PHYS 4000Y

Martin Ake 2022

Vector Fields, Electricity & Magnetism

ベクトル場と電気と磁気

NOTES | ノート

Vector Fields, Electricity & Magnetism

VECTOR CALCULUS
ベクトル解析

目次

1 Differentiation

1.1 Ordinary Differentiation

For $f = f(x)$,

dependent variable (f) · function (f) · independent variable (x)

DERIVATIVE

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

DIFFERENTIAL

$$df = \frac{df}{dx}\,dx$$

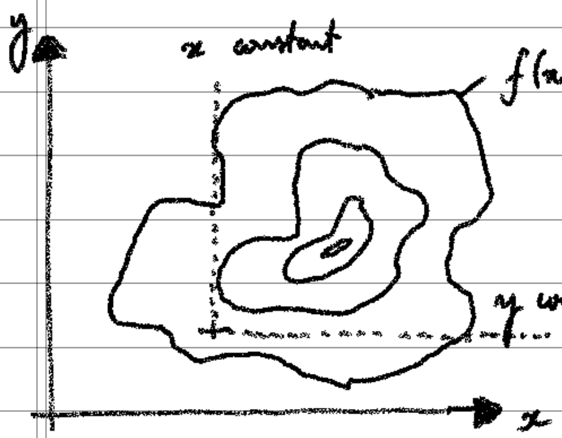
CHAIN RULE

$$f = f[x(t)]$$
$$\Rightarrow \frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt}$$

1.2 Partial Differentiation of Scalar Fields

For $f = f(x, y)$, a scalar field e.g. temperature

more than 1 independent variable (x, y)



PARTIAL DERIVATIVE

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}(x,y)\right)_y = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

y is constant

TOTAL DIFFERENTIAL

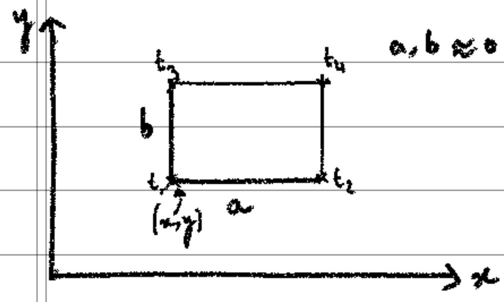
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

- "tangent-plane" approximation to Δf
- change in f for infinitesimal changes dx, dy

★ x, y don't need to be orthogonal, only independent

Geometric proof for second-order mixed derivatives:

CLAIRAULT'S THEOREM



$a, b \approx 0$

$$\Rightarrow \frac{\partial f}{\partial x} \simeq \frac{t_2 - t_1}{a}, \quad \frac{\partial f}{\partial y} \simeq \frac{t_3 - t_1}{b}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \simeq \frac{\frac{t_4 - t_3}{a} - \frac{t_2 - t_1}{a}}{b} = \frac{t_1 + t_4 - (t_2 + t_3)}{ab}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \simeq \frac{\frac{t_4 - t_2}{b} - \frac{t_3 - t_1}{b}}{a} = \frac{t_1 + t_4 - (t_2 + t_3)}{ab} \quad \square$$

1.3 PARTIAL DIFFERENTIATION OF VECTOR FIELDS

For $\vec{A}(x,y) = A_x(x,y)\hat{i} + A_y(x,y)\hat{j}$, i.e. 2D vector field, defining a vector at every x, y (e.g. velocity)

e.g. $\vec{A} = xy\hat{i} + (x^2+y^2)\hat{j}$ ← component is defined by scalar field

PARTIAL DERIVATIVE is defined as: partial derivative of each component

$$\frac{\partial \vec{A}}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{\vec{A}(x+\Delta x, y) - \vec{A}(x,y)}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{\begin{pmatrix} A_x(x+\Delta x, y) \\ A_y(x+\Delta x, y)\end{pmatrix} - \begin{pmatrix} A_x(x,y) \\ A_y(x,y)\end{pmatrix}}{\Delta x}\right] = \begin{pmatrix} \frac{\partial A_x}{\partial x} \\ \frac{\partial A_y}{\partial y}\end{pmatrix}$$

TOTAL DIFFERENTIAL: $\quad d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy$

1.4 EXACT DIFFERENTIAL

Given the particular differential: $A(x,y)\,dx + B(x,y)\,dy$,

we can say it is EXACT iff it gives the total differential of the same parent function f.

by Clairaut's Theorem, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \Rightarrow \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

☆ integral of an exact differential between 2 end states is independent of path taken

1.5 CHAIN RULE WITH PARTIAL DIFFERENTIATION

For $f = f(x,y)$ 3 cases:

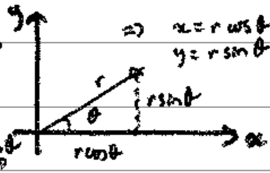
① $y = y(x)$ only 1 independent variable (x or y) $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\overbrace{\frac{dy}{dx}dx}^{dy} \Rightarrow \frac{df}{dx} = f_x + f_y\frac{dy}{dx}$

② $\begin{cases} x = x(t) \\ y = y(t)\end{cases}$ only 1 independent variable (t) $\quad df = \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt \Rightarrow \frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$

③ $\begin{cases} x = x(u,v) \\ y = y(u,v)\end{cases}$ multiple independent variables (u,v) $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$; $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$; $dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$

← 'CHANGE OF VARIABLES' $\quad \Rightarrow \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$; $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$

Ex.1 Plane Polar Co-ordinates



$\Rightarrow x = r\cos\theta,\ y = r\sin\theta$

For $x(r,\theta)$, $y(r,\theta)$, find $\left(\frac{\partial f}{\partial r}\right)_\theta$ and $\left(\frac{\partial f}{\partial \theta}\right)_r$

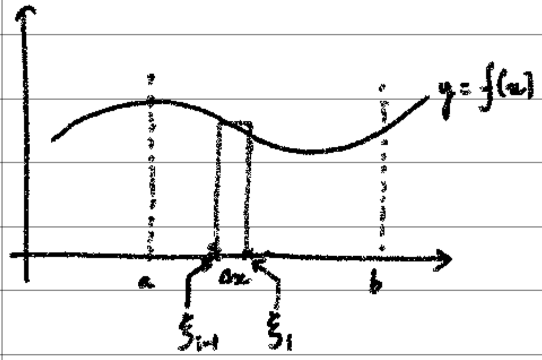
$f = x^2 + y^2 = r^2$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\overset{\cos\theta}{\cancel{\frac{\partial x}{\partial r}}} + \frac{\partial f}{\partial y}\cdot\overset{\sin\theta}{\cancel{\frac{\partial y}{\partial r}}} = 2x\cos\theta + 2y\sin\theta = 2r\,\cancel{(\cos^2\theta + \sin^2\theta)}^{1}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\overset{-r\sin\theta}{\cancel{\frac{\partial x}{\partial \theta}}} + \frac{\partial f}{\partial y}\cdot\overset{r\cos\theta}{\cancel{\frac{\partial y}{\partial \theta}}} = 2xr(-\sin\theta) + 2yr\cos\theta = 2r^2(-\sin\theta\cos\theta + \sin\theta\cos\theta) = 0$$

2 Integration

2.1 ORDINARY INTEGRATION   Riemann Sums



Integral over $[a, b]$ is limit of S, as $n \to \infty$:

$$S = \sum_{i=1}^{n} f(x_i)\left[\xi_i - \xi_{i-1}\right] \; : \; \xi_{i-1} < x_i < \xi_i$$

Riemann Sum — not all widths have to be equal; better to consider as a 'weighted sum'

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} S$$

- "area under surface"
- "weighted sum of lengths"

Ex.1 rod of uniform density ρ, uniform width w, variable thickness $t(x)$

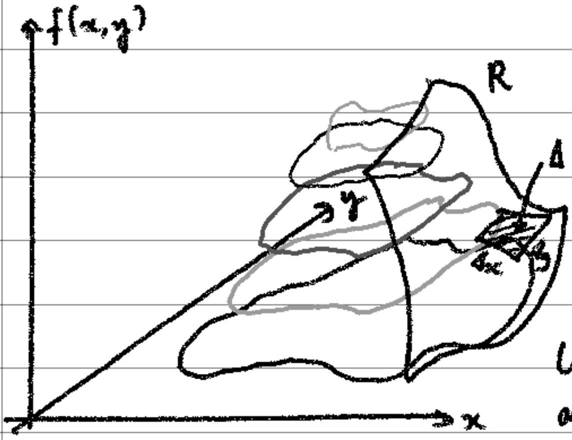
mass per unit length $\lambda(x)$:

$$\lambda(x) = \rho w \cdot t(x)$$

radius of gyration: radial distance to point with same M.O.I. if total mass of body there

LENGTH: $L = \int_0^L dx$; **MASS:** $m = \int_0^L \lambda\, dx$ (0th moment of λ) ; **C.O.M:** $\bar{x} = \frac{1}{m}\int_0^L \lambda x\, dx$ (1st moment of λ) ; **R.O.G:** $r_g^2 = \frac{1}{m}\int_0^L \lambda x^2\, dx$ (2nd moment of λ)

2.2 2-DIMENSIONAL INTEGRATION



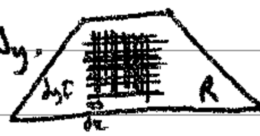
2D-Integral over Region R is limit of S, as $n \to \infty$:

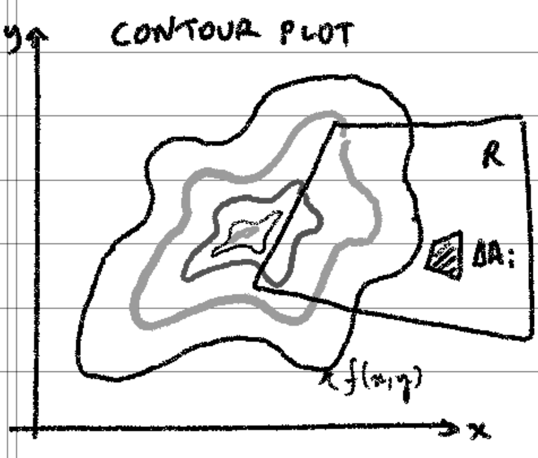
$$S = \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \; : \; \text{as } n \to \infty,\ \Delta A_i \to 0$$

$$\iint_R f(x,y)\,dA = \lim_{n \to \infty} S$$

- "volume under surface"
- "weighted sum of areas"

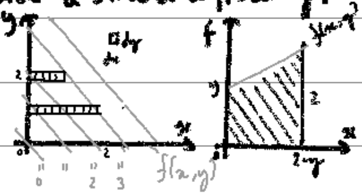
Considering R with uniform grid $\Delta x, \Delta y$, as $n \to \infty$: $\quad dA = dx\,dy$





Ex.2 $f(x,y) = x + y$; $R := \begin{cases} x = 0,\ y = 0 \\ y = 2 - x \end{cases}$

Find $\sum f(x,y)\,dx\,dy$. Consider a slice at a fixed y:



$$\Rightarrow dV = dy \sum f(x,y)\,dx = dy \int_{x=0}^{x=2-y} (x+y)\,dx$$

$$dV = dy\left[\frac{x^2}{2} + xy\right]_0^{2-y} = dy\left[2 - \frac{y^2}{2}\right] = A(y)\,dy \quad \therefore V = \int_{y=0}^{y=2} A(y)\,dy = \left[2y - \frac{y^3}{6}\right]_0^2 = \frac{8}{3}$$

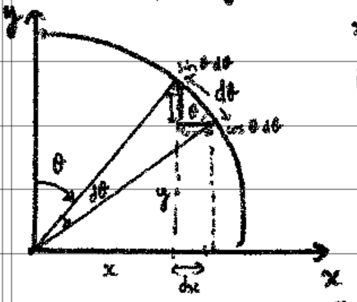
or $V = \int_{y=0}^{2}\left[\int_{x=0}^{2-y}(x+y)\,dx\right]dy$ * Consider order of limits; $dx\,dy$ OR $dy\,dx$

2.3 THE JACOBIAN MATRIX

PREAMBLE **Change of Variables in 1D Integration**

Ex 3. Area of a quadrant of a unit circle $\left(=\frac{\pi}{4}\right)$ $\quad A=\int_0^1 y\,dx=\int_0^1\sqrt{1-x^2}\,dx$

$x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$



Remember: • INTEGRAND: express $f(x)$ as $f[x(\theta)]$

$x=\sin\theta,\ f(x(\theta))=\sqrt{1-\sin^2\theta}=\cos\theta$

• LIMITS: express $x=\ldots$ as $\theta=\ldots$

$0<x<1 \equiv 0<\theta<\pi/2$

• DIFFERENTIAL: $dx=\frac{dx}{d\theta}d\theta$

$x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$ ☆ $dx=\boxed{\frac{dx}{d\theta}}d\theta$ — 1D Jacobian

"how much x changes as θ changes by $d\theta$"

• unevenly spaced
• same sum, different spacings

$$A=\int_0^1\sqrt{1-x^2}\,dx=\int_0^{\frac{\pi}{2}}\cos\theta\cdot\cos\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}(\cancel{\cos 2\theta}^{\,0}+1)\,d\theta=\frac{\pi}{4}$$

Change of Variables in 2D Integration

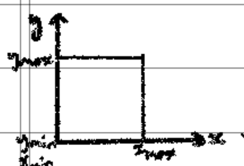
$$\iint_R f(x,y)\,dx\,dy=\iint_{R'} f[x(u,v),y(u,v)]\,|J|\,du\,dv$$

☆ Notation is reminder that it is a ratio of the partial derivatives of x, y, to u, v

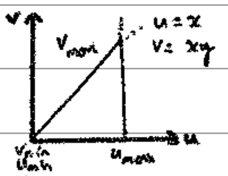
$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}x_u & y_u\\ x_v & y_v\end{vmatrix}$$

Remember: • INTEGRAND: express $f(x,y)$ as $f(x(u,v),y(u,v))$

• LIMITS: transform R_{xy} as R'_{uv} ☆ DRAW!



$\int_{y=y_{min}}^{y_{max}}\int_{x=x_{min}}^{x_{max}} \Rightarrow$



$\int_{v=v_{min}}^{v_{max}}\int_{u=u_{min}}^{u_{max}}$

☆ in xy or uv, may need to split into more than 1 integral

• DIFFERENTIAL:

$$dx\,dy=|J|\,du\,dv=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

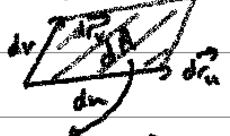
NOTE: $\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$

$$du\,dv=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|dx\,dy=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}dx\,dy$$

∴ compute easiest one!

2-D Jacobian Derivation

What area in the x, y plane does infinitesimal change du dv cover?



Area of Parallelogram $=\vec{a}\times\vec{b}=d\vec{r}_u\times d\vec{r}_v$

Position vector $\vec{r}$ is a vector field: $\vec{r}=\begin{pmatrix}x\\y\end{pmatrix}$

$$d\vec{r}=\frac{d\vec{r}}{du}du+\frac{d\vec{r}}{dv}dv=d\vec{r}_u+d\vec{r}_v$$

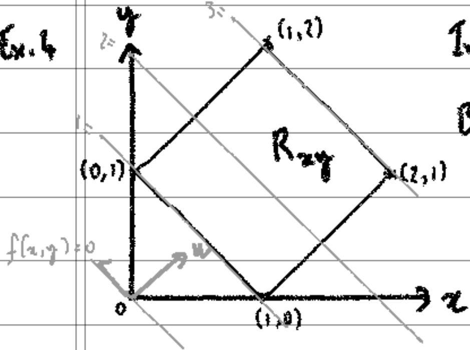
$$dA=|d\vec{A}| \text{ where } d\vec{A}=d\vec{r}_u\times d\vec{r}_v=\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}\,du\,dv$$

$$d\vec{r}_u=\frac{d}{du}\begin{pmatrix}x\\y\end{pmatrix}du=\begin{pmatrix}\frac{dx}{du}\\ \frac{dy}{du}\end{pmatrix}du\ ;\quad d\vec{r}_v=\frac{d}{du}\begin{pmatrix}x\\y\end{pmatrix}dv=\begin{pmatrix}\frac{dx}{dv}\\ \frac{dy}{dv}\end{pmatrix}dv$$

$$\therefore d\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{dx}{du} & \frac{dy}{du} & 0\\ \frac{dx}{dv} & \frac{dy}{dv} & 0\end{vmatrix}du\,dv=\hat{k}\begin{vmatrix}\frac{dx}{du} & \frac{dy}{du}\\ \frac{dx}{dv} & \frac{dy}{dv}\end{vmatrix}du\,dv=|J|\,du\,dv \quad \square$$

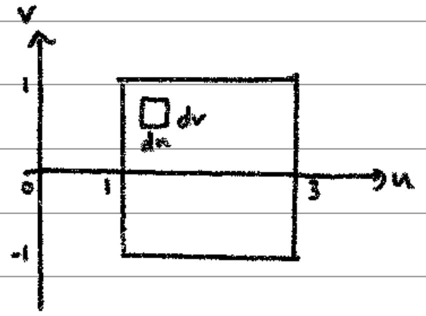
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

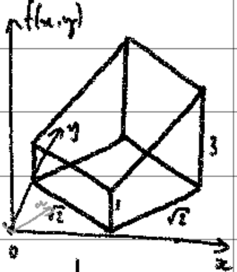
Ex.4



Integrate $f(x,y) = x+y$ over R_{xy}

Bounded by:
$$\begin{cases} \underbrace{x+y}_{u} = 1 & \Rightarrow u = 1 \\ x+y = 3 & \Rightarrow u = 3 \\ y-x = -1 & \Rightarrow v = -1 \\ \underbrace{y-x}_{v} = 1 & \Rightarrow v = 1 \end{cases}$$





$\therefore V = \frac{1}{2}(1+3)\sqrt{2}\cdot\sqrt{2} = 4$

$$I = \iint_{R_{xy}} f(x,y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=1-x}^{y=1+x}(x+y)\,dy\,dx + \int_{x=1}^{x=2}\int_{y=x-1}^{y=3-x}(x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{1-x}^{1+x} dx + \int_1^2 \left[xy + \frac{y^2}{2}\right]_{x+1}^{3-x} dx = \int_0^1 (2x^2+2x)\,dx + \int_1^2 (4+2x-2x^2)\,dx = 4$$

$$I = \iint_{R'_{uv}} f(x(u,v), y(u,v))\,|J|\,du\,dv = \int_{v=-1}^{v=1}\int_{u=1}^{u=3} u \cdot \frac{1}{2}\,du\,dv = \int_{-1}^{1}\left[\frac{u^2}{4}\right]_1^3 dv = [2v]_{-1}^{1} = 4 \quad \square$$

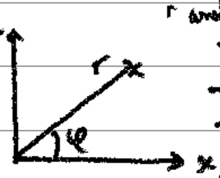
Geometric derivation of Jacobian:

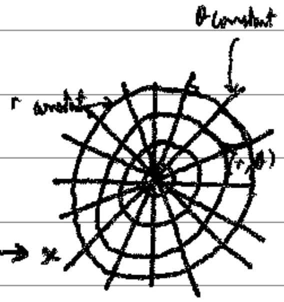
Area of Diamond in x-y plane (R_{xy}): $(\sqrt{2})^2 = 2$

Area of Diamond in u-v plane (R_{uv}): $(2)^2 = 4$

$\therefore$ an element $du\,dv$ covers an area $\underbrace{\tfrac{1}{2}}_{|J|}du\,dv$ in x-y plane

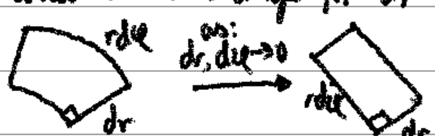
2.4 EX: PLANE POLAR CO-ORDINATES





$x = r\cos\varphi$, $y = r\sin\varphi$ $\Rightarrow$ $r = \sqrt{x^2+y^2}$, $\tan\varphi = \frac{y}{x}$

What is dA change for dr and dφ?



$\therefore dA = r\,dr\,d\varphi = |J|\,dr\,d\varphi$

$\Rightarrow |J| = r$

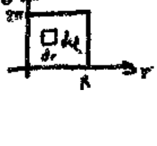
$$|J| = \left|\frac{\partial(x,y)}{\partial(r,\varphi)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r(\cos^2\varphi + \sin^2\varphi) = r$$

$$\therefore \iint_{F(A)} f(x,y)\,dx\,dy = \iint_A f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

difficult in Cartesian co-ordinates
has circular symmetry $\therefore$ use plane-polar co-ordinates

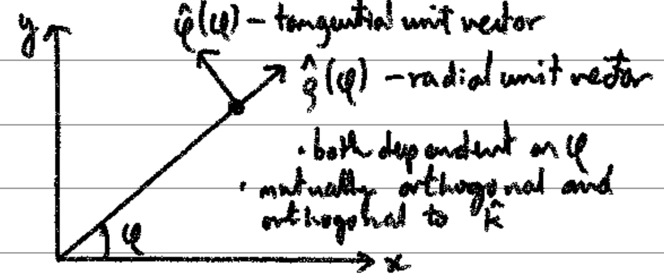
Ex.5 Disk of radius R, surface mass density $\sigma(x,y) = \frac{B}{\sqrt{x^2+y^2}}$. What is the mass of the disk?

$\sigma(x(r,\varphi), y(r,\varphi)) = \frac{B}{r}$ $\quad |J| = r$

R_{xy}:  $R'_{r\varphi}$: 

$$\Rightarrow I = \int_{\varphi=0}^{\varphi=2\pi}\int_{r=0}^{r=R}\frac{B}{r}\cdot r\,dr\,d\varphi = \int_{\varphi=0}^{2\pi}[Br]_0^R\,d\varphi = BR[\varphi]_0^{2\pi} = 2\pi BR$$

2.5 ↳ UNIT VECTORS IN PLANE POLAR CO-ORDINATES



Differentiate unit vectors by writing in x-y:

$\hat{\rho} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \Rightarrow \frac{d\hat{\rho}}{d\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} = \hat{\varphi}$

$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \Rightarrow \frac{d\hat{\varphi}}{d\varphi} = -\cos\varphi\,\hat{i} - \sin\varphi\,\hat{j} = -\hat{\rho}$

☆ $\frac{\partial\hat{\rho}}{\partial r} = 0$, $\frac{\partial\hat{\varphi}}{\partial r} = 0$ as they don't depend on r

DIFFERENTIATING VECTOR FIELDS IN PLANE-POLAR

general vector field: $\vec{A} = A_r(r,\varphi)\hat{\rho}(\varphi) + A\ (r,\varphi)\hat{\varphi}(r,\varphi)$

By the product rule: $\frac{d\vec{A}}{d\varphi} = \frac{\partial}{\partial\varphi}(A_r\hat{\rho}) + \frac{\partial}{\partial\varphi}(A_\varphi\hat{\varphi})$

Differentiating position vector $\vec{r} = r\hat{\rho}$ to derive Jacobian for plane polar co-ordinates:

Generally, $d\vec{r} = \frac{d\vec{r}}{dr}dr + \frac{d\vec{r}}{d\varphi}d\varphi$ and $|J| = \left|\frac{d\vec{r}}{dr} \times \frac{d\vec{r}}{d\varphi}\right|$

$$\left.\begin{aligned} \frac{d\vec{r}}{dr}(r,\varphi) &= \frac{\partial}{\partial r}(r\hat{\rho}) = \cancelto{1}{\frac{dr}{dr}}\hat{\rho} + r\cancelto{0}{\frac{d\hat{\rho}}{dr}} = \hat{\rho} \\ \frac{d\vec{r}}{d\varphi}(r,\varphi) &= \frac{\partial}{\partial\varphi}(r\hat{\rho}) = \cancelto{0}{\frac{dr}{d\varphi}}\hat{\rho} + r\cancelto{\hat{\varphi}}{\frac{d\hat{\rho}}{d\varphi}} = r\hat{\varphi} \end{aligned}\right\} \text{they are orthogonal} \therefore |J| = r$$

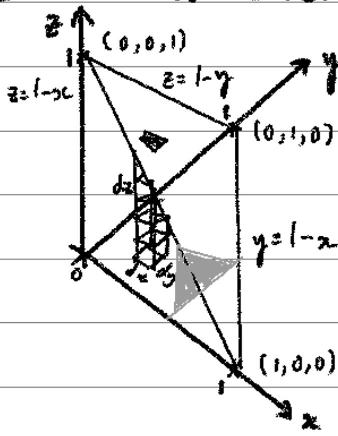
$$\frac{d\vec{r}}{dr} \times \frac{d\vec{r}}{d\varphi} = \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{k} \\ 1 & 0 & 0 \\ 0 & r & 0 \end{vmatrix} = r\hat{k} \quad \therefore |J| = r$$

3 3D Integration

3.1 3-Dimensional Integration in Cartesian Coordinates

Ex.1 Integrate $f(x,y,z) = \alpha x$ over tetrahedron

Bounded by: $\begin{cases} x=0,\ y=0,\ z=0 \\ x+y+z=1 \end{cases}$



Integrating in order dz, dy, dx, we have:

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \boxed{\int_{z=0}^{z=1-x-y} (\alpha x)\, dz}\, dy\, dx$$

weighting for 2D integral

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\alpha x z\right]_0^{1-x-y} dy\, dx = \alpha \int_{x=0}^{1} \int_{y=0}^{1-x} (x - x^2 - xy)\, dy\, dx$$

$$= \alpha \int_{x=0}^{1} \left[x(1-x)y - \frac{y^2}{2}x\right]_0^{1-x} dx = \alpha \int_{x=0}^{1} \boxed{\frac{x(1-x)^2}{2}}\, dx = \frac{\alpha}{24}$$

cross-section at constant x

3.2 The Jacobian Matrix in 3D

Change of Variables in 3D Integration

$$I = \iiint_{R_{xyz}} f(x,y,z)\, dx\, dy\, dz = \iiint_{R_{uvw}} f(x(u,v,w), y(u,v,w), z(u,v,w))\, |J|\, du\, dv\, dw$$

For $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$, we have:

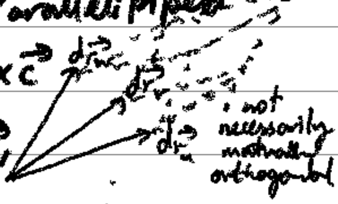
$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw = d\vec{r}_u + d\vec{r}_v + d\vec{r}_w$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \implies d\vec{r}_u = \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}\right) du, \text{ etc.}$$

Volume of Parallelipiped

$$dV_{uvw} = \vec{a} \cdot \vec{b} \times \vec{c}$$

$$= d\vec{r}_u \cdot d\vec{r}_v \times d\vec{r}_w$$



$$dV_{uvw} = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} du\, dv\, dw = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} du\, dv\, dw = \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{=|J|} du\, dv\, dw$$

(also referred to as ρ instead of r)

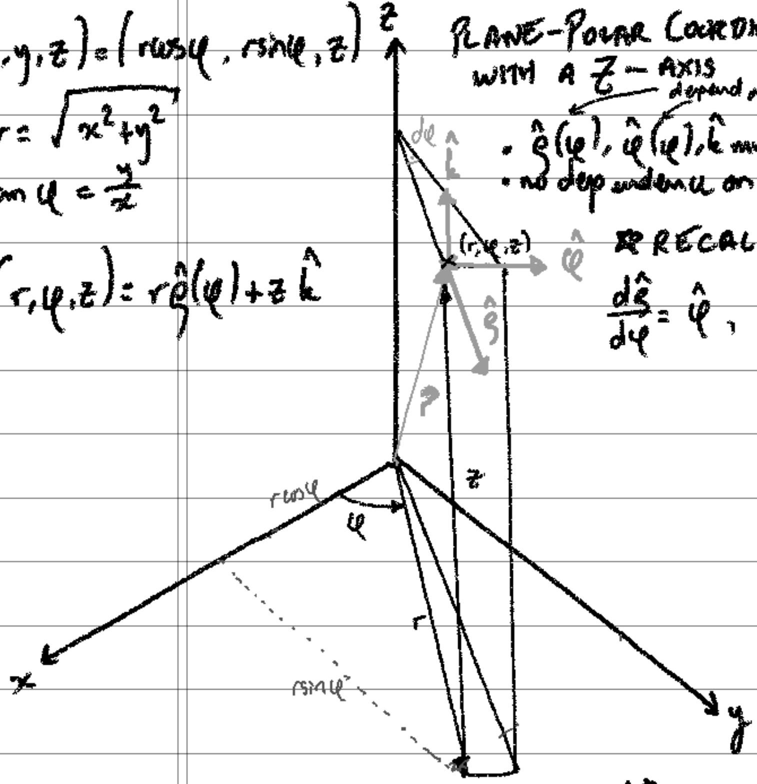
3.3 3-Dimensional Integration in Cylindrical Polar Coordinates

$(x,y,z) = (r\cos\varphi, r\sin\varphi, z)$

$r = \sqrt{x^2+y^2}$

$\tan\varphi = \frac{y}{x}$

$\vec{r}(r,\varphi,z) = r\hat{\rho}(\varphi) + z\hat{k}$



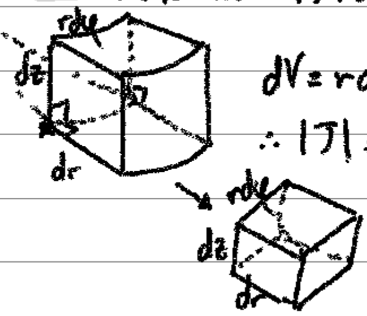
PLANE-POLAR COORDINATES WITH A Z-AXIS

- $\hat{\rho}(\varphi), \hat{\varphi}(\varphi), \hat{k}$ mutually $\perp$ (depend on φ)
- no dependence on r or z

RECALL:

$$\frac{d\hat{\rho}}{d\varphi} = \hat{\varphi}, \quad \frac{d\hat{\varphi}}{d\varphi} = -\hat{\rho}$$

Volume Element: $|J|\,dr\,d\varphi\,dz$



$dV = r\,dr\,d\varphi\,dz$

$\therefore |J| = r$

$$|J| = \begin{vmatrix} x_r & y_r & z_r \\ x_\varphi & y_\varphi & z_\varphi \\ x_z & y_z & z_z \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Find $d\vec{r}$ using $\hat{\rho}, \hat{\varphi}, \hat{k}$ to find dV

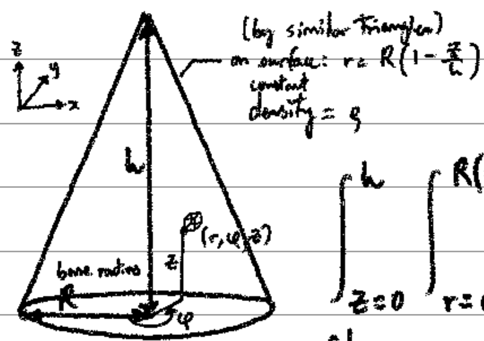
$\vec{r} = r\hat{\rho}(\varphi) + z\hat{k}$

$$d\vec{r} = \frac{d\vec{r}}{dr}dr + \frac{d\vec{r}}{d\varphi}d\varphi + \frac{d\vec{r}}{dz}dz$$

$$\underbrace{d\vec{r}_\rho = \frac{d\vec{r}}{dr}dr = dr\,\hat{\rho}, \quad d\vec{r}_\varphi = \frac{d\vec{r}}{d\varphi}d\varphi = r\,d\varphi\,\hat{\varphi}, \quad d\vec{r}_z = \frac{d\vec{r}}{dz}dz = dz\,\hat{k}}_{\text{orthogonal}}$$

orthogonal $\therefore dV = r\,dr\,d\varphi\,dz$

Ex.2 Vertical C.O.M. of solid cone



(by similar triangles) on surface: $r = R(1-\frac{z}{h})$

constant density $= \rho$

$$\bar{z} = \frac{\overbrace{\sum m_i z_i}^{\text{1st moment of volume}}}{\underbrace{\sum m_i}_{\text{0th moment of volume}}} = \frac{\cancel{\rho}\iiint z\,dV}{\cancel{\rho}\iiint dV} = \frac{\iiint z\,dV}{\frac{1}{3}\pi R^2 h}$$

$$\int_{z=0}^{h}\int_{r=0}^{R(1-\frac{z}{h})}\int_{\varphi=0}^{2\pi} z\,\underset{|J|}{\underset{\parallel}{R}}\,d\varphi\,dr\,dz = \int_{z=0}^{h}\int_{r=0}^{R} 2\pi z R\,dr\,dz$$

$$I = \int_{z=0}^{h} \pi z R^2\,dz = \int_{z=0}^{h} 2\pi R^2\left(1-\frac{z}{h}\right)^2 = \pi R^2\left[\frac{z^2}{2} - \frac{2z^3}{3h} + \frac{z^4}{4h^2}\right]_0^h = \frac{\pi R^2 h^2}{12}$$

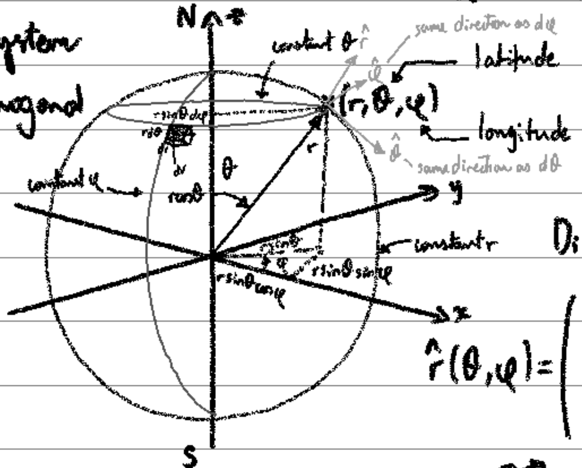
$$V = \pi\int_{z=0}^{h} R^2\left(1-\frac{z}{h}\right)^2 dz = \pi R^2\left[z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right]_0^h = \pi R^2\left[h - h + \frac{h}{3}\right] = \frac{\pi R^2 h}{3}$$

$$\bar{z} = \frac{I}{V} = \frac{\frac{\pi R^2 h^2}{12}}{\frac{\pi R^2 h}{3}} = \frac{h}{4}$$

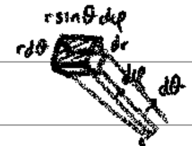
3.4 3-Dimensional Integration in Spherical Polar Coordinates

orthogonal system
$\hat{r}, \hat{\theta}, \hat{\varphi}$ are orthogonal

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$



Volume Element $\therefore dV = (r\sin\theta\, d\varphi)(r\, d\theta)\, dr$
$= r^2 \sin\theta\, dr\, d\theta\, d\varphi$



Differentiate unit vectors, expressing in $\hat{i}, \hat{j}, \hat{k}$:

$$\hat{r}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}; \hat{\theta}(\theta,\varphi) = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}; \hat{\varphi}(\varphi) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\vec{r} = r\,\hat{r}(\theta,\varphi) \Rightarrow d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \varphi} d\varphi$$

$$\frac{\partial \hat{r}}{\partial \theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \hat{\theta} \qquad \frac{\partial \hat{r}}{\partial \varphi} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} = \sin\theta\,\hat{\varphi}$$

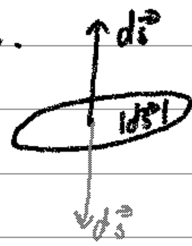
$$d\vec{r}_r = \frac{\partial \vec{r}}{\partial r} dr = dr\,\hat{r} \qquad d\vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} d\theta = r\,d\theta\,\hat{\theta} \qquad d\vec{r}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} d\varphi = r\sin\theta\, d\varphi\,\hat{\varphi}$$

mutually orth'l $\perp$ $\therefore dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

3.5 Surface Integrals

normal to the surface

On a 3D surface, define infinitesimal area element $d\vec{s}$ with area $|d\vec{s}|$.
* directions: in, out, up, down are matter of convention.

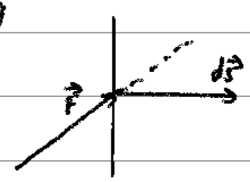


Used for calculating:

⟶ Area $= \iint_S |d\vec{s}|$ ← 3D surface

⟶ Total of a Scalar $= \iint_S \sigma |d\vec{s}|$ ← surface density
e.g. charge

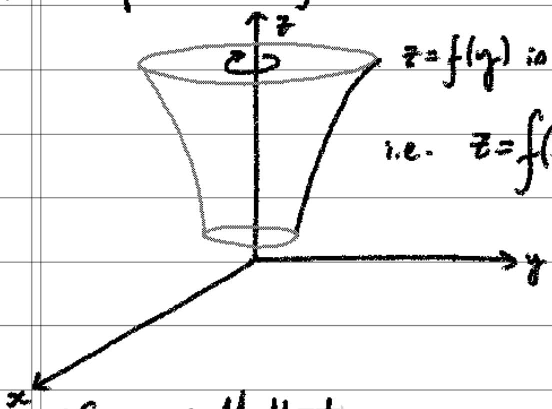
⟶ Flux $= \iint_S \underbrace{\vec{F} \cdot d\vec{s}}_{\text{rate of flow of energy flux through } d\vec{s}}$ ← any vector field



the rate of energy flow through unit area $\perp$r to $\vec{F}$

vector: Flux gives the magnitude / direction of the flow of [/]
scalar: Flux gives the surface integral of the perpendicular component of a vector field over its surface

Ex.3 Surface Area of Revolution



$z = f(y)$ is rotated

i.e. $z = f(r)$

Taking a slice of z:

$$dV = \pi r^2(z)\,dz \Rightarrow V = \pi \int r^2(z)\,dz$$

$$(ds)^2 = (dr)^2 + (dz)^2 \Rightarrow ds = dr\sqrt{1+\left(\frac{dz}{dr}\right)^2} = dz\sqrt{1+\left(\frac{dr}{dz}\right)^2}$$

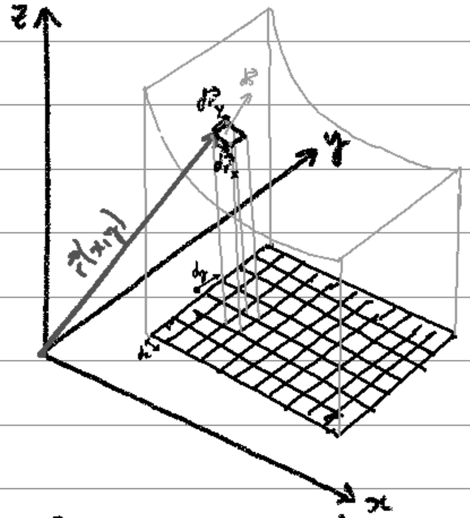
$$dA = 2\pi r(z)\,ds \Rightarrow A = 2\pi \int_{z=z_{min}}^{z_{max}} r(z)\sqrt{1+\left(\frac{dr}{dz}\right)^2}\,dz$$

General Method

$\vec{r}$ of a point can be defined with 2 co-ordinates e.g. $\vec{r} = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

surface defined: $z = f(x,y)$

$\therefore$ vector to surface: $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$



$$d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy$$

$$\therefore d\vec{r}_x = \vec{r}_x\,dx,\quad d\vec{r}_y = \vec{r}_y\,dy$$

$$d\vec{S} = \left(\vec{r}_x \times \vec{r}_y\right)dx\,dy = \vec{N}\,dx\,dy$$

☆ You might want $-d\vec{S}$ depending on the question

More generally, if we can express the vector to the surface in terms of only 2 variables u, v:

$$d\vec{S} = \left(\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\right)du\,dv$$

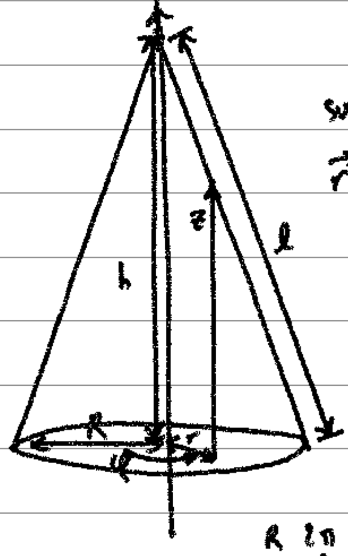
$z = f(x,y)$, $z = f(r,\varphi)$

e.g. $\vec{r} = r\hat{\rho}(\varphi) + z\hat{k}$

$\vec{r}(r,\varphi) = r\hat{\rho}(\varphi) + f(r,\varphi)\hat{k}$

$$\therefore d\vec{S} = \left(\frac{d\vec{r}}{dr} \times \frac{d\vec{r}}{d\varphi}\right)dr\,d\varphi$$

Ex.5 curved surface area of a cone $\left(= \pi R \ell\right)$, where $\ell = \sqrt{h^2 + r^2}$



SURFACE: $r = R\left(1 - \frac{z}{h}\right), \quad z = h\left(1 - \frac{r}{R}\right)$

$$\vec{r}(r, \varphi, z) = r\hat{\rho}(\varphi) + z\hat{k} \Rightarrow \vec{r}(r, \varphi) = r\hat{\rho}(\varphi) + h\left(1 - \frac{r}{R}\right)\hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \hat{\rho} - \frac{h}{R}\hat{k}, \quad \frac{\partial \vec{r}}{\partial \varphi} = r\hat{\varphi} \quad \left(\because \frac{\partial \hat{\rho}}{\partial \varphi} = \hat{\varphi}\right)$$

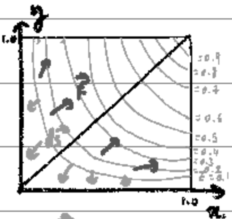
$$\Rightarrow \vec{N} = \vec{r}_r \times \vec{r}_\varphi = \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{k} \\ 1 & 0 & -\frac{h}{R} \\ 0 & r & 0 \end{vmatrix} = \begin{pmatrix} \frac{hr}{R} \\ 0 \\ r \end{pmatrix}$$

$$\therefore d\vec{s} = \left(\frac{h}{R}\hat{\rho} + \hat{k}\right) r\,dr\,d\varphi \Rightarrow |d\vec{s}| = \sqrt{1 + \frac{h^2}{R^2}}\, r\,dr\,d\varphi = \frac{\ell r}{R}\,dr\,d\varphi$$

$$\Rightarrow A = \iint_S |d\vec{s}| = \int_{r=0}^{R}\int_{\varphi=0}^{2\pi} \frac{\ell r}{R}\,d\varphi\,dr = \int_{r=0}^{R} 2\pi \frac{\ell r}{R}\,dr = \left[2\pi \frac{r^2 \ell}{2R}\right]_0^R = \pi R \ell \quad \square$$

Ex.4 Cartesian flux

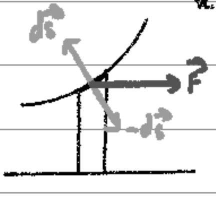
$z = f(x, y) = xy$ with $\vec{F} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$. Find the flux into the surface. $\iint_S \vec{F} \cdot d\vec{s}$ $\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases}$



$$\vec{r} = \begin{pmatrix} x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} x \\ z/x \\ z \end{pmatrix} \quad \because \vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \quad \therefore \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix} \Rightarrow ds = \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix} dx\,dy$$

※ $|\vec{N}| > 1$ means area $dx\,dy$ is larger on plane than the x-y plane

To get flux into the surface, we want $-\vec{F} \cdot d\vec{s}$



$$-\vec{F} \cdot d\vec{s} = \begin{pmatrix} x^2 \\ y^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y \\ x \\ -1 \end{pmatrix} = \begin{pmatrix} x^2 y \\ y^2 x \\ 0 \end{pmatrix} dx\,dy$$

$$\therefore \int_{y=0}^{1}\int_{x=0}^{1} (x^2 y + y^2 x)\,dx\,dy = \int_{y=0}^{1} \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^1 dy = \int_{y=0}^{1} \left(\frac{y}{3} + \frac{y^2}{2}\right) dy = \frac{1}{3}$$

4 Line Integrals

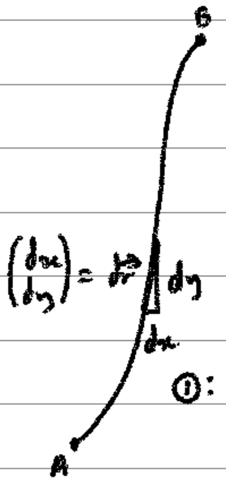
Integral along a path, a curve C in space.

e.g. $W = \int_C \vec{F}\cdot d\vec{r} = \lim_{\Delta r_i \to 0} \sum_i \vec{F}\cdot \Delta\vec{r}_i$ $\quad \int_C \overset{\text{scalar}}{\ell}\, d\vec{r} \quad \int_C \vec{F}\times d\vec{r}$

curve

$\Delta\vec{r}_i = \lim_{\Delta t\to 0}\left(\underbrace{\frac{\Delta r}{\Delta t}}_{\approx \frac{dr}{dt} = \vec{v}}\Delta t\right)$

4.1 EVALUATING LINE INTEGRALS (IN 2D)



Curve $y = y(x)$ or $x = x(y)$ and $\vec{F}(x,y) = \vec{F}_x(x,y)\,\hat{i} + \vec{F}_y(x,y)\,\hat{j}$

$$\int_A^B \vec{F}\cdot d\vec{r} = \sum_i \Big(\underset{①}{F_{x_i}(x_i,y_i)\,dx_i} + \underset{②}{F_{y_i}(x_i,y_i)\,dy_i}\Big)$$

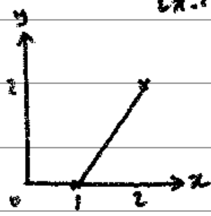
※ Using the curve, we can write each term solely using x, then integrate w.r.t x

Same is true in 3D, because point on a line is defined by only 1 co-ordinate

e.g. $z = x + y$, $y = x^2 \Rightarrow z = x + x^2 = \sqrt{y} + y$

①: $\displaystyle\int_{x_A}^{x_B} F_x(x, y(x))\,dx = \int_{y_A}^{y_B} F_x(x(y), y)\,\frac{dx}{dy}(y)\,dy$ $\quad$ ②: $\displaystyle\int_{y_A}^{y_B} F_y(x(y), y)\,dy = \int_{x_A}^{x_B} F_y(x, y(x))\,\frac{dy}{dx}(x)\,dx$

Ex.1 Linear Cartesian Example



Path from $A(1,0)$ to $B(2,2)$ is $y = 2x - 2$, $\vec{F} = \begin{pmatrix}2xy\\x^2\end{pmatrix}$ $\quad x = \frac{1}{2}y + 1$

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B F_x\,dx + \int_{A'}^{B'} F_y\,dy = \int_C 2xy\,dx + \int_C x^2\,dy = \int_1^2 2x(2x-2)\,dx + \int_0^2 \left(\frac{y}{2}+1\right)^2 dy = 8$$

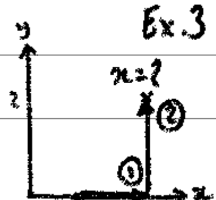
Ex.2 Parametric Cartesian Example

(useful for circles, θ)

$\begin{cases} x(t) = 2t + 1 \\ y(t) = 4t \end{cases}$ from $0 \le t \le \frac{1}{2}$, $\vec{F} = \begin{pmatrix}2xy\\x^2\end{pmatrix}$; $\quad dx = \frac{dx}{dt}dt = 2\,dt$, $\quad dy = \frac{dy}{dt}dt = 4\,dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C 2xy\,dx + \int_C x^2\,dy = \int_0^{1/2} 2(2t+1)(4t)\,2\,dt + \int_0^{1/2} (2t+1)^2\, 4\,dt = 8 \quad \square$$

Ex.3 Different path, same endpoints



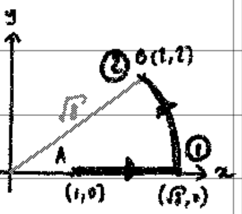
Along ①

$$\int \vec{F}\cdot d\vec{r} = \int_C \overset{I_2 = 0\ \because y=0}{\underset{I_1 = 0\ \because y = 0}{2xy\,dx}} + \int_C \overset{I_2 = 0\ \because dy = 0}{\underset{\text{still exists!}}{x^2\,dy}} = \int_0^2 4\,dy = [4y]_0^2 = 8$$

Along ②

independent of path taken

Ex.6 Plane Polar Co-ordinates. Different path, same end points.



① $\varphi = 0 \ , \ 1 \le r \le \sqrt{8}$ $\qquad \vec{F}\cdot d\vec{r} = 2xy\,dx + x^2\,dy \qquad x = r\cos\theta,\ y = r\sin\theta$

② $r = \sqrt{8} \ , \ 0 \le \varphi \le \frac{\pi}{4} \qquad dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \varphi}\,d\varphi$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \overset{I_1}{\int_C 3\cos^2\varphi \sin\varphi \; r^2\,dr} + \overset{I_2}{\int_C \cos\varphi\,(1 - 3\sin^2\varphi)\; r^3\,d\varphi}$$

① $\varphi = 0 \Rightarrow \sin\varphi = 0 \qquad \therefore I_1 = 0$

$\quad d\varphi = 0 \qquad \therefore I_2 = 0$

② $dr = 0 \qquad \therefore I_1 = 0$

③ $$\int_0^{\pi/4} \overset{(\sqrt{8})^3}{\cancel{r^3}}(\cos\varphi - 3\cos\varphi\sin^2\varphi)\,d\varphi = 8\sqrt{8}\left[\sin\varphi - \sin^3\varphi\right]_0^{\pi/4} = 8$$

independent of path taken

4.2 PATH INDEPENDENCE & CONSERVATIVE FIELDS

$y = 2x - 2$ $\qquad$ $y = 0 \ [1,2]$, $x = 2 \ [0,2]$

Ex.5 $\vec{F_1} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}, \ \vec{F_2} = \begin{pmatrix} 2xy \\ -x^2 \end{pmatrix}$ From $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ Path 1: / Path 2: ⌟

① Considering $\vec{F_1}$: $\int_C \vec{F_1}\cdot d\vec{r} = \int_C 2xy\,dx + \int_C x^2\,dy = I_1 + I_2$

Path 1: $\frac{10}{3} + \frac{14}{3} = 8$; Path 2: $0 + 8 = 8$

independent of path taken

② Considering $\vec{F_2}$: $\int_C \vec{F_2}\cdot d\vec{r} = \int_C 2xy\,dx - \int_C x^2\,dy = I_1 + I_2$

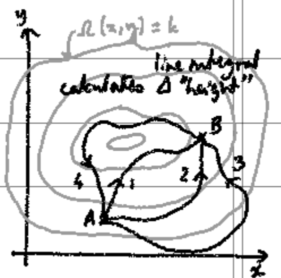
Path 1: $\frac{10}{3} - \frac{14}{3} = -\frac{4}{3}$ $\qquad$ Path 2: $0 - 8 = -8$

dependent of path taken $\quad$ parent function: $\Omega(x,y)$

$\vec{F_1}$ is an exact differential, $\vec{F_1} = P(x,y)\,dx + Q(x,y)\,dy \quad \therefore \frac{dP}{dy} = \frac{dQ}{dx}$

For an exact differential, the path doesn't matter; only the endpoints

i.e. $\int_C 2xy\,dx + \int_C x^2\,dy = \int_C d\Omega(x,y) \overset{\text{(by FTC 1)}}{=} \Omega_B(x,y) - \Omega_A(x,y)$



closed path / 'loop'

$$\int_{\text{Path 1}} \vec{F_1}\cdot d\vec{r} = \Omega_B - \Omega_A \ ; \ \int_{\text{Path 4}} \vec{F_1}\cdot d\vec{r} = \Omega_A - \Omega_B \Rightarrow \int_{\text{Path 1}} \vec{F_1}\cdot d\vec{r} + \int_{\text{Path 4}} \vec{F_1}\cdot d\vec{r} = \oint_C \vec{F_1}\cdot d\vec{r} = 0$$

$$\therefore \vec{F}\cdot d\vec{r} \text{ is exact} \Leftrightarrow \vec{F} = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} \ \exists\, \Omega(x,y,z) \Leftrightarrow \oint_C \vec{F}\cdot d\vec{r} = 0 \Leftrightarrow \vec{F} \text{ is conservative}$$

5 Gradient

Often denoted as: grad Ω or $\nabla\Omega$ — del operator, 'nabla' symbol

5.1 For scalar field $\Omega(x,y,z)$, the vector field $\vec{F} = \nabla\Omega$ is conservative ($\oint \vec{F}\cdot d\vec{r} = 0$)

$$\vec{F} = \begin{pmatrix} \frac{\partial\Omega}{\partial x} \\ \frac{\partial\Omega}{\partial y} \\ \frac{\partial\Omega}{\partial z} \end{pmatrix} = \text{grad}\,\Omega = \nabla\Omega \qquad \text{In general, } \nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix} \quad \begin{matrix} f: \mathbb{R}^n \to \mathbb{R} \\ \nabla f: \mathbb{R}^n \to \mathbb{R}^n \\ \text{at } p = (x_1, \dots, x_n) \end{matrix}$$

∇ is an operator,

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ Operating on a scalar field produces a vector field.

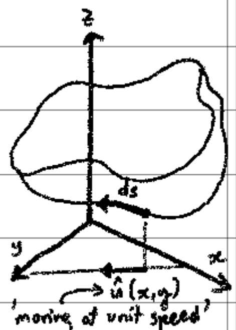
Here, Ω is the potential associated with vector field $\vec{F}$.

☆ Take care with signs associated. e.g. in gravitational fields, $g = -\nabla\phi$, but $\vec{F} = -m\vec{g}$.

5.2 ## DIRECTIONAL DERIVATIVES

at point P_0: $\left.\frac{d\Omega}{ds}\right|_{P_0,\hat{u}}$

Using $\nabla\Omega$, we can find the derivative of Ω in any direction



$$d\Omega = \nabla\Omega\cdot d\vec{r} = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz$$

$\left.\frac{d\Omega}{ds}\right|_{\hat{u}}$ = direction derivative in direction of $\hat{u}$

What is the direction in unit vector, $\hat{u}$?

= slope of slice by vertical plane $// \hat{u}$

By the chain rule: $\left.\frac{d\Omega}{ds}\right|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w|\,\cancel{|\hat{u}|}^{1}\cos\theta$ $\therefore$ max when $\cos\theta = 0$

$\Rightarrow \theta = 0 \Rightarrow \hat{u} = \text{dir}(\nabla\Omega)$

$\therefore \text{dir}(\nabla\Omega)$ is direction of fastest increase (maximum gradient)

Ex.1 choose $\hat{u} = \hat{k}$

$$\frac{d\Omega}{ds} = \begin{pmatrix} \frac{d\Omega}{dx} \\ \frac{\partial\Omega}{\partial y} \\ \frac{d\Omega}{\partial z} \end{pmatrix} \cdot \hat{k} = \frac{\partial\Omega}{\partial z}$$

At $p = (x,y,z)$, what direction $\hat{u}_{max}$ is $\frac{d\Omega}{ds}$ max?

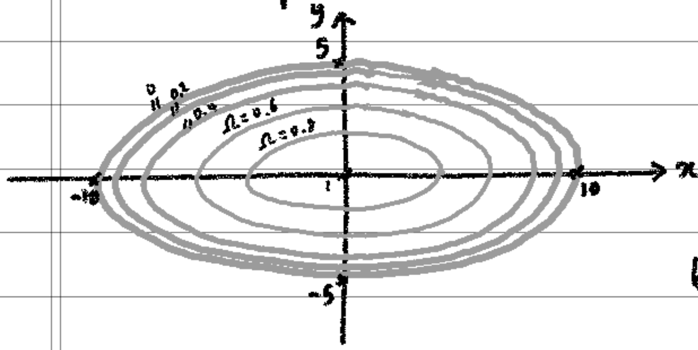
$$\hat{u}_{max} = \frac{\nabla\Omega}{|\nabla\Omega|}$$

What is the value of the steepest gradient?

$$\left(\frac{d\Omega}{ds}\right)_{max} = \nabla\Omega\cdot\hat{u}_{max} = \frac{\nabla\Omega\cdot\nabla\Omega}{|\nabla\Omega|} = |\nabla\Omega|$$

Ex. 2 2D elliptical hill (units in km)

$\Omega = z(x,y) = 1 - \frac{1}{100}x^2 - \frac{1}{25}y^2$



a) At $\binom{3}{-2}$, direction of max. gradient & value? (unit vector)

$\nabla z = \begin{bmatrix} -\frac{x}{50} \\ -\frac{2y}{25} \end{bmatrix} \Rightarrow \nabla z(3,2) = \begin{bmatrix} -0.06 \\ -0.16 \end{bmatrix}$

$\therefore$ magnitude: $\left| \begin{matrix} -0.06 \\ -0.16 \end{matrix} \right| \approx 0.17$

direction: $\frac{\nabla z(3,2)}{|\nabla z(3,2)|} = \begin{pmatrix} -\frac{0.06}{0.17} \\ -\frac{0.16}{0.17} \end{pmatrix}$

b) At $\binom{3}{-2}$, derivative in direction of $\binom{1}{1}$?

$\Rightarrow \hat{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$\therefore \nabla z(3,2) \cdot \hat{u} = \begin{pmatrix} -0.06 \\ -0.16 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \approx -0.155$ $\therefore$ downhill

5.3 Gradient in Other Co-ordinate Systems

In general, $d\Omega = \nabla\Omega \cdot d\vec{r}$

$\nabla\Omega$ in cylindrical polar co-ordinates

$$d\Omega = \frac{\partial\Omega}{\partial r}dr + \frac{\partial\Omega}{\partial\varphi}d\varphi + \frac{d\Omega}{dz}dz$$

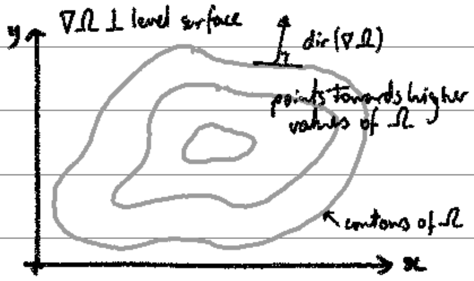
$$d\vec{r} = dr\,\hat{\varrho} + r\,d\varphi\,\hat{\varphi} + dz\,\hat{k} \quad \therefore \nabla\Omega(r,\varphi,z) = \frac{\partial\Omega}{\partial r}\hat{\varrho} + \frac{1}{r}\frac{\partial\Omega}{\partial\varphi}\hat{\varphi} + \frac{\partial\Omega}{\partial z}\hat{k} = \begin{pmatrix} \Omega_r \\ \frac{1}{r}\Omega_\varphi \\ \Omega_z \end{pmatrix}$$

$\nabla\Omega$ in spherical polar co-ordinates

$$d\Omega = \frac{\partial\Omega}{\partial r}dr + \frac{\partial\Omega}{\partial\varphi}d\varphi + \frac{d\Omega}{d\theta}d\theta$$

$$d\vec{r} = dr\,\hat{\varrho} + r\sin\theta\,d\varphi\,\hat{\varphi} + r\,d\theta\,\hat{\theta} \quad \therefore \nabla\Omega(r,\varphi,\theta) = \frac{\partial\Omega}{\partial r}\hat{\varrho} + \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial\varphi}\hat{\varphi} + \frac{1}{r}\frac{\partial\Omega}{\partial\theta}\hat{\theta}$$

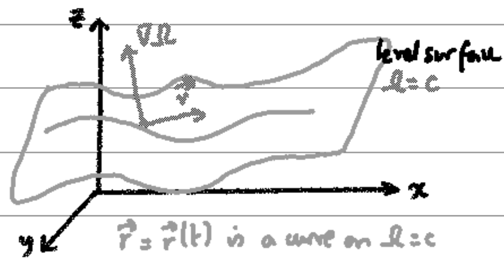
5.4 Gradient & Normal to a Surface



In 2D $\Omega(x,y)$, $\frac{d\Omega}{ds} = \nabla\Omega \cdot \hat{u}$. Along a contour, $\frac{d\Omega}{ds} = 0$

$\therefore \nabla\Omega \perp$ line of constant Ω

In 3D $\Omega(x,y,z)$, $\nabla\Omega \perp$ surfaces of constant Ω



Consider the 3D surface: $z = f(x,y)$

Let $\Omega(x,y,z) = f(x,y) - z$, then surface $\Omega = 0$

$\therefore$ vector normal to surface at (x,y,z) is $\vec{n} = \nabla\Omega(x,y,z)$

$\Rightarrow \Delta\Omega \approx \nabla\Omega \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$. Level when $\Delta\Omega = 0$

$\therefore$ tangent plane at (x_0, y_0, z_0) is: $\frac{\partial\Omega}{\partial x}(\underbrace{x - x_0}_{\Delta x}) + \frac{\partial\Omega}{\partial y}(\underbrace{y - y_0}_{\Delta y}) + \frac{\partial\Omega}{\partial z}(\underbrace{z - z_0}_{\Delta z}) = 0$

6 Divergence

Often referred to as: $div(\vec{B})$ or $\nabla . \vec{B}$

6.1 $\nabla . \vec{B} = \lim_{V \to 0} \left(\frac{1}{V} \oiint_S \vec{B} \cdot d\vec{S} \right)$

- not sensitive to rotation / translation
- sensitive to enlargements
- can be interpretted as "how much flow is expanding" or the source rate (i.e. amount of fluid added to a system/unit time/unit area)

→ closed surface, $d\vec{S}$ is out

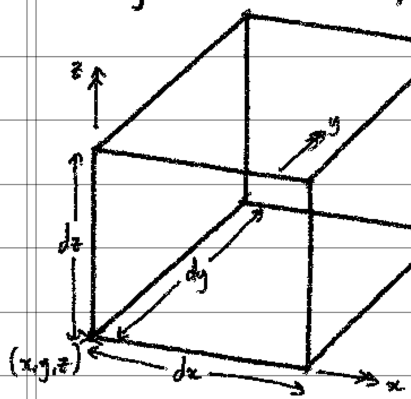
∴ divergence is a 'flux density'

→ operates on a vector field to produce a scalar field

{ gives the "quantity of the vector field's source at each point"
{ represents volume density of the outward flux from dV around a given point

Geometric Derivation in Carterian Co-ordinates

$\vec{B}(x,y,z) = B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$



We want $\sum_{i=1}^{6} F_i$ where F_i is the flux out over face i.

Compute F_i using tangent-plane approximation to $\vec{B}$ over each face.

⇒ F_i arbitrarily accurate as $dx, dy, dz \to 0$

FACE	$d\vec{S}$
① $\Delta x = 0$	$-\hat{\imath}\, dy\, dz$
② $\Delta x = dx$	$\hat{\imath}\, dy\, dz$
③ $\Delta y = 0$	$-\hat{\jmath}\, dx\, dz$
④ $\Delta y = dy$	$\hat{\jmath}\, dx\, dz$
⑤ $\Delta z = 0$	$-\hat{k}\, dx\, dy$
⑥ $\Delta z = dz$	$\hat{k}\, dx\, dy$

①: $\hat{\imath}$ component of vector field varies as: $B_x + \frac{\partial B_x}{\partial y} dy + \frac{\partial B_x}{\partial z} dz$

Since planar, ∴ average value at centre: $F_1 = (B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2})\, dy\, dz = -\bar{B}_{x_1} dy dz$

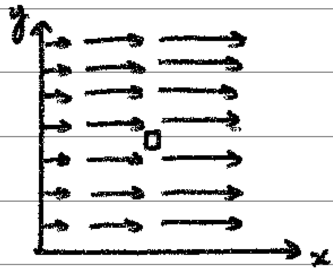
$F_2 = (\bar{B}_{x_1} + \frac{\partial \bar{B}_{x_1}}{\partial x} dx)\, dy\, dz \Rightarrow F_1 + F_2 = \frac{\partial}{\partial x}\left[B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2}\right] dx\, dy\, dz$

Analogous terms exist for $F_3 + F_4$ and $F_5 + F_6$

$$\nabla \cdot \vec{B} = \lim_{dx,dy,dz \to 0} \left[\frac{\sum_{i=1}^{6} F_i}{dx\, dy\, dz} \right] \Rightarrow \left[\frac{\partial}{\partial x}\left[B_x + \cancel{\frac{\partial B_x}{\partial y}\frac{dy}{2}} + \cancel{\frac{\partial B_x}{\partial z}\frac{dz}{2}}\right] + \frac{\partial}{\partial y}\left[B_y + \cancel{\frac{\partial B_y}{\partial x}\frac{dx}{2}} + \cancel{\frac{\partial B_y}{\partial z}\frac{dz}{2}}\right] + \frac{\partial}{\partial z}[\ldots] \right]$$

$$div\vec{B} = \nabla . \vec{B} = \frac{\partial B_x}{dx} + \frac{\partial B_y}{dy} + \frac{\partial B_z}{dz}$$

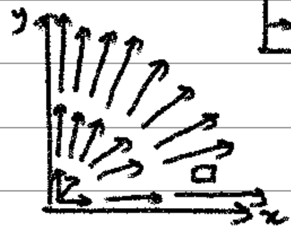
Ex.1 $\vec{B} = ax\,\hat{i}$



$$\nabla\cdot\vec{B} = \frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) + \frac{\partial}{\partial z}(B_z) = a \quad \text{(everywhere)}$$

$\therefore$ more flux flows out than in

Ex.2 $\vec{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\hat{r}$



$$\nabla\cdot\vec{B} = 1 + 1 + 1 = 3 \quad \text{(everywhere)}$$

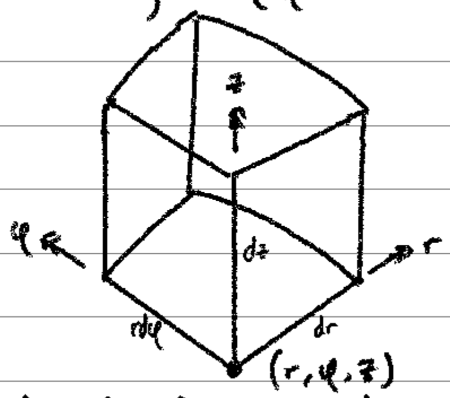
Ex.3 $\vec{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\nabla\cdot\vec{B} = 0 \quad \text{(everywhere)}$$

6.2 DIVERGENCE IN OTHER CO-ORDINATE SYSTEMS

Geometric Derivation in Cylindrical Co-ordinates

$\vec{B} = B_r\hat{\varrho} + B_\varphi\hat{\varphi} + B_z\hat{k}$



FACE		$d\vec{S}$
①	$\Delta r = 0$	$-r\,d\varphi\,dz\,\hat{\varrho}$
②	$\Delta r = dr$	$(r+dr)\,d\varphi\,dz\,\hat{\varrho}$
③	$\Delta\varphi = 0$	$-dr\,dz\,\hat{\varphi}$
④	$\Delta\varphi = d\varphi$	$dr\,dz\,\hat{\varphi}$
⑤	$\Delta z = 0$	$-\left[(r+dr)^2 - r^2\right]\frac{d\varphi}{2}\hat{k} = -r\,dr\,d\varphi\left[1+\frac{dr}{2r}\right]\hat{k}$
⑥	$\Delta z = dz$	$r\,dr\,d\varphi\left[1+\frac{dr}{2r}\right]\hat{k}$

As before, ignore variation over face: e.g. $\frac{\partial B_r}{\partial\varphi}\frac{d\varphi}{2}$ on ①

$F_1 = (-B_r)\,r\,d\varphi\,dz \qquad F_2 = \left(B_r + \frac{\partial B_r}{\partial r}dr\right)(r+dr)\,d\varphi\,dz$

$$\therefore\ F_1 + F_2 = \left[\frac{B_r}{r} + \frac{\partial B_r}{\partial r} + \frac{\partial B_r}{\partial r}\frac{dr}{r}\right] r\,dr\,d\varphi\,dz$$

$F_3 = (-B_\varphi)\,d\varphi\,dz \qquad F_4 = \left(B_\varphi + \frac{\partial B_\varphi}{\partial\varphi}d\varphi\right)dr\,dz$

$$\therefore\ F_3 + F_4 = \frac{1}{r}\left[\frac{\partial B_\varphi}{\partial r}\right] r\,dr\,d\varphi\,dz$$

$F_5 = (-B_z)\,r\,dr\,d\varphi\left(1+\frac{dr}{2r}\right) \qquad F_6 = \left(B_z + \frac{\partial B_z}{\partial z}dz\right) r\,dr\,d\varphi\left(1+\frac{dr}{2r}\right)$

$$\therefore\ F_5 + F_6 = \frac{\partial B_z}{\partial z}\left(1+\frac{dr}{2r}\right) r\,dr\,d\varphi\,dz$$

by product rule

$$\operatorname{div}\vec{B} = \nabla\cdot\vec{B} = \lim_{dr,d\varphi,dz\to 0}\frac{\sum F_i}{V} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi} + \frac{\partial B_z}{\partial z}$$

Geometric Derivation in Spherical Co-ordinates

This is left as an exercise for the reader.

$$\text{div}\vec{B} = \nabla.\vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

SIDENOTE

What does $\nabla.\vec{B}$ actually mean?

Cartesian:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{B} = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z$$

$$\nabla.\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

resembles dot product of 2 vectors

cylindrical:

$$\nabla = \hat{\varrho}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\varphi}\frac{\partial}{\partial\varphi} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{B} = \hat{\varrho}B_r + \hat{\varphi}B_\varphi + \hat{k}B_z$$

$$\nabla.\vec{B} = \frac{B_r}{r} + \frac{\partial B_r}{\partial r} + \frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi} + \frac{\partial B_z}{\partial z}$$

contains an extra term

$\nabla.\vec{B}$ is not the dot product of 2 vectors, but is an 'abuse of notation' shorthand for divergence. It means: "do differentiation first, then dot product"

This idea works for cylindrical & spherical co-ordinates, with both 'div' and 'curl'.

Curl $(\nabla\times\vec{B})$ means: "do differentiation first, then cross-product"

Analytical Derivation in Cylindrical Co-ordinates

$$\nabla = \frac{\partial}{\partial r}\hat{\varrho} + \frac{1}{r}\frac{\partial}{\partial\varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{k}, \quad \vec{B} = B_r\hat{\varrho} + B_\varphi\hat{\varphi} + B_z\hat{k}$$

$$\nabla.\vec{B} = \underbrace{\hat{\varrho}\cdot\frac{\partial\vec{B}}{\partial r}}_{①} + \frac{1}{r}\underbrace{\hat{\varphi}\cdot\frac{\partial\vec{B}}{\partial\varphi}}_{②} + \underbrace{\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}}_{③}$$

recall: $\frac{\partial\hat{\varrho}(\varphi)}{\partial\varphi} = \hat{\varphi}$, $\frac{\partial\hat{\varphi}(\varphi)}{\partial\varphi} = -\hat{\varrho}$

① $\frac{\partial\vec{B}}{\partial r} = \frac{\partial B_r}{\partial r}\hat{\varrho} + \frac{\partial B_\varphi}{\partial\varphi}\hat{\varphi} + \frac{\partial B_z}{\partial z}\hat{k} \Rightarrow \hat{\varrho}\cdot\frac{\partial\vec{B}}{\partial r} = \frac{\partial B_r}{\partial r}$

② $\frac{\partial\vec{B}}{\partial\varphi} = B_r\hat{\varphi} + \frac{\partial B_r}{\partial\varphi}\hat{\varrho} - B_\varphi\hat{\varrho} + \frac{\partial B_\varphi}{\partial\varphi}\hat{\varphi} + \frac{\partial B_z}{\partial\varphi}\hat{k} \Rightarrow \frac{1}{r}\hat{\varphi}\cdot\frac{\partial\vec{B}}{\partial\varphi} = \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi}$

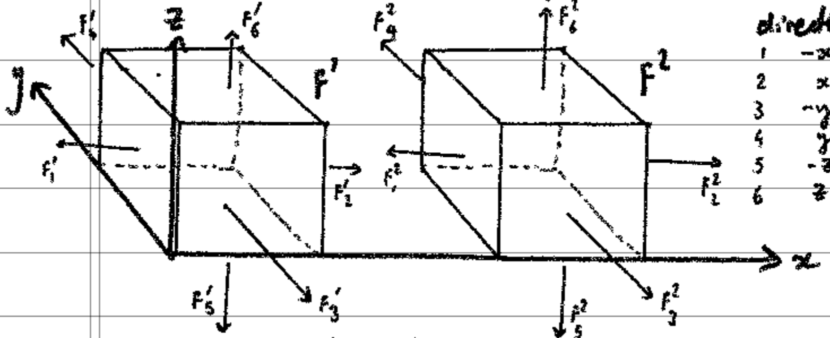
③ $\frac{\partial\vec{B}}{\partial z} = \frac{\partial B_r}{\partial z}\hat{\varrho} + \frac{\partial B_\varphi}{\partial z}\hat{\varphi} + \frac{\partial B_z}{\partial z}\hat{k} \Rightarrow \hat{k}\cdot\frac{\partial\vec{B}}{\partial z} = \frac{\partial B_z}{\partial z}$

Summing, we have:

by product rule

$$\text{div}\vec{B} = \frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi} + \frac{\partial B_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi} + \frac{\partial B_z}{\partial z}$$

6.3 DIVERGENCE / GAUSS' THEOREM



directions	
1	$-x$
2	x
3	$-y$
4	y
5	$-z$
6	z

N.B: $F_2^1 = -F_1^2$ (adjacent)

Considering 2 infinitesimal boxes, adjacent along x axis:

On each of the $\vec{F} = \vec{B} \cdot d\vec{S}$,

Box 1: $\sum_{i=1}^{6} F_i^1 = \nabla \cdot \vec{B}_1 \, dV$

Box 2: $\sum_{i=1}^{6} F_i^2 = \nabla \cdot \vec{B}_2 \, dV$

Push both boxes together, sum $\vec{B} \cdot d\vec{S}$ onto the surface:

$$\sum F = \underbrace{\sum_{i=1}^{6} F_i^1 + \sum_{i=1}^{6} F_i^2}\; - \cancelto{0}{\left(F_2^1 + F_1^2\right)} = \nabla \cdot B_1 dV + \nabla \cdot B_2 dV$$

$$\therefore \sum \vec{B} \cdot d\vec{S} = \sum_{j=1}^{2} \nabla B_j \, dV \quad \text{(creates new combined box)}$$

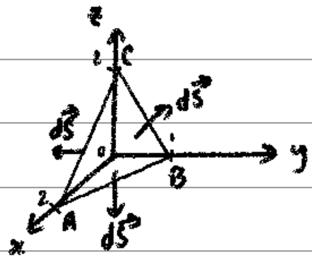
Adding more boxes to create macroscopic volume, we have:

$$\oiint_S \vec{B} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{B} \, dV \qquad \text{divergence theorem}$$

S ↳ closed surface

Ex.4 $\vec{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ tetrahedron is bounded by:

$x = 0$, $y = 0$, $z = 0$, $z = 2 - x - 2y$



$$\oiint_S \vec{B} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{B} \, dV$$

L.H.S: Easy to show on ΔAOB, ΔAOC, ΔBOC that $\vec{B} \cdot d\vec{S} = 0$

(closed surface) $\because d\vec{S}$ is in $-\hat{j}$ direction by $B_y = 0 \Rightarrow \vec{B} \cdot d\vec{S} = 0$

on ΔABC, $d\vec{S} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\hat{n}} dx\,dy \Rightarrow \vec{B} \cdot d\vec{S} = 2\,dx\,dy$

$$\therefore \iint_{\Delta ABC} \vec{B} \cdot d\vec{S} = 2 \times \text{area of } \Delta AOB = 2 \times 1 = 2$$

R.H.S: $\nabla \cdot \vec{B} = 3 \Rightarrow \iiint \nabla \cdot \vec{B} \, dV = 3 \times \text{area of } OABC$

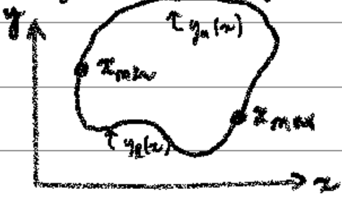
area of $OABC = \frac{1}{3} \times 2 \times 1$ $\therefore$ R.H.S $= 2 =$ L.H.S $\square$

7.1 GREEN'S THEOREM IN THE PLANE

can be any continuous functions

$$\oint P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{dQ}{dx} - \frac{dP}{dy}\,dx\,dy$$

line integral (anticlockwise on x-y plane) — double integral over enclosed region



PROOF: consider $I_1 = -\iint \frac{dP}{dy}\,dy\,dx$

$$-\int_{y=y_l(x)}^{y_u(x)} \frac{dP}{dy}(x,y) = -P(x,y_u) + P(x,y_l)$$

$$I_1 = -\int_{x_{max}}^{x_{min}} P(x,y_u)\,dx + \int_{x_{min}}^{x_{max}} P(x,y_l)\,dx = \oint P\,dx \quad \therefore \oint P\,dx = -\iint \frac{dP}{dy}\,dy\,dx$$

consider $I_2 = \iint_R \frac{dQ}{dx}\,dx\,dy$

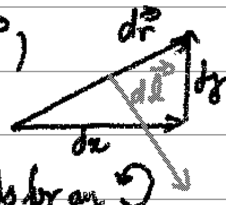
$$\int_{x=x_l(y)}^{x_r(y)} \frac{dQ}{dx}(x,y)\,dx = Q(x_r,y) - Q(x_l,y)$$

$$I_2 = \int_{y_{min}}^{y_{max}} Q(x_r,y)\,dy + \int_{y_{max}}^{y_{min}} Q(x_l,y)\,dy = \oint Q\,dy \Rightarrow \oint P\,dx + Q\,dy = \iint_R \frac{dQ}{dx} - \frac{dP}{dy}\,dx\,dy$$

Ex.1 $P\,dx + Q\,dy$ is an exact differential

$$\Rightarrow \frac{dP}{dy} = \frac{dQ}{dx} \quad \therefore \oint P\,dx + Q\,dy = \iint \left(\frac{dQ}{dx} - \frac{dP}{dy}\right)dx\,dy = 0 \quad \text{as expected}$$

Ex.2 Considering 'flux' out of a closed loop in 2D (2D equivalent of $\oiint \vec{B}\cdot d\vec{S}$)



vector field B, we want $\oint \vec{B}\cdot d\vec{l}$

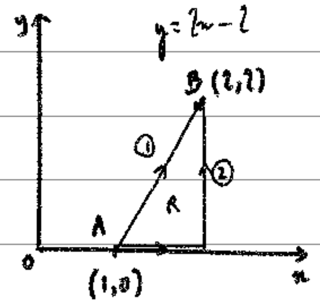
$d\vec{l}$ has a length of $d\vec{r}$, but is $\perp r$ and points outwards for an ↺ A.C.W

$d\vec{l} = dy\,\hat{\imath} - dx\,\hat{\jmath}$

Then, $\vec{B}\cdot d\vec{l} = (B_x\,\hat{\imath} + B_y\,\hat{\jmath})\cdot(dy\,\hat{\imath} - dx\,\hat{\jmath})$

$$\underline{\oint \vec{B}\cdot d\vec{l}} = \oint \overset{P}{\overset{\parallel}{-B_y}}\,dx + \overset{Q}{\overset{\parallel}{B_x}}\,dy = \iint \frac{dQ}{dx} - \frac{dP}{dy}\,dx\,dy = \underline{\iint \frac{dB_x}{dx} + \frac{dB_y}{dy}\,dx\,dy}$$

Ex.3 $$\oint \vec{F}\cdot d\vec{r} = \oint \overset{P}{\overset{\parallel}{F_x}}\,dx + \oint \overset{Q}{\overset{\parallel}{F_y}}\,dy = \iint \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\,dx\,dy$$ Stoke's Theorem in 2D



$$\vec{F}_2 = 2xy\hat{i} - x^2\hat{j}$$

Ex.4 path ① $$\int_A^B \vec{F}\cdot d\vec{r} = -\frac{4}{3} \quad \therefore \int_B^A \vec{F}\cdot d\vec{r} = \frac{4}{3}$$

path ② $$\int_A^B \vec{F}\cdot d\vec{r} = -8 \quad \therefore \oint \vec{F}\cdot d\vec{r} = -\frac{20}{3} = \oint \overset{P}{\overset{\parallel}{2xy}}\,dx - \overset{Q}{\overset{\parallel}{x^2}}\,dy$$

By Green's Theorem:

$$\oint \vec{F}\cdot d\vec{S} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = -\iint 4x\,dx\,dy \Rightarrow I = -\int_{x=1}^{2}\int_{y=0}^{2x-2} 4x\,dy\,dx = -\int_1^2 \left[4xy\right]_0^{2x-2} dx$$

$$= -\int_1^2 (8x^2 - 8x)\,dx = \left[-\frac{8}{3}x^3 + 4x^2\right]_1^2 = -\frac{20}{3} \quad \square$$

6.4 THE LAPLACIAN

$$\nabla\Omega = \begin{pmatrix} \frac{\partial\Omega}{\partial x} \\ \frac{\partial\Omega}{\partial y} \\ \frac{\partial\Omega}{\partial z} \end{pmatrix} \qquad \nabla^2(\Omega) = \nabla\cdot(\nabla\Omega) = \begin{pmatrix} \frac{\partial^2\Omega}{\partial x^2} \\ \frac{\partial^2\Omega}{\partial y^2} \\ \frac{\partial^2\Omega}{\partial z^2} \end{pmatrix}$$

"del squared" is called the Laplacian

e.g. Laplace's Equation
$$\nabla^2 u = 0$$

e.g. the wave equation
$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

in cylindrical co-ordinates:

$$\nabla^2\Omega = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Omega}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Omega}{\partial\varphi^2} + \frac{\partial^2\Omega}{\partial z^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Omega}{\partial r}\right) = \frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega}{\partial r}$$

in spherical co-ordinates:

$$\nabla^2\Omega = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Omega}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Omega}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right)$$

7 Curl

Often referred to as: curl $\vec{F}$, rot $\vec{F}$, $\nabla \times \vec{F}$

7.2 CURL DEFINITION $\nabla \times \vec{B}$

Curl is a vector that quantifies "circulation surface density"

$$(\nabla \times \vec{B}) \cdot \hat{n} = \lim_{A \to 0} \left(\frac{1}{A} \oint \vec{B} \cdot d\vec{r} \right)$$

where A: area

unit vector ⊥ to the area

R.H. rule

Analytical Derivation Considering in Cartesian, consider a loop of finite size and shrink.

in $\hat{k}$:
$$\frac{1}{A} \oint \vec{B} \cdot d\vec{r} = \frac{1}{A} \oint B_x\,dx + B_y\,dy \overset{\text{Green's Theorem}}{=} \frac{1}{A} \iint_R \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) dx\,dy$$

Now shrink:

$$\nabla \times \vec{B} \cdot \hat{k} = \lim_{A \to 0} \frac{\iint_R \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\,dx\,dy}{\iint_R dx\,dy} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

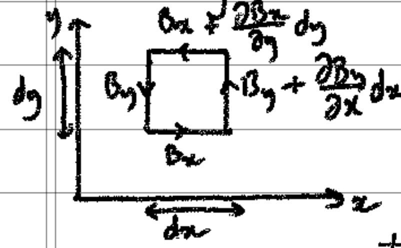
Similarly,

$$\nabla \times \vec{B} \cdot \hat{i} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \qquad \nabla \times \vec{B} \cdot \hat{j} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}$$

$$\therefore \nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \quad \text{where } \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \text{ and } \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

so it resembles a cross-product between 2 vectors. $\nabla \times \vec{B} = \hat{i} \times \frac{\partial \vec{B}}{\partial x} + \hat{j} \times \frac{\partial \vec{B}}{\partial y} + \hat{k} \times \frac{\partial \vec{B}}{\partial z}$

Geometric Derivation Considering the "circulation surface density", considering $\hat{k}$ on infinitesimally area $dx\,dy$



$$\oint \frac{\vec{B} \cdot d\vec{r}}{A} = \frac{B_x\,dx + \left(B_y + \frac{\partial B_y}{\partial x} dx\right) dy - \left(B_x + \frac{\partial B_x}{\partial y} dy\right) dx - B_y\,dy}{dx\,dy}$$

$$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

anticlockwise ↺ ↻ clockwise rotation (∴ −ve)

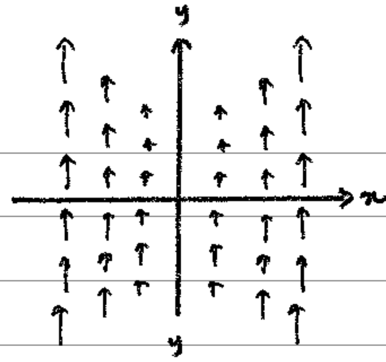
a conservative field

by Clairaut's Theorem

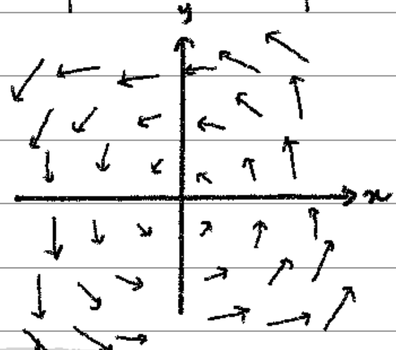
$$\nabla \times \nabla \Omega = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Omega}{\partial x} & \frac{\partial \Omega}{\partial y} & \frac{\partial \Omega}{\partial z} \end{vmatrix} = \hat{i} \left(\cancelto{0}{\frac{\partial^2 \Omega}{\partial y \partial z} - \frac{\partial^2 \Omega}{\partial z \partial y}} \right) + \hat{j} \left(\cancelto{0}{\frac{\partial^2 \Omega}{\partial z \partial x} - \frac{\partial^2 \Omega}{\partial x \partial z}} \right) + \ldots = 0$$

$\therefore \nabla \times \vec{B} = 0 \iff \vec{B}$ is a conservative field or "irrotational field"

Ex.5 $\vec{B} = \begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix} \Rightarrow \nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x \end{pmatrix}$



Ex.6 $\vec{B} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \Rightarrow \nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$



7.3 CURL IN OTHER CO-ORDINATE SYSTEMS

in cylindrical co-ordinates:

$\nabla = \frac{\partial}{\partial r}\hat{\rho} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{k}$,

$\vec{B} = B_r\hat{\rho} + B_\varphi\hat{\varphi} + B_z\hat{k}$

$\nabla \times \vec{B} = \frac{1}{r}\begin{vmatrix} \hat{\rho} & r\hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ B_r & rB_\varphi & B_z \end{vmatrix}$

in spherical co-ordinates:

$\nabla = \frac{\partial}{\partial r}\hat{\rho} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{\varphi} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$

$\vec{B} = B_r\hat{\rho} + B_\theta\hat{\theta} + B_\varphi\hat{\varphi}$

$\nabla \times \vec{B} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ B_r & rB_\theta & r\sin\theta B_\varphi \end{vmatrix}$

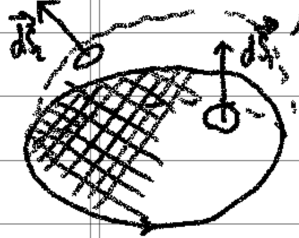
7.4 STOKE'S THEOREM

$$\underbrace{\oint_C \vec{B}\cdot d\vec{r} = \iint_S \nabla\times\vec{B}\cdot d\vec{S}}_{\text{R.H. rule}}$$

C: closed loop   S: open surface

$\nabla\times\vec{B}\cdot\hat{n} = \lim_{A\to 0}\frac{1}{A}\oint \vec{B}\cdot d\vec{r}$

$\therefore \nabla\times\vec{B}\cdot d\vec{S} = \oint \vec{B}\cdot d\vec{r}$ for infinitesimal loops



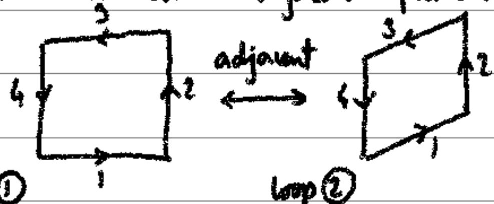
Any surface attached to the loop (think of butterfly net) :)

N.B. to be clear of orientation of the R.H. rule to "collapse" surface onto the loop

Sometimes we want a surface integral but a line integral or another surface is easier attached to the same loop.

(not necessarily in the same plane)

PROOF: Consider two adjacent infinitesimal square loops



(different planes)

loop ① $\sum_{i=1}^{4}\vec{B}\cdot d\vec{r} = \sum_{i=1}^{4} W_i^1 = (\nabla\times\vec{B}_1)\cdot\vec{dS_1}$

loop ② $\sum_{i=1}^{4}\vec{B}\cdot d\vec{r} = \sum_{i=1}^{4} W_i^2 = (\nabla\times\vec{B}_2)\cdot\vec{dS_2}$

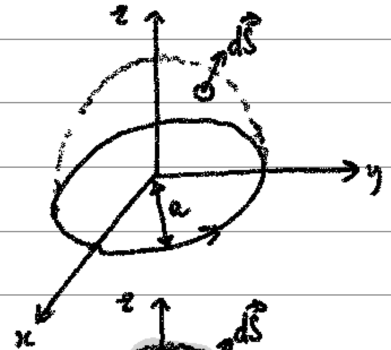
Join the loops together: but $W_2^1 = -W_4^2$

$\sum_{①+②}\vec{B}\cdot d\vec{r} = \sum_{i=1}^{4} W_i^1 + \sum_{i=1}^{4} W_i^2 \cancel{- W_2^1} \cancel{- W_4^2}$

$= (\nabla\times\vec{B}_1)\cdot\vec{dS_1} + (\nabla\times\vec{B}_2)\cdot\vec{dS_2} = \sum_{j=1}^{2}(\nabla\times B_j)\cdot\vec{dS_j}$

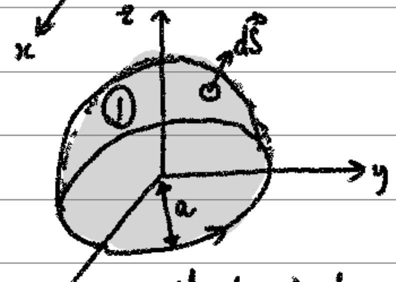
Add more loops to create macroscopic surface attached to a loop.

Ex.7 Hemisphere of radius a at $z > 0$



vector field: $\vec{B} = \begin{pmatrix} z \\ -y \\ -x \end{pmatrix}$, $d\vec{S} = a^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$

Want to evaluate: $\oint_C \vec{B} \cdot d\vec{r} = \iint_S \nabla \times \vec{B} \cdot d\vec{S}$



on the hemisphere

① $\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -y & -x \end{vmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, $\hat{j} \times \hat{r} = \sin\theta \sin\varphi$

$\therefore \iint_S \nabla \times \vec{B} \cdot d\vec{S} = 2a^2 \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} \sin^2\theta \sin\varphi \, d\varphi \, d\theta = 0$

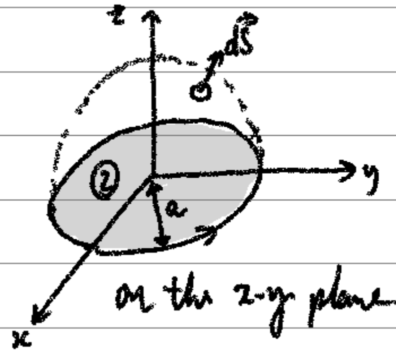
② $x = a\cos\varphi$ $\qquad$ $y = a\sin\varphi$

$\Rightarrow dx = -a\sin\varphi \, d\varphi$ $\qquad$ $\Rightarrow dy = a\cos\varphi \, d\varphi$

$\vec{B} \cdot d\vec{r} = B_x dx + B_y dy + \cancel{B_z dz}^{\,0} = \cancel{z dx}^{\,0} - y\,dy$

$= -a^2 \sin\varphi \cos\varphi \, d\varphi$

$\oint \vec{B} \cdot d\vec{r} = -a^2 \int_0^{2\pi} \sin\varphi \cos\varphi \, d\varphi = -\frac{a^2}{2} \left[\sin^2\varphi\right]_0^{2\pi} = 0$



on the x-y plane

③ Choose surface in x,y plane $\Rightarrow d\vec{S} = dS\,\hat{k}$

$\nabla \times \vec{B} \cdot d\vec{S} = 2\,dS\,\hat{j} \cdot \hat{k} = 0$

7.5 VECTOR IDENTITIES

① $\nabla \times (\nabla \Omega) = 0$ $\because$ if $\vec{B} = \nabla \Omega$, Ω is the parent function $\Rightarrow \vec{B}$ is conservative "irrotational"

② $\nabla \cdot (\nabla \times \vec{V}) = 0$ $\because$ $\frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = 0$

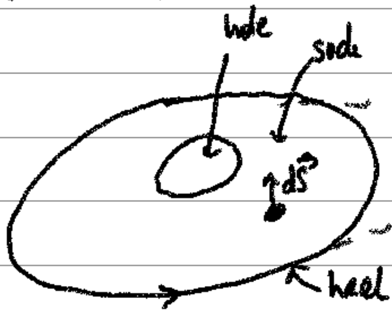
if $\vec{B} = \nabla \times \vec{V}$, $\vec{V}$ is the parent function $\Rightarrow \vec{B}$ is "solenoidal"

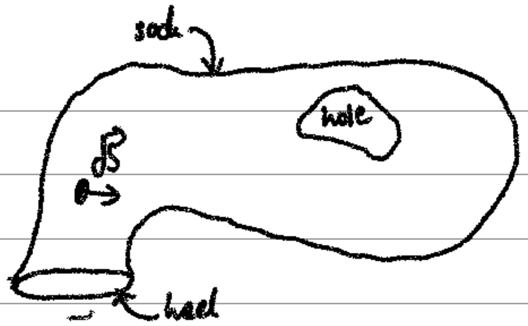
then $\vec{V}$ is the vector potential

③ HELMHALTZ THEOREM: $\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$ where $\nabla^2 \vec{B} = \begin{pmatrix} \nabla^2 B_x \\ \nabla^2 B_y \\ \nabla^2 B_z \end{pmatrix}$

Used to derive the wave equation

Ex.2 What is $\displaystyle\iint_{\substack{sock\\(ONLY)}} \nabla\times\vec{B}\cdot d\vec{S}$?





Now,

$$\oint_{heel}\vec{B}\cdot d\vec{r} = \iint_{sock}\nabla\times\vec{B}\cdot d\vec{S} + \iint_{hole}\nabla\times\vec{B}\cdot d\vec{S} \Rightarrow \iint_{sock}\nabla\times\vec{B}\cdot d\vec{S} = \oint_{heel}\vec{B}\cdot d\vec{r} - \overbrace{\oint_{hole}\vec{B}\cdot d\vec{r}}^{\iint_{hole}\nabla\times\vec{B}\cdot d\vec{S}}$$

Vector Fields, Electricity & Magnetism

ELECTROSTATICS
静電

目次

1 Electric Charge, Force, Field & Potential

INTRODUCTION

Maxwell's Equations

Gauss' Law: $\nabla \cdot \vec{E} = \frac{\rho_q}{\varepsilon_0}$

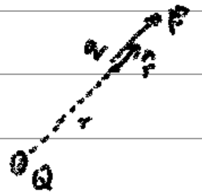
Gauss' Law for Magnetism: $\nabla \cdot \vec{B} = 0$ (no monopoles)

Faraday's Law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampère-Maxwell Law: $\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Electrostatic Circulation Law: $\nabla \times \vec{E} = 0$

1.1 FIELDS CAUSE FORCES



1.1.1 ELECTRIC CHARGE

Charge is quantized in units of $e = 1.6021 \times 10^{-19}$ C. Can be positive and negative charge.

1.1.2 COULOMB'S LAW

$\vec{F} = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{r}$ If Q and q are same signs, it is a repulsive force. Otherwise, there is an attractive force.

Force (vectors) linearly superpose.

1.1.4 ELECTRIC FIELD

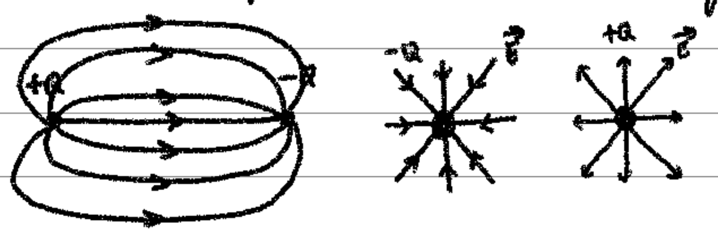
Electric field $\vec{E}$ due to Q at distance r along $\hat{r}$ is: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

Place charge q in the field. It experiences a force: $\vec{F} = q\vec{E}$

For the superposition principle, for n particles: $\vec{E} = \sum_{i=1}^{n} \vec{E}_i$

1.1.5 FIELD LINES

Depict the movement of positive charges relative to a positive charge. Thus, field lines always start at +ve ends and go to the -ve end; never crossing.



Tangent to $\vec{E}$ at every point in space.

$d\vec{\ell} \parallel E \therefore d\vec{\ell}(x,y,z) = \vec{E}(x,y,z)$

1.2 ELECTRIC POTENTIAL

1.2.1 POTENTIAL ENERGY

Consider charge Q, and 'test charge' q that experiences a force $\vec{F}_Q$. We wish to match F_Q with a F_{ext}. So, moving the charge (at constant speed), work is done by F_{ext}:

$$\vec{F}_{ext} + \vec{F}_Q = 0$$

Elemental work done by F_{ext} on q is: $\quad dW = \vec{F}_{ext} \cdot d\vec{l} = -\vec{F}_Q \cdot d\vec{l}$

The total work done by F_{ext} is found by integrating from A to B.

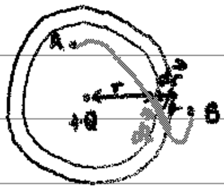
$$W = \int_A^B \vec{F}_{ext} \cdot d\vec{l} = -\int_A^B F_Q \cdot d\vec{l}$$

$\hat{r} \cdot d\vec{l} = \underbrace{|\hat{r}|}_{1}\underbrace{|d\vec{l}|\cos\theta}_{dr}$

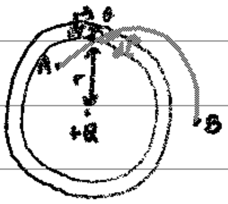
$\therefore \hat{r} \cdot d\vec{l} = dr$

The work done, W, is equal to the change in Potential Energy, ΔU, and so:

$$\Delta U = -\int_A^B F_Q \cdot d\vec{l} \quad \text{and we know } F_Q \text{ from Coulomb's law:}$$



$$\therefore dW = -\frac{qQ}{4\pi\varepsilon_0 r^2}\hat{r} \cdot d\vec{l} \qquad \therefore \text{work done depends on } \hat{r} \cdot d\vec{l}$$



Using the result that $\hat{r} \cdot d\vec{l} = dr$, work done depends on $\hat{r} \cdot d\vec{l}$.

$$\Delta U_{AB} = W = -\int_A^B \frac{qQ}{4\pi\varepsilon_0 r^2}\, dr = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Work done is only dependent on r, regardless of path taken.

If there are multiple charges, use superposition principle:

$$\Delta U_{AB} = -\int_A^B \vec{F}_{Q1} \cdot d\vec{l} - \int_A^B \vec{F}_{Q2} \cdot d\vec{l} - \dots - \int_A^B F_{QN} \cdot d\vec{l}$$

Each term depends only on distance from A to B $\therefore \Delta U_{AB}$ in total path independent.

1.2.2 POTENTIAL DIFFERENCE

Potential difference between A and B is ΔV_{AB} where:

$$\Delta V_{AB} = \frac{\Delta U_{AB}}{q} = -\int_A^B \vec{E} \cdot d\vec{l}$$

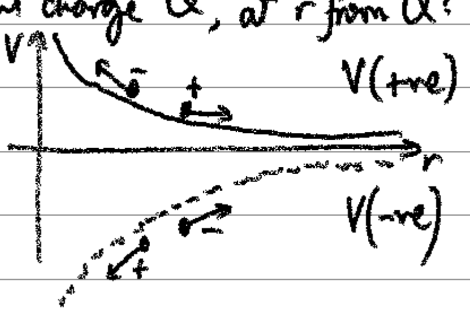
'gauge transformation'
-add a constant potential everywhere

ΔV_{AB} also path independent, not dependent on q with S.I unit $J.C^{-1}$.
Often defined as $V=0$ at $r=\infty$. Define potential V at P as external work to bring a charge of $+1C$ at constant speed from $r=\infty$ to P.

$$V = -\int_{\infty}^{P} \vec{E}\cdot d\vec{l}$$

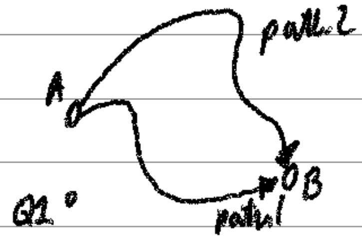
- Potential energy per unit charge.
- Scalar field, superposition principle applies.
- Without external forces applied: ⊕ charges to lower potentials, ⊖ charges to higher potentials

For a point charge Q, at r from Q:



$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

1.2.3 ELECTROSTATIC CIRCULATION LAW



$$\Delta V_{AB} = -\int_{A,\,path\,1}^{B} \vec{E}\cdot d\vec{l} = -\int_{A,\,path\,2}^{B} \vec{E}\cdot d\vec{l}$$

$$\Rightarrow -\int_{A,\,path\,1}^{B} \vec{E}\cdot d\vec{l} = \int_{B,\,path\,2}^{A} \vec{E}\cdot d\vec{l} \quad \therefore \int_{A,\,path\,1}^{B} \vec{E}\cdot d\vec{l} + \int_{A,\,path\,1}^{B} \vec{E}\cdot d\vec{l} = 0$$

$\therefore$ around any closed path, $\oint \vec{E}\cdot d\vec{l} = 0$ only true for $\vec{E}$ in electrostatics.

1.2.4 DIFFERENTIAL FORMS OF THE CIRCULATION LAW

By Stokes' Theorem: $$\oint_C \vec{E}\cdot d\vec{l} = \iint_S (\nabla\times\vec{E})\cdot d\vec{S} = 0$$

Applying to an infinitesimal surface element: $(\nabla\times\vec{E})\cdot d\vec{S} = 0$
Since $d\vec{S} \neq 0$, this means that at a point in space: $\nabla\times\vec{E} = 0$

Differential equation satisfied at all points in the field.

1.3 Electric Potential (continued)

1.3.1. Two Charges: Binding Energy

Starting with point charge Q_1, thus we can work out potential V at distance r, $V = \frac{Q_1}{4\pi\epsilon_0 r}$.
Move Q_2 from ∞ to separation r_{12} (using $\vec{F}_{ext}$ at constant speed). Change in potential is:

$$\Delta U = \int_\infty^P \vec{F}_{ext} \cdot d\vec{l} = -\int_\infty^P F_Q \cdot d\vec{l} = -Q_2 \int_\infty^P \vec{E} \cdot d\vec{l} = Q_2 V$$

Set $U=0$ at ∞, you can write $\Delta U = U(r) - \cancelto{0}{U_\infty} = U$

$$\therefore U = Q_2 \left(\frac{Q_1}{4\pi\epsilon_0 r_{12}} \right) = Q_2 V(r_{12})$$

This is the P.E of Q_2 in the potential of Q_1 at distance r_{12}.

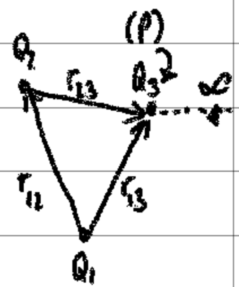
Similarly, we could write:

$$\Rightarrow U = Q_1 \left(\frac{Q_2}{4\pi\epsilon_0 r_{12}} \right) = Q_1 V(r_{12})$$

U is a property of the system. It is the <u>binding energy</u> – energy needed to remove a charge to ∞.

$1\,eV = 1.602 \times 10^{-19}\,V$

1.3.2. Potential Energy of a Set of Charges



Move Q_3 from $\infty \to P$ as before. The change in P.E is given by:

$$\Delta U = \int_\infty^P \vec{F}_{ext} \cdot d\vec{l} = -Q_3 \int_\infty^P \vec{E} \cdot d\vec{l} = Q_3 V_P$$

potential at P due to Q_1 and Q_2 $\therefore V_P = \frac{Q_1}{4\pi\epsilon_0 r_{13}} + \frac{Q_2}{4\pi\epsilon_0 r_{23}}$

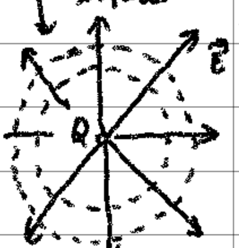
$$\Rightarrow U = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}} + \Delta U = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}} + \frac{Q_1 Q_3}{4\pi\epsilon_0 r_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 r_{23}} = \frac{1}{2}Q_1 V_1 + \frac{1}{2}Q_2 V_2 + \frac{1}{2}Q_3 V_3$$

Note that: V_1 is potential at position Q_1 due to the other charges.

In general for n charges: $U = \sum_{i=1}^{n} \frac{1}{2} Q_i V_i$ — potential at pos^n of Q_i due to other $n-1$ charges

1.3.3 Equipotential Surfaces

concentric spheres are equipotential surfaces



Recalling: $\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l} \Rightarrow dV_{AB} = -\vec{E} \cdot d\vec{l}$

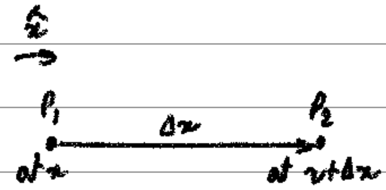
If $d\vec{l}$ is $\perp \vec{E} \Rightarrow dV_{AB} = 0$ Consequently $\vec{E} \perp$ 'equipotential surface'

For a uniform field where $\vec{E} = E\hat{x}$, field lines from higher V to lower V

$V(x) - V(0) = -\int_0^x \vec{E} \cdot d\vec{l} = -\int_0^x E\,dx = -Ex \quad \therefore V(x) = V(0) - Ex$

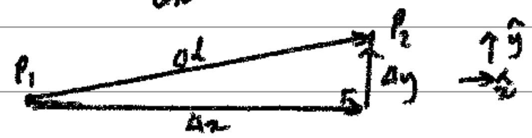
1.3.4. $\vec{E}$ and V

Consider the following 1D scenario:



$$V(P_2) = V(x+\Delta x) = V(x) + \Delta x \frac{dV}{dx} + \dots \approx V_1(P_1) + \Delta x \frac{dV}{dx}$$

Consider the following 2D scenario:



where the separation is: $\Delta \vec{l} = \Delta x \hat{x} + \Delta y \hat{y}$

$$\therefore V(P_2) = V(P_1) + \Delta x \frac{\partial V}{\partial x} + \Delta y \frac{\partial V}{\partial y}$$

Consider the 3D scenario, where $\Delta \vec{l} = \Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}$

$$\therefore V(P_2) = V(P_1) + \Delta x \frac{\partial V}{\partial x} + \Delta y \frac{\partial V}{\partial y} + \Delta z \frac{\partial V}{\partial z}$$

However, $\Delta V = -\vec{E} \cdot \Delta \vec{l} = -E_x \Delta x - E_y \Delta y - E_z \Delta z$

$$\Rightarrow E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

which can be expressed as:

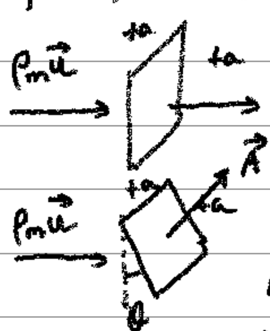
$$\nabla \times (\nabla A) = 0$$

Electrostatic circulation law equation is directly connected to equation with gradient in potential and electric field. Existence of a scalar potential gives circulation law directly.

1.4 GAUSS' LAW

1.4.1. ELECTRIC FLUX

The first of Maxwell's equations is Gauss' Law – charged particles are a source of (electric) field.



$\Delta m = \rho_m a^2 u \Delta t$

$\vec{A}$ – magnitude of surface area in direction of normal to surface

$\Delta m = \rho_m a^2 \cos\theta \, \vec{u} \, \Delta t$

Similarly, define electric flux Φ_E, where flow lines ⇔ field lines

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{S} \quad \text{(scalar quantity)}$$

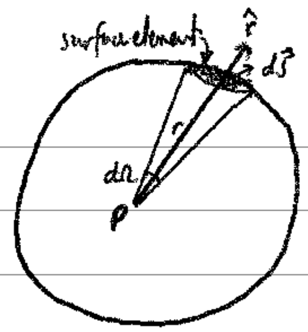
$\therefore \Delta m = \Delta t \, \rho_m \vec{u} \cdot \vec{A}$

→ $d\vec{S}$ points out of a closed surface by convention

For an arbitrary shape, using vector calculus:

→ thought of # $\vec{E}$ field lines through closed surface

$$\Delta m = \Delta t \iint_S \rho_m \vec{u} \cdot d\vec{S} \qquad \Phi_m = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \iint_S \rho_m \vec{u} \cdot d\vec{S}$$



1.4.2. SOLID ANGLES

ANGLES: a circle can be divided into 2π radians

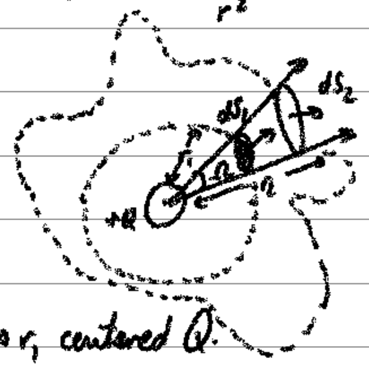
SOLID ANGLES: a sphere can be divided into 4π steradians

They help understand electric flux from a point charge.

$$d\Omega = \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

Considering surface element dS at distance r from P, then:

If $dS \parallel \hat{r}$: $d\Omega = \dfrac{dS}{r^2}$ (i.e. $\hat{r} \nearrow d\vec{S}$)



1.4.3. ELECTRIC FLUX DUE TO A POINT CHARGE

Given point charge Q, we have: $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

Calculating flux $\Phi_{E,1}$ through the spherical surface S_1, radius r_1 centered Q.

$|E| = E_{r1}$ is uniform and radial on the surface (of area $4\pi r_1^2$.)

★ $\therefore \; \Phi_{E,1} = 4\pi r_1^2 E_{r1} = \dfrac{Q}{\varepsilon_0}$

Now considering the flux $d\Phi_{E,1}$ through surface element dS_1, where $\vec{E}$ is radial:

$$d\Phi_{E,1} = \vec{E}_{r1}\cdot d\vec{S}_1 = \frac{\Phi_{E,1}}{4\pi r_1^2}dS_1 = \frac{\Phi_{E,1}}{4\pi}\left(\frac{dS}{r_1^2}\right) = \frac{\Phi_{E,1}}{4\pi}d\Omega = \frac{Q}{4\pi\varepsilon_0}d\Omega$$

Next, considering S_2, an arbitrary surface enclosing S_1. We have:

$$d\Phi_{E,2} = E_{r2}\cdot dS_2 = \frac{Q}{4\pi\varepsilon_0 r_2^2}\hat{r}\cdot d\vec{S}_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{d\vec{S}_2\cdot\hat{r}}{r_2^2}\right) = \frac{Q}{4\pi\varepsilon_0}d\Omega = d\Phi_{E,1}$$

Therefore, flux through two surface elements is the same, even though orientation of dS_2 is arbitrary (as same number of field lines through dS_1 and dS_2. Therefore, flux through any S_1 and S_2 is the same $\Rightarrow$ flux through any closed surface is <u>always</u> $\dfrac{Q}{\varepsilon_0}$.

N.B: (even if field lines cross surface multiple times.)

1.4.4. GAUSS' LAW (integral form of M1)

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{S} = \oint_S \vec{E}_1\cdot d\vec{S} + \oint_S \vec{E}_2\cdot d\vec{S} + \dots + \oint_S \vec{E}_n\cdot d\vec{S} = \frac{Q_1}{\varepsilon_0} + \frac{Q_2}{\varepsilon_0} + \dots + \frac{Q_N}{\varepsilon_0}$$

Since enclosed charge $Q_{enc} = Q_1 + Q_2 + \dots + Q_N$, $\therefore \oint_S \vec{E}\cdot d\vec{S} = \dfrac{Q_{enc}}{\varepsilon_0}$

⟶ all charges contribute to $\vec{E}$ at point in space, but only <u>some</u> to Gauss' law contribution.

⟶ calculating net flux, only charges <u>in</u> surface matter. Flux from outside does not contribute.

'electric Φ on the way in <u>cancels</u> electric Φ on the way out'.

1.5 DISTRIBUTED CHARGE

1.5.1. CHARGE DENSITY

Instead of point charges, we can conside regions of many non-uniformly distributed charges. We can think of this charge as spread out continuously as a 'charge density'.

For uniform charge density, if charge Q is evenly distributed over volume V: $\rho = \frac{Q}{V}$

For a non-uniform charge density, considering $\rho = \rho(x, y, z)$: $dQ = \rho \, dV$

Therefore, total charge can be found by integrating: $Q = \iiint_V \rho \, dV$ ← S.I. units: $C.m^{-3}$

If charge is spread over a thin layer, we can calculate σ: $Q = \iint_S \sigma \, dS$ ← S.I units: $C.m^{-2}$

1.5.2. GAUSS' LAW (W/ DISTRIBUTED CHARGE)

Previously, for point charges, we found: $\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0} = \sum_{i=1}^{n} \frac{Q_i}{\varepsilon_0}$

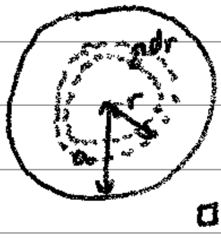
Consider each Q_i represents an infinitesimal dQ associated with dV where $dQ = \rho \, dV$.

$$\oint_{S \leftarrow \text{closed surface}} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_{V \leftarrow \text{volume enclosed by closed surface}} \rho \, dV \xRightarrow{\text{divergence theorem}} \int_V \nabla \cdot \vec{E} \, dV = \frac{1}{\varepsilon_0} \int_V \rho \, dV$$

Considering an infinitesimal volume, we have:

$$\nabla \cdot \vec{E} \, dV = \frac{\rho}{\varepsilon_0} dV \implies \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss' law in differential form)}$$

1.5.3. SPHERICALLY-SYMMETRIC CHARGE DISTRIBUTIONS



Consider situations with spherical symmetry, with ρ only varying with r, not φ or θ.
$\therefore \vec{E}$ is radial as $|E|$ varies by r, and electric flux Φ_E through spherical surface radius r is given by: $\Phi_E = 4\pi r^2 E$. For a uniformly-charged sphere: $\rho = \frac{Q}{\frac{4}{3}\pi a^3}$

- Charge enclosed by a sphere radius $r < a$ is: $Q_r = \frac{4}{3}\pi r^3 \rho = \frac{Q r^3}{a^3}$

Applying Gauss' law to a sphere, radius r, we find that: $4\pi r^2 E_r = \frac{Q r^3}{\varepsilon_0 a^3} \Rightarrow E_r = \frac{Q r}{4\pi \varepsilon_0 a^3}$

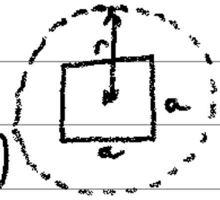
- Outside a uniformly-charged sphere, charge enclosed is simply Q.

$\therefore 4\pi r^2 E_r = \frac{Q}{\varepsilon_0} \Rightarrow E_r = \frac{Q}{4\pi \varepsilon_0 r^2}$, which resembles a point charge at O.

□ Outside a non-uniform, spherically-symmetric, charged sphere:
We only care about total enclosed charge Q in integration surface
$\therefore \vec{E}$ outside <u>any</u> spherically-symmetric distribution resembles point charge Q at the centre of this distribution.

• Inside a non-uniform, spherically-symmetric charged sphere, the electric field profile depends on charge distribution, by the volume integral over $0 \to r$. (see P.S.2)

□ Non-spherical charge distribution: e.g. a cube, side a
Total charge $= Q$ $\therefore$ total flux outside is Q/ε_0 $(r>a)$



$\vec{E}$ is non-uniform over sphere $\begin{cases} \text{stronger near cube corners} \\ \text{If } r \gg a \text{, cube approximates a point charge } (\propto \frac{1}{r^2}) \end{cases}$

1.5.4. FURTHER COMMENTS ON GAUSS' LAW

Recast $\vec{E}$ in terms of potential V as follows:

$$\nabla \cdot \vec{E} = -\nabla \cdot (\nabla V) = -\nabla^2 V \quad \Rightarrow \quad \underbrace{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}_{\text{Poisson's Equation}} \xRightarrow{\text{when } \rho = 0} \underbrace{\nabla^2 V = 0}_{\text{Laplace's Equation}}$$

1.6 ELECTROSTATICS & UNIQUENESS THEOREM

• $\nabla \times \vec{E} = 0$, $\vec{E} = -\nabla V$, $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ or $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

The uniqueness theorem states there is only <u>one</u> potential V satisfying Poisson's equation given the charge density $\rho(\vec{r})$ with specified boundary conditions. Electrostatic problems are most efficiently solved by Poisson's equation for V with appropriate boundary conditions and knowledge of charge distributions, then find $\vec{E}$.

2 Conductors, Capacitors, Dielectrics & Current

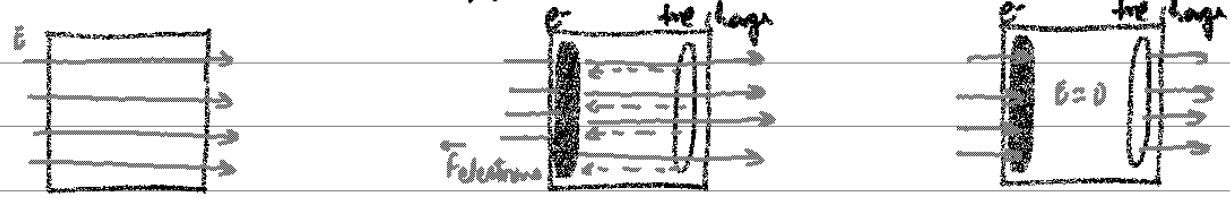
2.1 CONDUCTORS

2.1.1 BASIC PROPERTIES

i.e. metallic lattice

<u>Conductors</u> — allow free movement of charge particles (solid, and liquid eg. mercury)
<u>Insulators</u> — (or dielectrics) do not

- For conductors in static situations, $\vec{E} = 0$ inside the conductor.



1 Applying $\vec{E}$ to a conductor, free e^- experience $\vec{F}_{electrons}$, moving in response to field

2 e^- collect on L.H.S. of conductor, to give net -ve charge while a net +ve charge forms on R.H.S.

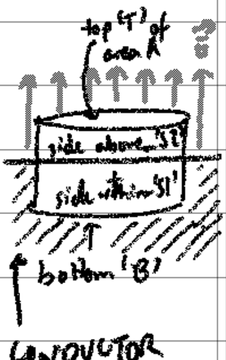
3 Charge separation has own field, shown by dashed red line, reducing electric field inside the conductor.

4 Continues until $\vec{E} = 0$ and no more force to redistribute any charge.

5 In equilibrium, surface charge, σ, a few angstroms ($\times 10^{-10}$m) thick forms. External field terminates on the surface charge.

Inside the conductor, $\vec{E} = 0$ so interior of conductor has same potential. The surface is also an equipotential — if not an electric field in surface layer would cause further charge redistribution until equalised potential. Equipotentials $\perp \vec{E}$ $\therefore$ field lines $\perp$ surface.

2.1.2 SURFACE CHARGE



Use Gauss' law to consider a "pill box" volume, i.e. cylindrical, on a conductor surface.

$$\therefore \oint_{pillbox} \vec{E}\cdot d\vec{S} = \underbrace{\iint_T \vec{E}\cdot d\vec{S}}_{\text{Non-zero integral}} + \underbrace{\iint_B \vec{E}\cdot d\vec{S}}_{=0} + \underbrace{\iint_{S1} \vec{E}\cdot d\vec{S}}_{=0} + \underbrace{\iint_{S2} \vec{E}\cdot d\vec{S}}_{=0}$$

Uniform electric field $\vec{E}$ points out through circular S.A. A. $\therefore \Phi_E = EA$

$\because \vec{E}$ in conductor $\Rightarrow \Phi_E = 0$

$\because \vec{E} \perp d\vec{S}$ $\Rightarrow \vec{E}\cdot\hat{n} = 0$

$\because \vec{E} = 0$ in conductor $\Rightarrow \Phi_E = 0$

Consequently, $\oint_{pillbox} \vec{E}\cdot d\vec{S} = EA$, where pillbox encloses total charge $\sigma A \Rightarrow E = \frac{\sigma}{\epsilon_0}$

2.1.3. AN ISOLATED, CHARGED CONDUCTING SPHERE

In 2.1.2, we only consider a flat surface. But what about a curved one?
Make the pillbox very small, s.t. $A \longrightarrow dA$. Therefore at small scales corresponding to dA, the surface element dS can be considered flat.
With an isolated conducting sphere of radius a, apply extra charge Q. This charge will redistribute within the conductor till the internal electric field is 0. So:

$\sigma = \frac{Q}{4\pi a^2}$. Using our result $E = \frac{\sigma}{\epsilon_0}$, we have $E = \frac{Q}{4\pi\epsilon_0 a^2}$ in agreement with 1.5.3.

2.1.4. ELECTROSTATIC SHIELDING

Inside an empty, arbitrarily-shaped cavity within a conductor, $\vec{E} = 0$. This is known as electrostatic shielding – the following argument can be made:

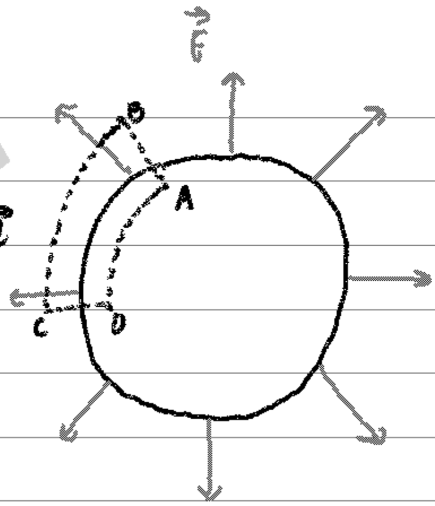
1. Consider closed surface S inside the conductor, containing the cavity.
   $\therefore \vec{E} = 0$ on this surface ($\because \vec{E} = 0$ inside the conductor). By Gauss' Law, $Q_{enc} = 0$.
2. Similarly, conclude net charge on inner surface $= 0$.
3. Positive and negative charges on different parts of the inner surface.
   - Electric field on the surface – charges redistribute till system is neutral.
   - $\oint \vec{E} \cdot d\vec{l} = 0$ Considering loop integral: $A \longrightarrow B$ in conductor, $B \longrightarrow A$ in cavity

   Therefore, we can write: $$\oint \vec{E} \cdot d\vec{l} = \underbrace{\oint_A^B \vec{E} \cdot d\vec{l}}_{\text{must equal } 0} + \underbrace{\oint_B^A \vec{E} \cdot d\vec{l}}_{= 0 \ \because \vec{E} = 0 \text{ inside the conductor}} = 0$$

   The only possibility: inner surface is an equipotential, neutral everywhere
4. No surface charge on inner surface, $\vec{E} = 0$ both inside cavity and conductor.

When an electric field is applied to a conductor which has a cavity, the cavity is shielded. This phenomenon is known as the 'Faraday Cage'.
Uses: MRI scan rooms, metallic-lined purses, defense & security.

2.1.5. Loop Integral Outside a Charged Conductor



$$\oint_{ABCD} \vec{E}\cdot d\vec{l} = 0 = \underbrace{\oint_{AB} \vec{E}\cdot d\vec{l}}_{+ve} + \underbrace{\oint_{BC} \vec{E}\cdot d\vec{l}}_{0} + \underbrace{\oint_{CD} \vec{E}\cdot d\vec{l}}_{-ve} + \underbrace{\oint_{DA} \vec{E}\cdot d\vec{l}}_{0}$$

$\oint_{AB}$: +ve (for path outside the conductor)

$\oint_{BC}$: 0 ($\vec{E}\cdot d\vec{l} = 0$ $\because \vec{E} \perp \widehat{BC}$)

$\oint_{CD}$: −ve (for path outside the conductor)

$\oint_{DA}$: 0 ($\vec{E} = 0$ inside the conductor)

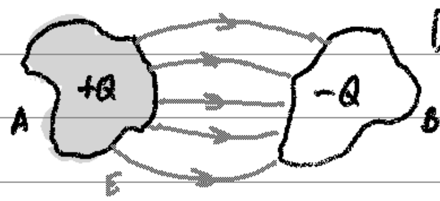
$\therefore \oint_{AB} \vec{E}\cdot d\vec{l} = -\oint_{CD} \vec{E}\cdot d\vec{l}$ $\therefore$ The charge density must be equal everywhere on the surface $\Rightarrow$ surface charge will spread out uniformly on surface

2.2 Capacitors

2.2.1 Ideal Capacitor

Consider an isolated pair of conductors, A and B, with equal and opposite charges $\pm Q$.
If isolated, all field lines from positive conductor end on the negative conductor.
Given potentials at each conductor, V_A and V_B, we have:

$$V = V_A - V_B$$



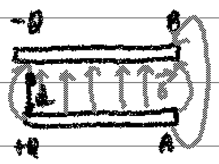
By the superposition principle, we have:

$$V \propto Q$$

$$\therefore \; Q = \underbrace{C}_{\text{capacitance}} V$$

Capacitors typically $C \ll 1F$, $Q \ll 1C$.

2.2.2 Parallel Plate Capacitor



Consider two large parallel plates, area A of charge $\pm Q$. Plates are separated by a distance $d \ll A$. Ignore edge effects to reduce to analysis of a 1D scenario.
Charge $\pm Q$ is spread uniformly over two inner plate surfaces.

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \Rightarrow V = V_A - V_B = Ed = \frac{Qd}{\varepsilon_0 A} \Rightarrow C = \frac{\varepsilon_0 A}{d}$$

$\therefore$ Capacitance depends on the area of the plates and their separation.

2.2.3 Energy Stored in a Capacitor

Capacitors store energy, as well as charges.

$$U = \frac{1}{2}\sum_i Q_i V_i \Rightarrow U = \frac{1}{2}QV_A - \frac{1}{2}QV_B \qquad \therefore U = \frac{1}{2}QV = \frac{1}{2}CV^2$$

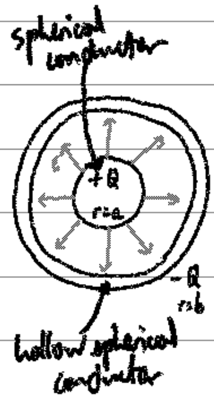
Given $Q = \varepsilon_0 AE$ and $V = Ed$,

$$\therefore U = \frac{1}{2}\varepsilon_0 AE(Ed) = \frac{1}{2}\varepsilon_0 AdE^2 \text{ where } Ad = \frac{\text{volume between}}{\text{the two plates}} = \frac{\text{volume occupied}}{\text{by the field } \vec{E}}$$

Therefore, we can define the energy density of $\vec{E}$, U_E, in the vacuum as: $U_E = \frac{1}{2}\varepsilon_0 E^2$

This energy is stored in the electric field between the plates.

2.2.4 Spherical Capacitors



Consider a capacitor consisting of two conductors: $\begin{cases} \text{a sphere, charge } +Q\text{, radius } a \\ \text{a sphere, charge } -Q\text{, radius } b \end{cases}$

Using Gauss' law applied to a surface between:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}. \qquad \text{For capacitors, we calculate } V = \overset{(+)}{V_a} - \overset{(-)}{V_b} = -\int_b^a \vec{E}\cdot d\vec{l}$$

$$\Rightarrow V = \int_a^b \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_a^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\therefore C = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

2.2.5 Capacitance of a Single-Charged Capacitor

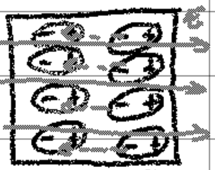
Considering an isolated, charged single conductor. As $b \to \infty$, we have:

$$C = \frac{4\pi\varepsilon_0}{1/a} = 4\pi\varepsilon_0 a$$

$\therefore$ Conductor evidently stores charge, energy is stored in electric field surrounding the conductor.

2.3 DIELECTRICS

2.3.1 POLARISATION



Consider the case where the dielectric is made of polar molecules (each with dipoles). Applying an electric field, dipoles rotate and align.
Consequently, there is a net negative surface charge on the left, and a net positive surface charge on the right edge. The centre of the dielectric cancels out ($Q_+ \approx Q_-$).
This argument applies for molecules of any material, even without intrinsic dipole moment.
In a dielectric electric field is reduced from:

$$\underbrace{E_D}_{\text{electric field in dielectric}} = \frac{\overbrace{E_0}^{\text{electric field in vacuum}}}{\underbrace{K}_{\text{dielectric constant}}}$$

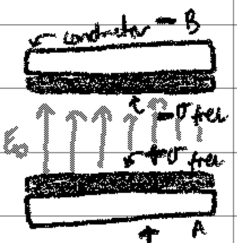
2.3.2 CHARGING DIELECTRIC & CONDUCTORS

Adding charge to a
- conductor : charges rearrange within conductor till equivalent Q is spread over the surface of the conductor
- dielectric : charges remain where placed

This is a major issue in electronics — energetic electrons from Van Allen radiation belts can impact and cause charging on geostationary communication satellites.
→ If charging very strong, deep dielectric discharge can occur damaging spacecraft.

2.3.3 DIELECTRICS IN CAPACITORS

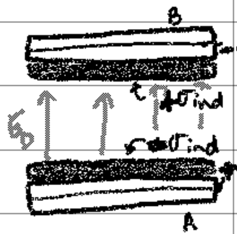
CHANGE IN ELECTRIC FIELD



Begin with two conducting plates separated by vacuum. There is a surface charge density $\pm\sigma_{free}$ on the inner side of each plate — the electric field between the plates is $\vec{E}_0$.

$$E_0 = \frac{\sigma_{free}}{\varepsilon_0}$$

Quickly filling the gap between the plates with a dielectric, the electric field $\vec{E}_0$ induces surface charges $\pm\sigma_{ind}$ on the dielectric with $\vec{E}_D$.



Solving $E_0 = KE_D$, we therefore have:

$$KE_D = \frac{\sigma_{free}}{\varepsilon_0} \Rightarrow E_D = \frac{\sigma_{free}}{K\varepsilon_0} = \frac{\sigma_{free}}{\varepsilon}$$

permittivity $= K\varepsilon_0$ (or $\varepsilon_r \cdot \varepsilon_0$)

Applying Gauss' law to the new 'pillbox' shape, we have flux out of this volume is $E_D A$ given a circular surface area of A, and $Q_{enc} = A(\sigma_{free} - \sigma_{ind})$.

$$\therefore E_D A = \frac{A(\sigma_{free} - \sigma_{ind})}{\varepsilon_0} \Rightarrow E_D = \frac{\sigma_{free} - \sigma_{ind}}{\varepsilon_0} \Rightarrow \frac{\sigma_{free}}{K\cancel{\varepsilon_0}} = \frac{\sigma_{free} - \sigma_{ind}}{\cancel{\varepsilon_0}} \quad \therefore \sigma_{free} = K(\sigma_{free} - \sigma_{ind})$$

$$\therefore \sigma_{ind} = \sigma_{free}\left(1 - \frac{1}{K}\right)$$

CHANGE IN CAPACITANCE

Occurs upon the presence of a dielectric. First, the capacitor is charged, and isolate the capacitor without a dielectric (i.e. a vacuum). Electric field $\vec{E_0}$ is inside the capacitor with charges $\pm Q_{free}$ on the plates.

Adding the dielectric, electric field is now $E_D = \frac{E_0}{K}$:

$$V_D = E_D d = \frac{E_0}{K} d = \frac{V_0}{K}.$$ The voltage drop decreases.

We can calculate the capacitance where initially: $C = \frac{Q_{free}}{V_0}$.

Q_{free} remains the same as the system is isolated, therefore new capacitance:

$$C_D = \frac{Q_{free}}{V_D} = \frac{K Q_{free}}{V_0} = KC$$ $\therefore$ capacitance increases by factor 'K' when the dielectric is inserted

i.e. the capacitor can hold the 'same charge' with a 'smaller voltage'

CHANGE IN ENERGY STORAGE

Depends on dielectric. Dielectric-filled capacitor stores more energy per given voltage.

$$U = \frac{1}{2} C_D V^2 = \frac{1}{2} K C_0 V^2 = K U_0 \qquad U_0 = \frac{1}{2} C_0 V^2$$

2.4 CURRENT & RESISTANCE

2.4.1 CURRENT DENSITY

Consider a material with many charged species: $1, 2, \dots, i, \dots, N$, each with a number density of n_i and charge q_i. Assume each species coherent moves with $\vec{v}_i$.

$\therefore$ current density, $\vec{J} = \sum_{i=1}^{N} n_i q_i \vec{v}_i$

It is often useful to reduce $\vec{J}$ to a statement about current I, which is the amount of charge passing through a surface per unit time: $I = \int_S \vec{J} \cdot d\vec{S}$

N.B: Current as a scalar must be used with care. Many assumptions and can cause misunderstandings. Not flexible enough to deal with many physical systems.

2.4.2. CONSERVATION OF CHARGE

If Q is the charge in volume V, and I is total current out of the volume:

$I = -\frac{dQ}{dt} = \oint_S \vec{J} \cdot d\vec{S}$ where S is a closed surface containing V.

By divergence theorem: $I = \oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J} \, dV$

Assuming charge is distributed continuously throughout the volume, we have:

$\therefore \frac{\partial}{\partial t} \int_V \rho \, dV + \int_V \nabla \cdot \vec{J} \, dV = 0 \quad \Leftarrow \quad Q = \int_V \rho \, dV$

Considering an infinitesimal volume dV:

$\frac{\partial \rho}{\partial t} dV + \nabla \cdot \vec{J} \, dV = 0 \xRightarrow{dV \text{ is finite}} \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

charge conservation in differential form

2.4.3. STEADY CURRENT & OHM'S LAW

Consider a conducting wire, cross-section A, length l, applying an electric field $\vec{E}$. Assuming a steady current density $\vec{j}$, uniform across the wire:

$$\underbrace{\vec{E} = \eta\vec{j}}_{\text{resistivity / } \Omega m} \qquad \text{Conductivity, } \sigma = \frac{1}{\eta}, \quad \therefore \vec{j} = \sigma\vec{E} \quad (\text{Ohm's Law})$$

Integrating along the length of the wire, we have:

$$\int \vec{E}\cdot d\vec{l} = \eta \int \vec{j}\cdot d\vec{l} \implies V = \eta \vec{j}\, l = \frac{\eta l}{A} I \Rightarrow V = IR$$

potential difference between two ends

$I = jA$ where A is cross-sectional area of wire

define as resistance $R = \frac{\eta l}{A}$

2.4.4. JOULE HEATING

In metallic conducting wire, treat e^- as moving and ions as at rest.

e^- are accelerated by $\vec{E}$, then decelerated by collisions with ion lattice.

→ on average, electrons move along with velocity $\bar{v}_e$ along the wire.

→ rate at which the applied force (due to $\vec{E}$) does work on each e^- is $\vec{F}\cdot\bar{v}_e$.

The power put into e^- per unit volume is:

$$\vec{F}\cdot\vec{\bar{v}}_e = n(-e\vec{E})\vec{\bar{v}}_e = \vec{j}\cdot\vec{E} = \eta j^2$$

number density e^- per unit V

For a conductor: length l, area A, total heating rate (power input) P is:

$$P = (\eta j^2) A l = \eta\left(\frac{I}{A}\right)^2 A l = \frac{\eta \cdot l}{A} I^2 = I^2 R$$

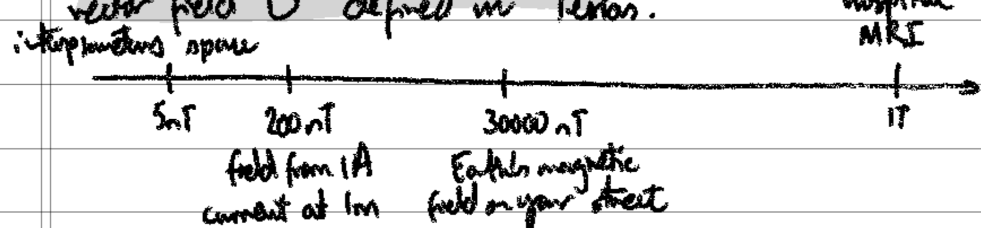
Vector Fields, Electricity & Magnetism

MAGNETISM
磁気

目次

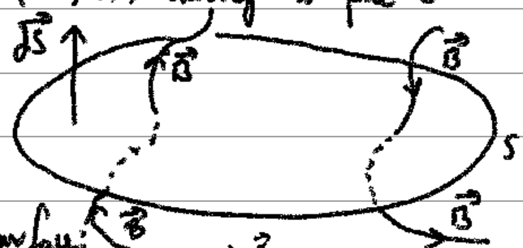
1 Magnetic Fields

vector field $\vec{B}$ defined in Teslas.



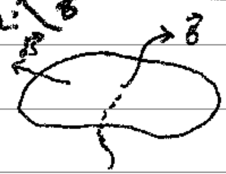
Define magnetic flux Φ (Wb, Tm^2) through surface S:

open surface $\Phi = \iint_S \vec{B} \cdot d\vec{S}$



Φ, the amount of flux 'threading' through the surface S.

Empirically, in a closed surface:



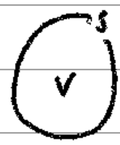
$$\oiint \vec{B} \cdot d\vec{S} = 0$$

closed

∴ every field line entering a closed volume leaves it again

N.B: Every field line is a loop, without a beginning or end (!)

For general field C, by divergence theorem, we have: $\int_V \nabla \cdot \vec{C}\, dV = \oiint_S \vec{C} \cdot d\vec{S}$



where surface S encloses volume V

∵ $\oiint \vec{B} \cdot d\vec{S} = 0$ for every S, we can make $S \to 0$

∴ $\nabla \cdot \vec{B} = 0$ (a Maxwell Equation) "No magnetic monopoles" $\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

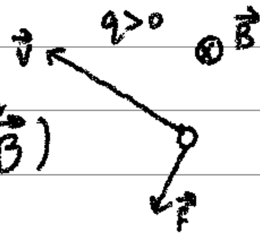
∴ Magnetic fields generated by charges in motion.

Forces on Moving Charges

⚠ Use RH rule

Charges in motion experience the Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

charge, velocity, magnetic field



→ $\vec{F}$ depends on $\vec{v}$ $\Rightarrow$ not conservative force

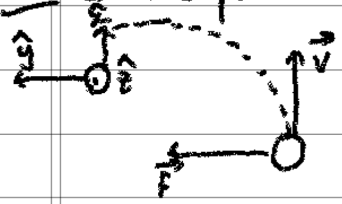
→ $\vec{F} \perp \vec{B}$ ∴ $\vec{B}$ not a line of force

→ Work done $dW = \vec{F} \cdot d\vec{r}$. ∵ $\vec{F} = \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} \times \vec{B} \perp d\vec{r}$ ∴ $\vec{F} \cdot d\vec{r} = 0$

$\Rightarrow$ magnetic fields do no work.

→ K.E. of a particle is unchanged, but it can accelerate.

Ex.1 Consider particle moving $\perp \vec{B}$ in a uniform magnetic field:



$\odot \vec{B}$

$q<0$ ⚠ N.B sign (!)

$\rightarrow \vec{F}$ is always $\perp \vec{v}$ and $\perp \vec{B}$

$\rightarrow |\vec{F}|$ is constant as $|\vec{v}|$ and $|\vec{B}|$ constant

$\Rightarrow$ circular motion

Ex.2 For a cyclotron, $F = m\omega^2 r = Bqv$, $v = r\omega$

$\Rightarrow \omega = \frac{qB}{m}$ $\rightarrow$ Cyclotron frequency independent of v and r

$\rightarrow$ all particles of same mass and charge $\Rightarrow$ same period of gyration

$\Rightarrow r = \frac{mv}{qB}$ radius of gyration (Larmor radius)

Gyration is in $\hat{x}$-$\hat{y}$ plane $\rightarrow$ what about motion in $\hat{z}$ plane?

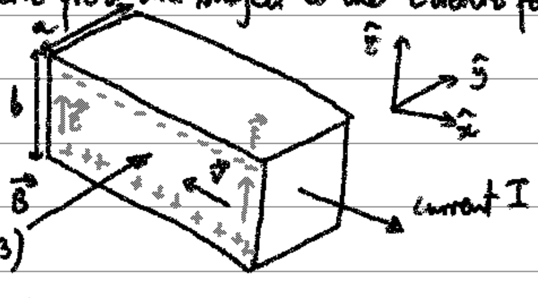
sense of gyration depends on sign of charge

$\rightarrow$ no force in this direction $\therefore$ speed unchanged

$\rightarrow$ if $v_z \neq 0$, we have helical motion.

Hall Effect

Charges moving in conductors in a magnetic field are subject to the Lorentz force – that induces a voltage across the conductor.



- Cross-sectional area: $A = ab$
- Current Density: $\vec{J} = \frac{I}{A}\hat{x}$ (in $A.m^{-2}$)
- Charge Carrier Number Density: n (m^{-3})

(N.B: in metals, $q<0$ i.e. electrons)

$\therefore \vec{J} = n\vec{v}q = \frac{I}{A}\hat{x}$, so velocity of charges $\vec{v} = \frac{I}{Anq}\hat{x}$

$$\boxed{\begin{aligned}\vec{E} &= -\vec{v}\times\vec{B} \\ \Rightarrow V &= \vec{E}d\end{aligned}}$$

Lorentz Force: $\vec{F} = q(E + \vec{v}\times\vec{B})$ deflects charges (in $\hat{z}$ direction for e^-)

$\rightarrow$ -ve charges deflect upwards, so there is a charge separation: more e^- at top, fewer e^- at bottom $\Rightarrow \vec{E}$ acts against deflection of e^-

Equilibrium when force on electrons in $\hat{z}$ direction is 0:

$\vec{F} = 0 \Rightarrow q(\vec{E} + \vec{v}\times\vec{B}) = 0 \Rightarrow \vec{E} = -\vec{v}\times\vec{B} = \frac{IB}{Anq}\hat{z}$

$\vec{E}d = V$ is voltage across the conductor due to an applied magnetic field

Forces on Wires

number density, cross-sectional area, length

$Q_{total} = q n A l$, $v = \vec{v}_d$ $\vec{F} = \vec{q}(\vec{v} \times \vec{B})$

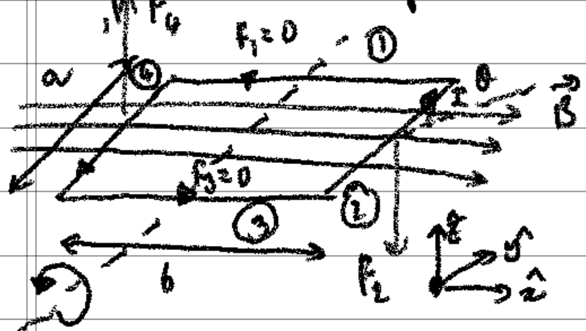
length vector in direction of $\vec{v}_d$

$$\Rightarrow \vec{F} = Q_{total}(\vec{v}_d \times \vec{B}) = qnAlv_dB = (qnAv_d)\vec{l} \times \vec{B} = I(\vec{l} \times \vec{B})$$

$$\therefore \vec{F} = I \int d\vec{s} \times \vec{B}$$

NB: For a closed loop, $\oint d\vec{s} = 0 \therefore \vec{F}_B = 0$

Torque on Current Loops



$$\vec{F}_{total} = \sum_{i=1}^{4} F_i = F_2 + F_4 = IaB\sin\theta\,\hat{k} - IaB\sin\theta\,\hat{k} = 0$$

$$\vec{\tau}_y = \vec{r} \times \vec{F} = -\frac{b}{2}\hat{x} \times (IaB\sin\theta\,\hat{z}) + \frac{b}{2}\hat{z} \times (-IaB\sin\theta\,\hat{z})$$

$$\vec{\tau} = bIaB\sin\theta\,\hat{y}$$

But, ab = Area $\therefore \vec{\tau} = I(\vec{A} \times \vec{B})$

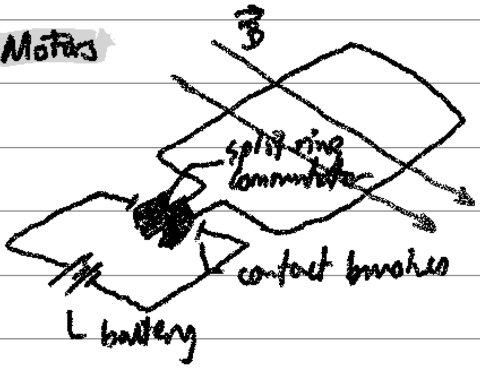
magnetic dipole moment

$\vec{\mu} = I\vec{A} \Rightarrow \vec{\tau} = \vec{\mu} \times \vec{B}$

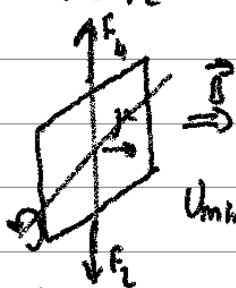
let $U_0 = 0$ — magnetic field potential energy

$$W = \int_{\theta_0}^{\theta} \tau \, d\theta' = \int_{\theta_0}^{\theta} \mu B \sin\theta' \, d\theta' = \mu B[\cancel{\cos\theta_0} - \cos\theta] \Rightarrow \overbrace{U = -\mu B\cos\theta}^{\text{magnetic field potential energy}}$$

Motors



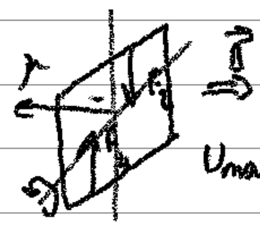
When $\theta = \pi/2$



$U_{min} = -\mu B$

parallel, stable equilibrium

$\theta = \frac{3\pi}{2}$



$U_{max} = +\mu B$

antiparallel, unstable equilibrium

Magnetic Force on a Dipole

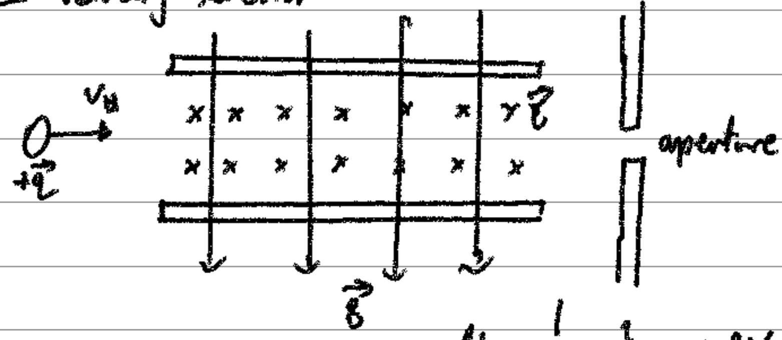
$\vec{F}_B = \nabla(\vec{\mu} \cdot \vec{B})$

non-uniform (?)

dipole

FORCES ON MOVING CHARGES

Ex.3 velocity selector



selected velocity with 0 deflection

$$\vec{F} = \vec{q}\,(\vec{E} + \vec{v} \times \vec{B})$$

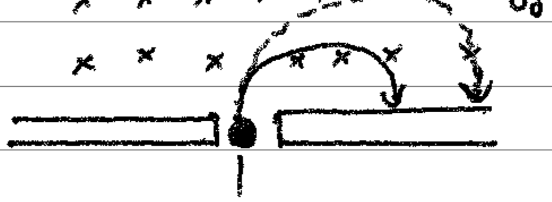
$$0 = \vec{q}\,(\vec{E} + v_x B)$$

$$\therefore v = -\frac{E}{B}$$

Also, $\frac{1}{2}mv^2 = q\Delta V \Rightarrow v = \sqrt{\frac{2q\Delta V}{m}}$

$$\therefore -\frac{E^2}{B^2} = \frac{2q\Delta V}{m} \quad \therefore \frac{m}{q} = -\frac{B^2}{E^2\, 2\Delta V}$$

Ex.4 Bainbridge mass spectrometer

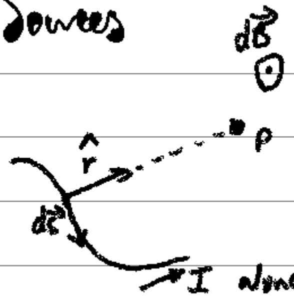


$$\frac{mv^2}{r} = B_0 qv \Rightarrow r = \frac{mv}{B_0 q}$$

$$\therefore r = \frac{m\frac{E}{B}}{B_0 q} \Rightarrow m = \frac{B B_0 q r}{E}$$

2 Magnetic Fields Sources

Biot-Savart Law.



$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}$$

field at point P

infinitesimal current source: $I\,d\vec{s}$

$\rightarrow I$ along conducting wire

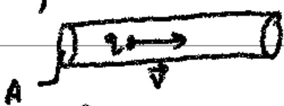
μ_0 – permeability of free space: $4\pi\times10^{-7}\ \text{T}\cdot\text{m/A}$

$$\vec{B} = \int_{wire} d\vec{B} = \frac{\mu_0 I}{4\pi}\int_{wire}\frac{d\vec{s}\times\hat{r}}{r^2}$$

⚠ N.B. vector integral, 3 integrals for each component of $\vec{B}$

Ex.1 and Ex.2 next page

Magnetic Field of Moving Point Charge



'n' charge carriers per unit volume

$\therefore I = nAq|\vec{v}|$

↑ current element, dI ∴ total charges: $dN = nA\,ds$

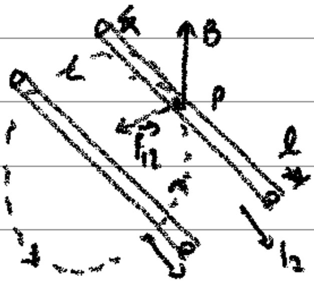
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{(nAq|\vec{v}|)\,d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{dN\,q\,\vec{v}\times\hat{r}}{r^2} \quad \therefore \vec{B} = \frac{\mu_0}{4\pi}\frac{Nq\,\vec{v}\times\hat{r}}{r^2}$$

$\because d\vec{s}\,/\!/\,\vec{v}$

$$\therefore B = \frac{\mu_0}{4\pi}\frac{q(\vec{v}\times\hat{r})}{r^2} \text{ for 1 charge}$$

$\left(\text{N.B.: } \frac{\hat{r}}{r^2} = \frac{\frac{|\vec{r}|}{r}}{r^2} = \frac{\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}x_i\\y_i\\z_i\end{pmatrix}}{r^3}\right)$

Force Between 2 // Wires



$\vec{v}\times\vec{B}$
$\Downarrow$
$/\!/ I$

at point P, $\vec{B} = \frac{-\mu_0 I_2}{2\pi a}\hat{j}$ (see Biot-Savart Law)

$$\vec{F} = I(\vec{\ell}\times\vec{B}) = I_1(\ell\hat{i})\times\left(\frac{-\mu_0 I_2}{2\pi a}\hat{j}\right) = -\frac{\mu_0 I_1 I_2 \ell}{2\pi a}\hat{k}$$

Force on Wire 2 by 1

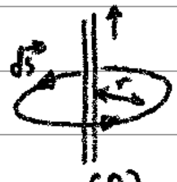
∴ attractive when I_1, I_2 //

Ampere's Law

– line integral of $\oint\vec{B}\cdot d\vec{s}$ around any closed Amperian loop is proportional to current enclosed in the loop

$$\oint_{closed\ loop}\vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

⚠ Symmetry (e.g. cylindrical), otherwise use BS law



In dir($\hat{\phi}$)

e.g. infinite straight wire

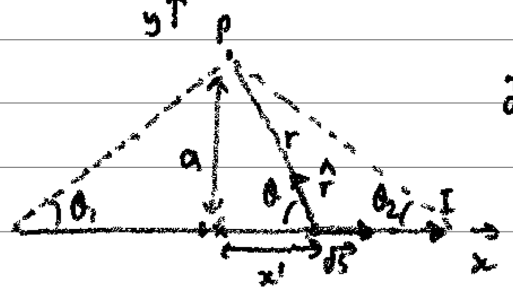
$$\vec{B} = \frac{\mu_0 I}{2\pi r}$$

$$d\vec{s} = dr\,\hat{r} + r\,d\varphi\,\hat{\varphi} + dz\,\hat{z}$$

$$\therefore \oint_{closed\ loop}\vec{B}\cdot d\vec{s} = \int_0^{2\pi}\frac{\mu_0 I}{2\pi \cancel{r}}\cancel{r}\,d\phi = \mu_0 I$$

BIOT–SAVART'S LAW

Ex.1 Force along a straight conducting wire



$x' = r\cos\theta\,\hat{i}$ $P: a\hat{j}$ Find source point & field point

$d\vec{s} = dx' = -r\sin\theta\,\hat{i}$

$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{a\hat{j} - x\hat{i}}{r} = \sin\theta\,\hat{j} - \cos\theta\,\hat{i}$ Find $\hat{r}$

$$d\vec{s}\times\hat{r} = \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ dx\sin\theta \end{pmatrix} = dx\sin\theta\,\hat{k}$$

$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{dx\sin\theta}{r^2}\,\hat{k}$ We know: $r = \frac{a}{\sin\theta}$, $r^2 = \frac{a^2}{\sin^2\theta} = a^2\csc^2\theta$

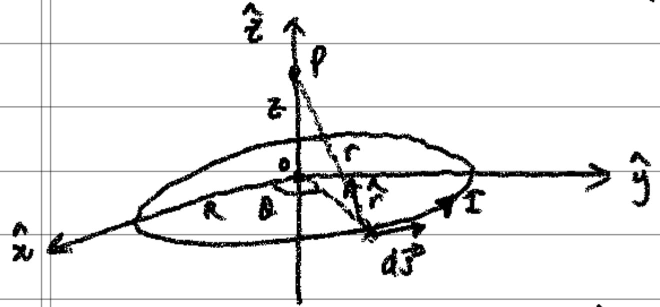
$\tan\theta = \frac{a}{x}$, $x = a\cot\theta$, $dx = -a\csc^2\theta\,d\theta$

$\vec{B}_k = \frac{\mu_0 I}{4\pi}\int \frac{-a\,\cancel{\csc^2\theta}\,\sin\theta}{\cancel{a^2\csc^2\theta}}\,d\theta$

(infinite length)
when $\theta_1, \theta_2 \to 0$

$= \frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\,d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)$ $\vec{B} = \frac{\mu_0 I}{2\pi a}$

Ex.2 Force around circular current loop



$\vec{r} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$

$d\vec{r} = -R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j}$ $d\vec{s} = \vec{r}\,d\theta$

$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{z\hat{k} - R\cos\theta\,\hat{i} - R\sin\theta\,\hat{j}}{\sqrt{R^2+z^2}}$

$$d\vec{s}\times\hat{r} = \frac{d\theta}{r}\begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -R\cos\theta \\ -R\sin\theta \\ z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} zR\cos\theta \\ zR\sin\theta \\ R^2 \end{pmatrix} d\theta = \frac{R}{r}\begin{pmatrix} \cos\theta \\ \sin\theta \\ R \end{pmatrix} d\theta$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{R(\cos\theta\,\hat{i} + \sin\theta\,\hat{j} + R\hat{k})}{r^3}\,d\theta$$

Integrate separately: w.r.t. $\hat{x}, \hat{y}, \hat{z}$:

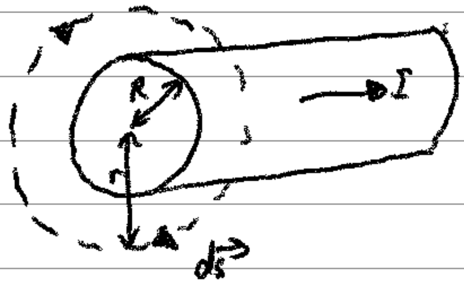
symmetry in x-y

$d\vec{B}_x = \frac{\mu_0 IR}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\cos\theta\,d\theta = 0$; $d\vec{B}_y = \frac{\mu_0 IR}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\theta\,d\theta = 0$

$d\vec{B}_z = \frac{\mu_0 IR^2}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}d\theta = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$ $\therefore \vec{F} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$

AMPERE'S LAW

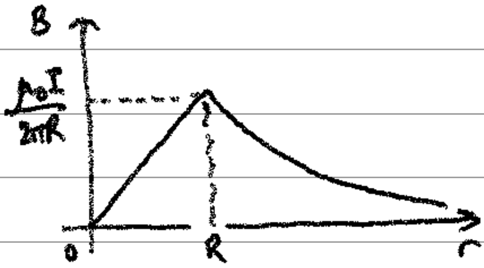
Ex.3 field of infinite straight wires carrying steady current, I



Outside of conductor:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I$$

$$\Rightarrow \vec{B}\oint d\vec{s} = \vec{B}(2\pi r) = \mu_0 I \quad \therefore \vec{B} = \frac{\mu_0 I}{2\pi r}$$

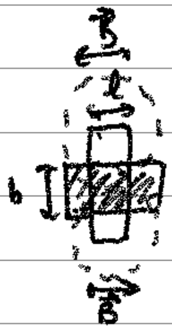


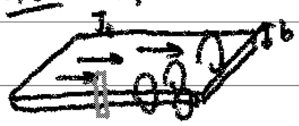
Inside of conductor:

$$I_{enc} = \frac{\pi r^2}{\pi R^2}$$

$$\Rightarrow \vec{B}(2\pi r) = \mu_0 \frac{\pi r^2}{\pi R^2} I \Rightarrow \vec{B} = \frac{\mu_0 I r}{2\pi R^2}$$

Ex.4 infinite current sheet, current density J_0, width b

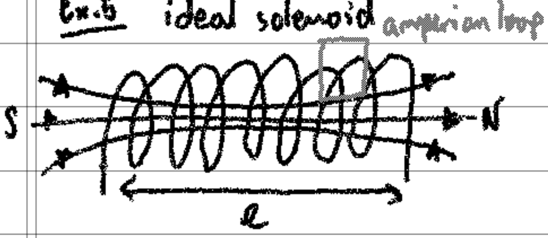




$$I_{enc} = \iint \vec{J}\cdot d\vec{A} = J_0(bl) \qquad \oint \vec{B}\cdot d\vec{s} = \mu_0 J_0 b l = B2l$$

$$\therefore \frac{\mu_0 J_0 b}{2} = \vec{B}$$

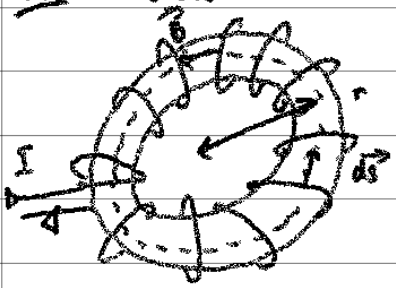
Ex.5 ideal solenoid



$$\oint \vec{B}\cdot d\vec{s} = Bl = \mu_0 \overbrace{NI}^{I_{enc}\text{ for }N\text{ loops}}$$

$$\therefore \vec{B} = \frac{\mu_0 N I}{l}$$

Ex.6 toroid with N turns

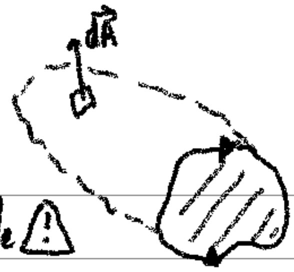


$$\oint \vec{B}\cdot d\vec{s} = B\oint d\vec{s} = B(2\pi r) = \mu_0 \overbrace{NI}^{\text{surface enclosed by }N\text{ loops}}$$

$$\therefore \vec{B} = \frac{\mu_0 N I}{2\pi r}$$

clockwise — dA $\otimes$

anti-clockwise — dA $\odot$



Gauss: 10^{-4} T

Faraday's Law : use RH corkscrew rule ⚠

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

closed loop — open surface

$$\varepsilon = -N\frac{d\phi}{dt}$$

for N loops

where $\phi = \vec{B}\cdot\vec{A} = \vec{B}\vec{A}\cos\theta$

$\Rightarrow \phi = \iint_S \vec{B}\cdot d\vec{A}$

↳ Kirchoff's Law is special case when $\frac{d\phi}{dt} = 0$

Lenz's Law - current induced in magnetic field that opposes change

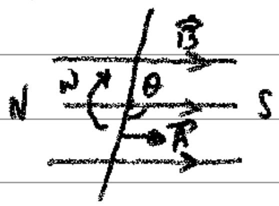
induce an e.m.f by
- → varying $|\vec{B}|$ over time $\frac{dB}{dt}$
- → varying $|\vec{A}|$ over time $\frac{dA}{dt}$
- → varying θ between $\vec{B}$ and $\vec{A}$ $\frac{d\theta}{dt}$

Motional E.M.F - induced if a conductor moves in magnetic field

$\varepsilon = \oint (\vec{v}\times\vec{B})\cdot ds$ for a conducting bar length l at constant $\vec{v}$,

$$\vec{\varepsilon} = -Bvl$$

Generators



$$\phi = \vec{B}\cdot\vec{A} = BA\cos\theta = BA\cos(\omega t)$$

$$\frac{d\phi}{dt} = -BA\omega\sin(\omega t)$$

N turns in the loop, total e.m.f generated across the two ends of loop

$$\varepsilon = -N\frac{d\phi}{dt} = NBA\omega\sin(\omega t)$$

$$I = \frac{|\varepsilon|}{R} = \frac{NBA\omega}{R}\sin(\omega t) \qquad P = I|\varepsilon| = \frac{(NBA\omega)^2}{R}\sin^2(\omega t)$$

torque ↳

$$\tau = \mu B\sin\theta = \mu B\sin(\omega t) \qquad P = \tau\omega = \mu B\omega\sin(\omega t)$$

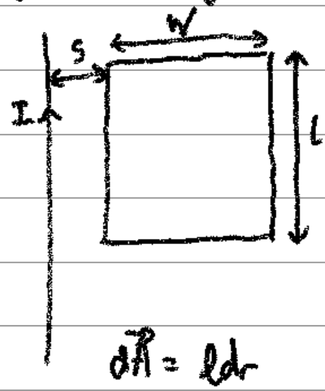
mechanical power to rotate loop

$$\mu = NIA = \frac{N^2A^2B\omega}{R}\sin(\omega t)$$

dipole moment

FARADAY'S LAW

Ex.7 Rectangular coil from a wire



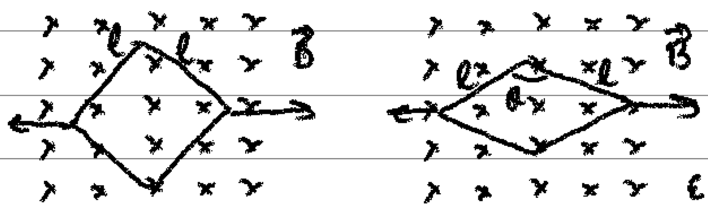
$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I \qquad \vec{B} = \frac{\mu_0 I}{2\pi r}$$

$$\therefore \Phi = \oint \vec{B}\cdot d\vec{A} = \frac{\mu_0 I l}{2\pi}\int_s^{s+w} \frac{dr}{r} = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{s+w}{s}\right)$$

$d\vec{A} = l\,dr$

$$\varepsilon = \frac{-d\Phi}{dt} = \frac{-\mu_0 l}{2\pi}\ln\left(\frac{s+w}{s}\right)\cdot\frac{dI}{dt}$$

Ex.8 Loop changing area



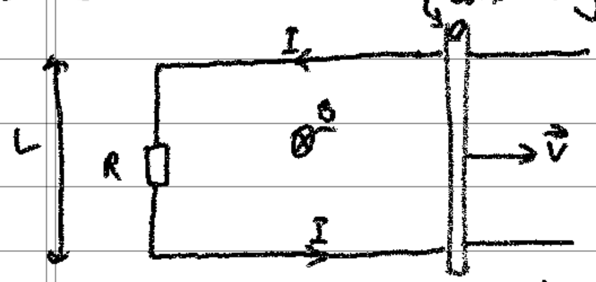
$$\Phi = BA = Bl^2\sin\theta$$

$$d\Phi = -Bl^2(1-\sin\theta)$$

$$\varepsilon = -\frac{d\phi}{dt} = +\frac{Bl^2(1-\sin\theta)}{\Delta t}$$

$$\therefore I = \frac{Bl^2(1-\sin\theta)}{R\Delta t}$$

Ex.9 Slidewire Generator — conducting rod is pushed at $\vec{v}$



$$\therefore \frac{d\vec{A}}{dt} = vL$$

$$\Rightarrow \frac{d\Phi}{dt} = \vec{B}\cdot\frac{d\vec{A}}{dt} = -vLB$$

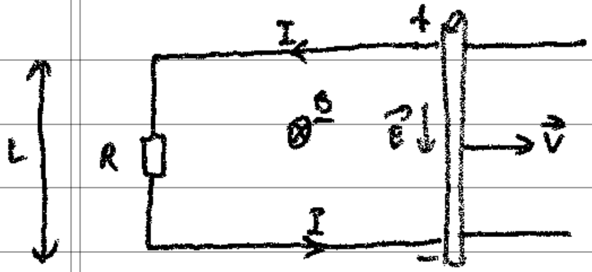
$$\therefore \varepsilon = -\frac{d\Phi}{dt} = +vLB \Rightarrow I = \frac{vLB}{R}$$

$$\circ\ P = I^2R = \frac{v^2L^2B^2}{R^2}R = \frac{v^2L^2B^2}{R}$$

Rod experiences Lorentz force due to flow of current:

$$F = I\vec{L}\times\vec{B} = \frac{BLv}{R}LB = \frac{L^2B^2v}{R} \Rightarrow P = Fv = \frac{B^2L^2v^2}{R} \text{ as before}$$

Motional Electric Fields



Consider Faraday's law to analyse charge as the rod moves:

$$\vec{F} = \vec{q}\,(\vec{E} + \vec{v} \times \vec{B})$$

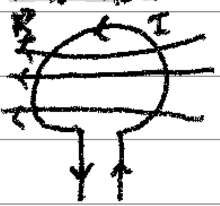
Electrons experience $\vec{F_e}$ downwards by RH rule, ∴ electrons build up on bottom

Charge imbalance => equal & opposite force to $\vec{F_e}$ (like Hall Effect)

∴ $\vec{E} = -\vec{v} \times \vec{B}$ ⇒ $\mathcal{E} = ED = -vBL$ along a rod

$I = \frac{\mathcal{E}}{R} = \frac{vLB}{R}$ around the circuit as before

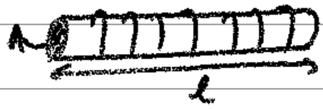
Inductors



Define: $\Phi = LI$ (inductance / H)
→ dependent on shape of loop
→ size / number of loops (↑ L)

∴ $\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

Inertia in electrical circuits, as inductors resist changes in current by producing a back e.m.f.



$\vec{B} = \frac{\mu_0 N I}{l}$ in a solenoid, 0 outside

∴ $\Phi = NBA = \frac{\mu_0 N^2 A I}{l}$ ⇒ $L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A I}{l I} = \frac{\mu_0 N^2 A}{l}$ where $l \gg A$

Energy Density of a Magnetic Field

suppose: inductor carries current I

$U = \frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_0 N^2 A}{l} I^2$ where $I = \frac{Bl}{\mu_0 N}$

∴ $U = \frac{1}{2}\frac{\mu_0 N^2 A}{l} B^2 \frac{l^2}{\mu_0^2 N^2} = \frac{1}{2}\frac{B^2}{\mu_0} Al$ where Al is volume of solenoid

⇒ B is constant inside, and 0 outside ∴ $\frac{B^2}{2\mu_0}$ is energy density

Maxwell's Equations

integral form

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot ds = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot ds = \mu_0 I + \underbrace{\varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}}_{\text{"displacement current"}}$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$] volume charge density

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$