2$ pounds. Since only 1200 pounds are available, $x_1$ and $x_2$ must satisfy

$$3x_1 + 2x_2 \leq 1200 \qquad \text{(fescue)}$$

Similarly, EverGreen needs 1 pound of rye per bag, QuickGreen needs 2 pounds per bag, and only 800 pounds of rye are available. Thus, the total amount of rye seed required is $x_1 + 2x_2$, and $x_1$ and $x_2$ must satisfy

$$x_1 + 2x_2 \leq 800 \qquad \text{(rye)}$$

As for the bluegrass, EverGreen requires 1 pound per bag and QuickGreen requires 1 pound per bag. Since 450 pounds are available,

$$x_1 + x_2 \leq 450 \qquad \text{(bluegrass)}$$

Of course, $x_1$ and $x_2$ cannot be negative, so $x_1$ and $x_2$ must also satisfy

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

The problem is summarized mathematically as

$$\begin{array}{lll} \text{Maximize} & 2x_1 + 3x_2 & \text{(profit function)} \\ \text{subject to} & 3x_1 + 2x_2 \leq 1200 & \text{(fescue)} \\ & x_1 + 2x_2 \leq 800 & \text{(rye)} \\ & x_1 + \phantom{2}x_2 \leq 450 & \text{(bluegrass)} \\ \text{and } x_1 \geq 0, x_2 \geq 0. & & \end{array}$$

**EXAMPLE 2** An oil refining company has two refineries that produce three grades of unleaded gasoline. Each day refinery A produces 12,000 gallons of regular, 4,000 gallons of premium, and 1,000 gallons of super gas, at a cost of \$3,500. Each day refinery B produces 4,000 gallons of regular, 4,000 gallons of premium, and 5,000 gallons of super gas, at a cost of \$3,000. An order is received for 48,000 gallons of regular, 32,000 gallons of premium, and 20,000 gallons of super gas. Set up a mathematical problem that determines the number of days each refinery should operate in order to fill the order at the least cost.

**Solution** Suppose that refinery A operates $x_1$ days and refinery B operates $x_2$ days. The cost of doing this is $3{,}500x_1 + 3{,}000x_2$ dollars. The problem is to find a production schedule $(x_1, x_2)$ that minimizes this cost and also ensures that the required gasoline is produced.

Since refinery A produces 12,000 gallons of regular gas each day and refinery B produces 4,000 gallons of regular each day, the total produced is $12{,}000x_1 + 4{,}000x_2$. The total should be at least 48,000 gallons. That is,

$$12{,}000x_1 + 4{,}000x_2 \geq 48{,}000$$

Similarly, for the premium gas,

$$4{,}000x_1 + 4{,}000x_2 \geq 32{,}000$$

and, for the super,

$$1{,}000x_1 + 5{,}000x_2 \geq 20{,}000$$

As in Example 1, $x_1$ and $x_2$ cannot be negative, so $x_1 \geq 0$ and $x_2 \geq 0$.

The problem is summarized mathematically as

$$\begin{array}{lll} \text{Minimize} & 3{,}500x_1 + 3{,}000x_2 & \text{(cost function)} \\ \text{subject to} & 12{,}000x_1 + 4{,}000x_2 \geq 48{,}000 & \text{(regular gas)} \\ & 4{,}000x_1 + 4{,}000x_2 \geq 32{,}000 & \text{(premium)} \\ & 1{,}000x_1 + 5{,}000x_2 \geq 20{,}000 & \text{(super)} \\ \text{and } x_1 \geq 0, x_2 \geq 0. & & \end{array}$$

The examples show how a linear programming problem involves finding the maximum (or minimum) of a linear function, called the **objective function**, subject to certain linear constraints. In many situations, the constraints take the form of linear inequalities and the variables are restricted to nonnegative values. Here is a precise statement of the so-called canonical form of a linear programming problem.

**DEFINITION**

Given $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ in $\mathbb{R}^m$, $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$, and an $m \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, the **canonical linear programming problem** is the following:

Find an $n$-tuple $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ to maximize

$$f(x_1, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}$$

and

$$x_j \ge 0 \quad \text{for } j = 1, \dots, n$$

This may be restated in vector-matrix notation as follows:

$$\text{Maximize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \tag{1}$$

$$\text{subject to the constraints } A\mathbf{x} \le \mathbf{b} \tag{2}$$

$$\text{and } \mathbf{x} \ge \mathbf{0} \tag{3}$$

where an inequality between two vectors applies to each of their coordinates.

Any vector $\mathbf{x}$ that satisfies (2) and (3) is called a **feasible solution**, and the set of all feasible solutions, denoted by $\mathscr{F}$, is called the **feasible set**. A vector $\overline{\mathbf{x}}$ in $\mathscr{F}$ is an **optimal solution** if $f(\overline{\mathbf{x}}) = \max_{\mathbf{x} \in \mathscr{F}} f(\mathbf{x})$.

The canonical statement of the problem is really not as restrictive as it might seem. To minimize a function $h(\mathbf{x})$, replace it with the problem of maximizing the function $-h(\mathbf{x})$. A constraint inequality of the sort

$$a_{i1}x_1 + \cdots + a_{in}x_n \ge b_i$$

can be replaced by

$$-a_{i1}x_1 - \cdots - a_{in}x_n \le -b_i$$

An equality constraint

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$$

can be replaced by two inequalities

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\le b_i \\ -a_{i1}x_1 - \cdots - a_{in}x_n &\le -b_i \end{aligned}$$

With an arbitrary canonical linear programming problem, two things can go wrong. If the constraint inequalities are inconsistent, then $\mathscr{F}$ is the empty set. If the objective function takes on arbitrarily large values in $\mathscr{F}$, then the desired maximum does not exist. In the former case, the problem is said to be **infeasible**; in the latter case, the problem is called **unbounded**.

**EXAMPLE 3** The problem

$$\begin{array}{ll} \text{Maximize} & 5x \\ \text{subject to} & x \le 3 \\ & -x \le -4 \\ & x \ge 0 \end{array}$$

is infeasible, since there is no $x$ such that $x \le 3$ and $x \ge 4$.

**EXAMPLE 4** The problem

$$\begin{array}{ll} \text{Maximize} & 5x \\ \text{subject to} & -x \le 3 \\ & x \ge 0 \end{array}$$

is unbounded. The values of $5x$ may be arbitrarily large, as $x$ is only required to satisfy $x \ge 0$ (and $x \ge -3$).

Fortunately, these are the only two things that can go wrong.

**THEOREM 6** If the feasible set $\mathscr{F}$ is nonempty and if the objective function is bounded above on $\mathscr{F}$, then the canonical linear programming problem has at least one optimal solution. Furthermore, at least one of the optimal solutions is an extreme point of $\mathscr{F}$.[1]

Theorem 6 describes when an optimal solution exists, and it suggests a possible technique for finding one. That is, evaluate the objective function at each of the extreme

[1]The feasible set is the solution of a system of linear inequalities. Geometrically, this corresponds to the intersection of a finite number of (closed) half-spaces, sometimes called a polyhedral set. Intuitively, the extreme points correspond to the "corner points," or vertices, of this polyhedral set. The notion of an extreme point is discussed more fully in Section 8.5.

A proof of Theorem 6 is in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), p. 171.

points of $\mathscr{F}$ and select the point that gives the largest value. This works well in simple cases such as the next two examples. The geometric approach is limited to two or three dimensions, but it provides an important visualization of the nature of the solution set and how the objective function interacts with the feasible set to identify extreme points.

**EXAMPLE 5** Maximize $f(x_1, x_2) = 2x_1 + 3x_2$

$$\begin{aligned} \text{subject to} \quad x_1 \qquad\quad &\leq 30 \\ x_2 &\leq 20 \\ x_1 + 2x_2 &\leq 54 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$.

**Solution** Figure 1 shows the shaded pentagonal feasible set, obtained by graphing each of the constraint inequalities. (For simplicity, points in this section are displayed as ordered pairs or triples.) There are five extreme points, corresponding to the five vertices of the feasible set. They are found by solving the appropriate pairs of linear equations. For example, the extreme point (14, 20) is found by solving the linear system $x_1 + 2x_2 = 54$ and $x_2 = 20$. The table below shows the value of the objective function at each extreme point. Evidently, the maximum is 96 at $x_1 = 30$ and $x_2 = 12$.

| $(x_1, x_2)$ | $2x_1 + 3x_2$ |
|---|---|
| (0, 0) | 0 |
| (30, 0) | 60 |
| (30, 12) | 96 ← |
| (14, 20) | 88 |
| (0, 20) | 60 |

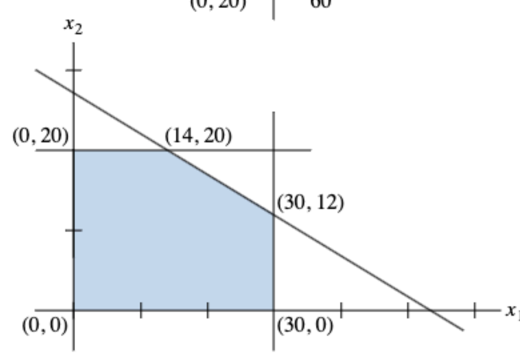



FIGURE 1

Another geometric technique that can be used when the problem involves two variables is to graph several *level lines* for the objective function. These are parallel lines, and the objective function has a constant value on each line. (See Fig. 2.) The values of the objective function $f(x_1, x_2)$ increase as $(x_1, x_2)$ moves from left to right. The level line farthest to the right that still intersects the feasible set is the line through the vertex (30, 12). Thus, the point (30, 12) yields the maximum value of $f(x_1, x_2)$ over the feasible set.

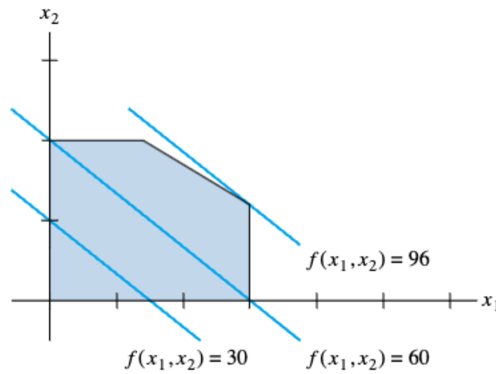



**FIGURE 2**

**EXAMPLE 6** Maximize $f(x_1, x_2, x_3) = 2x_1 + 3x_2 + 4x_3$

$$\begin{aligned} \text{subject to} \quad & x_1 + \phantom{2}x_2 + \phantom{4}x_3 \le 50 \\ & x_1 + 2x_2 + 4x_3 \le 80 \\ \text{and } x_1 \ge 0,\ & x_2 \ge 0,\ x_3 \ge 0. \end{aligned}$$

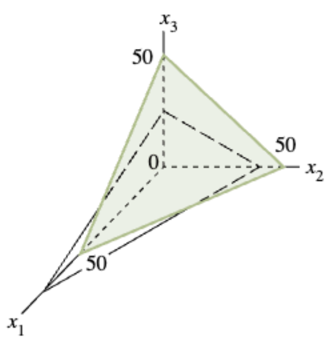



**FIGURE 3**

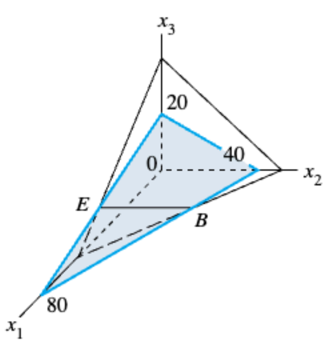



**FIGURE 4**

**Solution** Each of the five inequalities above determines a "half-space" in $\mathbb{R}^3$—a plane together with all points on one side of the plane. The feasible set of this linear programming problem is the intersection of these half-spaces, which is a convex set in the first octant of $\mathbb{R}^3$.

When the first inequality is changed to an equality, the graph is a plane that intercepts each coordinate axis 50 units from the origin and determines the equilateral triangular region shown in Fig. 3. Since $(0, 0, 0)$ satisfies the inequality, so do all the other points "below" the plane. In a similar fashion, the second (in)equality determines a triangular region on a plane (shown in Fig. 4) that passes somewhat closer to the origin. The two planes intersect in a line that contains segment $EB$.

The quadrilateral surface $BCDE$ forms a boundary of the feasible set, because it is below the equilateral triangular region. Beyond $EB$, however, the two planes change position relative to the origin, so the planar region $ABE$ forms another bounding surface for the feasible set. The vertices of the feasible set are the points $A$, $B$, $C$, $D$, $E$, and 0 (the origin). See Fig. 5, which has all sides of the feasible set shaded except the large "top" piece. To find the coordinates of $B$, solve the system

$$\begin{cases} x_1 + \phantom{2}x_2 + \phantom{4}x_3 = 50 \\ x_1 + 2x_2 + 4x_3 = 80 \\ \qquad\qquad\quad x_3 = \phantom{8}0 \end{cases} \Rightarrow \begin{cases} x_1 + \phantom{2}x_2 = 50 \\ x_1 + 2x_2 = 80 \end{cases}$$

Obtain $x_2 = 30$, and find that $B$ is $(20, 30, 0)$. For $E$, solve

$$\begin{cases} x_1 + \phantom{2}x_2 + \phantom{4}x_3 = 50 \\ x_1 + 2x_2 + 4x_3 = 80 \\ \qquad\; x_2 \qquad\;\; = \phantom{8}0 \end{cases} \Rightarrow \begin{cases} x_1 + \phantom{4}x_3 = 50 \\ x_1 + 4x_3 = 80 \end{cases}$$

Obtain $x_3 = 10$, and find that $E = (40, 0, 10)$.

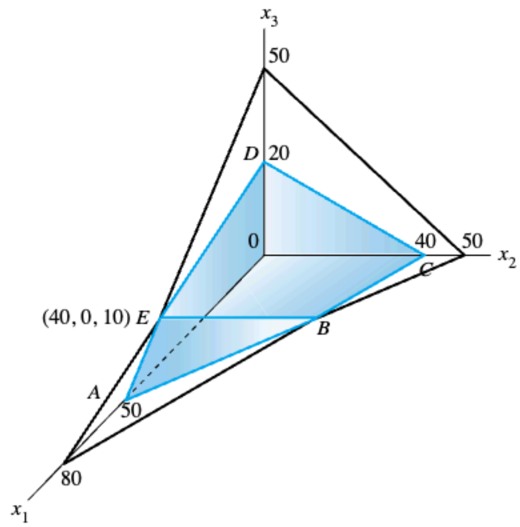



FIGURE 5

Now that the feasible set and its extreme points are clearly seen, the next step is to examine the objective function $f(x_1, x_2, x_3) = 2x_1 + 3x_2 + 4x_3$. The sets on which $f$ is constant are planes, rather than lines, all having $(2, 3, 4)$ as a normal vector to the plane. This normal vector has a direction different from the normal vectors $(1, 1, 1)$ and $(1, 2, 4)$ to the two faces $BCDE$ and $ABE$. So the level sets of $f$ are not parallel to any of the bounding surfaces of the feasible set. Figure 6 shows just the feasible set and a level set on which $f$ has the value 120. This plane passes through $C$, $E$, and the point $(30, 20, 0)$ on the edge of the feasible set between $A$ and $B$, which shows that the vertex $B$ is "above" this level plane. In fact, $f(20, 30, 0) = 130$. Thus the unique solution of the linear programming problem is at $B = (20, 30, 0)$.

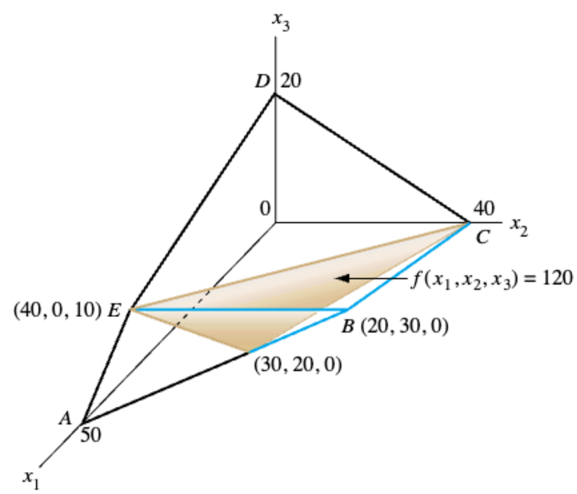



FIGURE 6

## PRACTICE PROBLEMS

1. Consider the following problem:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 - 2x_2 \geq -8 \\ & 3x_1 + 2x_2 \leq 24 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0, x_2 \geq 0.} \end{array}$$

Write this problem in the form of a canonical linear programming problem: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$. Specify $A$, $\mathbf{b}$, and $\mathbf{c}$.

2. Graph the feasible set for Practice Problem 1.
3. Find the extreme points of the feasible set in Practice Problem 2.
4. Use the answer to Practice Problem 3 to find the solution to the linear programming problem in Practice Problem 1.

## 9.2 EXERCISES

1. Betty plans to invest a total of \$12,000 in mutual funds, certificates of deposit (CDs), and a high yield savings account. Because of the risk involved in mutual funds, she wants to invest no more in mutual funds than the sum of her CDs and savings. She also wants the amount in savings to be at least half the amount in CD's. Her expected returns are 11% on the mutual funds, 8% on the CD's and 6% on savings. How much money should Betty invest in each area in order to have the largest return on her investments? Set this up as a linear programming problem in the following form: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Do *not* find the solution.

2. A dog breeder decides to feed his dogs a combination of two dog foods: Pixie Power and Misty Might. He wants the dogs to receive four nutritional factors each month. The amounts of these factors (a, b, c, and d) contained in 1 bag of each dog food are shown in the following chart, together with the total amounts needed.

| | a | b | c | d |
|---|---|---|---|---|
| Pixie Power | 3 | 2 | 1 | 2 |
| Misty Might | 2 | 4 | 3 | 1 |
| Needed | 28 | 30 | 20 | 25 |

The costs per bag are \$50 for Pixie Power and \$40 for Misty Might. How many bags of each dog food should be blended to meet the nutritional requirements at the lowest cost? Set this up as a linear programming problem in the following form: Minimize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Do *not* find the solution.

In Exercises 3–6, find vectors $\mathbf{b}$ and $\mathbf{c}$ and matrix $A$ so that each problem is set up as a canonical linear programming problem: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Do *not* find the solution.

3. $$\begin{array}{ll} \text{Maximize} & 3x_1 + 4x_2 - 2x_3 \\ \text{subject to} & x_1 + 2x_2 \leq 20 \\ & -3x_2 + 5x_3 \geq 10 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.} \end{array}$$

4. $$\begin{array}{ll} \text{Maximize} & 3x_1 + x_2 + 5x_3 \\ \text{subject to} & 5x_1 + 7x_2 + x_3 \leq 25 \\ & 2x_1 + 3x_2 + 4x_3 = 40 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.} \end{array}$$

5. $$\begin{array}{ll} \text{Minimize} & 7x_1 - 3x_2 + x_3 \\ \text{subject to} & x_1 - 4x_2 \geq 35 \\ & x_2 - 2x_3 = 20 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.} \end{array}$$

6. $$\begin{array}{ll} \text{Minimize} & x_1 + 5x_2 - 2x_3 \\ \text{subject to} & 2x_1 + x_2 + 4x_3 \leq 27 \\ & x_1 - 6x_2 + 3x_3 \geq 40 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.} \end{array}$$

In Exercises 7–10, solve the linear programming problems.

**7.** Maximize $80x_1 + 65x_2$
subject to
$$\begin{aligned} 2x_1 + \phantom{3}x_2 &\le 32 \\ x_1 + \phantom{3}x_2 &\le 18 \\ x_1 + 3x_2 &\le 24 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**8.** Minimize $5x_1 + 3x_2$
subject to
$$\begin{aligned} 2x_1 + 5x_2 &\ge 10 \\ 3x_1 + \phantom{5}x_2 &\ge 6 \\ x_1 + 7x_2 &\ge 7 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**9.** Maximize $2x_1 + 7x_2$
subject to
$$\begin{aligned} -2x_1 + \phantom{2}x_2 &\le -4 \\ x_1 - 2x_2 &\le -4 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**10.** Maximize $5x_1 + 12x_2$
subject to
$$\begin{aligned} x_1 - \phantom{2}x_2 &\le 3 \\ -x_1 + 2x_2 &\le -4 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

**11.** a. In a canonical linear programming problem, a nonnegative vector $\mathbf{x}$ is a feasible solution if it satisfies $A\mathbf{x} \le \mathbf{b}$.

b. A vector $\bar{\mathbf{x}}$ is an optimal solution of a canonical linear programming problem if $f(\bar{\mathbf{x}})$ is equal to the maximum value of the linear functional $f$ on the feasible set $\mathscr{F}$.

**12.** a. If a canonical linear programming problem does not have an optimal solution, then either the objective function is not bounded on the feasible set $\mathscr{F}$ or $\mathscr{F}$ is the empty set.

b. If $\bar{\mathbf{x}}$ is an optimal solution of a canonical linear programming problem, then $\bar{\mathbf{x}}$ is an extreme point of the feasible set.

**13.** Solve the linear programming problem in Example 1.

**14.** Solve the linear programming problem in Example 2.

**15.** The Benri Company manufactures two kinds of kitchen gadgets: invertible widgets and collapsible whammies. The production process is divided into three departments: fabricating, packing, and shipping. The hours of labor required for each operation and the hours available in each department each day are shown below.

| | Widgets | Whammies | Time available |
|---|---|---|---|
| Fabricating | 5.0 | 2.0 | 200 |
| Packing | .2 | .4 | 16 |
| Shipping | .2 | .2 | 10 |

Suppose that the profit on each widget is \$20 and the profit on each whammy is \$26. How many widgets and how many whammies should be made each day to maximize the company's profit?

Exercises 16–19 use the notion of a convex set, studied in Section 8.3. A set $S$ in $\mathbb{R}^n$ is convex if, for each $\mathbf{p}$ and $\mathbf{q}$ in $S$, the line segment between $\mathbf{p}$ and $\mathbf{q}$ lies in $S$. [This line segment is the set of points of the form $(1-t)\mathbf{p} + t\mathbf{q}$ for $0 \le t \le 1$.]

**16.** Let $\mathscr{F}$ be the feasible set of all solutions $\mathbf{x}$ of a linear programming problem $A\mathbf{x} \le \mathbf{b}$ with $\mathbf{x} \ge \mathbf{0}$. Assume that $\mathscr{F}$ is nonempty. Show that $\mathscr{F}$ is a convex set in $\mathbb{R}^n$. [*Hint*: Consider points $\mathbf{p}$ and $\mathbf{q}$ in $\mathscr{F}$ and $t$ such that $0 \le t \le 1$. Show that $(1-t)\mathbf{p} + t\mathbf{q}$ is in $\mathscr{F}$.]

**17.** Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The inequality $ax_1 + bx_2 \le c$ for some real number $c$ may be written as $\mathbf{v}^T\mathbf{x} < c$. The set $S$ of all $\mathbf{x}$ that satisfy this inequality is called a **closed half-space** of $\mathbb{R}^2$. Show that $S$ is convex. [See the Hint for Exercise 16.]

**18.** The feasible set in Example 5 is the intersection of five closed half-spaces. By Exercise 17, these half-spaces are convex sets. Show that the intersection of any five convex sets $S_1, \ldots, S_5$ in $\mathbb{R}^n$ is a convex set.

**19.** If $c$ is in $\mathbb{R}^n$ and if $f$ is defined on $\mathbb{R}^n$ by $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$, then $f$ is called a linear functional, and for any real number $d$, $\{\mathbf{x}\colon f(\mathbf{x}) = d\}$ is called a level set of $f$. (See level sets in Fig. 2 of Example 5.) Show that any such level set is convex.

## Solutions to Practice Problems

**1.** The first inequality has the wrong direction, so multiply by $-1$. This gives the following problem:

Maximize $2x_1 + x_2$
subject to
$$\begin{aligned} -x_1 + 2x_2 &\le 8 \\ 3x_1 + 2x_2 &\le 24 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

This corresponds to the canonical form

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

when

$$\mathbf{b} = \begin{bmatrix} 8 \\ 24 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$$

**2.** To graph the inequality $-x_1 + 2x_2 \leq 8$, first graph the corresponding equality $-x_1 + 2x_2 = 8$. The intercepts are easy to find: $(0, 4)$ and $(-8, 0)$. Figure 7 shows the straight line through these two points.

The graph of the inequality consists of this line together with all points on one side of the line. To determine which side, pick a point not on the line to see if its coordinates satisfy the inequality. For example, try the origin, $(0, 0)$. The inequality

$$-(0) + 2(0) \leq 8$$

is a true statement. Thus the origin and all other points below the line satisfy the inequality. As another example, substituting the coordinates of the point $(0, 8)$ into the inequality produces a false statement:

$$-(0) + 2(8) \leq 8$$

Thus $(0, 8)$ and all other points above the line do not satisfy the inequality. Figure 7 shows small arrows beneath the graph of $-x_1 + 2x_2 = 8$, to indicate which side is to be included.

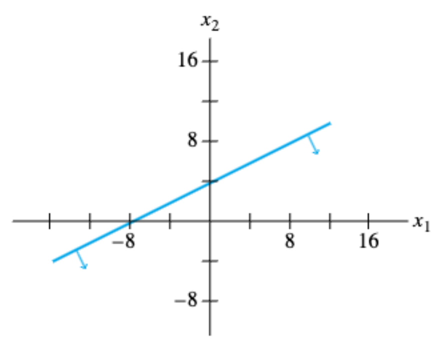



**FIGURE 7** Graph of $-x_1 + 2x_2 \leq 8$.

For the inequality

$$3x_1 + 2x_2 \leq 24$$

draw the graph of $3x_1 + 2x_2 = 24$, using the intercepts $(0, 12)$ and $(8, 0)$ or two other convenient points. Since $(0, 0)$ satisfies the inequality, the feasible set is on the side of the line containing the origin. The inequality $x_1 \geq 0$ gives the right half-plane, and the inequality $x_2 \geq 0$ gives the upper half-plane. All of these are graphed in Fig. 8, and their common solution is the shaded feasible set.

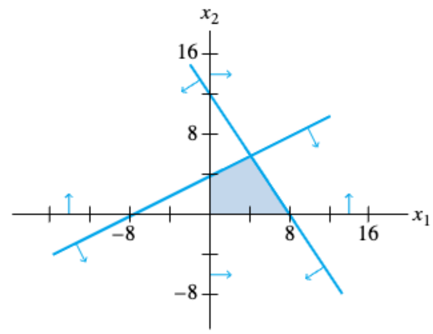



**FIGURE 8** Graph of the feasible set.

3. There are four extreme points in the feasible set:

   1. The origin: (0, 0)
   2. The $x_2$-intercept of the first inequality: (0, 4)
   3. The $x_1$-intercept of the second inequality: (8, 0)
   4. The intersection of the two inequalities.

   For the fourth extreme point, solve the system of equations $-x_1 + 2x_2 = 8$ and $3x_1 + 2x_2 = 24$ to obtain $x_1 = 4$ and $x_2 = 6$.

4. To find the maximum value of the objective function $2x_1 + x_2$, evaluate it at each of the four extreme points of the feasible set.

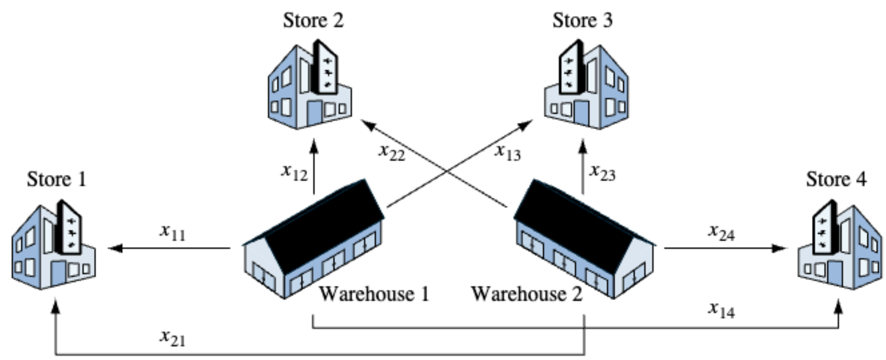
| | $2x_1 + x_2$ |
|---|---|
| (0, 0) | $2(0) + 1(0) = 0$ |
| (0, 4) | $2(0) + 1(4) = 4$ |
| (8, 0) | $2(8) + 1(0) = 16$ |
| (4, 6) | $2(4) + 1(6) = 14$ |

The maximum value is 16, attained when $x_1 = 8$ and $x_2 = 0$.

## 9.3 LINEAR PROGRAMMING—SIMPLEX METHOD

Transportation problems played an important role in the early days of linear programming, including the Berlin Airlift described in this chapter's Introductory Example. They are even more important today. The first example is simple, but it suggests how a problem of this type could involve hundreds, if not thousands, of variables and equations.

**EXAMPLE 1** A retail sales company has two warehouses and four stores. A particular model of outdoor hot tub is sold at all four stores, and each store has placed an order with company headquarters for a certain number of these hot tubs. Headquarters determines that the warehouses have enough hot tubs and can ship them immediately. The distances from the warehouses to the stores vary, and the cost of transporting a hot tub from a warehouse to a store depends on the distance. The problem is to decide on a shipping schedule that minimizes the total cost of shipping. Let $x_{ij}$ be the number of units (hot tubs) to ship from warehouse $i$ to store $j$.




Let $a_1$ and $a_2$ be the numbers of units available at warehouses 1 and 2, and let $r_1, \dots, r_4$ be the numbers of units requested by the various stores. Then the $x_{ij}$ must satisfy the equations

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} \qquad\qquad\qquad\qquad\qquad\qquad &\le a_1 \\
x_{21} + x_{22} + x_{23} + x_{24} &\le a_2 \\
x_{11} \qquad\qquad\qquad + x_{21} \qquad\qquad\qquad &= r_1 \\
x_{12} \qquad\qquad\qquad + x_{22} \qquad\qquad &= r_2 \\
x_{13} \qquad\qquad\qquad + x_{23} \qquad &= r_3 \\
x_{14} \qquad\qquad\qquad + x_{24} &= r_4
\end{aligned}$$

and $x_{ij} \ge 0$ for $i = 1, 2$ and $j = 1, \dots, 4$. If the cost of shipping one unit from warehouse $i$ to store $j$ is $c_{ij}$, then the problem is to minimize the function

$$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24}$$

subject to the four equalities and ten inequalities listed above.

The *simplex method*, discussed below, can easily handle problems the size of Example 1. To introduce the method, however, this section focuses mainly on the canonical linear programming problem from Section 9.2, in which the objective function must be maximized. Here is an outline of the steps in the simplex method.

1. Select an extreme point $\mathbf{x}$ of the feasible set $\mathcal{F}$.
2. Consider all the edges of $\mathcal{F}$ that join at $\mathbf{x}$. If the objective function $f$ cannot be increased by moving along any of these edges, then $\mathbf{x}$ is an optimal solution.
3. If $f$ can be increased by moving along one or more of the edges, then follow the path that gives the largest increase and move to the extreme point of $\mathcal{F}$ at the opposite end.
4. Repeat the process, beginning at step 2.

Since the value of $f$ increases at each step, the path will not go through the same extreme point twice. Since there are only a finite number of extreme points, this process will end at an optimal solution (if there is one) in a finite number of steps. If the problem is unbounded, then eventually the path will reach an unbounded edge at step 3 along which $f$ increases without bound.

The next five examples concern canonical linear programming problems in which each of the entries in the $m$-tuple $\mathbf{b}$ is *positive*:

$$\begin{aligned}&\text{Maximize } f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}\\ &\text{subject to the constraints } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Here $\mathbf{c}$ and $\mathbf{x}$ are in $\mathbb{R}^n$, $A$ is an $m \times n$ matrix, and $\mathbf{b}$ is in $\mathbb{R}^m$.

The simplex method begins by changing each constraint inequality into an *equality*. This is done by adding one new variable to each inequality. These new variables are not part of the final solution; they appear only in the intermediate calculations.

**DEFINITION** A **slack variable** is a nonnegative variable that is added to the smaller side of an inequality to convert it to an equality.

**EXAMPLE 2** Change the inequality

$$5x_1 + 7x_2 \leq 80$$

into the equality

$$5x_1 + 7x_2 + x_3 = 80$$

by adding the slack variable $x_3$. Note that $x_3 = 80 - (5x_1 + 7x_2) \geq 0$.

If $A$ is $m \times n$, the addition of $m$ slack variables in $A\mathbf{x} \leq \mathbf{b}$ produces a linear system with $m$ equations and $n + m$ variables. A solution to this system is called a **basic solution** if no more than $m$ of the variables are nonzero. As in Section 9.2, a solution to the system is called **feasible** if each variable is nonnegative. Thus, in a **basic feasible solution**, each variable must be nonnegative and at most $m$ of them can be positive. Geometrically, these basic feasible solutions correspond to the extreme points of the feasible set.

**EXAMPLE 3** Find a basic feasible solution for the system

$$\begin{aligned}2x_1 + 3x_2 + 4x_3 &\leq 60\\ 3x_1 + x_2 + 5x_3 &\leq 46\\ x_1 + 2x_2 + x_3 &\leq 50\end{aligned}$$

**Solution** Add slack variables to obtain a system of three *equations*:

$$\begin{aligned}2x_1 + 3x_2 + 4x_3 + x_4 \phantom{+ x_5 + x_6} &= 60\\ 3x_1 + x_2 + 5x_3 \phantom{+ x_4} + x_5 \phantom{+ x_6} &= 46\\ x_1 + 2x_2 + x_3 \phantom{+ x_4 + x_5} + x_6 &= 50\end{aligned} \tag{1}$$

There were only three variables in the original system, so a basic solution of (1) has at most three nonzero values for the variables. The following simple solution is called the basic feasible solution associated with (1):

$$x_1 = x_2 = x_3 = 0, \quad x_4 = 60, \quad x_5 = 46, \quad \text{and} \quad x_6 = 50$$

This solution corresponds to the extreme point $\mathbf{0}$ in the feasible set (in $\mathbb{R}^3$).

It is customary to refer to the nonzero variables $x_4, x_5$, and $x_6$ in system (1) as **basic variables** because each has a coefficient of 1 and occurs in only one equation.[1] The basic variables are said to be "in" the solution of (1). The variables $x_1, x_2$, and $x_3$ are said to be "out" of the solution. In a linear programming problem, this particular solution would probably not be optimal since only the slack variables are nonzero.

A standard procedure in the simplex method is to change the role a variable plays in a solution. For example, although $x_2$ is out of the solution in (1), it can be introduced "into" a solution by using elementary row operations. The goal is to **pivot** on the $x_2$ entry in the third equation of (1) to create a new system in which $x_2$ appears *only* in the third equation.[2]

First, divide the third equation in (1) by the coefficient of $x_2$ to obtain a new third equation:

$$\tfrac{1}{2}x_1 + x_2 + \tfrac{1}{2}x_3 \qquad\qquad + \tfrac{1}{2}x_6 = 25$$

Second, to equations 1 and 2 of (1) add multiples of this new equation that will eliminate $x_2$ from those equations. This produces the system

$$\begin{aligned}
\tfrac{1}{2}x_1 \qquad + \tfrac{5}{2}x_3 + x_4 \qquad - \tfrac{3}{2}x_6 &= -15 \\
\tfrac{5}{2}x_1 \qquad + \tfrac{9}{2}x_3 \qquad + x_5 - \tfrac{1}{2}x_6 &= 21 \\
\tfrac{1}{2}x_1 + x_2 + \tfrac{1}{2}x_3 \qquad\qquad + \tfrac{1}{2}x_6 &= 25
\end{aligned}$$

The basic solution associated with this new system is

$$x_1 = x_3 = x_6 = 0, \quad x_2 = 25, \quad x_4 = -15, \quad x_5 = 21$$

The variable $x_2$ has come into the solution, and the variable $x_6$ has gone out. Unfortunately, this basic solution is not feasible since $x_4 < 0$. This lack of feasibility was caused by an improper choice of a pivot equation. The next paragraph shows how to avoid this problem.

[1]This terminology generalizes that used in Section 1.2, where basic variables also had to correspond to pivot positions in a matrix *echelon* form. Here, the goal is not to solve for basic variables in terms of free variables, but to obtain a particular solution of the system when the nonbasic (free) variables are zero.

[2]To "pivot" on a particular term here means to transform its coefficient into a 1 and then use it to eliminate corresponding terms in *all* the other equations, not just the equations below it, as was done in Section 1.2.

In general, consider the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1k}x_k + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{i1}x_1 + \cdots + a_{ik}x_k + \cdots + a_{in}x_n &= b_i \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mk}x_k + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

and suppose the next step is to bring the variable $x_k$ into the solution by using equation $p$ to pivot on entry $a_{pk}x_k$. The basic solution corresponding to the resulting system will be feasible if the following two conditions are satisfied:

1. The coefficient $a_{pk}$ of $x_k$ must be positive. (When the $p$th equation is divided by $a_{pk}$, the new $b_p$ term must be positive.)
2. The ratio $b_p/a_{pk}$ must be the smallest among all the ratios $b_i/a_{ik}$ for which $a_{ik} > 0$. (This will guarantee that when the $p$th equation is used to eliminate the $x_k$ term from the $i$th equation, the resulting $b_i$ term will be positive.)

**EXAMPLE 4** Determine which row to use as a pivot in order to bring $x_2$ into the solution in Example 3.

**Solution** Compute the ratios $b_i/a_{i2}$:

$$\frac{b_1}{a_{12}} = \frac{60}{3} = 20, \qquad \frac{b_2}{a_{22}} = 46, \qquad \text{and} \qquad \frac{b_3}{a_{32}} = \frac{50}{2} = 25$$

Since the first ratio is the smallest, pivot on the $x_2$ term in the first equation. This produces the system

$$\begin{aligned} \tfrac{2}{3}x_1 + x_2 + \tfrac{4}{3}x_3 + \tfrac{1}{3}x_4 \qquad\qquad &= 20 \\ \tfrac{7}{3}x_1 \qquad + \tfrac{11}{3}x_3 - \tfrac{1}{3}x_4 + x_5 \qquad &= 26 \\ -\tfrac{1}{3}x_1 \qquad - \tfrac{5}{3}x_3 - \tfrac{2}{3}x_4 \qquad + x_6 &= 10 \end{aligned}$$

Now the basic feasible solution is

$$x_1 = x_3 = x_4 = 0, \quad x_2 = 20, \quad x_5 = 26, \quad x_6 = 10$$

A matrix format greatly simplifies calculations of this type. For instance, system (1) in Example 3 is represented by the augmented matrix

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 60 \\ 3 & 1 & 5 & 0 & 1 & 0 & 46 \\ 1 & 2 & 1 & 0 & 0 & 1 & 50 \end{array}\right] \end{array}$$

The variables are used as column labels, with the slack variables in color. Recall that the basic feasible solution associated with this matrix is

$$x_1 = x_2 = x_3 = 0, \quad x_4 = 60, \quad x_5 = 46, \quad x_6 = 50$$

The circled 3 in the $x_2$ column indicates that this entry will be used as a pivot to bring $x_2$ into the solution. (The ratio calculations in Example 4 identified this entry as the appropriate pivot.) Complete row reduction in column 2 produces the new matrix that corresponds to the new system in Example 4:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 20 \\ \frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 26 \\ -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 10 \end{array}\right] \end{array}$$

As in Example 4, the new basic feasible solution is

$$x_1 = x_3 = x_4 = 0, \quad x_2 = 20, \quad x_5 = 26, \quad x_6 = 10$$

The preceding discussion has prepared the way for a full demonstration of the simplex method, based on the constraints in Example 3. At each step, the objective function in Example 5 will drive the choice of which variable to bring into the solution of the system.

**EXAMPLE 5** Maximize $25x_1 + 33x_2 + 18x_3$

$$\begin{aligned} \text{subject to} \quad & 2x_1 + 3x_2 + 4x_3 \le 60 \\ & 3x_1 + \phantom{3}x_2 + 5x_3 \le 46 \\ & \phantom{3}x_1 + 2x_2 + \phantom{4}x_3 \le 50 \\ \text{and } x_j \ge 0 \text{ for } j = 1, \ldots, 3. \end{aligned}$$

**Solution** First, add slack variables, as before. Then change the objective *function* $25x_1 + 33x_2 + 18x_3$ into an *equation* by introducing a new variable $M$ given by $M = 25x_1 + 33x_2 + 18x_3$. Now the goal is to maximize the variable $M$, where $M$ satisfies the equation

$$-25x_1 - 33x_2 - 18x_3 + M = 0$$

The original problem is now restated as follows: Among all the solutions of the system of equations

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 \phantom{+ x_5 + x_6 + M} &= 60 \\ 3x_1 + x_2 + 5x_3 \phantom{+ x_4} + x_5 \phantom{+ x_6 + M} &= 46 \\ x_1 + 2x_2 + x_3 \phantom{+ x_4 + x_5} + x_6 \phantom{+ M} &= 50 \\ -25x_1 - 33x_2 - 18x_3 \phantom{+ x_4 + x_5 + x_6} + M &= 0 \end{aligned}$$

find a solution for which $x_j \ge 0$ ($j = 1, \ldots, 6$) and for which $M$ is as large as possible.

The augmented matrix for this new system is called the **initial simplex tableau**. It is written with two ruled lines in the matrix:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 0 & 60 \\ 3 & 1 & 5 & 0 & 1 & 0 & 0 & 46 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 50 \\ \hline -25 & -33 & -18 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The horizontal line above the bottom row isolates the equation corresponding to the objective function. This last row will play a special role in what follows. (The bottom row is used only to decide which variable to bring into the solution. Pivot positions are never chosen from the bottom row.) The column headings for the slack variables are in color, to remind us at the end of the calculations that only the original variables are part of the final solution of the problem.

Look in rows 1 to 3 of the tableau above to find the basic feasible solution. The *columns of the* $3 \times 3$ *identity matrix* in these three rows *identify the basic variables*—namely, $x_4$, $x_5$, and $x_6$. The basic solution is

$$x_1 = x_2 = x_3 = 0, \quad x_4 = 60, \quad x_5 = 46, \quad x_6 = 50, \quad M = 0$$

This solution is not optimal, however, since only the slack variables are nonzero. However, the bottom row implies that

$$M = 25x_1 + 33x_2 + 18x_3$$

The value of $M$ will rise when any of the variables $x_1, x_2$, or $x_3$ rises. Since the coefficient of $x_2$ is the largest of the three coefficients, bringing $x_2$ into the solution will cause the greatest increase in $M$.

To bring $x_2$ into the solution, follow the pivoting procedure outlined earlier. In the tableau above, compare the ratios $b_i/a_{i2}$ for each row except the last. They are $60/3$, $46/1$, and $50/2$. The smallest is $60/3$, so the pivot should be the entry 3 that is circled in the first row.

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\
\left[\begin{array}{rrrrrrr|r}
2 & 3 & 4 & 1 & 0 & 0 & 0 & 60 \\
3 & 1 & 5 & 0 & 1 & 0 & 0 & 46 \\
1 & 2 & 1 & 0 & 0 & 1 & 0 & 50 \\
\hline
-25 & -33 & -18 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

The result of the pivot operation is

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\
\left[\begin{array}{rrrrrrr|r}
\frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 20 \\
\frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 0 & 26 \\
-\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 0 & 10 \\
\hline
-3 & 0 & 26 & 11 & 0 & 0 & 1 & 660
\end{array}\right]
\end{array} \tag{2}$$

Now the columns of the $3 \times 3$ identity matrix are in columns 2, 5, and 6 of the tableau. So the basic feasible solution is

$$x_1 = x_3 = x_4 = 0, \quad x_2 = 20, \quad x_5 = 26, \quad x_6 = 10, \quad M = 660$$

Thus $M$ has increased from 0 to 660. To see if $M$ can be increased further, look at the bottom row of the tableau and solve the equation for $M$:

$$M = 660 + 3x_1 - 26x_3 - 11x_4 \tag{3}$$

Since each of the variables $x_j$ is nonnegative, the value of $M$ will increase only if $x_1$ increases (from 0). (Since the coefficients of $x_3$ and $x_4$ are both negative at this point, increasing one of them would *decrease* $M$.) So $x_1$ needs to come into the solution. Compare the ratios (of the augmented column to column 1):

$$\frac{20}{\frac{2}{3}} = 30 \qquad \text{and} \qquad \frac{26}{\frac{7}{3}} = \frac{78}{7}$$

The second ratio is smaller, so the next pivot should be $\frac{7}{3}$ in row 2.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 20 \\ \frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 0 & 26 \\ -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 0 & 10 \\ \hline -3 & 0 & 26 & 11 & 0 & 0 & 1 & 660 \end{array}\right] \end{array}$$

After pivoting, the resulting tableau is

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & \frac{88}{7} \\ 1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{78}{7} \\ 0 & 0 & -\frac{8}{7} & -\frac{5}{7} & \frac{1}{7} & 1 & 0 & \frac{96}{7} \\ \hline 0 & 0 & \frac{215}{7} & \frac{74}{7} & \frac{9}{7} & 0 & 1 & \frac{4854}{7} \end{array}\right] \end{array}$$

The corresponding basic feasible solution is

$$x_3 = x_4 = x_5 = 0, \quad x_1 = \tfrac{78}{7}, \quad x_2 = \tfrac{88}{7}, \quad x_6 = \tfrac{96}{7}, \quad M = \tfrac{4854}{7}$$

The bottom row shows that

$$M = \tfrac{4854}{7} - \tfrac{215}{7}x_3 - \tfrac{74}{7}x_4 - \tfrac{9}{7}x_5$$

The negative coefficients of the variables here show that $M$ can be no larger than $\frac{4854}{7}$ (because $x_3$, $x_4$, and $x_5$ are nonnegative), so the solution is optimal. The maximum value of $25x_1 + 33x_2 + 18x_3$ is $\frac{4854}{7}$, and this maximum occurs when $x_1 = \frac{78}{7}$, $x_2 = \frac{88}{7}$, and $x_3 = 0$. The variable $x_3$ is zero because in the optimal solution $x_3$ is a free variable, not a basic variable. Note that the value of $x_6$ is not part of the solution of the original problem, because $x_6$ is a slack variable. The fact that the slack variables $x_4$ and $x_5$ are zero means that the first two inequalities listed at the beginning of this example are both *equalities* at the optimal values of $x_1$, $x_2$, and $x_3$.

Example 5 is worth reading carefully several times. In particular, notice that a *negative* entry in the bottom row of any $x_j$ column will become a *positive* coefficient when that equation is solved for $M$, indicating that $M$ has not reached its maximum. See tableau (2) and equation (3).

In summary, here is the simplex method for solving a canonical maximizing problem when each entry in the vector **b** is positive.

### The Simplex Algorithm for a Canonical Linear Programming Problem

1. Change the inequality constraints into equalities by adding slack variables. Let $M$ be a variable equal to the objective function, and below the constraint equations write an equation of the form

$$\text{(objective function)} - M = 0$$

2. Set up the initial simplex tableau. The slack variables (and $M$) provide the initial basic feasible solution.
3. Check the bottom row of the tableau for optimality. If all the entries to the left of the vertical line are nonnegative, then the solution is optimal. If some are negative, then choose the variable $x_k$ for which the entry in the bottom row is as negative as possible.[3]
4. Bring the variable $x_k$ into the solution. Do this by pivoting on the positive entry $a_{pk}$ for which the nonnegative ratio $b_i/a_{ik}$ is the smallest. The new basic feasible solution includes an increased value for $M$.
5. Repeat the process, beginning at step 3, until all the entries in the bottom row are nonnegative.

Two things can go wrong in the simplex algorithm. At step 4, there might be a negative entry in the bottom row of the $x_k$ column, but no positive entry $a_{ik}$ above it. In this case, it will not be possible to find a pivot to bring $x_k$ into the solution. This corresponds to the case where the objective function is unbounded and no optimal solution exists.

The second potential problem also occurs at step 4. The smallest ratio $b_i/a_{ik}$ may occur in more than one row. When this happens, the next tableau will have at least one basic variable equal to zero, and in subsequent tableaus the value of $M$ may remain constant. Theoretically it is possible for an infinite sequence of pivots to occur and fail to lead to an optimal solution. Such a phenomenon is called **cycling**. Fortunately, cycling occurs only rarely in practical applications. In most cases, one may arbitrarily choose either row with a minimum ratio as the pivot.

**EXAMPLE 6** A health food store sells two different mixtures of nuts. A box of the first mixture contains 1 pound of cashews and 1 pound of peanuts. A box of the second mixture contains 1 pound of filberts and 2 pounds of peanuts. The store has available 30

[3]The goal of step 3 is to produce the greatest increase possible in the value of $M$. This happens when only one variable $x_k$ satisfies the conditions. Suppose, however, that the most negative entry in the bottom row appears in both columns $j$ and $k$. Step 3 says that either $x_j$ or $x_k$ should be brought into the solution, and that is correct. Occasionally, a few computations can be avoided by first using step 4 to compute the "smallest ratio" for both columns $j$ and $k$, and then choosing the column for which this "smallest ratio" is larger. This situation will arise in Section 9.4.

pounds of cashews, 20 pounds of filberts, and 54 pounds of peanuts. Suppose the profit on each box of the first mixture is \$2 and on each box of the second mixture is \$3. If the store can sell all of the boxes it mixes, how many boxes of each mixture should be made in order to maximize the profit?

**Solution** Let $x_1$ be the number of boxes of the first mixture, and let $x_2$ be the number of boxes of the second mixture. The problem can be expressed mathematically as

$$\begin{array}{lrl} \text{Maximize} & 2x_1 + 3x_2 & \\ \text{subject to} & x_1 \phantom{+ 2x_2} \le 30 & \text{(cashews)} \\ & x_2 \le 20 & \text{(filberts)} \\ & x_1 + 2x_2 \le 54 & \text{(peanuts)} \end{array}$$

and $x_1 \geq 0$, $x_2 \geq 0$.

This turns out to be the same problem solved graphically in Example 5 of Section 9.2. When it is solved by the simplex method, the basic feasible solution from each tableau corresponds to an extreme point of the feasible region. See Fig. 1.

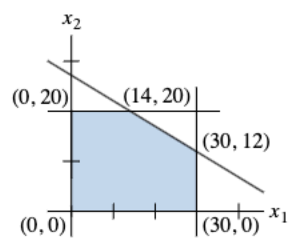


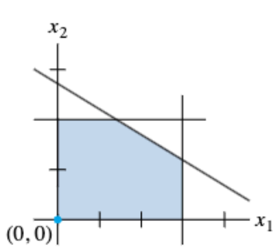

**FIGURE 1**

To construct the initial tableau, add slack variables and rewrite the objective function as an equation. The problem now is to find a nonnegative solution to the system

$$\begin{aligned} x_1 \phantom{+ 2x_2} + x_3 \phantom{+ x_4 + x_5 + M} &= 30 \\ x_2 + x_4 \phantom{+ x_5 + M} &= 20 \\ x_1 + 2x_2 \phantom{+ x_3 + x_4} + x_5 \phantom{+ M} &= 54 \\ -2x_1 - 3x_2 \phantom{+ x_3 + x_4 + x_5} + M &= 0 \end{aligned}$$

for which $M$ is a maximum. The initial simplex tableau is

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & & \end{array} \\ \left[\begin{array}{rrrrrr|r} 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 54 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

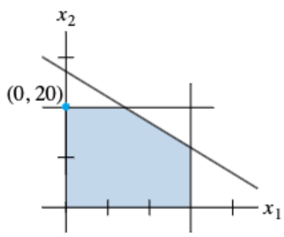



The basic feasible solution, where $x_1$, $x_2$, and $M$ are 0, corresponds to the extreme point $(x_1, x_2) = (0, 0)$ of the feasible region in Fig. 1. In the bottom row of the tableau, the most negative entry is $-3$, so the first pivot should be in the $x_2$ column. The ratios $20/1$ and $54/2$ show that the pivot should be the 1 in the $x_2$ column:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 54 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

After pivoting, the tableau becomes

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & -2 & 1 & 0 & 14 \\ \hline -2 & 0 & 0 & 3 & 0 & 1 & 60 \end{array}\right] \end{array}$$

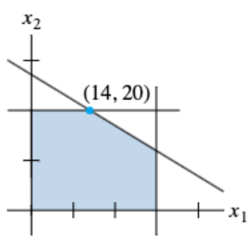



The basic feasible solution is now

$$x_1 = x_4 = 0, \quad x_2 = 20, \quad x_3 = 30, \quad x_5 = 14, \quad M = 60$$

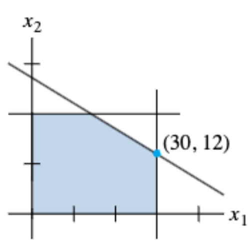
The new solution is at the extreme point $(x_1, x_2) = (0, 20)$ in Fig. 1. The $-2$ in the bottom row of the tableau shows that the next pivot is in column 1, which produces

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 2 & -1 & 0 & 16 \\ 0 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & -2 & 1 & 0 & 14 \\ \hline 0 & 0 & 0 & -1 & 2 & 1 & 88 \end{array}\right] \end{array}$$




This time $x_1 = 14$ and $x_2 = 20$, so the solution has moved across to the extreme point (14, 20) in Fig. 1, and the objective function has increased from 60 to 88. Finally, the $-1$ in the bottom row shows that the next pivot is in column 4. Pivoting on the 2 in the first row produces the final tableau:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 12 \\ 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & 96 \end{array}\right] \end{array}$$




Since all the entries in the bottom row are nonnegative, the solution now is optimal, with $x_1 = 30$ and $x_2 = 12$, corresponding to the extreme point (30, 12). The maximum profit of \$96 is attained by making 30 boxes of the first mixture and 12 boxes of the second. Note that although $x_4$ is part of the basic feasible solution for this tableau, its value is not included in the solution of the original problem, because $x_4$ is a slack variable.

## Minimization Problems

So far, each canonical maximizing problem involved a vector **b** whose coordinates were positive. But what happens when some of the coordinates of **b** are zero or negative? And what about a minimizing problem?

If some of the coordinates of **b** are zero, then it is possible for cycling to occur and the simplex method to fail to terminate at an optimal solution. As mentioned earlier, however, cycling does not generally happen in practical applications, and so the presence of zero entries in the right-hand column seldom causes difficulty in the operation of the simplex method.

The case when one of the coordinates of **b** is negative can occur in practice and requires some special consideration. The difficulty is that all the $b_i$ terms must be nonnegative in order for the slack variables to provide an initial basic feasible solution. One way to change a negative $b_i$ term into a positive term would be to multiply the inequality by $-1$ (before introducing slack variables). But this would change the direction of the inequality. For example,

$$x_1 - 3x_2 + 2x_3 \leq -4$$

would become

$$-x_1 + 3x_2 - 2x_3 \geq 4$$

Thus a negative $b_i$ term causes the same problem as a reversed inequality. Since reversed inequalities often occur in minimization problems, the following example discusses this case.

**EXAMPLE 7** Minimize $x_1 + 2x_2$

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 \geq 14 \\ & x_1 - x_2 \leq 2 \\ \text{and } x_1 \geq 0, & \ x_2 \geq 0. \end{aligned}$$

**Solution** The minimum of $f(x_1, x_2)$ over a set is the same as the maximum of $-f(x_1, x_2)$ over the *same* set. However, in order to use the simplex algorithm, the canonical *description* of the feasible set must use $\leq$ signs. So the first inequality above must be rewritten. The second inequality is already in canonical form. Thus the original problem is equivalent to the following:

$$\begin{aligned} \text{Maximize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -14 \\ & \phantom{-}x_1 - x_2 \leq 2 \\ \text{and } x_1 \geq 0, & \ x_2 \geq 0. \end{aligned}$$

To solve this, let $M = -x_1 - 2x_2$ and add slack variables to the inequalities, as before. This creates the linear system

$$\begin{aligned} -x_1 - x_2 + x_3 \phantom{+ x_4 + M} &= -14 \\ x_1 - x_2 \phantom{+ x_3} + x_4 \phantom{+ M} &= 2 \\ x_1 + 2x_2 \phantom{+ x_3 + x_4} + M &= 0 \end{aligned}$$

To find a nonnegative solution to this system for which $M$ is a maximum, construct the initial simplex tableau:

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 0 & -14 \\ 1 & -1 & 0 & 1 & 0 & 2 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding basic solution is

$$x_1 = x_2 = 0, \quad x_3 = -14, \quad M = 0$$

However, since $x_3$ is negative, this basic solution is not feasible. *Before the standard simplex method can begin, each term in the augmented column above the horizontal line must be a nonnegative number*. This is accomplished by pivoting on a negative entry.

In order to replace a negative $b_i$ entry by a positive number, find another negative entry in the same row. (If all the other entries in the row are nonnegative, then the problem has no feasible solution.) This negative entry is in the column corresponding to the variable that should now come into the solution. In this example, the first two columns both have negative entries, so either $x_1$ or $x_2$ should be brought into the solution.

For example, to bring $x_2$ into the solution, select as a pivot the entry $a_{i2}$ in column 2 for which the ratio $b_i/a_{i2}$ is the smallest nonnegative number. (The ratio is positive when both $b_i$ and $a_{i2}$ are negative.) In this case, only the ratio $(-14)/(-1)$ is nonnegative, so the $-1$ in the first row must be the pivot. After the pivot operations on column 2, the resulting tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 14 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ \hline -1 & 0 & 2 & 0 & 1 & -28 \end{array}\right]$$

Now each entry in the augmented column (except the bottom entry) is positive, and the simplex method can begin. (Sometimes it may be necessary to pivot more than once in order to make each of these terms nonnegative. See Exercise 15.) The next tableau turns out to be optimal:

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 8 \\ \hline 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & -20 \end{array}\right]$$

The maximum feasible value of $-x_1 - 2x_2$ is $-20$, when $x_1 = 8$ and $x_2 = 6$. So the *minimum* value of $x_1 + 2x_2$ is 20.

The final example uses the technique of Example 7, but the simplex tableau requires more preprocessing before the standard maximization operations can begin.

**EXAMPLE 8** Minimize $5x_1 + 3x_2$

$$\begin{aligned} \text{subject to} \quad 4x_1 + x_2 &\geq 12 \\ x_1 + 2x_2 &\geq 10 \\ x_1 + 4x_2 &\geq 16 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$.

**Solution** Convert the problem into a maximization problem, setting $M = -5x_1 - 3x_2$ and reversing the three main constraint inequalities:

$$-4x_1 - x_2 \leq -12, \quad -x_1 - 2x_2 \leq -10, \quad -x_1 - 4x_2 \leq -16$$

Add nonnegative slack variables, and construct the initial simplex tableau:

$$\begin{aligned} -4x_1 - x_2 + x_3 \qquad\qquad\qquad &= -12 \\ -x_1 - 2x_2 \qquad + x_4 \qquad\qquad &= -10 \\ -x_1 - 4x_2 \qquad\qquad + x_5 \qquad &= -16 \\ 5x_1 + 3x_2 \qquad\qquad\qquad + M &= 0 \end{aligned} \qquad \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} -4 & -1 & 1 & 0 & 0 & 0 & -12 \\ -1 & -2 & 0 & 1 & 0 & 0 & -10 \\ -1 & -4 & 0 & 0 & 1 & 0 & -16 \\ \hline 5 & 3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Before the simplex maximization process can begin, the top three entries in the augmented column must be nonnegative (to make the basic solution feasible). Pivoting on a negative entry to bring $x_1$ or $x_2$ into the solution will help. Trial and error will work. However, the fastest method is to compute the usual ratios $b_i/a_{ij}$ for all negative entries in rows 1 to 3 of columns 1 and 2. Choose as the pivot the entry with the *largest* ratio. That will make *all* the augmented entries change sign (because the pivot operation will *add* multiples of the pivot row to the other rows). In this example, the pivot should be $a_{31}$, and the new tableau is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 15 & 1 & 0 & -4 & 0 & 52 \\ 0 & 2 & 0 & 1 & -1 & 0 & 6 \\ 1 & 4 & 0 & 0 & -1 & 0 & 16 \\ \hline 0 & -17 & 0 & 0 & 5 & 1 & -80 \end{array}\right] \end{array}$$

Now the simplex maximization algorithm is available. The $-17$ in the last row shows that $x_2$ must be brought into the solution. The smallest of the ratios $52/15$, $6/2$, and $16/4$ is $6/2$. A pivot on the 2 in column 2 produces

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & -\frac{15}{2} & \frac{7}{2} & 0 & 7 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 3 \\ 1 & 0 & 0 & -2 & 1 & 0 & 4 \\ \hline 0 & 0 & 0 & \frac{17}{2} & -\frac{7}{2} & 1 & -29 \end{array}\right] \end{array}$$

The $-\frac{7}{2}$ in the last row shows that $x_5$ must be brought into the solution. The pivot is $\frac{7}{2}$ in column 5, and the new (and final) tableau is

$$\begin{array}{c}
\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 0 & \frac{2}{7} & -\frac{15}{7} & 1 & 0 & 2 \\
0 & 1 & \frac{1}{7} & -\frac{4}{7} & 0 & 0 & 4 \\
1 & 0 & -\frac{2}{7} & \frac{1}{7} & 0 & 0 & 2 \\
\hline
0 & 0 & 1 & 1 & 0 & 1 & -22
\end{array}\right]
\end{array}$$

The solution occurs when $x_1 = 2$ (from row 3), $x_2 = 4$, and $M = -22$, so the minimum of the original objective function is 22.

## The "Simplex" in the Simplex Algorithm

The geometric approach in Section 9.2 focused on the *rows* of a $2 \times n$ matrix $A$, graphing each inequality as a half-space in $\mathbb{R}^2$, and viewing the solution set as the intersection of half-spaces. In higher-dimensional problems, the solution set is again an intersection of half-spaces, but this geometric view does not lead to an efficient algorithm for finding the optimal solution.

The simplex algorithm focuses on the *columns* of $A$ instead of the rows. Suppose that $A$ is $m \times n$ and denote the columns by $\mathbf{a}_1, \dots, \mathbf{a}_m$. The addition of $m$ slack variables creates an $m$ by $n + m$ system of equations of the form

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n + x_{n+1}\mathbf{e}_1 + \cdots + x_{n+m}\mathbf{e}_m = \mathbf{b}$$

where $x_1, \dots, x_{n+m}$ are nonnegative and $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$. The initial basic feasible solution is obtained when $x_1, \dots, x_n$ are zero and $b_1\mathbf{e}_1 + \cdots + b_m\mathbf{e}_m = \mathbf{b}$. If $s = b_1 + \cdots + b_m$, then the equation

$$\mathbf{0} + \left(\frac{b_1}{s}\right)s\mathbf{e}_1 + \cdots + \left(\frac{b_m}{s}\right)s\mathbf{e}_m = \mathbf{b}$$

shows that $\mathbf{b}$ is in what is called the *simplex* generated by $\mathbf{0}, s\mathbf{e}_1, \dots, s\mathbf{e}_m$. For simplicity, we say that "$\mathbf{b}$ is in an $m$-dimensional simplex determined by $\mathbf{e}_1, \dots, \mathbf{e}_m$." This is the first simplex in the simplex algorithm.[4]

In general, if $\mathbf{v}_1, \dots, \mathbf{v}_m$ is any basis of $\mathbb{R}^m$, selected from the columns of the matrix $P = [\mathbf{a}_1 \cdots \mathbf{a}_n \quad \mathbf{e}_1 \cdots \mathbf{e}_m]$, and if $\mathbf{b}$ is a linear combination of these vectors with nonnegative weights, then $\mathbf{b}$ is in an $m$-dimensional simplex determined by $\mathbf{v}_1, \dots, \mathbf{v}_m$. A *basic* feasible solution of the linear programming problem corresponds to a particular *basis* from the columns of $P$. The simplex algorithm changes this basis and hence the corresponding simplex that contains $\mathbf{b}$, one column at a time. The various ratios

[4] If $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent vectors in $\mathbb{R}^m$, then the convex hull of the set $\{\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ is an $m$-dimensional simplex, $S$. (See Section 8.5.) A typical vector in $S$ has the form $c_0\mathbf{0} + c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m$, where the weights are nonnegative and sum to one. (Equivalently, vectors in $S$ have the form $c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m$, where the weights are nonnegative and their sum is at most one.) Any set formed by translating such a set $S$ is also called an $m$-dimensional simplex, but such sets do not appear in the simplex algorithm.

computed during the algorithm drive the choice of columns. Since row operations do not change the linear dependence relations among the columns, each basic feasible solution tells how to build **b** from the corresponding columns of $P$.

### PRACTICE PROBLEM

Use the simplex method to solve the following linear programming problem:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + x_2 \\ \text{subject to} & -x_1 + 2x_2 \le 8 \\ & 3x_1 + 2x_2 \le 24 \\ \text{and } x_1 \ge 0, x_2 \ge 0. & \end{array}$$

## 9.3 EXERCISES

In Exercises 1 and 2, set up the initial simplex tableau for the given linear programming problem.

**1.** $$\begin{array}{ll} \text{Maximize} & 21x_1 + 25x_2 + 15x_3 \\ \text{subject to} & 2x_1 + 7x_2 + 10x_3 \le 20 \\ & 3x_1 + 4x_2 + 18x_3 \le 25 \\ \text{and } x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. & \end{array}$$

**2.** $$\begin{array}{ll} \text{Maximize} & 22x_1 + 14x_2 \\ \text{subject to} & 3x_1 + 5x_2 \le 30 \\ & 2x_1 + 7x_2 \le 24 \\ & 6x_1 + x_2 \le 42 \\ \text{and } x_1 \ge 0, x_2 \ge 0. & \end{array}$$

For each simplex tableau in Exercises 3–6, do the following:

a. Determine which variable should be brought into the solution.

b. Compute the next tableau.

c. Identify the basic feasible solution corresponding to the tableau in part (b).

d. Determine if the answer in part (c) is optimal.

**3.** $$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 5 & 1 & 1 & 0 & 0 & 20 \\ 3 & 2 & 0 & 1 & 0 & 30 \\ \hline -4 & -10 & 0 & 0 & 1 & 0 \end{array}\right]$$

**4.** $$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} -1 & 1 & 2 & 0 & 0 & 4 \\ 1 & 0 & 5 & 1 & 0 & 6 \\ \hline -5 & 0 & 3 & 0 & 1 & 17 \end{array}\right]$$

**5.** $$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 20 \\ 2 & 1 & 0 & 1 & 0 & 16 \\ \hline -6 & -5 & 0 & 0 & 1 & 0 \end{array}\right]$$

**6.** $$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 5 & 8 & 1 & 0 & 0 & 80 \\ 12 & 6 & 0 & 1 & 0 & 30 \\ \hline 2 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$$

Exercises 7 and 8 relate to a canonical linear programming problem with an $m \times n$ coefficient matrix $A$ in the constraint inequality $A\mathbf{x} \le \mathbf{b}$. Mark each statement True or False, and justify each answer.

**7.** a. A slack variable is used to change an equality into an inequality.

b. A solution is feasible if each variable is nonnegative.

c. If one of the coordinates in vector **b** is positive, then the problem is infeasible.

**8.** a. A solution is called a basic solution if $m$ or fewer of the variables are nonzero.

b. The basic feasible solutions correspond to the extreme points of the feasible region.

c. The bottom entry in the right column of a simplex tableau gives the maximum value of the objective function.

Solve Exercises 9–14 by using the simplex method.

**9.** $$\begin{array}{ll} \text{Maximize} & 10x_1 + 12x_2 \\ \text{subject to} & 2x_1 + 3x_2 \le 36 \\ & 5x_1 + 4x_2 \le 55 \\ \text{and } x_1 \ge 0, x_2 \ge 0. & \end{array}$$

**10.** Maximize $5x_1 + 4x_2$
subject to
$$\begin{aligned} x_1 + 5x_2 &\le 70 \\ 3x_1 + 2x_2 &\le 54 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**11.** Maximize $4x_1 + 5x_2$
subject to
$$\begin{aligned} x_1 + 2x_2 &\le 26 \\ 2x_1 + 3x_2 &\le 30 \\ x_1 + x_2 &\le 13 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**12.** Maximize $2x_1 + 5x_2 + 3x_3$
subject to
$$\begin{aligned} x_1 + 2x_2 &\le 28 \\ 2x_1 + 4x_3 &\le 16 \\ x_2 + x_3 &\le 12 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

**13.** Minimize $12x_1 + 5x_2$
subject to
$$\begin{aligned} 2x_1 + x_2 &\ge 32 \\ -3x_1 + 5x_2 &\le 30 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

**14.** Minimize $2x_1 + 3x_2 + 3x_3$
subject to
$$\begin{aligned} x_1 - 2x_2 \quad\quad &\ge -8 \\ 2x_2 + x_3 &\ge 15 \\ 2x_1 - x_2 + x_3 &\le 25 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

**15.** Solve Example 7 by bringing $x_1$ into the solution (instead of $x_2$) in the initial tableau.

**16.** Use the simplex method to solve the linear programming problem in Section 9.2, Exercise 1.

**17.** Use the simplex method to solve the linear programming problem in Section 9.2, Exercise 15.

**18.** Use the simplex method to solve the linear programming problem in Section 9.2, Example 1.

## SOLUTION TO PRACTICE PROBLEM

Introduce slack variables $x_3$ and $x_4$ to rewrite the problem:

Maximize $2x_1 + x_2$
subject to
$$\begin{aligned} -x_1 + 2x_2 + x_3 \quad\quad &= 8 \\ 3x_1 + 2x_2 \quad\quad + x_4 &= 24 \end{aligned}$$
and $x_1 \ge 0, x_2 \ge 0$.

Then let $M = 2x_1 + x_2$, so that $-2x_1 - x_2 + M = 0$ provides the bottom row in the initial simplex tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} -1 & 2 & 1 & 0 & 0 & 8 \\ 3 & 2 & 0 & 1 & 0 & 24 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Bring $x_1$ into the solution (because of the $-2$ entry in the bottom row), and pivot on the second row (because it is the only row with a positive entry in the first column). The second tableau turns out to be optimal, since all the entries in the bottom row are positive. Remember that the slack variables (in color) are never part of the solution.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & \frac{8}{3} & 1 & \frac{1}{3} & 0 & 16 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 8 \\ \hline 0 & \frac{1}{3} & 0 & \frac{2}{3} & 1 & 16 \end{array}\right] \end{array}$$

The maximum value is 16, when $x_1 = 8$ and $x_2 = 0$. Note that this problem was solved geometrically in the Practice Problem for Section 9.2.

# 9.4 DUALITY

Associated with each canonical (maximization) linear programming problem is a related minimization problem, called the *dual* problem. In this setting, the canonical problem is called the *primal* problem. This section describes the dual problem and how it is solved, along with an interesting economic interpretation of the dual variables. The section concludes by showing how any matrix game can be solved using the primal and dual versions of a suitable linear programming problem.

Given vectors $\mathbf{c}$ in $\mathbb{R}^n$ and $\mathbf{b}$ in $\mathbb{R}^m$, and given an $m \times n$ matrix $A$, the canonical (primal) problem is to find $\mathbf{x}$ in $\mathbb{R}^n$ so as to maximize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. The dual (minimization) problem is to find $\mathbf{y}$ in $\mathbb{R}^m$ so as to minimize $g(\mathbf{y}) = \mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$:

| Primal Problem $P$ | Dual Problem $P^*$ |
|---|---|
| Maximize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ | Minimize $g(\mathbf{y}) = \mathbf{b}^T\mathbf{y}$ |
| subject to $A\mathbf{x} \leq \mathbf{b}$ | subject to $A^T\mathbf{y} \geq \mathbf{c}$ |
| $\mathbf{x} \geq \mathbf{0}$ | $\mathbf{y} \geq \mathbf{0}$ |

Observe that in forming the dual problem, the $c_i$ coefficients of $x_i$ in the objective function of the primal problem become the constants on the right-hand side of the constraint inequalities in the dual. Likewise, the numbers in the right-hand side of the constraint inequalities in the primal problem become the coefficients $b_j$ of $y_j$ in the objective function in the dual. Also, note that the direction of the constraint inequalities is reversed from $A\mathbf{x} \leq \mathbf{b}$ to $A^T\mathbf{y} \geq \mathbf{c}$. In both cases, the variables $\mathbf{x}$ and $\mathbf{y}$ are nonnegative.

**EXAMPLE 1** Find the dual of the following primal problem:

$$\begin{array}{ll} \text{Maximize} & 5x_1 + 7x_2 \\ \text{subject to} & 2x_1 + 3x_2 \leq 25 \\ & 7x_1 + 4x_2 \leq 16 \\ & \phantom{7}x_1 + 9x_2 \leq 21 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0,\ x_2 \geq 0.} \end{array}$$

**Solution**

$$\begin{array}{ll} \text{Minimize} & 25y_1 + 16y_2 + 21y_3 \\ \text{subject to} & \phantom{2}2y_1 + \phantom{1}7y_2 + \phantom{9}y_3 \geq 5 \\ & \phantom{2}3y_1 + \phantom{1}4y_2 + 9y_3 \geq 7 \\ \multicolumn{2}{l}{\text{and } y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.} \end{array}$$

Suppose that the dual problem above is rewritten as a canonical maximization problem:

$$\begin{array}{ll} \text{Maximize} & h(\mathbf{y}) = -\mathbf{b}^T\mathbf{y} \\ \text{subject to} & -A^T\mathbf{y} \leq -\mathbf{c} \quad \text{and} \quad \mathbf{y} \geq \mathbf{0}. \end{array}$$

Then the dual of *this* problem is

$$\begin{array}{ll} \text{Minimize} & F(\mathbf{w}) = -\mathbf{c}^T\mathbf{w} \\ \text{subject to} & (-A^T)^T\mathbf{w} \geq -\mathbf{b} \quad \text{and} \quad \mathbf{w} \geq \mathbf{0}. \end{array}$$

In canonical form, this minimization problem is equivalent to

$$\begin{array}{ll} \text{Maximize} & G(\mathbf{w}) = \mathbf{c}^T\mathbf{w} \\ \text{subject to} & A\mathbf{w} \leq \mathbf{b} \quad \text{and} \quad \mathbf{w} \geq \mathbf{0}. \end{array}$$

If $\mathbf{w}$ is replaced by $\mathbf{x}$, this problem is precisely the primal problem. So the dual of the dual problem is the original primal problem.

Theorem 7 below is a fundamental result in linear programming. As with the Minimax Theorem in game theory, the proof depends on certain properties of convex sets and hyperplanes.[1]

**THEOREM 7** **THE DUALITY THEOREM**

Let $P$ be a (primal) linear programming problem with feasible set $\mathcal{F}$, and let $P^*$ be the dual problem with feasible set $\mathcal{F}^*$.

a. If $\mathcal{F}$ and $\mathcal{F}^*$ are both nonempty, then $P$ and $P^*$ both have optimal solutions, say $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, respectively, and $f(\bar{\mathbf{x}}) = g(\bar{\mathbf{y}})$.

b. If one of the problems $P$ or $P^*$ has an optimal solution $\bar{\mathbf{x}}$ or $\bar{\mathbf{y}}$, respectively, then so does the other, and $f(\bar{\mathbf{x}}) = g(\bar{\mathbf{y}})$.

**EXAMPLE 2** Set up and solve the dual to the problem in Example 5 of Section 9.2.

**Solution** The original problem is to

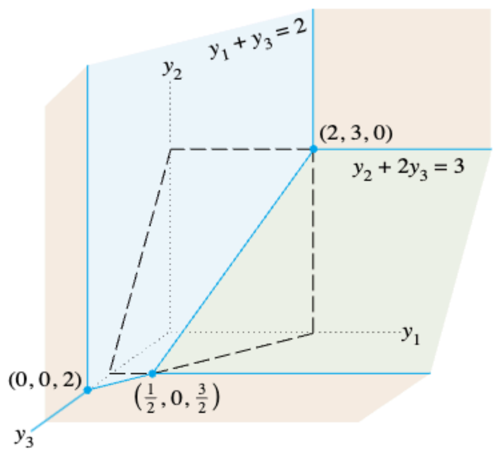
$$\begin{array}{ll} \text{Maximize} & f(x_1, x_2) = 2x_1 + 3x_2 \\ \text{subject to} & x_1 \phantom{+ 2x_2} \leq 30 \\ & \phantom{x_1 + 2}x_2 \leq 20 \\ & x_1 + 2x_2 \leq 54 \\ \multicolumn{2}{l}{\text{and } x_1 \geq 0,\ x_2 \geq 0.} \end{array}$$

Calculations in Example 5 of Section 9.2 showed that the optimal solution of this problem is $\bar{\mathbf{x}} = \begin{bmatrix} 30 \\ 12 \end{bmatrix}$ with $f(\bar{\mathbf{x}}) = 96$. The dual problem is to

$$\begin{array}{ll} \text{Minimize} & g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3 \\ \text{subject to} & y_1 \phantom{+ y_2} + \phantom{2}y_3 \geq 2 \\ & \phantom{y_1 +} y_2 + 2y_3 \geq 3 \\ \multicolumn{2}{l}{\text{and } y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.} \end{array}$$

[1] If the equation $A\mathbf{x} = \mathbf{b}$ has no nonnegative solution, then the sets $\{\mathbf{b}\}$ and $S = \{\mathbf{z} \in \mathbb{R}^m : \mathbf{z} = A\mathbf{x},\ \mathbf{x} \geq \mathbf{0}\}$ are disjoint. It is not hard to show that $S$ is a closed convex set, so Theorem 12 in Chapter 8 implies that there exists a hyperplane strictly separating $\{\mathbf{b}\}$ and $S$. This hyperplane plays a key role in the proof. For details, see Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), pp. 174–178.

The simplex method could be used here, but the geometric method of Section 9.2 is not too difficult. Graphs of the constraint inequalities (Fig. 1) reveal that $\mathscr{F}^*$ has three extreme points and that $\bar{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix}$ is the optimal solution. Indeed, $g(\bar{\mathbf{y}}) = 30(\frac{1}{2}) + 20(0) + 54(\frac{3}{2}) = 96$, as expected.




| $\mathbf{y}$ | $g(\mathbf{y})$ |
|---|---|
| $(0, 0, 2)$ | 108 |
| $(\frac{1}{2}, 0, \frac{3}{2})$ | 96 ← |
| $(2, 3, 0)$ | 120 |

**FIGURE 1** The minimum of $g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3$.

Example 2 illustrates another important property of duality and the simplex method. Recall that Example 6 of Section 9.3 solved this same maximizing problem using the simplex method. Here is the final tableau:

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 12 \\ 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & 96 \end{array}\right]$$

Notice that the optimal solution to the dual problem appears in the bottom row. The variables $x_3$, $x_4$, and $x_5$ are the slack variables for the first, second, and third equations, respectively. The bottom entry in each of these columns gives the optimal solution $\bar{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix}$ to the dual problem. This is not a coincidence, as the following theorem shows.

**THEOREM 7** **THE DUALITY THEOREM (CONTINUED)**

Let $P$ be a (primal) linear programming problem and let $P^*$ be its dual problem. Suppose $P$ (or $P^*$) has an optimal solution.

c. If either $P$ or $P^*$ is solved by the simplex method, then the solution of its dual is displayed in the bottom row of the final tableau in the columns associated with the slack variables.

**EXAMPLE 3** Set up and solve the dual to the problem in Example 5 in Section 9.3.

**Solution** The primal problem $P$ is to

$$\begin{array}{ll} \text{Maximize} & f(x_1, x_2, x_3) = 25x_1 + 33x_2 + 18x_3 \\ \text{subject to} & 2x_1 + 3x_2 + 4x_3 \le 60 \\ & 3x_1 + \phantom{3}x_2 + 5x_3 \le 46 \\ & \phantom{3}x_1 + 2x_2 + \phantom{4}x_3 \le 50 \end{array}$$

and $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

The dual problem $P^*$ is to

$$\begin{array}{ll} \text{Minimize} & g(y_1, y_2, y_3) = 60y_1 + 46y_2 + 50y_3 \\ \text{subject to} & 2y_1 + 3y_2 + \phantom{2}y_3 \ge 25 \\ & 3y_1 + \phantom{3}y_2 + 2y_3 \ge 33 \\ & 4y_1 + 5y_2 + \phantom{2}y_3 \ge 18 \end{array}$$

and $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$.

The final tableau for the solution of the primal problem was found to be

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \phantom{\frac{4854}{7}} \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & \frac{88}{7} \\ 1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{78}{7} \\ 0 & 0 & -\frac{8}{7} & -\frac{5}{7} & \frac{1}{7} & 1 & 0 & \frac{96}{7} \\ \hline 0 & 0 & \frac{215}{7} & \frac{74}{7} & \frac{9}{7} & 0 & 1 & \frac{4854}{7} \end{array}\right] \end{array}$$

The slack variables are $x_4, x_5$, and $x_6$. They give the optimal solution to the dual problem $P^*$. Thus,

$$y_1 = \frac{74}{7}, \quad y_2 = \frac{9}{7}, \quad \text{and} \quad y_3 = 0$$

Note that the optimal value of the objective function in the dual problem is

$$g\left(\frac{74}{7}, \frac{9}{7}, 0\right) = 60\left(\frac{74}{7}\right) + 46\left(\frac{9}{7}\right) + 50(0) = \frac{4854}{7}$$

which agrees with the optimal value of the objective function in the primal problem.

The variables in the dual problem have useful economic interpretations. For example, consider the problem of mixing nuts studied in Example 5 of Section 9.2 and

Example 6 of Section 9.3:

$$\begin{array}{lll} \text{Maximize} & f(x_1, x_2) = 2x_1 + 3x_2 & \\ \text{subject to} & x_1 \qquad\quad \le 30 & \text{(cashews)} \\ & \qquad\;\; x_2 \le 20 & \text{(filberts)} \\ & x_1 + 2x_2 \le 54 & \text{(peanuts)} \\ \multicolumn{3}{l}{\text{and } x_1 \ge 0,\ x_2 \ge 0.} \end{array}$$

Recall that $x_1$ is the number of boxes of the first mixture and $x_2$ is the number of boxes of the second mixture. Example 2 displayed the dual problem:

$$\begin{array}{ll} \text{Minimize} & g(y_1, y_2, y_3) = 30y_1 + 20y_2 + 54y_3 \\ \text{subject to} & y_1 \qquad + \;\; y_3 \ge 2 \\ & \qquad y_2 + 2y_3 \ge 3 \\ \multicolumn{2}{l}{\text{and } y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0.} \end{array}$$

If $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are optimal solutions of these problems, then by the Duality Theorem, the maximum profit $f(\bar{\mathbf{x}})$ satisfies the equation

$$f(\bar{\mathbf{x}}) = g(\bar{\mathbf{y}}) = 30\bar{y}_1 + 20\bar{y}_2 + 54\bar{y}_3$$

Suppose, for example, that the amount of cashews available was increased from 30 pounds to $30 + h$ pounds. Then the profit would increase by $h\bar{y}_1$. Likewise, if the amount of cashews was decreased by $h$ pounds, then the profit would decrease by $h\bar{y}_1$. So $\bar{y}_1$ represents the value (per pound) of increasing or decreasing the amount of cashews available. This is usually referred to as the **marginal value** of the cashews. Similarly, $\bar{y}_2$ and $\bar{y}_3$ are the marginal values of the filberts and peanuts, respectively. These values indicate how much the company might be willing to pay for additional supplies of the various nuts.[2]

**EXAMPLE 4** The final simplex tableau for the problem of mixing nuts was found (in Example 6 of Section 9.3) to be

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 12 \\ 1 & 0 & 1 & 0 & 0 & 0 & 30 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & 96 \end{array}\right] \end{array}$$

so the optimal solution of the dual is $\bar{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix}$. Thus the marginal value of the cashews is $\frac{1}{2}$, the marginal value of the filberts is 0, and the marginal value of the peanuts is $\frac{3}{2}$.

[2]The other entries in the final tableau can also be given an economic interpretation. See Saul I. Gass, *Linear Programming Methods and Applications*, 5th Ed. (Danvers, MA: Boyd & Fraser Publishing, 1985), pp. 173–177. Also see Goldstein, Schneider, and Siegel, *Finite Mathematics and Its Applications*, 6th Ed. (Upper Saddle River, NJ: Prentice Hall, 1998), pp. 166–185.

Note that the optimal production schedule $\bar{\mathbf{x}} = \begin{bmatrix} 30 \\ 12 \end{bmatrix}$ uses only 12 of the 20 pounds of filberts. (This corresponds to the slack variable $x_4$ for the filbert constraint inequality having value 8 in the final tableau.) This means that not all the available filberts are used, so there is no increase in profit from increasing the number of filberts available. That is, their marginal value is zero.

## Linear Programming and Matrix Games

Let $A$ be an $m \times n$ payoff matrix for a matrix game, as in Section 9.1, and assume at first that each entry in $A$ is positive. Let $\mathbf{u}$ in $\mathbb{R}^m$ and $\mathbf{v}$ in $\mathbb{R}^n$ be the vectors whose coordinates are all equal to one, and consider the following linear programming problem $P$ and its dual $P^*$. (Notice that the roles of $\mathbf{x}$ and $\mathbf{y}$ are reversed, with $\mathbf{x}$ in $\mathbb{R}^m$ and $\mathbf{y}$ in $\mathbb{R}^n$.)

$$\begin{array}{llll} P: & \text{Maximize} & \mathbf{v}^T\mathbf{y} \\ & \text{subject to} & A\mathbf{y} \le \mathbf{u} \\ & & \mathbf{y} \ge \mathbf{0} \end{array} \qquad \begin{array}{llll} P^*: & \text{Minimize} & \mathbf{u}^T\mathbf{x} \\ & \text{subject to} & A^T\mathbf{x} \ge \mathbf{v} \\ & & \mathbf{x} \ge \mathbf{0} \end{array}$$

The primal problem $P$ is feasible since $\mathbf{y} = \mathbf{0}$ satisfies the constraints. The dual problem $P^*$ is feasible since all the entries in $A^T$ are positive and $\mathbf{v}$ is a vector of 1's. By the Duality Theorem, there exist optimal solutions $\bar{\mathbf{y}}$ and $\bar{\mathbf{x}}$ such that $\mathbf{v}^T\bar{\mathbf{y}} = \mathbf{u}^T\bar{\mathbf{x}}$. Set

$$\lambda = \mathbf{v}^T\bar{\mathbf{y}} = \mathbf{u}^T\bar{\mathbf{x}}$$

Since the entries in $A$ and $\mathbf{u}$ are positive, the inequality $A\mathbf{y} \le \mathbf{u}$ has a nonzero solution $\mathbf{y}$ with $\mathbf{y} \ge \mathbf{0}$. As a result, the solution $\lambda$ of the primal problem is positive. Let

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda \qquad \text{and} \qquad \hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda$$

It can be shown (Exercise 23) that $\hat{\mathbf{y}}$ is the optimal mixed strategy for the column player $C$ and $\hat{\mathbf{x}}$ is the optimal mixed strategy for the row player $R$. Furthermore, the value of the game is equal to $1/\lambda$.

Finally, if the payoff matrix $A$ has some entries that are not positive, add a fixed number, say $k$, to each entry to make the entries all positive. This will not change the optimal mixed strategies for the two players, and it will add an amount $k$ to the value of the game. [See Exercise 25(b) in Section 9.1.]

**EXAMPLE 5** Solve the game whose payoff matrix is $A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$.

**Solution** To produce a matrix $B$ with positive entries, add 3 to each entry:

$$B = \begin{bmatrix} 1 & 4 & 5 \\ 6 & 5 & 3 \end{bmatrix}$$

The optimal strategy for the column player $C$ is found by solving the linear programming problem

$$\begin{array}{ll} \text{Maximize} & y_1 + y_2 + y_3 \\ \text{subject to} & y_1 + 4y_2 + 5y_3 \le 1 \\ & 6y_1 + 5y_2 + 3y_3 \le 1 \end{array}$$
and $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$.

Introduce slack variables $y_4$ and $y_5$, let $M$ be the objective function, and construct the initial simplex tableau:

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 4 & 5 & 1 & 0 & 0 & 1 \\ 6 & 5 & 3 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The three $-1$ entries in the bottom row are equal, so any of columns 1 to 3 can be the first pivot column. Choose column 1 and check the ratios $b_i/a_{i1}$. To bring variable $y_1$ into the solution, pivot on the 6 in the second row.

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \frac{19}{6} & \frac{9}{2} & 1 & -\frac{1}{6} & 0 & \frac{5}{6} \\ 1 & \frac{5}{6} & \frac{1}{2} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \hline 0 & -\frac{1}{6} & -\frac{1}{2} & 0 & \frac{1}{6} & 1 & \frac{1}{6} \end{array}\right] \end{array}$$

In the bottom row, the third entry is the most negative, so bring $y_3$ into the solution. The ratios $b_i/a_{i3}$ are $\frac{5}{6}/\frac{9}{2} = \frac{5}{27}$ and $\frac{1}{6}/\frac{1}{2} = \frac{1}{3} = \frac{9}{27}$. The first ratio is smaller, so pivot on the $\frac{9}{2}$ in the first row.

$$\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \frac{19}{27} & 1 & \frac{2}{9} & -\frac{1}{27} & 0 & \frac{5}{27} \\ 1 & \frac{13}{27} & 0 & -\frac{1}{9} & \frac{5}{27} & 0 & \frac{2}{27} \\ \hline 0 & \frac{5}{27} & 0 & \frac{1}{9} & \frac{4}{27} & 1 & \frac{7}{27} \end{array}\right] \end{array}$$

The optimal solution of the primal problem is

$$\bar{y}_1 = \frac{2}{27}, \quad \bar{y}_2 = 0, \quad \bar{y}_3 = \frac{5}{27}, \quad \text{with } \lambda = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = \frac{7}{27}$$

The corresponding optimal mixed strategy for $C$ is

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \begin{bmatrix} \frac{2}{7} \\ 0 \\ \frac{5}{7} \end{bmatrix}$$

The optimal solution of the dual problem comes from the bottom entries under the slack variables:

$$\bar{x}_1 = \frac{1}{9} = \frac{3}{27} \quad \text{and} \quad \bar{x}_2 = \frac{4}{27}, \quad \text{with } \lambda = \bar{x}_1 + \bar{x}_2 = \frac{7}{27}$$

which shows that the optimal mixed strategy for $R$ is

$$\hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$$

The value of the game with payoff matrix $B$ is $v = \dfrac{1}{\lambda} = \frac{27}{7}$, so the value of the original matrix game $A$ is $\frac{27}{7} - 3 = \frac{6}{7}$.

Although matrix games are usually solved via linear programming, it is interesting that a linear programming problem can be reduced to a matrix game. If the programming problem has an optimal solution, then this solution is reflected in the solution of the matrix game. Suppose the problem is to maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$, where $A$ is $m \times n$ with $m \le n$. Let

$$M = \begin{bmatrix} 0 & A & -\mathbf{b} \\ -A^T & 0 & \mathbf{c} \\ \mathbf{b}^T & -\mathbf{c}^T & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{s} = \begin{bmatrix} \bar{\mathbf{y}} \\ \bar{\mathbf{x}} \\ z \end{bmatrix}$$

and suppose that $M$ represents a matrix game and $\mathbf{s}$ is an optimal column strategy for $M$. The $(n+m+1)\times(n+m+1)$ matrix $M$ is skew-symmetric; that is, $M^T = -M$. It can be shown that in this case the optimal row strategy equals the optimal column strategy, the value of the game is 0, and the maximum of the entries in the vector $M\mathbf{s}$ is 0. Observe that

$$M\mathbf{s} = \begin{bmatrix} 0 & A & -\mathbf{b} \\ -A^T & 0 & \mathbf{c} \\ \mathbf{b}^T & -\mathbf{c}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}} \\ \bar{\mathbf{x}} \\ z \end{bmatrix} = \begin{bmatrix} A\bar{\mathbf{x}} - z\mathbf{b} \\ -A^T\bar{\mathbf{y}} + z\mathbf{c} \\ \mathbf{b}^T\bar{\mathbf{y}} - \mathbf{c}^T\bar{\mathbf{x}} \end{bmatrix} \le \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix}$$

Thus $A\bar{\mathbf{x}} \le z\mathbf{b}$, $A^T\mathbf{y} \ge z\mathbf{c}$, and $\mathbf{b}^T\bar{\mathbf{y}} \le \mathbf{c}^T\bar{\mathbf{x}}$. Since the column strategy $\mathbf{s}$ is a probability vector, $z \ge 0$. It can be shown that if $z > 0$, then $\bar{\mathbf{x}}/z$ is an optimal solution for the primal (maximization) problem for $A\mathbf{x} \le \mathbf{b}$, and $\bar{\mathbf{y}}/z$ is an optimal solution for the dual problem for $A^T\mathbf{y} \ge \mathbf{c}$. Also, if $\mathbf{z} = 0$, then the primal and dual problems have no optimal solutions.

In conclusion, the simplex method is a powerful tool in solving linear programming problems. Because a fixed procedure is followed, it lends itself well to using a computer for the tedious calculations involved. The algorithm presented here is not optimal for a computer, but many computer programs implement variants of the simplex method, and some programs even seek integer solutions. New methods developed in recent years take shortcuts through the interior of the feasible region instead of going from extreme point to extreme point. They are somewhat faster in certain situations (typically involving thousands of variables and constraints), but the simplex method is still the approach most widely used.

## PRACTICE PROBLEMS

The following questions relate to the Shady-Lane grass seed company from Example 1 in Section 9.2. The canonical linear programming problem can be stated as follows:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + 2x_2 \le 1200 \quad \text{(fescue)} \\ & \phantom{3}x_1 + 2x_2 \le \phantom{0}800 \quad \text{(rye)} \\ & \phantom{3}x_1 + \phantom{2}x_2 \le \phantom{0}450 \quad \text{(bluegrass)} \end{array}$$

and $x_1 \ge 0, x_2 \ge 0$.

1. State the dual problem.
2. Find the optimal solution to the dual problem, given that the final tableau in the simplex method for solving the primal problem is

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & -4 & 0 & 200 \\ 0 & 1 & 0 & 1 & -1 & 0 & 350 \\ 1 & 0 & 0 & -1 & 1 & 0 & 100 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1250 \end{array}\right]$$

3. What are the marginal values of fescue, rye, and bluegrass at the optimal solution?

## 9.4 Exercises

In Exercises 1–4, state the dual of the given linear programming problem.

1. Exercise 9 in Section 9.3
2. Exercise 10 in Section 9.3
3. Exercise 11 in Section 9.3
4. Exercise 12 in Section 9.3

In Exercises 5–8, use the final tableau in the solution of the given exercise to solve its dual.

5. Exercise 9 in Section 9.3
6. Exercise 10 in Section 9.3
7. Exercise 11 in Section 9.3
8. Exercise 12 in Section 9.3

Exercises 9 and 10 relate to a primal linear programming problem of finding $\mathbf{x}$ in $\mathbb{R}^n$ so as to maximize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Mark each statement True or False, and justify each answer.

9. a. The dual problem is to minimize $\mathbf{y}$ in $\mathbb{R}^m$ subject to $A\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.
   b. If both the primal and the dual problems are feasible, then they both have optimal solutions.
   c. If $\bar{\mathbf{x}}$ is an optimal solution to the primal problem and $\hat{\mathbf{y}}$ is a feasible solution to the dual problem such that $g(\hat{\mathbf{y}}) = f(\bar{\mathbf{x}})$, then $\hat{\mathbf{y}}$ is an optimal solution to the dual problem.
   d. If a slack variable is in an optimal solution, then the marginal value of the item corresponding to its equation is positive.
10. a. The dual of the dual problem is the original primal problem.
    b. If either the primal or the dual problem has an optimal solution, then they both do.
    c. If the primal problem has an optimal solution, then the final tableau in the simplex method also gives the optimal solution to the dual problem.
    d. When a linear programming problem and its dual are used to solve a matrix game, the vectors $\mathbf{u}$ and $\mathbf{v}$ are unit vectors.

Sometimes a minimization problem has inequalities only of the "$\geq$" type. In this case, replace the problem by its dual. (Multiplying the original inequalities by $-1$ to reverse their direction will not work, because the basic solution of the initial simplex tableau in this case will be infeasible.) In Exercises 11–14, use the simplex method to solve the dual, and from this solve the original problem (the dual of the dual).

11. Minimize $16x_1 + 10x_2 + 20x_3$
    subject to
    $$\begin{aligned} x_1 + x_2 + 3x_3 &\geq 4 \\ 2x_1 + x_2 + 2x_3 &\geq 5 \end{aligned}$$
    and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.
12. Minimize $10x_1 + 14x_2$
    subject to
    $$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 2x_1 + x_2 &\geq 4 \\ 3x_1 + x_2 &\geq 2 \end{aligned}$$
    and $x_1 \geq 0, x_2 \geq 0$.
13. Solve Exercise 2 in Section 9.2.
14. Solve Example 2 in Section 9.2.

Exercises 15 and 16 refer to Exercise 15 in Section 9.2. This exercise was solved using the simplex method in Exercise 17 of Section 9.3. Use the final simplex tableau for that exercise to answer the following questions.

**15.** What is the marginal value of additional labor in the fabricating department? Give an economic interpretation to your answer.

**16.** If an extra hour of labor were available, to which department should it be allocated? Why?

Solve the matrix games in Exercises 17 and 18 by using linear programming.

**17.** $\begin{bmatrix} 2 & 0 \\ -4 & 5 \\ -1 & 3 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -3 & 2 \end{bmatrix}$

**19.** Solve the matrix game in Exercise 9 in Section 9.1 using linear programming. This game and the one in Exercise 10 cannot be solved by the methods of Section 9.1.

**20.** Solve the matrix game in Exercise 10 in Section 9.1 using linear programming.

**21.** Bob wishes to invest \$35,000 in stocks, bonds, and gold coins. He knows that his rate of return will depend on the economic climate of the country, which is, of course, difficult to predict. After careful analysis, he determines the annual profit in dollars he would expect per hundred dollars on each type of investment, depending on whether the economy is strong, stable, or weak:

| | Strong | Stable | Weak |
|---|---|---|---|
| Stocks | 4 | 1 | −2 |
| Bonds | 1 | 3 | 0 |
| Gold | −1 | 0 | 4 |

How should Bob invest his money in order to maximize his profit regardless of what the economy does? That is, consider the problem as a matrix game in which Bob, the row player, is playing against the "economy." What is the expected value of his portfolio at the end of the year?

**22.** Let $P$ be a (primal) linear programming problem with feasible set $\mathscr{F}$, and let $P^*$ be the dual problem with feasible set $\mathscr{F}^*$. Prove the following:

a. If $\mathbf{x}$ is in $\mathscr{F}$ and $\mathbf{y}$ is in $\mathscr{F}^*$, then $f(\mathbf{x}) \le g(\mathbf{y})$. [*Hint*: Write $f(\mathbf{x})$ as $\mathbf{x}^T\mathbf{c}$ and $g(\mathbf{y})$ as $\mathbf{y}^T\mathbf{b}$. Then begin with the inequality $\mathbf{c} \le A^T\mathbf{y}$.]

b. If $f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$ for some $\hat{\mathbf{x}}$ in $\mathscr{F}$ and $\hat{\mathbf{y}}$ in $\mathscr{F}^*$, then $\hat{\mathbf{x}}$ is an optimal solution to $P$ and $\hat{\mathbf{y}}$ is an optimal solution to $P^*$.

**23.** Let $A$ be an $m \times n$ matrix game. Let $\bar{\mathbf{y}}$ and $\bar{\mathbf{x}}$ be the optimal solutions to the related primal and dual linear programming problems, respectively, as in the discussion prior to Example 5. Let $\lambda = \mathbf{u}^T\bar{\mathbf{x}} = \mathbf{v}^T\bar{\mathbf{y}}$, and define $\hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda$ and $\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda$. Let $R$ and $C$, respectively, denote the row and column players.

a. Show that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are mixed strategies for $R$ and $C$, respectively.

b. If $\mathbf{y}$ is any mixed strategy for $C$, show that $E(\hat{\mathbf{x}}, \mathbf{y}) \ge 1/\lambda$.

c. If $\mathbf{x}$ is any mixed strategy for $R$, show that $E(\mathbf{x}, \hat{\mathbf{y}}) \le 1/\lambda$.

d. Conclude that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal mixed strategies for $R$ and $C$, respectively, and that the value of the game is $1/\lambda$.

## Solutions to Practice Problems

**1.** Minimize $1200y_1 + 800y_2 + 450y_3$
subject to
$$3y_1 + y_2 + y_3 \ge 2$$
$$2y_1 + 2y_2 + y_3 \ge 3$$
and $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$.

**2.** The slack variables are $x_3, x_4$, and $x_5$. The bottom row entries in these columns of the final simplex tableau give the optimal solution to the dual problem. Thus $\bar{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

**3.** Slack variable $x_3$ comes from the constraint inequality for fescue. This corresponds to variable $y_1$ in the dual problem, so the marginal value of fescue is 0. Similarly, $x_4$ and $x_5$ come from rye and bluegrass, respectively, so their marginal values are both equal to 1.